Utilizing Conceptual Anchors in Mathematics to Promote Enduring Conceptual Knowledge and Procedural Skills

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Accessibility
Utilizing Conceptual Anchors in Mathematics to Promote Enduring Conceptual Knowledge and
Procedural Skills

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A Thesis in the Field of Mathematics for Teaching
for the Degree of Master of Liberal Arts in Extension Studies

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Abstract

Conceptual understanding and procedural knowledge in mathematics are at the core of what enables high school mathematics student to transfer successfully learning from one mathematical content area to another. While procedural knowledge, at its base level, can be memorized in a rote, mechanical manner, conceptual understanding requires an ability to connect a mathematical concept to multiple representations (e.g. algebraic, geometric, numerical). Mathematics educators aim to develop lasting mathematical understanding in their students so as to minimize misconceptions and prepare students for future studies. This thesis evaluated whether a student exploration could serve as a conceptual anchor designed to promote long-term understanding and enduring procedural mastery. Limited data from the research did not demonstrate benefits of the student exploration that served as a conceptual anchor. There is a possibility that the students who practiced rote procedures retained facility with those procedures better than the students who spent time with a mathematical exploration. However, there is no conclusive evidence that the students who demonstrated an enduring procedural mastery had a better conceptual understanding. Additionally, the students enrolled in honors-level courses tended to outperform the students enrolled in non-honors-level courses on questions pertaining to both procedural skills and conceptual understanding, but they did not retain those skills or that understanding any better than the students enrolled in non-honors-level courses. More research needs to be conducted to explore further the
question of how to promote enduring mathematical understanding in high school students.
Author’s Biographical Sketch

Josh Berberian graduated with a bachelor’s degree in mathematics from Brown University. He does not lament having Paul Lockhart as his professor for his mathematical analysis course, as Paul was a great teacher. While studying mathematics, Berberian stumbled upon taking an education course with Ted Sizer, which led to spending a semester taking courses at Bank Street College and student teaching at Deborah Meier’s Central Park East Secondary School. This is how, when, and where his interest in teaching and learning started.

Berberian has been teaching mathematics for one fortieth of a millennium. He began his career teaching middle school mathematics at Saint Thomas Choir School, under the mentorship of international guru of the student-centered classroom, Gordon Clem. After a brief stint at Poly Prep Country Day School, he landed at The Shipley School, where he has been for the past twenty-one years. While at Shipley, Berberian’s educational leadership has been shaped and cultivated by veteran teachers and administrators like Rob Robinson, Tom Nammack, Margaret van Steenwyk, and Steve Piltch. Berberian has led the Mathematics Department for over ten years, and he has been fortunate to surround himself with colleagues who are bright, supportive, committed, funny, and genuinely nice people. In addition to teaching mathematics, Berberian co-leads the mentoring program and was a founding leader of the faculty action research group that works to improve student equity and access.
Dedication

To everyone from whom I have learned—my teachers, my students, my children  
(Jake, Kat, and Elizabeth), and my parents.
Acknowledgments

First and foremost, I would like to thank my thesis director, Judah Schwartz, for his insight and support. His ability to pose mathematics problems that goad one’s curiosity is extraordinary. His failed retirement has been my great fortune as a graduate student.

I also want to thank my first Harvard Extension School (HES) teacher, Eric Connally, the author of the precalculus textbook that I use in class and someone who has put up with my questions and comments about pedagogy since my first day of class with him. Jeff Tecosky-Feldman is a neighbor, mathematics educator, and calculus textbook author who likewise deserves thanks for answering countless questions of mine over the past couple of decades. Of course, I have had many great teachers along the way at HES, including Brendan, Jameel, Kate, and Robbie.

I would be remiss if I did not express my deep gratitude to Andy Engelward, who directs a program that was the perfect educational journey for me, striking just the right balance of mathematics and education.

None of this experience would have been possible without the moral and financial support of Shipley’s Head of School, Steve Piltch. He has supported my growth as an educator for over two decades, and for that I will always be grateful. I am very appreciative of the student participants in this research study who took time out of their busy days to do math with me (just for fun!), as well.
Lastly, on the home front, I want to thank my amazing wife, Michelle. She happily and graciously kept everything running smoothly whenever I had to do homework, attend class (on campus and remotely), or write my thesis. She also made sure that I did not split any infinitives.
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Chapter I
Introduction

Secondary school mathematics educators strive to optimize classroom learning time so that they may incorporate higher-order tasks involving complex problem-solving. In order to do this, students must have a strong foundation with both their procedural knowledge and conceptual understanding of mathematics. The International Association for the Evaluation of Educational Achievement’s (IEA) Trends in International Mathematics and Science Study (TIMSS) discerns these cognitive domains as “knowing”, which includes recalling and computing, and “reasoning”, which includes analyzing and synthesizing (Martin & Mullis, 2014). Students who know how to do mathematics and who can reason mathematically are poised to find success in approaching challenging problems in their zone of proximal development. An excessive amount of scaffolding might be required for students lacking either the requisite skills or conceptual understanding.

Single classroom lessons with instruction focused on conceptual understanding have been found to create a greater retention of procedural and conceptual knowledge (Rittle-Johnson, Fyfe, & Loeher, 2016). This seems to indicate that a strong foundation of conceptual understanding is one key to helping students with “knowing” and “reasoning” in the context of mathematics. Yet, the Program of International Assessment (PISA) has documented a decline in 15-year-olds’ mathematical literacy in the United States since 2003, with the average score dropping 1.3% from 2003 to 2015 and the ninetieth percentile score dropping 2.2% (National Center for Educational Statistics, 2015). How,
then, can mathematics educators leverage contemporary research to help students retain
skills and understanding to improve achievement?

A Classroom Vignette

The following daily warm-up is posted on the board in a high school Precalculus
class composed of eleventh and twelfth grade students:

Calculate the distance between the points (2, 7) and (5, 3).

After five minutes there is clearly confusion among the students. Student
comments such as, “Do we add the coordinates and divide by two?’” “Isn’t there
something with a square root?” and “I have no idea what to do,” are overheard by the
teacher who is pacing anxiously around the room. The teacher is contemplating why this
review exercise, which was intended to take about a minute, is now taking ten minutes
without any of the students making forward progress. Surely, the students were taught the
distance formula several times in previous years. It is a topic that is taught at the middle
school level and again in Geometry. Why aren’t they remembering this? Why don’t some
mathematical concepts stick? It is time for a hint?

The teacher says to one group of three students, “You might consider drawing a
picture.”

Within seconds, without even taking the time to draw a picture, a student
exclaims, “Oh! This is the Pythagorean Theorem.” The student never drew a picture, but
it is clear that visualizing the problem provided him with the conceptual anchor to recall
the procedure required to solve the computational problem. With a picture, the distance
formula is reduced from a complicated equation to a more accessible geometric concept.
It is precisely this notion that conceptual anchors can promote enduring mathematical
understanding and retention of mathematical procedures—the “reasoning” and “knowing”—that is the motivating force for the research outlined in this thesis. If students can leverage conceptual anchors to retain mathematical understanding and procedures, then mathematics educators can utilize classroom time to pursue curriculum that enriches the learning experience instead of remediating prior understanding. Scaffolding students to solve interesting, non-routine problems is meaningful and engaging from both a teaching and learning perspective, whereas re-teaching a template approach to solving routine problems risks being boring and reducing mathematics to something that a computer can do.

Thesis Background

It has been widely accepted that procedural knowledge of mathematics is built on a strong conceptual understanding (National Council of Teachers of Mathematics, 2014). Given this perspective, one might consider starting all mathematical learning with a conceptual foundation on which to build procedural skills. Within the past decade, there has been a growing body of research in support of the iterative nature of procedural and conceptual knowledge (Rittle-Johnson, Schneider, & Star, 2015). Student improvement in one area of understanding (conceptual or procedural) can lead to improvement in the other. Indeed, the understanding of mathematical concepts and the skill to carry out mathematical procedures are considered to be integral components of mathematical proficiency (Mathematics Learning Study Committee, 2001). Not only can one improve mathematics skills and understanding through an iterative process of learning, but those skills and understandings develop at a deeper level (Star & Stylianides, 2013).
Given the symbiotic relationship between conceptual and procedural knowledge, this thesis aims to explore the impact of establishing conceptual anchors to promote enduring conceptual understanding and lasting procedural skills with algebraic topics. For the purpose of this research, the conceptual anchor for a specific topic was in the form of a dynamic online environment where students could explore the connection between multiple mathematical representations of a topic. More specifically, the dynamic investigation utilized was a web-based applet created using the software Geogebra. The applet was designed to allow students to engage in a self-guided investigation of multiple representation of mathematics with the ultimate goal of cultivating a conceptual understanding of the particular topic (see Appendix 1 for applet screenshots and description). The hope was that a self-realized conceptual understanding would lead to deeper, more meaningful, and more memorable learning than a teacher-directed lesson on its own.

The “math wars” started in the early 1990s, after the National Council of Teachers of Mathematics (1989) encouraged mathematics educators to stress deeper conceptual understanding in the classroom. This led many mathematics educators to reform the manner in which they had previously taught. Coupled with a primary focus on understanding mathematical connections and problem solving, these reform efforts de-emphasized procedures and algorithms. Supporters of the reform saw this as a way for students to construct a more indelible understanding of how mathematics works, as opposed to superficial memorization of rote procedures. The underlying theory being that if students understood “why” mathematics works, then they would have a context for knowing “how” mathematics works. For example, one might use algebra tiles to see how
terms in two binomials are distributed when multiplied, as opposed to memorizing FOIL to recall multiplying the terms by means of firsts, outers, inners, and lasts. Critics saw these reform efforts as eroding students’ abilities to carry out basic procedures and computations. Two-digit multiplication problems became onerous tasks with convoluted procedures. Opponents of the reform believed that deeper understanding came at the cost of automaticity of skills (Schoenfeld, 2004).

The Calculus Consortium based at Harvard (CCH) published the first edition of their reform calculus textbook in 1994 which, according to co-author Tecosky-Feldman, was designed to “rethink how the course might be rejuvenated by giving conceptual, graphical, and numerical applications equal weight with algebraic competence” (personal communication, July 14, 2018). Preliminary feedback from early adopters of the textbook asked for more problems to help students build procedural skills, as they felt there was not enough drill to help students develop their automaticity. These types of problems were eventually integrated into subsequent editions of the book. Tecosky-Feldman went on to say,

Students need to scaffold the conceptual learning by doing simple examples and gaining confidence and skill, before tackling tricky or unusual problems. Each edition of the text has added more skill problems, as well as conceptual true-false problems to help build understanding (personal communication, July 14, 2018).

Seemingly, a less dichotomous perspective of classroom learning that focused on a balance of procedural and conceptual knowledge emerged at the start of the twenty-first century. Research within the past two decades supports the notion that conceptual and procedural knowledge have a symbiotic relationship, whereby understanding in one area leads to better understanding in the other (Rittle-Johnson, Siegler, & Alibali, 2001; Rittle-Johnson, Schneider, & Star, 2015). In a study with sixth-grade students on
decimals, improved achievement was attained not only by iterating lessons between conceptual and procedural approaches, but also by interleaving the concepts and skills over time (Rittle-Johnson & Koedinger, 2009). Interleaving is the process of alternating learning between conceptual understanding and procedural skills in a manner such that they build off of each other. This is in contrast to blocking, whereby learning is done in discrete chunks. Therefore, it is important that mathematics educators optimize learning by determining how conceptual and procedural knowledge is best iterated and interleaved.

An important objective in lesson planning for secondary school mathematics teachers is to lay the groundwork for enduring understanding by anchoring procedural skills to a strong conceptual framework for students. This can be accomplished through well-planned direct instruction. More powerful, though, might be the use of a student exploration designed to scaffold the student’s construction or reconstruction of the conceptual underpinning (Fast, 1997). A student exploration should be designed with the goal of connecting multiple representations in order to establish a conceptual framework. For example, a linear function can be viewed algebraically as an equation, graphically as a line in the Cartesian plane, or numerically as a table of sample values. At the core of developing a deep conceptual understanding is the ability to recognize multiple representations of the same mathematical topic (e.g. algebraic, geometric, and numerical) (National Research Council, 2001). Combined with direct instruction, a student exploration can serve as a conceptual anchor which might encourage a longer-lasting retention, much like the situation described in The Classroom Vignette. The conceptual anchor is a recollection of memorable and meaningful mathematical discovery.
This thesis was designed to analyze the data collected from a research study on whether students could leverage conceptual anchors to promote enduring mathematical understanding, including both conceptual knowledge and procedural skills. The hope was to demonstrate that providing students with conceptual anchors will help improve their long-term understanding and procedural competence.

Thesis Overview

Researching methods of teaching that might serve to help students retain mathematical skills and understanding will ultimately improve student learning and, perhaps, attitudes about, engagement with, and persistence in mathematical problem-solving. This introductory chapter serves as a framework to contextualize the proposed research within recent trends in mathematics education. While there has been debate over best practices in the past, there seems to be widely accepted research that illustrates the benefit of iterated practice between learning that involves conceptual and procedural knowledge.

The next chapter will further delineate current studies on the topic of conceptual and procedural knowledge, explaining why student investigations might lead provide for a strong conceptual foundation of mathematical topics on which to build skills. A description of the proposed research methods is outlined, including the creation and use of a specific conceptual anchor, the distinction between the experimental and control group, and the assessment measures utilized to analyze the raw data. Subsequently, the results will be analyzed, comparing the effect of using a conceptual anchor on long-term student retention of skills and understanding.
Finally, the results will be discussed in the context of prominent research, indicating where the research in this thesis seems to be incongruent and other instances where the research is supported by the work of others. Research limitations that include a small sample size and confounding variables make it clear that future research is necessary to explore the use and impact of conceptual anchors more fully. A summary will provoke the reader to reconsider the role of student practice to gain deep and enduring conceptual understanding, as well as lasting procedural knowledge.
Chapter II
Research

The research for this thesis was designed to assess the utility of a conceptual anchor in promoting retention of conceptual understanding and procedural knowledge of polar coordinates. There is strong evidence that conceptual knowledge is a prerequisite on which to build procedural knowledge (Rittle-Johnson, Schneider, & Star, 2015). Laying a conceptual foundation enables students to understand why mathematics works the way it does which, in turn, promotes a better comprehension of procedures and algorithms. Could it be that a self-guided conceptual investigation will help students remember mathematical concepts and procedures over time?

In this study, the experimental group was given an online applet to investigate the relationship between Cartesian and polar coordinates prior to a teacher-directed lesson, whereas the control group was given the same teacher-directed lesson with additional practice problems to account for time on task across both groups. Students in both groups were then assessed on some questions pertaining to skills and other questions pertaining to understanding. The study occurred longitudinally, with assessments given one day, one week, and four weeks after the initial lesson.

Project Description

Procedural knowledge and conceptual understanding are at the heart of synthesizing and applying mathematical concepts. The definitions of both procedural and
conceptual knowledge have evolved over time and there is differing opinion on exactly what each measures, as well as how to measure it. Some contend that the skills and procedures of “how” mathematics works is purely superficial, and conceptual understanding of “why” mathematics work, by default, implies a deep comprehension of the topic. Others argue that skills and understanding knowledge can both be realized at superficial and deep levels (Baroody, Feil, & Johnson, 2007). For the sake of this thesis, procedural knowledge will refer to the “how” mathematics works and conceptual knowledge will refer to the “why.” More specifically, procedural knowledge includes the skills and algorithms that students use to solve routine mathematics problems, whereas conceptual knowledge is the foundational understanding of the ways in which mathematics works. Possessing conceptual knowledge helps one understand the structure of mathematical problems and can develop one’s capacity to make connections and explore more challenging mathematical concepts.

One construct is to consider procedural knowledge and conceptual understanding as being on a continuum with the habits of mind that go along with each developing independently in an iterative process (Rittle-Johnson, Siegler, & Alibali, 2001). Another, perhaps more relational, way of viewing the connection between procedural and conceptual knowledge is to consider a positive correlation, such as that presented in Figure 1 (Star, 2005; Baroody, Feil, & Johnson, 2007). There is not a dichotomy between procedural knowledge and conceptual understanding; instead, procedural knowledge and conceptual understanding help to inform and strengthen each other. In reference to Figure 1, it is optimal when a student possesses both procedural knowledge and conceptual understanding (colored green). When students possess a strong conceptual understanding
and procedural knowledge, they are poised to retain that knowledge and apply it to higher order problem-solving. Student who lack both (colored red), however, are at risk for limiting their opportunities for using and valuing mathematics. With limited conceptual understanding, procedures become rote and easily forgotten. With limited procedural knowledge, students may not have developed the necessary procedural fluency to choose efficient and effective problem-solving strategies (National Research Council, 2001).

Figure 1. Graph of Procedural Knowledge vs. Conceptual Understanding

*Adapted from Star (2005) and Baroody, Feil, & Johnson (2007)*

It seems plausible, then, that strengthening students’ procedural knowledge is a key component to developing their conceptual understanding and vice versa. It is widely accepted by many non-Western countries that student practice with a “sea of problems”
to bolster procedural skills leads to better conceptual understanding (Rittle-Johnson, Schneider, & Star, 2015). In fact, there might be some mathematical topics whereby starting with a strong procedural foundation is more beneficial than starting with learning the conceptual underpinnings. With a de-emphasis on algorithms and rote memorization in secondary mathematics education over the past two decades, it may be time to consider a more balanced approach that includes practicing procedural knowledge. In the context of a remedial college algebra course, students found educational value in practice (Stillson & Nag, 2009). To that end, fluency and flexibility are important aspects of procedural knowledge that can be practiced (Star, 2002).

Given that providing students with opportunities to practice procedural skills has the potential to impact conceptual knowledge and mathematical understanding, research suggests that online programs that offer practice, such as WeBWork, can be a useful tool in providing the necessary repetition. A study by Stillson and Alsup (2001) demonstrated that students were able to improve their mathematics understanding by including an online homework system to practice skills as part of their course of study. Boylan’s (2002) research suggests that the frequent and immediate feedback is the primary benefit of online practice, in addition to the ability to re-attempt problems until the skills are mastered. A mastery of skills can then lead to an improved conceptual understanding. In this sense, it is unclear as to whether one must start with a prerequisite conceptual understanding or a strong procedural knowledge to optimize success in mathematics. Again, different topics might benefit from careful consideration of the order in which procedures and concepts are taught.

Without conceptual understanding, though, procedural practice can be a
frustrating student exercise. Misconceptions can lead to procedural errors, and repeated practice with errors can serve to reinforce misconceptions. While some research indicates that computational errors are not necessarily the result of a lack of conceptual understanding (Byrnes & Wasik, 1991), there is other evidence that a strong conceptual understanding leads to greater computational and procedural proficiency (Mathematics Learning Study Committee, 2001). When students deeply understand mathematics, they are less prone to errors of magnitude and more apt to remember procedures because they have meaning. The use of a conceptual anchor can provide students an opportunity to refresh their conceptual understanding of a topic prior to procedural practice. If successful, this bolstering of conceptual understanding may lead to more productive reinforcement of the procedural skills.

It follows, then, that a thoughtfully created online applet designed for student investigation can serve as a vehicle for creating a conceptual anchor. The applet can provide students with a means for anchoring their understanding, in much the same way that the student in the introductory vignette was reminded of how the Pythagorean Theorem related to the distance formula. In that instance, a mere suggestion of geometric representation of the distance formula led to an immediate recollection of the formula, thus an image of the Pythagorean Theorem served as the conceptual anchor. Insomuch as an online applet has the potential to allow students to see mathematical connections as much and as often as they need, the online applet environment can serve as a simplified intellectual mirror (Schwartz, 1989). An intellectual mirror allows learners to elicit immediate feedback based on their actions within a specific exploratory environment in a
private, non-threatening, and non-judgmental manner. Creating and assessing the merits of such an environment will be the goal of this research.

Ultimately, this research aims to determine whether providing a vehicle for students to develop a conceptual anchor can promote enduring understanding and long-term skills retention of algebraic topics compared to strictly skills practice. If a well-crafted student exploration can serve as a meaningful and memorable conceptual anchor, then students will be able to retain and apply mathematical understanding better. This, in turn, can lead to learning more concepts, solving more complex problems, bolstering confidence, and cultivating a joy and appreciation for mathematics.

Research Methods

The broad research goals of this thesis intended to assess the ability of students to perform tasks that reflect two primary aspects of mathematical understanding, namely procedural skills and conceptual understanding. To differentiate between these two types of tasks, a framework was adopted using the Assessing Mathematical Understanding and Skill Effectively (AMUSE) schema. In summary, a skills performance task can be completed using a single mathematical approach (e.g. algebraic or geometric), whereas an understanding performance task necessarily requires the student to use and connect at least two mathematical approaches (e.g. algebraic and geometric) (Schwartz, n.d.). For instance, being able to relate a linear equation to a table of values or a graph of a line in the Cartesian place would indicate a basic understanding of mathematics. On the other hand, computing values of $x$ based on the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for given values of
$a, b, \text{ and } c$ without recognizing that the solutions represent the roots of a quadratic function that are symmetric with respect to a vertex in an example of a skill.

For the sake of this research, a lesson was designed to teach students how to convert between polar coordinates and Cartesian coordinates (see Appendix 2 for the detailed lesson plan). The topic was picked because it represents content which is relatively novel for most high school mathematics students. In this manner, there was less of a chance that student participants would be advantaged in completing the assessment tasks because of prior and repeated exposure to the material.

Student volunteers were solicited by means of email and a brief informational session at the start of their mathematics class. Great care was taken to ensure that students were informed of the completely voluntary nature of the research and that no preferential treatment would be afforded or withheld based on their decision to participate or not. Parents and guardians were also informed and given the opportunity to opt out. Upon volunteering for the study, students were provided with detailed information about the nature of the research and the time commitment. They were issued assent forms and reminded throughout the study that they had the option to stop participating without any recourse. Students were scheduled to participate during an optional school activity period as well as during a designated extra help time. Both of these times took place during the school day, and students were generous in forfeiting what might be considered free time in order to participate in the study.

An initial group of twenty-one students volunteered to participate in this research. The students who volunteered required a minimal completion of high school geometry in order to undertake the research assessment tasks most successfully. In order to convert
between polar coordinates and Cartesian coordinates it is beneficial to understand basic trigonometric ratios in right triangles, a topic that is covered in high school geometry. The student participants were high school students from the same suburban independent school in grades 9 through 12 who had completed Geometry. The student participants were enrolled in courses ranging from Algebra II through BC Calculus, including both standard-level and honors-level courses. Student participants who were enrolled in Precalculus and Calculus courses likely had been exposed to polar coordinates, albeit briefly, in their regular coursework; therefore, it was important to distribute these students evenly in the control group and experimental group so that the data would not be biased. Ideally, there would have been enough student participants available in a single course, such as Algebra II, so as to minimize the effect of previous exposure of the material on the data being collected.

The students were divided into two groups (A and B) using a matched random group assignment method whereby students enrolled in the same course and same level (standard or honors) were randomly assigned to a group (A or B) until the spots for the group were filled and then the remaining students were assigned to fill the remaining group (Becker, 2000). This type of group selection was used to help to eliminate any bias due to the course in which the student was enrolled. Due to schedule conflicts for some of the students, the group assignments had to be modified. The matching method was honored, although the assignment to Group A or Group B was not random for all students due to the considerations of accommodating specific students’ schedules.

Student participants in Group A started with an exploration using an online applet created using Geogebra (see Appendix 1 for applet screenshots and description). The
applet was designed for students to investigate the relationship between the polar and Cartesian coordinate systems, and it was accompanied by the reflection prompts in Figure 2 to help students gauge their own level of understanding. The applet, coupled with the reflection prompts, was designed to serve as an intellectual mirror.

1. How is $x$ related to $r$ and $\theta$? How is $y$ related to $r$ and $\theta$?

2. How is $r$ related to $x$ and $y$? How $\theta$ is related to $x$ and $y$?

3. Consider point A that has Cartesian coordinates $(x, y)$ and polar coordinates $(r, \theta)$. Name a point such that $x$, $y$, and $r$ are all integer values.

4. Consider point B that has Cartesian coordinates $(x, y)$ and polar coordinates $(r, \theta)$. Make an observation about $r$ and $\theta$ when $x$ and $y$ are equal.

Figure 2. Applet Reflection Prompts

*The reflection prompts accompanied the use of the applet to help guide student investigation in meaningful ways and help students self-assess their understanding of the exploration.*

Prompts 1 and 2 were designed to help students discover the relationships that underpin the conversion between polar and Cartesian coordinates. Prompts 3 and 4 were designed to encourage students to investigate and observe interesting connections between polar and Cartesian coordinates. This experience, namely the applet coupled with the learning prompts, served as the conceptual anchor for this mathematical topic.

Student participants in Group A were then provided with a direct instruction group lesson by the researcher. The lesson was geared toward connecting multiple mathematical representations (geometric and algebraic) of the topic.
In addition to the learning reflections prompts, other questions were asked at the conclusion of the lesson, as shown in Figure 3. The questions were asked as a means for students to reflect upon and summarize what learning they had done and to open that door to explore further learning opportunities if there was time.

1. What prior mathematics did you use in learning the material today?
2. Summarize what you learned in today’s lesson.
3. About what might you be interested in learning more?

Figure 3. Lesson Reflection Prompts

*The lesson reflection prompts accompanied the use of the applet to help the researcher gain a preliminary understanding as to the breadth and depth of the students’ comprehension of the lesson.*

Student participants in Group B, the control group, started with a direct instruction group lesson by the researcher. The lesson was geared toward connecting multiple mathematical representations of the topic. In this case, the connection was between two different geometrical representations (Cartesian and polar) and algebraic representations \((x^2 + y^2 = r^2, x = r \cdot \cos \theta, y = r \cdot \sin \theta, \text{ and } \arctan(\theta) = y/x)\). After the lesson, the group worked through several additional problems together and then the students were asked to complete the learning reflection prompts and summary questions that were answered by the students in Group A. The additional problems were designed to provide an equal amount of time on task for both Group A and Group B. The total time spent on these initial learning tasks was approximately forty minutes for both groups.
One day after the initial group lesson, students were asked to complete an assessment to evaluate their understanding and retention of the previous day’s lesson (see Appendix 3 for collection of assessment tasks). The assessment included four questions pertaining to skills (“how” mathematics works) and three questions pertaining to understanding (“why” mathematics works). The skills questions were written in a way that they could be addressed using a single mathematical perspective (in this case, either algebraic or geometric), whereas the questions for understanding were designed so that students would have to make connections between at least two different mathematical representations. Student participants were allotted twenty-five minutes to complete the assessment tasks, with most participants completing the tasks within fifteen minutes.

A similar reassessment of both groups occurred again one week and four weeks following the initial group lesson. Reassessment over a span of time including one day, one week, and four weeks was implemented to determine whether a conceptual anchor could serve to promote retention of conceptual understanding and procedural skills over a span of time.

Assessment tasks were scored using an adaptation of the rubric for scoring student performance from the Balanced Assessment for the Mathematics Curriculum team based at the Harvard Graduate School of Education (Harvard Group, 1995). Student responses for each assessment task (skills and understanding) were assigned a numeric grade ranging from 0 to 3, with each score corresponding to a demonstrated level of understanding outlined in Table 1.

In addition to a numeric score, the student responses were color coded in reference to how the students communicated their mathematical response, as shown in
Table 2. The coding does not necessarily reflect how the students understand the mathematical skills or concepts, but rather how they chose to communicate their ideas. One could certainly make an assumption that this choice of how to communicate is related to how the student understands the skills and concepts.

Table 1. Scoring Rubric.

<table>
<thead>
<tr>
<th>Score</th>
<th>Demonstrated level of understanding</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>student shows little evidence of skill or understanding</td>
</tr>
<tr>
<td>1</td>
<td>student shows a fragile skill or understanding</td>
</tr>
<tr>
<td>2</td>
<td>student shows an adequate level of skill or understanding</td>
</tr>
<tr>
<td>3</td>
<td>student shows a deep and robust level of skill or understanding</td>
</tr>
</tbody>
</table>

*Rubric for scoring the level of demonstrated understanding of student response for each assessment task (skills and understanding). Adapted from Harvard Group (1995).*

Through attrition, only twelve students completed all three learning assessments. The small sample size severely limited this research and applicability of the methods of statistical analysis of the data. Originally, it was hoped that a greater number of students would participate, thus allowing for analyses including a statistical t-test to compare the mean scores of the two populations. A t-test would have provided a more quantitative method for evaluating the hypothesis of this thesis. Since only twelve students completed all three learning assessment tasks, a simple comparison of the difference of mean scores for each aggregated group (Group A and Group B) over each time span (one day to one week and one day to four weeks) was used.
Although the limited data precluded the use of some statistical analysis, the results did exhibit differences in mean scores for specific groups and sub-groups of the study. The following chapter will detail those results, after which conclusion will be drawn and placed in the context of contemporary research.

Table 2. Color Coding.

<table>
<thead>
<tr>
<th>Color</th>
<th>Type of reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yellow</td>
<td>Algebraic</td>
</tr>
<tr>
<td>Blue</td>
<td>Geometric</td>
</tr>
<tr>
<td>Green</td>
<td>Algebraic and geometric</td>
</tr>
<tr>
<td>White</td>
<td>Answer with no reasoning</td>
</tr>
</tbody>
</table>

*Color coding rubric for denoting the type of student response.*
Chapter III
Results

Student participants were divided into two groups, an experimental group (Group A) and a control group (Group B). Students in Group A began their learning experience with a mathematical investigation via an online applet followed by a teacher directed lesson, whereas students in the control group, Group B, began with a teacher directed lesson. Student learning and retention was assessed by asking students to complete four skills tasks and two or three understanding tasks. The skills tasks were designed so that they could be answered by using a single mathematical representation, and the understanding tasks were designed to require the synthesis of at least two mathematical representations (Schwartz, n.d.). For the purpose of this research, the skills questions were likely to be answered using an algebraic or geometric representation, whereas the understanding questions required students to invoke a connection between two geometric representations (Cartesian and polar), perhaps also using an algebraic representation of each coordinate system.

Originally, twenty-one students volunteered to participate. Through attrition, fifteen students started the research and only twelve completed the initial lesson and three follow-up assessments of learning and retention. Of the participants who completed all of the assessments from the experimental group (Group A), three out of six (50%) were at that time enrolled in a mathematics course at the honors level. Of the participants who
completed all of the assessments from the control group (Group B), eight out of nine (89%) were at that time enrolled in a mathematics course at the honors level.

A summary of the results points to several possible conclusions. Students in the control group were given extra procedural practice and retained those skills better over time that the students in the experimental group who were not given the extra procedural practice. Students in the control group and students in the experimental group retained conceptual knowledge at very similar levels throughout the research. This might imply that the conceptual anchor did not provide a deep enough conceptual reference for the students in the experimental group to retain the knowledge. Although it was not the focus of this research, it is notable that student participants enrolled in an honors-level mathematics course outperformed student participants enrolled in a standard-level mathematics course in both procedural and conceptual assessment tasks throughout the duration of the research (see Appendix 4 for student scores).

Comparison of Group A and Group B

On the first assessment of learning, administered one day after the initial lesson, the aggregated score of the experimental group (Group A) participants \((n = 6)\) on the skills tasks was lower than the participants in the control group (Group B) \((n = 9)\), who had extra practice with skills type questions during the lesson (1.96 versus 2.39 on a three-point scale). The aggregated score on the understanding tasks of Group A participants was comparable, albeit slightly lower, to the aggregated score of Group B participants (1.67 versus 1.72).

On the second assessment of learning, administered one week after the initial lesson, the aggregated score of Group A participants \((n = 5)\) on the skills tasks was lower
than the participants in the control group, Group B \((n = 9)\), who had extra practice with skills type questions during the lesson (1.15 versus 2.42 on a three-point scale). This score differential is larger than the differential on the first assessment. The aggregated score on the understanding tasks of Group A participants was comparable, albeit slightly lower, to the aggregated score of Group B participants (1.53 versus 1.74).

On the third and final assessment of learning, administered four weeks after the initial lesson, the aggregated score of Group A participants \((n = 4)\) on the skills tasks was lower than the participants in the control group, Group B \((n = 8)\), who had extra practice with skills type questions during the lesson (0.81 versus 2.56 on a three-point scale). This score differential is larger than the differential on the first and second assessments. The aggregated score on the understanding tasks of Group A participants was comparable, albeit slightly lower, to the aggregated score of Group B participants (1.17 versus 1.29).

In terms of an overall trend from the first assessment to the final assessment, student participants in Group A saw an aggregated score drop on the skills tasks of 1.15 points. The student participants in the control group, Group B, who had extra practice with skills type questions during the lesson, saw an aggregated score increase of 0.17 points over the same four-week span. As for the understanding tasks, aggregated scores from the final assessment saw a comparable drop in each group relative to the first assessment. Group A’s aggregated scores dropped 0.50 points, whereas Groups B’s aggregated scores dropped 0.43 points.

Comparison of Students Enrolled in Honors and Regular Courses

On the first assessment of learning, administered one day after the initial lesson, the aggregated score of students enrolled in an honors-level course \((n = 11)\) on the skills
tasks was higher than the aggregated score of the participants enrolled in a standard-level course \((n = 4)\) (2.66 versus 1.00 on a three-point scale). The aggregated score on the understanding tasks of participants enrolled in an honors-level course was also higher than the aggregated score of the participants enrolled in a standard-level course (1.86 versus 1.25).

On the second assessment of learning, administered one week after the initial lesson, the aggregated score of students enrolled in an honors-level course \((n = 10)\) on the skills tasks was higher than the aggregated score of the participants enrolled in a standard-level course \((n = 4)\) (2.48 versus 0.69). The aggregated score on the understanding tasks of participants enrolled in an honors-level course was also higher than the aggregated score of the participants enrolled in a standard-level course (2.07 versus 0.67).

On the third and final assessment of learning, administered four weeks after the initial lesson, the aggregated score of students enrolled in an honors-level course \((n = 9)\) on the skills tasks was higher than the aggregated score of the participants enrolled in a standard-level course \((n = 3)\) (2.25 versus 1.17). The aggregated score on the understanding tasks of participants enrolled in an honors-level course was also higher than the aggregated score of the participants enrolled in a standard-level course (1.30 versus 1.11).

In terms of an overall trend from the first assessment to the final assessment, student participants enrolled in an honors-level course saw an aggregated score drop on the skills tasks of 0.41 points. The student participants enrolled in a standard-level course saw an aggregated score increase of 0.17 points over the same four-week span. As for the
understanding tasks, aggregated scores from the final assessment saw a drop in each group relative to the first assessment. Student participants enrolled in an honors-level course aggregated scores dropped 0.57 points, and student participants enrolled in a standard-level course aggregated scores dropped 0.14 points.

Algebraic Versus Geometric Responses

Although a student’s written response to a mathematical task is not necessarily indicative of how he or she thought about the task, it is reasonable to assume that there might be a connection. For the assessment tasks in this research, the student participants tended to communicate their ideas using an algebraic or geometric framework. Further discussion of this distinction will be found in the next chapter.

In addressing the first assessment of learning, administered one day after the initial lesson, students communicated using an algebraic method on 50% of the skills tasks, a geometric method on 3.3% of the skills tasks, a combination of algebraic and geometric methods on 10% of the skills tasks, and no apparent method on 36.7% of the skills tasks. Students communicated using an algebraic method on 23.3% of the understanding tasks, a geometric method on 36.7% of the understanding tasks, a combination of algebraic and geometric methods on 0% of the understanding tasks, and no apparent method on 40% of the understanding tasks.

In addressing the second assessment of learning, administered one week after the initial lesson, students communicated using an algebraic method on 51.8% of the skills tasks, a geometric method on 1.8% of the skills tasks, a combination of algebraic and geometric methods on 8.9% of the skills tasks, and no apparent method on 36.5% of the skills tasks. Students communicated using an algebraic method on 4.8% of the
understanding tasks, a geometric method on 38.1% of the understanding tasks, a combination of algebraic and geometric methods on 7.1% of the understanding tasks, and no apparent method on 50% of the understanding tasks.

In addressing the third and final assessment of learning, administered four weeks after the initial lesson, students communicated using an algebraic method on 54.1% of the skills tasks, a geometric method on 4.2% of the skills tasks, a combination of algebraic and geometric methods on 14.3% of the skills tasks, and no apparent method on 27.4% of the skills tasks. Students communicated using an algebraic method on 0% of the understanding tasks, a geometric method on 19.4% of the understanding tasks, a combination of algebraic and geometric methods on 0% of the understanding tasks, and no apparent method on 80.6% of the understanding tasks.
Chapter IV

Discussion

It was hoped that conceptual anchors could provide students with a significant improvement in their retention of procedural skills and conceptual understanding of mathematical content. When an online applet designed for exploration of polar coordinates was used to serve as a conceptual anchor, the limited data of this research suggest that it did not improve student retention of procedural skills and conceptual understanding over the students who did not use the online applet. Additionally, it appears that the students who did extra practice with the mechanics of polar coordinates retained those skills better than the students who used the online applet. As the online applet designed for student exploration was to provide a conceptual underpinning of the lesson, the results obtained in this thesis could be interpreted as running contrary to other, more prominent, research. Also, to be noted, the students enrolled in honors-level courses outscored the students in non-honors-level courses, particularly on skills tasks. This result might lead one to consider the merits and drawbacks of leveling mathematics courses.

The Results of Conceptual Anchors Versus Extra Practice

The results demonstrate a declining performance with tasks relating to both skills and understanding for students who were given a conceptual anchor in the form of an online applet and reflection prompts as an introduction to a lesson on polar coordinates. Students who were given extra procedural practice outperformed students who were
given a conceptual anchor in both skills tasks and understanding tasks throughout the four weeks of assessment. Most notably, though, was the comparison after four weeks, where the students given extra procedural practice retained those algorithms far better than the students who used the online applet.

It might be inferred from this trend that student retention of procedural skills benefits from extra practice with skills. Research on spaced repetition demonstrates that procedural practice over time promotes retention of skills. In turn, this retention and increased facility with mathematical skills serves as a foundation for students to pursue higher-level problem-solving (Kang, 2016; Rohrer, Dedrick, & Stershic, 2015). Since the students who explored polar coordinates with the online applet may not have had the opportunity to master their procedural skills during the initial lesson, those skills were not solidified and, subsequently, retained. A lack of skills knowledge may have led to student difficulties in retaining a conceptual understanding of polar coordinates, thereby accounting for lower scores on those tasks. Given these results, it appears the procedural practice can improve the retention of procedural skills, so it may be prudent to consider whether different mathematical topics benefit more from a conceptual foundation or a procedural foundation.

One exception to the overall general results was one student participant who used the conceptual anchor and scored at or above the control group average scores on both skills and understanding tasks on every assessment with the exception of the week four skills assessment. This student, at the time of the research, was enrolled in Calculus BC Honors, therefore having seen content on polar coordinates previously in his mathematics courses. For a student with previous exposure, the assessment tasks may not have seemed
novel, therefore making them easier to complete. The student’s mathematics studies at the honors-level may, too, account for his exceptional scores. The effect of being enrolled in an honors-level course is worth further consideration. On one hand, students in honors-level course may find success because of an aptitude for or positive attitude toward mathematics. Perhaps, too, students in honors-level courses have better long-term memory. On the other hand, it could be that the level of challenge and problem-solving stressed in an honors-level course cultivates an aptitude for or positive attitude toward mathematics in students.

Another possible explanation for the results of this research could be the disproportionate distribution of honors-level students in the research groups. While this was initially accounted for by using a matched random group assignment method, the attrition of student participants resulted in the group using the online applet consisting of 50% honors-level students and the control group consisting of 89% honors-level students. This imbalance may account for students in the control group performing better on the skills tasks because the honors-level students in that group might be more facile with procedural knowledge than students enrolled on standard-level courses. At the end of four weeks, the honors-level students far outscored standard-level students on the skills tasks, but only did marginally better on the understanding tasks. In a study of college students enrolled in an introductory-level physics class it was shown that honors-level students had many of the same misunderstandings regarding conceptual tasks as standard-level students did (Peters, 1982). This is re-affirming, given the comparable level of conceptual understanding demonstrated by honors- and standard-level students in this research. Again, this gives one pause to consider the cause and effect relationship
between enrollment in an honors-level mathematics course and student retention of procedural skills and conceptual understanding. If honors-level students tend to have similar misconceptions as standard-level students, then it seems that the main difference in retaining knowledge between honors-level and standard-level students is practice with procedural skills. Since procedural skills can be practiced and are often easy to assess for correctness, it might be beneficial for standard-level students to practice procedural skills repeatedly and frequently. A common pitfall, though, is when teachers work on developing only procedural skills with standard-level students. The key, of course, is that all students need practice with solving problems that require both procedural and conceptual knowledge. The amount and type of practice will vary from topic to topic and from student to student.

Given that the student participants have a range of different mathematical backgrounds, there is also the possibility that the students’ achievement on the research tasks was affected by past experiences with effective and ineffective teachers, as well as with different prior mathematics curricula. (Persson, 2015). While the school at which the student participants are enrolled takes pride in its dedicated and thoughtful teachers, there are some teachers that are more comfortable with classroom lessons and activities that involve conceptual understanding and other teachers that tend to focus on algorithms and procedures. This seems especially true for the elementary school teachers, where attitudes and approaches toward mathematics can impact students’ attitudes (Blazar & Kraft, 2017). The school at which the student participants are enrolled is also a large entry point for new students in the ninth grade. This means that students have had widely varying backgrounds and experiences in mathematics. Some student participants were
international students who demonstrated an exceptional level of procedural mastery on their incoming mathematics placement tests. Other student participants were previously enrolled in other area middle schools and demonstrated gaps in their algebraic skills and understanding on their incoming placement tests.

One way that future research might try to disentangle these confounding factors would be to repeat the study at a larger school with students who have all had consistent mathematical backgrounds and experiences. A large public high school with a minimal number of incoming students who are new to the school might be an ideal setting for such research. A standardized curriculum and a teacher who teaches multiple sections of the same course would also help to minimize some of the variability. While it is hard to calibrate students’ attitudes toward mathematics, a well-crafted entry questionnaire might be useful in correlating student attitudes toward learning mathematics to retention of conceptual and procedural knowledge. It might also give insight into the quality of experience that a student has had with previous teachers and curricula. In all, there are many confounding variables to consider, some of which are quantifiable and others which would require qualitative methods of analysis.

Relationship to Other Research

The results of this research seem to be in tension with much of the widely accepted research that indicates procedural skills are built on a foundation of strong conceptual understanding (NCTM, 2014). Other research also implies that initial student exploration designed to anchor a lesson leads to better understanding of mathematical content. According to Rittle-Johnson (2017), “Compared to children who solved the problems after the lesson, children who solved the unfamiliar problems before the lesson
gained more conceptual knowledge or procedural knowledge” (p. 4). In another study, post-test retention was statistically better when students experienced a lesson that was mostly conceptual in nature compared to a lesson that focused on a balance of conceptual and procedural skills (Rittle-Johnson, Fyfe, & Loeher, 2016). Indeed, calculus students who learned in a conceptual-based environment outperformed comparable students who learned in a procedure-based environment on conceptual tasks. Not only that, but the same calculus students who learned in a conceptual-based environment demonstrated a commensurate achievement on procedural tasks as the students who learned in a procedure-based environment. (Chappell & Killpatrick, 2003).

The data presented in this thesis imply that extra practice with mathematical procedures helps students retain procedural knowledge over time. Kanive, Nelson, Burns, & Ysseldyke (2013) support this finding with research they did with computer-based practice interventions in elementary school mathematics. In that study, students exhibited better recall of computational procedures with the aid of skills-based practice over students who had a conceptual intervention or no intervention, at all. Not surprisingly, practice helps students get better. It is fair to say that students in this study were not presented with ample opportunities to practice with problems that were either procedural or conceptual in nature, thus accounting for their declining retention. The students in the control group who were given limited extra practice with procedural skills were able to leverage that practice to retain those procedural skills better than students in the experimental group who did not have the extra practice.

It might be that different mathematical topics require different types of practice to lay the foundational groundwork. Given that research by Rittle-Johnson (2009, 2015,
2016, 2017) supports the bidirectional nature of improvement in both procedural and conceptual knowledge in mathematics, the research in this thesis seems to imply that some topics might be better suited for iterating from a procedural starting point. Converting Cartesian coordinates to polar coordinates might be a topic that is better suited for starting with the practice of procedural skills before focusing on conceptual understanding.

The Norwegian University of Science and Technology (NUST) (2013) found that student proficiency in one area of mathematics does not always translate into proficiency in another, and that students can strengthen their ability with practice. For example, a student with good spatial reasoning skills might find success in geometric content but struggle in algebra. Focused practice in algebra will help improve their algebraic understanding. The notion that one gets better at those things which one practices is also supported from a neurological standpoint in the same study. It seems reasonable, then, to expect students who practice procedural skills to improve their facility with and retention of procedural skills, and students who practice with problems that are conceptual in nature to find similar gains in their conceptual knowledge. The key, of course, is a balanced approach where students have ample opportunities to practice using both procedural and conceptual knowledge.

Many research efforts contradict the findings in this thesis, i.e. that students who engaged in procedural practice were found to have better retention of both conceptual and procedural knowledge over students who engaged in a self-guided conceptual exploration. With that said, the notion that practice improves retention is supported. The students who were afforded extra practice with solving procedural problems were able to
better solve procedural problems up to month later. The students who engaged in a self-guided conceptual exploration, though, may not have gained the requisite conceptual knowledge necessary to solidify their understanding. They may not have engaged fully in looking into the intellectual mirror to assess their understanding. This self-assessment is critical for determining whether the conceptual anchor served its purpose. Perhaps, students who are given the opportunity to master and then practice applying conceptual understanding might see long term retention benefits.

Implications of Results

Memorizing and implementing procedures and algorithms does not imply a conceptual understanding of mathematical content (nor does it refute that a student’s conceptual understanding exists). Certainly, effective mathematics educators strive to teach students deeply rooted understanding as well as keen facility with procedures and algorithms. The challenge is to offer students practice with and reinforcement of both conceptual and procedural thinking, while at the same time introducing new mathematical content and investigations.

Practice and mastery are key elements to promoting enduring conceptual knowledge and procedural skills. Of course, some students might need more practice than other students for certain topics, and this might be a situation where online practice platforms, like WeBWork, can be leveraged for practicing procedural skills. One of the advantages of online practice is that the feedback is immediate and private. One disadvantage, however, is that students must engage fully and deliberately to get the most out of that practice. Feedback is more complicated for practicing problems that require conceptual understanding, as it usually requires students to explain their reasoning.
verbally or in writing. Regardless of the mechanism for practice, deliberate and proper practice in mathematics is one way that students can improve retention of skills and understanding.

The research contained in this thesis implies that procedural practice was beneficial for the retention of those procedural skills for some students. On average, the students who had extra practice with procedural skills saw a score increase of 7% over four weeks, while student who did not have extra practice saw their scores on procedural problems drop, on average, by approximately 58%. While the research did not explore the effect of conceptual practice, it might be assumed that if students had been exposed to continued practice and mastery of conceptual problems then they might have retained that conceptual understanding over time.

Star (2005) discussed the importance of recognizing the quality of procedural knowledge in terms of being superficial or deep. While the understanding assessment tasks in this research were designed to probe for deep understanding, it could be argued that the procedural assessment tasks were designed at the superficial level. Indeed, the required procedural skills required to manipulate and solve equations such as \( x^2 + y^2 = r^2 \), \( x = r \cdot \cos \theta \), \( y = r \cdot \sin \theta \), and \( \arctan(\theta) = \frac{y}{x} \) are not particularly involved for students who are enrolled course at or beyond Algebra II.

The distinction between the type of knowledge (conceptual or procedural) versus the level (superficial or deep) is an important framework to consider (Star & Stylianides, 2013). The research in this thesis showed that students were able to retain superficial procedural knowledge if they had the opportunity to practice the procedures. The students using the conceptual anchor may have only gained a superficial understanding of polar
coordinates which did not translate well when assessed for whether they had retained a deep conceptual understanding. It seems apparent, then, that acquiring procedural and conceptual knowledge needs to be done at the deep level in order to retain these understandings.

Research Limitations

A major limitation of this research study was the limited number of student participants. The low number of participants prevented any sort of meaningful statistical analysis to be performed. The wide range of the students’ mathematical backgrounds, experiences, and attitudes also made comparisons hard to be drawn in a conclusive manner. For example, it might not be appropriate to compare the retention of procedural and conceptual knowledge between one student enrolled in a standard-level Algebra II course to another student enrolled in an honors-level Calculus course. A more desirable experimental setting might be a larger high school, where it would be possible to include upwards of fifty participants, all of whom were currently enrolled in the same mathematics course and who had similar mathematical backgrounds and experiences in prior mathematics courses. Similarly, a more desirable experiment would introduce a brand new mathematical topic that none of the participants have seen. While all students who completed the study were interested in participating, the study was ultimately an add-on activity for which they volunteered. The study was conducted during an activity period during the school day, and many of the students may have considered this free time to complete homework or socialize with peers. The students willingly participated in the lessons and completed the assessments, but there is a possibility that they did not take these activities seriously. This did not seem to be the case, although it remains a possible
limitation. There was no external motivation for the student participants to do their best, so the results largely hinged on each student’s internal motivation to do mathematics. Although twenty-one student participants initially volunteered, the fact that only twelve completed the initial lesson and three subsequent assessments seems to imply that there was a lack of motivation for at least some students. Overall, the research limitations certainly affected the quantity of data collected, which in turn limits the conclusions that may be drawn.

Considerations for Future Studies

This research focused on the effect of a conceptual anchor on students’ retention of procedural skills and conceptual understanding in mathematics. The limited data garnered from the research seem to suggest that a conceptual anchor did not aid in this retention, but the research does not necessarily refute this hypothesis, either. It remains plausible that continued practice with and subsequent mastery of conceptual understanding can lead to long-term retention of procedural and conceptual knowledge.

Continued research on this topic might include devising a new conceptual anchor for other mathematical tasks. Since some topics in mathematics might be better suited for conceptual anchors than other topics, a battery of conceptual anchors could be developed and the research could include an analysis of which topics benefit most from a conceptual anchor.

Another possibility for continued research could include providing students with intermittent practice with conceptual anchors, and investigating whether conceptual explorations have long-term positive effects on a student’s ability to explore novel mathematical problems. One of the shortcomings of this research appeared to be that it
was unclear as to whether students who used a conceptual anchor, in fact, mastered a conceptual understanding. Students can be susceptible to missing important mathematical connections, thereby decreasing the impact of a self-guided investigation (Rittle-Johnson, 2017). Students using the conceptual anchor demonstrated comparable levels of conceptual understanding to students who did not use the conceptual anchor throughout the research period. If students were given continued practice with the conceptual anchor, it is plausible that they would master and retain a deeper mathematical understanding. A future study might assign students in the experimental group to intermittent practice with the same conceptual anchor, as well as teacher intervention to help ensure mastery of the concepts.

A possible extension of the research would be to determine whether practicing mathematical explorations in one area of mathematics can lead to improved student insight when investigating mathematics in another area. In other words, does mathematical understanding, the ability to connect mathematics across two or more representations, in one domain lead to better understanding in other domains? We can understand this as seeing the “structure” of mathematical problems. As mathematician Paul Halmos (1916 – 2006) said, “The only way to learn mathematics is to do mathematics.” Research to support this claim could have an impact on the importance of conceptual practice with novel problems. An increased focus on learning tasks associated with conceptual practice could empower students to explore new mathematical concepts with increased confidence. In this sense, practice with various representations of mathematical topics could lead to student facility with different representation of other mathematical topics.
One Teacher’s Anecdote

As noted previously, The Calculus Consortium based at Harvard (CCH) was a motivating force in the mathematics reform movement when they published the first edition of their calculus textbook in 1994. The textbook implemented the “rule of three” (later to become the “rule of four”) for presenting mathematical topics, which focused on understanding calculus conceptually by emphasizing the connection of graphical, numerical, and analytical representations (Gleason & Hughes Hallett, 1992). The author and principal researcher of this thesis adopted this textbook in 1998, early in his career as a secondary school teacher of calculus. While the focus on connecting different representations of mathematics was challenging for the students, the teacher was confident that the increased conceptual understanding was worth the “growing pains.” In the beginning years of adoption, students often felt ill-prepared to contend with thinking about mathematics in ways other than a “plug and chug” approach. In the students’ defense, they had been trained to think about mathematics from solely a symbolic perspective. Students would often complain at the start of the year that what they were doing was not mathematics. They could not fathom that reading word problems and looking at tables and graphs was the very nature about what mathematics is. Often, by the end of the year, students would gain experience and confidence to make connections across mathematical representations and, subsequently, comment that they then understood calculus and how it could be applied in a variety of settings.

In the following years, the school at which this teacher worked adopted the precalculus textbook published by the CCH. This textbook, similar to the aforementioned calculus textbook, focused on presenting mathematics via the “rule of four,” now adding
verbal and written representations. Students entering the precalculus course now felt the growing pains of thinking conceptually. The upswing was that, over the course of the next few years, these students were well-prepared to tackle this same sort of conceptual thinking with their studies of calculus the following year. Specifically, the students had practice with conceptual thinking, just as they had with procedural thinking. In this sense, students who learned to think about mathematical topics in novel ways were better prepared to think about other mathematical topics in novel ways.

Although the preceding observations are purely anecdotal, it should be noted that there are student learning gains to be made from both procedural practice and conceptual practice. Research by Rittle-Johnson, Schneider, and Star (2015) supports the claim that gains in procedural knowledge lead to gains in conceptual knowledge and vice versa. While classroom learning time is not often a commodity that teachers of mathematics have in excess, it appears that time spent on practice, both procedural and conceptual, provides learning gains for students.
Chapter V
Conclusion

The “math wars” fueled the debate over the importance of conceptual understanding versus procedural skills for two decades. For all of the rich discussion, though, student achievement in mathematics has not improved according to some measures. From 1995 to 2015, secondary school seniors in the United States have seen a slight drop in performance on the Advanced Mathematics Achievement administered by the IEA (TIMSS), and the Program of International Assessment (PISA) has documented a decline in 15-year-old’s mathematical literacy in the United States since 2003 (National Center for Educational Statistics, 2015). It stands to reason that there is a gap between what mathematics educators are hoping students will learn and what students are retaining in terms of conceptual understanding and procedural skills.

An issue facing mathematics educators at the secondary school level is that students enter any given class with a wide range of mathematical skills and understandings. Some students, through no fault of their own, have not had access to quality learning opportunities in previous years of education (Persson, 2015). Mathematics educators are then faced with the challenge of helping students to remediate skills and bolster understanding of prerequisite material at the risk of having to eliminate new learning opportunities due to the time constraints of the academic year. Past NCTM President Matt Larson (2017) posted a reflection online urging mathematics educators to find time beyond scheduled class time to help students remediate procedural skills.
Textbooks often include skills review problems that can be assigned. However, if a student lacks conceptual understanding or possesses misconceptions, then it is unlikely that the student will be able to practice procedural skills with the intended outcome of reinforcing those skills. In fact, incorrectly practicing procedural skills might lead to reinforcing misconceptions. Likewise, a lack of facility with procedural skills can lead to lost opportunities in developing a deep conceptual understanding. Harvard Extension School instructor and textbook author, Eric Connally, commented,

I wholeheartedly believe in helping students reach a strong conceptual understanding of the various mathematical topics we cover . . . but it is often the case that a lack of basic algebra skills impedes progress. . . . The tension is overdoing drill at the expense of conceptual understanding, and vice versa. For every student (and perhaps every teacher) the balance will vary. (personal communication, August 3, 2018).

Ample evidence from previous research shows that there is an interdependence between conceptual and procedural learning gains. Furthermore, gains in one area are not only sufficient, but necessary, in order for students to achieve gains in the other area. This thesis set out to determine whether a student-guided conceptual exploration could be leveraged to promote enduring conceptual understanding and procedural skills. If this had been the case, then conceptual anchors would allow teachers to use more class time for enrichment instead of remediation of skills and concepts that were previously taught.

Oakley (2014) suggested that content mastery does not correlate simply and solely to conceptual understanding, but rather that mastery is a function of practice. The practice needs to be of both conceptual and procedural knowledge. She went on to say, “Understanding is key. But not superficial, light-bulb moment of understanding. In STEM, true and deep understanding comes with the mastery gained through practice” (Oakley, 2014). This is an apt synopsis of the research contained in this thesis. The
students who had extra practice with procedural skills saw benefits in retaining procedural skills. Perhaps, then, students who have extra practice applying conceptual reasoning would see benefits in retaining conceptual understanding.

In light of the research for this thesis, simply tasking students to explore polar coordinates in a conceptual manner, whereby connections between algebraic and graphical representations were highlighted, did not appear to promote enduring conceptual knowledge or procedural skills. The exploration was to serve as a conceptual anchor in hopes of providing students with a memorable experience that they could use as a lasting point of reference. The results indicate that the conceptual anchor did not aid in student retention of conceptual understanding or procedural skills. In fact, students in this study who were given extra practice in lieu of the conceptual exploration retained procedural skills longer and had a comparable conceptual understanding. The limitations of the research make it hard to draw meaningful conclusions, but it seems that students benefit from practice. It is plausible that different topics in mathematics may benefit from one type of practice (procedural or conceptual) over the other.

Of course, it is not surprising that students can benefit from practice with both procedural skills and conceptual understanding. The key, then, is to ensure that students have opportunities for practice, whether through iterating (Rittle-Johnson & Koedinger, 2009), interleaving spaced repetition (Kang, 2016), practicing skills using software after foundational concepts have been learned (Kanine et al., 2013), or practicing with applying conceptual understanding after foundational procedural skills have been learned.
Appendix 1

Geogebra Applet

An applet using Geogebra was created by Judah Scwhartz and adapted for this thesis by the researcher. The dynamic version of the applet can be found online at https://ggbm.at/U9Nypjyt. Below are screenshots of the applet with brief descriptions.

Figure 4. Geogebra Applet (screenshot 1)

This is the initial view of the applet when accessed by the student. The large white point can be dragged and the corresponding Cartesian values for $x$ and $y$ will change, as will the polar values for $r$ and $\theta$. Notice that the $x$ and $y$ coordinates are colored (green and blue, respectively) to correspond to the solid line segments that represent distances in the Cartesian plane and to the dashed line segments in the polar plane. Likewise, $r$ and $\theta$ are colored (yellow and purple, respectively) to correspond to the line segment and angle in the polar plane. This is all designed to help students see the connection between the two different visual representations and discover the mathematical relationships $x^2 + y^2 = r^2$ and $\theta = \arctan \left( \frac{y}{x} \right)$.
When the “Show Cartesian” box is clicked, the distances representing $x$ and $y$ are displayed in both the Cartesian plane and the polar plane.

When the “Show Polar” box is clicked, the distance and angle representing $r$ and $\theta$, respectively, are displayed in both the Cartesian plane and the polar plane.
Appendix 2

Lesson Plan

Objective: Students will be able to convert coordinates between polar and Cartesian (rectangular) form.

Lesson Notes, as copied and modified from *Functions Modeling Change* Instructor’s Guide (Connally, 2007):

Coordinates are used to specify location. Students are familiar with Cartesian coordinates that we denote by (x, y). However, there are other ways to express location. Polar coordinates express location by specifying the point’s distance from the origin and by the angle made between the x-axis and the line joining the origin to the point. Polar coordinates are denoted by (r, θ). For the purposes of this research, we will reference θ in degrees so as to minimize the any potential new concepts or topics.

Demonstrate how to plot the location of several points given in polar coordinates. For example, plot the points (2, 30°), (5, 90°), (4, 180°), (1, 0), (0, 45°).

Now discuss the idea of converting between coordinate systems. Plot the Cartesian point (3, 4). Now make a right triangle that shows r and θ (graphical representation). By using right-triangle trigonometry, the students can discover that $r^2 = 3^2 + 4^2$ and $\theta = \arctan(4/3)$. Generalize these formulas to an arbitrary point (a, b).

Next, start with a point (r, θ) in polar coordinates and have the students derive the formulas $x = r \cos \theta$ and $y = r \sin \theta$. Have the students practice converting points between the two systems. Students who are comfortable with trigonometry will probably see that
the choice of $\theta$ is not unique. Indeed, adding an integer multiple of $360^\circ$ to $\theta$ does not change the location of the point. Point out that the coordinates for a point do not have to be unique, they just have to specify a unique location.

Make sure the students understand that $r$ and $\theta$ work together to specify location. Suppose they wish to find polar coordinates for the point $(-2, 3)$. Observe that this point is in quadrant II. Solving $r^2 = (-2)^2 + 3^2$ yields $r = \pm \sqrt{13}$. Now, $\theta = \arctan(-3/2) = -56.31^\circ$. Thus, $\theta$ is in quadrant IV. If this value of $\theta$ is used, then $r$ must be negative. Hence one solution is $(-\sqrt{13}, -56.31^\circ)$. However, if $r$ must be positive, then $\theta$ must be shifted by $180^\circ$: $\theta + 180^\circ = 123.69^\circ$ radians. Hence, another polar representation for the point is $(\sqrt{13}, 123.69^\circ)$. In short, students should make sure their answer gives a point in the correct quadrant.

Summarize this section by reviewing the conversion formulas between the two coordinate systems using both the formulas and the graphical representation.
Appendix 3
Assessment Tasks

Skills Questions (1 day after lesson)

1. Point C has the Cartesian coordinates (3, 3). What are the polar coordinates of point C?
2. Point C has the Cartesian coordinates (-1, 2). What are the polar coordinates of point C?
3. Point P has the polar coordinates (4, 60°). What are the Cartesian coordinates of point P?
4. Point P has the polar coordinates (5, 150°). What are the Cartesian coordinates of point P?

Figure 7. Skills Questions (1 day after lesson)

Understanding Questions (1 day after lesson)

1. Two students, Radius and Theta, are discussing how to express the Cartesian point (2, 2) in polar coordinates.
   
   Radius: “I just learned about polar coordinates and love them. The Cartesian point (2, 2) has polar coordinates (2√2, 45°).”
   Theta: “I think that the point has polar coordinates (-2√2, 225°).”
   
   Who is correct? What do you think? Explain your reasoning.

2. A circle centered at the origin with a radius of 1 can be represented by the relationship $x^2 + y^2 = 1$ using Cartesian coordinates. How would you represent the same relationship using polar coordinates?

Figure 8. Understanding Questions (1 day after lesson)
Skills Questions (1 week after lesson)

1. Point C has the Cartesian coordinates (-1, 2). What are the polar coordinates of point C?

2. Point C has the Cartesian coordinates (-2, -2). What are the polar coordinates of point C?

3. Point P has the polar coordinates (5, 150°). What are the Cartesian coordinates of point P?

4. Point P has the polar coordinates (2, 30°). What are the Cartesian coordinates of point P?

Understanding Questions (1 week after lesson)

1. A line passing through both the points (0, 0) and (1,1) can be represented by the relationship \( \theta = 45° \) using polar coordinates. How would you represent the same relationship using Cartesian coordinates?

2. \( y = 10 \) and \( x = 10 \) are equations the represent graphs in the Cartesian plane. 
   
   \( r = 10 \) and \( \theta = 10° \) are equations that represent graphs in the polar plane. 
   Describe what each of these four graphs looks like.

3. \( y = x \) is the Cartesian equation of the graph of a line passing through the origin. 
   Describe what the graph of the polar equation looks like.
Figure 11. Skills Questions (4 weeks after lesson)

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<thead>
<tr>
<th>Question</th>
<th>Answer</th>
</tr>
</thead>
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<td>1. Point C has the Cartesian coordinates (-1, 2). What are the polar coordinates of point C?</td>
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</tr>
<tr>
<td>2. Point C has the Cartesian coordinates (0, 1). What are the polar coordinates of point C?</td>
<td></td>
</tr>
<tr>
<td>3. Point P has the polar coordinates (4, 60°). What are the Cartesian coordinates of point P?</td>
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</tr>
<tr>
<td>4. Point P has the polar coordinates (5, 135°). What are the Cartesian coordinates of point P?</td>
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Figure 12. Understanding Questions (4 weeks after lesson)

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<th>Question</th>
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<td>1. There are well over 400 city blocks in or around Philadelphia’s Center City. Explain why Cartesian coordinates might be more useful than polar coordinates to help you locate a building in Philadelphia.</td>
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<td>2. Middle Ground lighthouse in Long Island Sound is located offshore, about 6 miles from land. Suppose you are a lighthouse keeper. Explain why polar coordinates might be more useful than Cartesian coordinates in describing a ship’s location at sea relative to the lighthouse.</td>
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<td>3. A friend asks you what you know about polar and Cartesian coordinates. What do you tell her?</td>
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Table 3. Assessment 1 Student Scores

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Group A Mean Score | 2.00 | 1.67 | 2.17 | 2.00 | 1.83 | 1.50 | 1.96 | 1.67
Group B Mean Score | 2.56 | 2.00 | 2.56 | 2.44 | 1.89 | 1.56 | 2.39 | 1.72

| Honors Course Mean Score | 2.82 | 2.45 | 2.73 | 2.64 | 2.09 | 1.64 | 2.66 | 1.86
| Non-Honors Course Mean Score | 0.80 | 0.20 | 1.20 | 1.00 | 1.00 | 1.00 | 0.80 | 1.00

This table details individual student scores one day after the initial lesson, with mean scores for questions assessing skills and questions assessing understanding. Summary scores are aggregated for Group A (aplet group) and Group B (control group), as well for students enrolled in honors course and non-honors courses.
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| Group A Mean Score | 1.00 | 1.20 | 1.20 | 1.20 | 1.80 | 1.40 | 1.40 | 1.15 | 1.53 |
| Group B Mean Score | 2.11 | 2.00 | 2.67 | 2.89 | 1.89 | 2.00 | 1.33 | 2.42 | 1.74 |

| Honors Course Mean Score | 2.30 | 2.40 | 2.50 | 2.70 | 2.20 | 2.30 | 1.70 | 2.48 | 2.07 |
| Non-Honors Course Mean Score | 0.20 | 0.00 | 1.00 | 1.00 | 0.80 | 0.40 | 0.40 | 0.55 | 0.53 |

This table details individual student scores one week after the initial lesson, with mean scores for questions assessing skills and questions assessing understanding. Summary scores are aggregated for Group A (applet group) and Group B (control group), as well for students enrolled in honors course and non-honors courses.
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<td>9</td>
<td>Algebra II</td>
<td>Y</td>
<td>3.00</td>
<td>1.33</td>
</tr>
</tbody>
</table>

| Group A Mean Score | 0.50 | 0.25 | 1.25 | 1.25 | 1.50 | 1.00 | 1.00 | 0.81 | 1.17 |
| Group B Mean Score | 2.38 | 2.13 | 2.88 | 2.88 | 1.25 | 1.25 | 1.38 | 2.56 | 1.29 |

| Honors Course Mean Score | 2.11 | 1.78 | 2.56 | 2.56 | 1.33 | 1.22 | 1.33 | 2.25 | 1.30 |
| Non-Honors Course Mean Score | 0.50 | 0.50 | 1.25 | 1.25 | 1.00 | 0.75 | 0.75 | 0.88 | 0.83 |

This table details individual student scores four weeks after the initial lesson, with mean scores for questions assessing skills and questions assessing understanding. Summary scores are aggregated for Group A (applet group) and Group B (control group), as well for students enrolled in honors course and non-honors courses.
Table 6. Change in Student Scores Over Time

<table>
<thead>
<tr>
<th>Group</th>
<th>Student ID</th>
<th>Grade</th>
<th>Class</th>
<th>Honors</th>
<th>Change in Score (1 day to 1 week)</th>
<th>Change in Score (1 day to 4 weeks)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
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<td>Y</td>
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<td>-3.00</td>
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<tr>
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<td>Algebra II</td>
<td>Y</td>
<td>-1.00</td>
<td>-1.00</td>
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<td>03</td>
<td>10</td>
<td>Functions and Trig</td>
<td>N</td>
<td>-0.50</td>
<td>-0.50</td>
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<tr>
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<td>-0.25</td>
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<tr>
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<tr>
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<td>10</td>
<td>Algebra II</td>
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<td>-0.75</td>
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<tr>
<td>B</td>
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<td>11</td>
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<td>Y</td>
<td>0.00</td>
<td>0.00</td>
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<td>11</td>
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<tr>
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<td>-1.17</td>
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<tr>
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<td>10</td>
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<tr>
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<td>Algebra II</td>
<td>Y</td>
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<td>0.00</td>
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<tr>
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<td>Calculus BC</td>
<td>Y</td>
<td>-0.25</td>
<td>-1.00</td>
</tr>
<tr>
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<td>9</td>
<td>Algebra II</td>
<td>Y</td>
<td>0.50</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Group A Mean Change: -0.80, -0.17, -1.25, -0.58
Group B Mean Change: 0.03, 0.02, 0.22, -0.58

Honors Course Mean Change: -0.25, 0.17, -0.47, -0.70
Non-Honors Course Mean Change: -0.25, -0.47, 0.25, -0.04

This table details the change in individual student scores from one day after the lesson to one week after the lesson and four weeks after the lesson. Summary scores are aggregated for Group A (applet group) and Group B (control group), as well for students enrolled in honors course and non-honors courses.
References


