N = 4 topological strings

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N=4 Topological Strings

Nathan Berkovits

Mathematics Department, King's College
The Strand, London, WC2R 2LS, UK

Cumrun Vafa

Lyman Laboratory of Physics, Harvard University
Cambridge, MA 02138, USA

We show how to make a topological string theory starting from an $N = 4$ super-conformal theory. The critical dimension for this theory is $\hat{c} = 2$ ($c = 6$). It is shown that superstrings (in both the RNS and GS formulations) and critical $N = 2$ strings are special cases of this topological theory. Applications for this new topological theory include: 1) Proving the vanishing to all orders of all scattering amplitudes for the self-dual $N = 2$ string with flat background, with the exception of the three-point function and the closed-string partition function; 2) Showing that the topological partition function of the $N = 2$ string on the $K3$ background may be interpreted as computing the superpotential in harmonic superspace generated upon compactification of type II superstrings from 10 to 6 dimensions; and 3) Providing a new prescription for calculating superstring amplitudes which appears to be free of total-derivative ambiguities.

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1. Introduction

Topological quantum field theories in two dimensions have become increasingly important over the past few years. Their first appearance in string theory came from viewing $N = 2$ topological conformal theories as special vacua of bosonic strings which combine the ghosts with the matter fields in a non-trivial way, and where one of the twisted supercharges of the $N = 2$ theory plays the role of the $BRST$ operator Q and the other plays the role of the anti-ghost b. The critical case for this type of 'vacuum' for the bosonic string is when the $N = 2$ central charge is $\hat{c} = 3$ [1] . The most well-known examples of these theories come from twisting sigma models on Calabi-Yau threefolds, which was studied extensively in [2]. It was found there that the string field theory interpretation for these theories corresponds to a theory of gravity in three complex dimensions which was called the Kodaira-Spencer theory of gravity. The partition functions of this theory have the further property that they can be viewed as computing [2][3] the superpotential terms of the four-dimensional theory which would be obtained upon compactification of superstrings on the corresponding Calabi-Yau. This gave a physical reinterpretation for the meaning of $N = 2$ topological theories as isolating the F-term computations for standard superstring compactifications. Non-unitary models of $N = 2$ topological theories with $\hat{c} = 3$ are also interesting (one such model is equivalent to the $c = 1$ non-critical string [4]).

In a previous paper [5] , we used topological theories in a reverse way to show the equivalence of various strings: Roughly speaking, we untwisted the supersymmetry of the critical bosonic string and found that it has a superconformal symmetry with central charge corresponding to the critical dimension of $N = 1$ superstrings. We then coupled it to $N = 1$ supergravity and found that the theory is equivalent to the original bosonic string. In this way, we managed to imbed bosonic string vacua in a special class of vacua for $N = 1$ superstrings. Similarly, by 'untwisting' the critical $N = 1$ superstring (shifting some ghost spins), it was found to have an $N = 2$ superconformal symmetry which was then coupled to $N = 2$ supergravity and shown to be equivalent to the original $N = 1$ superstring. This gave an imbedding of $N = 1$ strings into $N = 2$ strings, suggesting that $N = 2$ strings have a very special role among all the strings. There has been further work along these directions [6].

The simplest vacuum of $N = 2$ strings corresponds to string propagation on a self-dual background in four dimensions with signature (2,2) or with Euclidean signature (4,0) [7]. The only degree of freedom is a scalar describing the Kähler class satisfying the Plebanski

equation. As was noted in [7] , this theory has many features in common with what one would expect of a topological theory. However attempts failed to fit it into the $N = 2$ topological theory coupled to gravity since the critical dimension for $N = 2$ topological theories is $\hat{c} = 3$, whereas here we need a critical dimension of $\hat{c} = 2$.

Our main goal in this paper is to show that there is a new class of topological theories based on the $N = 4$ superconformal algebra which includes the $N = 2$ strings, thus providing the elusive topological reformulation of $N = 2$ strings. The critical dimension for this theory turns out to be $\hat{c} = 2$.[1] To view the $N = 2$ strings as a special case of this string, one starts by noting that $N = 2$ matter with $\hat{c} = 2$ (which is the critical dimension of $N = 2$ strings) always has in addition an $N = 4$ superconformal symmetry. The matter piece with $N = 4$ is then twisted and the $N = 2$ ghosts are stripped off. Applying the rules of computation which we will define for this new $N = 4$ topological theory, one recovers the results for the $N = 2$ string amplitudes.

In a sense, this is reversing the arrow of imbedding that we had previously found for imbedding $N = 1$ strings into $N = 2$ strings. Namely, when applied to RNS strings imbedded into $N = 2$ strings, the steps we just mentioned give back the RNS field content untouched. Therefore, all we have to do is show that the rules of computation for RNS strings are the same as the rules of the $N = 4$ theory (note that RNS strings have a twisted $N = 4$ superconformal symmetry which follows from the fact that they have an $N = 2$ symmetry [5] and has $\hat{c} = 2$ with integral charges), and this is easily done. Moreover, it will be seen that this $N = 4$ reformulation of the amplitudes gives a definite prescription for computation which may resolve the ambiguity question for RNS strings.

There are further applications for this topological theory: First of all, it enables us to prove various vanishing theorems for the $N = 2$ strings on the flat (2,2) background. Secondly, if we consider $N = 2$ strings on $K3$, one would expect (since it is a topological theory and in analogy with the Calabi-Yau threefold case) to be computing superpotential terms for the six-dimensional theory obtained by compactifying superstrings on $K3$. This expectation is borne out, but moreover the superpotential terms that one computes *are automatically in the harmonic superspace* (as would be required for supersymmetry in six dimensions).

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[1] Other papers [8] have recently appeared on the subject of topological theories with $\hat{c} = 2$, however at the present time, we do not see a direct connection between these papers and our work.

The organization of this paper is as follows: In section 2, we introduce the new topological theory based on $N = 4$ superconformal symmetry. It is shown there that one obtains not just a single partition function F_g at each genus, but $(4g - 4) + 1$ of them assembled in the form of the coefficients of the homogeneous polynomial $F_g(u_1, u_2)$ of degree $4g - 4$. The u's are related to how we choose our $N = 2$ within the $N = 4$ algebra to define the string amplitudes, and are the usual harmonic superspace vielbeins. They are parametrized by the group $SU(2)$, and the coefficients of F_g tranform as a spin $j = 2g - 2$ representation of this $SU(2)$. We also discuss the notion of harmonicity of the partition function for the topological theory, which is a natural analog of the notion of holomorphicity for $N = 2$ topological theories (which was found to be anomalous [2]).

In section 3, we show how $N = 2$ strings are equivalent to this topological theory. We will find that there are $(4g - 4) + 1$ values of $N = 2$ instanton number where the partition function does not vanish and these are in one to one correspondence with the $(4g - 4) + 1$ partition functions computed using the topological prescription. Similar results are true for the correlation functions.

In section 4, we apply the topological prescription to $N = 2$ strings with self-dual backgrounds in four dimensions. We first consider the flat $R^{2,2}$ background and show that except for three point amplitudes and the closed-string partition function, everything else vanishes to all loops. We then consider Euclidean backgrounds of the form T^4. In particular, we study the T^4 partition function at one loop (for special moduli corresponding to $T^4 = T^2 \times T^2$) and explicitly check the harmonicity condition found in section 2. We find that at least in this case, there is no anomaly despite the fact that there could have been one.

In section 5, we show how the superstrings (both in the RNS and GS formulations) fit into this new topological theory. Since the superstring in either of these formulations can be described by an N=2 matter sector with $\hat{c} = 2$ (for the RNS string, the N=2 matter sector contains both the N=1 matter fields and the N=1 ghosts), the topological prescription described in the previous sections can be used to calculate superstring scattering amplitudes. Since the topological prescription does not contain ambiguities associated with the positions of picture-changing operators, the resulting scattering amplitudes appear to be free of the total-derivative ambiguities that plague conventional superstring computations.

In section 6, we show that just as the $N = 2$ topological theory on Calabi-Yau three-folds computes superpotential terms for the superstring compactified to four dimensions,

the $N = 4$ topological theory on $K3$ computes superpotential terms in harmonic super-space for the superstring compactified to six dimensions. The GS method for computation developed in [9] is shown to be an efficient means of establishing this correspondence for both the four-dimensional and the six-dimensional superpotential terms. Although the GS method is related to the RNS method by a field-redefinition [10], the advantage of the GS method is that there is no need to sum over spin structures since there are no square-root cuts and spacetime supersymmetry is manifest.

In section 7, we present our conclusions and suggestions for further study. In this section, we also note the possibility of constructing twisted space-time supersymmetry for self-dual $N = 2$ strings, which is particulary simple in the $N = 4$ topological framework. In appendix A, the field-redefinition which relates the $N = 2$ descriptions of the GS and RNS superstring is reviewed. In appendix B, we show that there is an arrow of imbeddings for topological theories. In particular, we show that $N = 2$ topological theories (which can be viewed as special bosonic string vacua) can be imbedded into $N = 3$ topological theories (which can be viewed as special $N = 1$ superstring vacua). Unfortunately, we are unable to construct an arrow of imbeddings that maps either of these topological theories into the ("big") $N = 4$ topological theory.

2.   A New Topological Theory

In this section, we describe a new class of topological strings based on $N = 4$ super-conformal symmetry. There are two types of $N = 4$ superconformal algebras, one "big" and the other "small". Here we are interested in the "small" type which has $SU(2)$ as its current [11]. We will be interested in both unitary and non-unitary realizations of this algebra[2] and the most natural unitary representations arise from sigma models on hyperkähler manifolds. The algebra can also be viewed as an $N = 2$ algebra $(T, G^{\pm}, J)$ with two additional currents of charge ± 2 , denoted by J^{++} and J^{--},[3] which together with J form an $SU(2)$ algebra. Moreover under this $SU(2)$, G^- and G^+ generate two new

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[2] In a unitary theory with the adjoint defined by $G^{+\dagger} = G^-$, $\tilde{G}^{+\dagger} = \tilde{G}^-$, $J^{++\dagger} = J^{--}$, and the rest of the currents self-adjoint, the representation of the algebra has a positive-definite norm.

[3]   Their existence for the hyperkähler manifolds follows from the existence of a unique $h^{2,0}$ cohomology.

supercurrents $\tilde{G}^+$ and $\tilde{G}^-$ which form pairs of doublets $(G^+, \tilde{G}^-)$ and $(\tilde{G}^+, G^-)$. In other words, we have the OPE's

$$J^{--}(z)J^{++}(0) \sim \frac{J(0)}{z}$$

$$J^{--}(z)G^+(0) \sim \frac{\tilde{G}^-(0)}{z} \qquad J^{++}(z)\tilde{G}^-(0) \sim \frac{-G^+(0)}{z}$$

$$J^{++}(z)G^-(0) \sim \frac{\tilde{G}^+(0)}{z} \qquad J^{--}(z)\tilde{G}^+(0) \sim \frac{-G^-(0)}{z}$$

$$J^{--}G^-(0) \sim J^{++}G^+ \sim J^{++}\tilde{G}^+ \sim J^{--}\tilde{G}^- \sim 0 \tag{2.1}$$

The OPE with J follows from the charges of the supercurrents which is denoted by the $\pm$ on their symbol. Moreover, the doublets have a non-singular OPE among themselves

$$G^+\tilde{G}^- \sim G^-\tilde{G}^+ \sim 0 \tag{2.2}$$

but have the following singular OPE between the elements of different doublets

$$G^+(z)G^-(0) \sim \frac{J(0)}{z^2} + \frac{2T(0) + \partial J(0)}{z}$$

$$\tilde{G}^+(z)\tilde{G}^-(0) \sim \frac{J(0)}{z^2} + \frac{2T(0) + \partial J(0)}{z}$$

$$G^+(z)\tilde{G}^+(0) \sim \frac{J^{++}(0)}{z^2} + \frac{\partial J^{++}(0)}{2z}$$

$$G^-(z)\tilde{G}^-(0) \sim \frac{J^{--}(0)}{z^2} + \frac{\partial J^{--}(0)}{2z} \tag{2.3}$$

The level of the $SU(2)$ current algebra is k and is proportional to the central charge $c = 6k$. For a unitary theory, k is an integer and one sees that in these cases, c takes the value expected for sigma models on a hyperkähler manifold of real dimension $4k$.

We would also like to discuss how to deform the theory in a manner that preserves the $N = 4$ superconformal structure. As is well known, [12] to deform an $N = 2$ superconformal theory, one needs chiral fields ϕ_i (i.e. killed by $G^+_{-1/2}$) of charge $+1$ and dimension $1/2$, and their conjugate $\overline{\phi}_i$. One adds to the action

$$S \to S + \int t^i G^- \phi_i + \overline{t}^i G^+ \overline{\phi}_i \tag{2.4}$$

where here and in the following, $G^\pm \phi$ means $\oint G^\pm(z)\phi(0)$. Note that here and in the following, we will sometimes concentrate only on left-movers since the equivalent argument

applies to the right-movers. In order for this deformation to preserve the $N = 4$ structure, we will have to make sure that the deformation is a singlet of the $SU(2)$ symmetry (since as discussed above, the $SU(2)$ currents and the $N = 2$ algebra together generate the $N = 4$ algebra). The deformation respects the $U(1)$ part of the $SU(2)$, so all we need to check are the invariances with respect to J^{++} and J^{--}.

If we look at the action of J^{++} on $G^-\phi_i$, we get two terms

$$\tilde{G}^+\phi_i + G^-(J^{++}\phi_i)$$

For these two terms to vanish, we thus require that

$$\tilde{G}^+\phi_i = 0 \tag{2.5}$$

and

$$\oint J^{++}\phi_i = 0$$

This means that ϕ_i, being killed by the raising operator and of charge 1, must be the top component of an $SU(2)$ doublet.

If we look at the action of J^{--} on $G^-\phi_i$, we only get one term

$$G^-(J^{--}\phi_i)$$

and this implies that $J^{--}\phi_i$ is killed by G^-, i.e., is antichiral. Together with the fact that the charge of $J^{--}\phi_i$ is -1, we see that we can expand it in terms of antichiral fields of charge -1 through some matrix $M_i^{\bar{j}}$:

$$\oint J^{--}\phi_i = M_i^{\bar{j}}\bar{\phi}_j \tag{2.6}$$

We thus see that ϕ_i is the upper component of an $SU(2)$ doublet with the lower component being an anti-chiral field. In a *unitary* $N = 4$ superconformal theory, a chiral field of charge $+1$ is automatically the top component of an $SU(2)$ doublet, with the lower component an anti-chiral field. However, since we are not necessarily dealing with a unitary theory, only chiral fields of charge $+1$ which satisfy this condition can be used to deform the theory. Conjugate statements of the above discussion of course hold true for $\bar{\phi}_i$.

Therefore, we have learned that the chiral/anti-chiral fields of interest are of charge $+1/-1$ and satisfy

$$G^+\phi_i = \tilde{G}^+\phi_i = 0, \qquad G^-\bar{\phi}_i = \tilde{G}^-\bar{\phi}_i = 0$$

6

$$J^{--}\phi_i = M_i^{\bar{j}}\bar{\phi}_j, \qquad J^{++}\bar{\phi}_i = -M_{\bar{i}}^j\phi_j, \quad M_i^{\bar{j}}M_{\bar{j}}^k = \delta_i^k \tag{2.7}$$

Note that these relations imply that we can use just chiral fields (or just anti-chiral fields) to deform the action since

$$G^+\bar{\phi}_i = M_{\bar{i}}^j G^+(J^{--}\phi_j) = M_{\bar{i}}^j(G^+J^{--})\phi_j = -M_{\bar{i}}^j\tilde{G}^-\phi_j \tag{2.8}$$

It is sometimes convenient to write the deformation of the action in the form

$$S \to S + \int t_1^i G^-\phi_i - t_2^i \tilde{G}^-\phi_i \tag{2.9}$$

where in principle, t_1^i and t_2^i are independent. In a unitary theory, they are not independent and must satisfy the condition that

$$t_2^i = (\bar{t}_1^{\bar{j}})M_{\bar{j}}^i \quad .$$

Suppose that we have an $N = 4$ theory with trivial $\tilde{G}^+$ cohomology (which often happens for non-unitary theories). Since a chiral field ϕ_i is annihilated by $\tilde{G}^+$ it can be solved in terms of another field

$$\phi_i = \tilde{G}^+V_i \tag{2.10}$$

where V_i is $U(1)$ neutral and has dimension 0. Since ϕ_i is a doublet of $SU(2)$ and $(\tilde{G}^+, G^-)$ transforms as a doublet, there are only two possibilities for V_i : Either it belongs to a singlet or a triplet of $SU(2)$. Let us assume it belongs to a singlet as will be the case in our examples. From (2.7) and the fact that $\{G^+_{-1/2}, \tilde{G}^+_{-1/2}\} = 0$, we deduce that G^+V_i is killed by $\tilde{G}^+$. We can thus construct out of V_i a new chiral field (by using the other G-current doublet):

$$\phi_i^{(1)} = G^+V_i \tag{2.11}$$

Note that $\phi_i^{(1)}$ is a chiral field of charge $+1$ and dimension $1/2$, is killed by $\tilde{G}^+$, and is a member of an $SU(2)$ doublet. So it is a candidate to deform the theory by. In fact, we can repeat this process for $\phi_i^{(1)}$ and construct a new chiral field $\phi_i^{(2)}$. This process can be continued ad-infinitum and we end up with an infinite collection of fields $\phi_i^{(n)}$ for $n > 0$ which were formed from the original chiral field ϕ_i. Note that if the G^+ cohomology were also trivial, we could have reversed this process. For example, $\phi_i^{(1)}$ being killed by G^+ can be solved as G^+V_i, and we can apply $\tilde{G}^+V_i$ to get a new chiral field, which is nothing but ϕ_i. In this way we would get an infinite collection of chiral fields $\phi_i^{(n)}$ for each integer

n. This process is reminiscent of the picture-changing operation of RNS which, as we will see later, is not an accident. Apriori, we are not guaranteed that all these fields are different. In fact, we will see that the $N = 2$ string with a self-dual background gives an example where all the $\phi_i^{(n)}$ are equal to each other up to multiplication by a c-number. But in cases where they are inequivalent, one would like to somehow identify them as in the picture-changing operation of RNS.

Having discussed some general aspects of $N = 4$ theories, we are now ready to consider their twisting. Since twisting involves coupling some current to a background gauge field identified with the spin-connection, we have to choose a $U(1)$ in $SU(2)$ for the twist. Once we do the twisting, the story is just the familiar twisting of $N = 2$ theories where we view an $N = 4$ theory as a special case of an $N = 2$ theory. In other words, twisting simply means that we shift the spin content of all the fields by half their $U(1)$ charge [13][14]. Thus the spin content of the fields which are affected are the following:

$$G^+, \tilde{G}^+ \to \text{spin } 1$$

$$G^-, \tilde{G}^- \to \text{spin } 2$$

$$J^{++} \to \text{spin } 0$$

$$J^{--} \to \text{spin } 2$$

Note that chiral fields ϕ_i of charge $+1$ and dimension $1/2$ in the twisted theory will have dimension zero.

Now we have to decide which fields in the twisted theory we would like to identify as 'physical'. In the case of $N = 2$ topological theories, the cohomology elements of G^+ form the physical spectrum. Moreover, any state which is G^+ of some other field is considered trivial and decouples from the theory. For the $N = 4$ theories under consideration, we have seen that the relevant chiral fields are not only killed by G^+ but also by $\tilde{G}^+$. Therefore the condition for a physical field should be that it is killed by both G_0^+ and $\tilde{G}_0^+$ (note that G's have integral Fourier coefficients in the twisted theory). It is important to note that

$$\{G_0^+, \tilde{G}_0^+\} = 0 \tag{2.12}$$

Of course a trivial way to form a field annihilated by both G^+ and $\tilde{G}^+$ is to consider $G^+\tilde{G}^+\chi$ for any χ (where we use (2.12)). This must be the condition for topological triviality. In other words we consider a field ϕ to be physical if

$$G^+\phi = \tilde{G}^+\phi = 0 \qquad \phi \sim \phi + G^+\tilde{G}^+\chi \tag{2.13}$$

8

To complete the story, we have to discuss how to couple this topological theory to gravity. In other words, what are the rules for integrating over the moduli of Riemann surfaces. In the $N = 2$ twisted theory, the story is rather simple: G^- having spin 2 plays the role of the b anti-ghost of bosonic strings, and the twisted $N = 2$ theory can be viewed as a generalization of a bosonic string vacuum. Before coupling to gravity, our theory can also be viewed as an $N = 2$ twisted theory. The only new ingredient is what we mean by a physical state, which should satisfy the condition (2.13).

If we were able to consider a *reduced* Hilbert space $\tilde{\mathcal{H}} \subset \mathcal{H}$ where $\tilde{\mathcal{H}}$ is the subspace of $\mathcal{H}$ killed by $\tilde{G}^+$, then the physical state condition (2.13) acting on this reduced Hilbert space would be the same as the condition for an $N = 2$ twisted physical field and we could use the same rules of computation. So the question is how to do this reduction. Using the fact that $(\tilde{G}^+)^2 = 0$, this can be done by simply inserting a $\oint \tilde{G}^+$ around each a-cycle on the Riemann surface, or in a manifestly modular-invariant way by combining with right-movers and integrating $\int d^2z \tilde{G}^+ \overline{\tilde{G}^+}$ over the surface (when $\tilde{G}^+$ is holomorphic, this surface integral reduces to integrals over the cycles in the usual way). We are therefore led to considering the partition function on genus g defined by the measure over moduli space $\mathcal{M}_g$

$$\langle |G^-(\mu_1)...G^-(\mu_{3g-3})|^2 \big[\int \tilde{G}^+ \overline{\tilde{G}^+} \big]^g \rangle$$

However this amplitude is identically zero. To see this, use $\oint \tilde{G}^+ J = -\tilde{G}^+$ to replace one of the $\tilde{G}^+$'s by the contour integral of $\oint \tilde{G}^+$ around J, and pull the contour of $\tilde{G}^+$ off the surface. Since the G^-'s and $\tilde{G}^+$'s have no singularity with it, we get zero! There is another reason why the above formula is not what we want. As discussed above (2.13) , a deformation is topologically trivial if it can be written as $G^+\tilde{G}^+\chi$. However in the above definition, it is easy to see that adding to the action $\tilde{G}^+\chi$ is already topologically trivial since we can pull the $\tilde{G}^+$ contour off of χ and get zero by the same reasoning as above.

So instead, we will define the topological partition function to be

$$F_g = \int_{\mathcal{M}_g} \langle |G^-(\mu_1)...G^-(\mu_{3g-3})|^2 \big[\int \tilde{G}^+ \overline{\tilde{G}^+} \big]^{g-1} \int J\bar{J} \rangle \tag{2.14}$$

which is no longer zero. Note that the contour of $\tilde{G}^+$ can no longer be pulled off the surface since it hits the J and gives back $\tilde{G}^+$ as the residue. For the same reason, adding $\tilde{G}^+\chi$ to the action may change the partition function. However if we consider adding $G^+\tilde{G}^+\chi$ to the action, then the $\tilde{G}^+$ contour can be pulled off of χ and converts the J to a $\tilde{G}^+$. Now

pulling the G^+ contour off of χ, we encounter no residues from $\tilde{G}^+$. From the G^-'s, we get residues which are the energy momentum tensor, thus giving us total derivatives in the moduli which at least formally (barring anomalies) integrate to zero. Thus (2.14) has the correct topological symmetry. Further motivation for the definition (2.14) will come in subsequent sections when we see that the partition functions of superstrings and N=2 strings can be viewed as special cases of it.

Before going on, it is useful for later applications to note that we can write F_g equivalently as

$$F_g = \int_{\mathcal{M}_g} \langle |G^-(\mu_1)...G^-(\mu_{3g-4})J^{--}(\mu_{3g-3})|^2 \left[\int \tilde{G}^+\overline{\tilde{G}^+} \right]^g \rangle \tag{2.15}$$

To obtain this from (2.14) , write G^- as the contour of $\tilde{G}^+$ around J^{--} and move $\oint \tilde{G}^+$ off the surface where it only hits J. This leaves a residue of $\tilde{G}^+$, which thus gives us (2.15).

Let us note that the charge violation in the genus g amplitude due to twisting is $\hat{c}(g-1)$. Since the definition (2.14) has a charge violation of $-2(g-1)$, we learn that the partition function is apriori non-vanishing when $\hat{c} = 2$ (for other values of $\hat{c}$, we need to insert charged operators to get a non-zero result). In this sense, $\hat{c} = 2$ *is the critical dimension for* $N = 4$ just as $\hat{c} = 3$ is the critical dimension for $N = 2$ topologically twisted theory. Note that any $N = 2$ theory with $\hat{c} = 2$ and integral $U(1)$ charges automatically has an $N = 4$ symmetry [11] where the spectral flow operator and its inverse, $\exp(\pm \int J)$, provide the raising and lowering operators of $SU(2)$. As far as unitary theories with a discrete spectrum, the only examples are based on sigma models on T^4 and on $K3$.

So far, we have avoided an important point. We have discussed how $N = 2$ superconformal symmetry can be realized in an $N = 4$ algebra and have used the $N = 2$ to discuss twisting and to define the topological amplitude. However we should also ask how many ways one can view an $N = 4$ theory as an $N = 2$ theory. If there is more than one way, as we will find, then there is more than one topological theory associated to a given $N = 4$ theory.

To pick an $N = 2$ in an $N = 4$ theory, we have to do two things: First we have to choose the $U(1)$ current J of $N = 2$ in the $SU(2)$ of the $N = 4$. Secondly, we have to decide which currents are G^+ and G^-. In fact, there is an $SU(2)$ worth of consistent choices for $G^\pm$. To see this, note that we have two doublets of G's and the $N = 4$ algebra is unmodified if we rotate them with each other by an $SU(2)$. To avoid confusion with the $SU(2)$ that is part of the current algebra of $N = 4$, we will call this the *flavor* $SU(2)_f$.

10

The $SU(2)$ which is part of the $N = 4$ algebra we shall call the *color $SU(2)_c$*. We write the flavor rotation as

$$\widehat{\tilde{G}^+}(u) = u_1\tilde{G}^+ + u_2 G^+$$

$$\widehat{G^-}(u) = u_1 G^- - u_2\tilde{G}^-$$

$$\widehat{\tilde{G}^-}(u) = u_2^*\tilde{G}^- - u_1^* G^-$$

$$\widehat{G^+}(u) = u_2^* G^+ + u_1^*\tilde{G}^+ \tag{2.16}$$

where

$$|u_1|^2 + |u_2|^2 = 1$$

Note that we are using the harmonic notation of [15] in which the complex conjugate of u_a is $\epsilon^{ab}u_b^*$ (i.e. $\overline{(u_1)} = u_2^*$ and $\overline{(u_2)} = -u_1^*$ where $*^2 = -1$). Furthermore, the SU(2) indices on u and u^* can be raised with the antisymmetric tensor ϵ^{ab}. For example, the normalization condition on the u's can be written as $u^a u_a^* = 1$.

Now let us see how different embeddings of the $N = 2$ algebra in $N = 4$ modify the twistings and amplitudes. As far as twisting, note that $SU(2)_c$ is an internal symmetry of the theory which rotates different choices of $U(1)$ into one another. Since the choice of twisting involves the choice for this $U(1)$, different twistings can be obtained by rotating the whole field space using $SU(2)_c$. In particular, any computation we do with any given choice is equivalent to a conjugate computation done with the conjugated $U(1)$. Thus the different choices of $U(1) \subset SU(2)_c$ do not lead to any new theories. In particular, this means that any computations done with the anti-topological twisting (where G^+ is dimension 2 and G^- is dimension 1) is equivalent to a computation with the topological twisting where the rotation in field space sends

$$G^+ \to \tilde{G}^-, \quad G^- \to \tilde{G}^+$$

$$\tilde{G}^+ \to -G^-, \quad \tilde{G}^- \to -G^+ \tag{2.17}$$

The $SU(2)_f$ is more interesting as far as getting a new theory. This is because $SU(2)_f$ is not realized by a symmetry (there is no current associated with it) and the theory we get with the rotated G's (2.16) in general results in a new theory. The definition of the flavor-rotated amplitude is given by (2.14) (or (2.15)), but now we use $(\widehat{\tilde{G}^+}(u), \widehat{G^-}(u))$,

instead of $(\tilde{G}^+, G^-)$ and we end up with $F_g(u_1, u_2)$. So the topological partition function is a homogeneous polynomial in u_1, u_2 of degree $4g - 4$:

$$F_g(u_1, u_2) = \sum_{n=-2g+2}^{2g-2} \frac{(4g-4)!}{(2g-2+n)!(2g-2-n)!} F_g^n u_1^{2g-2+n} u_2^{2g-2-n} \qquad (2.18)$$

$$= \int_{\mathcal{M}_g} \langle |\widehat{G^-}(\mu_1)...\widehat{G^-}(\mu_{3g-3})|^2 \Big[\int \widehat{\tilde{G}^+ \tilde{G}^+} \Big]^{g-1} \int J\bar{J} \rangle$$

Note that u_1 and u_2 transform as a doublet under $SU(2)_f$ where

$$J_f^+ = u_1 \frac{d}{du_2} + u_1^* \frac{d}{du_2^*}, \quad J_f^- = u_2 \frac{d}{du_1} + u_2^* \frac{d}{du_1^*}, \qquad (2.19)$$

$$J_f^3 = \frac{1}{2}\Big(u_1 \frac{d}{du_1} - u_2 \frac{d}{du_2} + u_1^* \frac{d}{du_1^*} - u_2^* \frac{d}{du_2^*}\Big).$$

So the F_g^n's form a spin $2g - 2$ dimensional representation of $SU(2)_f$ where n labels the J_3 eigenvalue of $SU(2)_f$. Thus taking the different imbeddings of the $N = 2$ in $N = 4$ theory would lead us not to a partition function, but to a partition vector F_g^n. Knowing F_g^n for all n allows us to compute the amplitudes for an arbitrary imbedding of $N = 2$ in a given $N = 4$ theory, where the imbedding is parametrized by the u_i's. The reason for the combinatorial factor in the definition of F_g^n above is that for a given power of u_1 and u_2, there are as many correlation functions to compute as indicated by the combinatorial factors. We will see below that, barring anomalies, all of these correlation functions are equal to F_g^n. Note also that we can do the left-moving and right-moving $SU(2)_f$ rotations seperately which would lead us to introducing another set of independent u's for the right-movers. In particular, F_g would be viewed as a $(2g - 2, 2g - 2)$ represention of the two $SU(2)_f$'s. As we have done in (2.18), we will continue suppressing the right-moving u's to avoid excessive notation.

It will sometimes be convenient to collect all the different F_g^n's into a single function by introducing a periodic variable θ and define[4]

$$F_g(\theta) = \sum_{n=2-2g}^{2g-2} \exp(in\theta) \, F_g^n.$$

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[4] As we will see later in the context of $N = 2$ strings, θ can be viewed as the θ-angle of the the U(1) gauge symmetry of the $N = 2$ strings.

This new parameter θ may be viewed as an extra "coupling constant" for the $N = 4$ topological strings. In this way, we get $SU(2)/U(1)$ inequivalent $N = 4$ topological theories, where the $U(1)$ is modded out because it can be undone by a shift in θ.

Let us show one basic property of F_g^n which also illustrates an application of $SU(2)_c$ symmetry. Consider the $SU(2)_c$ rotation which takes J to $-J$ and transforms the G's as in (2.17). Then conjugating the topological theory with this symmetry changes the topological twisting to the anti-topological twisting. Since this is a symmetry operation, it does not affect the topological amplitudes. We thus learn that

$$F_g^{n,\text{top.}} = F_g^{-n,\text{anti-top}} \tag{2.20}$$

where $F_g^{\text{anti-top}}$ is the amplitude defined in (2.18) but with the anti-topological twisting where $G^+ \leftrightarrow G^-$ and $\tilde{G}^+ \leftrightarrow \tilde{G}^-$. For a *unitary* $N = 4$ theory, the anti-topological theory is the complex cojugate of the topological theory implying that

$$F_g^n = \overline{F_g^{-n}}$$

So far, we have talked about the amplitude for genus $g > 1$. We have to note two special cases: genus 0 and genus 1. Genus 0 by itself has zero partition function by $SL(2, C)$ invariance, so we will discuss it when we talk about correlation functions. For $g = 1$, the definition is a simple generalization of the $N = 2$ partition function [16][2]. The main point is that we have an extra $\int J\overline{J}$. In other words, we have[5]

$$F_1 = \int d^2\tau \ \text{Tr}(-1)^{\text{F}} F_L^2 F_R^2 q^{H_L} \bar{q}^{H_R} \tag{2.21}$$

where $F_L = \oint J$ and $F_R = \oint \bar{J}$. If we take the derivative of F_1 with respect to the moduli t^i, we replace a single $F_L F_R$ in (2.21) with $|G^-|^2$ and an insertion of the chiral field ϕ_i. Since we have an extra $F_L F_R$ (which comes from $\int J\bar{J}$ upon using the Riemann identities), this is the natural generalization of (2.14) to genus 1. Note that if we had only the first power in F_L and F_R as in the $N = 2$ case, then (2.21) would have vanished for $N = 4$ theories by $SU(2)$ symmetry, since F_L is mapped to $-F_L$ by a rotation in $SU(2)_c$. Instead

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[5] In general, the zero point function at genus one may be ill-defined if there is a moduli-independent infinity due to the ground state contributions which needs to be deleted. For this reason, it is better to motivate the above definition of F_1 by noting that its derivatives with respect to the moduli generate the one point functions on the torus.

F_L^2 is proportional to the casimir when summed over all states of a given representation, and is not apriori zero.

We will now discuss the correlation functions for the topological theory. As discussed above, we need chiral fields satisfying (2.7). Note that the physical field condition of (2.7) is true no matter what u's we choose. We thus consider

$$F_{g,i_1...i_n}(u_1,u_2) = \int_{\mathcal{M}_{g,n}} \langle | \prod_{j=1}^{3g-3+n} \widehat{G^-}(\mu_i)|^2 [\int \widehat{\tilde{G}^+\tilde{G}^+}]^{g-1} \int J\bar{J} \quad \phi_{i_1}...\phi_{i_n} \rangle \qquad (2.22)$$

where chiral fields ϕ_i are inserted at arbitrary points (recall that they have dimension zero in the twisted theory), and we have chosen to treat the moduli of punctures and the Riemann surface symmetrically. Note that $F_{g,i_1...i_n}(u_1,u_2)$ is a homogeneous polynomial in u_i of degree $4g-4+n$ whose coefficients transform as a spin $2g-2+\frac{n}{2}$ representation of $SU(2)_f$.

Let us now discuss the amplitudes on the sphere. We will concentrate on the critical case with $\hat{c}=2$. The first non-vanishing case to consider, by $SL(2,C)$ invariance, is the three-point function. We need three fields each of dimension zero, but with a total charge of $+2$ to cancel the background charge of -2. Since the chiral fields have dimension 0 and charge 1 in the twisted theory, three of them would give zero by charge conservation. This is similar to the vanishing we would have gotten at higher genus had we inserted g $\tilde{G}^+$'s around the cycles. We solved this problem by deleting one of the $\tilde{G}^+$'s. We can do a similar thing here if the $\tilde{G}^+$ cohomology is trivial, as will turn out to be the case in many applications. As noted above (2.10) ,we can then write the chiral primary field ϕ_i in terms of

$$\phi_i = \tilde{G}^+ V_i$$

where V_i has charge 0 and dimension 0. The three point amplitude can then be defined as

$$C_{ijk} = \langle \phi_i \phi_j V_k \rangle \qquad (2.23)$$

Note that despite its appearance, C_{ijk} is symmetric in its indices. To see symmetry in the indices j and k, write $\phi_j = \tilde{G}^+ V_j$ and pull the $\tilde{G}^+$ contour off the surface. It only gives a contribution when it passes through V_k, converting it to ϕ_k and thereby establishing the symmetry. Similarly for $n > 3$, we define the tree amplitude to be

$$C_{i_1 i_2...i_n} = \int_{\mathcal{M}_{0,n}} \langle G^-(\mu_1)...G^-(\mu_{n-3})\phi_{i_1}...\phi_{i_{n-1}}V_n \rangle \qquad (2.24)$$

14

which is symmetric in its indices for the same reason.

In defining the above amplitude, we had to assume that the $\tilde{G}^+$ cohomology is trivial. Even if this is not the case, we can still define $n > 3$ point amplitudes on the sphere by noting that for $n > 3$, we can write V_n in the integrated form $G^- V_n$. Using $[J^{--}, \tilde{G}^+] = G^-$, the definition of $\tilde{G}^+ V_n = \phi_n$, and the fact that J^{--} annihilates V_n, we have

$$G^- V_n = J^{--} \phi_n$$

Thus we can define

$$C_{i_1 \ldots i_n} = \int_{\mathcal{M}_{0,n}} \langle G^-(\mu_1) \ldots G^-(\mu_{n-4}) \phi_{i_1} \ldots \phi_{i_{n-1}} \int J^{--} \phi_{i_n} \rangle \qquad (2.25)$$

The symmetry in indices follows from conformal invariance, which allows putting J^{--} at other ϕ_i. This definition of the genus zero correlation function has the advantage of making sense whether or not the $\tilde{G}^+$ cohomology is trivial. Moreover, it is the natural generalization of (2.15) to genus 0. However note that in general, we can only define the three point function when $\tilde{G}^+$ has trivial cohomology.

We can once again introduce the u_i's, in which case (2.25) will be a homogeneous polynomial of degree $n - 4$ for $n \geq 4$. This fits well with the higher genus correlation function which is a homogeneous polynomial of degree $4g - 4 + n$ in the u_i's.

Let us now discuss the relation of the correlation functions (2.22) to the partition function F_g where the action has been deformed as in (2.9). The relation is

$$F_{g, i_1 \ldots i_n}(u_1, u_2) = (u_a \frac{D}{D t_a^{i_1}}) \ldots (u_a \frac{D}{D t_a^{i_n}}) F_g(u_1, u_2) \qquad (2.26)$$

where D denotes an appropriate covariant derivative (see [2] for the definition of D and for the reason we need *covariant* derivatives). To get (2.26), one writes (2.22) in the form where the fields are put in the $(1, 1)$ picture $\widehat{G^- } \phi_i$ by taking the appropriate beltrami differentials.

For $N = 2$ topologically twisted theories, there is a notion of *holomorphicity*. Namely one formally expects the partition function to be independent of the anti-chiral deformation, i.e. $\bar{\partial}_i F_g = 0$, but there are anomalies which can be understood [2]. It is natural to discuss the analogous notion for the topological amplitude in the $N = 4$ topological theory. Note that the anti-chiral fields no longer decouple since $G^+ \bar{\phi}_i$ is not a trivial field in the new definition of the topological theory. This is also clear given the fact that we can write it as some combination of $\tilde{G}^- \phi_j$'s from (2.8).

However in the $N = 4$ topological theory, perturbation by $G^+\tilde{G}^+\chi$ is expected to be trivial up to total derivatives, which could in principle lead to anomalies due to boundary contributions. For this perturbation to be neutral (and thus not trivially zero), χ would have to be a charged field of dimension 1 and charge -2 in the twisted theory, or in the untwisted theory, it would correspond to a field of dimension 0 and charge -2. Such a χ does not exist in unitary theories as it would violate unitarity bounds (in unitary theories, the dimension is always greater than or equal to half the absolute value of the charge). However we will be interested in non-unitary theories as well, and in such cases there could exist such χ's and we could end up with such anomalies which deserve to be studied (note that the non-unitary case may be trickier than the one encountered for unitary theories in [2] because of potential contributions from negative energy states at the boundaries of moduli space).

Nevertheless, there is another notion of 'holomorphicity' which is applicable even to unitary theories. One can formally argue that

$$\epsilon_{ab}\frac{d}{du_a}\frac{D}{Dt_b^i}F_g(u_1, u_2) = 0, \qquad \epsilon_{ab}\frac{d}{du_a}\frac{D}{Dt_b^{*i}}F_g(u_1, u_2) = 0 \qquad (2.27)$$

where the first equation is evaluated using the deformed action

$$S + \int t_1^i G^-\phi_i - t_2^i\tilde{G}^-\phi_i$$

and the second equation is evaluated using the deformed action

$$S + \int t_2^{*i}G^+\overline{\phi}_i + t_1^{*i}\tilde{G}^+\overline{\phi}_i.$$

We will sometimes refer to (2.27) as the "harmonicity" condition. The two equations are consequences of one another since from (2.8), $M_i^j t_1^{*i} = t_1^j$ and $M_i^j t_2^{*i} = t_2^j$ (we are using the same "harmonic" notation for the t's as we did for the u's in (2.16)). It will be more convenient to prove the second equation involving t^*'s.

To prove this equation, it is convenient to expand F_g in powers of u's. We can then rewrite the above equation as the statement that inserting the operator $\int G^+\overline{\phi}_i$ in F_g^n is equivalent to inserting the operator $\int \tilde{G}^+\overline{\phi}_i$ for F_g^{n-1} (as usual, we are ignoring the right-movers in our discussion to reduce the notation) . Let us see why this is true modulo total derivatives in moduli.

From F_g^n, we get $(2g-2+n)!(2g-2-n)!/(4g-4)!$ times a sum of contributions of the form

$$\langle G^-(\mu_1)...G^-(\mu_r)\tilde{G}^-(\mu_{r+1})...\tilde{G}^-(\mu_{3g-3})$$

$$\tilde{G}^+(z_1)...\tilde{G}^+(z_s)G^+(z_{s+1})...G^+(z_{g-1})J(z_g)\rangle$$

where $r+s = 2g-2+n$. In other words $2g-2+n$ is the number of one type of doublet, $2g-2-n$ is the number of the other type, and all allowed values of r and s subject to the above condition occur as part of the definition of F_g^n. We will now see that up to total derivatives in moduli (which we are going to ignore for the moment), all different allowed values of r, s give the same result. Since there are $(4g-4)!/(2g-2+n)!(2g-2-n)!$ such terms in F_g^n, F_g^n is normalized to be equal to any one of these terms (up to total derivatives).

To see that all allowed values of r, s give the same result, write one of the G^-'s (to simplify the notation, say the first one) as the contour of $\tilde{G}^+$ (which is a current of dimension 1) around the current J^{--}, and contour deform $\tilde{G}^+$. When the contour hits $\tilde{G}^-$'s, we get residues of the energy momentum tensor which lead to total derivatives that we are ignoring. Otherwise, it has no other residues except with J which gives us

$$\langle J^{--}(\mu_1)...G^-(\mu_r)\tilde{G}^-(\mu_{r+1})...\tilde{G}^-(\mu_{3g-3})$$

$$\tilde{G}^+(z_1)...\tilde{G}^+(z_s)G^+(z_{s+1})...G^+(z_{g-1})\tilde{G}^+(z_g)\rangle$$

Now we can write one of the G^+'s (say the last one) as the contour of G^+ around J. Contour deforming the G^+ we see that, again up to total derivatives, it only picks up a residue when it hits J^{--} which gives us

$$\langle \tilde{G}^-(\mu_1)...G^-(\mu_r)\tilde{G}^-(\mu_{r+1})...\tilde{G}^-(\mu_{3g-3})$$

$$\tilde{G}^+(z_1)...\tilde{G}^+(z_s)G^+(z_{s+1})...J(z_{g-1})\tilde{G}^+(z_g)\rangle$$

We thus see that we have changed $r \to r-1$ and $s \to s+1$, which is what we wished to show.

Now we consider inserting the operator $G^+\overline{\phi}_i$ in F_g^n. By the above argument, all we have to show is that we can convert it to $\tilde{G}^+\overline{\phi}_i$, while at the same time changing the number $r+s$ by -1. By contour deforming G^+ off of $\overline{\phi}_i$, we can put G^+ on the J using the above procedure. Next, we pull one of the $\tilde{G}^+$'s (by writing it as the contour of $\tilde{G}^+$ around

J) off the surface, which hits only ϕ_i to give us $\tilde{G}^+\overline{\phi}_i$. We have therefore established (2.27) up to total derivatives.

Of course, it is still possible that (2.27) has anomalies due to boundary contributions, a phenomenon which does take place in the $N = 2$ topological theory coupled to gravity. It would be rather interesting to carefully study the potential existence of similar anomalies in this case.

Let us consider the genus 1 generalization of (2.27). In this case, since the partition function has no u dependence, we should replace F_1 in (2.27) with the one point function on a torus, $F_{1,i}(u_1, u_2)$. Using definition (2.26) for $F_{1,i}$, we therefore need to prove up to total derivatives that

$$\epsilon_{ab}\frac{d}{du_a}\frac{d}{dt_b^{*j}}F_{1,i} = \epsilon_{ab}\frac{d}{du_a}\frac{d}{dt_b^{*j}}(u_c\frac{d}{dt_c^i})F_1 = \epsilon_{ab}\frac{d}{dt_b^{*j}}\frac{d}{dt_a^i}F_1 = 0 \qquad (2.28)$$

for all i and j. We shall be rather brief in our proof as the arguments are similar to the above.

Start with $\partial_{t_1^i}F_1$ which involves

$$G^-(z_1)J(z_2)\phi_i(z_3)$$

and consider the $\partial_{t_2^{*j}}$ derivative of it, which gives

$$-G^-(z_1)G^+(z_2)\phi_i(z_3)\overline{\phi}_j(z_4)$$

Now we write G^- as a contour integral of $\tilde{G}^+$ around J^{--} and pull the contour off the surface, leaving only the contribution from the residue at $\overline{\phi}_j$ which is

$$J^{--}(z_1)G^+(z_2)\phi_i(z_3)\tilde{G}^+\overline{\phi}_j(z_4)$$

Finally, writing G^+ as the contour of G^+ around J and pulling the G^+ contour off the surface (and noting there is no contribution from the $\overline{\phi}_j$ terms due to (2.8)), we are left with

$$-\tilde{G}^-(z_1)J(z_2)\phi_i(z_3)\tilde{G}^+\overline{\phi}_j(z_4)$$

which is just the definition of $\partial_{t_1^{*j}}\partial_{t_2^i}F_1$ and so we are done. Note that equation (2.28) can also be rewritten using (2.8) as

$$\partial_i\overline{\partial}_{\overline{j}}F_1 = M_i^{\overline{i}}M_{\overline{j}}^j\overline{\partial}_{\overline{i}}\partial_j F_1 \qquad (2.29)$$

18

where holomorphic derivatives refer to the first component of t and anti-holomophic derivatives refer to the second component of t^*. Again, there was room for total derivative terms contributing to (2.29) and we have not considered them. However we shall see in a later section that in the context of a special example, the potential anomalies do not appear. It would be interesting to study potential holomorphic anomalies more carefully and see if they are always absent.

Before closing this section, we will show that when the theory is unitary and all chiral fields are primary, the topological correlation functions can be expressed in a slightly different form which will be useful for some later applications. Consider (2.22) and study its dependence on the position of the $\widehat{\tilde{G}^+}$'s before integration over the Riemann surface. Since $\widehat{\tilde{G}^+}$ is a holomorphic current, the dependence on its position is going to be meromorphic. Since the chiral fields are primary, $\widehat{\tilde{G}^+}$ has no singularities at the ϕ_i's. Furthermore, since the theory is unitary, $\widehat{\tilde{G}^+}$ has no unphysical poles coming from the negative-energy fields. Therefore, the only potential singularity as a function of its position could come from the pole at J^{--}. But the amplitude for the residue of this pole, which involves replacing J^{--} with $\widehat{G^-}$, is zero because we can now write any of the $\widehat{\tilde{G}^+}$'s as a contour of $\widehat{\tilde{G}^+}$ around J and pull the contour off the surface.

Thus we see that as a function of the g positions, v_i, of the $\tilde{G}^+$'s, we have a holomorphic, totally anti-symmetric object which transforms as one forms. This implies that the dependence of the correlations on v_i is given by $\det\omega^i(v_j)$ where ω^i are the holomorphic 1-forms. Multiplying and dividing by this factor and integrating over the positions of v_i, we learn that

$$F_g = \det \mathrm{Im}\tau \int_{\mathcal{M}_g} \left|\det\omega^i(v_j)\right|^{-2} \cdot$$

$$\left\langle \left|G^-(\mu_1)...J^{--}(\mu_{3g-3+N})\right|^2 \prod_{i=1}^{g} \left|G^+(v_i)\right|^2 \phi_1...\phi_N \right\rangle \tag{2.30}$$

where $\det \mathrm{Im}\tau$ arises from $\int |\det\omega^i(z_j)|^2$, τ is the period matrix, and the v_i's can be any g points on the surface.

3. $N = 2$ String as an $N = 4$ Topological String

We have seen that the critical $N = 4$ topological string has $\hat{c} = 2$. We have also noted that every $N = 2$ superconformal theory with $\hat{c} = 2$ (and with integral $U(1)$ charges) automatically gives rise to a critical $N = 4$ superconformal theory. Given the fact that

every $N = 2$ superconformal theory with $\hat{c} = 2$ can be viewed as a background for $N = 2$ strings, it is natural to ask what is the relation between these two string theories. In this section, we will show that these two strings are indeed equivalent.

Before going into the detail of the proof of this equivalence, let us outline some aspects of the proof which may be helpful in following the arguments. $N = 2$ strings require integrating over $N = 2$ supergeometries which in genus g has the usual $3g - 3$ bosonic moduli, $4g - 4$ fermionic moduli, and g additional bosonic moduli correponding to integrating over $U(1)$ flat connections. Moreover, we have to sum over all possible $U(1)$ instanton numbers. It turns out that the only non-vanishing partition functions have instanton numbers n_I satisfying $2 - 2g \leq n_I \leq 2g - 2$. The rest vanish due to superghost zero modes.

We will show that

$$F_g^n = A_g^n \tag{3.1}$$

where F_g^n is the topological partition function defined in (2.18) and A_g^n is the N=2 string partition function on a surface of genus g and instanton number n (note that the instanton number is identified with the eigenvalue of J_3 flavor where u_1 has charge $+1$ and u_2 has charge -1).[6]

Note that for a given $N = 2$ vacuum, we have an $SU(2)$ worth of inequivalent ways of defining its $N = 2$ fermionic generators, just as in the topological case. Labeling the different choices with the harmonic variables u_i, we will show that

$$F_g(u_1, u_2) = \widehat{A}_g^{2g-2}(u_1, u_2) \tag{3.2}$$

where $\widehat{A}_g^{2g-2}(u_1, u_2)$ is the N=2 string partition function at instanton-number $2g - 2$ when the N=2 BRST operator is constructed using $\widehat{G}^+$ and $\widehat{G}^-$ as the matter part of the two

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[6] In the case of $N = 2$ strings (for a fixed realization of the $N = 2$ algebra), the complete answer for the partition function at genus g is given by

$$A_g(\theta) = \sum_{n_I} \exp(in_I \theta) A_g^{n_I}$$

where θ is the instanton θ-angle for the $U(1)$ field of the $N = 2$ string. This suggests that in the $N = 4$ topological formulation, we should also consider the corresponding sum as the definition of the genus g amplitude, where we think of θ as an additional coupling constant. This means that there are not an $SU(2)_f$ worth of different theories associated to a given $N = 4$ superconformal theory, but instead, an $SU(2)_f/U(1)$ worth of different theories where θ labels the $U(1) \subset SU(2)_f$.

fermionic generators (although $\widehat{G}^+$ depends on u^*, it will be found that $\widehat{A}_g^{2g-2}$ is actually independent of u^*).

The first step in the equivalence proof will be to show that (3.1) holds when $n = 2g-2$, i.e. $F_g^{2g-2} = A_g^{2g-2}$. After checking that the ghost partition function cancels out (with a judicious choice of where the various operators are inserted), we will be left with a twisted matter theory and some additional operators which reproduces (2.15). The second step will be to show that for a different realization of $N = 2$ (parametrized by u), $\widehat{A}_g^{2g-2}(u_1, u_2)$ is a polynomial in u where the coefficients are the A_g^n's calculated using the original unrotated $N = 2$ superconformal generators. By comparing with the dependence of $F_g(u_1, u_2)$ on the u's in (2.18), this proves (3.1) for all values of n. We now proceed with constructing the proof in detail.

The $N = 2$ string prescription for calculating the partition function on a genus g surface of instanton number n_I is given by :

$$A_g^{n_I} = \prod_{i=1}^{g} \int d^2 M_i \prod_{j=1}^{3g-3} \int d^2 m_j$$

$$< |(\int_{a_i} \tilde{b})(\int \mu_j b))(Z^-)^{2g-2+n_I}(Z^+)^{2g-2-n_I} I^{n_I} \tilde{c}(x_0)|^2 > \qquad (3.3)$$

where M_i are the U(1) moduli which take values in the Jacobian variety $\mathcal{C}^g/(\mathcal{Z}^g + \tau \mathcal{Z}^g)$, m_j are the Teichmuller parameters, $\int_{a_i}$ is an integration around the ith a-cycle, μ_j are the beltrami differentials for the Teichmuller parameters, $(\tilde{b}, \tilde{c})$ are the U(1) ghosts (the $\tilde{c}$ ghost can be inserted anywhere on the surface since only its zero mode contributes), the N=2 superconformal ghosts are bosonized as $(\gamma^+, \beta^-) = (\eta^+ e^{\phi^+}, \partial_z \xi^- e^{-\phi^+})$ and $(\gamma^-, \beta^+) = (\eta^- e^{\phi^-}, \partial_z \xi^+ e^{-\phi^-})$, the picture-changing operators are

$$Z^- = \{Q, \xi^-\} = e^{\phi^+}[G^- + (b - \frac{1}{2}\partial_z \tilde{b})\eta^- e^{\phi^-} - \tilde{b}\partial_z \eta^- e^{\phi^-}] + c\partial_z \xi^-,$$

$$Z^+ = \{Q, \xi^+\} = e^{\phi^-}[G^+ + (b + \frac{1}{2}\partial_z \tilde{b})\eta^+ e^{\phi^+} + \tilde{b}\partial_z \eta^+ e^{\phi^+}] + c\partial_z \xi^+,$$

$I = \exp(\int J_{total}) = \exp(\phi^- - \phi^+ + \int J + \tilde{c}\tilde{b})$ is the BRST invariant instanton-number-changing operator (or spectral flow operator), and n_I is the instanton number of the U(1) gauge field. Note that the relative number of picture-changing operators depends on the instanton number because the number of zero modes of $\beta^\pm, \gamma^\pm$ (which carry U(1) charge) depend on the instanton number.

The above amplitude vanishes for $|n_I| > 2g - 2$ since for $n_I > 2g - 2$, there are zero modes of γ^- which can not be absorbed (this is similar to the vanishing of tree amplitudes with less than three external vertex operators because of the zero modes of the c ghost). Note that there are no inverse-picture-changing operators in the N=2 string which could absorb the γ^+ zero modes. Furthermore, note that the $c\tilde{b}$ term in I can be ignored in calculations since there are no extra $\tilde{c}$'s available to absorb the $\tilde{b}$.

In order to end up with the topological prescription, we will first choose $n_I = 2g - 2$ and twist the N=2 algebra so that the conformal weight of all fields is shifted by half their U(1) charge (e.g., G^- now has conformal weight 2 and G^+ has conformal weight 1). Twisting the algebra is equivalent to removing $g - 1$ I's from the above expression for the scattering amplitude (note that the locations of the I's are irrelevant since moving I is equivalent to shifting the U(1) moduli). Furthermore, the locations of the picture-changing operators should be chosen so that the path integrals over the N=2 ghosts cancel out. Note that when $n_I = 2g - 2$, the integrand of $A_g^{n_I}$ is independent of the positions of the Z^-'s since $\partial_z Z^- = \{Q, \partial_z \xi^-\}$ and there are no η^+'s available to absorb the ξ^-. This is a special feature that is only true for $n_I = 2g - 2$ since otherwise, Z^+'s are present which contain η^+'s.

A convenient choice for the locations of the picture-changing operators is to sew in $3g - 4$ of the Z^-'s using the first $3g - 4$ beltrami differentials μ_j's, and to sew I^{g-1} with the last beltrami differential (for notational convenience, we shall assume that this beltrami differential is concentrated at the point w). The remaining g Z^-'s should be located at points $v_1, ..., v_g$ satisfying $[v_1 + ... + v_g - gw]_i = M_i$ where the divisor $[x - y]$ is defined as $[x - y]_i = \int_x^y \omega_i$ and ω_i are the g abelian one-forms satisfying $\int_{a_j} \omega_i = \delta_{ij}$ and $\int_{b_j} \omega_i = \tau_{ij}$.

In this special picture, the above scattering amplitude takes the form:

$$A_g^{2g-2} = \prod_{i=1}^{g} \int d^2 M_i \prod_{j=1}^{3g-3} \int d^2 m_j$$

$$< |(\int_{a_i} \tilde{b}) \prod_{k=1}^{3g-4} (\int \mu_k b) \delta(\int \mu_k \beta^-)(\int \mu_k G^-)\mu_{3g-3}(w)b(w)I^{g-1}(w)$$

$$Z^-(v_1)...Z^-(v_g)\tilde{c}(x_0)|^2 > .$$

The next step is to change variables from M_i to v_i using the Jacobian factor $\det(\omega_k(v_i))$. Since the integrand evaluated at $M_i = [v_1 + ... + v_g - gw]_i$ is equivalent

22

to inserting $I(v_1)...I(v_g)I^{-g}(w)$ and evaluating at $M_i = 0$, the above amplitude can be written as:

$$A_g^{2g-2} = \prod_{i=1}^{g} \int d^2 v_i |\det(\omega_k(v_i))|^2 \prod_{j=1}^{3g-3} \int d^2 m_j$$

$$< |(\int_{a_i} \tilde{b}) \prod_{k=1}^{3g-4} (\int \mu_k b) \delta(\int \mu_k \beta^-)(\int \mu_k G^-) \mu_{3g-3}(w) b(w) I^{-1}(w)|^2$$

$$\prod_{l=1}^{g} I(v_l) Z^-(v_l) \tilde{c}(x_0)|^2 > \tag{3.4}$$

evaluated at $M_i = 0$.

Since there are no available η^+'s or ξ^+'s to absorb ξ^-'s and η^-'s, $I(v_i)Z^-(v_i)$ can be replaced by $e^{\phi^-} \tilde{G}^+(v_i)$. In the above expression, it is easy to check that the path integral over the (b, c) ghosts cancels the path integral over the (β^-, γ^+) ghosts. The path integral over the (β^+, γ^-) fields is

$$\{Z([v_1 + ... + v_g - w - \Delta])\}^{-1} = \{Z_1 det\omega^j(v_k)\}^{-1}$$

where $(Z_1)^{-\frac{1}{2}}$ is the partition function for a chiral boson and Δ is the Riemann class. Since the $(\tilde{b}, \tilde{c})$ path integral contributes

$$\prod_{i=1}^{g} \int_{a_i} Z([y_1 + ... + y_g - x_0 - \Delta]) = Z_1,$$

the product of these path integrals cancels the $det\omega^j(v_k)$ factor in (3.4).

So the scattering amplitude is:

$$A_g^{2g-2} = \prod_{i=1}^{g} \int d^2 v_i \prod_{j=1}^{3g-3} \int d^2 m_j$$

$$< |\tilde{G}^+(v_1)...\tilde{G}^+(v_g) \prod_{k=1}^{3g-4} (\int \mu_k G^-) \mu_{3g-3}(w) J^{--}(w)|^2 >$$

which is just the topological prescription defined in (2.15).

It is easy to check that this proof for the partition function can be repeated for N-point scattering amplitudes. For vertex operators $\hat{V}_i = |c \exp(-\phi^+ - \phi^-)|^2 V_i$ where V_i is an N=2 primary field, the N=2 string correlation function at instanton-number n_I is

$$A_{g,1,...,N}^{n_I} = \prod_{i=1}^{g} \int d^2 M_i \prod_{j=1}^{3g-3+N} \int d^2 m_j$$

23

$$< |(\int_{a_i} \tilde{b})(\int \mu_j b))|^2 \cdot$$

$$|(Z^-)^{2g-2+N+n_I}(Z^+)^{2g-2+N-n_I} I^{n_I} \tilde{c}(x_0)|^2 \hat{V}_1(z_1)...\hat{V}_N(z_N) > .$$

At instanton-number $n_I = 2g - 2 + N$, $A^{2g-2+N}_{g,1,...,N}$ is equal to the topological expression

$$F^{2g-2+N}_{g,N} = \prod_{i=1}^{g} \int d^2 v_i \prod_{j=1}^{3g-3+N} \int d^2 m_j < |\tilde{G}^+(v_1)...\tilde{G}^+(v_g)|^2 \cdot$$

$$| \prod_{k=1}^{3g-4+N} (\int \mu_k G^-)\mu_{3g-3+N}(w)J^{--}(w)|^2 \phi_1(z_1)...\phi_N(z_N) >$$

where ϕ_i is the N=2 chiral field defined by $\phi_i = |\tilde{G}^+|^2 V_i$.

Although the above equivalence proof for the $N = 2$ strings and $N = 4$ topological prescription has been stated at the level of amplitudes, it is straightforward to check that the equivalence proof also holds at the level of integrands. In other words,

$$\prod_{i=1}^{g} \prod_{j=1}^{3g-4} < |(\int_{a_i} \tilde{b})(\int \mu_j b))(\mu(w)b(w))(Z^-)^{4g-4} I^{2g-2}(y_0)\tilde{c}(x_0)|^2 >$$

$$= \prod_{i=1}^{g} \prod_{j=1}^{3g-4} | \det(\omega_k(v_i))|^{-2} < |\tilde{G}^+(v_1)...\tilde{G}^+(v_g)(\int \mu_j G^-)(\mu_{3g-3}(w)J^{--}(w))|^2 >$$

where the U(1) moduli in the non-topological expression is equal to the divisor $[v_1 + ... + v_g - w + (2 - 2g)y_0 + \Delta]$ in the second expression. Since the topological integrand for unitary theories has been shown in (2.30) to be independent of the v_i locations for $g > 1$ (or more precisely, it depends on v_i like $\det(\omega_i(v_j))$), this immediately implies for a unitary N=2 string that the integrand at instanton number $n_I = 2g - 2$ is independent of the U(1) moduli when $g > 1$.

We next consider other instanton numbers. Since there are total-derivative ambiguities when both Z^+'s and Z^-'s are present in the non-topological calculation (there is a singularity when Z^+ and Z^- collide, so it is clear that the integrand of the non-topological amplitude can not be independent of their locations), the integrand of the non-topological expression can not be equal to the integrand of a topological amplitude. However, as was shown in the previous section, the topological partition function can be expressed as

$$F^{top}_g = \sum_{n=2-2g}^{2g-2} \frac{(4g-4)!}{(2g-2+n)!(2g-2-n)!} F^n_g (u_1)^{2g-2+n}(u_2)^{2g-2-n},$$

where equation (2.14) is for F_g^{2g-2}. We will now prove that the coefficients, F_g^n, of this polynomial provide an unambiguous definition of the $N = 2$ string partition function on a surface of instanton number n for $2 - 2g \leq n \leq 2g - 2$.

The first step in the proof is to start with an N=2 string which has been rotated by the SU(2) flavor generators, so the matter part of its two superconformal generators are $\widehat{G^+}$ and $\widehat{G^-}$ where $\widehat{G^+} = u_2^* G^+ - u_1^* \tilde{G}^+$ and $\widehat{G^-} = u_1 G^- + u_2 \tilde{G}^-$ (note that the OPE of $\widehat{G^+}$ with $\widehat{G^-}$ is the same as the OPE of G^+ with G^-). Since this replaces G^- by $\widehat{G^-}$ and $\tilde{G}^+$ by $\widehat{\tilde{G}^+} = u_1 \tilde{G}^+ - u_2 G^+$, it is clear that the resulting partition function at instanton number $n_I = 2g - 2$ is precisely $F_g(u_1, u_2)$. We will now show that this partition function of the flavor-rotated N=2 string at $n_I = 2g - 2$ is related to the partition function of the unrotated string at $n_I < 2g - 2$.

The rotated partition function at $n_I = 2g - 2$ is given by the non-topological expression:

$$\widehat{A}_g^{2g-2}(u_1, u_2) = \prod_{i=1}^{g} \int d^2 M_i \prod_{j=1}^{3g-3} \int d^2 m_j$$

$$< |(\int_{a_i} \tilde{b})(\int \mu_j b))(\widehat{Z}^-)^{4g-4} I^{2g-2} \tilde{c}(x_0)|^2 >$$

where $\widehat{Z}^- = \{\widehat{Q}, \xi^-\}$ and $\widehat{Q}$ is the flavor-rotated Q which has the matter part of the superconformal generators, G^+ and G^-, replaced with $\widehat{G^+}$ and $\widehat{G^-}$. The next step in the proof is to perform the unitary transformation:

$$\xi^- \to u_1 \xi^- + u_2 \xi^+ \tilde{I}^{-1}, \quad \eta^- \to u_1 \eta^- + u_2 \eta^+ \tilde{I}^{-1},$$

$$\xi^+ \to u_2^* \xi^+ - u_1^* \xi^- \tilde{I}, \quad \eta^+ \to u_2^* \eta^+ - u_1^* \eta^- \tilde{I}, \tag{3.5}$$

where $\tilde{I} = \exp(\phi^- - \phi^+ - J) = I - \tilde{c}bI$ (note that $\tilde{I}$ is not BRST invariant since we have removed the $\tilde{b}$ dependence). The infinitesimal version of this transformation is generated by

$$J^+ = \int \xi^+ \eta^+ \tilde{I}^{-1}, \quad J^- = \int \xi^- \eta^- \tilde{I},$$

$$J^3 = \int (\xi^- \eta^+ - \xi^+ \eta^- + \partial_z(\phi^+ - \phi^-) + J_{color}).$$

It is easy to check that under this transformation, $\widehat{Q} \to Q$, and therefore,

$$\widehat{Z}^- \to \{Q, \; u_1 \xi^- + u_2 \xi^+ \tilde{I}^{-1}\}$$

25

$$= u_1 Z^- + u_2 Z^+ I^{-1} + u_2 c \xi^+ \partial_z (I^{-1})$$

where we have ignored terms proportional to $\tilde{b}$ since they can not contribute to the scattering amplitude. Furthermore, since moving the location of the I's is the same as shifting the U(1) moduli, the term proportional to $\partial_z(I^{-1})$ can be dropped since it is a total derivative in the U(1) moduli.

So after performing this unitary transformation, the rotated partition function at $n_I = 2g - 2$ can be expressed as

$$\widehat{A}_g^{2g-2}(u_1, u_2) = \prod_{i=1}^{g} \int d^2 M_i \prod_{j=1}^{3g-3} \int d^2 m_j$$

$$< |(\int_{a_i} \tilde{b})(\int \mu_j b))(u_1 Z^- + u_2 Z^+ I^{-1})^{4g-4} I^{2g-2} \tilde{c}(x_0)|^2 >,$$

which is equal to

$$\sum_{n=2-2g}^{2g-2} \frac{(4g-4)!}{(2g-2+n)!(2g-2-n)!} A_g^n (u_1)^{2g-2+n} (u_2)^{2g-2-n}$$

where A_g^n is the non-rotated partition function at instanton number n which is defined in (3.3). We have therefore proven our claim that the various coeffcients in the topological partition function F_g provide an unambiguous definition of the $N = 2$ string partition function at all instanton numbers. It is straightforward to generalize this proof for N-point scattering amplitudes and find a similar relation between the topological prescription and the $N = 2$ string prescription at different instanton numbers.

There is one subtle point in this equivalence proof which needs to be discussed. In defining $N = 2$ string scattering amplitudes, we need to integrate over the negative-energy ghost fields ϕ^+ and ϕ^-. As was shown in [17] (although this reference only discusses the N=1 ghosts, it is trivial to generalize their prescription to the N=2 case), a convenient way to perform this integration is to sew $\int \eta^+$ and $\int \eta^-$ around each of the a-cycles, and to restrict the momentum $\int (\partial_z \phi^\pm - \eta^\pm \xi^\mp)$ to be zero through the internal loops. In order for our proof to be correct, we must check that this prescription is unchanged by the transformation of (3.5). Fortunately, this is the case since

$$\int \eta^- \int \eta^+ \rightarrow \int (u_1 \eta^- + u_2 \eta^+ \tilde{I}^{-1}) \int (u_2^* \eta^+ - u_1^* \eta^- \tilde{I}) = \int \eta^- \int \eta^+$$

(we have used that $\int \eta^+ \int \eta^+ = 0$ and that $\tilde{I}$'s can be moved around without changing the amplitude). Similarly, it is easy to check that $\int (\partial_z \phi^\pm - \eta^\pm \xi^\mp)$ is unchanged by the transformation.

Note that the integrand of the topological amplitude is equal to the integrand of the $N = 2$ string amplitude only for the terms corresponding to the instanton numbers $n_I = 2g - 2$ and $n_I = 2 - 2g$ (the $n_I = 2 - 2g$ term is related to the $n_I = 2g - 2$ term by replacing Z^- with Z^+ and twisting the N=2 algebra in the opposite direction). For all other n_I, only the integrated amplitudes are guaranteed to be equal (this must be true since if $|n_I| < 2g - 2$, the integrand of $A_g^{n_I}$ in the non-topological prescription is only defined up to total derivatives). So even though F_g is independent of the v_i locations for all values of u, we are only able to say for instanton-numbers $n_I = \pm(2g - 2)$ that the non-topological partition functions are independent of the U(1) moduli.

Since we have found a relation between flavor-rotated amplitudes at instanton number $n_I = 2g - 2$, $\widehat{A}_g^{2g-2}$, and unrotated amplitudes at other instanton-numbers, A_g^n, we can ask what is the relation between the flavor-rotated amplitudes at different instanton numbers. Let us define $\widehat{A}_g^n(u, u^*)$ to be the flavor-rotated amplitude at instanton-number n (i.e., $\widehat{A}_g^n$ is the non-topological amplitude at $n_I = n$ using $\widehat{G^-}$ and $\widehat{G^+}$ as the matter part of the N=2 superconformal generators).

We can use our knowledge of $\widehat{A}_g^{2g-2}$ to check that at $u_1 = 1$ and $u_2 = 0$,

$$(u_a^* \frac{d}{du_a})^m \widehat{A}_g^{2g-2} = \frac{(4g-4)!}{(4g-4-m)!} A_g^{2g-2-m} \tag{3.6}$$

Note that at this value of u, $\widehat{G}^+ = G^+$ and $\widehat{G}^- = G^-$ so $\widehat{A}_g^{2g-2-m}$ is equal to A_g^{2g-2-m}. However, $u_a^* \frac{d}{du_a}$ commutes with the SU(2) flavor rotations of (2.19) which transform u_a and u_a^* as independent doublets. So if (3.6) is valid at $u_1 = 1$ and $u_2 = 0$, it must be valid for all values of u, and therefore

$$(u_a^* \frac{d}{du_a})^m \widehat{A}_g^{2g-2} = \frac{(4g-4)!}{(4g-4-m)!} \widehat{A}_g^{2g-2-m}. \tag{3.7}$$

This immediately implies that

$$(u_a^* \frac{d}{du_a}) \widehat{A}_g^n = (2g - 2 + n) \widehat{A}_g^{n-1}, \quad (u_a \frac{d}{du_a^*}) \widehat{A}_g^n = (2g - 2 - n) \widehat{A}_g^{n+1}.$$

It is also easy to show that the harmonicity equations defined in (2.27) are equivalent to the equations:

$$u_a^* \frac{d}{dt_a} \widehat{A}_g^n = u_a \frac{d}{dt_a} \widehat{A}_g^{n-1}, \quad u_a^* \frac{d}{dt_a^*} \widehat{A}_g^n = u_a \frac{d}{dt_a^*} \widehat{A}_g^{n-1}$$

for $2 - 2g < n \leq 2g - 2$.

4. The Self-Dual String

In this section, we shall apply the topological method to the calculation of scattering amplitudes for the N=2 string which describes self-dual gravity in two complex dimensions [7]. It will be seen that the topological description of the amplitudes is much more convenient to work with as there are no $N = 2$ ghosts around. This will in particular allow us to show that all amplitudes except for three point amplitudes vanish to all loops. Moreover, in the context of the self-dual string, we will be able to interpret the u variables as describing the twistor space.

The simplest background for the $N = 2$ string is R^4 with signature $(2, 2)$ [18][19]. To couple a conformal theory to the $N = 2$ string, we need an $N = 2$ superconformal theory, which in particular requires a complex structure in the target-space. This is easily done by viewing R^4 as C^2, and labeling the coordinates as $(x_j, \bar{x}_{\bar{j}})$ with $j = 1, 2$. The 'Lorentz group' for this $N = 2$ string is thus the $U(1, 1)$ subgroup of $SO(2, 2)$ which preseves the complex structure. The fermionic right-moving variables of this $N = 2$ string will be denoted by ψ_j^+ and $\psi_{\bar{j}}^-$, and the fermionic left-moving variables by $\bar{\psi}_j^+$ and $\bar{\psi}_{\bar{j}}^-$. The non-zero components of the spacetime metric are $\eta^{1\bar{1}} = -\eta^{2\bar{2}} = 1$.

The right-moving twisted N=2 superconformal generators are given by

$$L = \partial_z x_j \partial_z \bar{x}_{\bar{j}} + \psi_{\bar{j}}^- \partial_z \psi_j^+, \quad G^+ = \psi_j^+ \partial_z \bar{x}_{\bar{j}}, \quad G^- = \psi_{\bar{j}}^- \partial_z x_j, \quad J = \psi_j^+ \psi_{\bar{j}}^-.$$

Note that after twisting, ψ_j^+ has dimension 0 while $\psi_{\bar{j}}^-$ has dimension 1.

Since this N=2 stress-tensor has $\hat{c} = 2$, it can be used to construct an N=4 stress-tensor in the manner described in section 2. The additional generators are

$$\tilde{G}^+ = \epsilon^{jk} \psi_j^+ \partial_z x_k, \quad \tilde{G}^- = \epsilon^{\bar{j}\bar{k}} \psi_{\bar{j}}^- \partial_z \bar{x}_{\bar{k}}, \quad J^{++} = \epsilon^{jk} \psi_j^+ \psi_k^+, \quad J^{--} = \epsilon^{\bar{j}\bar{k}} \psi_{\bar{j}}^- \psi_{\bar{k}}^-.$$

Note that the currents form an SU(1,1) "color" symmetry rather than an SU(2) (the SU(1,1) OPE's are those of (2.1) and (2.3) but with J^{--} and $\tilde{G}^-$ replaced with $-J^{--}$ and $-\tilde{G}^-$). Similarly, the "flavor" symmetry of these N=4 generators is SU(1,1) instead of SU(2).

In defining the $N = 2$ generators, we had to choose a complex structure on R^4, and as is well known from twistor constructions of self-dual metrics, the possibility for doing this is parametrized by $SU(2)/U(1)$ for the $(4, 0)$ signature and $SU(1, 1)/U(1)$ for the $(2, 2)$ signature. This is precisely the freedom we have in defining an $N = 2$ algebra starting from the R^4 theory which has $N = 4$ symmetry (taking into account the θ angle which was

discussed in a footnote in the previous section). In order to keep track of this $SU(1,1)_f$ flavor symmetry, it is convenient to define a 3-parameter family of N=4 generators as in section 2. These generators are:

$$\widehat{\tilde{G}^+}(u) = u_1 \epsilon^{jk} \psi_j^+ \partial_z x_k + u_2 \psi_{\bar{j}}^+ \partial_z \bar{x}_{\bar{j}}, \quad \widehat{G^-}(u) = u_1 \psi_{\bar{j}}^- \partial_z x_j + u_2 \epsilon^{\bar{j}\bar{k}} \psi_{\bar{j}}^- \partial_z \bar{x}_{\bar{k}},$$

$$\widehat{\tilde{G}^-}(u) = u_2^* \epsilon^{\bar{j}\bar{k}} \psi_{\bar{j}}^- \partial_z \bar{x}_{\bar{k}} + u_1^* \psi_{\bar{j}}^- \partial_z x_j, \quad \widehat{G^+}(u) = u_2^* \psi_j^+ \partial_z \bar{x}_{\bar{j}} + u_1^* \epsilon^{jk} \psi_j^+ \partial_z x_k,$$

where $|u_1|^2 - |u_2|^2 = 1$. Note that we are using the notation that $\overline{u_a} = \epsilon^{ab} \eta_{a\bar{a}} u_b^*$ so $\overline{u_1} = u_2^*$ and $\overline{u_2} = u_1^*$.

Although it is not obvious in this notation, the u's keep track of not only the SU(1,1) flavor symmetry, but also the choice of complex structure in the original SO(2,2) space-time.[7] In order to make this more transparent, it is convenient to write the spacetime x variables in SO(2,2) vector notation. This can be done by defining $x_j = x^\mu \sigma_\mu^{1k} \epsilon_{kj}$ where

$$\sigma_0^{jk} = i\delta^{jk}, \quad \sigma_1^{jk} = i\sigma_x^{jk}, \quad \sigma_2^{jk} = i\sigma_y^{jk}, \quad \sigma_3^{jk} = \sigma_z^{jk}.$$

Note that the σ^μ matrices are defined such that $\overline{(x_j)} = \bar{x}_{\bar{j}} = x^\mu (\sigma_\mu^{2k}) \eta_{\bar{j}k}$.

We can therefore write the generators in SO(2,2) notation as

$$\widehat{\tilde{G}^+}(u) = u_i \psi_j^+ \partial_z x^{ij}, \quad \widehat{G^-}(u) = u_i \psi_{\bar{j}}^- \partial_z x^{ik} \epsilon_{jk},$$

$$\widehat{\tilde{G}^-}(u) = u_i^* \psi_{\bar{j}}^- \partial_z x^{ik} \epsilon_{jk}, \quad \widehat{G^+}(u) = u_i^* \psi_j^+ \partial_z x^{ij},$$

where $x^{ij} = x^\mu \sigma_\mu^{ij}$. Note that SO(2,2) is equivalent to $SU(1,1) \times SU(1,1)$, and the first and second indices of x_{ij} transform as doublets under the first and second SU(1,1)'s. Therefore, just as the u's keep track of how the SU(1,1) flavor symmetry of the N=4 string is broken, they also keep track of how the first SU(1,1) of SO(2,2) is broken.

Although one might attempt to integrate over the u variables in the style of [15] and thereby try to recover SO(2,2) invariance, this does not seem like the right thing to do from the string point of view. Since the u variables label different choices of complex structure

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[7] N=4 worldsheet supersymmetry was also used by Siegel in [20] to describe the self-dual string in manifest $SO(2,2)$ notation. However unlike [20] , we are not attempting to quantize the $N = 4$ strings, but instead are using the u variables to choose a complex structure in the target-space, thereby allowing quantization as an $N = 2$ string. Although Siegel has claimed to find target-space supersymmetry using his method of quantization[21], we have yet to find such target-space supersymmetry in the $N = 2$ self-dual string.

for the background, integrating over the u's would be like integrating over different choices of backgrounds for the string theory. This would be like trying to do second-quantized string field theory, instead of following the usual first-quantized procedure of calculating scattering amplitudes in a fixed background. However as discussed in the previous section, if we know the amplitudes for a given choice of complex structure, we can obtain the amplitudes for other choices by the fact that the instanton corrections lead to a partition function which transforms according to a representation of SU(2) (or (SU(1,1)), where instanton number labels the J_3 eigenvalue of the flavor SU(2) (or SU(1,1)).

Even though we have developed the above discussion in the context of flat space, the story can be stated more generally: Suppose we have a self-dual Ricci-flat metric on a manifold. Then there is a sphere (or disc in the (2,2) signature) worth of ways to choose a complex structure on the manifold for which the Ricci-flat metric is a Kähler metric, thus giving rise to an $N = 2$ string vacuum (see [22] for a recent discussion on this and its relation to $K3$ moduli).

Let us now return to the flat case and consider computing the scattering amplitudes. The vertex operators for the self-dual string will be constructed out of the dimension zero primary fields $V(k) = e^{ik^\mu x_\mu}$ where $k_\mu k^\mu = 0$, and out of the zero-momentum dimension $1/2$ chiral fields $\phi_j^+ = \psi_j^+$. As was described in the previous sections, the topological prescription for the scattering amplitudes of these vertex operators is given by:

$$F_{g,1,...,N} = \prod_{i=1}^{g} \int d^2 v_i \prod_{j=1}^{3g-3+N} \int d^2 m_j$$

$$< |\widehat{\widetilde{G}^+}(v_i) \prod_{k=1}^{3g-4+N} (\int \mu_k \widehat{G^-}) \int \mu_{3g-3+N} J^{--}|^2 \phi_1^+(z_1)...\phi_N^+(z_N) >$$

where for states with non-zero momentum, $\phi^+ = [\widehat{\widetilde{G}^+}, V(k)] = u_i \psi_j^+ k^{ij} e^{ik^\mu x_\mu}$.

Note that for genus 0, we can use the definition of amplitudes given in (2.24). In particular, the three point function is given by

$$\langle \phi_1^+ \phi_2^+ V_3 \rangle$$

which easily reproduces what one expects for the three point function of self-dual strings [7].

4.1. Vanishing Theorems

We will now discuss why certain amplitudes vanish for the self-dual string in flat space. The first type of vanishing theorem will concern scattering amplitudes with less than three external states. The proofs will use the fact that scattering amplitudes are invariant under all SO(2,2) Lorentz-transformations if the u_i variables are transformed like SU(2) spinors. The open-string partition function is a polynomial of degree $4g - 4$ in u_i and since there are no momentum factors to contract with the SU(2) index of u_i, this amplitude must vanish.[8] The one-point function vanishes since the vertex operator is proportional to the momentum, and therefore is zero by momentum conservation. For the two-point function, there is one independent momentum k^μ satisfying $k^\mu k_\mu = 0$. Since there is no way to construct an SO(2,2) scalar out of k^μ's, u_i^R's and u_i^L's which is at least linear in k^μ, the two-point function must also vanish. Note that the vanishing theorems for the one and two-point functions are valid for both open and closed N=2 strings.

The second type of vanishing theorem that will be proven is that for N-point functions of arbitrary genus, the amplitude vanishes unless $k_\mu^r k^{\mu\,s} = 0$ for all $1 \le r, s \le N$ where k_μ^r is the momentum of the rth external state. For three-point functions, this is not a restriction since it is implied by conservation of momentum and the mass-shell condition. However for more than three external states, it implies that the amplitude vanishes unless all particles are moving in the same self-dual plane. Since scattering amplitudes should be analytic in momenta, this means that all amplitudes other than three-point amplitudes are identically zero to all orders.

The trick to proving this second type of vanishing theorem is to realize that the primary field $V(k_\mu) = e^{ik_\mu x^\mu}$ satisfies the identity $[\int G^+, V] = h(k_\mu)[\int \tilde{G}^+, V]$ where $|h(k_\mu)|^2 = 1$. This identity is true since

$$[\int G^+, V] = \psi_j^+ \bar{k}_{\bar{j}} e^{ik_\mu x^\mu} = h(k_\mu) \epsilon^{jl} \psi_j^+ k_l e^{ik_\mu x^\mu} = h(k_\mu)[\int \tilde{G}^+, V]$$

where $h(k_\mu) = \bar{k}_1/k_2$ (for convenience, we have reverted to the original SU(1,1) notation). Note that since $k^\mu k_\mu = k_1 \bar{k}_1 - k_2 \bar{k}_2 = 0$, $\bar{k}_2/k_1 = \bar{k}_1/k_2$. In fact for $k \ne 0$, the G^+ and

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[8] For the closed-string partition function, one has both right-handed u_i^R's and left-handed u_i^L's, so it is possible to construct the SO(2,2) scalar $(u_i^R u_j^L \epsilon^{ij})^{4g-4}$. For this reason, we can only prove that the closed-string partition function vanishes when $u_i^R u_j^L \epsilon^{ij} = 0$ (for example, it vanishes if the right and left-handed instanton numbers are not equal). We would like to thank Hirosi Ooguri for pointing this out to us.

$\tilde{G}^+$ cohomology is trivial and so we can apply the construction of section 2 (see (2.11)) to obtain a whole ladder of observables $\phi^{(n)}$. What we are finding here is that all $\phi^{(n)}$ constructed in this way are proportional:

$$\phi^{(n)} = h^n \phi^{(0)} \tag{4.1}$$

We will now show that unless $h_r = h_s$ for $1 \leq r, s \leq N$, the amplitude vanishes. Since $h_r = h_s$ implies $\bar{k}_2^r / k_1^r = \bar{k}_1^s / k_2^s$ and $\bar{k}_1^r / k_2^r = \bar{k}_2^s / k_1^s$, it also implies that $k_\mu^r k^{s\,\mu} = \bar{k}_{\bar{j}}^r k_j^s + k_j^r \bar{k}_{\bar{j}}^s = 0$. In other words, the amplitude vanishes unless all the Mandelstam variables are zero. Since for more than three external states, we can define an amplitude with vanishing Mandelstam variables as a limit of one with non-vanishing Mandelstam variables, we conclude that all N-point functions for $N \neq 3$ vanish to all loops. For the three-point function, the Mandelstam variables are identically zero so we cannot use this argument to prove vanishing, and in fact it is known to be non-vanishing at tree [7] and one loop [23] .

However our symmetry argument teaches us something also about three-point amplitudes. We can view the momentum of a massless particle in $U(1,1)$ notation as two complex numbers of equal norm, one denoting the 'energy' and the other denoting the 'momentum'. Note moreover that the h we have defined above is simply the sum of the phases of these two complex numbers. For a three-point amplitude, conservation of energy-momentum implies that we have two equal triangles, one on the energy plane and the other on the momentum plane. It was found in [7] (and similarly in [23]) that the three point amplitude is non-vanishing only when the two triangles have opposite orientation. This actually follows from the fact that h_r is the same for the three particles.

We will now prove that the amplitude vanishes when any two of the h's are not equal. Suppose that h_1 and h_2 are not equal. Then the scattering amplitude is given by

$$F_{g,1,\dots,N} = \prod_{i=1}^{g} \int d^2 v_i \prod_{j=1}^{3g-3+N} \int d^2 m_j$$

$$< |\prod_{i=1}^{g} \widehat{\tilde{G}^+}(v_i) \prod_{k=1}^{3g-4+N} (\int \mu_k \widehat{G^-})(\int \mu_{3g-3+N} J^{--})|^2 \phi_1^+(z_1)...\phi_N^+(z_N) >$$

where

$$\phi_1^+ = \widehat{\tilde{G}^+}\widehat{G^+}V_1 = \widehat{G^+}(u_1 \tilde{G}^+ + u_2 G^+)e^{ik_\mu^1 x^\mu} = (u_1 h_1^{-1} + u_2)\widehat{\tilde{G}^+}G^+V_1$$

and $\phi_2^+ = \widehat{\tilde{G}^+}\widehat{\tilde{G}^+}V_2$.

Now pull $\widehat{\tilde{G}^+}$ off of V_2 and onto J^{--}, which is the only place it has a singularity. The amplitude is then

$$F_{g,1,...,N} = \prod_{i=1}^{g} \int d^2v_i \prod_{j=1}^{3g-3+N} \int d^2m_j$$

$$(u_1 h_1^{-1} + u_2) < |\prod_{i=1}^{g} \widehat{\tilde{G}^+}(v_i) \prod_{k=1}^{3g-4+N} (\int \mu_k \widehat{G^-})|^2 (\int \mu_{3g-3+N}\widehat{G^-} \int \bar{\mu}_{3g-3+N}\bar{J}^{--})$$

$$\widehat{\tilde{G}^+}G^+V_1(z_1)\widehat{\tilde{G}^+}V_2(z_2)\phi_3^+(z_3)...\phi_N^+(z_N) > .$$

The next step is to pull G^+ off of V_1 and onto V_2. Although G^+ also has singularities at the $\widehat{G^-}$'s, the residues of these poles are total derivatives in the $3g - 3 + N$ modular parameters.[9]

Finally, pull $\widehat{\tilde{G}^+}$ off of the last beltrami differential and onto V_1, which is the only place it has a singularity. The resulting expression for the scattering amplitude is

$$F_{g,1,...,N} = \prod_{i=1}^{g} \int d^2v_i \prod_{j=1}^{3g-3+N} \int d^2m_j$$

$$(u_1 h_1^{-1} + u_2) < |\prod_{i=1}^{g} \widehat{\tilde{G}^+}(v_i) \prod_{k=1}^{3g-4+N} (\int \mu_k \widehat{G^-})(\int \mu_{3g-3+N}J^{--})|^2$$

$$\widehat{\tilde{G}^+}\widehat{G^+}V_1(z_1)\widehat{\tilde{G}^+}G^+V_2(z_2)\phi_3^+(z_3)...\phi_N^+(z_N) > .$$

Since $G^+V_2 = (u_1 h_2^{-1} + u_2)^{-1}\widehat{G^+}V_2$, we get

$$F_{g,1,...,N} = \frac{u_1 h_1^{-1} + u_2}{u_1 h_2^{-1} + u_2}F_{g,1,..,N}.$$

By replacing G^+ in the above argument with $\tilde{G}^+$, one can similarly prove that

$$F_{g,1,...,N} = \frac{u_1 + u_2 h_1}{u_1 + u_2 h_2}F_{g,1,...,N}.$$

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[9]   In this proof of the second type of vanishing theorem, we will ignore all potential surface term contributions. This is presumably justified since we can analytically continue in momenta and we have derivatives on moduli parameters for both left- and right-movers [12]. The story may be more subtle for hetertotic $N = 2$ strings.

It is obvious that for both of these equations to be true, either $h_1 = h_2$ or $F_{g,1,\ldots,N}$ must vanish.

Note that the above techniques can also be used to prove that $F_{g,1,\ldots,N}$ for the self-dual string obeys the identity

$$(h\, d/du_1 - d/du_2)F_{g,1,\ldots,N}(u_1, u_2) = 0,$$

and therefore $F^n_{g,1,\ldots,N} = hF^{n+1}_{g,1,\ldots,N}$ where $h = \bar{k}^r_1/k^r_2$.

4.2. Meaning of Vanishing Theorems for Self-dual Backgrounds

Having proven some vanishing theorems, we will now discuss their physical meaning[10]. The vanishing of an amplitude usually implies the existence of symmetries. It is important in this context to recall the Coleman-Mandula theorem [24] which states that if we have a higher-spin symmetry in a theory, it is either a two dimensional thoery or it is a free theory. The idea of the proof is to look at two-particle scattering and to show that the higher-spin symmetry forces the amplitude to be zero except for forward or backward scattering. In more than two dimensions, if the theory is not free, this would imply a non-analytic S-matrix which is ruled out. We can thus have a non-trivial theory only in two dimensions. Indeed as is well known, all integrable models are two-dimensional theories of this type which have an infinite number of higher-spin symmetries.

Here, for the $N = 2$ self-dual string, we have proven that scattering amplitudes vanish for four and higher-point functions. In fact, our proof resembled the steps in the Coleman-Mandula theorem since we first showed that the amplitudes vanish unless the Mandelstam variables are all zero, and we then used analyticity of the S-matrix to prove that the amplitudes always vanish. But the self-dual $N = 2$ string is nevertheless not a free theory! The explanation for this is that we have signature $(2, 2)$ and not $(3, 1)$. For a theory with signature $(3, 1)$, vanishing of the four-point amplitude implies vanishing of the three-point amplitude by unitarity arguments. However for signature $(2, 2)$, we have seen that four-point amplitudes can vanish in accord with having higher-spin symmetries and the Coleman-Mandula theorem, but this does not imply vanishing of the three-point amplitude (for which there is a phase space, unlike the signature $(3, 1)$ case).

We should thus expect a higher-spin symmetry in the self-dual string. In fact, an infinite symmetry group is expected as in two-dimensional integrable models since most (if

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[10] Some of the above vanishing theorems were anticipated in [19].

not all) integrable models in two dimensions are reductions of self-dual equations in four dimensions. As was discussed in [7], the result of [25] shows that the loop group of area-preserving diffeomorphisms is a group which maps one solution to other solutions. This transformation is space-time dependent and may be viewed as corresponding to higher-spin transformations. It would be very important to flesh out this symmetry in a more conventional form. The above vanishing proof may give some hints as to how one may realize these symmetries more concretely.

4.3. One loop Partition Function

So far, we have talked about a very simple self-dual string background, namely R^4. There are other interesting classes of such backgrounds [7], for example $T^*\Sigma$ (the cotangent of a Riemann surface) with signature $(2,2)$ or Euclidean backgrounds of signature $(4,0)$. As for compact Euclidean theories, there are only two possibilities: T^4 and $K3$. As will be discussed in section 6, even the Euclidean backgrounds will be important for string theory (they can be viewed as the internal part of superstrings compactified to six dimensions). In the Euclidean theories, the only physical fields are the moduli of the manifold (there are no propagating modes), so the most relevant computation is the dependence of the partition function on the moduli. To get a feeling for the Euclidean theories, we will consider in this section the one loop partition function for the case of T^4 (and the subcases $T^2 \times T^2$, $R^2 \times T^2$ and R^4). We will check at least for these cases if it is valid to ignore the total derivative terms in deriving (2.27) .

For the one-loop partition function on T^4, we have four left-moving and four right-moving fermionic zero modes. Therefore the F_L^2 and F_R^2 in the definition of the one loop amplitude (2.21) will absorb them. Since the fermionic and bosonic oscillators cancel out, all that is left to contribute to (2.21) is the extrema of classical solutions which gives the usual Narain sum [26]. Let $\Gamma^{4,4}$ denote a self-dual $(4,4)$ Narain lattice with a decomposition to left and right momenta (p_L, p_R). Then we have

$$F_1^{T^4} = \int d^2\tau \sum_{(p_L,p_R)\in\Gamma^{4,4}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}$$

where as in (2.21) ,we subtract the ground state contribution to get a convergent answer, i.e. the $(p_L, p_R) = 0$ part is deleted from the sum.

For the theory with $T^2 \times T^2$, we should restrict to lattices which decompose to $\Gamma^{2,2} \oplus \Gamma^{2,2} = \Gamma^{4,4}$. The moduli of each T^2 is determined by a pair of complex variables (σ, ρ)

which take values in the upper half plane modulo an $SL(2, Z)$ action and an exchange of σ with ρ [27]. We will define

$$Z(\sigma, \rho) = \sum_{p_L, p_R} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2}$$

where

$$(p_L, p_R) = \frac{1}{\sqrt{2\mathrm{Im}\sigma\mathrm{Im}\rho}}(n_1 + m_1\sigma + \rho(n_2 + m_2\sigma); n_1 + m_1\sigma + \bar{\rho}(n_2 + m_2\sigma))$$

for integer n_i, m_i. Then for F_1 on $T^2 \times T^2$ we have

$$F_1^{T^2 \times T^2} = \int d^2\tau \ Z(\sigma_1, \rho_1)Z(\sigma_2, \rho_2) \tag{4.2}$$

We can also take the limit where the torus degenerates to obtain the theory on $T^2 \times R^2$ and R^4 (or $R^{2,2}$) with the results

$$F_1^{T^2 \times R^2} = \int \frac{d^2\tau}{\tau_2} Z(\sigma, \rho) = \log(\sigma_2\rho_2|\eta(\sigma)\eta(\rho)|^2)$$

$$F_1^{R^4} = \int \frac{d^2\tau}{\tau_2^2} = \frac{\pi}{3}$$

(the first integral is discussed in [28]). It is satisfying to see that the $N = 2$ string computation of the above in [7] agrees with the topological answer given by (2.21).

We would now like to check the *harmonicity* condition that was derived in section 2 (eq. (2.29)). This also serves to check in at least one concrete example whether or not there are any boundary terms coming from total derivatives that we have ignored in our derivation of (2.27) . We will consider the case of $T^2 \times T^2$, and for concreteness, we will fix σ_1 and σ_2 to a constant (say i) and concentrate on the dependence of (4.2) on ρ_j where $j = 1, 2$ labels the tori.

To check equation (2.29) , we first note that the non-vanishing components of $M_i^{\bar{j}}$ and $M_{\bar{i}}^j$ are given by [11]

$$M_{\rho_1}^{\bar{\rho}_2} = M_{\rho_1}^{\rho_2} = \frac{\mathrm{Im}\rho_2}{\mathrm{Im}\rho_1} \qquad M_{\rho_2}^{\bar{\rho}_1} = M_{\rho_2}^{\rho_1} = \frac{\mathrm{Im}\rho_1}{\mathrm{Im}\rho_2}$$

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[11] This can be easily derived using the tt^* equations of [29].

From this, we see that equation (2.29) for $\partial_{\rho_1}\bar{\partial}_{\rho_2}$ is identically satisfied and the only equation to check is

$$\partial_{\rho_1}\partial_{\bar{\rho}_1}F_1 = \frac{(\mathrm{Im}\rho_2)^2}{(\mathrm{Im}\rho_1)^2}\partial_{\rho_2}\partial_{\bar{\rho}_2}F_1 \tag{4.3}$$

Using the identity (noted in [30])

$$\partial_\rho\bar{\partial}_\rho Z(\sigma,\rho) = \frac{\mathrm{Im}\tau}{(\mathrm{Im}\rho)^2}\partial_\tau\bar{\partial}_\tau(\mathrm{Im}\tau\ Z(\sigma,\rho)) \tag{4.4}$$

we learn that the left hand side of (4.3) is given by

$$\partial_{\rho_1}\partial_{\bar{\rho}_1}F_1 = \int d^2\tau\partial_{\rho_1}\bar{\partial}_{\rho_1}Z(\sigma_1,\rho_1)Z(\sigma_2,\rho_2)$$

$$= \int d^2\tau\frac{\mathrm{Im}\tau}{(\mathrm{Im}\rho_1)^2}\partial_\tau\bar{\partial}_\tau(\mathrm{Im}\tau Z(\sigma_1,\rho_1))Z(\sigma_2,\rho_2)$$

Now assuming that integration by parts can be done (which we will justify in a moment), the last equation becomes

$$\frac{1}{(\mathrm{Im}\rho_1)^2}\int d^2\tau\ \mathrm{Im}\tau\ Z(\sigma_1,\rho_1)\partial_\tau\bar{\partial}_\tau(\mathrm{Im}\tau Z(\sigma_2,\rho_2))$$

$$= \frac{(\mathrm{Im}\rho_2)^2}{(\mathrm{Im}\rho_1)^2}\partial_{\rho_2}\partial_{\bar{\rho}_2}F_1$$

where to get the last line, we once again use (4.4), this time in the opposite direction.

We have thus proven (4.3) modulo checking the validity of integration by parts that we used. This requires checking that

$$\int d^2\tau\left[\partial\bar{\partial}(\mathrm{Im}\tau Z_1)(\mathrm{Im}\tau Z_2) - (\mathrm{Im}\tau Z_1)\partial\bar{\partial}(\mathrm{Im}\tau Z_2)\right] = 0. \tag{4.5}$$

One has to be careful with the possible boundary contributions because even though Z_i goes to 1 exponentially fast as $\tau \to \infty$, the prefactor $\mathrm{Im}\tau \to \infty$. To show (4.5) is satisfied, it is convenient to use the language of differential forms and rewrite the left hand side of (4.5) as

$$\int d\left[\bar{\partial}(\mathrm{Im}\tau Z_1)(\mathrm{Im}\tau Z_2) + (\mathrm{Im}\tau Z_1)\partial(\mathrm{Im}\tau Z_2)\right]$$

$$= \left[\bar{\partial}(\mathrm{Im}\tau Z_1)(\mathrm{Im}\tau Z_2) + (\mathrm{Im}\tau Z_1)\partial(\mathrm{Im}\tau Z_2)\right]\Big|_{\tau\to\infty}$$

$$= \frac{-1}{2i}(\mathrm{Im}\tau - \mathrm{Im}\tau + \text{exp. small})|_{\tau\to\infty} = 0.$$

This proves the validity of (4.5) and completes the proof of (4.3). This is a strong case for the absence of anomalies in general. Note in particular that in the $N = 2$ one-loop topological amplitude [2] ,even the torus example has anomalies.

5. Topological description of the superstring

It was recently shown that both the RNS and GS versions of the ten-dimensional superstring can be described by a critical N=2 string. The N=2 description of the RNS superstring is obtained by using the results of [5] to embed the usual N=1 RNS string in an N=2 string. The N=2 description of the GS superstring is related by a field-redefinition[10] to this N=2 description of the RNS string (the field-redefinition and the N=2 $\hat{c} = 2$ stress-tensors for these two descriptions of the superstring can be found in Appendix A).

Since the topological methods of section 2 apply to any critical N=2 string, we can now ask what is the resulting topological prescription for the superstring. In one sense, the topological prescription is closely related to the original $N = 1$ RNS prescription for the superstring since the $N = 2$ ghosts which were added to the $N = 1$ RNS string to make it into an $N = 2$ string are removed in the topological theory. Nevertheless, we shall find that the topological prescription has an advantage over the conventional $N = 1$ prescription since it does not appear to suffer from total-derivative ambiguities. However there is a technical point to overcome before this is fully established. The technical point arises because one of the N=2 matter fields in the superstring has negative-energy (this field is related to the superdiffeomorphism ghost of the $N = 1$ superstring).

We will begin by discussing the topological prescription for the RNS version of the superstring. As descibed in section 2, the topological amplitude is a polynomial in two variables which can be expressed as

$$F_g = \sum_{n=2-2g}^{2g-2} \frac{(4g-4)!}{(2g-2+n)!(2g-2-n)!} F_g^n (u_1)^{2g-2+n} (u_2)^{2g-2-n}.$$

We will show that F_g^{2g-2} is equivalent to the usual N=1 prescription for RNS amplitudes when the sum of the pictures of the vertex operators is equal to $2g-2$ (similar observations have been recently made in [31]). The other F_g^n's give a new prescription for calculating RNS superstring amplitudes when the sum of the pictures of the vertex operators is equal to n. Since the topological amplitude has no total-derivative ambiguities, these new pre-scriptions may be useful in resolving the ambiguities that plague the conventional N=1 RNS description of the superstring.

We shall first review the conventional $N = 1$ rules for calculating RNS superstring scattering amplitudes[32]. Aside from having to insert enough $N = 1$ picture-changing operators to get a non-zero result, the only subtlety in the $N = 1$ rules comes from the

functional integration over the $N = 1$ superdiffeomorphism ghosts, β and γ. Because the ϕ field (which comes from bosonizing β and γ as $\beta = \partial_z \xi \exp(-\phi)$ and $\gamma = \eta \exp(\phi)$) has negative-energy, the naive functional integral over its zero modes does not converge. The explanation of this divergence is that each physical state is represented by an infinite set of vertex operators which are related by picture-changing operators, and therefore the naive BRST cohomology contains infinite copies of each physical state.

To get a finite answer, the states propagating in internal loops should be restricted to a fixed picture. This can be accomplished by inserting the operator $\prod_{i=1}^{g} \delta(\int_{a_i}(\partial_z \phi + \xi \eta) - P_i)$ which restricts the picture of the state propagating through the ith a-cycle to be equal to P_i. Furthermore, because the ξ zero mode is not present in the (β, γ) ghosts, one should insert a ξ field anywhere on the surface and sew η fields around each a-cycle in order to reproduce the zero-mode structure of the original unbosonized system. As was shown in [17], these rules correctly reproduce the usual RNS scattering amplitudes for arbitrary genus.[12] It was also proven that the resulting expressions for the scattering amplitudes are independent of the choice of the P_i's.

To get the $N = 4$ topological theory for RNS strings, we define the $N = 4$ generators as

$$J^{++} = c\eta, \quad J^{--} = b\xi, \quad J = cb + \eta\xi, \tag{5.1}$$

$$G^- = b, \quad \tilde{G}^+ = \eta,$$

$$G^+ = \gamma G_m + c(T_m - \frac{3}{2}\beta\partial_z\gamma - \frac{1}{2}\gamma\partial_z\beta - b\partial_z c) - \gamma^2 b + \partial^2 c + \partial_z(c\xi\eta),$$

$$\tilde{G}^- = b(e^{\phi}G_m + \eta e^{2\phi}\partial b - c\partial\xi) - \xi(T_m - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}\gamma\partial\beta + 2b\partial c - c\partial b) + \partial^2 \xi$$

where T_m and G_m are the matter parts of the $N = 1$ stress-tensor for the RNS string with central charge $c = 15$. These N=4 generators can be obtained in the usual way from the critical N=2 stress-tensor used to embed the RNS string in an N=2 string. Note that $G^+ = J_{BRST} + \partial^2 c + \partial_z(c\xi\eta)$ where $Q_{N=1} = \int J_{BRST}$, and $\tilde{G}^- = [Q_{N=1}, b\xi]$.

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[12] There is one ambiguous point in reference [17] which should be mentioned. In proving that the above rules reproduce the usual RNS scattering amplitudes, the authors of [17] use the fact that $\sum_{k\epsilon \mathcal{Z}^g} \exp(iky) = \delta(y)$. This is of course true if y takes values on the g-torus $\mathcal{R}^g/\mathcal{Z}^g$, however in this context, y takes values in the Jacobian variety $\mathcal{C}^g/(\mathcal{Z}^g + \tau\mathcal{Z}^g)$. The problem is caused by trying to treat the right and left-moving sectors independently, which means that one is trying to construct a δ-function which depends on y but not on $\bar{y}$.

We will now show that the topological prescription for calculating F_g^{2g-2} precisely reproduces the conventional N=1 rules for RNS scattering amplitudes when the sum of the pictures of the vertex operators is equal to $2g - 2$. Since the N=2 matter sector of the topological string also contains the negative-energy field ϕ, one gets a naive divergence from its functional integral even in the topological prescription. This is because the cohomology of $\tilde{G}^+$ is trivial and as discussed in section 2, there is therefore an infinite set of chiral N=2 fields for each physical state of the RNS string (note that $[G^+, W] = [\tilde{G}^+, W] = 0$ implies that $[G^+, W'] = [\tilde{G}^+, W'] = 0$ where $W' = \{G^+, \xi W\}$). So as in the $N = 1$ prescription, we need to restrict the picture of the states propagating through the internal loops by inserting the operator $\prod_{i=1}^{g} \delta(\int_{a_i} (\partial_z \phi + \xi \eta) - P_i)$.

To calculate the term F_g^{2g-2} in F_g, we need to sew G^-'s with $3g - 4$ of the Beltrami differentials, sew J^{--} with the last Beltrami differential, and insert $[\int d^2 z (\tilde{G}^+ \bar{\tilde{G}}^+)]^g$. To get the right-moving part of this insertion, we should write

$$[\int d^2 z (\tilde{G}^+ \bar{\tilde{G}}^+)]^g = [\sum_{i=1}^{g} (\int_{a_i} dz \tilde{G}^+ \int_{b_i} d\bar{z} \bar{\tilde{G}}^+ - \int_{b_i} dz \tilde{G}^+ \int_{a_i} d\bar{z} \bar{\tilde{G}}^+)]^g$$

$$= g! \prod_{i=1}^{g} (\int_{a_i} dz \tilde{G}^+ \int_{b_i} d\bar{z} \bar{\tilde{G}}^+ - \int_{b_i} dz \tilde{G}^+ \int_{a_i} d\bar{z} \bar{\tilde{G}}^+).$$

It is easy to check that if $\tilde{G}^+$ is holomorphic and $\bar{\tilde{G}}^+$ is anti-holomorphic, this last expression is proportional to $det(Im\tau)|\prod_{i=1}^{g} \int_{a_i} dz \tilde{G}^+|^2$, and therefore the holomorphic part is obtained by sewing $\tilde{G}^+$ around the g a-cycles.[13] But using the N=4 generators defined in (5.1) , $\tilde{G}^+ = \eta$, $G^- = b$, and $J^{--} = b\xi$. We therefore reproduce the N=1 prescription of [17] where the ξ field is inserted on one of the Beltrami differentials. Since there are no explicit N=1 picture-changing operators in the topological prescription, all ϕ charge must come from the vertex operators which means that the sum of their pictures must equal $2g - 2$.

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[13] The $det(Im\tau)$ factor is cancelled by the functional integral over the ϕ and (ξ, η) fields which contributes $det(Im\tau)^{-1}$. Because the restriction on the ϕ momentum removes the quadratic part of the soliton action for these fields, there is no compensating contribution of $det(Im\tau)^{+1}$ coming from the sum over soliton configurations. This explains why the functional integral contains dependence on $Im\tau$, unlike the functional integrals for ordinary chiral bosons.

The condition on the sum of the pictures of the vertex operators can be understood by observing that $G^{\pm}$ carries zero picture while $\tilde{G}^{\pm}$ carries ± 1 picture (the picture-counting operator is $R = \int(\partial_z\phi + \xi\eta)$). Therefore for F_g^n to be non-zero,

$$\sum_{r=1}^{N} p_r = (n - g + 2) - 1 + (g - 1) = n$$

where p_r is the picture of the rth vertex operator, $(n - g + 2)$ is the number of $\tilde{G}^{+}$'s minus the number of $\tilde{G}^{-}$'s in the calculation of F_g^n, -1 is the picture of J^{--}, and $(g - 1)$ is the anomaly in R on a genus g surface. We therefore conjecture that F_g^n provides an unambiguous definition of the RNS scattering amplitude when the sum of the pictures of the vertex operators is equal to n.

One point which needs to be addressed is the functional integral over the negative-energy ϕ field. For F_g^{2g-2}, this functional integral was regularized by inserting around the internal loops the operator

$$\delta(\int(\partial_z\phi + \xi\eta) - P_i) = \delta(R - P_i) \tag{5.2}$$

for some integer or half-integer P_i. For the general F_g^n, it is not obvious that this regularization will give the correct scattering amplitude. Although R commutes with $G^{\pm}$, it does not commute with $\widehat{G^{\pm}}$ and is therefore not invariant under SU(2) flavor rotations.

However it will now be argued that the precise form of R is irrelevant for the calculation, and different choices for R give the same result. The only requirement on R is that the δ-function insertion fixes the momentum of the ϕ field, thereby reducing the cohomology to a single copy of each physical state. The argument is that since the scattering amplitude has been proven to be independent of the choice of the P_i's we can replace P_i by the momentum of any field other that $\xi\eta$ or ϕ, and the amplitude will not change. For example, we can replace P_i by $\int bc + P_i'$, so the new regularization is given by inserting the operator

$$\delta(\int(\partial_z\phi + \xi\eta - bc) - P_i').$$

If we do the functional integration over the ϕ and (η, ξ) fields before we do the integration over the (b, c) fields, it is clear that we can treat the zero mode $\int bc$ as a constant.

In fact, we could now reverse the roles of (b, c) and (η, ξ) and first do the functional integration over the ϕ and (b, c) fields. Since $\int \eta\xi$ now acts as a constant, we can remove

this constant by shifting $P_i' \to P_i' - \int \xi\eta$. So the resulting regularization prescription would be to insert the operator

$$\delta(\int (\partial_z \phi - bc) - P_i') = \delta(R' - P_i').$$

Note that $R' = \int(\partial_z\phi - bc)$ satisfies the commutation relations $[R', G^{\pm}] = \pm G^{\pm}$ and $[R', \tilde{G}^{\pm}] = 0$, so it is related to R by an SU(2) flavor rotation (R' is the $N = 1$ ghost-number operator). So we have argued that the precise form of the regularization for the ϕ field does not affect the scattering amplitude, and we can therefore use the operator insertion of (5.2) for all F_g^n. To be honest, we should note that the above argument is only as rigorous as the ambiguous point of reference [17] which was mentioned in a previous footnote. It would therefore be nice to perform explicit calculations using different regularizations for the ϕ field and to explicitly check if the resulting expressions are independent of the choice of regularization.

We will now discuss the topological prescription for the GS version of the superstring. Since we will also discuss the GS superstring in the following section, we will limit our discussion here to those aspects that are not covered later in the paper. For further details on the non-topological description of the uncompactified ten-dimensional GS superstring, see [9], while for details on the GS superstring compactified on a Calabi-Yau manifold, see reference [33].

Since the RNS description of the superstring is related to the GS description by a field redefinition, all aspects of the topological prescription are easily translated from RNS language into GS language. For example, as shown in Appendix A, the picture-counting operator in the RNS description is translated into the R-parity operator in the GS description. This means that F_g^n in the GS topological prescription calculates the term in the scattering amplitude which has an R-anomaly equal to n. Therefore after compactifying down to four dimensions, the scattering amplitude of N^+ chiral fermions, N^- anti-chiral fermions, and M vector bosons is given by F_g^n where $n = \frac{1}{2}(N^+ - N^-)$ (note that the gth-order term in this scattering amplitude vanishes if $|N^+ - N^-| > g - 1$). For compactifications down to six dimensions, R-parity is defined by breaking the internal SU(2) of the spinors down to U(1). As will be discussed in the following section, this preserves the SO(5,1) Lorentz invariance but breaks half of the eight spacetime supersymmetries. For the uncompactified superstring in ten dimensions, R-parity is defined by breaking the SO(9,1) Lorentz invariance down to $SU(4) \times U(1)$.

The only element of the GS topological prescription that requires further discussion is the functional integration over the negative-energy field. In the four and six-dimensional compactifications, this field was called ρ, while in the uncompactified GS superstring, it was called $(h^+ - h^-)$. As in the RNS topological prescription, we need to regularize the functional integral by inserting a delta-function of an operator that restricts the momentum of the negative-energy field through the internal loops. However, we should not use a delta-function of the R-parity operator (which is related to the RNS picture-counting operator by the field-redefinition) since this operator does not commute with spacetime supersymmetry. Instead, we should choose an operator which commutes with spacetime supersymmetry but still restricts the momentum of the negative-energy field.[14] For the reasons stated earlier, the resulting scattering amplitude should not depend on our choice, however it would be nice to verify this fact with an explicit comparison using different operators.

6. Topological amplitudes and superstring compactifications

It was previously shown that for the superstring compactified on a six-dimensional Calabi-Yau manifold, certain spacetime scattering amplitudes corresponding to superpotential terms can be computed exactly and can be expressed as topological amplitudes of the $(N = 2, \hat{c} = 3)$ superconformal field theory representing the Calabi-Yau manifold [2][3]. Since we now have a prescription for calculating topological amplitudes of an $(N = 4, \hat{c} = 2)$ superconformal field theory, it is natural to conjecture that these new topological amplitudes correspond to certain spacetime scattering amplitudes coming from superpotential terms when the superstring is compactified on the four-dimensional $K3$ manifold (or T^4 for the one loop correction). As will be shown in this section, the conjecture is correct.

Because the target-space described by the superstring on $K3$ is spacetime supersymmetric in six dimensions, it is convenient to use six-dimensional Green-Schwarz variables to

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[14] In previous work on non-topological multiloop calculations for the uncompactified GS superstring, it was shown that the functional integral over the $h^\pm$ fields was well-defined if the U(1) moduli, M_i, was restricted to satisfy the condition that $M_i = [\sum_{r=1}^{m} c_r z_r]_i$ where z_r are the locations of the fields $\exp(c_r h^-(z_r))$ and $[x - y]_i = \int_x^y \omega_i$. This condition on the U(1) moduli can be understood as coming from the insertion of the operator $\prod_{i=1}^{g} \delta(\int_{a_i} \partial_z h^+ - M_i)$ which restricts the momentum of the negative-energy field through the internal loops. Note that because the negative-energy field in the GS formalism transforms under U(1) transformations, its momentum is not integer or half-integer valued, but is shifted from an integer value by the U(1) moduli.

perform the computation of the superstring scattering amplitudes. These Green-Schwarz variables are related to the conventional Ramond-Neveu-Schwarz variables by a field redefinition which can be found in Appendix A. Since this field redefinition preserves all OPE's, the computations using the Green-Schwarz variables should be equivalent to the computations using the Ramond-Neveu-Schwarz variables. The GS description of the superstring contains N=2 worldsheet supersymmetry, so one might ask how it can be equivalent to the N=1 RNS description of the superstring. The answer is that the N=1 RNS superstring (or any critical N=1 string) can be embedded in an N=2 string[5], and the GS description is related by the above field redefinition to the N=2 version of the RNS string [10](in fact, this was how the $N = 1 \rightarrow N = 2$ embedding was discovered).

There are three advantages to using the GS variables instead of the RNS variables. Firstly, because the GS variables are all GSO projected (i.e., they have no square-root cuts with the spacetime-supersymmetry generators), there is no need to sum over spin structures when performing multiloop computations. Secondly, all states which are independent of the compactification moduli (i.e., the graviton, gravitino, etc.) can be represented by vertex operators constructed entirely out of spacetime GS fields. And thirdly, manifest spacetime-supersymmetry can be preserved at all stages in the calculation.

Because the GS techniques will not be familiar to most readers, we will begin by reproducing the Calabi-Yau topological amplitudes as a warm-up exercise for the $K3$ case. In other words, we will show using four-dimensional GS techniques that a gth order term in the scattering amplitude of $2g$ four-dimensional supermultiplets is given by a topological amplitude of the $(N = 2, \hat{c} = 3)$ superconformal field theory representing the Calabi-Yau manifold. After completing this warm-up exercise, we will show using six-dimensional GS techniques that a gth order term in the scattering amplitude of $4g$ six-dimensional supermultiplets is given by a topological amplitude of the $(N = 4, \hat{c} = 2)$ superconformal field theory representing the $K3$ manifold. We will only discuss the Type II closed string case in which the supermultiplets describe supergravity fields, however it should be possible to generalize to the heterotic case where the supermultiplets describe super-Yang-Mills fields.

6.1. GS superstring on CY 3-fold

As was discussed in [33], the worldsheet variables of the four-dimensional GS superstring consist of the spacetime variables, x^m ($m = 0$ to 3), the right-moving fermionic variables, θ^α and $\theta^{*\dot{\alpha}}$ ($\alpha, \dot{\alpha} = 1$ to 2), the conjugate right-moving fermionic variables, p_α

and $p^*_{\dot\alpha}$, and one right-moving boson ρ. The chiral boson ρ is identified with $\rho \pm 2\pi$ (in two-dimensional Minkowski space, ρ is imaginary valued and $i\rho$ takes values on a circle of radius 1) and is related to R-transformations of four-dimensional superspace. For the Type II GS superstring, one also has the left-moving fermionic fields, $\bar\theta^\alpha, \bar\theta^{*\dot\alpha}$, $\bar p_\alpha$, $\bar p^*_{\dot\alpha}$, and one left-moving boson, $\bar\rho$. In this section, an asterisk will mean anti-chiral whereas a bar will mean left-moving.

In conformal gauge, the worldsheet action for these fields is:

$$\int d^2z[\tfrac{1}{2}\partial_{\bar z}x^m\partial_z x_m + p_\alpha\partial_{\bar z}\theta^\alpha + p^*_{\dot\alpha}\partial_{\bar z}\theta^{*\dot\alpha} + \tfrac{1}{2}\partial_{\bar z}\rho\partial_z\rho + \bar p_\alpha\partial_z\bar\theta^\alpha + \bar p^*_{\dot\alpha}\partial_z\bar\theta^{*\dot\alpha} + \tfrac{1}{2}\partial_{\bar z}\bar\rho\partial_z\bar\rho].$$

The free-field OPE's for these worldsheet variables are

$$x^m(y)x^n(z) \to -\eta^{mn}\log|y-z|, \quad \rho(y)\rho(z) \to -\log(y-z),$$

$$p_\alpha(y)\theta^\beta(z) \to \frac{\delta^\beta_\alpha}{y-z}, \quad p^*_{\dot\alpha}(y)\theta^{*\dot\beta}(z) \to \frac{\delta^{\dot\beta}_{\dot\alpha}}{y-z},$$

$$\bar p_\alpha(y)\bar\theta^\beta(z) \to \frac{\delta^\beta_\alpha}{\bar y-\bar z}, \quad \bar p^*_{\dot\alpha}(y)\bar\theta^{*\dot\beta}(z) \to \frac{\delta^{\dot\beta}_{\dot\alpha}}{\bar y-\bar z}, \quad \bar\rho(y)\bar\rho(z) \to -\log(\bar y-\bar z).$$

Note that the chiral boson ρ can not be fermionized since $e^{\rho(y)}\, e^{\rho(z)} \to e^{2\rho(z)}(y-z)^{-1}$ while $e^{\rho(y)}\, e^{-\rho(z)} \to (y-z)$. It has the same behavior as the negative-energy field ϕ that appears when bosonizing the RNS ghosts $\gamma = \eta e^\phi$ and $\beta = \partial\xi e^{-\phi}$[34].

These worldsheet GS variables form a representation of an $N = 2$ superconformal algebra with $c = -3$. The generators of this algebra are given by:

$$L_{d=4} = \frac{1}{2}\partial_z x^m\partial_z x_m + p_\alpha\partial_z\theta^\alpha + p^*_{\dot\alpha}\partial_z\theta^{*\dot\alpha} + \frac{1}{2}\partial_z\rho\partial_z\rho \tag{6.1}$$

$$G^-_{d=4} = e^\rho(d)^2, \quad G^+_{d=4} = e^{-\rho}(d^*)^2, \quad J_{d=4} = \partial_z\rho,$$

where

$$d_\alpha = p_\alpha + i\theta^{*\dot\alpha}\partial_z x_{\alpha\dot\alpha} - \frac{1}{2}(\theta^*)^2\partial_z\theta_\alpha + \frac{1}{4}\theta_\alpha\partial_z(\theta^*)^2,$$

$$d^*_{\dot\alpha} = p^*_{\dot\alpha} + i\theta^\alpha\partial_z x_{\alpha\dot\alpha} - \frac{1}{2}(\theta)^2\partial_z\theta^*_{\dot\alpha} + \frac{1}{4}\theta^*_{\dot\alpha}\partial_z(\theta)^2,$$

and $(d)^2$ means $\epsilon^{\alpha\beta}d_\alpha d_\beta$. It is straghtforward to check [35] that d_α and $d^*_{\dot\alpha}$ commute with the $(N = 2, d = 4)$ spacetime supersymmetries which are generated by

$$q_\alpha = \int dz[p_\alpha - i\theta^{*\dot\alpha}\partial_z x_{\alpha\dot\alpha} - \frac{1}{4}(\theta^*)^2\partial_z\theta_\alpha], \tag{6.2}$$

$$q_{\dot\alpha}^* = \int dz^- [p_{\dot\alpha}^* - i\theta^\alpha \partial_z x_{\alpha\dot\alpha} - \frac{1}{4}(\theta)^2 \partial_z \theta^*{}_{\dot\alpha}],$$

$$\bar q_\alpha = \int d\bar z [\bar p_\alpha - i\theta^{*\dot\alpha} \partial_{\bar z} x_{\alpha\dot\alpha} - \frac{1}{4}(\bar\theta^*)^2 \partial_{\bar z} \bar\theta_\alpha],$$

$$\bar q_{\dot\alpha}^* = \int d\bar z [\bar p_{\dot\alpha}^* - i\bar\theta^\alpha \partial_{\bar z} x_{\alpha\dot\alpha} - \frac{1}{4}(\bar\theta)^2 \partial_{\bar z} \bar\theta^*_{\dot\alpha}].$$

The generator of R-parity transformations in superspace is given by:

$$R = \int dz(\partial_z \rho + \frac{1}{2}(p_\alpha \theta^\alpha - p_{\dot\alpha}^* \theta^{*\dot\alpha})), \tag{6.3}$$

$$\bar R = \int d\bar z(\partial_{\bar z}\bar\rho + \frac{1}{2}(\bar p_\alpha \bar\theta^\alpha - \bar p_{\dot\alpha}^* \bar\theta^{*\dot\alpha})).$$

As will be shown in Appendix A, the R-weight of a GS vertex operator is equal to the picture of the corresponding RNS vertex operator. Therefore the sum of the R-weights of the GS vertex operators is equal to the instanton-number of the surface that contributes to the scattering amplitude. It is easy to check that the N=2 tensor of equation (6.1) commutes with the spacetime-supersymmetry and R-parity generators.

Since the Calabi-Yau manifold is descibed by an N=2 superconformal field theory with $c = 9$, the combined system of the four-dimensional GS superstring and the Calabi-Yau manifold is described by an N=2 superconformal field theory with $c = 6$. The generators of the corresponding N=2 algebra are given by:

$$L_{GS} = L_{d=4} + L_{CY}, \quad G_{GS}^- = G_{d=4}^- + G_{CY}^-, \tag{6.4}$$

$$G_{GS}^+ = G_{d=4}^+ + G_{CY}^+, \quad J_{GS} = J_{d=4} + J_{CY},$$

where $[L_{CY}, G_{CY}^-, G_{CY}^+, J_{CY}]$ are the $(N = 2, c = 9)$ generators describing the Calabi-Yau manifold and $[L_{d=4}, G_{d=4}^-, G_{d=4}^+, J_{d=4}]$ are the $(N = 2, c = -3)$ generators defined in (6.1) .

As was described in reference [33], this N=2 tensor is related by a field redefinition to the N=2 tensor obtained by embedding the N=1 RNS string into an N=2 string (see appendix A). An advantage of the GS variables is that it allows the N=2 tensor to be split into two pieces, one of which is independent of the internal manifold. As will be shown below, this simplifies the computation of scattering amplitudes.

All physical states of the superstring are represented by vertex operators of the form $\hat V = |ce^{-\phi^+ - \phi^-}|^2 V$ where V is an N=2 primary field which is constructed entirely out

46

of matter fields and is dimension (0,0). In other words, L and $G^\pm$ have only $(y-z)^{-1}$ singularities with V, $\bar{L}$ and $\bar{G}^\pm$ have only $(\bar{y}-\bar{z})^{-1}$ singularities with V, while J and $\bar{J}$ have no singularities with V. To obtain vertex operators in other pictures, one can attach arbitrary combinations of $Z^\pm$, $\bar{Z}^\pm$, $I^\pm$, and $\bar{I}^\pm$ onto $\hat{V}$.

For example, if V depends only on the four-dimensional GS fields and is independent of the Calabi-Yau manifold,

$$|Z^- Z^+|^2 \hat{V} = |c \ (d^{*\dot\alpha} \ \nabla^2 \nabla^*_{\dot\alpha} + d^a \ \nabla^{*2} \nabla_\alpha +$$

$$\partial_z \theta^{*\dot\alpha} \ \nabla^*_{\dot\alpha} + \Pi^{\alpha\dot\alpha} \ \nabla_\alpha \nabla^*_{\dot\alpha}) \ + \gamma \ e^{-\rho} \bar{d}^{\dot\alpha} \ \nabla^*_{\dot\alpha}|^2 V$$

where $\nabla_\alpha V = [\int dz \ d_\alpha \ , V]$ and $\nabla^*_{\dot\alpha} V = [\int dz \ d^*_{\dot\alpha} \ , V]$. Since $c\bar{c}$ can be replaced with $\int d^2 z$, the vertex operator can be written in integrated form as:

$$U = \int d^2 z \ |\bar{d}^{\dot\alpha} \ \nabla^2 \nabla^*_{\dot\alpha} + d^a \ \nabla^{*2} \nabla_\alpha \tag{6.5}$$

$$+ \frac{1}{2}(\partial_z \theta^{*\dot\alpha} \ \nabla^*_{\dot\alpha} + \partial_z \theta^\alpha \ \nabla_\alpha + \Pi^{\alpha\dot\alpha} \ (\nabla_\alpha \nabla^*_{\dot\alpha} - \nabla^*_{\dot\alpha} \nabla_\alpha))|^2 V.$$

For the massless supermultiplet of the closed superstring which describes the four-dimensional supergravity fields, the vertex operator is constructed from the primary field $V(\theta, \theta^*, \bar\theta, \bar\theta^*, x)$. The constraints of being N=2 primary imply that

$$(\nabla)^2 V = (\nabla^*)^2 V = (\bar\nabla)^2 V = (\bar\nabla^*)^2 V = \partial^m \partial_m V = 0$$

where $\nabla_\alpha = \partial_{\theta^\alpha} + i\theta^{*\dot\alpha}\partial_{\alpha\dot\alpha}$, $\nabla^*_{\dot\alpha} = \partial_{\theta^{*\dot\alpha}} + i\theta^\alpha \partial_{\alpha\dot\alpha}$, $\bar\nabla_\alpha = \partial_{\bar\theta^\alpha} + i\bar\theta^{*\dot\alpha}\partial_{\alpha\dot\alpha}$, $\bar\nabla^*_{\dot\alpha} = \partial_{\bar\theta^{*\dot\alpha}} + i\bar\theta^\alpha \partial_{\alpha\dot\alpha}$. The gauge transformations of V which leave the integrated vertex operator unchanged are

$$\delta V = (\nabla)^2 \Lambda + (\nabla^*)^2 \Lambda^* + (\bar\nabla)^2 \bar\Lambda + (\bar\nabla^*)^2 \bar\Lambda^*.$$

The remaining unconstrained components describe the physical massless fields of the superstring which are independent of the Calabi-Yau manifold. For example, the graviton and axion are described by $g_{mn} + b_{mn} = \sigma_m^{\alpha\dot\alpha}\sigma_n^{\beta\dot\beta}\nabla_\alpha\nabla^*_{\dot\alpha}\bar\nabla_\beta\bar\nabla^*_{\dot\beta}V$, the chiral graviphoton by $T_{\alpha\beta} = (\nabla^*)^2 \nabla_\alpha (\bar\nabla^*)^2 \bar\nabla_\beta V$, etc.).

Since the GS formalism is a critical N=2 string theory, we can use (2.22) to calculate superstring scattering amplitudes. Therefore the genus g scattering amplitude for $2g$ massless supermultiplets is given by:

$$A_g = |\int du \sum_{n_I=2-2g}^{2g-2} (u_2^*)^{2g-2+n_I}(u_1^*)^{2g-2-n_I}|^2 \tag{6.6}$$

$$\prod_{j=1}^{3g-3} \int d^2 m_j \prod_{i=1}^{g} \int d^2 v_i < |\prod_{i=1}^{g-1} \widehat{\tilde{G}^+}(v_i) J(v_g)(\int \mu_j \widehat{G^-})|^2 U_1...U_{2g} >,$$

where

$$U_i = \int d^2 z |d^{*\dot{\alpha}} (\nabla)^2 \nabla^*_{\dot{\alpha}} + d^a (\nabla^*)^2 \nabla_\alpha$$

$$+\frac{1}{2}(\partial_z \theta^{*\dot{\alpha}} \nabla^*_{\dot{\alpha}} + \partial_z \theta^\alpha \nabla_\alpha + \Pi^{\alpha\dot{\alpha}} (\nabla_\alpha \nabla^*_{\dot{\alpha}} - \nabla^*_{\dot{\alpha}} \nabla_\alpha))|^2 V_i,$$

$$\widehat{\tilde{G}^+} = u_1(e^{2\rho+J_{CY}}(d)^2 + e^\rho \tilde{G}^{++}_{CY}) + u_2(e^{-\rho}(d^*)^2 + G^+_{CY}), \tag{6.7}$$

$$\widehat{G^-} = u_1(e^\rho(d)^2 + G^-_{CY}) + u_2(e^{-2\rho-J_{CY}}(d^*)^2 + e^{-\rho}\tilde{G}^{--}_{CY}),$$

$\tilde{G}^{\pm\pm}_{CY}$ is the residue of the pole in the OPE of $e^{\pm J_{CY}}$ with $G^\mp_{CY}$, and U comes from (6.5) . The integration over u with the insertion of $(u_2^*)^{2g-2+n_I}(u_1^*)^{2g-2-n_I}$ picks out the contribution of instanton number n_I to the amplitude since up to an overall normalization,

$$\int du(u_1^{2g-2+m}u_2^{2g-2-m})(u_2^*)^{2g-2+n}(u_1^*)^{2g-2-n} = \delta_{mn}(2g-2+m)!(2g-2-m)!. \tag{6.8}$$

So the combinatorial factor in (2.18) is appropriately cancelled. Summing over n_I includes the contributions of all possible R-parity anomalies to the scattering amplitude with the instanton θ-angle set equal to zero (it is clear that shifting the θ-angle is equivalent to rotating the vertex operators by a phase proportional to their R-weight). Note that the integration over u is done independently in the right and left-moving sectors.

The claim is that the piece of this amplitude that describes the scattering of $2g-2$ chiral graviphotons and two gravitons is

$$\int d^2\theta d^2\bar{\theta}(W_{\alpha\beta}W^{\alpha\beta})^g T_{CY} \tag{6.9}$$

where $W_{\alpha\beta}$ is the chiral field strength of $(N=2, d=4)$ supergravity (the lowest component of $W_{\alpha\beta}$ is the chiral graviphoton and the $\theta\bar{\theta}$ component is the Riemann tensor) and T_{CY} is the topological partition function of the Calabi-Yau manifold. Note that it makes sense to define a chiral superspace integral in (6.9) since $\nabla_{\dot{\alpha}}W_{\alpha\beta} = \nabla_{\dot{\alpha}}T_{CY} = 0$. The chirality of T_{CY} comes from the holomorphicity condition which says that (up to possible anomalies), T_{CY} is independent of $\bar{t}_i$ if the action is deformed as in (2.4) . Since the massless deformations of the Calabi-Yau manifold described by t_i are four-dimensional chiral scalar superfields, $\nabla_{\dot{\alpha}}T_{CY} = (\nabla_{\dot{\alpha}}t_i)\partial_{t_i}T_{CY} = 0$.

The first step in proving (6.9) is to realize that since the chiral graviphoton carries R-weight $+\frac{1}{2}$ and the graviton carries R-weight 0, the relevant term in (6.6) is the piece

with $n_I = g - 1$ (recall that the sum of the R-weights of the vertex operators is equal to the instanton number of the surface that contributes to the scattering amplitude). We therefore have $3g - 3 - m$ G^-'s, m $\tilde{G}^-$'s, $g - 1 - m$ G^+'s, and m $\tilde{G}^+$'s contributing in (6.6) . Furthermore, because of the background charge, these operators need to contribute $1 - g$ units of ρ charge and $3g - 3$ units of J_{CY} charge. It is easy to check that this is only possible if each G^- contributes G_{CY}^-, each $\tilde{G}^-$ contributes $e^{-2\rho - \int J_{CY}}(d^*)^2$, each G^+ contributes $(d^*)^2 e^{-\rho}$, and each $\tilde{G}^+$ contributes $\tilde{G}_{CY}^{++} e^\rho$.

It is convenient to regularize the functional integral over the negative-energy by constraining $\int (\partial_z \rho - p_1^* \theta^{*1})$ to be zero through the internal loops. This has the effect of fixing the v_i's such that the ones that go with $\tilde{G}^+$ are sewn in with the m beltrami differentials that contribute $\tilde{G}^-$. The remaining $g - m$ v_i's can be inserted at arbitrary points on the surface (with this choice of the v_i locations, ρ always occurs in the combinations $\rho - i\sigma$ where $i\partial_z \sigma = p_1^* \theta^{*1}$). The resulting expression for the $n_1 = g - 1$ piece of (6.6) is:

$$F_g^{1-g} = \prod_{j=1}^{3g-3} \int d^2 m_j \, det(Im\tau) |det\omega^k(v_i)|^2$$

$$< |\prod_{i=m+1}^{g-1} e^{-\rho}(d^*)^2(v_i) J(v_g) \prod_{j=1}^{m}(\int \mu_j G_{CY}^- e^{-\rho}(d^*)^2) \prod_{k=m+1}^{3g-3}(\int \mu_k G_{CY}^-)|^2$$

$$U_1...U_{2g} >,$$

where $v_1...v_m$ coincide with the first m beltrami differentials. The

$$det(Im\tau)|det\omega^k(v_i)|^2 \tag{6.10}$$

factor comes from the fact that the ρ regularization contributes

$$det(Im\tau) \prod_{i=1}^{g} \delta([\sum_j a_j y_j - \sum_k b_k z_k]_i)$$

where $[y - z]_i = \int_z^y \omega_i$, y_j is the position of $e^{a_j \rho(y_j)}$ and z_k is the position of $e^{ib_k \sigma(z_k)}$ (note that $\theta^{*1} = e^{-i\sigma}$ and $p_1^* = e^{i\sigma}$). It is easy to check that only the $\int d^2 z_i |d^a(\nabla^*)^2 \nabla_\alpha|^2 V = \int d^2 z_i d^\alpha \bar{d}^\beta W_{\alpha\beta}$ term contributes in U_i of (6.5) because we need at least $2g$ zero modes of d^α and $\bar{d}^\alpha$.

To perform the functional integral over the ρ, $d^{*\dot{\alpha}}$ and $\theta^{*\dot{\alpha}}$ fields, it is useful to pull G^+ off one of the vertex operators and circle it around $J(v^g)$ (otherwise the naive functional integral is ill-defined since one gets $0/0$). The resulting expression for the scattering amplitude is:

$$F_g^{1-g} = \prod_{j=1}^{3g-3} \int d^2 m_j det(Im\tau) |det\omega^k(v_i)|^2$$

$$< | \prod_{i=m+1}^{g} e^{-\rho}(d^*)^2(v_i) \prod_{j=1}^{m} (\int \mu_j G_{CY}^- e^{-\rho}(d^*)^2) \prod_{k=m+1}^{3g-3} (\int \mu_k G_{CY}^-)|^2$$

$$(\prod_{r=1}^{2g-1} \int d^2 z_r d^\alpha \bar{d}^\beta W_{\alpha\beta}) \int d^2 z_{2g} |e^\rho d^a \nabla_\alpha|^2 V > .$$

It is easy to check that the functional integrals over the ρ and (p_1^*, θ^{*1}) fields contribute $|Z_1|^{-2}[det(Im\tau)]^{-1}$ where $(Z_1)^{-\frac{1}{2}}$ is the partition function for a chiral boson. The functional integral over the (p_2^*, θ^{*2}) fields contributes $|Z_1 det\omega_j(v_k)|^2$. The $(p_\alpha, \theta^\alpha)$ functional integrals, when integrated over the vertex operator locations, give an overall factor of $|Z_1|^4(det[Im\tau])^2$ and contract the spinor indices of the W's. Finally, the functional integral over the four x^m's give a factor of $|Z_1|^{-4}(det[Im\tau])^{-2}$. Putting all of this together, one finds

$$F_g^{1-g} = \int d^2\theta d^2\theta^* d^2\bar{\theta} d^2\bar{\theta}^* (W^{\alpha\beta} W_{\alpha\beta})^{g-1} W^{\gamma\delta} \nabla_\gamma \bar{\nabla}_\delta V$$

$$\prod_{j=1}^{3g-3} \int d^2 m_j < | \prod_{j=1}^{3g-3} \int \mu_j G_{CY}^-|^2 >,$$

where the superspace integration comes from the zero mode of the functional integral over the θ's and θ^*'s. Performing the integrations over θ^* and $\bar{\theta}^*$ (which turns $\nabla_\gamma \bar{\nabla}_\delta V$ into $W_{\gamma\delta}$), one proves the claim that

$$F_g^{1-g} = \int d^2\theta d^2\bar{\theta} (W_{\alpha\beta} W^{\alpha\beta})^g T_{CY} \tag{6.11}$$

where $T_{CY} = \prod_{j=1}^{3g-3} \int d^2 m_j < | \prod_{j=1}^{3g-3} \int \mu_j G_{CY}^-|^2 > .$

6.2. GS superstring in K3 Background

After completing the warm-up exercise with a Calabi-Yau background, we are now ready to use GS techniques to study superstring amplitudes in a $K3$ background. In the Calabi-Yau case, we were able to preserve manifest the SO(3,1) Poincaré invariance, as well

as the 8 spacetime supersymmetries of the four-dimensional Type II superstring. In the $K3$ case, we will be able to preserve manifest the full SO(5,1) Poincaré invariance, but only 8 of the 16 spacetime-supersymmetries of the six-dimensional Type II superstring. Although it may be possible to modify the six-dimensional GS formalism so that it preserves manifest all 16 of the spacetime supersymmetries, we do not see how to do this at the present time. Nevertheless, the six-dimensional GS formalism is still much more convenient than the RNS formalism for the reasons stated at the beginning of this section.

The worldsheet variables of the six-dimensional GS superstring consist of the space-time variables, x^m ($m = 0$ to 5), the right-moving fermionic variables, θ^α ($\alpha = 1$ to 4), the conjugate right-moving fermionic variables, p_α, and two right-moving bosonic variables, ρ and σ. Under the SO(5,1) super-Poincaré transformations, m is a vector index whereas α is a spinor index (note that a vector index can be represented as an anti-symmetric product of two spinor indices). As in the four-dimensional GS superstring, ρ is imaginary-valued like the ϕ boson in the bosonized N=1 superconformal ghosts. The chiral boson σ, on the other hand, is real-valued like that obtained from bosonizing a pair of chiral fermions. The background charge in the untwisted theory is +2 for both of these chiral bosons (after twisting $L \to L + \frac{1}{2}\partial_z J$, the background charge for these bosons is +3). For the Type II GS superstring, one also has the left-moving fermionic variables, $\bar{\theta}^\alpha$, $\bar{p}_\alpha$, and the left-moving bosonic variables, $\bar{\rho}$, $\bar{\sigma}$. The field redefinition that relates these six-dimensional GS variables to the conventional RNS variables can be found in Appendix A.

In conformal gauge, the worldsheet action for these fields is:

$$\int d^2z [\frac{1}{2}\partial_{\bar{z}}x^m\partial_z x_m + p_\alpha\partial_{\bar{z}}\theta^\alpha + \frac{1}{2}\partial_{\bar{z}}\rho\partial_z\rho - \frac{1}{2}\partial_{\bar{z}}\sigma\partial_z\sigma +$$

$$\bar{p}_\alpha\partial_z\bar{\theta}^\alpha + +\frac{1}{2}\partial_{\bar{z}}\bar{\rho}\partial_z\bar{\rho} - \frac{1}{2}\partial_{\bar{z}}\bar{\sigma}\partial_z\bar{\sigma}].$$

The free-field OPE's for these worldsheet variables are

$$x^m(y)x^n(z) \to -\eta^{mn}\log|y - z|,$$

$$\rho(y)\rho(z) \to -\log(y - z), \qquad \sigma(y)\sigma(z) \to +\log(y - z),$$

$$p_\alpha(y)\theta^\beta(z) \to \frac{\delta_\alpha^\beta}{y - z}, \quad \bar{p}_\alpha(y)\bar{\theta}^\beta(z) \to \frac{\delta_\alpha^\beta}{\bar{y} - \bar{z}},$$

$$\bar{\rho}(y)\bar{\rho}(z) \to -\log(\bar{y} - \bar{z}), \quad \bar{\sigma}(y)\bar{\sigma}(z) \to +\log(\bar{y} - \bar{z}).$$

In terms of these GS variables, the untwisted $c = 6$ N=2 stress-tensor is:

$$L_{GS} = \frac{1}{2}\partial_z x^m \partial_z x_m + p_\alpha \partial_z \theta^\alpha + \frac{1}{2}\partial_z \rho \partial_z \rho - \frac{1}{2}\partial_z \sigma \partial_z \sigma + \partial_z^2(\rho + i\sigma) + L_{K3}$$

$$G_{GS}^- = e^{-i\sigma - 2\rho}(p)^4 + e^{-\rho}\epsilon_{\alpha\beta\gamma\delta}p^\alpha p^\beta \partial_z x^{\gamma\delta} +$$

$$e^{i\sigma}(\partial_z x^m \partial_z x_m + 2p_\alpha \partial_z \theta^\alpha + \partial_z(\rho + i\sigma)\partial_z(\rho + i\sigma)) + G_{K3}^-,$$

$$G_{GS}^+ = e^{-i\sigma} + G_{K3}^+,$$

$$J_{GS} = -\partial_z(\rho + i\sigma) + J_{K3}, \tag{6.12}$$

where $(p)^4$ means $\epsilon^{\alpha\beta\gamma\delta}p_\alpha p_\beta p_\gamma p_\delta$, and $[L_{K3}, G_{K3}^-, G_{K3}^+, J_{K3}]$ forms a $c = 6$ N=2 stress-tensor which describes the $K3$ manifold. As in the Calabi-Yau case, this $(N = 2, c = 6)$ stress-tensor is related to the N=2 tensor of the RNS string by a field redefinition (see Appendix A), and the GS version has the advantage that it splits the N=2 tensor into a six-dimensional piece and a $K3$ piece.

The spacetime supersymmetry generators can be written as:

$$q_\alpha^1 = \int dz \; p_\alpha, \quad q_\alpha^2 = \int dz[e^{-i\sigma - \rho}p_\alpha - i\epsilon_{\alpha\beta\gamma\delta}\theta^\beta \partial_z x^{\gamma\delta}].$$

$$\bar{q}_\alpha^1 = \int d\bar{z} \; \bar{p}_\alpha, \quad \bar{q}_\alpha^2 = \int d\bar{z}[e^{-i\bar{\sigma} - \bar{\rho}}\bar{p}_\alpha - i\epsilon_{\alpha\beta\gamma\delta}\bar{\theta}^\beta \partial_{\bar{z}} x^{\gamma\delta}].$$

which satisfy the anti-commutation relations $\{q_\alpha^i, q_\beta^j\} = \epsilon^{ij}\epsilon_{\alpha\beta\gamma\delta}\int dz \; \partial_z x^{\gamma\delta}$ and $\{\bar{q}_\alpha^i, \bar{q}_\beta^j\} = \epsilon^{ij}\epsilon_{\alpha\beta\gamma\delta}\int d\bar{z} \; \partial_{\bar{z}} x^{\gamma\delta}$. Because there is no internal SU(2) index on θ^α and $\bar{\theta}^\alpha$, only half of the spacetime supersymmetries are manifestly preserved in the six-dimensional GS formalism (the spacetime supersymmetries generated by q_α^1 and $\bar{q}_\alpha^1$ are manifest since θ^α and $\bar{\theta}^\alpha$ only appear in the form $\partial_z \theta^\alpha$ and $\partial_{\bar{z}}\bar{\theta}^\alpha$). Note however that scattering amplitudes will of course be invariant under all of the spacetime supersymmetries. The reason why the internal SU(2) cannot be manifestly preserved in the N=2 GS string is because a U(1) direction has to be chosen to define R-parity. The generators of the six-dimensional R-parity transformations are:

$$R = \int dz[\partial_z \rho + \frac{1}{2}p_\alpha \theta^\alpha], \tag{6.13}$$

$$\bar{R} = \int d\bar{z}[\partial_{\bar{z}}\bar{\rho} + \frac{1}{2}\bar{p}_\alpha \bar{\theta}^\alpha].$$

Note that the N=2 tensor (6.12) is invariant under all of the spacetime supersymmetries and R-parity transformations.

As in the four-dimensional case, all physical states of the six-dimensional GS superstring can be represented by vertex operators of the form $\hat{V} = |ce^{-\phi^+ - \phi^-}|^2 V$ where V is an N=2 primary field. For the massless fields independent of the $K3$ background, V is constructed just out of the six-dimensional GS fields. Because it has dimension (0,0) and has no singularities with J, V only depends on σ and ρ in the combinations

$$V = \sum_{m,n=-\infty}^{+\infty} e^{m(i\sigma + \rho) + n(i\bar{\sigma} + \bar{\rho})} V_{m,n}(x, \theta, \bar{\theta}).$$

Furthermore, since V has only a pole singularity with G^+ and $\bar{G}^+$, $V_{m,n}=0$ for $m > 1$ or $n > 1$.

Vertex operators in integrated form are obtained by hitting $\hat{V}$ with $Z^+ Z^-$ and removing the c ghost. Since $\{\bar{G}^+, [G^+, V]\} = e^{\rho + \bar{\rho}} V_{1,1}$, it is easily shown that the integrated form of the above vertex operators is:

$$U = \int d^2z |e^{-i\sigma - \rho} \epsilon^{\alpha\beta\gamma\delta} p_\alpha \ (\nabla_\beta \nabla_\gamma \nabla_\delta) + \epsilon^{\alpha\beta\gamma\delta} p_\alpha \ (\nabla_\beta \partial_{\gamma\delta}) \tag{6.14}$$

$$+ e^{i\sigma + \rho} (\partial_z x^m \ \partial_m + \partial_z \theta^\alpha \ \nabla_\alpha)|^2 V_{1,1}$$

where $\nabla_\alpha = \frac{d}{d\theta^\alpha}$. Note that $[G^+, V]$ having only a single pole with G^- implies the on-shell constraints

$$(\nabla)^4 V_{1,1} = \epsilon^{\alpha\beta\gamma\delta} \nabla_\alpha \nabla_\beta \partial_{\gamma\delta} V_{1,1} = \partial_m \partial^m V_{1,1} = 0. \tag{6.15}$$

The remaining unconstrained components of $V_{1,1}$ describe the physical massless fields of the superstring which are independent of the $K3$ moduli. For example, the graviton and axion are described by $g_{mn} + b_{mn} = \sigma_m^{\alpha\beta} \sigma_n^{\gamma\delta} \nabla_\alpha \nabla_\beta \bar{\nabla}_\gamma \bar{\nabla}_\delta V_{1,1}$, and the four graviphotons by $u^{*i} \bar{u}^{*j} T_{ij}^{\alpha\bar{\alpha}} = |\epsilon^{\alpha\beta\gamma\delta}(u_1^* \nabla_\beta \nabla_\gamma \nabla_\delta + u_2^* \nabla_\beta \partial_{\gamma\delta})|^2 V_{1,1}$. Note that the four graviphotons have $(\pm\frac{1}{2}, \pm\frac{1}{2})$ R-parity, so we can define the chiral graviphoton to be $T_{22}^{\alpha\bar{\alpha}}$ which has $(+\frac{1}{2}, +\frac{1}{2})$ R-parity.

Suppose we are scattering $4g$ supermultiplets on a genus g surface. Then the $N = 4$ topological amplitude is

$$A = |\int du \sum_{n_I=1}^{2g-2} (u_1^*)^{2g-2+n_I} (u_2^*)^{2g-2-n_I}|^2 \tag{6.16}$$

$$\prod_{j=1}^{3g-3} \int d^2 m_j \prod_{i=1}^{g} \int d^2 v_i < |\prod_{i=1}^{g-1} \widehat{G^+}(v_i) J(v_g)(\int \mu_j \widehat{G^-})|^2 U_1 ... U_{4g} >,$$

where

$$U_i = \int d^2 z_i |e^{-i\sigma-\rho}\epsilon^{\alpha\beta\gamma\delta}p_\alpha \ (\nabla_\beta\nabla_\gamma\nabla_\delta) + \epsilon^{\alpha\beta\gamma\delta}p_\alpha \ (\nabla_\beta\partial_{\gamma\delta}) +$$

$$+e^{i\sigma+\rho}(\partial_z x^m \ \partial_m + \partial_z\theta^\alpha \ \nabla_\alpha)|^2 V_{1,1},$$

$$\widehat{G^+} = u_1 e^{J_{K3}}(e^{-3\rho-2i\sigma}(p)^4 + \ e^{-2\rho-i\sigma}\epsilon_{\alpha\beta\gamma\delta}p^\alpha p^\beta \partial_z x^{\gamma\delta} +$$

$$e^{-\rho}(\partial_z x^m \partial_z x_m + 2p_\alpha\partial_z\theta^\alpha + \partial_z(\rho+i\sigma)\partial_z(\rho+i\sigma))) $$

$$+ \ u_1 e^{-\rho-i\sigma}\tilde{G}^+_{K3} + u_2(e^{-i\sigma} + G^+_{K3}),$$

$$\widehat{G^-} = u_1(e^{-2\rho-i\sigma}(p)^4 + \ e^{-\rho}\epsilon_{\alpha\beta\gamma\delta}p^\alpha p^\beta \partial_z x^{\gamma\delta} +$$

$$e^{+i\sigma}(\partial_z x^m \partial_z x_m + 2p_\alpha\partial_z\theta^\alpha + \partial_z(\rho+i\sigma)\partial_z(\rho+i\sigma)) + G^-_{K3})$$

$$+u_2(e^{-J_{K3}+\rho} + e^{\rho+i\sigma}\tilde{G}^-_{K3}). \tag{6.17}$$

The claim is that the piece of this amplitude that describes the scattering of $4g - 4$ chiral graviphotons and 4 gravitons is

$$\int d^4\theta_+ \int d^4\bar\theta_+ \int du \int d\bar u (\epsilon\bar\epsilon)^g [\widehat{W}^{\alpha\bar\alpha}_{++}]^{4g} F_g(u,\bar u) \tag{6.18}$$

where $\theta^\alpha_- = u^b \theta^\alpha_b$ and $\theta^\alpha_+ = u^{*b}\theta^\alpha_b$ are the usual harmonic superspace variables constructed out of the SO(5,1) spinors with internal SU(2) index θ^α_b and $\theta^{*\alpha}_b$ (similarly for $\bar\theta^\alpha_-$ and $\bar\theta^\alpha_+$), $\widehat{W}^{\alpha\bar\alpha}_{++}(u^*, \bar u^*, \theta_+, \bar\theta_+)$ is the analytic field strength in harmonic superspace which satisfies the on-shell conditions

$$\frac{d}{d\theta^\beta_-}\widehat{W}^{\alpha\bar\alpha} = \frac{d}{d\bar\theta^\beta_-}\widehat{W}^{\alpha\bar\alpha} = u^*_b\frac{d}{du_b}\widehat{W}^{\alpha\bar\alpha} = \bar u^*_b\frac{d}{d\bar u_b}\widehat{W}^{\alpha\bar\alpha} = 0 \tag{6.19}$$

(the lowest component of W_{++} is $u^*_i\bar u^*_j$ times the graviphoton T_{ij} and the $\theta_+\bar\theta_+$ component is the Riemann tensor), the $(4g, 4g)$ spinor indices of $[W]^{4g}$ are contracted with $g \ \epsilon_{\alpha\beta\gamma\delta}$'s and $g \ \bar\epsilon_{\bar\alpha\bar\beta\bar\gamma\bar\delta}$'s in all possible ways, and $F_g(u,\bar u)$ is the topological partition function of (2.18) for the $N = 2$ string on the $K3$ manifold.

It will now be shown that the harmonicity condition (2.27) for F_g allows the harmonic superspace integral of (6.18) to make sense. First note that F_g is a function of degree $(4g - 4, 4g - 4)$ in $(u, \bar u)$, so it carries U(1) number $(-4g + 4, -4g + 4)$. When combined with the $(4g, 4g)$ U(1) number of the W's and the $(-4, -4)$ U(1) number of the $\theta_+\bar\theta_+$

integration, the harmonic integrand has the correct U(1) charge. Second, note that the superspace integral only makes sense if

$$\frac{d}{d\theta_-^\beta} \int du \int d\bar{u} (\epsilon\bar{\epsilon})^g [\widehat{W}_{++}^{\alpha\bar{\alpha}}]^{4g} F_g(u, \bar{u}) \tag{6.20}$$

is zero. Because of (2.27) (we are ignoring possible anomalies),

$$\frac{d}{dt_a^i} F_g = \frac{d}{du_a} Y_g^i$$

for some Y_g^i where t_a^i are the six-dimensional scalar hypermultiplets that describe the massless deformations of the $K3$ manifold (the equation of motion for the scalar hypermultiplet is $\partial_{\theta_a^\beta} t_b^i = \delta_b^a f_\beta^i$ for some f_β^i). Using the equations of motion for W_{++} in (6.19) , (6.20) is equal to

$$\int du \int d\bar{u} (\epsilon\bar{\epsilon})^g [\widehat{W}_{++}^{\alpha\bar{\alpha}}]^{4g} \frac{d}{d\theta_-^\beta} F_g$$

$$= \int du \int d\bar{u} (\epsilon\bar{\epsilon})^g [\widehat{W}_{++}^{\alpha\bar{\alpha}}]^{4g} (u_a^* \frac{d}{d\theta_a^\beta} t_b^i) \frac{d}{dt_b^i} F_g$$

$$= \int du \int d\bar{u} (\epsilon\bar{\epsilon})^g [\widehat{W}_{++}^{\alpha\bar{\alpha}}]^{4g} u_a^* f_\beta^i \frac{d}{du_a} Y_g^i$$

$$= \int du \int d\bar{u} (u_a^* \frac{d}{du_a}) (\epsilon\bar{\epsilon})^g [\widehat{W}_{++}^{\alpha\bar{\alpha}}]^{4g} f_\beta^i Y_g^i. \tag{6.21}$$

But by the rules of harmonic integration, (6.21) is zero so we have checked that the harmonic superspace integral in (6.18) makes sense.

To prove (6.18) , one first uses the fact that the chiral graviphoton has R-weight $(+\frac{1}{2}, +\frac{1}{2})$ and the graviton has R-weight 0, and therefore the only term that contributes is that of instanton number $(n_I = 2g - 2, \bar{n}_I = 2g - 2)$. This means that we have just G^-'s and $\tilde{G}^+$'s, and since we need $2g - 2$ units of J_{K3} charge, the only contribution from $\widehat{G^-}$ is $u_1 G_{K3}^-$ and the only contribution from $\widehat{\tilde{G}^+}$ is $u_1 e^{-\rho - i\sigma} \tilde{G}_{K3}^+$. In order that $\tilde{G}_{K3}^+$ can not be pulled off the surface, $J(v_g)$ must contribute J_{K3}. Also, in order to have enough p_α's and enough $e^{\rho+i\sigma}$'s, only the term

$$\int d^2 z_i |\epsilon_{\alpha\beta\gamma\delta} p_\alpha (\nabla_\beta \partial_{\gamma\delta})|^2 V_{1,1} = \int d^2 z_i p_\alpha \bar{p}_\beta W_{2,2}^{\alpha\beta}$$

contributes to U_i of (6.14) . Note that $W_{22}^{\alpha\beta}$ is defined as $W_{++}^{\alpha\beta}$ when $u_1^* = 0$, $u_2^* = 1$, $\theta_+^\alpha = \theta^\alpha$, and $\bar{\theta}_+^\alpha = \bar{\theta}^\alpha$.

So the relevant term in the scattering amplitude is

$$F_g^{2-2g} = \prod_{j=1}^{3g-3} \int d^2 m_j \prod_{i=1}^{g} \int d^2 v_i$$

$$< |\prod_{i=1}^{g-1} e^{-\rho-i\sigma} \tilde{G}_{K3}^+(v_i) J_{K3}(v_g) \prod_{j=1}^{3g-3} (\int \mu_j G_{K3}^-)|^2 \prod_{r=1}^{4g} \int d^2 z_i p_\alpha \bar{p}_\beta W_{22}^{\alpha\beta} > .$$

It is convenient to regularize the functional integral over the negative-energy ρ field by constraining $\int \partial_z (\rho + i\sigma)$ to be zero through the internal loops.[15] With this regularization, the functional integrals over ρ and σ contribute $|Z_1|^{-2}[det(Im\tau)]^{-1}$ where $(Z_1)^{-\frac{1}{2}}$ is the partition function for a chiral boson. The functional integral over the six x's give a contribution of $|Z_1|^{-6}(det[Im\tau])^{-3}$, and the functional integral over the p^α's and θ^α's (after integrating over the locations of the vertex operators) contributes $|Z_1|^8(det[Im\tau])^4$ and a product of g $\epsilon_{\alpha\beta\gamma\delta} \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}$'s. Putting all this together, one finds

$$F_g^{2-2g} = \prod_{j=1}^{3g-3} \int d^2 m_j \prod_{i=1}^{g} \int d^2 v_i \int d^4\theta d^4\bar{\theta} \tag{6.22}$$

$$< |\prod_{i=1}^{g-1} \tilde{G}_{K3}^+(v_i) J_{K3}(v_g) \prod_{j=1}^{3g-3} (\int \mu_j G_{K3}^-)|^2 > (\epsilon\bar{\epsilon})^g [W_{22}^{\alpha\bar{\alpha}}]^{4g},$$

where the superspace integration comes from the zero mode of the functional integral over the θ's and $\bar{\theta}$'s.

Since $W_{22}^{\alpha\bar{\alpha}}$ is just one component of $W_{++}^{\alpha\bar{\alpha}}$, internal SU(2) invariance implies that $(W_{22}^{\alpha\bar{\alpha}})^{4g}$ appears as one of the components of $(W_{++}^{\alpha\bar{\alpha}})^{4g}$. From equation (6.16), it will now be argued that the scattering amplitude of (6.22) is one component of the amplitude

$$\int d^4\theta_+ \int d^4\bar{\theta}_+ \int du \int d\bar{u}(\epsilon\bar{\epsilon})^g [W_{++}^{\alpha\bar{\alpha}}]^{4g} \tag{6.23}$$

$$\prod_{j=1}^{3g-3} \int d^2 m_j \prod_{i=1}^{g} \int d^2 v_i < |\prod_{i=1}^{g-1} \widehat{\tilde{G}_{K3}^+}(v_i) J_{K3}(v_g) \prod_{j=1}^{3g-3} (\int \mu_j \widehat{G_{K3}^-})|^2 >,$$

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[15] Since ρ only occurs in the combination $\rho + i\sigma$, the locations of the v_i's are not restricted by this regularization. So we can either remove the integration over the v_i's and insert $det(Im\tau)|det\omega^k(v_i)|^2$ as in (6.10), or we can leave the integration unchanged. These two prescriptions are equivalent as was shown in (2.30).

thereby proving the claim of (6.18) .

The other contributions to $W_{++}^{\alpha\bar{\alpha}}$ come from the term

$$e^{-i\sigma-\rho}\epsilon^{\alpha\beta\gamma\delta}p_\alpha(\nabla_\beta\nabla_\gamma\nabla_\delta)$$

in U_i of equation (6.14) . In other words,

$$W_{++}^{\alpha\bar{\alpha}} = |u_1^*(\nabla_\beta\nabla_\gamma\nabla_\delta) + u_2^*(\nabla_\beta\partial_{\gamma\delta})|^2 V_{1,1} \tag{6.24}$$

where θ_+^α is defined as $u_2^*\theta^\alpha$ (note that all θ^α dependence comes from the u_2^* term because $(\nabla)^4 V_{1,1} = 0$ by the on-shell condition of (6.15)).

Because non-chiral graviphotons can have $-\frac{1}{2}$ R-parity, the scattering amplitude for $4g$ W_{++}'s involves surfaces of all instanton numbers. However, let us consider only the following contributions of $\widehat{\widetilde{G}^+}$ and $\widehat{G^-}$ in (6.17) :

$$\widehat{\widetilde{G}^+} = u_1\tilde{G}_{K3}^+ e^{-\rho-i\sigma} + u_2 G_{K3}^+, \quad \widehat{G^-} = u_1 G_{K3}^- + u_2\tilde{G}_{K3}^- e^{\rho+i\sigma}$$

(when $n_I \neq |2 - 2g|$, the other pieces of $\widehat{\widetilde{G}^+}$ and $\widehat{G^-}$ may contribute to terms in (6.16) which are unrelated to the scattering of $4g - 4$ chiral graviphotons). Since the functional integrals over the six-dimensional GS fields are the same as in the $W_{11}^{\alpha\bar{\alpha}}$ case (the $e^{\pm(\rho+i\sigma)}$ factors cancel each other out), the only remaining part of (6.23) that needs to be explained is the u dependence.

We have to show that the harmonic integration over u gives the correct scattering amplitude at each value of the R-anomaly. If the sum of the R-parities of the vertex operators is equal to n_I, then we need a factor of $(u_1^*)^{2g-2-n_I}(u_2^*)^{2g-2+n_I}$ in order to select F_g^n from the topological amplitude (recall (6.8)). But since each graviphoton in $W^{\alpha\bar{\alpha}}$ contributes $|(u_1^*)^{\frac{1}{2}-R}(u_2^*)^{\frac{1}{2}+R}|^2$ where R is its R-weight, the integration over u assigns the correct n_I to the scattering amplitude.

7. Conclusion

We have formulated a new topological string theory based on $N = 4$ superconformal symmetry which has critical dimension $\hat{c} = 2$. Each such topological theory comes in a family parametrized by $S^2 = SU(2)/U(1)$ and the partition function at each genus g forms a representation of spin $2g - 2$ for this $SU(2)$. Since the $N = 2$ string with matter and ghosts decoupled is equivalent to this new topological theory, we can use the topological

reformulation to gain insight into the structure of some $N = 2$ string vacua. There are two basic classes of examples of $N = 2$ string vacua: One corresponds to four-dimensional self-dual geometries which in signature $(2, 2)$ has only one physical propagating particle. The other class of examples corresponds to the ten-dimensional superstring which, in both the RNS and GS formulations, has a twisted $N = 4$ superconformal symmetry with $\hat{c} = 2$.

Using the $N = 4$ topological description, we proved certain vanishing theorems for the $N = 2$ self-dual string on R^4. In particular, we showed to all loops that only the three-point function and the closed-string partition function are non-vanishing, which strongly hints at an integrable structure for the self-dual string. We also studied two Euclidean self-dual Ricci-flat backgrounds: T^4 and $K3$. It was shown that the partition function of the $N = 2$ string on $K3$ computes superpotential terms in harmonic superspace generated by compactifying the type II superstring to six dimensions on $K3$. This is very much analogous to the fact that $N = 2$ topological theories on Calabi-Yau compute superpotential terms generated by compactifying the superstring to four dimensions on a Calabi-Yau manifold.

We also used the topological reformulation of superstrings to show how the $N = 4$ topological string theory provides a definite prescription for computation which may resolve the question of ambiguities for RNS strings.

There are a number of things left to do: Even though we have formulated the computation of the $N = 2$ string partition function in a convenient topological language, we do not yet know how to compute it explicitly. We believe that this should be possible analogous to what occurs for the Kodaira-Spencer theory of gravity. Another question to resolve is the *mathematical meaning* of the topological partition function we are computing. For the partition function of $N = 2$ topological strings, we are counting the number of holomorphic maps from the Riemann surface to the Calabi-Yau in one version, while in another version, we are studying a quantum variation of Hodge structure. Similar interpretations would be desirable for $N = 4$ topological strings. However the situation is bound to be more complicated simply because it includes both versions of the $N = 2$ topological theory at the same time.

Another issue worth investigating is whether or not there are analogs of holomorphic anomalies for $N = 4$ topological strings. We explicitly checked one example (T^4) where such anomalies could have existed, but we found that they are absent.

As far as RNS strings, we have found a definite prescription for computation which appears to avoid ambiguities. It would be interesting to explicitly calculate the superstring partition function using this new prescription and see how the $N = 4$ topological reformulation removes the ambiguities.

Another application of the new topological theory is to shed new light on the relationship between the worldsheet supersymmetries and target-space supersymmetries of the GS superstring. It has been conjectured by many physicists that the correct description of the GS superstring in d spacetime dimensions should be with a string theory containing $d - 2$ worldsheet supersymmetries[36]. In such a description, all of the local fermionic Siegel-symmetries (which are necessary to get to light-cone gauge) can be substituted with worldsheet supersymmetries. Until now, this description was realized at the quantum level only for the $N = 2$ worldsheet-supersymmetric description of the four-dimensional GS superstring[33]. But with the results of this paper, the $N = 4$ worldsheet-supersymmetric description of the six-dimensional GS superstring can now also be realized at the quantum level. Of course, the most interesting case of the ten-dimensional GS superstring is yet to be solved, however it suggests that there may be an $N = 8$ topological string theory waiting to be discovered.

It is quite possible that our new topological theory leads to other insights. For example, if one considers self-dual strings, it has been speculated [7] that there is a twisted supersymmetry in the target-space which would explain why the physical modes are concentrated on self-dual backgrounds as appears in four-dimensional topological theories [37]. In the topological reformulation of the $N = 2$ string, one indeed has a twisted $N = 1$ spacetime supersymmetry in the target space.[16] The four supercharges are the zero modes of G^+, $\tilde{G}^+$, and $\psi_{\bar{i}}$, which have the non-vanishing anti-commutators

$$\{G^+, \psi_{\bar{i}}\} = \partial x_{\bar{i}}$$
$$\{\tilde{G}^+, \psi_{\bar{i}}\} = \epsilon_{\bar{i}\bar{j}} \eta^{j\bar{j}} \partial x_j.$$

It is amusing that the most successful approach in computations for four-dimensional topological theories has been based on considering Kähler backgrounds (which is automatic for $N = 2$ strings), and deforming the theory while preserving $N = 1$ topological symmetry (which is similar to what one has here for $N = 2$ strings). Clearly, many interesting discoveries remain to be made in this area.

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[16] Because this spacetime supersymmetry is twisted, it does not generate new physical fermionic states. However, the twisted supersymmetry algebra may be realized in the string field theory by including the ghost fields.

8. Appendix A: The relationship between the GS and RNS formalisms

In this appendix, we will describe the relation between the GS formalism and the RNS formalism of the superstring. Because the GS superstring has N=2 superconformal invariance, it is related to the N=2 embedding of the N=1 RNS superstring. As was discussed in reference [5], this embedding is described by the following (N=2,c=6) stress-energy tensor constructed out of the RNS fields:

$$L = \frac{1}{2}\partial_z x^m \partial_z x_m + \psi^m \partial_z \psi_m + L_{RNS} - \frac{3}{2}\beta \partial_z \gamma - \frac{1}{2}\gamma \partial_z \beta - \frac{3}{2}b\partial_z c + \frac{1}{2}c\partial_z b + \frac{1}{2}\partial_z(\xi\eta),$$

$$G^- = b, \tag{8.1}$$

$$G^+ = \gamma(\psi^m \partial_z x_m + G^+_{RNS} + G^-_{RNS})+$$

$$c(\frac{1}{2}\partial_z x^m \partial_z x_m + \psi^m \partial_z \psi_m + L_{RNS} - \frac{3}{2}\beta \partial_z \gamma - \frac{1}{2}\gamma \partial_z \beta - b\partial_z c) - \gamma^2 b + \partial^2_- c + \partial_z(c\xi\eta),$$

$$J = cb + \eta\xi,$$

where x^m and ψ^m are the spacetime RNS matter fields, $[b,c]$ and $[\beta = \partial_z \xi e^{-\phi}, \gamma = \eta e^{\phi}]$ are the twisted RNS ghost fields, and $[L_{RNS}, G^+_{RNS}, G^-_{RNS}, J_{RNS}]$ is the N=2 stress-energy tensor coming from the compactification (we will assume that the compactification preserves at least $(N = 1, d = 4)$ spacetime supersymmetry so the internal manifold has $N = 2$ worldsheet supersymmetry).

There is an OPE-preserving field redefinition which maps GS fields into GSO-projected combinations of RNS fields (GSO-projected means that there are no square-root cuts with the spacetime-supersymmetry generators) and maps the N=2 tensor of the GS string into the N=2 tensor of equation (8.1) . Since the N=2 tensor of equation (8.1) contains both N=1 RNS matter and ghost fields, this field redefinition maps GS matter fields into RNS matter and ghost fields. This is expected since spacetime-supersymmetry transforms GS matter fields into matter fields but mixes RNS matter and ghost fields together.

Using the N=2 rules of computation (either in the non-topological form with N=2 ghosts, or in the topological form discussed in this paper), the two superstring formalisms should give the same scattering amplitudes since they are related by a field redefinition. However, the actual calculations will look different since the GS fields are GSO projected

before doing the path integral.[17] Therefore in the GS formalism, there are no square-root cuts at any stage in the calculation and there is no need to sum over spin structures. This allows the calculations to be performed in a manifestly spacetime-supersymmetric manner. Up to now in ten-dimensional GS calculations, it has only been possible to manifestly preserve an $SU(4) \times U(1)$ subgroup (or an $SO(5,1) \times U(1)$ subgroup) of the original $SO(9,1)$ Lorentz-invariance. However, if the compactification itself breaks the $SO(9,1)$ down to $SO(5,1) \times U(1)$ or smaller (as in the Calabi-Yau compactification to four dimensions or the $K3$ compactification to six dimensions where the internal $SU(2)$ is broken to $U(1)$), this is not a disadvantage.

We will first discuss the relationship between GS and RNS fields for the four-dimensional case when the superstring is compactified on a Calabi-Yau manifold, and will then discuss the relationship for the six-dimensional case when the superstring is compactified on $K3$. For the ten-dimensional uncompactified case, see reference [10].

For the four-dimensional case, first define a "chiral" set of GS variables by performing the unitary transformation,

$$\tilde{\Phi} = \exp(-\int dz[i\partial_z x_{\alpha\dot{\alpha}}\theta^\alpha\theta^{*\dot{\alpha}} + e^{-\rho}(\theta)^2 G_{CY}])$$

$$\Phi \ \exp(\int dz[i\partial_z x_{\alpha\dot{\alpha}}\theta^\alpha\theta^{*\dot{\alpha}} + e^{-\rho}(\theta)^2 G_{CY}]),$$

where Φ includes all GS fields defined in Section 6.1. In terms of these chiral GS variables, $G_{GS}^+ + G_{CY}^+$ is simply $e^{\tilde{\rho}}(\tilde{p})^2$. The field redefinition from these chiral GS variables to the RNS variables is:

$$\tilde{x}_{GS}^m = x_{RNS}^m, \quad \partial_z\tilde{\rho} = -3\partial_z\phi + cb + 2\xi\eta - J_{CY}^{RNS}, \tag{8.2}$$

$$\tilde{\theta}^\alpha = c\xi e^{-\frac{1}{2}(3\phi + \int^z J_{CY}^{RNS})}S^\alpha, \quad \tilde{\theta}^{*\dot{\alpha}} = e^{\frac{1}{2}(\phi + \int^z J_{CY}^{RNS})}S^{\dot{\alpha}},$$

$$\tilde{p}_\alpha = b\eta e^{\frac{1}{2}(3\phi + \int^z J_{CY}^{RNS}])}S_\alpha, \quad \tilde{p}_{\dot{\alpha}}^* = e^{-\frac{1}{2}(\phi + \int^z J_{CY}^{RNS})}S_{\dot{\alpha}},$$

$$\tilde{L}_{CY} = L_{CY}^{RNS} + \frac{3}{2}(\partial_z\phi + \eta\xi)^2 - (\partial_z\phi + \eta\xi)J_{CY}^{RNS}, \quad \tilde{G}_{CY} = e^\phi\eta G_{CY}^{RNS}$$

$$\tilde{\bar{G}}_{CY} = e^{-\phi}\xi\bar{G}_{CY}^{RNS}, \quad \tilde{J}_{CY} = J_{CY}^{RNS} + 3(\partial_z\phi + \eta\xi),$$

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[17] A similar situation exists in the GS and RNS light-cone formulations of the superstring [38]. Although a field-redefinition based on SO(8) triality relates the light-cone GS fields with the light-cone RNS fields, the amplitude calculations look very different in the two formulations since in the GS light-cone formulation, there are no square-root cuts and no sum over spin-structures.

where S^α is the SO(3,1) chiral spinor constructed out of the four ψ fields as $S^\alpha = e^{\frac{1}{2}\int^z(\pm\psi^0\psi^1\pm\psi^2\psi^3)}$ with an even number of $+$ signs, and $S^{\dot\alpha}$ is the anti-chiral spinor constructed like S^α but with an odd number of $+$ signs.

In addition to mapping the N=2 GS tensor of equation (6.4) onto the N=2 RNS tensor of equation (8.1) , this field redefinition maps the integrated GS vertex operators (in the picture $|Z^+Z^-|^2\hat{V}$ where they have no N=2 ghost dependence) onto integrated RNS vertex operators and maps the GS spacetime-supersymmetry generators of eqn. (6.2) onto the RNS spacetime-supersymmetry generators

$$q_\alpha = \int dz[\; b\eta e^{\frac{1}{2}(3\phi+\int^z J^{RNS}_{CY})}S_\alpha$$

$$-e^{\frac{\phi}{2}}\{\; \psi_m\partial_z x^m + G^{RNS}_{CY} + \bar{G}^{RNS}_{CY}\; ,\; e^{\frac{1}{2}\int^z J^{RNS}_{CY}}S_\alpha\;\}\;],$$

$$q^*_{\dot\alpha} = \int dz[e^{-\frac{1}{2}(\phi+\int^z J^{RNS}_{CY})}S_{\dot\alpha}].$$

Since $R = \int dz(\partial_z\rho - \frac{1}{2}(\theta^\alpha p_\alpha - \theta^{*\dot\alpha}\bar{p}_{\dot\alpha}))$ gets mapped onto the RNS operator $\int dz(\xi\eta - \partial_z\phi)$, the R-weight of a GS operator is equal to the picture of the corresponding RNS operator. In other words, the vertex operators for chiral fermions in the GS formalism get mapped onto Ramond states in the $+\frac{1}{2}$ picture, while the vertex operators for anti-chiral fermions get mapped onto Ramond states in the $-\frac{1}{2}$ picture. In the N=1 RNS formalism, this identification of chirality and ghost number would be inconsistent since it would force amplitudes to vanish unless they had a fixed number of chiral minus anti-chiral external states. In the N=2 formalism, however, there is no inconsistency because of the sum over instanton number which picks up contributions from different RNS pictures.

For the six-dimensional case on $K3$, the field redefinition relating the six-dimensional GS variables and the conventional RNS variables is:

$$\tilde{x}^m_{GS} = x^m_{RNS}, \quad \tilde{\theta}^\alpha = e^{\frac{1}{2}(\phi+\int^z J^{RNS}_{K3})}S^\alpha, \quad \tilde{p}_\alpha = e^{-\frac{1}{2}(\phi+\int^z J^{RNS}_{K3})}S_\alpha,$$

$$i\partial_z\tilde{\sigma} = cb, \quad \partial_z\tilde{\rho} = -2\partial_z\phi + \xi\eta - J^{RNS}_{K3},$$

$$\tilde{L}_{K3} = L^{RNS}_{K3} + (\partial_z\phi + \eta\xi)^2 - (\partial_z\phi + \eta\xi)J^{RNS}_{K3}, \quad \tilde{G}^-_{K3} = e^\phi\eta G^{-\,RNS}_{K3}$$

$$\tilde{G}^+_{K3} = e^{-\phi}\xi G^{+\,RNS}_{K3}, \quad \tilde{J}_{K3} = J_{K3,RNS} + 2(\partial_z\phi + \eta\xi) \tag{8.3}$$

where S^α is the four-component SO(5,1) chiral spinor constructed out of the six ψ fields as $S^\alpha = e^{\frac{1}{2}\int^z(\pm\psi^0\psi^1\pm\psi^2\psi^3\pm\psi^4\psi^5)}$ with an even number of $+$ signs, S_α is the four-component

anti-chiral spinor constructed like S^α but with an odd number of $+$ signs, and the GS fields with a tilde are related to those without a tilde by the relation:

$$\tilde{\Phi} = \exp(-\int dz e^{i\sigma} G^+_{K3}) \; \Phi \; \exp(\int dz e^{i\sigma} G^+_{K3}).$$

As in the Calabi-Yau case, this six-dimensional field redefinition maps the N=2 GS tensor of equation (6.12) into the N=2 RNS tensor of equation (8.1) and maps the GS vertex operators (in integrated form) into the corresponding RNS vertex operators. The generator of R-transformations, $\int dz[\partial_z \rho + \frac{1}{2} p_\alpha \theta^\alpha]$, is once again mapped into the RNS operator $\int dz(\xi\eta - \partial_z\phi)$, so R-weight is still equivalent to RNS picture.

9. Appendix B: The equivalence proof of N=2 and N=3 topological amplitudes

In this paper, we have seen that $N = 1$ RNS amplitudes are naturally phrased in the language that we have developed for the new $N = 4$ topological string, and in fact, this was a primary motivation for defining the new theory. In this sense, viewing $N = 2$ strings (when the matter and ghosts decouple) as an $N = 4$ topological theory is reversing the arrow of imbedding discovered in [5], where it was shown that for backgrounds where the ghost and matter do not mix, there is a hierarchy of imbeddings of $N = 0 \subset N = 1 \subset N = 2$. However, one may be interested in more general $N = 2$ string backgrounds where the matter is mixed with the $N = 2$ ghosts (even though none is presently known, there are no reasons to doubt their existence). These backgrounds cannot be computed via the new topological theory under discussion. At any rate, it is natural to try to generalize the hierarchy of [5] to include strings with more interesting matter and ghost mixings.

A nice class of such mixed strings are the topological strings themselves. In particular, the $N = 2$ topological string may be viewed as a generalized vacuum for the $N = 0$ bosonic string where the matter and ghosts are mixed. Similarly, $N = 3$ topological strings may be viewed as a generalized vacuum for $N = 1$ superstrings where the matter and ghosts are mixed (see [39] for related discussions). Also, one would expect that $N = 4$ topological theory could be viewed as a generalized $N = 2$ string background (see the end of this appendix for some comments on such a background).[18] One could therefore try to

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[18] We should note here that there are two different $N = 4$ algebras: the smaller one with $SU(2)$ symmetry discussed in this paper, and the bigger one (the 'new' $N = 4$ algebra) with $SU(2) \times SU(2)$ symmetry group which includes the flavor $SU(2)$ also in the algebra. We have shown in this paper that the special class of $N = 2$ string backgrounds which do not mix the matter with the ghosts are given by the topological string based on the small $N = 4$ algebra. The more general $N = 2$ string backgrounds are presumably related to the bigger $N = 4$ algebra.

construct a topological ladder of hierarchy, mirroring what was found in [5]: $(N = 2)_{top} \subset$ $(N = 3)_{top} \subset (N = 4)_{top}^{big}$ where $(N = 4)_{top}^{big}$ refers to a topological theory based on the $SU(2) \times SU(2)$ version of the $N = 4$ algebra. In this appendix, we will explicitly construct the first imbedding and will comment on some difficulties in constructing the second one.

The N=0 topological partition function at genus g for an $N = 2$ twisted superconformal field theory is given by

$$\prod_{j=1}^{3g-3} d^2 m_j < |(\int \mu_j G^-)|^2 > . \tag{9.1}$$

where μ_j is the beltrami differential for m_j and G^- is the dimension 2 superconformal generator of the twisted N=2 theory (the other N=2 generators are L of dimension 2, G^+ of dimension 1, and J of dimension 1). This partition function can be understood from the $N = 0$ perspective as the vacuum amplitude $\prod_{j=1}^{3g-3} d^2 m_j < |(\int \mu_j b^-)|^2 >$ since in the standard technique for getting an N=2 stress-tensor from an N=0 conformal field theory, G^- is simply b. Ignoring "improvement" terms constructed out of U(1) currents, the other elements of the N=2 stress-tensor are $G^+ = J_{BRST}$ (where $Q_{N=0} = \int dz J_{BRST}$), $J = bc$, and $L = L_{matter} + L_{ghost}$. Note that the topological partition function of equation (9.1) vanishes unless $\hat{c} = 3$ ($c = 9$) by U(1) charge conservation (the charge violation must be equal to $\hat{c}(g - 1)$).

The N=1 topological partition function at genus g for an $N = 3$ twisted superconformal field theory is given by

$$\prod_{j=1}^{3g-3} d^2 m_j < |(\int \mu_j \hat{G}^-) \prod_{k=1}^{2g-2} \delta(\hat{J}^-) \hat{G}^3(y_k)|^2 > . \tag{9.2}$$

where $(\hat{G}^-, \hat{G}^3, \hat{G}^+)$ are the superconformal generators of dimension $(2, 3/2, 1)$, $(\hat{J}^-, \hat{J}^3, \hat{J}^+)$ are the SO(3) currents of dimension $(3/2, 1, 1/2)$, and $\hat{F}$ is the fermionic generator of dimension $1/2$ (the twisted N=3 tensor is related to the untwisted tensor by $\hat{L} \rightarrow \hat{L} + \frac{1}{2}\partial_z \hat{J}^3, \hat{G}^3 \rightarrow \hat{G}^3 + \partial_z \hat{F}$). We will put hats on all $N = 3$ generators in order to distinguish them from $N = 2$ generators. The partition function of equation (9.2) can be understood from the N=1 perspective as the vacuum amplitude

$$\prod_{j=1}^{3g-3} d^2 m_j < |(\int \mu_j b^-) \prod_{k=1}^{2g-2} \delta(\beta)(G_{matter} + G_{ghost})(y_k)|^2 >$$

64

since in the standard technique for getting an $N = 3$ stress-tensor from an $N = 1$ super-conformal field theory, $\hat{G}^- = b$, $\hat{G}^3 = G_{matter} + G_{ghost}$, $\hat{J}^- = \beta$. Ignoring "improvement" terms, the other elements of the N=3 stress-tensor are $\hat{L} = L_{matter} + L_{ghost}$, $\hat{G}^+ = J_{BRST}$, $\hat{J}^+ = j_{BRST}$ (where $Q_{N=1} = \int dz J_{BRST} = \int dz \{G_{N=1}, j_{BRST}\}$), $\hat{J}^3 = cb - \beta\gamma$, and $\hat{F} = \beta c$. Unless $\hat{c} = 1$ ($c = 3$), the topological partition function (9.2) vanishes by conservation of J^3 charge.

Now suppose we are given an N=2 system with $c = 9$. It was recently discovered in [40] that any ($N = 2, c = 9$) stress-tensor can be embedded into an ($N = 3, c = 3$) stress-tensor. It will now be shown that the $N = 1$ topological partition function of the resulting ($N = 3, c = 3$) system is equal to the $N = 0$ topological partition function of the original ($N = 2, c = 9$) system, and therefore the N=0 and N=1 topological theories coming from these two systems are equivalent.

As was shown in [40], there exists an embedding of any $N = 2$ system with central charge c into an $N = 3$ system with central charge $c - 6$. This embedding does not require a U(1) current, and when $c = 6$, it reduces to an $N = 3$ subset of the $N = 2 \rightarrow N = 4$ embedding discovered in [41]. In $N = 2$ superspace notation, this embedding is given by

$$\hat{T} = T - \frac{1}{2}\partial_z(BC) + (DC)(\bar{D}B) + (\bar{D}C)(DB),$$

$$\hat{G} = B + CT + C(DC)(\bar{D}B) + C(\bar{D}C)(DB) - B(\bar{D}C)(DC) - \frac{c-6}{3}[D, \bar{D}]C$$

$$+ \frac{c-6}{6}\sum_{n=1}^{\infty}(-C(DC)(\bar{D}\partial_z C) + C(\bar{D}C)(D\partial_z C) - 2(DC)(\bar{D}C)[D, \bar{D}]C$$

$$- (2n-1)C(D\bar{D}C)(\bar{D}DC))[(DC)(\bar{D}C)]^{n-1}, \tag{9.3}$$

where $D = \partial_\theta - \frac{1}{2}\bar{\theta}\partial_z$ and $\bar{D} = \partial_{\bar{\theta}} - \frac{1}{2}\theta\partial_z$ are the usual $N = 2$ fermionic derivatives, $\theta_{12} \equiv \theta_1 - \theta_2$, $\bar{\theta}_{12} \equiv \bar{\theta}_1 - \bar{\theta}_2$, $z_{12} \equiv z_1 - z_2 + \frac{1}{2}(\theta_1\bar{\theta}_2 + \bar{\theta}_1\theta_2)$, $B = \tilde{b} + \theta\beta^+ - \bar{\theta}\beta^- + \theta\bar{\theta}b$ and $C = c + \theta\gamma^+ + \bar{\theta}\gamma^- + \theta\bar{\theta}\tilde{c}$ are N=2 superfields satisfying the OPE:

$$C(Z_1)B(Z_2) \rightarrow \theta_{12}\bar{\theta}_{12}/z_{12},$$

$T(Z) = J + \theta G^+ + \bar{\theta}G^- + \theta\bar{\theta}L$ is the untwisted $N = 2$ stress-tensor with central charge c satisfying the OPE

$$T(Z_1)T(Z_2) \rightarrow$$

$$\frac{\frac{c}{3} + \theta_{12}\bar{\theta}_{12}T(Z_2)}{z_{12}^2} + \frac{-\theta_{12}DT(Z_2) + \bar{\theta}_{12}\bar{D}T(Z_2) + \theta_{12}\bar{\theta}_{12}\partial_z T(Z_2)}{z_{12}},$$

65

and

$$(\hat{T}, \hat{G}) = (\hat{J}^3 + \theta\hat{G}^+ + \bar{\theta}\hat{G}^- + \theta\bar{\theta}\hat{L}, \hat{F} + \theta\hat{J}^+ + \bar{\theta}\hat{J}^- + \theta\bar{\theta}G^3)$$

forms an untwisted $N = 3$ stress-tensor with central charge $c - 6$ satifying the OPE's:

$$\hat{T}(Z_1)\hat{T}(Z_2) \rightarrow \frac{\frac{c-6}{3} + \theta_{12}\bar{\theta}_{12}\hat{T}(Z_2)}{z_{12}^2} + \frac{-\theta_{12}D\hat{T} + \bar{\theta}_{12}\bar{D}\hat{T}(Z_2) + \theta_{12}\bar{\theta}_{12}\partial_z\hat{T}(Z_2)}{z_{12}},$$

$$\hat{T}(Z_1)\hat{G}(Z_2) \rightarrow \frac{\theta_{12}\bar{\theta}_{12}\hat{G}(Z_2)}{2z_{12}^2} + \frac{-\theta_{12}D\hat{G}(Z_2) + \bar{\theta}_{12}\bar{D}\hat{G}(Z_2) + \theta_{12}\bar{\theta}_{12}\partial_z\hat{G}(Z_2)}{z_{12}},$$

$$\hat{G}(Z_1)\hat{G}(Z_2) \rightarrow \frac{\frac{4c-24}{3} + 2\theta_{12}\bar{\theta}_{12}\hat{T}(Z_2)}{z_{12}^2}.$$

In a previous paper, we showed that a critical N=0 string could be embedded into a critical N=1 string and the non-topological N=0 and N=1 rules for computing the scattering amplitudes gave the same result. Although the $N = 2 \rightarrow N = 3$ embedding of (9.3) is much more complicated than the $N = 0 \rightarrow N = 1$ embedding discussed in [5], the same techniques can be used to prove equivalence of the relevant topological partition functions. The only crucial property of the $N = 2 \rightarrow N = 3$ embedding is that the ghost-number -1 contribution to $\hat{G}$ is B (in components, this means that $\hat{G}^3 = b + ...$, $\hat{J}^+ = \beta^+ + ...$, $\hat{J}^- = \beta^- + ...$, and $\hat{F} = \tilde{b} + ...$). Furthermore note that $\hat{T}$ is simply the sum of T and the stress-tensor for a (B, C) system of conformal weight $(1/2, -1/2)$.

After twisting $\hat{T}$ and $\hat{G}$ by sending $\hat{L} \rightarrow \hat{L} + \frac{1}{2}\partial_z J^3$ and $\hat{G}^3 \rightarrow \hat{G}^3 + \partial_z F$, we can express the $N = 1$ topological partition function of equation (9.2) in terms of T, B, and C. Because of BC conservation, the only term that contributes to G^3 is b, the only term that contributes to $\hat{G}^-$ is G^-, and the only term that contributes to $\delta(J^-)$ is $\delta(\beta^-)$. So the N=1 topological partition function takes the form:

$$\prod_{j=1}^{3g-3} d^2m_j < |(\int \mu_j G^-) \prod_{k=1}^{2g-2} \delta(\beta^-)b(y_k)|^2 > . \tag{9.4}$$

It will now be shown that after integrating over the B and C fields, this expression reduces to the N=0 topological partition function of the original $(N = 2, c = 9)$ system.

The non-zero modes of these fields cancel each other out since after twisting, (b, c) and (β^-, γ^+) each have weights $(3/2, -1/2)$, while (β^+, γ^-) and $(\tilde{b}, \tilde{c})$ each have weights $(1/2, 1/2)$ (before twisting, the conformal weights of (b, c) are $(3/2, -1/2)$, of $(\beta^\pm, \gamma^\mp)$ are $(1, 0)$, and of $(\tilde{b}, \tilde{c})$ are $(1/2, 1/2)$). The $2g - 2$ zero modes of the b and β^- fields come from

the G^3 and $\delta(J^-)$ factors in equation (9.2) and cancel each other out. Since there are no zero modes needed for the other (B, C) fields, expression (9.4) reduces to

$$\prod_{j=1}^{3g-3} d^2 m_j < |(\int \mu_j G^-)|^2 >,$$

which is just the N=0 topological partition function (9.1) for the original N=2 system.

Since this equivalence proof of the $N = 1$ and $N = 0$ topological partition functions closely resembles the equivalence proof of the $N = 1$ and $N = 0$ non-topological scattering amplitudes in [5], it is natural to ask if there is also a topological analog for the equivalence proof of $N = 2$ and $N = 1$ non-topological scattering amplitudes in [5]. In fact, there is a natural definition of an $N = 2$ topological partition function if one has a 'big' twisted $N = 4$ superconformal field theory of central charge $c = 0$. This definition is

$$\prod_{j=1}^{3g-3} d^2 m_j < |(\int \mu_j G^-) \prod_{k=1}^{2g-2} \delta(J_1^-) G_1^3 \delta(J_2^-) \hat{G}_2^3(y_k) \prod_{m=1}^{g-1} F^-|^2 >$$

where (G^-, G_1^3, G_2^3, G^+) are the superconformal generators of weight $(2, 3/2, 3/2, 1)$, $(J_1^-, J_1^3, J_1^+, J_2^-, J_2^3, J_2^+)$ are the $SO(4)$ currents of weight $(3/2, 1, 1/2, 3/2, 1, 1/2)$, (F^-, F_1^3, F_2^3, F^+) are the fermionic generators of weight $(1, 1/2, 1/2, 0)$, and H is the bosonic generator of weight 0. By conservation of $J_1^3 + J_2^3$ charge, this partition function vanishes unless $c = 0$.

Since the $N = 1$ topological partition function requires an $N = 2$ superconformal field theory with $c = 3$, we are therefore looking for an embedding that takes an $(N = 3, c = 3)$ superconformal field theory into an $(N = 4, c = 0)$ superconformal field theory. Unfortunately, all of the embeddings that do not require " improvement" terms with U(1) currents map $N = 3$ stress-tensors with central charge c into $N = 4$ stress-tensors with the same central charge (this is essentially because the $N = 3$ ghost system has $c = 0$). So with our present knowledge of embeddings, we are unable to find a topological analog to the $N = 1 \rightarrow N = 2$ non-topological embedding of [5].

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