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## Citation

Bershadsky, Michael, Zurab Kakushadze, and Cumrun Vafa. 1998. "String Expansion as Large N Expansion of Gauge Theories." *Nuclear Physics B* 523 (1-2): 59-72. [https://doi.org/10.1016/s0550-3213\(98\)00272-7](https://doi.org/10.1016/s0550-3213(98)00272-7).

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# String Expansion as Large $N$ Expansion of Gauge Theories

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## Abstract

We consider string perturbative expansion in the presence of D-branes imbedded in orbifolded space-time. In the regime where the string coupling is weak and  $\alpha' \rightarrow 0$ , the string perturbative expansion coincides with 't Hooft's large  $N$  expansion. We specifically concentrate on theories with  $d = 4$  and  $\mathcal{N} = 0, 1, 2, 4$ , and use world-sheet orbifold techniques to prove vanishing theorems for the field theory  $\beta$ -functions to all orders in perturbation theory in the large  $N$  limit. This is in accord with recent predictions.

March 8, 1998

## 1. Introduction

One of the most interesting ideas in gaining insight into the structure of gauge theory is ‘t Hooft’s idea of considering a large  $N$  limit [1]. It was noticed in [1] that in this limit the gauge theory diagrams organize themselves in terms of Riemann surfaces, where addition of each extra handle on the surface corresponds to suppression by  $\frac{1}{N^2}$ . In fact, this similarity led ‘t Hooft to speculate about possible connections with the “dual” model perturbation expansion—which today we call string theory perturbation.

Even though the idea sounded promising, no direct connection between the two was made for a long time. The first concrete connection came with the beautiful work of Witten [2] where it was shown that at least in the context of topological strings, with boundaries mapped to topological versions of  $N$  D3-branes, the string expansion was actually the same as the large  $N$  expansion of the 3d Chern-Simons gauge theory. In particular, diagram by diagram string theory expansion was mapped to large  $N$  expansion of gauge theory, in the sense of ‘t Hooft.

In a seemingly unrelated development, it was noticed [3] (for a recent discussion, see, *e.g.*, [4]) that string theory perturbation techniques is a very useful way of summing up various field theory Feynman diagrams. The basic idea here is to consider a limit  $\alpha' \rightarrow 0$ , where string theory reduces to its massless modes, and try to extract the contribution of gauge fields and matter in the corresponding string theory diagram. This direction has been studied extensively, with various interesting applications. These applications suggest that even if we are just interested in gauge theories, the string perturbation techniques are very powerful and should not be overlooked.

The basic idea of this paper is to combine these two approaches. Namely, we consider type II strings in the presence of a large number  $N$  of D-branes and consider a limit where  $\alpha' \rightarrow 0$  while keeping  $\lambda = N\lambda_s$  fixed, where  $\lambda_s$  is the type II string coupling. Note that in this context a world-sheet with  $g$  handles and  $b$  boundaries is weighted with

$$(N\lambda_s)^b \lambda_s^{2g-2} = \lambda^{2g-2+b} N^{-2g+2} .$$

After we identify  $\lambda_s = g_{YM}^2$ , this is the same as large  $N$  expansion considered by ‘t Hooft. Note that for this expansion to make sense we have to consider the limit where  $N \rightarrow \infty$  while fixing  $\lambda$  at a small value  $\lambda < 1$ . If  $\lambda > 1$  then no matter how large  $N$  is, for sufficiently many boundaries the higher genus terms would be relevant, and we lose the genus expansion of large  $N$ . This, of course, is the same as the domain of validity of string

perturbation theory. Therefore, one should expect a surface expansion in large  $N$  *only* for small  $\lambda$ . In this limit we can map the string diagrams directly to (specific sums of) large  $N$  Feynman diagrams. Note in particular that the genus  $g = 0$  planar diagrams dominate in the large  $N$  limit.

In the large  $N$  limit we are still left with the free parameter  $\lambda$  and there are two natural regimes of parameters to consider. As just discussed the case which makes contact with large  $N$  analysis of 't Hooft and string perturbation theory is small  $\lambda$ . It is also interesting to ask what happens for large  $\lambda$ . This is a very non-trivial question and is beyond the domain of validity of 't Hooft's large  $N$  analysis or string theory perturbation techniques. This is precisely the case considered recently in several very interesting papers [5][6][7][8][9][10], and has also been studied in related works [11]. This is a limit where one expects an effective supergravity description to take over.

We will be considering the limit where  $\lambda$  is small. We shall see that the string world-sheet techniques allow us to prove certain statements about the gauge theory in this limit, which in principle should be properties of Feynman diagrams. But since string theory organizes Feynman diagrams in a very economical way, it turns out that the proof is obvious only in the string theory setup. Using string theory perturbation techniques we establish part of the conjectures in [12] (extending the work in [13]) in connection with constructing four dimensional conformal field theories<sup>1</sup> (including the case with no supersymmetries). More precisely, we *prove* that in the large  $N$  limit the  $\beta$ -functions of all theories considered in [12] are identically zero. We also gain insight into possible  $1/N$  corrections in this context. The vanishing of the  $\beta$ -functions was proved up to two loops in the  $\mathcal{N} = 1$  examples in [12], and up to one-loop level in the  $\mathcal{N} = 0$  case. Given how cumbersome such Feynman diagram computations are it is quite pleasant to observe the power of string perturbation techniques (in the context of orbifolds) in proving such statements. Moreover, in doing so we also gain insight into the conditions imposed in the orbifold construction in [12] (and, in particular, *why* the orbifold groups considered in [12] should act in the regular representation when acting on gauge degrees of freedom). In fact we will be able to show more. Namely, we show that any correlation computation for these theories reduces as  $N \rightarrow \infty$  to the corresponding computation in the  $\mathcal{N} = 4$  theory. We will also see why the  $\frac{1}{N}$  corrections will be different from those of the  $\mathcal{N} = 4$  theory.

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<sup>1</sup> For a review of field theory discussions of this subject, see, *e.g.*, [14]. For a recent attempt for constructing finite gauge theories via orientifolds, see, *e.g.*, [15].

The remainder of this paper is organized as follows. In section 2 we describe construction of gauge theories via type IIB D3-branes in orbifold backgrounds which lead to theories that are (super)conformal in the large  $N$  limit. In section 3 we show that in these theories the  $\beta$ -functions as well as anomalous scaling dimensions vanish to all orders in the large  $N$  limit. In section 4 we discuss subleading corrections at large  $N$ . We point out that, subject to certain assumptions, one may also be able to prove that  $\mathcal{N} = 1$  theories may be superconformal even at finite  $N$ . We also discuss the issues that need to be understood for checking such statements for the  $\mathcal{N} = 0$  case.

## 2. Setup

We start with type IIB string theory with  $N$  parallel D3-branes where the space transverse to the D-branes is  $\mathcal{M} = \mathbf{R}^6/\Gamma$ . The orbifold group  $\Gamma = \{g_a \mid a = 1, \dots, |\Gamma|\}$  ( $g_1 = 1$ ) must be a finite discrete subgroup of  $Spin(6)$ . If  $\Gamma \subset SU(3)$  ( $SU(2)$ ), we have  $\mathcal{N} = 1$  ( $\mathcal{N} = 2$ ) unbroken supersymmetry, and  $\mathcal{N} = 0$ , otherwise.

We will confine our attention to the cases where type IIB on  $\mathcal{M}$  is a modular invariant theory<sup>2</sup>. The action of the orbifold on the coordinates  $X_i$  ( $i = 1, \dots, 6$ ) on  $\mathcal{M}$  can be described in terms of  $SO(6)$  matrices:  $g_a : X_i \rightarrow (g_a)_{ij} X_j$ . The world-sheet fermionic superpartners of  $X_i$  transform in the same way. We also need to specify the action of the orbifold group on the Chan-Paton charges carried by the D3-branes. It is described by  $N \times N$  matrices  $\gamma_a$  that form a representation of  $\Gamma$ . Note that  $\gamma_1$  is the identity matrix and  $\text{Tr}(\gamma_1) = N$ .

The D-brane sector of the theory is described by an *oriented* open string theory. In particular the world-sheet expansion corresponds to summing over oriented Riemann surfaces with arbitrary genus  $g$  and arbitrary number of boundaries  $b$ , where the boundaries of the world-sheet are mapped to the D3-brane world-volume. Moreover we consider various “twists” corresponding to orbifold sectors, around the cycles of the Riemann surface. The choice of “twists” corresponds to a choice of homomorphism of the fundamental group of the Riemann surface with boundaries to  $\Gamma$ .

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<sup>2</sup> This is always the case in the supersymmetric case. For the non-supersymmetric case this is also true if  $\not\exists \mathbf{Z}_2 \subset \Gamma$ . If  $\exists \mathbf{Z}_2 \subset \Gamma$ , then modular invariance requires that the set of points in  $\mathbf{R}^6$  fixed under the  $\mathbf{Z}_2$  twist has real dimension 2.

For example, consider one-loop vacuum amplitude ( $g = 0, b = 2$ ). The corresponding graph is an annulus whose boundaries lie on D3-branes. The one-loop partition function in the light-cone gauge is given by

$$\mathcal{Z} = \frac{1}{2|\Gamma|} \sum_a \text{Tr} (g_a (1 + (-1)^F) e^{-2\pi t L_0}) , \quad (2.1)$$

where  $F$  is the fermion number operator,  $t$  is the real modular parameter on the cylinder, and the trace includes sum over the Chan-Paton factors. The states in the Neveu-Schwarz (NS) sector are space-time bosons and enter the partition function with weight  $+1$ , whereas the states in the Ramond (R) sector are space-time fermions and contribute with weight  $-1$ .

Note that the elements  $g_a$  acting in the Hilbert space of open strings will act both on the left-end and the right-end of the open string. In particular  $g_a$  corresponds to  $\gamma_a \otimes \gamma_a$  acting on the Chan-Paton indices. The individual terms in the sum in (2.1) therefore have the following form:

$$(\text{Tr}(\gamma_a))^2 \mathcal{Z}_a , \quad (2.2)$$

where  $\mathcal{Z}_a$  are characters corresponding to the world-sheet degrees of freedom. The “untwisted” character  $\mathcal{Z}_1$  is the same as in the  $\mathcal{N} = 4$  theory for which  $\Gamma = \{1\}$ . The information about the fact that the orbifold theory has reduced supersymmetry is encoded in the “twisted” characters  $\mathcal{Z}_a, a \neq 1$ .

Here we are interested in constructing finite gauge theories. Since  $\mathcal{N} = 4$  gauge theories are finite, we can hope to obtain finite gauge theories (at least in the large  $N$  limit) with reduced supersymmetry by arranging for the twisted sector contributions to the  $\beta$ -functions and anomalous scaling dimensions to be absent. The above discussion suggests one natural way of possibly achieving this goal. Consider representations of  $\Gamma$  such that

$$\text{Tr}(\gamma_a) = 0 \quad \forall a \neq 1 . \quad (2.3)$$

In section 3 we will show that gauge theories corresponding to such representations are indeed finite in the large  $N$  limit to all orders in perturbation theory. In fact, we will see in the next subsection that cancellation of tadpoles in the closed string channel requires this trace condition. Moreover, we will show that this trace condition fixes the representation  $\gamma$  to be a sum of copies of the regular representation of  $\Gamma$ .

## 2.1. Tadpole cancellation

In this section we investigate the conditions under which the oriented open string theory that describes the D-brane sector in the above framework is finite in the *ultraviolet*. Ultraviolet finiteness guarantees absence of anomalies, and is related to tadpole cancellation in the closed string channel. Here we discuss the one-loop tadpole cancellation conditions which as we will see impose constraints on the Chan-Paton matrices  $\gamma_a$ .

The characters  $\mathcal{Z}_a$  in (2.2) are given by

$$\mathcal{Z}_a = [\eta(e^{-2\pi t})]^{-2-d_a} (\mathcal{X}_a(e^{-2\pi t}) - \mathcal{Y}_a(e^{-2\pi t})) . \quad (2.4)$$

Here  $d_a$  is the real dimension of the set of points fixed under the twist  $g_a$ . Two of the  $\eta$ -functions come from the oscillators corresponding to the space-time directions filled by D3-branes (and the time-like and longitudinal contributions are absent due to the light-cone gauge). The other  $d_a$   $\eta$ -functions come from the oscillators corresponding to the directions transverse to the D-branes. Finally, the characters  $\mathcal{X}_a$ ,  $\mathcal{Y}_a$  correspond to the contributions of the world-sheet fermions, as well as the world-sheet bosons with  $g_a$  acting non-trivially on them (for  $a \neq 1$ ):  $\mathcal{X}_a$  arises in the trace  $\text{Tr}(g_a e^{-2\pi t L_0})$ , whereas  $\mathcal{Y}_a$  arises in the trace  $\text{Tr}(g_a (-1)^F e^{-2\pi t L_0})$  (see (2.1)). We will not need their explicit form here.

The contributions to the one-loop vacuum amplitude corresponding to  $\mathcal{Z}_a$  are (proportional to)

$$\int_0^\infty \frac{dt}{t^3} (\text{Tr}(\gamma_a))^2 \mathcal{Z}_a = \mathcal{A}_a - \mathcal{B}_a ,$$

where

$$\begin{aligned} \mathcal{A}_a &= (\text{Tr}(\gamma_a))^2 \int_0^\infty \frac{dt}{t^3} [\eta(e^{-2\pi t})]^{-2-d_a} \mathcal{X}_a(e^{-2\pi t}) , \\ \mathcal{B}_a &= (\text{Tr}(\gamma_a))^2 \int_0^\infty \frac{dt}{t^3} [\eta(e^{-2\pi t})]^{-2-d_a} \mathcal{Y}_a(e^{-2\pi t}) . \end{aligned} \quad (2.5)$$

These integrals<sup>3</sup> are generically divergent as  $t \rightarrow 0$  reflecting presence of tadpoles. To extract these divergences we can change variables  $t = 1/\ell$  so that the divergences correspond

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<sup>3</sup> For space-time supersymmetric theories the total tadpoles vanish:  $\mathcal{A}_a - \mathcal{B}_a = 0$ . (The entire partition function vanishes as the numbers of space-time bosons and fermions are equal.) For consistency, however, we must extract individual contributions  $\mathcal{A}_a$  and  $\mathcal{B}_a$  and make sure that they cancel as well. Thus, cancellation of the tadpoles in  $\mathcal{B}_a$  is required for consistency of the untwisted and twisted R-R four-form (to which D3-branes couple) equations of motion.

to  $\ell \rightarrow \infty$ :

$$\begin{aligned}\mathcal{A}_a &= (\text{Tr}(\gamma_a))^2 \int_0^\infty \frac{d\ell}{\ell^{d_a/2}} \sum_{\sigma_a} A_{\sigma_a} e^{-2\pi\ell\sigma_a} , \\ \mathcal{B}_a &= (\text{Tr}(\gamma_a))^2 \int_0^\infty \frac{d\ell}{\ell^{d_a/2}} \sum_{\rho_a} B_{\rho_a} e^{-2\pi\ell\rho_a} .\end{aligned}\tag{2.6}$$

The closed string states contributing to  $\mathcal{A}_a$  ( $\mathcal{B}_a$ ) in the transverse channel are the NS-NS (R-R) states with  $L_0 = \bar{L}_0 = \sigma_a(\rho_a)$  (and  $A_{\sigma_a}(B_{\rho_a}) > 0$  is the number of such states). Note that the massive states with  $\sigma_a(\rho_a) > 0$  do not lead to divergences as  $\ell \rightarrow \infty$ . The ground states in the R-R sectors are massless. Note that in  $\ell \rightarrow \infty$  limit the value of  $d_a$  in the prefactor  $\ell^{-d_a/2}$  determines the divergence property of the integral. Note that  $d_a$  is the real dimension of the set of points fixed under the twist  $g_a$ . We thus get divergences in  $\mathcal{B}_a$  for large  $\ell$  for  $d_a \leq 2$ . In such a case we must make sure that the corresponding  $\text{Tr}(\gamma_a) = 0$ . Given the orientability of  $\Gamma$  the allowed values of  $d_a$  are 0, 2, 4, 6. For  $d_a = 0, 2$  tadpole cancellations thus require  $\text{Tr}(\gamma_a) = 0$ . For the case  $d_a = 4$  the corresponding twisted NS-NS closed string sector will contain tachyons. This will lead to a tachyonic tadpole in  $\mathcal{A}_a$  unless  $\text{Tr}(\gamma_a) = 0$  in the  $g_a$  twisted sector. For  $d_1 = 6$ , there is no divergence in  $\mathcal{B}_1$  and thus we have no restriction for  $\text{Tr}(\gamma_1) = N$ . We therefore conclude that to cancel all tadpoles it is necessary that

$$\text{Tr}(\gamma_a) = 0 \quad \forall a \neq 1 .\tag{2.7}$$

Since the untwisted NS-NS closed string sector does not contain tachyons, no further constraint arises on  $\text{Tr}(\gamma_1) = N$  (so that the number of D3-branes is arbitrary). Thus we conclude that one-loop tadpole cancellation is possible if and only if the constraint (2.7) is satisfied. This is precisely the condition (2.3) discussed in the beginning of this section.

Thus, we see that the condition on  $\text{Tr}(\gamma_a)$  is necessary and sufficient to guarantee one-loop ultraviolet finiteness and consistency of the corresponding string theory. For illustrative purposes, to see what can go wrong if we relax this condition, let us consider the following example. Let  $\mathcal{M} = \mathbf{C}^3/\Gamma$ , where the action of  $\Gamma$  on the complex coordinates  $X_\alpha$  ( $\alpha = 1, 2, 3$ ) on  $\mathcal{M}$  is that of the  $Z$ -orbifold:  $g : X_\alpha \rightarrow \omega X_\alpha$  (where  $g$  is the generator of  $\Gamma$ , and  $\omega = e^{2\pi i/3}$ ). Next, let us choose the representation of  $\Gamma$  when acting on the Chan-Paton charges as follows:  $\gamma_g = \text{diag}(\mathbf{I}_{N_1}, \omega \mathbf{I}_{N_2}, \omega^2 \mathbf{I}_{N_3})$  (where  $N_1 + N_2 + N_3 = N$ , and  $\mathbf{I}_m$  is an  $m \times m$  identity matrix). The massless spectrum of this model is  $\mathcal{N} = 1$   $U(N_1) \otimes U(N_2) \otimes U(N_3)$  gauge theory with the following matter content:

$$3(\mathbf{N}_1, \bar{\mathbf{N}}_2, \mathbf{1}) , 3(\mathbf{1}, \mathbf{N}_2, \bar{\mathbf{N}}_3) , 3(\bar{\mathbf{N}}_1, \mathbf{1}, \mathbf{N}_3) .$$



Note that this spectrum is anomalous (the non-Abelian gauge anomaly does not cancel) unless  $N_1 = N_2 = N_3$ . On the other hand,  $\text{Tr}(\gamma_g) = 0$  if and only if  $N_1 = N_2 = N_3$ . Thus, we see that the constraint (2.7) derived from the tadpole cancellation conditions is necessary in this case to have a consistent gauge theory.

Here we should mention that not all the choices of  $\gamma_a$  that do not satisfy (2.7) lead to such apparent inconsistencies. Consider for example the same  $\Gamma$  as above but with  $\gamma_g = \mathbf{I}_N$ . The massless spectrum of this model is  $\mathcal{N} = 1$   $U(N)$  gauge theory with no matter, so it is anomaly free. It would be interesting to see to what extent such theories can be studied using the present string theory construction.

## 2.2. Regular representation

In the examples constructed in [12], generalizing the construction of [13] in attempts for constructing conformal field theories in four dimensions, the orbifold action on the gauge degrees of freedom was chosen to be an  $n$ -fold copy of the *regular* representation of  $\Gamma$ . Here we prove that tadpole cancellation conditions for  $\gamma_a$  (2.7) correspond to having an  $n$ -fold copy of the regular representation of  $\Gamma$ . Conversely, if  $\gamma_a$  form an  $n$ -fold regular representation of  $\Gamma$ , then the condition (2.7) is satisfied.

Recall that the regular representation corresponds to considering the vector space which is identified with  $\{|g_a\rangle\}$  for elements  $g_a \in \Gamma$ . The action of the group in the regular representation is given by

$$\gamma_b |g_a\rangle = |g_a g_b\rangle .$$

Let us consider  $g_b \neq 1$ . Then it is clear that in this representation we have  $\text{Tr}(\gamma_b) = 0$  since for all  $g_a$  we have  $g_a g_b \neq g_a$ . Also note that  $\text{Tr}(\gamma_1) = |\Gamma|$ . If we consider  $n$  copies of this representation we will have the condition that the trace of all elements are zero except for the identity element whose trace is  $n|\Gamma|$ . This is the same condition as (2.3).

Next, we show that (2.7) implies that  $\gamma_a$  form an  $n$ -fold copy of regular representation of  $\Gamma$ . First, let us establish that  $\text{Tr}(\gamma_1) = n|\Gamma|$  for some integer  $n$ , *i.e.*, that the dimension of the representation is an integer multiple of  $|\Gamma|$ . To show this note that the number of times  $n_1$  the trivial representation appears in  $\gamma$  must be an integer, and this is given by

$$n_1 = \frac{1}{|\Gamma|} \text{Tr} \left( \sum_a \gamma_a \right) = \frac{1}{|\Gamma|} \text{Tr}(\gamma_1) .$$

Denoting  $n_1$  by  $n$ , we thus conclude that  $\text{Tr}(\gamma_1) = n|\Gamma|$ . Now recall from representation theory of groups that the characters (*i.e.*, traces) of elements in a representation uniquely

fix the representation [16]. We thus conclude that if the condition (2.7) is satisfied we have  $n$  copies of the regular representation of  $\Gamma$ .

The regular representation decomposes into a direct sum of all irreducible representations  $\mathbf{r}_i$  of  $\Gamma$  with degeneracy factors  $n_i = |\mathbf{r}_i|$ . The gauge group is ( $N_i \equiv nn_i$ )

$$G = \otimes_i U(N_i) .$$

The matter consists of Weyl fermions (and scalars) transforming in bi-fundamentals  $(\mathbf{N}_i, \overline{\mathbf{N}}_j)$  according to the decomposition of the tensor product of  $\mathbf{4}$  ( $\mathbf{6}$ ) of  $Spin(6)$  with the corresponding representation (see [12] for details).

In [12] it was shown (using Feynman diagram techniques) that the one-loop  $\beta$ -functions vanish for these gauge theories. Moreover, in  $\mathcal{N} = 1$  cases it was shown that the one-loop anomalous scaling dimensions for matter fields also vanish. This implies that the two-loop  $\beta$ -functions also vanish in the  $\mathcal{N} = 1$  theories.

In section 3 we will show that in the large  $N$  limit all of these theories are finite to all loops. The proof there will crucially depend on the fact that the representation for the Chan-Paton matrices  $\gamma_a$  satisfies the condition (2.7).

### 3. Large $N$ limit and finiteness

In this section we consider perturbative expansion of the theories constructed in section 2 which satisfy the condition (2.3). We will work in the full string theory framework which as we will see is much simpler than the Feynman diagram techniques. At the end we will take  $\alpha' \rightarrow 0$  limit which amounts to reducing the theory to the gauge theory subsector. In this way we will be able to show directly that at large  $N$  these theories are finite (including the theories without supersymmetry). This is in accord with the arguments in [13] corresponding to the region where  $\lambda$  is large. However [13] uses the non-trivial conjecture in [7] whereas our arguments are more elementary and apply to the region where  $\lambda$  is small. Moreover, we will show that in this limit computation of any correlation function in these theories reduces to the corresponding computation in the  $\mathcal{N} = 4$  Yang-Mills theory. We will then discuss the subleading  $\frac{1}{N}$  corrections. We will heuristically argue that at least for the  $\mathcal{N} = 1$  case, the theories under consideration may remain superconformal even for finite  $N$ . The story with  $\mathcal{N} = 0$  theories appears to be more involved due to complications associated with closed string tachyons.

### 3.1. Conditions for finiteness

As we discussed in section 2, the gauge group in the theories we are considering is  $G = \otimes_i U(N_i) (\subset U(N))$ . To consider finite theories we delete the  $U(1)$  factors (for which there are matter fields charged under them) and consider  $G = \otimes_i SU(N_i)$ . Our goal in this section is to show that in the large  $N$  limit this non-Abelian gauge theory is conformal (including the  $U(1)$ 's would not have affected this analysis as it is subleading in  $N$ ).

To establish that a non-Abelian gauge theory is conformal we need to check three points: (i) gauge coupling non-renormalization which amounts to computing two-point correlators involving gauge bosons; (ii) vanishing of anomalous scaling dimensions (wavefunction non-renormalization) which amounts to computing two-point correlators involving matter fields; (iii) non-renormalization of Yukawa (three-point) and quartic scalar (four-point) couplings. As far as perturbation theory is concerned the third point needs to be checked only in non-supersymmetric theories for in  $\mathcal{N} = 1$  theories we have perturbative non-renormalization theorem for the superpotential.

There are two classes of diagrams we need to consider: (i) diagrams without handles; (ii) diagrams with handles. The latter correspond to closed string loops and are subleading in the large  $N$  limit. We will therefore discuss these contributions only when we turn to subleading  $\frac{1}{N}$  corrections.

The diagrams without handles can be divided into two classes: (i) planar diagrams where all the external lines are attached to the same boundary; (ii) non-planar diagrams where the external lines are attached to at least two different boundaries. The latter are subleading in the large  $N$  limit. We will in fact show the stronger statement that all the planar diagrams in the orbifolded theory give correlations which are *identical* to that of the parent  $\mathcal{N} = 4$  theory, by showing that all the diagrams containing twisted boundary conditions identically vanish in this limit.

### 3.2. Planar diagrams

In the case of planar diagrams we have  $b$  boundaries with  $M$  external lines attached to one of them while others are free. As noted before to leading order in large  $N$  we will always consider the external lines attached to one boundary as depicted in Fig.1. The amplitude consists of summing over all possible twisted boundary conditions (homomorphism of the fundamental group of the planar diagram to  $\Gamma$ ). This is simply how string theory ensures that the states contributing to the amplitudes are properly projected by the action of the

orbifold group. The trivial boundary conditions correspond to the same amplitudes as in the  $\mathcal{N} = 4$  case (modulo factors of  $1/\sqrt{|\Gamma|}$  which can be reabsorbed by a redefinition of  $\lambda$ ). We will now show that for the diagrams under consideration for all other boundary conditions the amplitudes are identically zero! It turns out that all we need to do is to carefully consider what Chan-Paton factors we get for each string diagram.

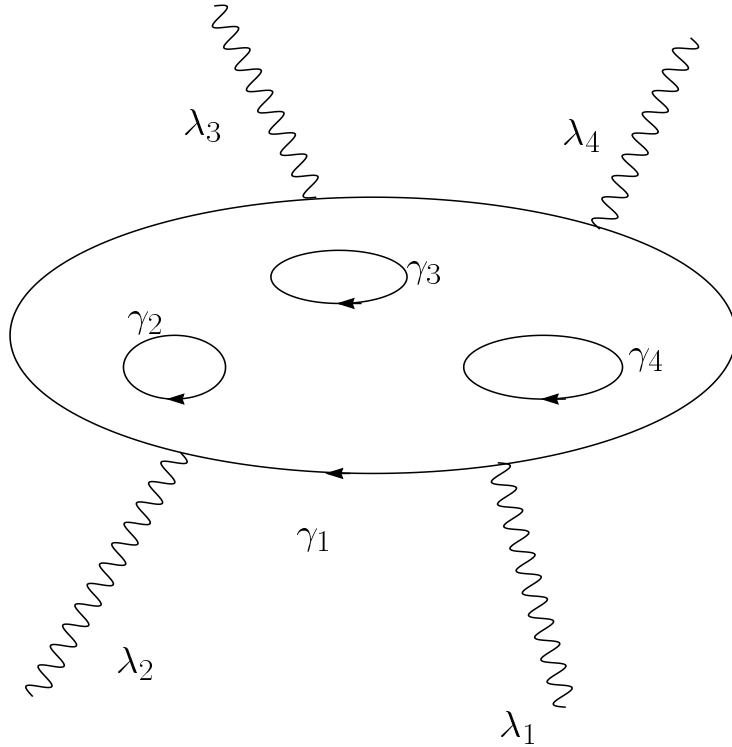


Fig.1. A planar diagram.

Here we need to specify the twists on the boundaries. A convenient choice (consistent with that made for the annulus amplitude (2.1)) is<sup>4</sup>

$$\gamma_{a_1} = \prod_{s=2}^b \gamma_{a_s} , \quad (3.1)$$

where  $b$  is the total number of boundaries,  $\gamma_{a_1}$  corresponds to the outer boundary, and  $\gamma_{a_s}$ ,  $s = 2, \dots, b$ , correspond to inner boundaries.

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<sup>4</sup> Here some care is needed in the cases where  $\Gamma$  is non-Abelian and we will have to choose base points on the world-sheet to define the twists. However the argument we give is unmodified also in this case.

Let  $\lambda_r$ ,  $r = 1 \dots M$ , be the Chan-Paton matrices corresponding to the external lines. Then the planar diagram with  $b$  boundaries has the following Chan-Paton group-theoretic dependence:

$$\sum \text{Tr}(\gamma_{a_1} \lambda_1 \dots \lambda_M) \prod_{s=2}^b \text{Tr}(\gamma_{a_s}) ,$$

where the sum involves all possible distributions of  $\gamma_{a_s}$  twists that satisfy the condition (3.1) as well as permutations of  $\lambda_r$  factors (note that the  $\lambda$ 's here are the states which are kept after projection and so they commute with the action of  $\gamma$ 's). The important point here is that unless all twists  $\gamma_{a_s}$  are trivial for  $s = 2, \dots, b$ , the above diagram vanishes by the virtue of (2.7). But then from (3.1) it follows that  $\gamma_{a_1}$  is the identity element as well. This is exactly what we wished to prove. We have thus established that to leading order in  $N$  all the amplitudes of the orbifolded theories agree with the corresponding ones for  $\mathcal{N} = 4$  case (with a simple rescaling of coupling), and in particular the  $\beta$ -functions and anomalous scaling dimensions all vanish.

### 3.3. Non-planar diagrams without handles

We will now consider the extension of the previous considerations to non-planar diagrams at  $g = 0$ , *i.e.*, diagrams obtained by attaching vertex operators to different boundaries.

Let us start with 2-point functions. Consider a non-planar diagram with an arbitrary number of boundaries and two external lines attached to two distinct boundaries. In order for the amplitude not to be zero we need the twist along the other boundaries to be trivial. So the only possibility is that if the two boundaries with external lines have the same twists (with opposite orientations for the boundaries). Thus the Chan-Paton group-theoretic dependence of this diagram is given by

$$\sum_a \text{Tr}(\gamma_a \lambda_1) \text{Tr}(\gamma_a \lambda_2) .$$

If  $\lambda_1$  or  $\lambda_2$  correspond to charged fields (such as *non-Abelian* gauge bosons), then this expression vanishes. Indeed, for the adjoint of  $SU(N_i)$  we have  $\text{Tr}(\lambda_{1,2}) = 0$ . Note that this is not the case for the neutral fields, such as  $U(1)$  gauge bosons and neutral matter fields (if any). This is indeed the source of the running of

the coupling constants for  $U(1)$ 's (and potential source for anomalous scaling dimensions for neutral fields). Thus, we see that non-planar diagrams without handles do not contribute to the *non-Abelian* gauge coupling renormalization and anomalous scaling dimensions of charged matter fields to all loops. Even though these diagrams are only a subset of subleading corrections at large  $N$ , the fact that they vanish for all theories under consideration (including the non-supersymmetric ones) is a hint that they may actually be finite even for finite  $N$ . This issue will be discussed further below.

We can also extend the above argument to three point functions. In particular the Yukawa couplings are the same as those of the corresponding  $\mathcal{N} = 4$  theory. But if we consider higher than three point functions at genus 0 and put them at various boundaries, there is no argument why the twisted boundary conditions would not contribute. This thus shows that not all the correlation functions are going to be the same as the corresponding  $\mathcal{N} = 4$  theory in the subleading corrections. In particular, the quartic self-interaction of bosons for the  $\mathcal{N} = 0$  case may *a priori* be different from that of the  $\mathcal{N} = 4$  case, by attaching two pairs of bosons to two boundaries, for the non-planar diagrams in subleading order in  $N$ . The Chan-Paton group-theoretic dependence of this diagram is given by

$$\sum \text{Tr}(\gamma_a \lambda_1 \lambda_2) \text{Tr}(\gamma_a \lambda_3 \lambda_4) .$$

Note that this is the case where by factorization property of string theory it can be represented as two disc diagrams connected by a long thin tube corresponding to a closed string exchange.

We can *a priori* expect twisted closed string states, including tachyons, propagating in the tube to contribute. The tachyons, at least naively, may lead to infrared divergences. (In section 4 we discuss some aspects of this.) The twisted massless states may also contribute to the infrared divergence. Note that the contributions corresponding to  $a = 1$  still look like the  $\mathcal{N} = 4$  correlators and are therefore finite. As for the twisted sector contributions, it is unclear whether they can be finite in the  $\alpha' \rightarrow 0$  limit.

#### 4. Subleading corrections at large $N$ : diagrams with handles

So far we have talked about leading order diagrams at large  $N$  as well as some subleading contributions depending on where we insert the external states to the planar diagram. We have shown that the beta functions for all these cases vanish, with the possible exception of quartic scalar coupling in the  $\mathcal{N} = 0$  theories in the subleading order in  $N$ . Here we wish to see what can be said by including other subleading corrections in  $N$  which come from including handles.

Note that now we can have also twists around the handles which make contributions to amplitudes which are different from the  $\mathcal{N} = 4$  case (which would correspond to the case without handles). Thus, *a priori* all amplitudes may have  $1/N$  corrections which are distinct from those of the corresponding  $\mathcal{N} = 4$  theory.

Let us however ask if the  $\beta$ -functions will be zero or not, *i.e.*, let us see if the theories under consideration will be conformal at subleading order in  $N$ , when we include handles.

Here we note that renormalization in effective field theory is due to string theory infrared divergences corresponding to massless modes [17]. Thus, if a given diagram does not contain infrared divergences it will not contribute into the field-theoretic renormalization of the corresponding correlator. If the diagram is non-zero, however, there is a *finite* (and *independent* of the energy scale) renormalization. Such renormalizations might change the value of the fixed point couplings, but not the fact that we have vanishing beta functions.

The case where  $\Gamma \subset SU(2)$ , which gives rise to  $\mathcal{N} = 2$ , is finite, because there are no higher loop correction to  $\beta$ -functions in these theories. We are thus primarily interested in the  $\mathcal{N} = 0, 1$  cases. Let us consider the  $\mathcal{N} = 1$  case first. We have already argued that all the  $\beta$ -functions vanish for  $g = 0$  diagrams with arbitrary boundaries. To consider what happens by including handles we consider a disc with arbitrary number of holes and with  $g$  handles attached to it. Let us consider two point functions (which in the  $\mathcal{N} = 1$  case is sufficient for checking perturbative finiteness). In this case both vertex operators are attached to the same boundary (otherwise the trace would vanish). We are looking at potential sources of infrared divergences in such amplitudes. Since

we have already argued that the disc with arbitrary boundaries by itself does not have divergent contributions, any potential divergences may come from the infrared divergences in the integration over the handle moduli, or from corners of moduli where handles approach boundaries of the disc.

The integration over the handle moduli will not have any infrared divergences, as that would correspond to massless tadpoles for closed type IIB string theory in a background with  $\mathcal{N} = 1$  supersymmetry. Thus the only potential source of divergence is when handles approach boundary points.

We will now present evidence why these may also be zero (modulo subtleties having to do with neutral fields discussed below). Let us first consider the case of a disc with a single handle attached (Fig. 2) and consider contributions to the  $\beta$ -function for gauge theory (say, from two point function of gauge fields).

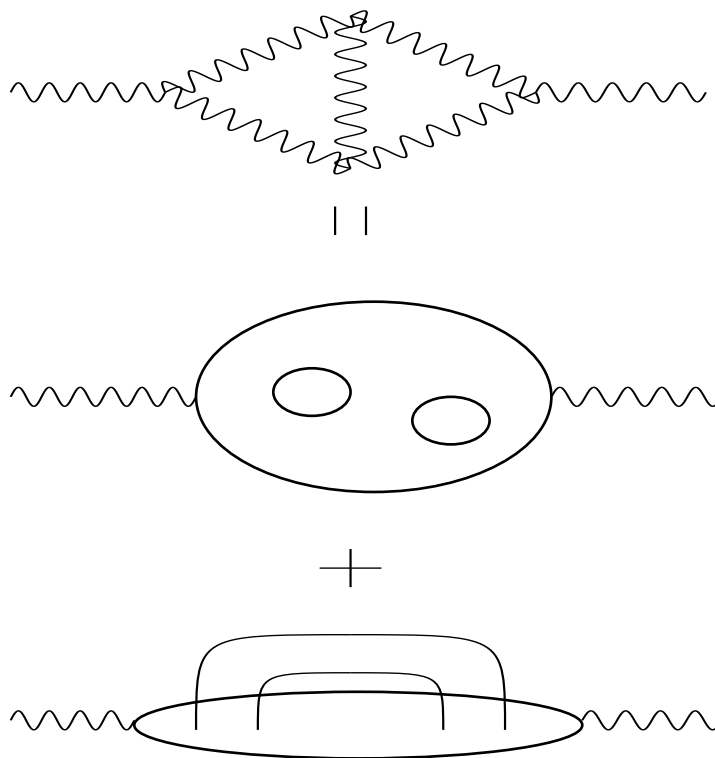


Fig.2. Two-loop order in large  $N$  limit.

We consider two points on the boundary of the disc, corresponding to insertion of vertex operators of the gauge fields. In this case the only potential



source of divergence arises when the handle approaches the boundary (at the insertion points or otherwise). However we will now show there is no such divergence. It was shown in [12] that in the  $\mathcal{N} = 1$  case the  $\beta$ -functions vanish to two loops for any  $N$ . When translating this to large  $N$  order we have two contributions, one from a  $g = 0$  diagram with three boundaries, and the other with a disc and one handle attached. Thus the sum of these diagrams will give zero contribution to  $\beta$ -functions. On the other hand, we have already shown that the one corresponding to the  $g = 0$  diagram does not contribute to the  $\beta$ -functions. This shows, therefore, that the disc with one handle attached will also not contribute to the  $\beta$ -functions.

If we have more handles, from what we have said, it is essentially clear that each individual handle approaching boundaries will not give any contributions to the  $\beta$ -function. The question of what happens when some number of them approach each other and the boundary is not completely clear. However it is likely that even in this case, using factorization properties of string amplitudes and the structure of boundary of moduli spaces one may be able to prove that they are zero. However, we have to note<sup>5</sup> that in these computations we are still *including* all the neutral fields, and in particular the  $U(1)$ 's. However as already noted the  $U(1)$  couplings do run and the scaling dimensions of neutral fields are non-zero even in the limit where we ignore handles (but subleading in  $N$ ). What this means is that if we drop all the neutral fields from the spectrum of field theory, in order to get a finite theory, we will get some shifts of the order of  $1/N^2$  in the values of the coupling constants at the fixed points inherited from the parent  $N = 4$  theory.

For the  $\mathcal{N} = 0$  case the argument is less clear. Here we do have infrared divergences from the closed string sector by itself. In particular there is no reason why one point amplitude of the dilaton tadpole is zero, and that itself will be a potential source of contributions to  $\beta$ -functions. The tachyon is another potential source for infinities. Nonetheless, one may formally expect them to be irrelevant for the gauge theory discussion, because we are taking the limit  $\alpha' \rightarrow 0$  and so the square of tachyon mass is  $-\infty$ . One would expect to formally subtract them off in gauge theory discussions, along with massive modes of

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<sup>5</sup> We would like to thank Andrei Johansen for pointing this out to us.

strings (perhaps a way to make this precise is to consider  $\alpha' \rightarrow i\epsilon$ ). Clearly more work remains to be done in this direction to settle the exact finiteness of the  $\mathcal{N} = 0$  theories.

This work is supported by NSF grant PHY-92-18167. The research of M.B. is supported in addition by NSF 1994 NYI award and the DOE 1994 OJI award. The work of Z.K. was supported in part by the grant NSF PHY-96-02074, and the DOE 1994 OJI award. Z.K. would also like to thank Albert and Ribena Yu for financial support.

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