Mirror Symmetry, Autoequivalences, and Bridgeland Stability Conditions

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Yu-Wei Fan

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Abstract

The present thesis studies various aspects of Calabi–Yau manifolds, including mirror symmetry, systolic geometry, and dynamical systems.

We construct the mirror operation of Atiyah flop in symplectic geometry. We construct the mirror metric of the Weil–Petersson metric on the complex moduli space of Calabi–Yau manifolds, in terms of derived categories and Bridgeland stability conditions.

We propose a new generalization of Loewner’s torus systolic inequality from the perspective of Calabi–Yau geometry, and prove a generalized systolic inequality for generic K3 surfaces.

We study the dynamical systems formed by autoequivalences on the derived categories of Calabi–Yau manifolds, and find the first counterexamples of Kikuta–Takahashi’s conjecture on categorical entropy.
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Introduction

Calabi–Yau manifolds have very rich geometric structures and have been studied extensively in mathematics and physics. The present thesis studies three different aspects of Calabi–Yau manifolds: mirror symmetry, systolic geometry, and dynamical systems.

Mirror symmetry was first introduced by string theorists. It predicts that Calabi–Yau manifolds come in pairs, in which the complex geometry of one is equivalent to the symplectic geometry of the other, and vice versa. One mathematical formulation of this prediction was proposed by Kontsevich [65].

**Conjecture 0.0.1** (Homological mirror symmetry conjecture [65]). Let $X$ be a Calabi–Yau manifold. There exists a Calabi–Yau manifold $Y$ such that there are equivalences between triangulated categories

$$
\mathcal{D}^b\text{Coh}(X) \cong \mathcal{D}^\pi\text{Fuk}(Y) \quad \text{and} \quad \mathcal{D}^b\text{Coh}(Y) \cong \mathcal{D}^\pi\text{Fuk}(X).
$$

Here $\mathcal{D}^b\text{Coh}(X)$ and $\mathcal{D}^\pi\text{Fuk}(X)$ denote the derived category of coherent sheaves and the derived Fukaya category of $X$, respectively.

Roughly speaking, the objects in the derived category $\mathcal{D}^b\text{Coh}(X)$ are complexes of holomorphic vector bundles on $X$, and the objects in the derived Fukaya category $\mathcal{D}^\pi\text{Fuk}(X)$ are Lagrangian submanifolds in $X$ with certain extra data. Homological mirror symmetry conjecture has been proved in many cases, see for instance [82, 88, 90].

Mirror symmetry phenomenon naturally leads to the following question.

**Question 0.0.2.** Given an “object” (e.g. operation, geometric structure, etc.) in complex geometry, can one construct its counterpart in symplectic geometry, or vice versa?
Chapter 1 and Chapter 2 contain results along this line of thought.

Chapter 1: Mirror of Atiyah flop

Flops are fundamental operations in birational geometry. Among them, Atiyah flop is the simplest and the most well-known one. An Atiyah flop $\hat{X} \to X \leftarrow \hat{X}^+$ contracts a $(-1, -1)$-rational curve $C$ in a complex threefold $\hat{X}$ and resolve the resulting conifold singularity with another $(-1, -1)$-rational curve $C^+$.

The goal of Chapter 1 is to construct the mirror of Atiyah flop in symplectic geometry under mirror symmetry.

**Theorem 0.0.3** (= Proposition 1.1.1). Given a symplectic sixfold $(Y, \omega)$ and a Lagrangian three-sphere $S \subset Y$, we construct another symplectic sixfold $(Y^+, \omega^+)$ with a corresponding Lagrangian three-sphere $S^+ \subset Y^+$, together with a symplectomorphism $f^{(Y,S)} : (Y, \omega) \to (Y^+, \omega^+)$. It has the property that $f^{(Y^+,S^+)} \circ f^{(Y,S)} = \tau_S^{-1}$, where $\tau_S$ is the Dehn twist along the Lagrangian sphere $S$.

The symplectomorphism $f^{(Y,S)}$ in Theorem 0.0.3 is the mirror of Atiyah flop. The contracted $(-1, -1)$-rational curve in algebraic geometry corresponds to the Lagrangian three-sphere in symplectic geometry. Bridgeland [13] shows that threefolds related by a flop are derived equivalent: $D^b(\hat{X}) \cong D^b(\hat{X}^+)$. The property $f^{(Y^+,S^+)} \circ f^{(Y,S)} = \tau_S^{-1}$ is mirror to the fact that the composition of two flop functors $D^b(\hat{X}) \to D^b(\hat{X}^+) \to D^b(\hat{X})$ is the inverse of the Seidel–Thomas spherical twist [89] by $O_C(1)$.

Let $D_{\hat{X}/X} \subset D^b(\hat{X})$ be the subcategory which consists of objects supported on $C$. This subcategory captures the local geometry of the flopping curve. Then Bridgeland’s equivalence restricts to an equivalence $D_{\hat{X}/X} \cong D_{\hat{X}^+/X}$. It is proved by Chan–Pomerleano–Ueda [21] that $D_{\hat{X}/X}$ is equivalent to certain derived Fukaya category $D^b\mathcal{F}_Y$. We prove the following compatibility result between the Atiyah flop and our mirror Atiyah flop.

**Theorem 0.0.4** (= Proposition 1.5.4). The symplectomorphism $f^{(Y,S)}$ induces an equivalence between the derived Fukaya categories $D^b\mathcal{F}_Y \cong D^b\mathcal{F}_{Y^+}$. Moreover, the equivalence is the same as the composition $D^b\mathcal{F}_Y \cong D_{\hat{X}/X} \cong D_{\hat{X}^+/X} \cong D^b\mathcal{F}_{Y^+}$, where the first and third equivalences are given by Chan–Pomerleano–Ueda [21], and the second equivalence is Bridgeland’s flopping equivalence.
Note that unlike the Atiyah flop which produces different complex manifold, its mirror $f^{(Y,S)}$ is a symplectomorphism. It is not very surprising since symplectic geometry is much softer than complex geometry. On the other hand, we can endow more structures so that the effect of mirror Atiyah flop can be seen. One way to do so is to consider the Bridgeland stability conditions [14] on $D^bF_Y$.

**Theorem 0.0.5 (= Theorem 1.1.3).** Let $Y = \{u_1v_1 = z + q, u_2v_2 = z + 1, z \neq 0\}$ be the deformed conifold and $\Omega_Y = dz \wedge du_1 \wedge du_2$ be a holomorphic volume form on $Y$. Then there exists a collection $\mathcal{P}$ of graded special Lagrangian submanifolds which defines a geometric stability condition $(Z, \mathcal{P})$ on $D^bF_Y$. Moreover, the mirror Atiyah flop $f^{(Y,S)}$ defines another geometric stability condition $(Z^+, \mathcal{P}^\dagger)$ with respect to $(f^{(Y,S)})^*\Omega_Y$. Finally, $(Z, \mathcal{P})$ and $(Z^+, \mathcal{P}^\dagger)$ are related by a wall-crossing in the space of Bridgeland stability conditions $\text{Stab}(D^bF_Y)$, which matches with Toda’s wall-crossing in $\text{Stab}(\mathcal{D}_{X/X})$ on the mirror [98].

Another way to see the effect of mirror Atiyah flop is by equipping the symplectic sixfold $Y$ with a Lagrangian fibration. See Theorem 1.1.2 for more details.

**Chapter 2: Mirror of Weil–Petersson metric**

The moduli space of complex structures $\mathcal{M}_{\text{cpx}}(Y)$ on a Calabi–Yau manifold $Y$ has a canonical Kähler metric, the Weil–Petersson metric. The existence of such a natural metric often implies strong results that one can not obtain by purely algebraic methods. In the case of Calabi–Yau threefold, this metric provides a fundamental differential geometric tool, the special Kähler geometry, to study mirror symmetry.

The goal of Chapter 2 is to construct the mirror object of the Weil–Petersson metric. Under mirror symmetry, $\mathcal{M}_{\text{cpx}}(Y)$ should be identified with the so-called “stringy Kähler moduli space” $\mathcal{M}_{\text{Kah}}(X)$ of a mirror Calabi–Yau manifold $X$. When $\dim(X) \leq 2$, $\mathcal{M}_{\text{Kah}}(X)$ is well-defined via Bridgeland stability conditions by a work of Bayer–Bridgeland [8]. When $\dim(X) \geq 3$, it is conjectured by Bridgeland [14, 16] that there is an embedding $\mathcal{M}_{\text{Kah}}(X) \hookrightarrow \text{Aut}(D^b(X))\backslash\text{Stab}(D^b(X))/\mathbb{C}$. In other words, the stringy Kähler moduli space is encoded in the space of Bridgeland stability conditions on the derived category.
\( \mathcal{D}^b(X) \). Our strategy therefore is to first define the Weil–Petersson geometry on the space of Bridgeland stability conditions, then restrict to the stringy Kähler moduli space.

**Definition 0.0.6** (= Definition 2.3.3). For any Calabi–Yau category \( \mathcal{D} \), we define the Weil–Petersson metric on \( \text{Stab}^+ (\mathcal{D}) / \mathbb{C} \) for an appropriate subset \( \text{Stab}^+ (\mathcal{D}) \subset \text{Stab}(\mathcal{D}) \). The metric descends to the double quotient space \( \text{Aut}(\mathcal{D}) \backslash \text{Stab}^+ (\mathcal{D}) / \mathbb{C} \).

We compute several low-dimensional examples to justify our definition of Weil–Petersson metric:

**Theorem 0.0.7** (= Example 2.3.5 and Theorem 2.1.1). We compute the following examples of Weil–Petersson metric on the space of Bridgeland stability conditions.

- Let \( E \) be an elliptic curve. Then \( \mathcal{M}_{\text{Kah}}(E) \cong \text{Aut}(\mathcal{D}^b(E)) \backslash \text{Stab}^+ (\mathcal{D}^b(E)) / \mathbb{C} \cong \text{PSL}(2, \mathbb{Z}) \backslash \mathbb{H} \); our Weil–Petersson metric coincides with the Poincaré metric on \( \mathbb{H} \).

- Let \( A = E \times E \) be the self-product of a generic elliptic curve. Then \( \mathcal{M}_{\text{Kah}}(A) \cong \overline{\text{Aut}_{\text{CY}}(\mathcal{D}^b(A))} \backslash \text{Stab}^+ (\mathcal{D}^b(A)) / \mathbb{C} \cong \text{Sp}(4, \mathbb{Z}) \backslash \mathcal{H}_2 \) is the Siegel modular variety; our Weil–Petersson metric coincides with the Bergman metric on \( \text{Sp}(4, \mathbb{Z}) \backslash \mathcal{H}_2 \).

The Weil–Petersson metric on \( \text{Aut}(\mathcal{D}) \backslash \text{Stab}^+ (\mathcal{D}) / \mathbb{C} \) is a degenerate metric in general. However, the Weil–Petersson metric on the complex moduli \( \mathcal{M}_{\text{cpx}}(Y) \) is non-degenerate. Hence we expect the non-degeneracy condition can be used to characterize the stringy Kähler moduli space \( \mathcal{M}_{\text{Kah}}(X) \).

**Conjecture 0.0.8** (= Conjecture 2.3.6). For \( \dim(X) \geq 3 \), there exists an embedding of the stringy Kähler moduli space

\[
i : \mathcal{M}_{\text{Kah}}(X) \hookrightarrow \text{Aut}(\mathcal{D}^b(X)) \backslash \text{Stab}^+ (\mathcal{D}^b(X)) / \mathbb{C}.
\]

Moreover, the pullback of our Weil–Petersson metric is a non-degenerate Kähler metric on \( \mathcal{M}_{\text{Kah}}(X) \). It should be identified with the Weil–Petersson metric on the complex moduli space \( \mathcal{M}_{\text{cpx}}(Y) \) of a mirror manifold of \( Y \).
Chapter 3: Systolic inequality on K3 surfaces

Systolic geometry studies the least length of a non-contractible loop $\text{sys}(M, g)$ in a Riemannian manifold $M$. Loewner’s torus systolic inequality states that $\text{sys}(T^2, g)^2 \leq \frac{2}{\sqrt{3}} \text{vol}(T^2, g)$ holds for any metric on the two-torus. We propose the following question that naturally generalizes Loewner’s torus systolic inequality from the perspective of Calabi–Yau geometry.

**Question 0.0.9** (= Question 3.1.2). Let $Y$ be a Calabi–Yau manifold, and let $\omega$ be a symplectic form on $Y$. Does there exist a constant $C > 0$ such that

$$\min_{L:\text{sLag}} \left| \int_L \Omega \right|^2 \leq C \cdot \left| \int_Y \Omega \wedge \overline{\Omega} \right|$$

holds for any holomorphic top form $\Omega$ on $Y$? Here “sLag” denotes the special Lagrangian submanifolds in $Y$ with respect to $\omega$ and $\Omega$.

Motivated by the connection between flat surfaces and stability conditions, as well as the conjectural description of Bridgeland stability conditions on Fukaya category by Bridgeland and Joyce [14, 16, 55], we define the categorical analogue of $\min_{L:\text{sLag}} \left| \int_L \Omega \right|$, which can also be regarded as the categorical analogue of systole (see Table 3.1).

**Definition 0.0.10** (= Definition 3.1.3). Let $\mathcal{D}$ be a triangulated category, and $\sigma$ be a Bridgeland stability condition on $\mathcal{D}$. Its systole is defined to be

$$\text{sys}(\sigma) := \min \{ \left| Z_{\sigma}(E) \right| : E \text{ is } \sigma-\text{semistable} \}.$$ 

Using the idea in Chapter 2, we also define the categorical analogue of holomorphic volume $\left| \int_Y \Omega \wedge \overline{\Omega} \right|$ of a Calabi–Yau manifold $Y$.

**Definition 0.0.11** (= Definition 3.2.7). Let $\{E_i\}$ be a basis of the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$ and let $\sigma = (Z, \mathcal{P})$ be a Bridgeland stability condition on $\mathcal{D}$. Its volume is defined to be

$$\text{vol}(\sigma) := \left| \sum_{i,j} \chi^{ij} Z(E_i) Z(E_j) \right|,$$

where $(\chi^{ij}) = (\chi(E_i, E_j))^{-1}$ is the inverse matrix of the Euler pairings.
Then one can ask the following question, which should give the same answer as Question 0.0.9 assuming mirror symmetry.

**Question 0.0.12.** Let \( X \) be a Calabi–Yau manifold and \( \mathcal{D} = \mathcal{D}^{\mathbb{R}} \text{Coh}(X) \) be its derived category of coherent sheaves. Does there exist a constant \( C > 0 \) such that

\[
\text{sys}(\sigma)^2 \leq C \cdot \text{vol}(\sigma)
\]

holds for any \( \sigma \in \text{Stab}^*(\mathcal{D}) \)? Here \( \text{Stab}^*(\mathcal{D}) \) is a subset of \( \text{Stab}(\mathcal{D}) \) whose double quotient by \( \text{Aut}(\mathcal{D}) \) and \( \mathbb{C} \) gives the stringy Kähler moduli space \( \mathcal{M}_{\text{Kah}}(X) \) of \( X \).

Question 0.0.9 and Question 0.0.12 are mirror to each other in the sense that

- in Question 0.0.9, the symplectic form \( \omega \) is fixed, and the question is asking for the supremum of the ratio \( \text{sys}^2/\text{vol} \) among all \( \Omega \in \mathcal{M}_{\text{cpx}}(Y) \); while

- in Question 0.0.12, the complex structure \( \Omega \) is fixed, and the question is asking for the supremum of the ratio \( \text{sys}^2/\text{vol} \) among all \( [\sigma] \in \text{Aut}(\mathcal{D}) \backslash \text{Stab}^*(\mathcal{D}) / \mathbb{C} \cong \mathcal{M}_{\text{Kah}}(X) \).

Note that the ratio \( \text{sys}(\sigma)^2/\text{vol}(\sigma) \) is invariant under the \( \text{Aut}(\mathcal{D}) \)-action and the free \( \mathbb{C} \)-action.

The main result in this chapter is the following theorem.

**Theorem 0.0.13 (= Theorem 3.1.5).** Let \( X \) be a K3 surface of Picard rank one, with \( \text{Pic}(X) = \mathbb{Z}H \) and \( H^2 = 2n \). Then

\[
\text{sys}(\sigma)^2 \leq (n + 1) \text{vol}(\sigma)
\]

holds for any \( \sigma \in \text{Stab}^*(\mathcal{D}) \).

**Chapter 4: Entropy of autoequivalences**

The topological entropy \( h_{\text{top}}(f) \) of an automorphism \( f \) is an important dynamical invariant that measures the complexity of the dynamical system formed by the iterations of \( f \). A fundamental theorem by Gromov–Yomdin [44, 45, 106] states that if \( X \) is a compact Kähler
manifold and \( f : X \to X \) is a holomorphic surjective map, then

\[
h_{\text{top}}(f) = \log \rho(f^*).\]

Here \( f^* \) denotes the induced linear map on \( H^*(X; \mathbb{C}) \), and \( \rho(f^*) \) is its spectral radius.

Let \( \mathcal{D} \) be a triangulated category and let \( \Phi : \mathcal{D} \to \mathcal{D} \) be an endofunctor. The notion of categorical entropy \( h_{\text{cat}}(\Phi) \) has been introduced by Dimitrov–Haiden–Katzarkov–Kontsevich [29]. It was shown by Kikuta–Takahashi [63] that if \( \mathcal{D} = \mathcal{D}b\text{Coh}(X) \) and \( \Phi = L\mathcal{L}f^* \) is induced by an automorphism \( f \) on \( X \), then \( h_{\text{cat}}(\mathcal{L}f^*) = h_{\text{top}}(f) \). They made the following conjecture which can be viewed as the categorical analogue of Gromov–Yomdin theorem.

**Conjecture 0.0.14** (Kikuta–Takahashi [63]). Let \( X \) be a smooth projective variety over \( \mathbb{C} \) and let \( \Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(X) \) be an autoequivalence. Then

\[
h_{\text{cat}}(\Phi) = \log \rho([\Phi]).\]

Here \( [\Phi] \) denotes the induced linear map on the numerical Grothendieck group of \( \mathcal{D}^b(X) \).

The main result of this chapter is the discovery of the first counterexamples of Conjecture 0.0.14.

**Theorem 0.0.15** (= Theorem 4.1.1 and Proposition 4.1.4). Let \( X \) be a Calabi–Yau hypersurface in \( \mathbb{C}\mathbb{P}^{d+1} \) and \( d \geq 4 \) be an even integer. Consider the autoequivalence \( \Phi := \mathcal{T}_{\mathcal{O}_X} \circ (-(\otimes \mathcal{O}(-1))) \) on \( \mathcal{D}^b(X) \), where \( \mathcal{T}_{\mathcal{O}_X} \) is the spherical twist [89] with respect to the structure sheaf \( \mathcal{O}_X \). Then

\[
h_{\text{cat}}(\Phi) > 0 = \log \rho([\Phi]).
\]

This gives a counterexample to Conjecture 0.0.14. In fact, \( h_{\text{cat}}(\Phi) \) is the unique positive real number \( \lambda > 0 \) satisfying \( \sum_{k \geq 1} \frac{\chi(\mathcal{O}(k))}{\lambda^k} = 1 \).

One can consider spherical twists as “hidden symmetries” on a Calabi–Yau manifold \( X \): they are not induced by automorphisms on \( X \), but rather correspond to *Dehn twists along Lagrangian spheres on a mirror Calabi–Yau* \( Y \) under mirror symmetry [89]. Theorem 0.0.15 shows that because of the existence of hidden symmetries on Calabi–Yau manifolds, the
autoequivalences on the derived category \( D^b(X) \) contain much richer dynamical information than the automorphisms on \( X \).

On the other hand, it still is interesting to characterize the autoequivalences that satisfy Conjecture 0.0.14. Along this direction, we show that the \( \mathbb{P}^d \)-twists defined by Huybrechts–Thomas [52] satisfy Conjecture 0.0.14. Note that \( \mathbb{P}^d \)-twists are mirror to Dehn twists along Lagrangian complex projective spaces.

**Theorem 0.0.16** (= Theorem 4.5.5). Let \( X \) be a smooth projective variety of dimension \( 2d \) over \( \mathbb{C} \), and let \( P_\mathcal{E} \) be the \( \mathbb{P}^d \)-twist [52] of a \( \mathbb{P}^d \)-object \( \mathcal{E} \in D^b(X) \). Then

\[
h_{\text{cat}}(P_\mathcal{E}) = 0 = \log \rho([P_\mathcal{E}])
\]

Hence Conjecture 0.0.14 holds for \( \mathbb{P}^d \)-twists.
Chapter 1

Mirror of Atiyah flop

1.1 Introduction

Flop is a fundamental operation in birational geometry. By the work of Kollár [64], any birational transformation of compact threefolds with nef canonical classes and $Q$-factorial terminal singularities can be decomposed into flops.

Atiyah flop is the most well-known among many different kinds of flops. It contracts a $(-1, -1)$ curve and resolves the resulting conifold singularity by a small blow-up, producing a $(-1, -1)$ curve in another direction, see Figure 1.1.

Figure 1.1: The Atiyah flop.

In mirror symmetry, complex and symplectic geometries are dual to each other. Flop is an important operation in complex geometry. It is natural to ask whether there is a mirror

\footnote{Co-authored with Hansol Hong, Siu-Cheong Lau and Shing-Tung Yau. Reference: [35].}
operation in symplectic geometry. In this chapter we focus on the mirror of Atiyah flop.

SYZ mirror symmetry of a conifold singularity is well-known by the works of [46, 20, 76, 2, 21, 56]. A conifold singularity is given by $u_1 v_1 = u_2 v_2$ in $\mathbb{C}^4$. There are two different choices of anti-canonical divisors which turn out to be mirror to each other, namely $D_1 = \{u_2 v_2 = 1\}$ and $D_2 = \{(u_2 - 1)(v_2 - 1) = 0\}$. Consider the resolved conifold $O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1)$, with the divisor $D_2$ deleted. Its SYZ mirror is given by the deformed conifold $\{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^\times : u_1 v_1 = z + q, u_2 v_2 = z + 1\}$. Here $q$ is the Kähler parameter of the resolved conifold, namely $q = e^{-A}$ where $A$ is the symplectic area of the $(-1, -1)$ curve in the resolved conifold. The deformed conifold contains a Lagrangian sphere whose image in the $z$-coordinate projection is the interval $[-1, -q] \subset \mathbb{C}$. The Lagrangian sphere is mirror to the holomorphic sphere in the resolved conifold.

Now take the Atiyah flop. The Kähler moduli of the resolved conifold is the punctured real line $\mathbb{R} - \{0\}$, consisting of two Kähler cones $\mathbb{R}_+$ and $\mathbb{R}_-$ of the resolved conifold and its flop respectively. $A$ serves as the standard coordinate and flop takes $A \in \mathbb{R}_+$ to $-A \in \mathbb{R}_-$. Thus the Atiyah flop amounts to switching $A$ to $-A$, or equivalently $q$ to $q^{-1}$. As a result, the SYZ mirror changes from $\{u_1 v_1 = z + q, u_2 v_2 = z + 1\}$ to $\{u_1 v_1 = z + q^{-1}, u_2 v_2 = z + 1\}$.

However the above two manifolds are symplectomorphic to each other, and hence they are just equivalent from the viewpoint of symplectic geometry. Unlike Atiyah flop in complex geometry, the mirror operation does not produce a new symplectic manifold. It is not very surprising since symplectic geometry is much softer than complex geometry.

In contrast to complex geometry, the mirror flop is just a symplectomorphism rather than a new symplectic manifold. First observe that this symplectomorphism is non-trivial (Section 1.4.1).

**Proposition 1.1.1.** Given a symplectic threefold $(X, \omega)$ and a Lagrangian three-sphere $S \subset X$, we have another symplectic threefold $(X^\dagger, \omega^\dagger)$ with a corresponding Lagrangian three-sphere $S^\dagger \subset X^\dagger$, together with a symplectomorphism $f^{(X, S)} : (X, \omega) \to (X^\dagger, \omega^\dagger)$. It has the property that $f^{(X^\dagger, S^\dagger)} \circ f^{(X, S)} = \tau_S^{-1}$, where $\tau_S$ is the Dehn twist along the Lagrangian sphere $S$.

We shall regard $X$ and $X^\dagger$ as the same symplectic manifold using the above symplecto-
morphism $f^{(X,S)}$.

We need to endow a symplectic threefold with additional geometric structures in order to make it more rigid, so that the effect of the mirror flop can be seen. In the above local case, $\{u_1v_1 = z + q, u_2v_2 = z + 1\}$ and $\{u_1v_1 = z + q^{-1}, u_2v_2 = z + 1\}$ simply have different complex structures. However in general requiring the existence of a complex structure on a symplectic manifold would be too restrictive. Friedman [37] and Tian [97] showed that there are topological obstructions to complex smoothing of conifold points; Smith-Thomas-Yau [92] found the mirror statement for topological obstructions to Kähler resolution of conifold points.

In this chapter, we consider two kinds of geometric structures, namely Lagrangian fibrations, and Bridgeland stability conditions on the derived Fukaya category. First consider a symplectic threefold $X$ equipped with a Lagrangian fibration $\pi : X \to B$. Let $S \subset X$ be a Lagrangian sphere. We assume that $\pi$ around $S$ is given by a local model of Lagrangian fibration on the deformed conifold, where $S$ is taken as the vanishing sphere under a conifold degeneration, see Definition 1.4.3. We call such a fibration to be conifold-like around $S$. Then we make sense of the mirror flop by doing a local surgery around $S$ and obtain another Lagrangian fibration $\pi^\dagger : X \to B$. ($X$ and $X^\dagger$ have been identified by the above symplectomorphism $\rho_{X,S}$.)

**Theorem 1.1.2.** Given a symplectic threefold $(X, \omega)$ with a Lagrangian fibration $\pi : X \to B$ which is conifold-like around a Lagrangian three-sphere $S \subset X$, there exists another Lagrangian fibration $\pi^\dagger : X \to B$ with the following properties.

1. $\pi^\dagger$ is also conifold-like around $S$.

2. The images of $S$ under $\pi$ and $\pi^\dagger$ are the same, denoted by $S$. They are one-dimensional affine submanifolds in $B$ away from discriminant locus.

3. $\pi^\dagger = \pi$ outside a tubular neighborhood of $S$. In particular the affine structures on $B$ induced from $\pi$ and $\pi^\dagger$ are identical away from a neighborhood of $S$.

4. The induced orientations on $S$ from $\pi$ and $\pi^\dagger$ are opposite to each other.
We call the change from $\pi$ to $\pi^+$ to be the A-flop of a Lagrangian fibration along $S$. As a compact example, consider the Schoen’s Calabi–Yau, which admits a conifold-like Lagrangian fibration around certain Lagrangian spheres by the work of Gross [47] and Castaño-Bernard and Matessi [76]. Then we can apply the A-flop to obtain other Lagrangian fibrations.

More generally we can consider the effect of A-flop along $S$ on Lagrangian submanifolds other than Lagrangian torus fibers. Given a Lagrangian submanifold $L \subset X$ which has $T^2$-symmetry around $S$ (see Definition 1.4.7), we can construct another Lagrangian submanifold $L^+$ (which also has $T^2$-symmetry around $S$) which we call to be the A-flop of $L$, with the property that $(L^+)^+$ equals to the inverse Dehn twist of $L$ along $S$.

Then we can take A-flop of special Lagrangian submanifolds with respect to a certain holomorphic volume form (if it exists). Formally we start with a Bridgeland stability condition $(Z, P)$ [14] on the derived Fukaya category, where $Z$ is a homomorphism of the K group to $\mathbb{C}$, and $P$ is a collection of objects in the derived Fukaya category which are said to be stable. A stability condition $(Z, P)$ is said to be geometric if there exists a holomorphic volume form $\Omega$ such that $Z$ is given by the period $\int \Omega$ and $P$ is a collection of graded special Lagrangians with respect to $\Omega$. A-flop should be understood as a change of stability conditions $(Z, P) \mapsto (Z^+, P^+)$.

In this chapter we realize the above for the local deformed conifold in Section 1.6. We obtain the following theorem in Section 1.6.6.

**Theorem 1.1.3.** Let $X$ be the deformed conifold $\{u_1v_1 = z + q, u_2v_2 = z + 1, z \neq 0\}$ (where $q \neq 1$). Equip $X$ with the holomorphic volume form $\Omega = dz \wedge du_1 \wedge du_2$. There exists a collection $P$ of graded special Lagrangians which defines a geometric stability condition $(Z, P)$ on $X$. Moreover the flop $(Z^+, P^+)$ also defines a geometric stability condition with respect to $(f^{(X,S)})^*\Omega_{X^+}$ where $f^{(X,S)} : X \rightarrow X^+ = \{u_1v_1 = z + 1, u_2v_2 = z + 1/q : z \neq 0\}$ is the symplectomorphism in Proposition 1.1.1 (and $\Omega_{X^+} = dz \wedge du_1 \wedge du_2$ on $X^+$).

Stability conditions for the derived Fukaya category were constructed for the $A_n$ case by Thomas [94], for certain local Calabi–Yau threefolds associated to quadratic differentials by
Bridgeland–Smith [17, 91], and for punctured Riemann surfaces with quadratic differentials by Haiden–Katzarkov–Kontsevich [48]. In this chapter we construct stability conditions on the derived Fukaya category of the deformed conifold by applying the mirror functor construction in [24, 23]; in the mirror side we use the results of Nagao–Nakajima [74] about stability conditions on the noncommutative resolved conifold (see Theorem 1.6.9).

**Theorem 1.1.4** (see Theorem 1.6.4). The mirror construction in [23] applied to the deformed conifold $X$ produces the noncommutative resolved conifold $\mathcal{A}$ given by Equation (1.15). In particular, there is a natural equivalence of triangulated categories

$$\Psi : D^b \mathcal{F} \rightarrow D^b_{\text{nil mod}} \mathcal{A}$$  \hspace{1cm} (1.1)

where $\mathcal{F}$ is a subcategory of $\text{Fuk}(X)$ generated by Lagrangians spheres, and $D^b_{\text{nil mod}} \mathcal{A}$ is a subcategory of $D^b \text{mod} \mathcal{A}$ consisting of modules with nilpotent cohomology.

The relation between the mirror construction in [23] and the SYZ construction is summarized in Figure 1.2. The SYZ construction uses Lagrangian torus fibration coming from degeneration to the large complex structure limit. The noncommutative mirror construction in [23] uses Lagrangian vanishing spheres coming from degeneration to the conifold point.

![Figure 1.2](image)

**Figure 1.2:** The local conifold is self-mirror. More precisely for the local conifold, a resolution and its flop are equivalent, so the upper hemisphere should be identified with the lower hemisphere.

We shall prove that stable modules in $D^b_{\text{nil mod}} \mathcal{A}$ with respect to a certain stability condition can be obtained as transformations of special Lagrangians under (1.1); as a result the corresponding stability condition on $D^b \mathcal{F}$ is geometric.
1.2 Review on flops and Bridgeland stability conditions

In this section, we recall the results by Toda which relate flops with wall-crossings in the space of Bridgeland stability conditions on certain triangulated categories. For more details and proofs, see [99].

1.2.1 Bridgeland stability conditions and crepant small resolutions

Let \( f : \hat{Y} \to Y \) be a crepant small resolution in dimension three and \( C \) the exceptional locus, which is a tree of rational curves \( C = C_1 \cup \cdots \cup C_N \).

Define the triangulated subcategory \( D_{\hat{Y}/Y} \subseteq D^b(\hat{Y}) \) to be
\[
D_{\hat{Y}/Y} := \{ E \in D^b(\hat{Y}) \mid \text{Supp}(E) \subseteq C \}. \tag{1.2}
\]

Let \( \text{Per}(\hat{Y}/Y) \subseteq D^b(\hat{Y}) \) \((p = 0, -1)\) be the abelian categories of perverse coherent sheaves introduced by Bridgeland [13], and

\[
\text{Per}(D_{\hat{Y}/Y}) := \text{Per}(\hat{Y}/Y) \cap D_{\hat{Y}/Y}.
\]

Proposition 1.2.1 ([28]). The abelian categories \( ^0\text{Per}(D_{\hat{Y}/Y}) \) and \( ^{-1}\text{Per}(D_{\hat{Y}/Y}) \) are the hearts of certain bounded t-structures on \( D_{\hat{Y}/Y} \), and are finite-length abelian categories. The simple objects in \( ^0\text{Per}(D_{\hat{Y}/Y}) \) and \( ^{-1}\text{Per}(D_{\hat{Y}/Y}) \) are \( \omega_C[1], \mathcal{O}_{C_1}(-1), \ldots, \mathcal{O}_{C_N}(-1) \) and \( \mathcal{O}_C, \mathcal{O}_{C_1}(-1)[1], \ldots, \mathcal{O}_{C_N}(-1)[1] \) respectively.

Theorem 1.2.2 ([13][22]). Let \( g : \hat{Y} \to Y \) be the flop of \( f \), and \( \phi : \hat{Y} \to \hat{Y}^+ \) be the canonical birational map. Then the Fourier-Mukai functor with the kernel \( \mathcal{O}_{\hat{Y} \times \hat{Y}^+} \in D^b(\hat{Y} \times \hat{Y}^+) \) is an equivalence
\[
\Phi_{\hat{Y} \to \hat{Y}^+} : D^b(\hat{Y}) \xrightarrow{\cong} D^b(\hat{Y}^+).
\]
This equivalence restricts to an equivalence \( D_{\hat{Y}/Y} \xrightarrow{\cong} D_{\hat{Y}^+/Y} \) and takes \( ^0\text{Per}(\hat{Y}/Y) \) to \( ^{-1}\text{Per}(\hat{Y}^+/Y) \).

Such an equivalence is called standard in [99].

Let \( \text{FM}(\hat{Y}) \) be the set of pairs \( (W, \Phi) \), where \( W \to Y \) is a crepant small resolution, and \( \Phi : D^b(W) \to D^b(\hat{Y}) \) can be factorized into standard equivalences and the auto-equivalences
given by tensoring line bundles. For each \((W, \Phi) \in \text{FM}(\hat{Y})\), there is an associated open subset

\[ U(W, \Phi) \subset \text{Stab}_n(\hat{Y}/Y) \]

of the space of normalized Bridgeland stability conditions on \(\mathcal{D}_{\hat{Y}/Y}\). A Bridgeland stability condition on \(\mathcal{D}_{\hat{Y}/Y}\) is called normalized if the central charge \(Z([O_x])\) of the skyscraper sheaf at each \(x \in C\) is \(-1\).

Assume in addition that there is a hyperplane section in \(Y\) containing the singular point such that its pullback in \(\hat{Y}\) is a smooth surface, Toda proved the following theorem.

**Theorem 1.2.3** ([99]). Let \(\text{Stab}_n^0(\hat{Y}/Y)\) be the connected component of \(\text{Stab}_n(\hat{Y}/Y)\) containing the standard region \(U(\hat{Y}, \Phi = \text{id}_{\mathcal{D}_{\hat{Y}}})\). Define the following union of chambers

\[ \mathcal{M} := \bigcup_{(W, \Phi) \in \text{FM}(\hat{Y})} U(W, \Phi). \]

Then \(\mathcal{M} \subset \text{Stab}_n^0(\hat{Y}/Y)\), and any two chambers are either disjoint or equal. Moreover, \(\overline{\mathcal{M}} = \text{Stab}_n^0(\hat{Y}/Y)\).

In other words, we can obtain the whole connected component \(\text{Stab}_n^0(\hat{Y}/Y)\) from the standard region \(U(\hat{Y}, \text{id})\) by sequence of flops and tensoring line bundles.

### 1.2.2 The conifold

Let \(Y = \text{Spec } \mathbb{C}[[x, y, z, w]]/(xy - zw)\) and \(f : \hat{Y} \to Y\) be the blowing up at the ideal \((x, z)\). As computed in [99],

\[ \text{Stab}_n^0(\hat{Y}/Y)/\text{Aut}^0(\mathcal{D}_{\hat{Y}/Y}) \cong \mathbb{P}^1 - \{ 3 \text{ points} \}. \]

Let \(\hat{Y}^+ \to Y\) be the blowing up at the other ideal \((x, w)\). Then the three removed points correspond to the large volume limit points of \(\hat{Y}\) and \(\hat{Y}^+\), and the conifold point.

More precisely, \(\mathbb{P}^1 - \{ 3 \text{ points} \}\) is obtained by gluing the upper and lower half complex planes \(\mathbb{H}, \mathbb{H}^+\), and the real line with the origin removed. The hearts of the Bridgeland stability conditions in \(\mathbb{H}\) and \(\mathbb{H}^+\) are given by \(\text{Coh}_{\hat{Y}/Y}\) and \(\text{Coh}_{\hat{Y}^+/Y}\) respectively. The
heart of the Bridgeland stability conditions on the real line is given by the perverse heart

\[ \Per^0(D_{\hat{Y}/Y}) \cong \Per(D_{\hat{Y}^+ / Y}). \]

Let \( C, C^+ \) be the exceptional curves of \( \hat{Y} \to Y, \hat{Y}^+ \to Y \) respectively. Then the equivalence \( D_{\hat{Y}/Y} \to D_{\hat{Y}^+ / Y} \) satisfies

1. \( \Phi(O_C(-1)) = O_{C^+}(-1)[1] \).
2. \( \Phi(O_C(-2)[1]) = O_{C^+} \).
3. For \( x \in C \), the cohomology of \( E := \Phi(O_x) \in D_{\hat{Y}^+ / Y} \) vanish except for \( H^0(E) = O_{C^+} \) and \( H^{-1}(E) = O_{C^+}(-1) \).

One can observe the following wall-crossing phenomenon: the skyscraper sheaves \( O_x \in D_{\hat{Y}/Y} \) are stable objects with respect to the stability conditions on the upper half plane \( H \), but are unstable in \( H^\dagger \). In fact, its image under \( \Phi \) is a two term complex \( E \) that fits into the following exact triangle:

\[ O_{C^+}(-1)[1] \to E \to O_{C^+} \]

(1.3)

Note that the usual skyscraper sheaf at a point in \( C^\dagger \) can be obtained by switching the first and the third terms in (1.3).

\textbf{Remark 1.2.4.} It is well-known that if \( C \) is a \((-1,-1)\)-curve, then the ‘flop-flop’ functor is the same as the inverse of the spherical twist by \( O_C(-1) \), i.e. \( \Phi_{\hat{Y} \to \hat{Y}^+} \circ \Phi_{\hat{Y}^+ \to \hat{Y}} = T_{O_C(-1)} \). Proposition 1.1.1 is the mirror statement of this fact.

1.3 Review on the SYZ mirror of the conifold

SYZ mirror construction for toric Calabi–Yau manifolds was carried out in [20] using the wall-crossing techniques of [6]. The reverse direction, namely SYZ construction for blow-up of \( V \times C \) along a hypersurface in a toric variety \( V \) was carried out by [2]. In this section we recall the construction for the conifold \( Y = \{(u_1, v_1, u_2, v_2) \in C^4 : u_1v_1 = u_2v_2 \} \) as a special
case in [20, 2]. The statement is that $Y - \{u_2v_2 = 1\}$ is mirror to $Y - (\{u_2 = 1\} \cup \{v_2 = 1\})$.

The study motivates the definition of A-flop for Lagrangian fibrations in the next section.

The resolved conifold $\hat{Y} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ is obtained from a small blowing-up of the conifold point $(u_1, v_1, u_2, v_2) = 0$. It is a toric manifold equipped with a toric Kähler form. We have the $T^2$-action on $\hat{Y}$ given by $(\lambda_1, \lambda_2) \cdot (u_1, v_1, u_2, v_2) = (\lambda_1 u_1, \lambda_1^{-1} v_1, \lambda_2 u_2, \lambda_2^{-1} v_2)$, and we denote the corresponding moment map by $(\mu_1, \mu_2) : \hat{Y} \to \mathbb{R}^2$. Then from the works of Ruan [83], Gross [46] and Goldstein [42], there is a Lagrangian fibration

$$(\mu_1, \mu_2, |zw - 1|) : \hat{Y} \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}.$$ 

It serves as one of the local models of Lagrangian fibrations which were used by Castaño-Bernard and Matessi [75, 76] to build up global fibrations from a tropical base manifold.

![Figure 1.3: The base and discriminant loci of Lagrangian fibrations in conifold transition.](image)

The discriminant locus of this fibration is contained in the hyperplane

$$\{(x_1, x_2, x_3) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0} : x_3 = 1\},$$

see the top left of Figure 1.3. This hyperplane is known as the wall for open Gromov–Witten invariants of torus fibers as it contains images of holomorphic discs of Maslov index zero. By studying wall-crossing of holomorphic discs emanated from infinity divisors (of a compactification of $\hat{Y}$), [20] constructed the SYZ mirror of $\hat{Y} - \{zw = 1\}$.

**Theorem 1.3.1** (A special case in [20] and [2]). The SYZ mirror of $\hat{Y} - \{u_2v_2 = 1\}$ is

$$\{(u_1, v_1, u_2, v_2) : u_1, v_1 \in \mathbb{C}, u_2, v_2 \in \mathbb{C}^\times : u_1v_1 = 1 + u_2 + v_2 + qu_2v_2\}$$
where \( q = \exp - (\text{complexified symplectic area of the zero section } \mathbb{P}^1 \text{ of } \hat{Y}). \)

Take the change of coordinates \( \tilde{u}_2 = q^{1/2} u_2 + 1/q^{1/2}, \tilde{v}_2 = q^{1/2} v_2 + 1/q^{1/2}. \) (Here we have fixed a square root of \( q. \) Then the equation becomes \( u_1 v_1 = u_2 v_2 + 1 - 1/q \) and the divisors are \( u_2 = 1/q^{1/2} \) and \( v_2 = 1/q^{1/2}. \) Further rescaling \((u_1, v_1, u_2, v_2)\) by \( q^{1/2}, \) the SYZ mirror is the deformed conifold

\[
\hat{Y} = \{(u_1, v_1, u_2, v_2) \in \mathbb{C}^4 : u_1 v_1 = u_2 v_2 + (q - 1)\}
\]

with the divisor \( \{(u_2 - 1)(v_2 - 1) = 0\} \) deleted. To conclude, we have the mirror pair \( \hat{Y} - \{ u_2 v_2 = 1 \} \) and \( \hat{Y} - \{(u_2 - 1)(v_2 - 1) = 0\}. \)

Taking the Atiyah flop of the \((-1, -1)\) curve in \( \hat{Y} \) amounts to switching \( q \) to \( 1/q. \) As a result, the mirror of \( \hat{Y} - \{ z w = 1 \} \) changes from

\[
\{u_1 v_1 = u_2 v_2 + (q - 1)\} - \{(u_2 - 1)(v_2 - 1) = 0\}
\]

to

\[
\{u_1 v_1 = u_2 v_2 + (1/q - 1)\} - \{(u_2 - 1)(v_2 - 1) = 0\}
\]

under flop on \( \hat{Y}. \) However changing equation just results in a symplectomorphism. Thus unlike the flop of a \((-1, -1)\) curve, the mirror flop (of a Lagrangian vanishing sphere in conifold degeneration) does ‘nothing’ to the symplectic manifold. We need additional geometric structures to detect the mirror flop. For this local model it is obvious that they can be distinguished by complex structures. In general we would like to consider geometric structures in the symplectic category. This will be further studied in the next section.

We can also consider a different relative Calabi–Yau so that Lagrangian spheres can be seen more easily. First rescale \((u_1, v_1, u_2, v_2)\) so that \( \tilde{Y} \) is given as

\[
u_1 v_1 - u_2 v_2 = q^{1/2} - q^{-1/2}.
\]

Rewrite \( \tilde{Y} \) as a double conic fibration,

\[
\tilde{Y} = \{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^5 : u_1 v_1 = z + q^{1/2}, u_2 v_2 = z + q^{-1/2}\}.
\]
It is equipped with the standard symplectic form from $C^5$. If we flop $\hat{Y}$, the mirror $\tilde{Y}$ becomes $\{u_2v_2 = z + q^{-1/2}; u_1v_1 = z + q^{1/2}\}$. We take the complement $\tilde{Y} - \{z = c\}$ where $c \in C - \{-q^{1/2}, -q^{-1/2}\}$.

We have the Lagrangian fibration

$$(x_1, x_2, x_3) = (|u_1|^2 - |v_1|^2, |u_2|^2 - |v_2|^2, |z - c|) : \tilde{Y} \to \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$$

where the boundary divisor is exactly $\{z = c\}$. The discriminant loci are $\{x_1 = 0, x_3 = |q^{1/2} + c|\}$ and $\{x_2 = 0, x_3 = |q^{-1/2} + c|\}$ contained in the walls $\{x_3 = |q^{1/2} + c|\}$ and $\{x_3 = |q^{-1/2} + c|\}$ respectively, see the top right of Figure 1.3. By [2, Theorem 11.1] (or SYZ in [68] by Minkowski decompositions), the resulting SYZ mirror is the following.

**Theorem 1.3.2** (A special case in [2] and [68]). The SYZ mirror of $\tilde{Y} - \{z = c\}$ is $\hat{Y} - (\{u_2 = 1\} \cup \{v_2 = 1\})$.

Denote $a = -q^{-1/2}$ and $b = -q^{1/2}$, and without loss of generality assume that $a, b$ are real, $c = 0$ and $a < b < 0$. Consider the Fukaya category of $\tilde{Y} - \{z = 0\}$ generated by the two Lagrangian spheres $S_1$ and $S_2$, where

$$S_0 = \{z = -t, |u_1| = |v_1|, |u_2| = |v_2| : a \leq t \leq b\},$$

$$S_1 = \{z = \exp(t \zeta_1 + (1 - t) \zeta_0) : t \in [0, 1], |u_1| = |v_1|, |u_2| = |v_2|\}$$

where $\zeta_0 = \log |a| - \pi i$ and $\zeta_1 = \log |b| + \pi i$. $S_0$ and $S_1$ are oriented by $dt \wedge d\theta_1 \wedge d\theta_2$ where $\theta_1, \theta_2$ are the arguments of $u_1, u_2$ respectively. They are special Lagrangians and in particular graded by a suitable holomorphic volume form. (We shall go back to this point in more detail in Section 1.5.) Figure 1.4 shows $S_0$ in the picture of double conic fibration.

Chan–Pomerleano–Ueda [21] proved homological mirror symmetry for the mirror pair $(\tilde{Y} - \{z = 0\}, \hat{Y} - (\{u_2 = 1\} \cup \{v_2 = 1\})$ making use of the SYZ transformation. The result is the following.

**Theorem 1.3.3** (Theorem 1.2 and 1.3 of [21]). There is an equivalence between the derived wrapped Fukaya category of $\tilde{Y}_0 := \tilde{Y} - \{z = 0\}$ and the derived category of coherent sheaves of $\hat{Y}_0 := \hat{Y} - (\{u_2 = 1\} \cup \{v_2 = 1\})$. 

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\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Lagrangian $S^3$ seen from the double conic fibration.}
\end{figure}

**Remark 1.3.4.** The spheres $S_0$ and $S_1$ here were denoted as $S_1$ and $S_0$ in [21] respectively.

In Section 1.5 and 1.6, $\tilde{Y}_0$ will be denoted as $X_{t=0}$ which appears as a member in a family of symplectic manifolds $X_t$.

Restricting to the Fukaya subcategory consisting of $S_0, S_1$, we have the equivalence between $D^b(S_0, S_1)$ and $\mathcal{D}_{\tilde{Y}/Y}$ (1.2). We will revisit this equivalence in Section 1.5 (see Theorem 1.5.3 for more details on the equivalence). Then we will compare the flop on B-side and the corresponding operation on A-side (to be constructed below) using this.

On the other hand, we can take the approach of [23] to construct the noncommutative mirror of $\tilde{Y}_0$. From homological mirror symmetry between $\tilde{Y}_0$ and its noncommutative mirror, we obtain stability conditions on the derived Fukaya category generated by $S_0$ and $S_1$ in Section 1.6. We will show that stable objects are special Lagrangian submanifolds.

### 1.4 A-flop in symplectic geometry

#### 1.4.1 Mirror of Atiyah flop as a symplectomorphism

Let $(X, \omega)$ be a symplectic threefold and $S$ a Lagrangian sphere of $X$. By Weinstein neighborhood theorem, a neighborhood of $S \subset X$ can always be identified symplectomor-
phically with a neighborhood of $S \subset T^*S$, which can be identified with \{(u_1, v_1, u_2, v_2) \in \mathbb{C}^4 : u_1v_1 - u_2v_2 = \epsilon\} = \{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^5 : u_1v_1 = z + \epsilon, u_2v_2 = z\} for some $\epsilon > 0$, where $\omega$ is given by the restriction of the standard symplectic form on $\mathbb{C}^4$. This identification is adapted to conifold degeneration at the limit $\epsilon \to 0$.

In other words, we take a conifold-like chart in the following sense.

**Definition 1.4.1.** A conifold-like chart around $S$ is $(U, i)$, where $U$ is an open neighborhood of $S$ and $i : U \hookrightarrow \mathbb{C}^x \times \mathbb{C}^4$ is a symplectic embedding (where $\mathbb{C}^x \times \mathbb{C}^4$ is equipped with the standard symplectic form) such that the following holds.

1. The image of $U$ under the embedding is given by

\[
\begin{aligned}
  u_1v_1 &= z - a, \\
  u_2v_2 &= z - b
\end{aligned}
\]  

for some real numbers $a < b$, where $|z - \frac{a + b}{2}| < R$ for some fixed $R > \frac{b - a}{2}$, and $|u_1|^2 - |v_1|^2 < L$, $|u_2|^2 - |v_2|^2 < L$ for some fixed $L > 0$. Here $z$ is the coordinate of $\mathbb{C}^x$ and $u_1, v_1, u_2, v_2$ are the coordinates of $\mathbb{C}^4$. We will also denote the image by $U$ for simplicity.

2. The Lagrangian sphere $i(S)$ is given by \{|u_1| = |v_1|, |u_2| = |v_2|, z \in [a, b]\}.

We will simply identify $U$ with its image under $i$. Let

\[
V = \left\{ u_1v_1 = z - a, u_2v_2 = z - b, \left| z - \frac{a + b}{2} \right| \leq R - \epsilon, |u_i|^2 - |v_i|^2 \leq L - \epsilon \text{ for } i = 1, 2 \right\} \subset U,
\]

\[
V' = \left\{ u_1v_1 = z - a, u_2v_2 = z - b, \left| z - \frac{a + b}{2} \right| < R - 2\epsilon, |u_i|^2 - |v_i|^2 < L - 2\epsilon \text{ for } i = 1, 2 \right\} \subset V
\]

for $\epsilon > 0$ sufficiently small. We have a diffeomorphism from $U - V$ to the corresponding open subset of

\[
U^+ := \left\{ u_1v_1 = z - b, u_2v_2 = z - a, \left| z - \frac{a + b}{2} \right| < R, \text{ for } i = 1, 2 \right\}
\]

defined by $z \mapsto z$, $(u_1, v_1) \mapsto \left( \frac{z - b}{z - a} \right)^{1/2} (u_1, v_1)$, $(u_2, v_2) \mapsto \left( \frac{z - a}{z - b} \right)^{1/2} (u_2, v_2)$. (We choose a branch of the square root. It is well-defined since $a, b \notin U - V$.) By Moser argument, we
can cook up a symplectomorphism isotopic to this diffeomorphism. Thus we have fixed a symplectomorphism \( \rho \) from \( U - V \) to an open subset of \( U^\dagger \).

In analogous to a flop along a \((-1, -1)\) curve in complex geometry, we define another symplectic threefold \((X', \omega')\) by gluing \((X - V, \omega)\) with a suitable open subset of \( U^\dagger \) by the above symplectomorphism \( \rho \) on \( U - V \). By construction (an open subset of) \( U^\dagger \) is a conifold chart of \( X^\dagger \) around the Lagrangian \( S^\dagger \subset U^\dagger \) defined by \( \{|u_1| = |v_1|, |u_2| = |v_2|, z \in [a, b]\} \).

However, unlike the flop along a \((-1, -1)\) holomorphic sphere, \((X', \omega')\) is just symplectomorphic to the original \((X, \omega)\), since the gluing map \( \rho \) can be extended to \( U \to U^\dagger \). Let

\[
\tilde{\psi}_\pm : \{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^5 : u_1v_1 = z - a, u_2v_2 = z - b\} \to \{u_1v_1 = z - b, u_2v_2 = z - a\}
\]

be defined by \((u_1, v_1, u_2, v_2, z) \mapsto (\pm u_1, \pm v_1, \pm u_2, \pm v_2, -z + a + b)\) respectively. It commutes with the \( T^2 \) action

\[
(\lambda_1, \lambda_2) \cdot (u_1, v_1, u_2, v_2, z) = (\lambda_1 u_1, \lambda_1^{-1} v_1, \lambda_2 u_2, \lambda_2^{-1} v_2, z)
\]

and hence descends to the symplectic reduction, which is simply rotating the \( z \)-plane by \( \pi \) around \((a + b)/2\). Let \( \psi \) be the restriction of \( \tilde{\psi}_+ \) to \( V' \). Then we have a symplectomorphism \( U \to U^\dagger \) by interpolating between the gluing map \( \rho \) and \( \psi \) in the region \( R - 2\epsilon < |z - \frac{a+b}{2}| < R - \epsilon \), \( L - 2\epsilon < |u_1|^2 - |v_1|^2 \). Namely we take a diffeomorphism which equals to \( \psi \) on \( V' \), and is given by

\[
z \mapsto e^{\pi i f(|z-(a+b)/2|)}(z - (a + b)/2) + (a + b)/2
\]

\[
(u_1, v_1) \mapsto \left(\frac{e^{\pi i f(|z-(a+b)/2|)}(z - (a + b)/2) + (a - b)/2}{z - a}\right)^{1/2} (u_1, v_1)
\]

\[
(u_2, v_2) \mapsto \left(\frac{e^{\pi i f(|z-(a+b)/2|)}(z - (a + b)/2) + (b - a)/2}{z - b}\right)^{1/2} (u_2, v_2)
\]

on \( U - V \). Here \( f(r) \) is a decreasing function valued in \([0, 1]\) which equals to 1 for \( r < R - 2\epsilon \) and equals to 0 for \( r > R - \epsilon \). The square root \( z^{1/2} \) is taken for the branch \( 0 < \text{arg}(z) \leq \pi \).
By Moser argument we have a symplectomorphism isotopic to this, and \( \rho \) is the restriction to \( U - V \).

In conclusion, given a symplectic manifold \( (X, \omega) \) and a conifold-like chart around a Lagrangian sphere \( S \), we have a symplectomorphism \( f^{(X,S)} : (X, \omega) \to (X^+, \omega^+) \) by a surgery in analogous to flop in complex geometry. The operation does not produce a new symplectic manifold because symplectic geometry is too soft.

If we do the operation twice, we obtain \( X^{++} \) which is canonically identified with \( X \) as follows. \( X^{++} \) is glued from \( X - V \) and \( U = U^{++} \) by \( \rho^+ \circ \rho \). The composition of \( z \mapsto z, (u_1, v_1) \mapsto \left( \frac{z-b}{z-a} \right)^{1/2} (u_1, v_1), (u_2, v_2) \mapsto \left( \frac{z-g}{z-t} \right)^{1/2} (u_2, v_2) \) and \( z \mapsto z, (u_1, v_1) \mapsto \left( \frac{z-g}{z-t} \right)^{1/2} (u_1, v_1), (u_2, v_2) \mapsto \left( \frac{z-b}{z-a} \right)^{1/2} (u_2, v_2) \) is simply identity. Hence the gluing \( \rho^+ \circ \rho = \text{Id} \) and \( X^{++} = X \).

Below we see that doing the above operation twice produces the Dehn twist along the Lagrangian sphere \( S \), which induces a non-trivial automorphism on the Fukaya category.

**Proposition 1.4.2** (same as Proposition 1.1.1). \( f^{(X^+, S)} \circ f^{(X, S)} : X \to X^{++} = X \) equals to the inverse of the Dehn twist of \( X \) along \( S \).

**Proof.** \( f^{(X^+, S)} \circ f^{(X, S)} : X \to X^{++} \) is given as follows. Write \( X = X^{++} = (X - V) \cup_{\text{Id}} U \). The map is identity on \( X - V \). In \( V' \subset U \) it is given by \( \psi^2 \) which maps \( u_i \mapsto -u_i, v_i \mapsto -v_i, z \mapsto z \), and in particular is the antipodal map on the three-sphere

\[
\{u_1v_1 = z - a, u_2v_2 = z - b, z \in [b, a], |u_i| = |v_i| \text{ for } i = 1, 2 \} \subset V'.
\]

On \( U - V' \) it is isotopic to

\[
z \mapsto e^{2\pi i f(|z-(a+b)/2|)}(z - (a + b)/2) + (a + b)/2
\]

\[
(u_1, v_1) \mapsto \left( \frac{e^{2\pi i f(|z-(a+b)/2|)}(z - (a + b)/2) + (b - a)/2}{z - a} \right)^{1/2} (u_1, v_1)
\]

\[
(u_2, v_2) \mapsto \left( \frac{e^{2\pi i f(|z-(a+b)/2|)}(z - (a + b)/2) + (a - b)/2}{z - b} \right)^{1/2} (u_2, v_2),
\]

where \( f(r) \) is a decreasing function valued in \([0, 1]\) which equals to 1 for \( r < R - 2\epsilon \) and equals to 0 for \( r > R - \epsilon \). The square root \( z^{1/2} \) is taken for the branch where \( 0 < \arg(z) \leq 2\pi \).
Thus we see that it is the inverse of the Dehn twist.

\[ \square \]

### 1.4.2 Lagrangian fibrations

We see from the last section that the mirror of the Atiyah flop surgery does not produce a new symplectic manifold unfortunately. We need additional geometric structures in order to distinguish \( X^\dagger \) from \( X \). In this section we consider Lagrangian fibrations. Conceptually it can be understood as a ‘real polarization’, playing the role of the complex polarization (namely the complex structure) for flop of a \((-1,-1)\) curve.

From now on we identify \( X \) and \( X^\dagger \) as the same symplectic manifold using the symplectomorphism \( f^{(X,S)} \).

Let \( \pi : X \to B \) be a Lagrangian torus fibration. We consider a conifold degeneration of \( X \) with a vanishing sphere \( S \), such that the Lagrangian fibration around \( S \) is like the one on the deformed conifold [46, 42].

**Definition 1.4.3.** Assume the notations in Section 1.4.1. A Lagrangian fibration \( \pi \) is said to be of conifold-like if we have the commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{i} & i(U) \\
\downarrow \pi & & \downarrow (|z-c|^2(|u_1|^2-|v_1|^2), |u_2|^2-|v_2|^2)) \\
f(U) & \cong & I \times (-L,L) \times (-L,L)
\end{array}
\]

where \( |c - \frac{a+b}{2}| > R \) and \( |c - a| \neq |c - b| \), and \( I \) is a certain open interval.

**Theorem 1.4.4** (Theorem 1.1.2 in the Introduction). Given a symplectic threefold \((X, \omega)\) with a Lagrangian fibration \( \pi : X \to B \) which is conifold-like around a Lagrangian three-sphere \( S \subset X \), there exists a Lagrangian fibration \( \pi^{\dagger} : X \to B \) with the following properties.

1. \( \pi^{\dagger} \) is also conifold-like around \( S \).

2. The images of \( S \) under \( \pi \) and \( \pi^{\dagger} \) are the same, denoted by \( S \). It is a one-dimensional affine submanifold in \( B \) away from discriminant locus.
3. \( \pi^\dagger = \pi \) outside a neighborhood \( V \supset S \) and \( V \subset U \). In particular the affine structures on \( B \) induced from \( \pi \) and \( \pi^\dagger \) are identical away from a neighborhood of \( S \).

4. From above, there is a canonical correspondence between orientations of regular fibers of \( \pi \) and that of \( \pi^\dagger \). Fix an orientation of torus fibers of \( \pi \) over the image of \( U \), and an orientation of a regular fiber of \( \pi|_S \) (which is topologically \( T^2 \)). Then the induced orientations on \( S \) through \( \pi \) and \( \pi^\dagger \) are opposite to each other.

Proof. \( \pi^\dagger \) is constructed from the symplectomorphism \( f : X \to X^\dagger \) given in the last subsection. Namely we glue the Lagrangian fibration of \( X - V \) with the Lagrangian fibration of \( U^\dagger \) by \( \rho \). This gives a Lagrangian fibration on \( X^\dagger \), and hence on \( X \) by the symplectomorphism \( f \). It is constructed directly as follows.

For each fixed \( u_1, u_2, v_1, v_2 \), take the diffeomorphism \( \phi_{u_1, u_2, v_1, v_2} \) on \( \{ z \in \mathbb{C} : |z - (a + b)/2| < R \} \) defined by

\[
\phi_{u_1, u_2, v_1, v_2}(z) = e^{\pi i f \left( \left( \frac{|z - (a + b)/2|}{R} \right)^2 + \left( \frac{|u_1|^2 - |v_1|^2}{L} \right)^2 + \left( \frac{|u_2|^2 - |v_2|^2}{L} \right)^2 \right)} (z - (a + b)/2) + (a + b)/2 \ 
\]

(1.5)

where \( f(r) \) is a decreasing function valued in \([0, 1]\) which equals to 1 for \( r < 1 - 2\epsilon \) and equals to 0 for \( r > 1 - \epsilon \). Thus \( \phi_{u_1, u_2, v_1, v_2} \) is identity on

\[
\left\{ (u_1, v_1, u_2, v_2, z) \in U : \left( \frac{|z - (a + b)/2|}{R} \right)^2 + \left( \frac{|u_1|^2 - |v_1|^2}{L} \right)^2 + \left( \frac{|u_2|^2 - |v_2|^2}{L} \right)^2 > 1 - \epsilon \right\}.
\]

Define a fibration \( U \to I \times (-L, L) \times (-L, L) \) by

\[
\left( |\phi_{u_1, u_2, v_1, v_2}(z) - c|, \frac{1}{2}(|u_1|^2 - |v_1|^2), \frac{1}{2}(|u_2|^2 - |v_2|^2) \right).
\]

For \( |u_1| = |v_1|, |u_2| = |v_2| \), the resulting level curves of \( |\phi(0, 0, z) - c| \) are depicted in Figure 1.5. Since the fibration is \( T^2 \)-equivariant and any curve on the plane is Lagrangian, it is a Lagrangian fibration by symplectic reduction. Moreover it agrees with the original Lagrangian fibration \( \pi \) on \( U - V \). Hence we can glue this with the original Lagrangian fibration on \( X - V \), and obtain another Lagrangian fibration \( \pi^\dagger : X \to B \).

By definition \( \pi = \pi^\dagger \) away from \( V \). In the neighborhood defined by \( \left( \frac{|z - (a + b)/2|}{R} \right)^2 + \)
Figure 1.5: The symplectic reduction of Lagrangian torus fibers before and after the A-flop.

\[
\left( \frac{|u_1|^2 - |v_1|^2}{L} \right)^2 + \left( \frac{|u_2|^2 - |v_2|^2}{L} \right)^2 < 1 - 2\epsilon, \text{ the fibration is simply}
\]
\[
\left( z - (a + b - c), \frac{1}{2}(|u_1|^2 - |v_1|^2), \frac{1}{2}(|u_2|^2 - |v_2|^2) \right)
\]

which is also conifold-like around \( S \). The image of \( S \) under either \( \pi \) and \( \pi^\dagger \) is the interval \([b - c, a - c] \times \{0\} \times \{0\} \). Since the fibration around \( S \) is compatible with the symplectic reduction of the \( T^2 \)-action on \((u_1, v_1, u_2, v_2)\), the second and third coordinates \((b_2, b_3) = \left( \frac{1}{2}(|u_1|^2 - |v_1|^2), \frac{1}{2}(|u_2|^2 - |v_2|^2) \right) \) serve as the action coordinates of the base of the Lagrangian fibration. Thus the image \( S \subset \{b_2 = b_3 = 0\} \) is an affine submanifold.

Since \( \pi = \pi^\dagger \) outside \( V \subset U \) and every torus fiber of \( \pi \) and \( \pi^\dagger \) has non-empty intersection in \( X - V \), the orientations can be canonically identified. We have induced orientations on the fibers of \(|z - c|\) and also fibers of \( |\phi_{u_1,u_2,v_1,v_2}(z) - c| \). The induced orientation on \( S \) from \( \pi \) (or \( \pi^\dagger \) resp.) is such that \( \omega(u, v) > 0 \) where \( u \) is a tangent vector along the orientation of \( S \) and \( v \) (or \( v' \) resp.) is a tangent vector along the orientation of a fiber of \(|z - c|\) (or a fiber of \( |\phi_{u_1,u_2,v_1,v_2}(z) - c| \) resp.). It follows that \( v = -v' \) (up to scaling by a positive number) and hence the two induced orientations on \( S \) are opposite to each other.

To distinguish from the usual notion of flop in complex geometry, we call \( \pi^\dagger \) to be the \textit{A-flop} of the Lagrangian fibration \( \pi \) (where ‘A’ stands for the ‘symplectic side’ in mirror
symmetry). It is the mirror operation of Atiyah flop.

In analogous to foliations, we identify two Lagrangian fibrations if they are related by diffeomorphisms as follows.

**Definition 1.4.5.** Two Lagrangian fibrations $\pi_1, \pi_2 : X \to B$ are said to be equivalent if there exists a symplectomorphism $\Phi : X \to X$ and a diffeomorphism $f : B \to B$ such that $f \circ \pi_1 = \pi_2 \circ \Phi$.

The following easily follows from construction.

**Proposition 1.4.6.** If we make different choices of $(U, i)$ and the function $f$ in the proof of Theorem 1.4.4, the resulting Lagrangian fibrations are equivalent.

### 1.4.3 A-flop on Lagrangian submanifolds

We can also consider the effect of A-flop on Lagrangian submanifolds other than torus fibers. We restrict to the following kind of Lagrangian submanifolds.

**Definition 1.4.7.** Let $S$ be a Lagrangian sphere in $X$. A Lagrangian submanifold $L \subset X$ is said to have $T^2$-symmetry around $S$ if there exists a conifold-like chart $(U, (u_1, v_1, u_2, v_2, z))$ around $S$ such that the image of $L \cap U$ under $z$ is a curve and the fiber at each point is given by $|u_i|^2 - |v_i|^2 = c_i(z)$ for some real-valued function $c_i$ on the curve, $i = 1, 2$.

Given $L$ with $T^2$-symmetry with respect to a conifold-like chart $(U, (u_1, v_1, u_2, v_2, z))$ around $S$, we define $L^\dag$ as follows. Recall the diffeomorphism $\phi_{u_1, u_2, v_1, v_2}$ on $\{ z \in \mathbb{C} : |z - (a + b)/2| < R \}$ in Equation (1.5). Write the image of $L \cap U$ under $z$ as a level curve $f(z) = 0$ of a real-valued function $f$. Then $L^\dag$ is given as

$$\{(u_1, v_1, u_2, v_2, z) \in U : f(\phi_{u_1, u_2, v_1, v_2}(z)) = 0, |u_i|^2 - |v_i|^2 = c_i(z) \text{ for } i = 1, 2\}$$

in $U$ and equals to $L$ outside $U$. Since the image of $L^\dag$ is a curve in the symplectic reduction by $T^2$, it is a Lagrangian submanifold with $T^2$-symmetry. Note that $L^\dag$ and $L$ can be topologically different from each other.

By construction we have
Proposition 1.4.8. \((L^+)^+\) equals to the inverse of the Dehn twist applied to \(L\).

If we have a stability condition \((Z, P)\) on the Fukaya category generated by Lagrangians with \(T^2\)-symmetry around \(S\), then A-flop should give another stability condition \((Z^+, P^+)\) where \(Z^+(L^+) = Z(L)\). In the next two sections we will restrict to the deformed conifold and carry out this construction explicitly.

1.4.4 Examples

Deformed conifold

Consider the deformed conifold \(X = \{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^\times : u_1v_1 = z - a, u_2v_2 = z - b\}\) where \(a < b < 0\). (We have taken away the divisor \(z = 0\).) We can take the flop of the Lagrangian fibration
\[
\pi = (|u_1|^2 - |v_1|^2, |u_2|^2 - |v_2|^2, |z|)
\]
which is special with respect to the holomorphic volume form \(\Omega = d \log z \wedge du_1 \wedge du_2\). The base of the fibration is \(\mathbb{R}^2 \times \mathbb{R}_{>0}\). The discriminant locus of the fibration is \(\{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\} \times \{|b|\}\). After the flop, the discriminant locus becomes \(\{(0, t, |\phi_{0,t,0,0}(a)|) : t \in \mathbb{R}\} \cup \{(t, 0, |\phi_{0,t,0,0}(b)|) : t \in \mathbb{R}\}\) where \(\phi\) is given in Equation (1.5). For \(t \ll 1\), \(\phi_{0,t,0,0}(a) = b\) and \(\phi_{0,t,0,0}(b) = a\); for \(t\) big enough, \(\phi_{0,t,0,0}(a) = a\) and \(\phi_{0,t,0,0}(b) = b\). The base and discriminant locus are shown in Figure 1.6. The new fibration \(\pi^+\) is no longer special with respect to \(\Omega\); however it is equivalent to the corresponding special Lagrangian fibration on \(X^+ = \{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^\times : u_1v_1 = z - b, u_2v_2 = z - a\}\) with respect to \(\Omega^+\) (defined by the same expression as \(\Omega\)).

We have a family of complex manifolds defined by
\[
\begin{align*}
u_1v_1 &= z - \left(\frac{a + b}{2} + \frac{b - a}{2}e^{\pi i (1 + s)}\right), \\
u_2v_2 &= z - \left(\frac{a + b}{2} + \frac{b - a}{2}e^{\pi i s}\right)
\end{align*}
\]
for \(s \in [0, 1]\) joining \(X\) and \(X^+\). They are depicted in Figure 1.7. Each member has a special
Lagrangian fibration defined by the same formula as \( \pi \) above. Before (or after) the moment \( s = \frac{1}{2} \), the Lagrangian fibrations are all equivalent. Approaching the moment \( s = \frac{1}{2} \), two singular Lagrangian fibers collide into one and the Lagrangian fibration changes. Thus \( s = \frac{1}{2} \) is the ‘wall’. It can also been seen clearly from the base, see the top right of Figure 1.3. At the moment \( s = \frac{1}{2} \), the two discriminant loci (which are two lines in different directions) collides.

The torus fibers of \( \pi \) and \( \pi^\dagger \) are different objects in the Fukaya category. Namely consider a fiber \( T = \{|z| = k, |u_1| = |v_1|, |u_2| = |v_2|\} \) of \( \pi \) where \( |b| < k < |a| \) and the corresponding fiber \( T^\dagger = \{|\phi(z)| = k, |u_1| = |v_1|, |u_2| = |v_2|\} \) of \( \pi^\dagger \). We shall see in Section 1.5 that \( T \), which is special Lagrangian with respect to \( \Omega \), is a surgery \( S_1\#S_0 \) for
a morphism in \( \text{Mor}(S_1, S_0) \), while \( T^t \), which is special Lagrangian with respect to \( \Omega^t \), is a surgery \( S_0\#{S_1} \) for a morphism in \( \text{Mor}(S_0, S_1) \). \( S_0, S_1 \) are Lagrangian spheres defined by \( S_0 = \{ z = -t, \|u_1\| = \|v_1\|, \|u_2\| = \|v_2\| : a \leq t \leq b \} \) and \( S_1 = \{ z = 1 + \exp(t\zeta_1 + (1 - t)\zeta_0) \) for \( t \in [0, 1], \|u_1\| = \|v_1\|, \|u_2\| = \|v_2\| \} \) where \( \zeta_0 = \log |a| - i\pi \) and \( \zeta_1 = \log |b| + i\pi \).

**Deformed orbifolded conifold**

For \( k \geq l \geq 1 \), the orbifolded conifold \( O_{k,l} \) is the quotient of the conifold \( \{ u_1v_1 = u_2v_2 \} \subset \mathbb{C}^4 \) by the abelian group \( \mathbb{Z}_k \times \mathbb{Z}_l \), where the primitive roots of unity \( \zeta_k \in \mathbb{Z}_k \) and \( \zeta_l \in \mathbb{Z}_l \) act by

\[
(u_1, v_1, u_2, v_2) \mapsto (\zeta_k u_1, \zeta_k^{-1} v_1, u_2, v_2), \quad \text{and} \quad (x, y, z, w) \mapsto (u_1, v_1, \zeta_l u_2, \zeta_l^{-1} v_2).
\]

In equations

\[
O_{k,l} = \left\{ u_1v_1 = (z - 1)^k, \ u_2v_2 = (z - 1)^l \right\} \subset \mathbb{C}^5.
\]

It is a toric Gorenstein singularity whose fan is the cone over the rectangle \( [0, k] \times [0, l] \subset \mathbb{R}^2 \).

For the purpose of constructing a Lagrangian torus fibration with only codimension-two discriminant loci, we shall delete the divisor \( \{ z = 0 \} \subset O_{k,l} \) and obtain \( X_0 = \left\{ (u_1, u_2, v_1, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^\times : u_1v_1 = (z - 1)^k, \ u_2v_2 = (z - 1)^l \right\} \).

We shall consider smoothings of \( X_0 \), which correspond to the Minkowski decompositions of the rectangle \( [0, k] \times [0, l] \) into \( k \) copies of \( [0, 1] \times \{0\} \) and \( l \) copies of \( \{0\} \times [0, 1] \) [4]. Explicitly a smoothing is given by

\[
X = \{ (u_1, u_2, v_1, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^\times : u_1v_1 = f(z), \ u_2v_2 = g(z) \}
\]

where \( f(z) \) and \( g(z) \) are polynomials of degree \( k \) and \( l \) respectively, such that the roots \( r_i \) and \( s_j \) of \( f(z) \) and \( g(z) \) respectively are pairwise-distinct and non-zero. For later purpose we shall assume \( |r_i|, |s_j| \) are all pairwise distinct.

\( X \) admits a double conic fibration \( X \rightarrow \mathbb{C}^\times \) by projecting to the \( z \)-coordinate. There is also a natural Hamiltonian \( T^2 \)-action on \( X \) given by \( (s, t) \cdot (u_1, v_1, u_2, v_2, z) := (su_1, s^{-1} v_1, tu_2, t^{-1} v_2, z) \) for \( (s, t) \in T^2 \subset \mathbb{C}^2 \). The symplectic reduction of \( X \) by the \( T^2 \)-action is identified with \( \mathbb{C}^\times \),
the base of the double conic fibration. Using the construction of Goldstein [42] and Gross [46], we have the Lagrangian fibration

\[ \pi : X \to B := \mathbb{R}^2 \times (0, \infty) \]

\[ \pi(u_1, v_1, u_2, v_2, z) = \left( \frac{1}{2}(|u_1|^2 - |v_1|^2), \frac{1}{2}(|u_2|^2 - |v_2|^2), |z| \right) . \]

The map to the first two coordinates is the moment map of the Hamiltonian \( T^2 \)-action. We denote the coordinates of \( B \) by \( b = (b_1, b_2, b_3) \). The discriminant locus is given by the disjoint union of lines

\[ \bigcup_{i=1}^{k} \{b_1 = 0, b_3 = |r_i| \} \cup \bigcup_{j=1}^{l} \{b_2 = 0, b_3 = |s_j| \} \subset B , \]

and the fibers are special Lagrangians in the same phase \( \pi/2 \) with respect to the volume form \( \Omega := du_1 \wedge du_2 \wedge d \log z \) (Proposition 3.17).

Now let \( a = r_1 \) and \( b = s_1 \). Assume that \( |a| \neq |b| \); zero and all other roots \( r_i, s_j \) lie outside the disc \( |z - (a + b)/2| \). Let \( S_0 \) be the Lagrangian matching sphere corresponding to the straight line segment joining \( a \) and \( b \). Then the above Lagrangian fibration is conifold-like around \( S \). The flop of this is equivalent to the corresponding Lagrangian fibration on \( X^\dagger = \{u_1v_1 = f^\dagger(z), u_2v_2 = g^\dagger(z)\} \), where \( f^\dagger \) and \( g^\dagger \) are polynomials with sets of roots \( \{s_1, r_2, \ldots, r_k\} \) and \( \{r_1, s_2, \ldots, s_l\} \) respectively; and the Lagrangian fibration is

\[ \pi^\dagger(u_1, v_1, u_2, v_2, z) = \left( \frac{1}{2}(|u_1|^2 - |v_1|^2), \frac{1}{2}(|u_2|^2 - |v_2|^2), |z| \right) : X^\dagger \to B . \]

The Lagrangian fibration \( \pi^\dagger \) is no longer special with respect to \( \Omega \) on \( X \); however it is (equivalent to) a special Lagrangian fibration with respect to \( \Omega^\dagger = du_1 \wedge du_2 \wedge d \log z \) on \( X^\dagger \).

**Schoen’s Calabi–Yau**

Given a compact simple integral affine threefold \( B \) with singularities \( \Delta \), Castaño-Bernard and Matessi [75] constructed a symplectic manifold \( X \) together with a Lagrangian fibration \( X \to B \) inducing the given affine structure. It is achieved by gluing local models of Lagrangian fibrations around \( \Delta \) with the Lagrangian fibration over the affine manifold.
B − Δ. In particular their construction can be applied to Schoen’s Calabi–Yau [76]. The Lagrangian fibration is conifold-like, and so the mirror flop defined here can be applied.

Schoen’s Calabi–Yau is given by the fiber product of two elliptic fibrations on $K3$ surfaces over the base $\mathbb{P}^1$. The affine base manifold (which is topologically $S^3$) of Schoen’s Calabi–Yau was found by Gross [47, Section 4]. Section 9.2 of [76] constructed a conifold degeneration of the affine base forming nine conifold points simultaneously.

We quickly review their construction here. Consider the following polyhedral decomposition of $S^3$. Take six copies of triangular prisms

$$\text{Conv}\{(0,0,0), (0,1,0), (1,0,0), (0,0,1), (0,1,1), (1,0,1)\},$$

three of them are labeled as $\sigma_j$ and three of them are labeled as $\tau_k$ for $j, k \in \mathbb{Z}_3$. Take nine copies of cubes

$$\text{Conv}\{(0,0,0), (0,1,0), (1,0,0), (1,1,0), (0,0,1), (1,0,1), (0,1,1), (1,1,1)\}$$

and label them as $\omega_{jk}$. See Figure 1.8. The top triangular face of $\sigma_j$ is glued to the bottom triangular face of $\sigma_{j+1}$ ($j \in \mathbb{Z}_3$), and so topologically $\bigcup_{j=1}^{\mathbb{Z}_3} \sigma_j$ forms a solid torus. Similarly do the same thing for $\tau_k$ so that $\bigcup_{k=1}^{\mathbb{Z}_3} \tau_k$ forms another solid torus. For the nine cubes, glue the top face of $\omega_{jk}$ with the bottom face of $\omega_{j+1,k}$ for $j \in \mathbb{Z}_3$, and glue the right face of $\omega_{jk}$ with the left face of $\omega_{j,k+1}$ for $k \in \mathbb{Z}_3$. This topologically forms a two-torus times an interval. Finally glue the front face of $\omega_{jk}$ with the $j$-th square face of $\tau_k$, and glue the back face of $\omega_{jk}$ with the $k$-th square face of $\sigma_j$. Here the square faces of $\sigma_j$ and $\tau_k$ are ordered counterclockwisely. This forms $S^3$ as gluing of two solid tori along their boundaries.

The fan structure at every vertex of the polyhedral decomposition is that of $\mathbb{P}^2 \times \mathbb{P}^1$. Together with the standard affine structure of each polytope, this gives $S^3$ an affine structure with singularities. The discriminant locus is given by the dotted lines shown in Figure 1.8. Note that each dotted line in a square face of a prism indeed has multiplicity three. Thus the discriminant locus is a union of 24 circles counted with multiplicities. Moreover the dotted lines in cubes form three horizontal and three vertical circles, intersecting with each
other at nine points. These are the nine conifold singularities (which are positive nodes).

By gluing local models of Lagrangian fibrations around discriminant locus with the Lagrangian fibration from the affine structure away from discriminant locus, [76] produced a symplectic manifold which is homeomorphic to the Schoen’s Calabi–Yau. Moreover by using the results on symplectic resolution of Smith–Thomas–Yau [92] and complex smoothing of Friedman [37] and Tian [97], they showed that the existence of certain tropical two-cycles containing a set of conifold points ensure that the nodes can be simultaneously resolved (and smoothened in the mirror side). In particular all the nine nodes in this example can be resolved simultaneously.

In the smoothing the three horizontal and three vertical circles which form part of the discriminant locus are moved apart so that they no longer intersect with each other. This gives a symplectic manifold $X$ together with a Lagrangian fibration. The corresponding affine base coincides with the one in the previous work of Gross [47, Section 4].

The local model for each conifold point in this example is the Lagrangian fibration $\left( |u_1|^2 - |v_1|^2, |u_2|^2 - |v_2|^2, |z| \right)$ on $\{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^\times : u_1 v_1 = z - a = u_2 v_2 \}$ for $a < 0$; the local model for its smoothing is the fibration defined by the same expression on $\{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^\times : u_1 v_1 = z - a, u_2 v_2 = z - b \}$ for $a < b < 0$. A Lagrangian fibration corresponding to the simultaneous smoothing can be constructed by gluing these
local models. In particular $X$ and the fibration are conifold-like around each of the vanishing spheres corresponding to the nine conifold points. Hence we can perform A-flop around each of these spheres and obtain new Lagrangian fibrations. The operation can be understood as link surgery in the base $S^3$.

Note that we cannot always keep the circles $A_i, B_j$ in constant levels in the A-flop. For instance, suppose $A_i$ and $B_j$ are contained in the planes in levels $a_i, b_j$ respectively with $a_1 < a_2 < a_3 < b_1 < b_2 < b_3$. (These planes have normal vectors pointing to the right if drawn in Figure 1.8.) Now we perform the A-flop along the vanishing sphere between levels $a_1$ and $b_1$. The resulting fibration is equivalent to the one with these circles in levels $a_2 < a_3 < b_1 < a'_1 < b_2 < b_3$ where $a'_1$ is the new level of $A_1$. At this stage all these circles are still kept in constant levels. Now let’s do the A-flop along the vanishing sphere between levels $a_2$ and $b_3$. Then the resulting fibration cannot have all these circles in constant levels: if they were in constant levels, then $a_2 < b_1 < a'_1 < b_3 < a_2$, a contradiction!

1.5 Derived Fukaya category of the deformed conifold

In Example 1.4.4, we consider a path of complex structures on the deformed conifold (with a fixed symplectic form) given by the equations

$$X_s = \left\{ u_1v_1 = z - \left( \frac{a + b}{2} + \frac{b - a}{2} e^{\pi i (1+s)} \right), u_2v_2 = z - \left( \frac{a + b}{2} + \frac{b - a}{2} e^\pi is \right), z \neq 0 \right\}$$

for $s \in [0, 1]$. ($X_{s=0}$ and $X_{s=1}$ were denoted as $X$ and $X^\dagger$ in 1.4.4, respectively.) This deformation of complex structures parametrized by $s$ is SYZ mirror to the flop operation on the resolved conifold. In the last section we realized this operation as surgery of a Lagrangian fibration.

In this section, we study the effect of deformation of complex structures (together with holomorphic volume forms) on special Lagrangians. This would motivate us to consider A-flop on stability conditions of the derived Fukaya category.

Recall from Section 1.3 that we have two Lagrangian spheres $S_0$ and $S_1$ in $X_{s=0}$. Moreover,
there is a sequence of Lagrangian spheres \( \{ S_n : n \in \mathbb{Z} \} \) in \( X_{s=0} \) which corresponds to a collection of non-trivial matching paths in the base of the double conic fibration \( X_{s=0} \to \mathbb{C}^\times \).

We depict these spheres in the universal cover of \( \mathbb{C}^\times (\exists z) \) as shown in Figure 1.9.

![Figure 1.9: Sequence of Lagrangian spheres in \( X_{s=0} \).](image)

**Definition 1.5.1.** \( \mathcal{F} \) is defined to be the full subcategory of \( \text{Fuk}(X_{s=0}) \) generated by \( S_0 \) and \( S_1 \).

The main purpose of this section is to prove the following.

**Theorem 1.5.2.** Regular Lagrangian torus fibers of \( \pi \) which have non-empty intersection with \( S_0 \), as well as the Lagrangian spheres \( S_i \), are contained in \( \mathcal{F} \).

The theorem follows from Proposition 1.5.7 and 1.5.9 below.

The torus fibers and spheres \( S_i \) are special Lagrangians with respect to the holomorphic volume form

\[
\Omega := d \log \bar{z} \wedge du_1 \wedge du_2
\]

on \( X_{s=0} \). Here we used \( d \log \bar{z} \) instead of \( d \log z \) to match the ordering of phases on both sides of the mirror\(^2\). In particular, we measure the angle in clockwise direction for phases of \( S_i \). The diagram in the right side of Figure 1.9 compares the phases of \( S_i \)'s, where \( S_0 \) has the

\[^2\text{In order to match the phase inequality in the mirror side, we can either impose the mirror functor to be contravariant, or use the complex structure induced by the conjugate volume form } d \log \bar{z} \wedge du_1 \wedge du_2 \text{ like here. All } S_i \text{ are still special Lagrangians under this volume form, and we have the phase inequalities } \theta(S_i) > \theta(S_j) \text{ for } 0 < i < j \text{ or } i < j < 1. \text{ This matches the ordering of the phases of stable objects in an exact triangle of the mirror B-side convention. Namely for an exact triangle } L_1 \to L_1 \# L_2 \to L_2 \text{ where } L_i \text{ are special Lagrangians, their phases should satisfy } \theta(L_1) \leq \theta(L_1 \# L_2) \leq \theta(L_2).\]

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biggest phase in our convention. In Section 1.6 we will see that taking these to be stable objects defines a Bridgeland stability condition on the derived Fukaya category.

Moreover each $S_i$ corresponds to another Lagrangian sphere $S_i^+$ in $\mathcal{F}$, the flop of $S_i$ constructed in Section 1.4.3. The Lagrangian torus fibers of $\pi^+$ and $S_i^+$ are special with respect to the pull-back holomorphic volume form from $X_{s=1}$, and they define another Bridgeland stability condition. In fact, we have $S_i^+ = \rho^{-1}(S'_{-i})$ where \{S'_n : n \in \mathbb{Z}\} is the set of new special Lagrangian spheres in $X_{s=1}$ which map to straight line segments by $z$-projection as in Figure 1.10.

\[\begin{array}{c}
S'_{-2} \\
S'_{-1} \\
S_0' \\
S_1' \\
S_2'
\end{array}\]

Figure 1.10: Special Lagrangian spheres $S_i'$ in $X_{s=1}$.

For later use we orient these spheres as follows. In conic fiber direction, each $S_i$ restricts to a 2-dimensional torus \{|u_1| = |v_1|, |u_2| = |v_2|\}. We fix the orientation on the fiber torus to be $d\theta_1 \wedge d\theta_2$ where $\theta_i$ are the arguments of $u_i$ respectively. We orient their matching paths as in the right side of Figure 1.9.

Set $L_0 := S_0$ and $L_1 := S_1$. They are distinguished objects in the set $\{S_n\}$ of Lagrangian spheres in the sense that they have minimal/maximal slopes (or phases) as well as they generate $D^b \mathcal{F}$. We will study Lagrangian Floer theory of $L_0$ and $L_1$ intensely in Section 1.6.1.
Recall from Section 1.2 that $\mathcal{D}_{\hat{Y}/Y}$ is the subcategory of $\mathcal{D}^b(\hat{Y})$ generated by $\mathcal{O}_C(-1)[1]$ and $\mathcal{O}_C$. [21] proved the following equivalence of subcategories of $\text{Fuk}(X_{s=0})$ and $\mathcal{D}^b(\hat{Y})$.

**Theorem 1.5.3.** ([21]) There is an equivalence $\mathcal{D}^b\mathcal{F} \simeq \mathcal{D}_{\hat{Y}/Y}$, sending

$$L_0 \mapsto \mathcal{O}_C(-1)[1] \quad L_1 \mapsto \mathcal{O}_C.$$ (1.7)

Using the chain model of Abouzaid [1], they explicitly computed the $A_{\infty}$-structure of the endomorphism algebras of $L_0 \oplus L_1$ to conclude that

$$\text{End}(L_0 \oplus L_1) \simeq \text{End}(\mathcal{O}_C(-1)[1] \oplus \mathcal{O}_C).$$ (1.8)

See [21, Section 5, 7] for more details.

In this chapter, we shall use either the Morse–Bott model in [38] or pearl trajectories [11, 90] to study Lagrangian torus fibers and the noncommutative mirror functor. They are conceptually easier to understand.

![Diagram](image)

**Figure 1.11:** Transformation of $L_0$ and $L_1$ by the symplectomorphism $\rho$.

The A-flop can be realized by the symplectomorphism $\rho$ from $X_{s=0}$ to $X_{s=1}$ given in 1.4.4 (see Figure 1.7). Figure 1.11 shows how $\rho$ acts on $L_i$, where the third diagram describes the moment at which $L_0$ and $L_1$ happen to have the same phases. Observe that $X_{s=1}$ (1.6) is obtained from $X_{s=0}$ by swapping two sets of coordinates $(u_1, v_1)$ and $(u_2, v_2)$. However, swapping the coordinates is different from the symplectomorphism that gives A-flop, as its effect on $z$-plane shows.
As in Figure 1.11, \( \rho \) sends \( L_0 \) and \( L_1 \) to special Lagrangian spheres in \( X_{s=1} \) which we denote by \( L_0' \) and \( L_1' \) respectively. Let \( \mathcal{F}' \) denote the Fukaya subcategory of \( X_{s=1} \) consisting of \( L_0' \) and \( L_1' \). There is a natural functor \( \rho_* \) : \( \mathcal{F} \rightarrow \mathcal{F}' \) induced by the symplectomorphism \( \rho \). On the other hand, we can repeat the same argument as in the proof of Theorem 1.5.3 to see that

\[
\mathcal{D}^b \mathcal{F}' \simeq \mathcal{D}_{Y/Y} \quad \text{with} \quad L_0' \mapsto \mathcal{O}_{C^*}(-1) \quad \text{and} \quad L_1' \mapsto \mathcal{O}_{C^*}(-2)[1].
\]

Notice that this identification is coherent with the fact that \( L_0' \) is somewhat similar to the orientation reversal of \( L_0 \), whereas \( L_1 \mapsto L_1' \) can be understood as a change of winding number with respect to \( z = 0 \).

In fact, we have

\[
\text{End}(L_0' \oplus L_1') \simeq \text{End}(L_0 \oplus L_1)
\]

(1.9)

as two set of objects are related by a symplectomorphism, and

\[
\text{End}(\mathcal{O}_{C^*}(-1) \oplus \mathcal{O}_{C^*}(-2)[1]) \simeq \text{End}(\mathcal{O}_{C}(-1)[1] \oplus \mathcal{O}_{C})
\]

(1.10)

due to the flop functor (see 1.2.2). It directly implies that the functor \( \rho_* \) induced by the symplectomorphism is mirror to the flop functor through the identification of A and B side categories via [21]. Namely,

**Proposition 1.5.4.** We have a commutative diagram of equivalences:

\[
\begin{array}{ccc}
\mathcal{D}^b \mathcal{F} & \simeq & \mathcal{D}_{Y/Y} \\
\rho_* \downarrow & & \Phi \\
\mathcal{D}^b \mathcal{F}' & \simeq & \mathcal{D}_{Y'/Y}
\end{array}
\]

(1.11)

*Proof.* It obviously commutes on the level of objects by the construction. (1.8), (1.9) and (1.10) imply that the diagram also commutes on morphism level.

We shall study how the symplectomorphism \( \rho \) or its induced functor \( \rho_* \) acts on various geometric objects in \( \mathcal{D}^b \mathcal{F} \). For that, we should examine what kind of geometric objects are actually contained in \( \mathcal{D}^b \mathcal{F} \).
1.5.1 Geometric objects of $D^b\mathcal{F}$

Let us first prove that any torus fiber intersecting $L_0$ and $L_1$ is isomorphic to a mapping cone $\text{Cone}(L_0 \xrightarrow{\alpha} L_1)$ for some degree-1 element $\alpha \in HF(L_0, L_1)$ in the derived Fukaya category. In particular this will imply that the category $D^b\mathcal{F}$ contains those torus fibers as objects.

One can choose the gradings on $L_i$ such that $HF^*(L_0, L_1) = H^*(S^1_a)[-1] \oplus H^*(S^1_b)[-1]$. Here, we use the Morse–Bott model, where $S^1_a$ and $S^1_b$ denotes the intersection loci over $z = a$ and $z = b$, respectively. Both $S^1_a$ and $S^1_b$ are isomorphic to a circle. Thus degree 1 elements in $HF(L_0, L_1)$ are given by linear combinations of (Poincaré duals of) fundamental classes of $S^1_a$ and $S^1_b$. The cone $\text{Cone}(L_0 \xrightarrow{\alpha} L_1)$ can be identified with a boundary deformed object $(L_0 \oplus L_1, \alpha)$ (see [38] or [86]).

Let $L_c := (a < c < b)$ be a Lagrangian torus that intersects $L_0$ at $z = c$. This condition determines $L_c$ uniquely, as components of $L_c$ in double conic fiber direction should satisfy the same equation as those of $L_0$ and $L_1$. We orient $L_c$ as in Figure 1.9 in $z$-plane components, and use the standard one (from $d\theta_1 d\theta_2$ as for $S_i$) along the conic fiber directions. $L_c$ cleanly intersects $L_0$ and $L_1$ along 2-dimensional tori which we denote by $T_0 := L_c \cap L_0$ and $T_1 := L_c \cap L_1$. One can see that $CF(L_c, L_0) = C^*(T_0)$ and $CF(L_1, L_c) = C^*(T_1)$ for suitable choice of a grading on $L_c$. Similarly, $CF(L_0, L_c) = C^*(T_0)[-1]$ and $CF(L_c, L_1) = C^*(T_1)[-1]$.

Let $U_{\rho_1, \rho_2, \rho_z}$ be a unitary flat line bundle on $L_c$ whose holonomies along circles in the double conic fibers are $\rho_1$ and $\rho_2$ and that along the circle in $z$-plane is $\rho_z$.

**Lemma 1.5.5.** If $(\rho_1, \rho_2) \neq (1, 1)$, then

$$HF(L_0, (L_c, U_{\rho_1, \rho_2, \rho_z})) = 0, \quad HF(L_1, (L_c, U_{\rho_1, \rho_2, \rho_z})) = 0. \quad (1.12)$$

**Proof.** One can simply use the Morse–Bott model for each of cohomology groups in (1.12). Each of this group is simply a singular cohomology of the intersection loci, equipped with twisted differential. Since the intersection loci are 2-dimensional torus in the double conic fiber, the twisting is determined by $(\rho_1, \rho_2)$. Here we only have classical differential, as there is no holomorphic strip between $L_i$ and $T_c$. One can easily check that the cohomology
vanishes if the twisting is nontrivial.

Alternatively, one can perturb Lagrangians to have transversal intersections as in Figure 1.12 to see that the Floer differential has coefficients $\rho_1 - 1$ and $\rho_2 - 1$, which are nonzero for nontrivial $(\rho_1, \rho_2)$.

![Figure 1.12: Perturbation of $L_c$, $L_i$ and polygons contributing to $m_1^0$ on $CF(L, (L_c, \rho))$.](image)

The lemma implies that $(L_c, U_{\rho_1, \rho_2, \rho_z})$ has no Floer theoretic intersection with $L_0$ or $L_1$ unless $\rho_1 = \rho_2 = 1$. From now on, we will only consider flat line bundles of the type $U_{0,0,\rho_z}$ on Lagrangian torus fibers, which will be written as $U_{\rho_z}$ instead of $U_{0,0,\rho_z}$ for notational simplicity. Let $P_0 := PD[T_0] \in CF^*(L_c, L_0) = C^*(T_0)$ and $P_1 := PD[T_1] \in CF^*(L_1, L_c) = C^*(T_1)$. We also set $\alpha_a := PD[S^1_a] \in CF^*(L_0, L_1)$ and $\alpha_b := PD[S^1_b] \in CF^*(L_0, L_1)$. Notice that $\deg \alpha_a = \deg \alpha_b = 1$ whereas $\deg P_0 = \deg P_1 = 0$.

**Lemma 1.5.6.** $P_0 \in CF^0((L_c, U_{\rho_z}), (L_0 \oplus L_1, \alpha))$ and $P_1 \in CF^0((L_0 \oplus L_1, \alpha), (L_c, U_{\rho_z}))$ are cycles with respect to $m_1^{0,\alpha}$ and $m_1^{\alpha,0}$ respectively if and only if $\alpha$ is given as $\lambda_a \alpha_a + \lambda_b \alpha_b \in CF^1(L_0, L_1)$ with $\lambda_a : \lambda_b = (T^{\alpha_1} \rho_2 : T^{\alpha_2})^3$ where $\Delta_1$ and $\Delta_2$ are triangles shaded in Figure 1.13.

---

It is harmless to put $T = e^{-1}$ since only finitely many polygons contribute to $A_\infty$-structures. Nevertheless we will keep the notation $T$ to highlight contributions from nontrivial holomorphic polygons.
Proof. We will prove the lemma for $P_1$ only, and the proof for $P_0$ is similar. We pick a point $\times$ as in Figure 1.13 for representative of $U_{\rho_z}$ so that when boundary of a holomorphic polygon passes this point, the corresponding $m_k$-operation is multiplied by $\rho_z^{\pm 1}$ depending on the orientation. (More precisely, the point $\times$ represent 2-dimensional subtorus in $L_c$ lying over this point, which is called a hyper-torus and used to fix the gauge of a flat line bundle in [25].)

Observe that two holomorphic triangles $\Delta_1$ and $\Delta_2$ shown in Figure 1.13 contribute to the following operations:

$$m_2(\lambda_a \alpha_a, P_1) = \lambda_a T^{\omega(\Delta_1)} \bar{P}_0, \quad m_2(\lambda_b \alpha_b, P_1) = -\rho_z \lambda_b T^{\omega(\Delta_2)} \bar{P}_0. \quad (1.13)$$

where $\bar{P}_0$ is $PD[T_0]$ regarded as an element of $CF(L_0, L_c) = C^*(T_0)[-1] \subset CF((L_0 \oplus L_1, \alpha), (L_c, U_{\rho_z}))$ (note that $\deg \bar{P}_0 = 1$). We do not provide the precise sign rule here since it is not crucial in our argument. Indeed we can assume that two operations in (1.13) produces outputs with the opposite signs by replacing $\lambda_b$ to $-\lambda_b$ if necessary.

Therefore we see that

$$m_1^{a,0}(P_1) = \sum_k m_k(\alpha, \cdots, \alpha, P_1) = \left(\lambda_a T^{\omega(\Delta_1)} - \rho_z \lambda_b T^{\omega(\Delta_2)}\right) \bar{P}_0 = 0$$

if and only if $\lambda_a$ and $\lambda_b$ have the ratio as given in the statement. \qed
We next prove that $P_0$ and $P_1$ in Lemma 1.5.6 give isomorphisms between two objects $(L_c, U_{P_c})$ and $\text{Cone}(L_0 \xrightarrow{\alpha} L_1)$ where $\alpha$ is chosen as in Lemma 1.5.6. Here, it is enough to present the ratio between $\lambda_a$ and $\lambda_b$, as the mapping cone does not depend on the scaling of $\alpha$ by an element in $\mathbb{C}^\times$ (or in $\Lambda \setminus \{0\}$ if we do not substitute $T$ by $e^{-1}$).

**Proposition 1.5.7.** We have $(L_c, U_{P_c}) \cong \text{Cone}(L_0 \xrightarrow{\alpha} L_1)$ in the derived Fukaya category of $X_{s=0}$ where $\alpha = \lambda_a a_a + \lambda_b a_b$ is chosen as in Lemma 1.5.6.

*Proof.* Let us fix $\lambda_a$ and $\lambda_b$ to be $r_z T_w(D_2)$ and $T_w(D_1)$ for simplicity. We claim that

$$m_2^0,0,0(P_1, P_0) = C(1_L + 1_L), \quad m_2^0,0,0(P_0, P_1) = C1_{L_c}$$

for some common constant $C$. (One should rescale $P_0$ and $P_1$ to get strict identity morphisms.) To see this, pick generic points "*" on $L_0$, $L_1$ as in Figure 1.13, whose number of appearance in the boundary of holomorphic discs determines the coefficient of $1_{L_i}$. The same triangles in the proof of Lemma 1.5.6 now contribute as

$$m_2^0,0,0(P_1, P_0) = \lambda_a T^{\omega(\Delta_1)} 1_{L_1} + \rho_z \lambda_b T^{\omega(\Delta_2)} 1_{L_0} = \rho_z T^{\omega(\Delta_1) + \omega(\Delta_2)} (1_{L_0} + 1_{L_1})$$

Here two contributions add up contrary to (1.13). In fact, the relative signs are completely determined by $z$-directions since all the Lagrangians share the other directions, and one can use the sign rule due to Seidel [87] for $z$-plane components.

It is easy to check that the computation does not depend on the choice of generic points "*" (it is essentially because $P_0$ and $P_1$ are cycles). Likewise, $\Delta_2$ contributes to

$$m_2^0,0,0(P_0, P_1) = \rho_z \lambda_b T^{\omega(\Delta_2)} 1_{L_c} = \rho_z T^{\omega(\Delta_1) + \omega(\Delta_2)} 1_{L_c}.$$  

In particular, Proposition 1.5.7 implies the following exact sequence in the derived Fukaya category

$$L_1 \to (L_c, U_{P_c}) \to L_0 \xrightarrow{[1]}.$$  

$\square$
**Remark 1.5.8.** Analogously, the following gives an exact triangle in $\mathcal{D}_{\tilde{Y}/Y}$:

$$\mathcal{O}_C \to \mathcal{O}_y \to \mathcal{O}_C(-1)[1] \to $$

for a point $y$ in $C$, or equivalently $\mathcal{O}_y \cong \text{Cone}(\mathcal{O}_C(1) \to \mathcal{O}_C)$ for some degree 1 morphism. Note that $\mathcal{O}_y$ is a SYZ mirror of one of torus fibers $L_c$ (with a flat line bundle $U_{r_c}$). Proposition 1.5.7 and the above exact triangle shows that the equivalence (1.7) sends torus fibers to skyscraper sheaves over points in $C$.

By symmetric argument (or by the triangulated structure on $D^b\text{Fuk}(X_{s=0})$), one also has

$$\text{Cone}(L_0[1] \xrightarrow{\beta} L_c) \cong L_1$$

in $D^b\text{Fuk}(X_{s=0})$ where $\deg(\beta) = 1$. Here, $[1]$ can be though of as taking orientation reversal of the z-component of $L_0$ (or, more precisely, such change of grading). A similar statement holds true for other Lagrangian spheres \{S_m : m \in \mathbb{Z}\} (Figure 1.9).

![Figure 1.14](image-url)  
**Figure 1.14:** Triangles contributing to $m_1$ and $m_2$ on $\text{CF}(S_m[1] \oplus S_{m+1}, L_c)$ and $\text{CF}(L_c, S_m[1] \oplus S_{m+1})$.

**Proposition 1.5.9.** Lagrangian sphere $S_i$ for any $i$ can be obtained from taking successive cones from $S_0$ and $S_1$. More precisely, one has the following:
• for $m \geq 1$,
  \[ S_{m+1} \cong \text{Cone}(L_c \to S_m), \]

• for $n \leq 0$,
  \[ S_{n-1} \cong \text{Cone}(S_n[1] \to L_c[1]). \]

Proof. We only prove the first identity, and the proof for the second can be performed in a similar manner. One can easily check that $L_c = \text{Cone}(S_m[1] \xrightarrow{\alpha} S_{m+1})$ (for $n \leq 0$) by the same argument as in the proof of Lemma 1.5.6, where $\alpha$ is a degree 1 morphism from $S_{n-1}$ to $S_n[1]$. The contributing pair of triangles are as shown in Figure 1.14, and hence we should take into account the relative areas of these two triangles together with the location of $c \in (a, b)$, when we choose $\alpha$. We omit the details as it is completely parallel to the proof of Lemma 1.5.6.

From $L_c = \text{Cone}(S_m[1] \xrightarrow{\alpha} S_{m+1})$, we have an exact triangle in the derived Fukaya category $S_{m+1} \to L_c \to S_m[1] \xrightarrow{[1]}$ or equivalently $S_m \to S_{m+1} \to L_c$, which implies $S_{m+1} = \text{Cone}(L_c \to S_m)$ for some degree 1 morphism.

We conclude that $D^b\mathcal{F}$ contains a sequence of Lagrangian spheres $\{S_n : n \in \mathbb{Z}\}$ and $\mathbb{P}^1 \setminus \{0, \infty\}$-family of Lagrangian tori parametrized by $(c, \rho_z)$, where two missing points 0 and $\infty$ are presumably corresponding to two singular torus fibers. Indeed, $D^b\mathcal{F}$ contains the cones $\text{Cone}(L_0 \xrightarrow{\lambda_{a\rho_z}} L_1), \text{Cone}(L_0 \xrightarrow{\lambda_{b\rho_z}} L_1)$, and we believe that they are isomorphic to two singular fibers that pass through $z = a$ and $z = b$, respectively. Although a similar argument as in the proof of Proposition 1.5.7 seem to go through, we do not spell this out here due to technical reasons.

Notice that the above Lagrangian submanifolds are all special, thus they are expected to be stable objects in the Fukaya category. Later in Section 1.6, we will study their transformations into noncommutative resolution of the conifold $Y$ to give stable quiver representations.

Remark 1.5.10. By the equivalence in Theorem 1.5.3, Lagrangians spheres correspond to line bundle
on the exceptional curve $C$ (or their shifts) as follows:

\[
S_m \mapsto \mathcal{O}_C(m - 1) \quad \text{for } m \geq 1 \\
S_n \mapsto \mathcal{O}_C(n - 1)[1] \quad \text{for } n \leq 0.
\]

One can easily check this comparing the cone relations in Proposition 1.5.9 and exact sequences consisting of line bundles and skyscraper sheaves on $C$.

### 1.5.2 Mirror to perverse point sheaves

In this section, we describe how torus fibers (intersecting $L_0$ and $L_1$) are affected by A-flop. We will see that they behave precisely in the same way as skyscraper sheaves supported at points in $C(\subset \hat{Y})$. Note that points in $\hat{Y}$ are mirror to torus fibers in SYZ point of view, and those in $C$ are mirror to torus fibers that intersect $L_i$. Thus it is natural to expect that those torus fibers are transformed to unstable objects (i.e. non-special Lagrangians) which can be written as mapping cones analogous to (1.3).

![Figure 1.15: Transformation of torus fibers under $\rho$.](image)

**Proposition 1.5.11.** The functor $\rho_*$ sends $(L_c, \rho_z)$ to $\text{Cone}(L'_0 \xrightarrow{\alpha'} L'_1)$ for a degree 1 morphism $\alpha'$.
Proof. The proof is the same as that of Lemma 1.5.6, except that $\alpha'$ in $HF(L'_{0,1})$ now represent outer (non-convex) angle in the z-plane picture. Alternatively, since $\rho_*$ sends cones to cones, and $\rho_*(L_0) = L'_0, \rho_*(L_1) = L'_1, we have$

$$\rho_*(L_c, \rho_z) = \rho_*(\text{Cone}(L_0 \xrightarrow{\alpha} L_1)) = \text{Cone}(\rho(L_0) \xrightarrow{\rho(\alpha)} \rho(L_1)) = \text{Cone}(L'_0 \xrightarrow{\alpha'} L'_1)$$

where $\alpha'$ is a degree one morphism from $L'_0$ to $L'_1$, and occupies outer angles (after z-projection) as in Figure 1.15.

\[ \square \]

Remark 1.5.12. Note that we have an exact triangle

$$L'_1 \rightarrow \rho_*(L_c, \rho_z) \rightarrow L'_0 \xrightarrow{[1]}$$

from Proposition 1.5.11. Thus $L'_1$ can be thought of as a subobject of $\rho_*(L_c, \rho_z)$. $L'_1$ has bigger phase than $\rho_*(L_c, \rho_z)$, which is another way to explain unstability of $\rho_*(L_c, \rho_z)$.

Likewise, flop sends most of Lagrangians spheres in $\{S_m : m \in \mathbb{Z}\}$ to non-special objects. In fact, it is easy to see from the picture that $S_0$ and $S_1$ are the only spheres in this family that remain special after A-flop. We conclude that the equivalence $\rho_*$ does not preserve the set of special Lagrangians, and hence the A-flop can be thought of as a nontrivial change of holomorphic volume form, while keeping the symplectic structure as its induced from a symplectomorphism. We will revisit this point of view in 1.6.6.

1.6 Non-commutative mirror functor for the deformed conifold and stability conditions

So far, we have studied A-flop on the smoothing $X_{s=0} (1.6)$ of the conifold mostly in SYZ perspective comparing with its SYZ mirror, the resolve conifold (taken away a divisor). In this section we will consider a certain quiver algebra as another mirror to $X_{s=0}$, which is well-known to be a noncommutative crepant resolution of the conifold. The relation between
noncommutative resolution and commutative one will be explained later (see Remark 1.6.3). The mirror quiver category will enable us to study stability conditions more explicitly.

There is a natural way to obtain the above quiver as a formal deformation space of the object $L = L_0 \oplus L_1$ (recall $L_0 = S_0$ and $L_1 = S_1$ are Lagrangian spheres with maximal/minimal phases) in the Fukaya category. By the result in [24], such a construction comes with an $A_\infty$-functor from a Fukaya category to the category of quiver representations. We will construct geometric stability conditions using the functor, and examine A-flop on these stability conditions.

We begin with an explicit computation of the $A_\infty$-structure on $CF(L, L)$, which is crucial to describe formal deformation space of $L$.

1.6.1 Floer cohomology of $L$

As our Lagrangian $L$ is given as a direct sum, $CF(L, L)$ consists of four components:

$$CF(L, L) = CF(L_0, L_0) \oplus CF(L_1, L_1) \oplus CF(L_0, L_1) \oplus CF(L_1, L_0).$$

The first two components are both isomorphic to the cohomology of the three-sphere as graded vector spaces, and hence have degree-0 and degree-3 elements only. These elements will not be used for formal deformations. We only take degree-1 elements for deformations, so that the $\mathbb{Z}$-grading is preserved.

Recall that $L_0$ and $L_1$ intersect along two disjoint circles. There are several computable models for $CF(L_0, L_1)$ and $CF(L_1, L_0)$ provided in [1]. Explicit computation was given in [21] using one of these models, which we spell out here.

**Theorem 1.6.1.** (See [21, Theorem 7.1].) The $A_\infty$-structure on $CF(L, L)$ are given as follows. As vector spaces,

$$CF(L_i, L_i) = \Lambda \langle 1_{L_i} \rangle \oplus \Lambda \langle [pt]_{L_i} \rangle \text{ for } i = 0, 1$$

$$CF(L_0, L_1) = \Lambda \langle X \rangle \oplus \Lambda \langle Z \rangle \oplus \Lambda \langle \bar{Y} \rangle \oplus \Lambda \langle \bar{W} \rangle$$

$$CF(L_1, L_0) = \Lambda \langle Y \rangle \oplus \Lambda \langle W \rangle \oplus \Lambda \langle \bar{X} \rangle \oplus \Lambda \langle \bar{Z} \rangle.$$
with degrees of generators given as

\[
\begin{align*}
\text{deg}1_{L_i} &= 0, \quad \text{deg}[pt]_{L_i} = 3, \\
\text{deg}X &= \text{deg}Y = \text{deg}Z = \text{deg}W = 1, \\
\text{deg}X &= \text{deg}Y = \text{deg}Z = \text{deg}W = 2.
\end{align*}
\]

We have \(m_1 \equiv 0\) and \(m_2 \equiv 0\). The only nontrivial operations are

\[
\begin{align*}
-m_2(X, \tilde{X}) &= -m_2(Z, \tilde{Z}) = m_2(\tilde{Y}, Y) = m_2(\tilde{W}, W) = [pt]_{L_0}, \\
m_2(X, X) &= m_2(\tilde{Z}, Z) = -m_2(\tilde{Y}, \tilde{Y}) = -m_2(W, W) = [pt]_{L_1}, \\
m_3(X, Y, Z) &= -m_3(Z, Y, X) = \tilde{W}, \quad m_3(Y, Z, W) = -m_3(W, Z, Y) = \tilde{X}, \\
m_3(Z, W, X) &= -m_3(X, W, Z) = \tilde{Y}, \quad m_3(W, X, Y) = -m_3(Y, X, W) = \tilde{Z}.
\end{align*}
\]

and those determined by the property of the unit \(1_{L_i}\).

In what follows, we take an alternative way to compute \(A_\infty\)-structure on \(CF(L, L)\) hiring pearl trajectories, which is more geometric in the sense that it shows explicitly the holomorphic disks (attached with Morse trajectories) contributing to the \(A_\infty\)-operations. This will also help us to have geometric understanding of various computations to be made later, although most of the proof will rely on algebraic arguments.

First we choose a generic Morse function \(f_i\) on \(L_i\) with minimum and maximum only for \(i = 0, 1\). We denote these critical points by \(1_{L_i}, [pt]_{L_i}\) by obvious analogy, where \(\text{deg}1_{L_i} = 0\) and \(\text{deg}[pt]_{L_i} = 3\). Then \(CF(L_i, L_i)\) is defined to be the Morse complex of \(f_i\), which is nothing but the 2-dimensional vector space generated by \(1_{L_i}, [pt]_{L_i}\).

The Morse trajectories of \(f_i\) are described as follows. Recall that the two Lagrangians \(L_0\) and \(L_1\) intersect along two disjoint circles which we denoted by \(S^1_a\) and \(S^1_b\) (lying over \(z = a\) and \(z = b\), respectively). We want to argue that generically, there is a unique gradient flow line in each \(L_i\) that runs from \(S^1_a\) to \(S^1_b\) and vice versa. We may assume that there are no critical points on \(S^1_a\) or \(S^1_b\) by genericity.

Let \(W^- (S^1_b)\) be the unstable manifold of \(S^1_b\) with respect to the Morse function \(f_i\) on \(L_i\), namely

\[
W^- (S^1_b) = \left\{ x \in L_0 : \phi(t) \in S^1_b \text{ for some } t \geq 0 \right\}
\]
where \( q^t \) is the flow of \( \nabla f_i \). Including the maximum (1\( L_i \) in our notation), the unstable manifold of \( S_1^1 \) is topologically a disk that bounds \( S_1^1 \). Now observe that two circles \( S_a^1 \) and \( S_b^1 \) form a Hopf link in \( L_i \). Therefore, generically \( S_a^1 \) intersects the disk \( W^- (S_b^1) \cup \{ max \} \) at one point as shown in Figure 1.16. Therefore we see that there is a unique trajectory flowing from \( S_a^1 \) to \( S_b^1 \). This is the only property of the Morse functions \( f_0 \) and \( f_1 \) which we will use later.

**Figure 1.16:** A trajectory of the Morse function \( f_i \) from \( S_a^1 \) to \( S_b^1 \).

For the other components of \( CF(L, L) \), we perturb \( L_1 \) in double conic fiber direction as in Figure 1.17, so that \( L_0 \) and \( L_1 \) intersect each other transversely at four different points after perturbation. Therefore, both \( CF(L_0, L_1) \) and \( CF(L_1, L_0) \) are generated by these four points, which we denote as follows.

\[
CF(L_0, L_1) = \Lambda \langle X \rangle \oplus \Lambda \langle Z \rangle \oplus \Lambda \langle \bar{Y} \rangle \oplus \Lambda \langle \bar{W} \rangle
\]

\[
CF(L_1, L_0) = \Lambda \langle Y \rangle \oplus \Lambda \langle W \rangle \oplus \Lambda \langle \bar{X} \rangle \oplus \Lambda \langle \bar{Z} \rangle
\]

with degrees of generators given as

\[
\deg X = \deg Y = \deg Z = \deg W = 1, \quad \deg \bar{X} = \deg \bar{Y} = \deg \bar{Z} = \deg \bar{W} = 2.
\]

Here, \( X \) and \( \bar{X} \) are represented by the same point, but regarded as elements in \( CF(L_0, L_1) \) and \( CF(L_1, L_0) \) respectively, and similar for \( Y, Z, W \). In fact, they can be thought of as
Poincare dual to each other. Obviously, they are all cycles (i.e. $m_1$-closed) since opposite strips (pairs of strips on cylinders in Figure 1.17) cancel pairwise. Therefore $CF(L, L)$ comes with a trivial differential.

Now we are ready to spell out $A_\infty$-algebra structure on $CF(L, L)$ in terms of the above model. Recall from [11, 90] that $A_\infty$-operation counts the configurations which consist of several holomorphic disks (pearls) joined by gradient trajectories as shown in Figure 1.17. The constant disk at $X$ (and $\bar{X}$) attached with flows to $[pt]_{L_i}$ contributes as $m_2(X, \bar{X}) = [pt]_{L_0}$. Similarly, we have

$$m_2(X, \bar{X}) = m_2(Z, \bar{Z}) = -m_2(\bar{Y}, Y) = -m_2(W, W) = -[pt]_{L_0},$$

$$m_2(\bar{X}, X) = m_2(\bar{Z}, Z) = -m_2(Y, \bar{Y}) = -m_2(W, \bar{W}) = [pt]_{L_1}.$$ 

Other $m_2$’s are either determined by properties of units $1_{L_i}$, or zero by degree reason.

![Figure 1.17: Pearl trajectory contributing to $m_3(X, Y, Z)$.](image)

Computation of $m_3$ involves more complicated pearl trajectories. We give an explicit picture for one of those trajectories, and the rest can be easily found in a similar way. In Figure 1.17, one can see a pearl trajectory consisting of two bigons connected by a gradient flow, which contributes to $m_3(X, Y, Z)$ with output $\bar{W}$. The red colored connecting flow in Figure 1.17 is precisely the gradient trajectory in Figure 1.16, and hence the corresponding moduli is isolated. The pearl trajectory degenerates into a Morse tree when perturbing $L_0$. 

back to the original position (see Figure 1.18), which one of models in [1] takes into account.

Figure 1.18: Morse trees for $m_3(X, Y, Z)$.

Consequently, we have the following complete list of nontrivial $m_3$-operations:

$$m_3(X, Y, Z) = -m_3(Z, Y, X) = \tilde{W}, \quad m_3(Y, Z, W) = -m_3(W, Z, Y) = \tilde{X},$$

$$m_3(Z, W, X) = -m_3(X, W, Z) = \tilde{Y}, \quad m_3(W, X, Y) = -m_3(Y, X, W) = \tilde{Z}.$$  

We remark that the symplectic area of a pearl trajectory becomes zero after taking limit back to the original clean intersection situation (so that it degenerates into a Morse tree), which explains why there are no $T$ appearing in the above computations. If one wants to keep working with the perturbed picture, one can simply rescale generators so that the coefficients of $m_3$ to be still 1.

Note that the $A_\infty$-structure computed in this way precisely coincides with the one given in Theorem 1.6.1.

1.6.2 Construction of mirror from L

As in [23], we take formal variables $x, y, z, w$ and consider the deformation parameter $b = xX + yY + zZ + wW$, and solve the following Maurer–Cartan equation:

$$m_1(b) + m_2(b, b) + m_3(b, b, b) + \cdots = 0. \quad (1.14)$$
where $m_k$ with inputs involving $x, y, z, w$ are defined simply by pulling out the coefficient of Floer generators to the front, i.e.,

$$m_k(x_1X_1, \cdots, x_kX_k) = (-1)^k x_k x_{k-1} \cdots x_1 m_k(X_1, \cdots, X_k).$$

Here $(-1)^k$ is determined by usual Koszul sign convention, and in particular, is positive when $x_i X_i$ is one of $\{xX, yY, zZ, wW\}$.

**Remark 1.6.2.** In [23], “weak” Maurer–Cartan equations were mainly considered (i.e. the right hand side of (1.14) replaced by $\lambda \cdot 1_\mathbf{L}$), but in our example, weak Maurer–Cartan equation cannot have solutions unless $\lambda = 0$ due to degree reason.

By Theorem 1.6.1, the Maurer–Cartan equation (1.14) is equivalent to

$$(zyx - xyz)\bar{W} + (wzy - yzw)\bar{X} + (xwz - zwx)\bar{Y} + (ywx - wxy)\bar{Z} = 0.$$ 

Therefore, $b$ is a solution of 1.14 if and only if $(x, y, z, w)$ is taken from the path algebra (modulo relations) $A$ of the following quiver with the potential:

$$Q : \quad \begin{array}{c}
\begin{tikzcd}
 v_0 \arrow{r}{x} \arrow{d}[swap]{z} & v_1 \arrow{d}[swap]{y} \arrow{r}{w} \\
 & v_1
\end{tikzcd}
\end{array}$$

$$\Phi = (xyzw)_{cyc} - (wzyx)_{cyc}$$

where the vertex $v_i$ corresponds to the object $L_i$ and arrows correspond to degree 1 morphisms between $L_i$'s (or their associated formal variables). See [23, Section 6] for more details. The quiver algebra has the following presentation:

$$A : = \frac{\Gamma Q}{\langle \partial_x \Phi, \partial_y \Phi, \partial_z \Phi, \partial_w \Phi \rangle} = \frac{\Gamma Q}{(xyz - zyx, yzw - zwx, zwx - xwz, wxy - ywx)}. \quad (1.15)$$

**Remark 1.6.3.** $(Q, \Phi)$ is well-known to be the noncommutative crepant resolution of the conifold (see for instance [27]). In fact, one can easily check that the subalgebra of $A$ consisting of loops based at one of vertices is isomorphic to the function algebra of the conifold. For e.g., if we set $\alpha = xy, \beta = xw, \gamma = zy, \delta = zw$, then the relations among $x, y, z, w$ force $\alpha, \beta, \gamma, \delta$ to commute with each other and to satisfy $\alpha \delta = \beta \gamma$. 

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By [23], we have an (triangulated) $A_{\infty}$-functor from the Fukaya category to $D^b\text{Mod}\ A$ (here, $D^b\text{Mod}\ A$ denotes dg-enhanced triangulated category)

$$\varPhi_L : D^b\mathcal{F} \to D^b\text{Mod}\ A.$$ 

In the rest of the section, we will use the degree shift $\Psi := \varPhi_L[3]$ instead of $\varPhi_L$ itself, in order to have the images of the geometric objects in $D^b\mathcal{F}$ lying in the standard heart $\text{Mod}\ A$ of $D^b\text{Mod}\ A$. This point will be clearer after the computation of $\Psi(S_i)$ in the next section.

We will also see that the functor is an equivalence onto a certain subcategory of $D^b\text{Mod}\ A$.

### 1.6.3 Transformation of $L_0$ and $L_1$ and their central charges

We first compute the images of $L_0$ and $L_1$ themselves under $\Psi$. $\Psi(L_0)$ is simply a chain complex over $A$ given by $CF(L, L_0)$, which is a direct sum

$$CF(L, L_0) = CF(L_0, L_0) \oplus CF(L_1, L_0) + \langle 1_{L_0}, [pt]_{L_0} \rangle \oplus \langle Y, W, \bar{X}, \bar{Z} \rangle$$

with a differential $m^b_1$. Recall that $m^b_1(p) = \sum_k m_k(b, \cdots, b, p)$.

We have already made essential computations for $m^b_1$ (see Theorem 1.6.1). Using the previous computation, the list of $m^b_1$ acting on the generators is given as follows:

- $m^b_1(1_{L_0}) = yY + wW$
- $m^b_1(Y) = xW\bar{Z} - zw\bar{X}$
- $m^b_1(W) = z\bar{X} - xy\bar{Z}$
- $m^b_1(\bar{X}) = x[pt]_{L_0}$
- $m^b_1(\bar{Z}) = z[pt]_{L_0}$
- $m^b_1([pt]_{L_0}) = 0$

Therefore $[pt]_{L_0}$ is the only nontrivial class in the cohomology. Moreover, $x[pt]_{L_0}$ and $z[pt]_{L_0}$ are zero in the cohomology, and hence we obtain a finite dimensional representation of $(Q, \Phi)$ over $\mathbb{C}$. Consequently, $m^b_1$-cohomology of $\Psi(L_0) \in D^b\text{Mod}\ A$ is 1-dimensional vector space (over $\mathbb{C}$) supported over the vertex $v_0$, generated by the class $[pt]_{L_0}$. We remark
that this vector space sits in degree 0 part due to the shift \( \Psi = \Psi^L[3] \).

Likewise, \( \Psi(L_1) \) gives a 1-dimensional C-vector space supported over \( v_1 \) after taking \( m^1 \)-cohomology. In particular, the image of \( \Psi \) lies in a subcategory \( D^b \text{mod} \, A \) consisting of objects with finite dimensional cohomology.

**Theorem 1.6.4.** \( \Psi: D^b F! \rightarrow D^b \text{mod} \, A \) is a fully faithful embedding, which sends \( L_0 \) and \( L_1 \) to their corresponding vertex simples. Moreover, it is an equivalence onto \( D^b_{\text{nil}} \text{mod} \, A \), the full subcategory of \( D^b \text{mod} \, A \) consisting of objects with nilpotent cohomologies.

**Proof.** As \( D^b_{\text{nil}} \text{mod} \, A \) is a full subcategory of \( D^b \text{mod} \, A \), it suffices to prove that

\[
\Psi: D^b F! \rightarrow D^b_{\text{nil}} \text{mod} \, A
\]

is an equivalence. Since the image of generators \( L_0 \) and \( L_i \) are vertex simples which are nilpotent over \( A \), \( \Psi \) lands on \( D^b_{\text{nil}} \text{mod} \, A \).

We prove that the morphism level functor on \( \text{hom}(L_i, L_j) \) \( (i, j = 0, 1) \) induces isomorphisms of cohomology groups. Without loss of generality, it is enough to consider \( \text{hom}(L_0, L_0) \) and \( \text{hom}(L_0, L_1) \). By [23, Theorem 6.10], we know that both

\[
\Psi_1: \text{hom}(L_0, L_0) \rightarrow \text{hom}(\Psi(L_0), \Psi(L_0))
\]

\[
\Psi_1: \text{hom}(L_0, L_1) \rightarrow \text{hom}(\Psi(L_0), \Psi(L_1))
\]

induce injective maps on the level of cohomology. Thus, it is enough to check that

\[
\dim \text{Ext}(\Psi(L_0), \Psi(L_0)) = 2 \quad \text{and} \quad \dim \text{Ext}(\Psi(L_0), \Psi(L_1)) = 4.
\]

On the other hand, it is known by [28] (see [93] also) that \( D^b_{\text{nil}} \text{mod} \, A \) is equivalent to \( \mathcal{D}_{Y/Y} \) (1.2) with vertex simples corresponding to \( \mathcal{O}_C \) and \( \mathcal{O}_C(-1)[1] \). Therefore, the computation of endomorphisms of \( \mathcal{O}_C \oplus \mathcal{O}_C(-1)[1] \) due to [21, Section 5] finishes the proof. \( \Box \)

From now on, we take \( D^b_{\text{nil}} \text{mod} \, A \) to be the target category of \( \Psi \).

Let \( z_i \) be the central charge of \( L_i \), namely \( z_i := \int_{L_i} \Omega \). We define a central charge on quiver representations as follows. For a representation \( V := (V_0, V_1) \) of \( (Q, \Phi) \) (with some
maps between $V_0$ and $V_1$ which we omitted),

$$Z(V) := z_0 \dim V_0 + z_1 \dim V_1. \quad (1.16)$$

We define a dimension vector of a representation $V$ by $\dim(V) := (\dim V_0, \dim V_1) \in Z_{\geq 0}^2$ for later use. For general objects in $D^b\text{mod} \mathcal{A}$, $V_i$ above should be replaced by the corresponding cohomology.

**Proposition 1.6.5.** The object level functor $\Psi_0 : \text{Obj} (D^b \mathcal{F}) \to \text{Obj} (D^b_{\text{nil}} \text{mod} \mathcal{A})$ is a central charge preserving map.

**Proof.** The statement is obviously true for $\mathbb{L}_0$ and $\mathbb{L}_1$ as they are mapped to modules with 1-dimensional cohomology supported at the corresponding vertices. Since the central charges on both sides are additive (i.e. they are the maps from the $K$-groups) and $\Psi$ is a triangulated functor, the statement directly follows. \qed

In particular, special Lagrangians $\mathbb{L}_0$ and $\mathbb{L}_1$ are sent to simple and hence stable objects on quiver side.

### 1.6.4 Stables on quiver side

Set $\zeta_i$ to be the argument of $z_i$ taken in $(0, \pi]$ for $i = 0, 1$. Since $\mathbb{L}_0$ and $\mathbb{L}_1$ are special Lagrangians, $\zeta_i$ is nothing but the phase of $\mathbb{L}_i$. According to our convention (see the discussion below Theorem 1.5.2), we have $\zeta_0 > \zeta_1$.

Nagao and Nakajima used the following notion of stability for the abelian category $\text{mod} \mathcal{A}$.

**Definition 1.6.6.** We define stability of quiver representations of $(Q, \Phi)$ as follows.

1. We define the phase function $\zeta$ on $\text{mod} \mathcal{A}$ by

$$\zeta(V) = \zeta_0 \dim V_0 + \zeta_1 \dim V_1 \in \mathbb{R}$$

for $V \in \text{mod} \mathcal{A}$. 

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2. An object $V$ of $\text{mod} \mathcal{A}$ is said to be stable (semistable resp.) if for any subobject $W$ of $V$,

$$\zeta(W) < \zeta(V) \quad (\zeta(W) \leq \zeta(V) \text{ resp.}).$$

It is elementary to check that $\arg Z(V) \in (0, \pi]$ for $V \in \text{mod} \mathcal{A}$ induces the equivalent stability on the abelian category $\text{mod} \mathcal{A}$ as Definition 1.6.6. In fact, one can easily check that two quantities have the same ordering relations.

Lemma 1.6.7. For $V, W \in \text{mod} \mathcal{A}$, $\zeta(V) \leq \zeta(W)$ if and only if $\arg Z(V) \leq \arg Z(W)$.

The proof is elementary, and we omit here. In particular, the set of stable objects remains the same even if we use $\arg Z(V)$ in place of $\zeta(V)$.

Remark 1.6.8. One advantage of using $Z(V)$ (rather than $\zeta$) is that it can be lifted to a Bridgeland stability condition on $D^b_{\text{nil}} \text{mod} \mathcal{A}$. In fact, one can check that it is equivalent to the perverse stability on $D_{\hat{\mathcal{Y}}/\mathcal{Y}}$ via the identification $D^b_{\text{nil}} \text{mod} \mathcal{A} \cong D_{\hat{\mathcal{Y}}/\mathcal{Y}}$. [74, Remark 4.6] gives a correspondence between stable objects in the heart of each category.

We briefly review the classification of stable objects in $\text{mod} \mathcal{A}$ following [74]. We first set up the notation as follows. We define $\mathcal{A}$-module $V_{\pm}(m)$ by $(C^m, C^{m \pm 1})$ together with the maps corresponding arrows given as in the left two columns in (1.17) where right (left, resp.) arrows are $x, z$ ($y, w$, resp.). Likewise, $V^\dagger_{\pm}(m)$ denotes $\mathcal{A}$-module $(C^m, C^{m \mp 1})$ which can be visualized as the right two columns in (1.17). Up to isomorphism, one can assume that all arrows act by the identity map from $C$ to itself.
We are now ready to state the classification result by Nagao-Nakajima.

**Theorem 1.6.9.** ([74, Theorem 4.5]) Stable modules $W$ in $\text{mod} A$ are classified as follows:

1. when $\xi_0 > \xi_1$
   - $V_+(m)$ ($m \geq 1$)
   - $A$-module $W$ with $\dim W = (1, 1)$ parametrized by $\hat{Y}$
   - $V_-(n)$ ($n \geq 0$)

2. when $\xi_0 < \xi_1$
   - $V'_+(m)$ ($m \geq 1$)
   - $A$-module $W$ with $\dim W = (1, 1)$ parametrized by $\hat{Y}^+$
   - $V'_-(n)$ ($n \geq 0$)

In particular, $\{\xi_1 = \xi_2\}$ gives a wall, and the wall structure consists of only two chambers.

In order to precisely match the pictures in [74], one should locate the vertex $v_1$ to the left in the quiver diagram. For instance, the vertex simple at the left vertex in [74] corresponds to $O_C$ whereas in our case $v_1$ (sitting on the right) represents the Lagrangian $L_1 (= S_1)$ which is mirror to $O_C$. See [74, Remark 4.6].
Remark 1.6.10. Modules with dimension vector $(1, 1)$ in (1) and (2) of Theorem 1.6.9 can be presented as

\[
\begin{array}{c}
\text{C} \\
\xymatrix{
\ar[r]^{x} & \text{C} \\
\ar[r]_{z} & \text{C} \\
\ar[r]_{y} & \text{C} \\
\ar[r]^{w} & \text{C}
}
\end{array}
\] (1.18)

for some $(x, y, z, w) \in \mathbb{C}^4$. Note that (1.18) is nilpotent if and only if either $x = z = 0$ or $y = w = 0$. Notice that (1.18) has three dimensional deformation (scaling actions of arrows $x, y, z, w$ up to overall rescaling) whereas $V_{\pm}(k)$ is rigid for all $k$. Later, we will see that (1.18) is mirror to a Lagrangian torus fibers whose first Betti number is 3, and $V_{\pm}(k)$ is mirror to a Lagrangian sphere.

In what follows, we shall show that transformations of special Lagrangians in $D^b F$ by our functor $\Psi$ recovers all the stable representations which are nilpotent.

1.6.5 Transformation of special Lagrangians in $D^b F$ before/after flop

We next compute the transformation of other geometric objects in $D^b F = D^b (L_0, L_1)$ under the mirror functor $\Psi$ induced by $L = L_0 \oplus L_1$. Recall from 1.5.1 that this category contains Lagrangian spheres and torus fibers (intersecting spheres) as geometric objects.

We begin with a torus fiber $L_c$ ($a < c < b$) in $X$ that intersects each of $L_0$ and $L_1$ along $\mathbb{T}^2$ at $z = c$. (see 1.5.1). Suppose $L_c$ is also equipped with a flat line bundle $U$ whose holonomy along a circle in $z$-direction is $\rho$. Recall that we only consider $U$ with trivial holonomies along both of double conic fiber directions as otherwise they would not belong to the category.

Lemma 1.6.11. The transformation of $(L_c, \rho)$ by $\Psi$ is the representation of $(Q, \Phi)$ (after taking cohomology) given as

\[
\begin{array}{c}
\text{C} \\
\xymatrix{
\ar[r]^{\lambda_2} & \text{C} \\
\ar[r]_{\lambda_1} & \text{C} \\
\ar[r]_{\alpha_0} & \text{C} \\
\ar[r]^{\alpha_1} & \text{C}
}
\end{array}
\] (1.19)
where \([\lambda_1 : \lambda_2]\) parametrizes the exceptional curve \(C\) in \(\hat{Y}\), and the maps in the other direction (i.e. actions of \(y\) and \(w\)) are zero.

**Proof.** Recall from Proposition 1.5.7 that \((L_c, \rho)\) is a mapping cone \(\text{Cone}(L_0 \xrightarrow{\alpha} L_1)\) for nonzero \(\alpha \in HF^1(L_0, L_1)\) uniquely determined up to scaling. i.e. the following defines an exact triangle in \(D^bF\)

\[
L_1 \rightarrow (L_c, \rho_z) \rightarrow L_0[1]. \tag{1.20}
\]

Since the functor \(\Psi\) is a triangulated equivalence and \(\alpha \neq 0\), \(\Psi(L_c, \rho_z)\) is also a nontrivial extension of \(\Psi(L_1)\) and \(\Psi(L_0)\). It is elementary exercise to show that all nontrivial extensions of these two representations which are nilpotent should be of the form given in (1.19). Moreover, since we have \(HF^1(L_0, L_1) \cong \text{Ext}^1(\Psi(L_0), \Psi(L_1))\) by the morphism level functor of \(\Psi\), we see that \(\Psi(L_c, \rho)\) gives all possible extensions as \(\alpha\) varies, or equivalently \(c\) and \(\rho\) (in \((L_c, \rho)\)) vary. Note that the family of such \((L_c, \rho)\) precisely parametrizes points in the exceptional curve \(C\) by SYZ mirror construction due to [21]. \(\square\)

We remark that the cones \(\text{Cone}(L_0 \xrightarrow{\lambda_1 a_i} L_1)\) and \(\text{Cone}(L_0 \xrightarrow{\lambda_0 b_i} L_1)\) in \(D^bF\) (which are supposedly singular torus fibers) are sent to the representations of the same form with one of \(\lambda_i\) being zero, which together with \(\Psi(L_c, \rho)\) completes the \(\mathbb{P}^1\)-family of stable representations.

**Geometric argument**

We provide a more geometric computation of \(\Psi(L_c, \rho)\) making use of pearl trajectory model introduced in 1.6.1. As shown in Figure 1.12, \(L_c\) intersect \(L_0 \cup L_1\) at eight different points after perturbation. Let

\[
L_0 \cap L_c := \{a_{00}, a_{01}, a_{10}, a_{11}\}
\]

\[
L_1 \cap L_c := \{b_{00}, b_{01}, b_{10}, b_{11}\}
\]

(see Figure 1.12) where \(\deg a_{ij} = i + j + 1\) and \(\deg b_{ij} = i + j\). Note that the degrees for generators in \(L_0 \cap L_c\) are shifted by 1 due to Floer theoretic grading from intersections in \(z\)-direction.
for the above generators can be computed as follows:

\[
\begin{align*}
m_1^b(b_{00}) &= (T^{\omega(\Delta_2)} x - \rho T^{\omega(\Delta_2)} z) a_{00} + xy b_{01} + yz b_{10}, & m_1^b(a_{00}) &= yxa_{01} + wza_{10} \\
m_1^b(b_{01}) &= (\rho T^{\omega(\Delta_2)} z - T^{\omega(\Delta_1)} x) a_{01} - zw b_{11}, & m_1^b(a_{01}) &= -wza_{11} \\
m_1^b(b_{10}) &= (\rho T^{\omega(\Delta_2)} z - T^{\omega(\Delta_1)} x) a_{10} + xy b_{11}, & m_1^b(a_{10}) &= yxa_{11} \\
m_1^b(b_{11}) &= (T^{\omega(\Delta_1)} x - \rho T^{\omega(\Delta_2)} z) a_{11}, & m_1^b(a_{11}) &= 0
\end{align*}
\]

Here, the coefficients of the form \((T^{\omega(\Delta_1)} x - \rho T^{\omega(\Delta_2)} z)\) in front of \(a_{ij}\) (in \(m_1^b(b_{ij})\)) is from the pair of holomorphic polygons projecting to the shaded triangles \(\Delta_1\) and \(\Delta_2\) in \(z\)-plane shown in Figure 1.12. (They are the same triangles appearing in the proof of Lemma 1.5.6.) Other terms are contributed by pearl trajectories consisting of two 2-gons joined by a gradient flow, which have the same shape as the one contributing to \(m_3\) drawn in Figure 1.17.

Therefore the \(m_1^b\)-cohomology is generated by \([a_{11}]\) as \(A\)-module, and we have \(T^{\omega(\Delta_1)} x[a_{11}] = \rho T^{\omega(\Delta_2)} z[a_{11}]\) (since their difference is \(m_1^b(b_{11})\)). Moreover, these are the only nontrivial scalar multiplication since \(w z[a_{11}] = y x[a_{11}] = 0\). In particular, we see that \(\lambda_1, \lambda_2\) in (1.19) satisfy \(\lambda_1 : \lambda_2 = \rho T^{\omega(\Delta_2)} : T^{\omega(\Delta_1)}\).

On the other hand, \(\Psi\) transforms Lagrangian spheres \(S_k\) into the following stable representations.

**Proposition 1.6.12.** The images of spheres \(\{S_k : k \in \mathbb{Z}\}\) in \(D^b F\) (after taking cohomology) are given as follows:

1. For \(m \geq 1\), \(\Psi(S_m) = V_+(m)\),

2. For \(n \leq 0\), \(\Psi(S_n) = V_-(|n|)\).

where \(V_\pm(k)\) are as in (left two columns of) (1.17).

**Proof.** We will only prove (1), and the proof of (2) can be done in a similar manner. The statement is true for \(m = 1\) by Theorem 1.6.4. We will proceed by induction. Let us assume that it is true for \(m\). By Proposition 1.5.9, we have

\[
S_{m+1} \cong \text{Cone}(L_c \xrightarrow{\delta} S_m)
\]
for some \( a \) which implies the exact triangle \( S_m \to S_{m+1} \to L_c \xrightarrow{[1]} \) in \( D^b F \). Since \( a \) is a nonzero element in the Floer cohomology, \( S_{m+1} \) is a nontrivial extension of \( S_m \) and \( L_c \). We see that \( \Psi(S_{m+1}) \) is an extension of \( V_+(m) \) and \( \Psi(L_c) = C \xrightarrow{\lambda} C \). For simplicity, we choose a suitable \( L_c \) such that \( \lambda_1 = -\lambda_2 \), and hence, after rescaling two arrows act as \( id \) and \( -id \) respectively.

On the other hand, we already know one nontrivial extension of these two modules, which is nothing but \( V_+(m + 1) \). To see this, observe that the map \( \sigma : V_+(m) \to V_+(m + 1) \) defined by

\[
\begin{align*}
\sigma : \\
e_i &\mapsto e_i + e_{i+1}, \\
f_j &\mapsto f_j + f_{j+1}
\end{align*}
\]

is an injective \( A \)-module map, where \( e_i \) (resp. \( f_j \)) denotes the standard basis of \( V_+(m) \) spanning the \( i \)-th (resp. \( j \)-th) component \( C \) over \( v_0 \) (resp. \( v_1 \)) for \( 1 \leq i \leq m - 1 \) (resp. \( 1 \leq j \leq m \)). See (1.21) below.

\[
\begin{align*}
\langle e_1 \rangle &= C \\
\langle e_2 \rangle &= C \\
\vdots & \\
\langle e_{m-2} \rangle &= C \\
\langle e_{m-1} \rangle &= C \\
V_+(m) & \xrightarrow{x} C = \langle f_1 \rangle \\
& \xrightarrow{z} C = \langle f_2 \rangle \\
& \xrightarrow{z} C = \langle f_{m-1} \rangle \\
& \xrightarrow{z} C = \langle f_m \rangle \\
V_+(m + 1) & \xrightarrow{id} C
\end{align*}
\]

Now the cokernel of \( \sigma \) is spanned by \([e_1]\) and \([f_1]\), and \( x[e_1] = [xe_1] = [f_1] \) and \( z[e_1] = [ze_1] = [f_2] = -[f_1] \) since \( f_1 + f_2 \) is in the image of \( \sigma \). Therefore, we have

\[
0 \to V_+(m) \xrightarrow{\sigma} V_+(m + 1) \to \left( C \xrightarrow{\text{id or } -id} C \right) \to 0.
\]
Moreover, $V_+(m+1)$ is the only nontrivial extension of $V_+(m)$ and $\mathbb{C} \xrightarrow{\text{id}} \mathbb{C}$ since

$$\dim \ Ext \left( \mathbb{C} \xrightarrow{\text{id}} \mathbb{C}, V_+(m) \right) \cong \dim HF^1(L_c, S_m) = 1.$$ (Here, we used the fact that $\Psi$ is an equivalence, and the induction hypothesis that $\Psi(S_m) = V_+(m)$.) We conclude that $S_{m+1}$ should map to $V_+(m+1)$ by $\Psi$. $\square$

**Geometric argument**

Alternatively, one can compute the image of spheres under $\Psi$ directly by holomorphic disk counting (or more precisely computing $m^b_1$ on $CF(L, S_i)$). We give a brief sketch of the computation for $S_m$ for $m \geq 1$. On $z$-plane, the projection of $S_m$ intersects the interval $(a, b)$ ($m-1$)-times (not including the end points $a$ and $b$). Here, we perturb $L_0, L_1$ and $S_m$ along the fiber direction as in the proof of Lemma 1.6.11 so that they mutually intersect transversely.

Let us denote these $m-1$ points in $(a, b)$ by $c_1, c_2, \ldots, c_{m-1}$ as shown in Figure 1.19. These are the only locations where one has the highest degree intersections (i.e. degree 3 elements in $CF(L, S_m)$ that map to degree 0 elements by $\Psi = \Psi^L[3]$). Cohomology long exact sequence tells us that it is enough to consider these elements, as cohomologies of $\Psi(S_0)$ and $\Psi(S_1)$ are supported only at this degree. We remark that the intersection $L \cap S_m$ occurring at $z=a$ and $z=b$ only produces degree 0 and 1 elements in the Floer complex, which are not in the highest degree.

Denote these highest degree intersection points by

$$\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_{m-1}$$

where $\tilde{c}_i$ projects down to $c_i$. (Obviously there is only one highest degree intersection point over each $\tilde{c}_i$ after perturbation.) There are pairs of triangles whose $z$-projections are given as shaded region in Figure 1.19. Each of these pair contributes to $m^b_1$ with the same input, say $r_i$ as in the figure, and gives rise to

$$m^b_1(r_i) = z\tilde{c}_i - x\tilde{c}_{i+1}. \quad (1.22)$$
Figure 1.19: Contributing holomorphic polygons to (the differential of) $\Psi(S_m)$.

Also, there are pearl trajectories with a similar shape to the ones contributing to formal deformation ($m_3$) of $L$, which lies over the interval $[a, b]$. These trajectories induce

$$m_1^b(p_i) = \pm yx \tilde{c}_i, \quad m_1^b(q_i) = \pm wz \tilde{c}_i$$  \hspace{1cm} (1.23)

where $p_i$ and $q_i$ are the two degree 2 intersection points lying over $c_i$.

Set $e_i := [\tilde{c}_i]$ which belong to the $v_0$-component of the resulting quiver representation. (1.22) implies

$$ze_i = xe_{i+1},$$

and we denote this element by $f_i$ which belongs to $v_1$-component. We also set $f_1 := xe_1$ and $f_m := ze_{m-1}$. Combining (1.22) and (1.23) implies all other actions of $A$ are trivial. Therefore the resulting cohomology has precisely the same as $V_\pm(m)$ as a $A$-module.

**Effect of A-flop**

The images of $L_0 = S_0$ and $L_1 = S_1$ under the symplectomorphism $\rho : X_{s=0} \to X_{s=1}$ gives another Lagrangian spheres $S'_0$ and $S'_1$ (see Figure 1.11). In addition, we have a new sequence of special Lagrangian spheres $\{S'_k : k \in \mathbb{Z}\}$ depicted in Figure 1.10. Readers are
warned that $S'_k$ is not a image of $S_k$ under A-flop unless $k = 0$ or $k = 1$. Note that phases of other spheres lie between those of $S'_0$ and $S'_1$ due to our choice of gradings (or orientations) as in Figure 1.10. (Recall that we measure the phase angles in clockwise direction.)

We perform the same mirror construction making use of $L'_0 = S'_0$ and $L'_1 = S'_1$ which obviously produce the same quiver with the potential. Only difference is that now the ordering of the phases of $L'_0$ and $L'_1$ are switched. Namely, in this case, we have $\zeta'_0 < \zeta'_1$ where $\zeta'_i$ is a phase of $L'_i$. Thus one can naturally expect to obtain stable representations in (2) of Theorem 1.6.9 by applying the resulting functor $\Psi' : D^b(S'_0, S'_1) \to D^b_{\text{nil}} A$ to $\{S'_k\}$ and new torus fibers (which can be represented as straight vertical lines in Figure 1.10).

**Proposition 1.6.13.** Torus fibers intersecting $S'_0$ and $S'_1$ are transformed under $\Psi'$ into

$$C \xleftarrow{\lambda_1 / \lambda_2} C$$

after taking $m^!$-cohomology, where $[\lambda_1 : \lambda_2]$ parametrizes the exceptional curve $C^!$ in $\hat{Y}^!$. The images of spheres $\{S'_k : k \in \mathbb{Z}\}$ in $D^b Y$ (after taking cohomology) are given as follows:

1. For $m \geq 1$, $\Psi(S'_m) = V^+_+(m)$,
2. For $n \leq 0$, $\Psi(S'_n) = V^+_0(|n|)$.

where $V^+_\pm(k)$ are as in (right two columns of) (1.17).

The proof is essentially the same as that of Proposition 1.6.12, and we will not repeat here.

**1.6.6 Bridgeland stability on $D^b Y$**

As mentioned in Remark 1.6.8, $D^b_{\text{nil}} A$ admits a Bridgeland stability through the isomorphism $D^b_{\text{nil}} A \cong D_{\hat{Y} / Y}$. Therefore, $D^b Y$ also admits a Bridgeland stability condition $(Z, P)$ by pulling-back the one on $D^b_{\text{nil}} A$ via the equivalence $\Psi$. By Proposition 1.6.5, we see that the pull-back stability on $D^b Y$ is geometric in the sense that its central charge is given by the period $\int \Omega_{s=0}$. Moreover, the discussion in 1.6.5 tells us that the special Lagrangian spheres $S_k$ and tori (that intersects spheres) are stable objects in the heart.
After applying A-flop, we consider the subcategory $\mathcal{F}^{'}$ generated by $S'_0$ and $S'_1$ in $X_{s=1}$.

By the same reason, $D^b \mathcal{F}^{'}$ admits a Bridgeland stability condition whose stable objects (in the heart) are special Lagrangian spheres $S'_i$’s and new torus fibers in $X_{s=1}$. Note that their inverse image under $\rho : X_{s=0} \to X_{s=1}$ are $\{S_i^\dagger\}$, the A-flop of the spheres $\{S_i\}$ in $X_{s=0}$, which were discussed in Section 1.4.3.

By pulling back this stability condition on $D^b \mathcal{F}^{'}$ via $\rho_*$, we get a new stability condition $(Z^\dagger, P^\dagger)$ on $D^b \mathcal{F}$ (whose central charge comes from $\rho^* \Omega_{s=1}$) with the set of stable objects

$$P^\dagger = \{L^\dagger : L \text{ is stable with respect to } (Z, P)\}.$$

This proves Theorem 1.1.3.
Chapter 2

Mirror of Weil–Petersson metric\textsuperscript{1}

2.1 Introduction

The present chapter investigates some differential geometric aspects of the space of Bridgeland stability conditions on a Calabi–Yau triangulated category.

The motivation of our study comes from mirror symmetry. It is classically known that the complex moduli space $\mathcal{M}_{\text{cpx}}(Y)$ of a projective Calabi–Yau manifold $Y$ comes equipped with a canonical Kähler metric, called the Weil–Petersson metric. The existence of such a natural metric often implies strong results that one cannot obtain by purely algebraic methods. In the case of Calabi–Yau threefold, this metric provides a fundamental differential geometric tool, known as the special Kähler geometry, to study mirror symmetry. In light of duality between complex geometry and Kähler geometry of a mirror pair of Calabi–Yau manifolds, a natural problem is to construct the mirror object of the Weil–Petersson geometry, which should be defined on the stringy Kähler moduli space $\mathcal{M}_{\text{Kah}}(X)$ of a mirror Calabi–Yau manifold $X$ of $Y$. However, there is yet no mathematical definition of $\mathcal{M}_{\text{Kah}}(X)$ at present.

In the celebrated work [14], Bridgeland introduced and studied stability conditions on a triangulated category with the hope of rigorously defining $\mathcal{M}_{\text{Kah}}(X)$. He conjectured and confirmed in several important cases [8, 14, 15, 16] that the string theorists’ stringy Kähler

\textsuperscript{1}Co-authored with Atsushi Kanazawa and Shing-Tung Yau. Reference: [36].
moduli space $\mathcal{M}_{\text{Kah}}(X)$ admits an embedding into the double quotient

$$\text{Aut}(\mathcal{D}_X) \backslash \text{Stab}(\mathcal{D}_X) / \mathbb{C}$$

of the space $\text{Stab}(\mathcal{D}_X)$ of Bridgeland stability conditions on the bounded derived category $\mathcal{D}_X = \mathbb{D}^b\text{Coh}(X)$ of coherent sheaves on $X$.

The purpose of this chapter is to provide a step toward differential geometric study of $\mathcal{M}_{\text{Kah}}(X)$ via the space of Bridgeland stability conditions. Although we are mostly interested in geometric situations, we will use a more general categorical language in this chapter. On careful comparison of the two sides of Kontsevich’s homological mirror symmetry $\mathbb{D}^b\text{Coh}(X) \cong \mathbb{D}^b\text{Fuk}(Y)$, we will give a provisional definition of Weil–Petersson geometry and propose a conjecture which refine a previously known one. We will also provide some supporting evidence by computing basic geometric examples. A key example is the following, where our Weil–Petersson metric coincides with the classical Bergman metric on a Siegel modular variety (notations will be explained later).

**Theorem 2.1.1** (Theorem 2.4.8, Corollary 2.4.12). *Let $A$ be the self-product $E_\tau \times E_\tau$ of an elliptic curve $E_\tau$. Then there is an identification

$$\overline{\text{Aut}_{\text{CY}}(\mathcal{D}_A)} \backslash \text{Stab}_X^+(\mathcal{D}_A) / \mathbb{C}^\times \cong \text{Sp}(4, \mathbb{Z}) \backslash \mathfrak{h}_2.$$*

Moreover, the Weil–Petersson metric on the stringy Kähler moduli space $\overline{\text{Aut}_{\text{CY}}(\mathcal{D}_A)} \backslash \text{Stab}_X^+(\mathcal{D}_A) / \mathbb{C}^\times$ is identified with the Bergman metric on the Siegel modular variety $\text{Sp}(4, \mathbb{Z}) \backslash \mathfrak{h}_2$.

This result is compatible with the mirror duality between $A$ and the principally polarized abelian surface. In fact, the complex moduli space of the latter is given by $\text{Sp}(4, \mathbb{Z}) \backslash \mathfrak{h}_2$. Another example is a quintic threefold $X \subset \mathbb{P}^4$. Assuming that a conjectural Bridgeland stability condition exists, we will observe that the Weil–Petersson metric is given by a quantum deformation of the Poincaré metric near the large volume limit.

It is worth noting that some aspects of the ideas in this chapter were presented in a series of Wilson’s works [101, 104, 105] on metrics on the complexified Kähler cones. In fact, his works are also motivated by mirror symmetry since the complexified Kähler cone...
is expected to give a local chart of $\mathcal{M}_{\text{Kah}}(X)$ near a large volume limit. For example, in the case of a Calabi–Yau 3-fold $X$, the curvature of the so-called asymptotic Weil–Petersson metric was shown to be closely related to the trilinear form on $H^2(X, \mathbb{Z})$ [101, 105].

On the other hand, an advantage of our approach taken in this chapter is the fact that our Weil–Petersson metric is global and makes perfect sense away from large volume limits, in contrast to Wilson’s local study. As a matter of fact, the global aspects of the moduli space are of special importance in recent study of mirror symmetry and string theory.

However, the real difficulty of the subject is to understand the precise relation between the stringy Kähler moduli space $\mathcal{M}_{\text{Kah}}(X)$ and the space of Bridgeland stability conditions. Nevertheless, as an application of our work, we find a new condition on the conjectural embedding of $\mathcal{M}_{\text{Kah}}(X)$ into a quotient of the space of Bridgeland stability conditions, namely the pullback of the Weil–Petersson metric on $\mathcal{M}_{\text{Kah}}(X)$ should be non-degenerate (Conjecture 2.3.6). This simply means that the mirror identification $\mathcal{M}_{\text{Kah}}(X) \cong \mathcal{M}_{\text{cpx}}(Y)$ respects the Weil–Petersson geometries.

This chapter is organized as follows. In Section 2.2, we provide basic backgrounds on the Bridgeland stability conditions and twisted Mukai pairings. In Section 2.3, after a brief review and reformulation of the classical Weil–Petersson geometry, we translate it in the context of Bridgeland stability conditions on a Calabi–Yau triangulated category. Section 2.4 is the heart of the chapter. We carry out detailed calculations for two classes of abelian surfaces to justify our proposal. Lastly, we discuss the case of a quintic 3-fold, and comment on further research directions.

We use the following notations and conventions throughout this chapter. For an abelian group $A$, we denote by $A_{\text{fd}}$ the quotient of $A$ by its torsion subgroup. We write $A_K := A \otimes_{\mathbb{Z}} K$ for a field $K$. $\text{ch}(-)$ denotes the Chern character and $\text{Td}_X$ denotes the Todd class of $X$. We write $\Re(z)$ and $\Im(z)$ for the real and imaginary parts of $z$ respectively. A Calabi–Yau manifold is a complex manifold whose canonical bundle is trivial. Throughout the chapter, we work over complex numbers $\mathbb{C}$.
2.2 Bridgeland stability conditions

2.2.1 Review: Bridgeland stability conditions

In the seminal work [14], Bridgeland introduced the notion of stability conditions on a triangulated category $D$, which was motivated from M.R. Douglas’ work on $\Pi$-stability in physics. We recall the definition and some basic properties of Bridgeland stability conditions that will be needed in later discussions.

Throughout the article, we assume that $D$ is essentially small, linear over $\mathbb{C}$, and is of finite type. The last condition means that for any pair of objects $E, F \in D$, the $\mathbb{C}$-vector space $\bigoplus_{i \in \mathbb{Z}} \text{Hom}_D(E, F[i])$ is of finite-dimensional.

The Euler form $\chi$ on the Grothendieck group $K(D)$ is given by the alternating sum

$$\chi(E, F) := \sum_i (-1)^i \dim_{\mathbb{C}} \text{Hom}_D(E, F[i]).$$

The numerical Grothendieck group $N(D) := K(D)/K(D)_{\perp \chi}$ is defined to be the quotient of $K(D)$ by the null space of the Euler pairing $\chi$.

We assume that $D$ is numerically finite, that is, $N(D)$ is of finite rank. One large class of examples of such triangulated categories is provided by the bounded derived category of coherent sheaves $\text{D}^b\text{Coh}(X)$ of a smooth projective variety $X$.

Now we recall the definition of Bridgeland stability conditions on a triangulated category $D$. We will only consider the Bridgeland stability conditions that are full and numerical.

**Definition 2.2.1 ([14]).** A (full numerical) stability condition $\sigma = (Z, P)$ on a triangulated category $D$ consists of:

- a group homomorphism $Z : N(D) \to \mathbb{C}$, and

- a collection of full additive subcategories $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$ of $D$,

such that:

1. If $0 \neq E \in P(\phi)$, then $Z(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$. 

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2. \( \mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1] \).

3. If \( \phi_1 > \phi_2 \) and \( A_i \in \mathcal{P}(\phi_1) \), then \( \text{Hom}(A_1, A_2) = 0 \).

4. For every \( 0 \neq E \in \mathcal{D} \), there exists a (unique) collection of distinguished triangles

\[
0 = E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{k-1} \rightarrow E
\]

such that \( B_i \in \mathcal{P}(\phi_i) \) and \( \phi_1 > \phi_2 > \cdots > \phi_k \). Denote \( \phi^+_i(E) := \phi_i \) and \( \phi^-_i(E) := \phi_k \). The mass of \( E \) is defined to be \( m_{\sigma}(E) := \sum |Z(B_i)| \).

5. (Support property [66]) There exists a constant \( C > 0 \) and a norm \( \| \cdot \| \) on \( \mathcal{N}(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{R} \) such that

\[
\|E\| \leq C|Z(E)|
\]

for any semistable object \( E \).

The group homomorphism \( Z \) is called the central charge, and the nonzero objects in \( \mathcal{P}(\phi) \) are called the semistable objects of phase \( \phi \). The additive subcategories \( \mathcal{P}(\phi) \) actually are abelian, and the simple objects of \( \mathcal{P}(\phi) \) are said to be stable.

We denote \( \text{Stab}(\mathcal{D}) \) the space of (full numerical) Bridgeland stability conditions on \( \mathcal{D} \). There is a nice topology on \( \text{Stab}(\mathcal{D}) \) introduced by Bridgeland, which is induced by the generalized distance:

\[
d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi_{\sigma_2}^-(E) - \phi_{\sigma_1}^-(E)|, |\phi_{\sigma_2}^+(E) - \phi_{\sigma_1}^+(E)|, |\log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)}| \right\} \in [0, \infty].
\]

The forgetful map

\[
\text{Stab}(\mathcal{D}) \rightarrow \text{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C}), \quad \sigma = (Z, \mathcal{P}) \mapsto Z
\]

is a local homeomorphism [14, 66]. Hence \( \text{Stab}(\mathcal{D}) \) is a complex manifold.

There are two important group actions on the space of Bridgeland stability conditions \( \text{Stab}(\mathcal{D}) \):
• The group of autoequivalences $\text{Aut}(\mathcal{D})$ on the triangulated category $\mathcal{D}$ acts on $\text{Stab}(\mathcal{D})$ as isometries with respect to the generalized metric: Let $\Phi \in \text{Aut}(\mathcal{D})$ be an autoequivalence,

$$\sigma = (Z, P) \mapsto \Phi \cdot \sigma := (Z \circ [\Phi]^{-1}, P'),$$

where $[\Phi]$ is the induced automorphism on $\mathcal{N}(\mathcal{D})$, and $P'(\phi) := \Phi(P(\phi))$.

• The abelian group of complex numbers $\mathbb{C}$ acts freely on $\text{Stab}(\mathcal{D})$: Let $z = x + iy \in \mathbb{C}$ be a complex number,

$$\sigma = (Z, P) \mapsto z \cdot \sigma := (e^z Z, P''),$$

where $P''(\phi) := P(\phi - \frac{y}{\pi})$.

These two actions commute with each other. Note that there actually is a larger group $\mathbb{C} \subset \text{GL}^+(2, \mathbb{R})$ acting on the space of stability conditions $\text{Stab}(\mathcal{D})$, which “shears” the central charges and also commutes with the $\text{Aut}(\mathcal{D})$-actions.

Remark 2.2.2. Let $\text{D}^b\text{Fuk}(Y)$ be the derived Fukaya category of a Calabi–Yau manifold $Y$. We fix a holomorphic volume form $\Omega$ of $Y$. It is a folklore conjecture (c.f. [16]) that there exists a Bridgeland stability condition on $\text{D}^b\text{Fuk}(Y)$ with central charge given by the period integral

$$Z(L) = \int_L \Omega.$$

Moreover, the special Lagrangian submanifolds of phase $\phi$ are the semistable objects of phase $\phi$ with respect to this stability condition.

2.2.2 Central charge via twisted Mukai pairing

Let $X$ be a smooth projective variety. Motivated by work of Mukai in the case of K3 surfaces, Căldăraru defined the the Mukai pairing on $H^*(X; \mathbb{C})$ as follows [26]: for $v, v' \in H^*(X; \mathbb{C})$,

$$\langle v, v' \rangle_{\text{Muk}} := \int_X e^{c_1(X)/2} \cdot v \cdot v'.$$
Here $v = \sum_j v_j \in \oplus_j H^j(X; \mathbb{C})$ and its Mukai dual $v^\vee = \sum_j \sqrt{-1}^j v_j \in H^*(X; \mathbb{C})$. Note that the above Mukai paring differs from Mukai’s original one [73] for K3 surfaces by a sign. We define a twisted Mukai vector of $E \in \mathcal{D}_X = \mathbb{D}^b\text{Coh}(X)$ by
\[
v_\Lambda(E) := \text{ch}(E)\sqrt{\text{Td}_X} \exp(\sqrt{-1}\Lambda)
\]
for any $\Lambda \in H^*(X; \mathbb{C})$ such that $\Lambda^\vee = -\Lambda$. The usual Mukai vector is the special case where $\Lambda = 0$. A twisted Mukai pairing is compatible with the Euler pairing; by the Hirzebruch–Riemann–Roch theorem,
\[
\chi(E, F) = \int_X \text{ch}(E')\text{ch}(F) \text{Td}_X = \langle v_\Lambda(E), v_\Lambda(F) \rangle_{\text{Muk}}.
\]
A geometric twisting $\Lambda$ compatible with the integral structure on the quantum cohomology was introduced by Iritani [53] and Katzarkov–Kontsevich–Pantev [59]. We shall use a reformulation due to Halverson–Jockers–Lapan–Morrison [49] in the following. First let us recall a familiar identity from complex analysis
\[
\frac{z}{1 - e^{-z}} = \frac{z/2}{\sinh(z/2)} = \frac{e^{z/2}}{\Gamma(1 + \frac{z}{2\pi\sqrt{-1}})}\Gamma(1 - \frac{z}{2\pi\sqrt{-1}}),
\]
where $\Gamma(z)$ is the Gamma function. The power series in the LHS induces the Todd class $\text{Td}_X$. We then consider a square root of the Todd class by writing
\[
\sqrt{\frac{z}{1 - z}} \exp(\sqrt{-1}\Lambda(z)) = e^{z/4}\Gamma(1 + \frac{z}{2\pi\sqrt{-1}}),
\]
and solve it for $\Lambda(z)$, where $z$ is real, as
\[
\Lambda(z) = \Im(\log(1 + \frac{z}{2\pi\sqrt{-1}})) \\
= \frac{\gamma z}{2\pi} + \sum_{j \geq 1} (-1)^j \frac{\zeta(2j + 1)}{2j + 1} \left( \frac{z}{2\pi} \right)^{2j}
\]
where $\gamma$ is Euler’s constant. Since the constant term of $\Lambda(z)$ is zero, we may use it to define an additive characteristic class $\Lambda_X$, called the log Gamma class. Note that $\Lambda_X^\vee = -\Lambda_X$ as

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only odd powers of \( z \) appear in \( \Lambda(z) \). In the Calabi–Yau case, we can explicitly write it as

\[
\Lambda_X = -\frac{\zeta(3)}{(2\pi)^3} c_3(X) + \frac{\zeta(5)}{(2\pi)^5} (c_5(X) - c_2(X)c_3(X)) + \ldots
\]

For K3 and abelian surfaces, there is no effect of twisting as \( \Lambda_X = 0 \). For Calabi–Yau 3-folds, the modification is precisely given by the first term, which is familiar in period computations in the B-model side. We define \( v_X(E) \) to be the twisted Muaki vector of an object \( E \in \mathcal{D}_X \) associated to the log Gamma class \( \Lambda_X \), i.e.

\[
v_X(E) := \text{ch}(E)\sqrt{\text{Td}_X} \exp(\sqrt{-1}\Lambda_X).
\]

Let \( X \) be a projective Calabi–Yau manifold equipped with a complexified Kähler parameter

\[
\omega = B + \sqrt{-1}\kappa \in H^2(X;\mathbb{C}),
\]

where \( \kappa \) is a Kähler class. Let also \( q = \exp(2\pi\sqrt{-1}\omega) \). We define \( \exp_*(\omega) \) by

\[
\exp_*(\omega) := 1 + \omega + \frac{1}{2!} \omega * \omega + \frac{1}{3!} \omega * \omega * \omega + \cdots.
\]

where \( * \) denotes the quantum product. It is conjectured (c.f. [54]) that near the large volume limit, which means that \( \int_C \mathfrak{S}(\omega) \gg 0 \) for all effective curve \( C \subset X \), there exists a Bridgeland stability condition on \( \mathcal{D}_X \) with central charge of the form

\[
Z(E) = -\langle \exp_*(\omega), v_X(E) \rangle_{\text{Muk}}. \tag{2.1}
\]

Then the asymptotic behavior of the above central charge near the large volume limit is given by

\[
Z(E) \sim -\int_X e^{-\omega}v_X(E) + O(q). \tag{2.2}
\]

The existence of a Bridgeland stability condition with the asymptotic central charge given by the leading term of the above expression has been proved in various important examples including K3 surfaces and abelian surfaces [15], as well as abelian threefolds [10, 69].
2.3 Weil–Petersson geometry

2.3.1 Classical Weil–Petersson geometry

We review some basics of the classical Weil–Petersson geometry on the complex moduli space \( \mathcal{M}_{\text{cpx}}(Y) \) of a projective Calabi–Yau \( n \)-fold \( Y \). In fact, such a (possibly degenerate) metric can be defined on the complex moduli space of a polarized Kähler–Einstein manifold by the Kodaira–Spencer theory, but we will use a period theoretic method for a Calabi–Yau manifold (see for example [96]). It fits naturally into the framework of the Hodge theory and gives us a connection to the Bridgeland stability conditions.

First, we consider the following vector bundle on the complex moduli space \( \mathcal{M}_{\text{cpx}}(Y) \)

\[
\mathcal{H} = R^n \pi_* \mathbb{C} \otimes \mathcal{O}_{\mathcal{M}_{\text{cpx}}(Y)} \rightarrow \mathcal{M}_{\text{cpx}}(Y).
\]

It comes equipped with a natural Hodge filtration \( \{ F^i \mathcal{H} \} \) of weight \( n \). By the Calabi–Yau assumption, the first piece of the filtration defines a holomorphic line bundle \( F^n \mathcal{H} \rightarrow \mathcal{M}_{\text{cpx}}(Y) \), which is called the vacuum bundle. For a nowhere vanishing local section \( \Omega \) of the vacuum bundle, the quantity

\[
K_{\text{cpx}}^{\text{WP}}(z) := - \log \left( (\sqrt{-1})^n \int_Y \Omega_z \wedge \overline{\Omega}_z \right)
\]

defines a local smooth function \( K_{\text{cpx}}^{\text{WP}} \), known as the Weil–Petersson potential, on the complex moduli space \( \mathcal{M}_{\text{cpx}}(Y) \). Then the Weil–Petersson metric on \( \mathcal{M}_{\text{cpx}}(Y) \) is defined to be the Hessian metric \( \frac{\sqrt{-1}}{2} \partial \bar{\partial} K_{\text{cpx}}^{\text{WP}} \). A fundamental fact is that the Hessian metric is non-degenerate and provides a canonical Kähler metric on the complex moduli space \( \mathcal{M}_{\text{cpx}}(Y) \).

Let \( \chi(L_1, L_2) = \chi(HF^*(L_1, L_2)) \) be the Euler pairing on the derived Fukaya category \( D^b \text{Fuk}(Y) \). The following identity is useful for computing the Weil–Petersson potential.

**Proposition 2.3.1.** Provided that there exist formal sums of Lagrangian submanifolds \( \{ L_i \} \) representing a basis of \( H_n(Y; \mathbb{Z})_{\text{tf}} \), we have

\[
K_{\text{WP}}^{\text{cpx}}(z) = - \log \left( (\sqrt{-1})^{-n} \sum_{i,j} \chi^{i,j} \int_{L_i} \Omega_z \int_{L_j} \overline{\Omega}_z \right),
\]

(2.3)
where \((\chi^{ij}) = (\chi(L_i, L_j))^{-1}\) is the inverse matrix.

Proof. Let \(\{A_i\}\) be a basis of \(H_n(Y; \mathbb{Z})_{\text{ht}}\). We define \((\gamma^{ij}) = (A_i \cdot A_j)^{-1}\) to be the inverse of the intersection matrix. Then by expanding \(\Omega_z\) and \(\overline{\Omega}_z\) by the dual basis of \(\{A_i\}\), we can rewrite the Weil–Petersson potential as

\[
K_{\text{WP}}^\text{cpx}(z) = -\log \left( (\sqrt{-1})^{n^2} \sum_{i,j} \gamma^{ij} \int_{A_i} \Omega_z \int_{A_j} \overline{\Omega}_z \right).
\]

On the other hand, for Lagrangian submanifolds \(L_1, L_2 \subseteq Y\), the identity

\[
[L_1] \cdot [L_2] = (\sqrt{-1})^{n(n+1)} \chi(L_1, L_2),
\]

is standard in the Lagrangian Floer theory (see for example [40, Section 4.3]). This completes the proof.

An advantage of the new expression (Equation (2.3)) is that it is not only categorical but also Hodge theoretic in the sense that it is written in terms of period integrals.

### 2.3.2 Weil–Petersson geometry on \(\text{Stab}_N(D)\)

Motivated by Remark 2.2.2 and Proposition 2.3.1, we shall propose a definition of Weil–Petersson geometry on a suitable quotient of the space of Bridgeland stability conditions on a Calabi–Yau triangulated category \(D\) of dimension \(n \in \mathbb{N}\), i.e. for every pair of objects \(E\) and \(F\), there is a natural isomorphism

\[
\text{Hom}_{D}^\sigma(E, F) \cong \text{Hom}_{D}^\sigma(F, E[n])^\vee.
\]

An important consequence is that the Euler form on \(N(D)\) is (skew-)symmetric if \(n\) is even (odd).

Let \(\{E_i\}\) be a basis of the numerical Grothendieck group \(N(D)\). We define a bilinear form \(\mathfrak{b} : \text{Hom}(N(D), \mathbb{C})^{\otimes 2} \to \mathbb{C}\) by

\[
\mathfrak{b}(\mathcal{Z}_1 \otimes \mathcal{Z}_2) := \sum_{i,j} \chi^{ij} \mathcal{Z}_1(E_i) \mathcal{Z}_2(E_j),
\]
where \((\chi^{ij}) := (\chi(E_i, E_j))^{-1}\). It is an easy exercise to check that the bilinear form \(b\) is independent of the choice of a basis.

**Definition 2.3.2.** We define the subset \(\text{Stab}_N^+(D) \subset \text{Stab}_N(D)\) by

\[
\text{Stab}_N^+(D) := \{\sigma = (Z, P) \mid b(Z, Z) = 0, (\sqrt{-1})^{-n}b(Z, \overline{Z}) > 0\}.
\]

The first condition is vacuous when \(n\) is odd as the bilinear form \(b\) is skew-symmetric. Such conditions have been studied in the case of K3 surfaces (a dual description via the Mukai pairing) under the name of reduced stability conditions [15]. We note that \(\text{Stab}_N^+(D)\) is an analogue of a period domain in the Hodge theory, and the natural free \(\mathbb{C}\)-action on \(\text{Stab}_N(D)\) preserves the subset \(\text{Stab}_N^+(D)\).

**Definition 2.3.3.** Let \(s = (Z_\sigma, P_\sigma)\) be a local holomorphic section of the \(\mathbb{C}\)-torsor \(\text{Stab}_N^+(D) \to \text{Stab}_N^+(D)/\mathbb{C}\), then

\[
K_{WP}(\tilde{s}) := -\log \left((\sqrt{-1})^{-n}b(Z_\sigma, \overline{Z_\sigma})\right)
\]

defines a local smooth function on \(\text{Stab}_N^+(D)/\mathbb{C}\). We call it the Weil–Petersson potential on \(\text{Stab}_N^+(D)/\mathbb{C}\).

**Proposition 2.3.4.** The complex Hessian \(\frac{\sqrt{-1}}{2} \partial \overline{\partial} K_{WP}\) of the Weil–Petersson potential \(K_{WP}\) does not depend on the choice of a local section \(s\). Moreover, it descends to the double quotient space

\[
\text{Aut}(D) \backslash \text{Stab}_N^+(D)/\mathbb{C}
\]

away from singular loci.

**Proof.** The first assertion is standard. The second assertion follows from the fact that autoequivalences are compatible with the Euler pairing, and the induced actions on the numerical Grothendieck group \(N(D)\) send a basis to another basis. Therefore the local sections which are identified by elements of \(\text{Aut}(D)\) differ only by multiplying local holomorphic functions, and thus the well-definedness of the metric follows from that of \(b\). \(\Box\)

The situation is particularly interesting when \(n\) is odd, as \(\text{Stab}_N(D)\) naturally carries
a holomorphic symplectic structure. Given a symplectic basis \( \{ E_i, F_i \} \) of \( \mathcal{N}(\mathcal{D}) \), the skew-symmetric bilinear form \( b : \text{Hom}(\mathcal{N}(\mathcal{D}), \mathbb{C}) \times \mathbb{C} \) is simply

\[
Z_1 \otimes Z_2 \mapsto \sum_i \left( Z_1(E_i)Z_2(F_i) - Z_1(F_i)Z_2(E_i) \right),
\]

which provides a nowhere vanishing holomorphic 2-form on \( \text{Stab}_{\mathcal{N}}(\mathcal{D}) \).

**Example 2.3.5.** As a sanity check, we shall carry out the above construction for the derived category \( \mathcal{D}_X = \text{DbCoh}(X) \) of an elliptic curve \( X \). Since the action of \( \text{GL}^+(2, \mathbb{R}) \) on \( \text{Stab}_{\mathcal{N}}(\mathcal{D}_X) \) is free and transitive [14, Theorem 9.1], we have

\[
\text{Stab}_{\mathcal{N}}^+(\mathcal{D}_X) = \text{Stab}_{\mathcal{N}}(\mathcal{D}_X) \cong \text{GL}^+(2, \mathbb{R}) \cong \mathbb{C} \times \mathbb{H},
\]

as a complex manifold. Thus the double quotient is

\[
\text{Aut}(\mathcal{D}_X) \text{Stab}_{\mathcal{N}}^+(\mathcal{D}_X) / \mathbb{C} \cong \text{PSL}(2, \mathbb{Z}) \setminus \mathbb{H}.
\]

This is indeed the Kähler moduli space of the elliptic curve \( X \). The normalized central charge at \( \tau \in \mathbb{H} \) is given by

\[
Z(E) = -\text{deg}(E) + \tau \cdot \text{rank}(E).
\]

Hence the Weil–Petersson potential is

\[
K_{WP}(\tau) = -\log \left( (\sqrt{-1})^{-1} \left( Z(O_p)Z(O_E) - Z(O_E)Z(O_p) \right) \right)
\]

\[
= -\log(3(\tau)) - \log 2.
\]

This is the Poincaré potential on \( \mathbb{H} \) and it descends to the Kähler moduli space \( \text{PSL}(2, \mathbb{Z}) \setminus \mathbb{H} \).

### 2.3.3 Refining conjecture

Let \( X \) be a projective Calabi–Yau \( n \)-fold. Then \( \text{Stab}_{\mathcal{N}}(\mathcal{D}_X) \) can be considered as an extended version of the stringy Kähler moduli space \( \mathcal{M}_{\text{Kah}}(X) \) [16, Section 7.1]. It is akin to the big quantum cohomology rather than the small quantum cohomology in the sense that the tangent space of \( \mathcal{M}_{\text{Kah}}(X) \) is \( H^{1,1}(X) \) while that of \( \text{Stab}_{\mathcal{N}}(\mathcal{D}_X) \) is \( \oplus_p H^{p,p}(X) \). It is
conjectured by Bridgeland [14] that there should exist an embedding of \( \mathcal{M}_{\text{Kah}}(X) \) into

\[
\operatorname{Aut}(DX) \backslash \text{Stab}_{\mathcal{N}}(DX)/\mathbb{C}.
\]

Note that when \( n \) is odd, the double quotient is a holomorphic contact space thanks to the holomorphic symplectic structure on \( \text{Stab}_{\mathcal{N}}(DX) \).

Motivated by mirror symmetry and classical Weil–Petersson geometry, especially the fact that Weil–Petersson metric is non-degenerate on \( \mathcal{M}_{\text{cpx}}(X) \), we can now propose the following, which refines the previous conjecture.

**Conjecture 2.3.6.** There exists an embedding of the stringy Kähler moduli space

\[
i : \mathcal{M}_{\text{Kah}}(X) \hookrightarrow \operatorname{Aut}(DX) \backslash \text{Stab}_{\mathcal{N}}^{+}(DX)/\mathbb{C}.
\]

The complex Hessian of the pullback \( i^*K_{\text{WP}} \) of the Weil–Petersson potential \( K_{\text{WP}} \) defines a Kähler metric on \( \mathcal{M}_{\text{Kah}}(X) \), i.e. non-degenerate. Moreover, it is identified with the Weil–Petersson metric on the complex moduli space \( \mathcal{M}_{\text{cpx}}(Y) \) of a mirror manifold \( Y \) under the mirror map \( \mathcal{M}_{\text{Kah}}(X) \cong \mathcal{M}_{\text{cpx}}(Y) \). When \( n = 3 \), the image of \( \mathcal{M}_{\text{Kah}}(X) \) is locally a holomorphic Legendre variety.

We checked that the conjecture holds for the elliptic curves (Example 2.3.5) and will provide more supporting evidence in the next section. It is worth noting that the real difficulty lies in providing a mathematical definition of the stringy Kähler moduli space \( \mathcal{M}_{\text{Kah}}(X) \). One potential application of the above conjecture is that, we can make use of the non-degeneracy condition on the Weil–Petersson metric to characterize \( \mathcal{M}_{\text{Kah}}(X) \).

### 2.4 Computation

We begin our discussion with Calabi–Yau surfaces, for which there is a mathematical definition of stringy Kähler moduli space via the Bridgeland stability conditions [8, Section 7]. Our computation heavily relies on existing deep results, mainly due to Bridgeland, and we do not claim originality. The purpose of this section is to back up our conjecture by
2.4.1 Self-product of elliptic curve

We consider the self-product $A := E_{\tau} \times E_{\tau}$ of an elliptic curve $E_{\tau} := \mathbb{C} / (\mathbb{Z} + \tau \mathbb{Z})$ for a generic $\tau \in \mathcal{H}$. We denote by $\text{NS}(A) := H^2(A, \mathbb{Z}) \cap H^{1,1}(A)$ the Néron–Severi lattice of $A$. Before considering the space of stability conditions, let us take a look at the set of complexified Kähler forms $\omega \in \text{NS}(A) \mathbb{C}$. Let $dz_1$ be a basis of $H^{1,0}(E_{\tau})$ of the first $E_{\tau}$ factor of $A$, and $dz_2$ similarly. Then a complexified Kähler form $\omega$ can be expressed as

$$\omega = \sqrt{-1} (\rho dz_1 \wedge d\bar{z}_1 + \tau dz_2 \wedge d\bar{z}_2 + \sigma (dz_1 \wedge d\bar{z}_2 - d\bar{z}_1 \wedge dz_2))$$

such that the imaginary part $\Im(\omega)$ is a Kähler form. The real part $\Re(\omega)$ is often called a B-field. Let $\mathfrak{H}_g$ be the Siegel upper half-space of degree $g$ defined by

$$\mathfrak{H}_g := \{ M \in M(g, \mathbb{C}) \mid M^t M = \Im(M) > 0 \}.$$ 

In this abelian surface example, we do not fix a polarization, but we vary it in a 3-dimensional space $\mathfrak{H}_2$ as follows.

**Proposition 2.4.1 ([57]).** Let $A_g := E_{\tau}^g$ be the self-product of $g$ copies of an elliptic curve $E_{\tau}$. The set of complexified Kähler forms can be identified with the Siegel upper-half space $\mathfrak{H}_g$ of genus $g$. In the $g = 2$ case, the identification is given by the assignment $\omega \mapsto M_\omega := \begin{bmatrix} \rho & \sigma \\ \sigma & \tau \end{bmatrix}$.

**Proof.** The $g = 1$ case is standard. Suppose that $g = 2$. It suffices to show that $\Im(M_\omega) > 0$. Since $\omega$ is a complexified Kähler form, we have

$$\text{tr}(\Im(M_\omega)) = \Im(\rho) + \Im(\tau) = \int_{E_{\tau} \times \text{pt}} \Im(\omega) + \int_{\text{pt} \times E_{\tau}} \Im(\omega) > 0$$

and

$$\det(\Im(M_\omega)) = \Im(\rho) \Im(\tau) - \Im(\sigma)^2 = \Im(\omega)^2 > 0,$$

and thus $M_\omega \in \mathfrak{H}_2$. 

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On the other hand, since $A_2$ contains no rational curves, the Kähler cone coincides with
the connected component of the positive cone in $\text{NS}(S)$ which contains a Kähler class. This
readily proves the assertion for $g = 2$. Since we do not need the higher genus case, we leave
a proof to the reader (c.f. [57, Section 6]).

We now recall some notations in [15]. A result of Orlov [77, Proposition 3.5] shows that
every autoequivalence of $\mathcal{D}_A$ induces a Hodge isometry of the lattice $H^*(A; \mathbb{Z})$
equipped with the Mukai pairing. Hence there is a group homomorphism

$$\delta : \text{Aut}(\mathcal{D}_A) \rightarrow \text{Aut}H^*(A; \mathbb{Z}).$$

The kernel of the homomorphism will be denoted by $\text{Aut}^0(\mathcal{D}_A)$.

Let $\Omega \in H^2(A; \mathbb{C})$ be the class of a nonzero holomorphic two-form on $A$. The sublattice
$$\mathcal{N}(A) := H^*(A; \mathbb{Z}) \cap \Omega^\perp \subset H^*(A; \mathbb{C})$$
can be identified with $\mathcal{N}(\mathcal{D}_A) = H^0(A; \mathbb{Z}) \oplus \text{NS}(A) \oplus H^4(A; \mathbb{Z})$ and has signature $(3, 2)$. In fact, since the complex moduli $\tau \in \mathbb{H}$ is generic, $\mathcal{N}(\mathcal{D}_A) \cong \mathbb{U} \oplus \mathbb{Z}^2$ as a lattice. Here $\mathbb{U}$ is
the hyperbolic lattice, and $\mathbb{Z}^2$ denotes an integral lattice of rank 1 with the Gram matrix $(2)$.

We define a subset $\mathcal{P}(A) \subset \mathcal{N}(\mathcal{D}_A)_{\mathbb{C}}$ consisting of vectors $\mathcal{U} \in \mathcal{N}(\mathcal{D}_A)_{\mathbb{C}}$ whose real
and imaginary parts span a negative definite 2-plane in $\mathcal{N}(\mathcal{D}_A)_{\mathbb{R}}$. This subset has two
connected components. We denote by $\mathcal{P}^+(A)$ the component containing vectors of the form
$\mathcal{U}_\omega := \exp(\omega)$ for a complexified Kähler class $\omega \in \text{NS}(A)_{\mathbb{C}}$.

Let us review some results on the space of Bridgeland stability conditions on algebraic
surfaces following [15]. The central charge of a numerical stability condition is of the form

$$Z_{\mathcal{U}}(E) = -\langle \mathcal{U}, v_A(E) \rangle_{\text{Muk}} = -\langle \mathcal{U}, \text{ch}(E) \rangle_{\text{Muk}}$$

for some $\mathcal{U} \in \mathcal{N}(\mathcal{D}_A)_{\mathbb{C}}$. When $\mathcal{U} = \mathcal{U}_\omega$ for some complexified Kähler class $\omega$, one can con-
struct a stability condition with central charge $Z_{\mathcal{U}_\omega}$ using the tilting theory and Bogomolov
inequality. Moreover, such stability conditions are geometric in the sense that all skyscraper
sheaves are stable and of the same phase. We denote by $\text{Stab}^+(A) \subset \text{Stab}(A)$ the connected
Theorem 2.4.2 ([15]). Let $A$ be an abelian surface over $\mathbb{C}$.

1. The forgetful map $\pi$ sending a stability condition to the associated vector $\mathcal{U} \in \mathcal{N}(\mathcal{D}_A)_\mathbb{C}$ maps onto the open subset $\mathcal{P}^+(A) \subset \mathcal{N}(\mathcal{D}_A)_\mathbb{C}$. Moreover, the map

$$\pi : \text{Stab}^+(A) \to \mathcal{P}^+(A).$$

is the universal cover of $\mathcal{P}^+(A)$ with the group of deck transformations generated by the double shift functor $[2] \in \text{Aut}(\mathcal{D}_A)$.

2. The action of $\text{Aut}(\mathcal{D}_A)$ on $\text{Stab}(A)$ preserves the connected component $\text{Stab}^+(A)$.

3. The group $\text{Aut}^0(\mathcal{D}_A)$ is generated by the double shift functor $[2]$, together with twists by elements of $\text{Pic}^0(A)$, and pullbacks by automorphisms of $A$ acting trivially on $H^*(A; \mathbb{Z})$. Note that twists by elements of $\text{Pic}^0(A)$ and pullbacks by automorphisms of $A$ acting trivially on $H^*(A; \mathbb{Z})$, act trivially on $\text{Stab}^+(A)$.

4. There exists a short exact sequence of groups

$$1 \to \text{Aut}^0(\mathcal{D}_A) \to \text{Aut}(\mathcal{D}_A) \to \text{Aut}^+H^*(A; \mathbb{Z}) \to 1,$$

where $\text{Aut}^+H^*(A; \mathbb{Z}) \subset \text{Aut}H^*(A; \mathbb{Z})$ is the index 2 subgroup consisting of elements which do not exchange the two components of $\mathcal{P}(A)$.

Following our proposal in the previous section, it is natural to consider the following subset of $\text{Stab}^+(A)$.

$$\text{Stab}^+_\chi(A) := \{(\mathcal{Z}, \mathcal{P}) \in \text{Stab}^+(A) \mid b(\mathcal{Z}, \mathcal{Z}) = 0, \ -b(\mathcal{Z}, \overline{\mathcal{Z}}) > 0\}$$

(c.f. Definition 2.3.2). Hence we need to compute the bilinear form $b$. We start with a lemma.

Lemma 2.4.3. Let $X$ be a smooth projective variety of dimension $n$. Recall the twisted Mukai vector

$$v_X(E) = \text{ch}(E)\sqrt{\text{Td}_X} \exp(\sqrt{-1}\Lambda_X).$$

Then
1. Assume that $X$ is Calabi–Yau, then $\langle v, w \rangle_{\text{Muk}} = (-1)^n \langle w, v \rangle_{\text{Muk}}$ for any $v, w \in H^{2*}(X; \mathbb{C})$.

2. Let $\{E_i\}$ be a basis of the numerical Grothendieck group $N(D_X)$. Then

$$\sum_{i,j} \langle v, v_X(E_i) \rangle_{\text{Muk}} \cdot \chi^{ij} \cdot \langle v_X(E_j), w \rangle_{\text{Muk}} = \langle v, w \rangle_{\text{Muk}},$$

where $(\chi^{ij}) = (\chi(E_i, E_j))^{-1}$.

**Proof.** The first assertion follows directly from the Serre duality. The second assertion is a simple linear-algebraic fact which follows from the identity $\chi(E_i, E_j) = \langle v_X(E_i), v_X(E_j) \rangle_{\text{Muk}}$.

Note that Lemma 2.4.3 is purely algebraic and thus holds in a categorical setting as well.

We now compute the bilinear form $b$, with central charge of the form

$$\mathcal{Z}_{\text{U}}(E) = -\langle \mathcal{U}, v_X(E) \rangle_{\text{Muk}},$$

where $\mathcal{U} \in H^*(X; \mathbb{C})$.

**Lemma 2.4.4.** $b(\mathcal{Z}_{\text{U}_1}, \mathcal{Z}_{\text{U}_2}) = (-1)^n \langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{\text{Muk}}$.

**Proof.** This is a simple application of Lemma 2.4.3. We have

$$b(\mathcal{Z}_{\text{U}_1}, \mathcal{Z}_{\text{U}_2}) = \sum_{i,j} \chi^{ij} \cdot \langle \mathcal{U}_1, v_X(E_i) \rangle_{\text{Muk}} \cdot \langle \mathcal{U}_2, v_X(E_j) \rangle_{\text{Muk}}$$

$$= (-1)^n \sum_{i,j} \chi^{ij} \cdot \langle \mathcal{U}_1, v_X(E_i) \rangle_{\text{Muk}} \cdot \langle v_X(E_j), \mathcal{U}_2 \rangle_{\text{Muk}}$$

$$= (-1)^n \langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{\text{Muk}},$$

Using Lemma 2.4.4, we can determine which stability conditions lie in the subset $\text{Stab}^+(A) \subset \text{Stab}_A^+(A)$.

**Proposition 2.4.5.** A stability condition $(\mathcal{Z}_\mathcal{U}, \mathcal{P}) \in \text{Stab}^+_A(A)$ lies in $\text{Stab}^+_X(A)$ if and only if $\langle \mathcal{U}, \mathcal{U} \rangle = 0$ and $-\langle \mathcal{U}, \overline{\mathcal{U}} \rangle > 0$. This condition is equivalent to that $\mathcal{U}$ is of the form $\overline{\mathcal{U}}_\omega = c \exp(\omega)$ for some constant $c \in \mathbb{C}$ and complexified Kähler class $\omega$.  

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Proof. The first assertion follows directly from Lemma 2.4.4. The second assertion follows from an explicit calculation of the Mukai pairing on surfaces.

Note that central charge of the form

\[ Z_{\mathcal{U}_\omega}(E) = -\langle \mathcal{U}_\omega, v_A(E) \rangle_{\text{Muk}} \]

where \( \mathcal{U}_\omega = \exp(\omega) \) for a complexified Kähler class \( \omega \), has been discussed in physics literatures, see Section 2.2.2. One can prove that a stability condition with such a central charge always lies in \( \text{Stab}^+_N(D) \).

Proposition 2.4.6. 1. The central charge \( Z_{\mathcal{U}_\omega} \) satisfies

\[ b(Z_{\mathcal{U}_\omega}, Z_{\mathcal{U}_\omega}) = 0, \quad (\sqrt{-1})^{-n} b(Z_{\mathcal{U}_\omega}, \overline{Z_{\mathcal{U}_\omega}}) > 0. \]

2. The Weil–Petersson potential at a stability condition with central charge \( Z_{\mathcal{U}_\omega} \) is given by

\[ K_{WP}(\omega) = -\log(\Im(\omega)^n) - \log \left( \frac{2^n}{n!} \right). \]

Proof. 1. By applying Lemma 2.4.4, we have

\[ (\sqrt{-1})^{-n} b(Z_{\mathcal{U}_\omega}, \overline{Z_{\mathcal{U}_\omega}}) = (\sqrt{-1})^n \langle \mathcal{U}_\omega, \overline{\mathcal{U}_\omega} \rangle_{\text{Muk}} \]

\[ = \frac{(\sqrt{-1})^n}{n!} (\omega + \overline{\omega})^n \]

\[ = \frac{2^n}{n!} \Im(\omega)^n > 0. \]

The last inequality is a consequence of the fact that \( \omega \) is a complexified Kähler class.

The equality \( b(Z_{\mathcal{U}_\omega}, Z_{\mathcal{U}_\omega}) = 0 \) also follows from the above by replacing \( \overline{Z_{\mathcal{U}_\omega}} \) by \( Z_{\mathcal{U}_\omega} \).

2. The assertion follows from the above explicit computation.

Motivated by Proposition 2.4.5, we define the following subset \( \mathcal{R}^+(A) \) of \( \mathcal{P}^+(A) \).

\[ \mathcal{R}^+(A) := \{ \mathcal{U} \in \mathcal{P}^+(A) \mid \langle \mathcal{U}, \overline{\mathcal{U}} \rangle_{\text{Muk}} = 0, \quad -\langle \mathcal{U}, \overline{\mathcal{U}} \rangle_{\text{Muk}} > 0 \}. \]
By Theorem 2.4.2 (1), the forgetful map \( \text{Stab}_X^+(D_A) \to R^+(A) \) is a covering map with the group of deck transformation generated by the double shift functor \([2] \in \text{Aut}(D_A)\).

**Lemma 2.4.7.** \( R^+(A)/\mathbb{C}^\times \cong \Omega_2 \) as a complex manifold. Thereby we have an identification

\[
\langle [2] \rangle \text{Stab}_X^+(D_A)/\mathbb{C}^\times \cong \Omega_2.
\]

**Proof.** The quotient

\[
R^+(A)/\mathbb{C}^\times \cong \{ C\Omega \in \mathbb{P}(N(D_A)_C) \mid \langle \Omega, \overline{\Omega} \rangle_{\text{Muk}} = 0, -\langle \Omega, \overline{\Omega} \rangle_{\text{Muk}} > 0 \}
\]

is the symmetric domain of type IV_3. The assertion follows from the standard identification IV_3 \cong III_2 of the symmetric domains. More explicitly, it is given by the tube domain realization \( \Omega_2 \to R^+(A)/\mathbb{C}^\times : \omega \mapsto [\Omega_\omega] \).

We now recall the definition of **Calabi–Yau autoequivalences** following the work of Bayer and Bridgeland [8]. Define

\[
\text{Aut}^+_\text{CY} H^+(A) \subset \text{Aut}^+H^+(A)
\]

to be the subgroup of Hodge isometries which preserve the class of holomorphic 2-form \([\Omega] \in \mathbb{P}H^+(A;\mathbb{C})\). Any such isometry restricts to give an isometry of \( N(D_A) \). In fact,

\[
\text{Aut}^+_\text{CY} H^+(A) \subset \text{Aut}N(D_A)
\]

is the subgroup of index two which do not exchange the two components of \( \mathcal{P}(A) \).

An autoequivalence \( \Phi \in \text{Aut}(D_A) \) is said to be **Calabi–Yau** if the induced Hodge isometry \( \delta(\Phi) \) lies in \( \text{Aut}^+_\text{CY} H^+(A) \). We denote \( \text{Aut}_{\text{CY}}(D_A) \subset \text{Aut}(D_A) \) the group of Calabi–Yau autoequivalences. By Theorem 2.4.2 (4), there exists a short exact sequence

\[
1 \to \text{Aut}^0(D_A) \to \text{Aut}_{\text{CY}}(D_A) \to \text{Aut}^+_\text{CY} H^+(A) \to 1.
\]

We write \( \text{Aut}^0_{\text{tri}}(D_A) \subset \text{Aut}^0(D_A) \) for the subgroup generated by twists by elements of \( \text{Pic}^0(A) \) and pullbacks by automorphisms of \( A \) acting trivially on \( H^+(A;\mathbb{Z}) \). Recall from
Theorem 2.4.2 (3) that $\text{Aut}^0_{\text{tri}}(D_A)$ acts trivially on $\text{Stab}^+(D_A)$. We define 

$$\overline{\text{Aut}}_{\text{CY}}(D_A) := \text{Aut}_{\text{CY}}(D_A)/\text{Aut}^0_{\text{tri}}(D_A).$$

Then $\overline{\text{Aut}}_{\text{CY}}(D_A)$ acts on $\text{Stab}^+_X(D_A)$, and there is a short exact sequence

$$1 \rightarrow \langle [2] \rangle \rightarrow \overline{\text{Aut}}_{\text{CY}}(D_A) \rightarrow \text{Aut}^+_X H^*(A) \rightarrow 1.$$

**Theorem 2.4.8.** The covering map $\pi$ induces an isomorphism

$$\overline{\text{Aut}}_{\text{CY}}(D_A)\backslash \text{Stab}^+_X(D_A)/C^\times \cong \text{Sp}(4, \mathbb{Z})\backslash \mathfrak{h}_2$$

between the double quotient of $\text{Stab}^+_X(D_A)$ and the Siegel modular variety $\text{Sp}(4, \mathbb{Z})\backslash \mathfrak{h}_2$. We will call it the stringy Kähler moduli space of $A$.

**Proof.** From the previous discussions, we have

$$\overline{\text{Aut}}_{\text{CY}}(D_A)\backslash \text{Stab}^+_X(D_A)/C^\times \cong \text{Aut}^+_X H^*(A)\backslash \mathfrak{h}_2.$$

The action of $\text{Aut}^+_X H^*(A)$ on $\mathfrak{h}_2$ is purely lattice theoretic. As an abstract group $\text{Aut}^+_X H^*(A) \cong \text{O}^+(U^\oplus 2 \oplus \langle 2 \rangle)$. By a fundamental result [43, Lemma 1.1] of Gritsenko and Nikulin, it can be identified with the standard $\text{Sp}(4, \mathbb{Z})$-action on the Siegel upper-half space $\mathfrak{h}_2$. \qed

**Remark 2.4.9.** It is shown in [57] that $A_g$ is mirror symmetric to a principally polarized abelian surface of dimension $g$. Theorem 2.4.8 is thereby compatible with the fact that the Siegel modular variety $\text{Sp}(2g, \mathbb{Z})\backslash \mathfrak{h}_g$ is precisely the complex moduli space of principally polarized abelian surfaces of dimension $g$. For $g > 2$, we expect that $\text{Sp}(2g, \mathbb{Z})\backslash \mathfrak{h}_g$ is covered by a similar double quotient of a suitable subset of $\text{Stab}(D_{A_g})$.

There exists a canonical metric on the Siegel modular variety $\text{Sp}(4, \mathbb{Z})\backslash \mathfrak{h}_2$, namely the Bergman metric. It is known to be a complete Kähler–Einstein metric. The main theorem of this section is to show that the Bergman metric coincides with the Weil–Petersson metric defined by Definition 2.3.3.

**Proposition 2.4.10 ([84]).** The Bergman kernel $K_{\text{Ber}} : \mathfrak{h}_g \times \mathfrak{h}_g \to \mathbb{C}$ of the Siegel upper half-space
$\mathfrak{g}_g$ of degree $g$ is given by

$$K_{\text{Ber}}(M, N) = -\text{tr}(\log(-\sqrt{-1}(M - N))).$$

The Bergman metric is defined to be the complex Hessian of the Bergman potential

$$K_{\text{Ber}}(M) := K_{\text{Ber}}(M, M) = -\text{tr}(\log(2\Im(M))).$$

**Theorem 2.4.11.** The Weil–Petersson potential on $\text{Stab}^+_{\mathcal{N}}(\mathcal{D}_A)/\mathbb{C}$ coincides with the Bergman potential of the Siegel upper half-plane $\mathfrak{g}_2$ up to a constant.

**Proof.** By Proposition 2.4.5, the central charge of a stability condition in $\text{Stab}^+_{\mathcal{N}}(\mathcal{D}_A)$ is of the form $Z(E) = -c(\tilde{U}_\omega, \nu_A(E))$ for some complexified Kähler class $\omega$. So we can apply the calculation of the Weil–Petersson potential in Proposition 2.4.6.

The key idea is to use the identification of $\omega$ and $M_\omega$ provided in Proposition 2.4.1. Then the two Kähler potentials are related as follows:

$$K_{\text{WP}}(\omega) = -\log(\Im(\omega)^2) - \log 2$$

$$= -\log(\det(\Im(M_\omega))) - \log 2$$

$$= -\text{tr}(\log(2\Im(M_\omega))) + \log 2$$

$$= K_{\text{Ber}}(M_\omega) + \log 2.$$

This completes the proof. $\square$

**Corollary 2.4.12.** The Weil–Petersson metric on the stringy Kähler moduli space is identified with the Bergman metric on $\text{Sp}(4, \mathbb{Z}) \backslash \mathfrak{g}_2$ via the isomorphism in Theorem 2.4.8.

### 2.4.2 Split abelian surfaces

Now let $A$ be a split abelian surface, that is, $A \cong E_{\tau_1} \times E_{\tau_2}$ for elliptic curves $E_{\tau_1}$ and $E_{\tau_2}$. Such a splitting is unique provided that $E_{\tau_1}$ and $E_{\tau_2}$ are generic, or equivalently $\text{NS}(A) \cong U$.

Discussions in the previous section carries over for the split abelian surface $A$. The set of complexified Kähler forms is identified with $\mathbb{H} \times \mathbb{H}$, which is diagonally embedded in $\mathfrak{g}_2$. 

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(c.f. Proposition 2.4.1). It is precisely the symmetric domain of type IV\(_2\) associated to \(U^{\oplus 2}\).

On the other hand, it is known (c.f. [50, Proposition 2.6]) that

\[
\text{Aut}^+_\text{CY}(A) \cong \text{O}^+(U^{\oplus 2}) \cong \text{P}(\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})) \rtimes \mathbb{Z}_2,
\]

where \(\text{P}(\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z}))\) represents the quotient group of \(\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})\) by the involution \((A, B) \mapsto (-A, -B)\) and the semi-direct product structure is given by the generator of \(\mathbb{Z}_2\) acting on \(\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})\) by exchanging the two factors.

**Theorem 2.4.13.** There is an identification

\[
\overline{\text{Aut}}_{\text{CY}}(D_A) \backslash \text{Stab}_X^+(D_A) / \mathbb{C}^\times \cong \text{P}(\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})) \rtimes \mathbb{Z}_2 \backslash (\mathbb{H} \times \mathbb{H})
\]

Moreover, the Weil–Petersson metric on the stringy Kähler moduli space \(\overline{\text{Aut}}_{\text{CY}}(D_A) \backslash \text{Stab}_X^+(D_A) / \mathbb{C}^\times\) is identified with the Bergman metric on the Siegel modular variety \(\text{P}(\text{SL}(2, \mathbb{Z}) \times \text{SL}(2, \mathbb{Z})) \rtimes \mathbb{Z}_2 \backslash (\mathbb{H} \times \mathbb{H})\).

This observation is compatible with self-mirror symmetry for the split abelian surfaces. In fact a lattice polarized version of the global Torelli Theorem asserts that the complex moduli space of split abelian surfaces are given by the above Siegel modular variety.

**Remark 2.4.14.** A similar computation can be carried out for M-polarized K3 surfaces for the lattice \(M \cong U^{\oplus 2} \oplus (-2)\) or \(U^{\oplus 2}\). The main difference is that there are spherical objects in the derived category \(\mathcal{D}_X\) of a K3 surface \(X\) and we need to remove the union of certain hyperplanes from \(P^+\). Moreover, the subgroup of \(\text{Aut}^0(\mathcal{D}_X)\) which preserves the connected component \(\text{Stab}^+(\mathcal{D}_X)\) acts freely on \(\text{Stab}^+(\mathcal{D}_X)\). So one does not need to take the quotient of the group of Calabi–Yau autoequivalences by \(\text{Aut}^0(\mathcal{D})\) as in the abelian surface case (c.f. [15]).

**2.4.3 Abelian variety**

Let \(X\) be an abelian variety of dimension \(n\). Since there is no quantum corrections and the Chern classes are trivial, the expected central charge at the complexified Kähler moduli
\( \omega \in H^2(X; \mathbb{C}) \) is given by

\[
Z_{\Omega_\omega}(E) = -\langle \Omega_\omega, v_X(E) \rangle_{\text{Muk}} = -\int_X e^{-\omega} \text{ch}(E).
\]

The existence of Bridgeland stability condition with this central charge is known for \( n \leq 3 \).

By Proposition 2.4.6, the Weil–Petersson potential is

\[
K_{\text{WP}}(\tau) = -\log(\Im(\omega)^n) - \log \frac{2^n}{n!}.
\]

Fix a polarization \( H \). We think of \( \omega = \tau H \) for \( \tau \in \mathbb{H} \) as a slice of the stringy Kähler moduli space \( \mathcal{M}_{\text{Kah}}(X) \). Then the Weil–Petersson metric on \( \mathbb{H} \) is essentially the Poincaré metric. This example is a toy model in the sense that there is no quantum correction.

The above observation is compatible with Wang’s mirror result [102, Remark 1.3], which says that in the case of infinite distance, the Weil–Petersson metric is asymptotic to a scaling of the Poincaré metric.

### 2.4.4 Quintic threefold

Although the existence of a Bridgeland stability condition for a quintic threefold \( X \subset \mathbb{P}^4 \) has not yet been proven, we can still compute the Weil–Petersson potential using the central charge in Equation (2.1) near the large volume limit.

Let \( \tau H \in H^2(X; \mathbb{C}) \) be the complexified Kähler class, where \( H \) is the hyperplane class and \( \tau \in \mathbb{H} \). First we observe that

\[
\exp_{\tau}(\tau H) = 1 + \tau H + \frac{\tau^2}{2} \left(1 + \frac{1}{5} \sum_{d \geq 1} N_d d^3 q^d \right) H^2 + \frac{\tau^3}{6} \left(1 + \frac{1}{5} \sum_{d \geq 1} N_d d^3 q^d \right) H^3,
\]

where \( q = e^{2\pi \sqrt{-1} \tau} \) and \( N_d^X \) denotes the genus 0 Gromov–Witten invariant of \( X \) of degree \( d \), and we use the quantum product

\[
H \ast H = \Phi(q) H^2 = \frac{1}{5} \left(5 + \sum_{d \geq 1} N_d^X q^d d^3 \right) H^2.
\]
Then the central charge computes to be
\[ Z(E) = - \langle \exp_*(\tau H), \mathbb{v}_X(E) \rangle_{\text{Muk}} \]
\[ = - \int_X e^{-\tau H} \mathbb{v}_X(E) + \frac{\zeta(3) \chi(X)}{(2\pi)^3} \left( \frac{\tau^2}{10} H^2 \mathfrak{c}_1(E) - \frac{\tau^3}{6} \mathfrak{c}_0(E) \right) \sum_{d \geq 1} N_d^X d^3 q^d, \]
where \( \chi(X) \) is the topological Euler number of \( X \). Near the large volume limit, the Weil–Petersson potential is given by
\[ K_{\text{WP}}(\tau) = - \log \left( H^3 \left( \Phi(q) \left( \frac{\tau^3}{6} + \frac{\tau^2}{2} \right) - \Phi(q) \left( \frac{\tau^3}{6} + \frac{\tau^2}{2} \right) \right) \right) - 2 \log \left( \frac{\zeta(3) \chi(X)}{(2\pi)^3} \right) \]
\[ \sim - \log \left( \frac{4}{3} H^3 \Im(\tau)^3 \right) - 2 \log \left( \frac{\zeta(3) \chi(X)}{(2\pi)^3} \right) + O(q). \]

Therefore the Weil–Petersson metric of a quintic threefold is a quantum deformation of the Poincaré metric on \( \mathbb{H} \) as expected. In particular, for sufficiently small \( q \), it is non-degenerate and the Weil–Petersson distance to the large volume limit is infinite. When there is no B-field, i.e. \( \tau \in \sqrt{-1} \mathbb{R} \), the correction term \( O(q) \) is explicitly given by \( \log(\Phi(q)) \).

**Remark 2.4.15 ([18]).** The stringy Kähler moduli space \( \mathcal{M}_{Kah}(X) \) of a quintic Calabi–Yau threefold \( X \subset \mathbb{P}^4 \) is expected to be identified with the suborbifold
\[ \left[ \{ z \in \mathbb{C} \mid z^5 \neq 1 \} / \mathbb{Z}_5 \right] \subset \left[ \mathbb{P}^1 / \mathbb{Z}_5 \right]. \]
The point \( z = \infty \) is the large volume limit, the point \( z^5 = 1 \) is the conifold point, and the point \( z = 0 \) is the Gepner point. We expect the following properties of the Weil–Petersson metric on \( \mathcal{M}_{Kah}(X) \).

1. The Weil–Petersson distance to the conifold point, which corresponds to a quintic threefold with a conifold singularity, should be finite. This is based on a result of Wang [102] on the mirror complex moduli, which asserts that if a Calabi–Yau variety has at worst canonical singularities, then it has finite Weil–Petersson metric along any smoothing to Calabi–Yau manifolds.

2. The Weil–Petersson metric at the Gepner point should be an orbifold metric. This is because the auto-equivalence
\[ \Phi(-) = T_{O_X} \circ \left( (-) \otimes O_X(H) \right), \]
where \( T_{O_X} \) denotes the Seidel–Thomas spherical twist with respect to \( O_X \), at the Gepner...
point satisfies the relation $\Phi^5 = [2]$. This descends to $\Phi^5 = \text{id}$ on $K(D_X)$. On other hand, the calculations of Candelas–de la Ossa–Green–Parkes [18] shows that the Weil–Petersson curvature tends to $+\infty$ as we approach the Gepner point.

It is interesting to investigate the interplay among the geometry of a Calabi–Yau threefold $X$, the cubic intersection form on $H^2(X;\mathbb{Z})$, and curvature properties of the Weil–Petersson metric near a large volume limit [101, 103, 105].

On the other hand, probably a more alluring research direction is to examine the Weil–Petersson metric away from a large volume limit, where central charges are not of the form (2.1), as the metric is inherently global. For instance, the Weil–Petersson metric around a Gepner point may be studied via matrix factorization categories via the Orlov equivalence [78]

$$\text{D}^b\text{Coh}(X) \cong \text{HMF}(W),$$

where $\text{HMF}(W)$ is the homotopy category of a graded matrix factorization of the defining equation $W$ of the quintic 3-fold $X$. Toda studied stability conditions, called the Gepner type stability conditions, conjecturally corresponding to the Gepner point [100].
Chapter 3

Systolic inequality on K3 surfaces\(^1\)

3.1 Introduction

Let \((M, g)\) be a Riemannian manifold. Its systole \(\text{systole}(M, g)\) is defined to be the least length of a non-contractible loop in \(M\). In 1949, Charles Loewner proved that for any metric \(g\) on the two-torus \(T^2\),

\[ \text{syst}(T^2, g)^2 \leq \frac{2}{\sqrt{3}} \text{vol}(T^2, g). \]

Moreover, the equality can be attained by the flat equilateral torus. There are various generalizations of Loewner’s tours systolic inequality. We refer to [58] for a survey on the rich subject of systolic geometry.

The first goal of the present chapter is to propose a new generalization of Loewner’s torus systolic inequality from the perspective of Calabi–Yau geometry. We start with an observation in the case of two-torus. Suppose that the torus is flat \(T^2 \cong \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}\), and is equipped with the standard complex structure \(\Omega = dz\) and symplectic structure \(\omega = dx \wedge dy\). Then the shortest non-contractible loops must be given by straight lines, which are nothing but special Lagrangian submanifolds with respect to the complex and symplectic structures.

\(^1\)Reference: [34].
Hence, under these assumptions, one can rewrite Loewner’s torus systolic inequality as:

\[
\inf_{L: \text{sLag}} \left\| \int_L dz \right\|^2 \leq \frac{1}{\sqrt{3}} \left\| \int_{T^2} dz \wedge d\bar{z} \right\|.
\]

Here “sLag” denotes the compact special Lagrangian submanifolds in \( T^2 \) with respect to \( \omega \) and \( \Omega = dz \). The key observation is that the quantities in both sides of the above inequality can be generalized to any Calabi–Yau manifold.

We propose the following definition of systole of a Calabi–Yau manifold, with respect to its complex and symplectic structures.

**Definition 3.1.1.** Let \( Y \) be a Calabi–Yau manifold, equipping with a symplectic form \( \omega \) and a holomorphic top form \( \Omega \). Then its systole is defined to be

\[
\text{sys}(Y, \omega, \Omega) := \inf_{L: \text{sLag}} \left\| \int_L \Omega \right\|.
\]

Here “sLag” denotes the compact special Lagrangian submanifolds in \((Y, \omega, \Omega)\).

With this definition, we propose the following question that naturally generalizes Loewner’s torus systolic inequality for Calabi–Yau manifolds.

**Question 3.1.2.** Let \( Y \) be a Calabi–Yau manifold and \( \omega \) be a symplectic form on \( Y \). Does there exist a constant \( C > 0 \) such that

\[
\text{sys}(Y, \omega, \Omega)^2 \leq C \cdot \left\| \int_Y \Omega \wedge \overline{\Omega} \right\|
\]

holds for any holomorphic top form \( \Omega \) on \( Y \)?

Note that the ratio \( \left\| \int_L \Omega \right\|^2 / \left\| \int_Y \Omega \wedge \overline{\Omega} \right\| \) has been considered in the context of attractor mechanism in physics [31, 67, 72], which is of independent interest.

To answer Question 3.1.2, one needs to characterize the middle homology classes of the Calabi–Yau manifold \( Y \) that can support special Lagrangian submanifolds. This is a difficult problem in general. Nevertheless, if \( Y \) has a mirror Calabi–Yau manifold \( X \), then special Lagrangian submanifolds on \( Y \) conjecturally are corresponded to certain Bridgeland semistable objects in the derived category of coherent sheaves on \( X \) [16]. The good news is that when \( X \) is a K3 surface, there is a algebro-geometric result (Proposition 3.4.2) which
characterizes the classes that support Bridgeland semistable objects. This motivates us to turn Question 3.1.2 into its mirror question (Question 3.1.4) and answer it in the case of K3 surfaces (Theorem 3.1.5). The formulation of the mirror question involves the categorical analogues of systole and volume, which we now describe.

The second goal of the present chapter is to propose the definitions of categorical systole and categorical volume, in terms of Bridgeland stability conditions on triangulated categories. For the definition of categorical systole, there are two sources of motivation: the correspondence between flat surfaces and stability conditions, and the conjectural description of stability conditions on the Fukaya categories of Calabi–Yau manifolds.

To give a Bridgeland stability condition on a triangulated category $D$, one needs to declare a set of objects in $D$ to be semistable, and assign a complex number (central charge $Z$) to each object in $D$. The correspondence between flat surfaces and stability conditions was studied by Gaiotto–Moore–Neitzke [39], Bridgeland–Smith [17], and Haiden–Katzarkov–Kontsevich [48]. Roughly speaking, under this correspondence, saddle connections on flat surfaces are corresponded to semistable objects, and their lengths are corresponded to the absolute values of central charges.

On the other hand, the systole of a flat surface is defined to be the length of its shortest saddle connection. Based on the aforementioned correspondence, we propose the following definition of systole of a Bridgeland stability condition.

**Definition 3.1.3 (= Definition 3.2.1).** Let $D$ be a triangulated category, and $\sigma$ be a Bridgeland stability condition on $D$. Its systole is defined to be

$$\text{sys}(\sigma) := \min \{|Z_\sigma(E)| : E \text{ is } \sigma-\text{semistable}\}.$$  

This definition is supported by another source of motivation, the conjectural description of stability conditions on the Fukaya categories of Calabi–Yau manifolds. Let $Y$ be a Calabi–Yau manifold and $\omega$ be a symplectic form on $Y$. One can associate a triangulated category, the derived Fukaya category $D^\tau_{\text{Fuk}}(Y, \omega)$, to the pair $(Y, \omega)$. Roughly speaking, the objects in the Fukaya category are Lagrangian submanifolds with some extra data.
It is conjectured by Bridgeland [16] and Joyce [55] that a holomorphic top form $\Omega$ on $Y$ should give a Bridgeland stability condition on $D^\pi\text{Fuk}(Y, \omega)$, with central charges given by period integrals of $\Omega$ along Lagrangians, and special Lagrangians are semistable objects. Assuming this conjecture, Definition 3.1.1 coincides with Definition 3.1.3 for $D = D^\pi\text{Fuk}(Y, \omega)$ and $\sigma$ is given by a holomorphic top form $\Omega$.

We summarize the correspondences in the following table.

<table>
<thead>
<tr>
<th>Surface $S$</th>
<th>Calabi-Yau $(Y, \omega)$</th>
<th>Triangulated category</th>
</tr>
</thead>
<tbody>
<tr>
<td>abelian differentials</td>
<td>holomorphic top forms</td>
<td>stability conditions</td>
</tr>
<tr>
<td>saddle connections</td>
<td>special Lagrangians</td>
<td>semistable objects</td>
</tr>
<tr>
<td>lengths</td>
<td>period integrals</td>
<td>central charges</td>
</tr>
<tr>
<td>$\text{sys}(S)$</td>
<td>$\text{sys}(Y, \omega, \Omega)$</td>
<td>$\text{sys}(\sigma)$</td>
</tr>
<tr>
<td>$\left</td>
<td>\int_Y \Omega \wedge \Omega \right</td>
<td>$</td>
</tr>
</tbody>
</table>

*Table 3.1: Correspondence among flat surfaces, holomorphic top forms on Calabi-Yau manifolds, and stability conditions on triangulated categories.*

In a previous joint work with Kanazawa and Yau [36], we study the categorical analogue of the holomorphic volume $\left| \int_Y \Omega \wedge \Omega \right|$ of Calabi-Yau manifolds. It is again motivated from the conjectural description of stability conditions on their Fukaya categories. We recall the definition of categorical volume $\text{vol}(\sigma)$ in Section 3.2.

The third goal of the present chapter is to propose the mirror question of Question 3.1.2 under mirror symmetry, in terms of Bridgeland stability conditions. Mirror symmetry conjecture states that Calabi-Yau manifolds come in pairs, in which the complex geometry of one is equivalent to the symplectic geometry of the other, and vice versa. One mathematical formulation of the conjecture, the homological mirror symmetry conjecture, was proposed by Kontsevich [65] in 1994. It states that if $(X, \omega_X, \Omega_X)$ and $(Y, \omega_Y, \Omega_Y)$ is a mirror pair of Calabi-Yau manifolds, then there are equivalences between triangulated categories

$$D^b\text{Coh}(X, \Omega_X) \cong D^\pi\text{Fuk}(Y, \omega_Y) \quad \text{and} \quad D^b\text{Coh}(Y, \Omega_Y) \cong D^\pi\text{Fuk}(X, \omega_X).$$

Under mirror symmetry, the complex moduli space $M_{\text{cpx}}(Y)$ should be identified with the stringy Kähler moduli space $M_{\text{Kah}}(X)$. The mathematical formulation of stringy Kähler
moduli space was proposed by Bridgeland [16]. The proposed definition is that $\mathcal{M}_{Kah}(X)$ is the double quotient $\text{Aut}(\mathcal{D}) \backslash \text{Stab}^*(\mathcal{D}) / \mathbb{C}$ of certain subset $\text{Stab}^*(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ of the space of stability conditions on the derived category of coherent sheaves $\mathcal{D} = \mathcal{D}^b\text{Coh}(X)$ on $X$.

We can now formulate the mirror question of Question 3.1.2 in terms of Bridgeland stability conditions.

**Question 3.1.4.** Let $X$ be a Calabi–Yau manifold and $\mathcal{D} = \mathcal{D}^b\text{Coh}(X, \Omega)$ be its derived category of coherent sheaves. Does there exist a constant $C > 0$ such that

$$\text{sys}(\sigma)^2 \leq C \cdot \text{vol}(\sigma)$$

holds for any $\sigma \in \text{Stab}^*(\mathcal{D})$?

Question 3.1.2 and Question 3.1.4 are mirror to each other in the sense that

- in Question 3.1.2, the symplectic form $\omega$ is fixed, and the question is asking for the supremum of the ratio $\text{sys}^2 / \text{vol}$ among all $\Omega \in \mathcal{M}_{\text{cpx}}(Y)$; while

- in Question 3.1.4, the complex structure $\Omega$ is fixed, and the question is asking for the supremum of the ratio $\text{sys}^2 / \text{vol}$ among all $[\sigma] \in \text{Aut}(\mathcal{D}) \backslash \text{Stab}^*(\mathcal{D}) / \mathbb{C} \cong \mathcal{M}_{Kah}(X)$. Note that the ratio $\text{sys}(\sigma)^2 / \text{vol}(\sigma)$ is invariant under the $\text{Aut}(\mathcal{D})$-action and the free $\mathbb{C}$-action by Lemma 3.2.3 and Lemma 3.2.8.

The subset $\text{Stab}^*(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$ in Question 3.1.4 is well-defined for $\dim(X) \leq 2$. For $X$ an elliptic curve, $\text{Stab}^*(\mathcal{D}) = \text{Stab}(\mathcal{D})$ is the whole space of Bridgeland stability conditions. For $X$ a K3 surface, there is a precise definition of $\text{Stab}^*(\mathcal{D})$ by Bayer and Bridgeland [8] which we will recall in Section 3.4. However, when $\dim(X) \geq 3$, there is no general characterization of the subset $\text{Stab}^*(\mathcal{D}) \subset \text{Stab}(\mathcal{D})$. We refer to [16, 36] for some attempts.

The fourth goal, which is the main result of the present chapter, is to give an affirmative answer to Question 3.1.4 for K3 surfaces of Picard rank one.
Theorem 3.1.5 (\(=\) Theorem 3.4.1). Let \(X\) be a K3 surface of Picard rank one, with \(\text{NS}(X) = \mathbb{Z}H\) and \(H^2 = 2n\). Then

\[
\text{sys}(\sigma)^2 \leq (n + 1)\text{vol}(\sigma)
\]

holds for any \(\sigma \in \text{Stab}^*(\mathcal{D}^b\text{Coh}(X))\).

We now sketch the proof of Theorem 3.1.5. The key algebro-geometric input is a result of Bayer and Macrì [9, Theorem 6.8], which gives rise to a numerical criterion of classes that support Bridgeland semistable objects (Proposition 3.4.2). This will show that the categorical systole and categorical volume of a stability condition on \(\mathcal{D} = \mathcal{D}^b\text{Coh}(X)\) only depend on its central charge. Then one can use the description of \(\text{Stab}^*(\mathcal{D})\) in Bayer–Bridgeland [8] to turn the systolic inequality into the following equivalent lattice-theoretic problem.

Question 3.1.6. Show that

\[
\min_{(s,d,r) \in \mathbb{Z}^3 \atop s \leq nd^2 + 1 \atop (s,d,r) \neq (0,0,0)} |s + 2n(\beta + i\omega)d + n(\beta + i\omega)^2r|^2 \leq 4(n + 1)n\omega^2
\]

holds for any \(\beta \in \mathbb{R}\) and \(\omega > 0\).

Note that \((\beta, \omega)\) in the inequality parametrizes the Bridgeland stability conditions lie in \(\text{Stab}^*(\mathcal{D})\), and \((s,d,r)\) parametrizes the classes that support Bridgeland semistable objects.

To answer Question 3.1.6, the most difficult part is to handle the case when \(\omega\) is very close to zero. This is done in Lemma 3.4.4. We then obtain the main result.

We also study the notion of spherical systoles of stability conditions. It is similar to the notion of categorical systole except we only consider the masses of spherical objects (Remark 3.4.5). The spherical systole and the categorical systole coincides for stability conditions on the derived category of an elliptic curve. However, this is not true in general. In particular, we show that the systolic inequality does not hold for spherical systoles of stability conditions on the derived categories of K3 surface of Picard rank one (Proposition 3.4.6).

This chapter is organized as follows. In Section 3.2, we introduce the notions of categorical systole and categorical volume. In Section 3.3, we give an affirmative answer to
Question 3.1.4 for elliptic curves, which matches with Loewner’s torus systolic inequality under mirror symmetry as expected. In Section 3.4, we give an affirmative answer to Question 3.1.4 for K3 surfaces of Picard rank one by proving Theorem 3.1.5. In Section 3.5, we indicate some directions for further studies.

3.2 Categorical systole and categorical volume

We refer to Section 2.2.1 for review on Bridgeland stability conditions.

3.2.1 Categorical systole

In the Introduction, we explained the two sources of motivation of our definition of categorical systole: the connection between flat surfaces and stability conditions, and the conjectural description of stability conditions on Fukaya categories of Calabi–Yau manifolds.

We recall the proposed definition of categorical systole, and investigate some basic properties.

**Definition 3.2.1.** Let \( \sigma \) be a Bridgeland stability condition on \( \mathcal{D} \). Its systole is defined to be

\[
\text{sys}(\sigma) := \min \{|Z_\sigma(E)| : E \text{ is } \sigma\text{-semistable}\}.
\]

**Remark 3.2.2.** The support property (Definition 2.2.1) of stability conditions guarantees that the systole \( \text{sys}(\sigma) \) is a positive number and is attained by the absolute value of the central charge of a stable object.

Below is an easy lemma.

**Lemma 3.2.3.** Let \( \sigma \) be a Bridgeland stability condition on \( \mathcal{D} \). Then

1. \( \text{sys}(\sigma) = \min_{E: \sigma\text{-stable}} |Z_\sigma(E)| = \min_{E \neq 0 \in \mathcal{D}} m_\sigma(E) \). Here \( m_\sigma(E) \) is the mass of \( E \) with respect to the stability condition \( \sigma \) (Definition 2.2.1).

2. \( \text{sys}(\Phi \cdot \sigma) = \text{sys}(\sigma) \) for any autoequivalence \( \Phi \in \text{Aut}(\mathcal{D}) \).

3. \( \text{sys}(z \cdot \sigma) = e^x \cdot \text{sys}(\sigma) \) for any complex number \( z = x + iy \in \mathbb{C} \).
Proof. The first statement follows directly from the definition of mass. The second and third statements follow from the definition of the actions of Aut(D) and C. \qed

**Remark 3.2.4.** By Lemma 3.2.3 (2), the categorical systole defines a real-valued function on the quotient space Stab(D)/Aut(D). It is shown in Bridgeland–Smith [17] that this quotient space is isomorphic to the space of certain meromorphic quadratic differentials with simple zeros on a marked Riemann surface, provided D is the CY3 triangulated category associated to the surface. Under this correspondence, the categorical systole is related to the period integrals $\int \sqrt{\phi}$ along the saddle connections of the quadratic differential $\phi$.

**Proposition 3.2.5.** Categorical systole defines a continuous function on the space of Bridgeland stability conditions: 

$$\text{Stab}(D) \rightarrow \mathbb{R}_{> 0}, \quad \sigma \mapsto \text{sys}(\sigma).$$

Proof. Let $\sigma_1, \sigma_2 \in \text{Stab}(D)$ be two Bridgeland stability conditions with $d(\sigma_1, \sigma_2) < \epsilon$. By Remark 3.2.2, $\text{sys}(\sigma_1) = m_{\sigma_1}(E)$ for some object $E \in D$. Thus

$$\log \text{sys}(\sigma_1) = \log m_{\sigma_1}(E) > \log m_{\sigma_2}(E) - \epsilon \geq \log \text{sys}(\sigma_2) - \epsilon.$$ 

Similarly, we have $\log \text{sys}(\sigma_2) > \log \text{sys}(\sigma_1) - \epsilon$. Hence $\log \text{sys}(\sigma)$ is a continuous function on Stab(D), and so is sys(\sigma). \qed

**Example 3.2.6 (Derived category of A2-quiver).** Let $Q$ be the $A_2$-quiver $(\cdot \rightarrow \cdot)$ and $D = \mathcal{D}^b(\text{Rep}(Q))$ be the bounded derived category of the category of representations of Q. Let $E_1$ and $E_2$ be the simple objects in $\text{Rep}(Q)$ with dimension vectors $\text{dim}(E_1) = (1,0)$ and $\text{dim}(E_2) = (0,1)$. There is one more indecomposable object $E_3$ that fits into the exact sequence

$$0 \rightarrow E_2 \rightarrow E_3 \rightarrow E_1 \rightarrow 0.$$ 

Let $\sigma = (Z, P)$ be a stability condition on $D$ with $z_1 := Z(E_1)$ and $z_2 := Z(E_2)$. Suppose that $P(0,1) = \text{Rep}(Q)$. Then

- When arg($z_1$) < arg($z_2$), the only $\sigma$-stable objects are $E_1$ and $E_2$ up to shiftings. Thus $\text{sys}(\sigma) = \min\{|z_1|, |z_2|\}$. 

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When \( \arg(z_1) > \arg(z_2) \), the only \( \sigma \)-stable objects are \( E_1, E_2 \) and \( E_3 \) up to shiftings. Thus \( \text{sys}(\sigma) = \min\{|z_1|, |z_2|, |z_1 + z_2|\} \).

Note that in order to compute the categorical systoles of stability conditions in different chambers, one needs to compute the central charges of different sets of dimensional vectors. However, it is not the case for the derived categories of elliptic curves and K3 surfaces, see the proof of Theorem 3.3.1 and Proposition 3.4.2.

### 3.2.2 Categorical volume

Motivated by the discussion in Section 2.3.2, we define the notion of categorical volume of Bridgeland stability conditions. It is the categorical analogue of the holomorphic volume \( \int_Y \Omega \wedge \Omega^1 \) of a compact Calabi-Yau manifold \( Y \) with holomorphic top form \( \Omega \).

**Definition 3.2.7.** Let \( \{E_i\} \) be a basis of the numerical Grothendieck group \( \mathcal{N}(D) \) and let \( \sigma = (Z, \mathcal{P}) \) be a Bridgeland stability condition on \( D \). Its volume is defined to be

\[
\text{vol}(\sigma) := \left| \sum_{i,j} \chi^{ij} Z(E_i) \overline{Z(E_j)} \right|,
\]

where \( \chi^{ij} = (\chi(E_i, E_j))^{-1} \) is the inverse matrix of the Euler pairings.

One can easily check that the above definition is independent of the choice of the basis \( \{E_i\} \). The following lemma is also straightforward.

**Lemma 3.2.8.** Let \( \sigma \) be a Bridgeland stability condition on \( D \). Then

1. \( \text{vol}(\Phi \cdot \sigma) = \text{vol}(\sigma) \) for any autoequivalence \( \Phi \in \text{Aut}(D) \).

2. \( \text{vol}(z \cdot \sigma) = e^{2x} \cdot \text{vol}(\sigma) \) for any complex number \( z = x + iy \in \mathbb{C} \).

**Remark 3.2.9.** Although the categorical volume can be defined for stability conditions on any triangulated category \( D \), its geometric meaning is not clear unless \( D \) comes from compact Calabi-Yau geometry. Below is an example of Calabi-Yau triangulated category for which the categorical volume vanishes for some stability conditions.
Example 3.2.10 (CY3-category of $A_2$-quiver). Let $Q$ be the $A_2$-quiver $(\cdot \to \cdot)$ and $\Gamma Q$ be the Ginzburg Calabi–Yau–3 dg-algebra associated with $Q$. See [41, 60] for the definition of $\Gamma Q$. Let $\mathcal{D}(\Gamma Q)$ be the derived category of dg-modules over $\Gamma Q$ and $\mathcal{D} = \mathcal{D}_{\text{id}}(\Gamma Q)$ be the full subcategory of $\mathcal{D}(\Gamma Q)$ consisting of dg-modules with finite dimensional total cohomology.

By Keller [60, Theorem 6.3], the category $\mathcal{D}$ is a Calabi–Yau–3 category, i.e., there is a natural isomorphism $\text{Hom}(E, F) \cong \text{Hom}(F, E[3])^*$ for any $E, F \in \mathcal{D}$. By Smith [91], there is an embedding of $\mathcal{D}$ into the Fukaya category of certain quasi-projective Calabi–Yau threefold.

The numerical Grothendieck group $N(\mathcal{D})$ is generated by two spherical objects $S_1, S_2$, each associated with a vertex in the $A_2$-quiver. The spherical objects satisfy:

- $\text{Hom}_\mathcal{D}(S_1, S_1) = \text{Hom}_\mathcal{D}(S_2, S_2) = \mathbb{C} \oplus \mathbb{C}[-3]$.
- $\text{Hom}_\mathcal{D}(S_1, S_2) = \mathbb{C}[-1], \text{Hom}_\mathcal{D}(S_2, S_1) = \mathbb{C}[-2]$.

Let $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{D}$ with $z_1 := Z(S_1)$ and $z_2 := Z(S_2)$. Then its categorical volume is

$$\text{vol}(\sigma) = |z_1 \overline{z_2} - z_2 \overline{z_1}| = 2|\text{Im}(z_1 \overline{z_2})|,$$

which vanishes if $z_1 \overline{z_2} \in \mathbb{R}$. This can happen if $z_1$ and $z_2$ are of the same phase, i.e., the stability condition $\sigma$ sits on a wall in $\text{Stab}(\mathcal{D})$.

3.3 Elliptic curves case

In this section, we give an affirmative answer to Question 3.1.4 in the case of elliptic curves. Let $\mathcal{D} = \mathcal{D}^{b}\text{Coh}(E)$ be the derived category of an elliptic curve $E$. In this case, the $\text{GL}^+(2; \mathbb{R})$-action on the space of Bridgeland stability conditions $\text{Stab}(\mathcal{D})$ is free and transitive [14, Theorem 9.1]. Hence,

$$\text{Stab}(\mathcal{D}) \cong \text{GL}^+(2; \mathbb{R}) \cong \mathbb{C} \times H,$$

and the double quotient

$$\text{Aut}(\mathcal{D}) \backslash \text{Stab}(\mathcal{D}) / \mathbb{C} \cong \text{PSL}(2, \mathbb{Z}) \backslash H.$$
is indeed the Kähler moduli space of elliptic curve. Thus we should take \( \text{Stab}^*(D) = \text{Stab}(D) \) in Question 3.1.4 to be the whole space of stability conditions.

**Theorem 3.3.1.** Let \( D = \mathcal{D}^b\text{Coh}(E) \) be the derived category of an elliptic curve \( E \). Then

\[
\text{sys}(\sigma)^2 \leq \frac{1}{\sqrt{3}} \cdot \text{vol}(\sigma)
\]

holds for any \( \sigma \in \text{Stab}(D) \).

One can consider this inequality as the mirror of Loewner’s torus systolic inequality in the introduction.

**Proof.** Firstly, by Lemma 3.2.3 and Lemma 3.2.8, one notices that the ratio

\[
\frac{\text{sys}(\sigma)^2}{\text{vol}(\sigma)}
\]

is invariant under the free \( \mathbb{C} \)-action on the space of stability conditions. Hence we only need to compute the ratio on the quotient space \( \text{Stab}(D)/\mathbb{C} \cong \mathbb{H} \).

The quotient space \( \text{Stab}(D)/\mathbb{C} \cong \mathbb{H} \) can be parametrized by the *normalized stability conditions* as follows. Let \( \tau = \beta + i\omega \in \mathbb{H} \) where \( \beta \in \mathbb{R} \) and \( \omega > 0 \). The associated normalized stability condition \( \sigma_\tau \) is given by:

- **Central charge:** \( Z_\tau(F) = -\deg(F) + \tau \cdot \text{rk}(F) \).
- For \( 0 < \phi \leq 1 \), the set of (semi)stable objects \( \mathcal{P}_\tau(\phi) \) consists of the slope-(semi)stable coherent sheaves whose central charge lies in the ray \( \mathbb{R}_{>0} \cdot e^{i\pi\phi} \).
- For other \( \phi \in \mathbb{R} \), define \( \mathcal{P}_\tau(\phi) \) by the property \( \mathcal{P}_\tau(\phi + 1) = \mathcal{P}_\tau(\phi)[1] \).

Note that there is no wall-crossing phenomenon in the elliptic curve case, i.e., for any Bridgeland stability condition on \( \mathcal{D}^b\text{Coh}(E) \), the set of all Bridgeland (semi)stable objects is the same as the set of all slope-(semi)stable coherent sheaves up to shiftings. This makes the computation of categorical systole easier.

To compute the systole of \( \sigma_\tau \), by Lemma 3.2.3 (1), we need to know the central charges of all the stable objects of \( \sigma_\tau \), which are in this case the slope-stable coherent sheaves. Recall
that if $F$ is a slope-stable coherent sheaf on $E$, then it is either a vector bundle or a torsion sheaf. The slope-stable vector bundles on an elliptic curve $E$ are well-understood, see for instance [5, 81].

- Let $F$ be an indecomposable vector bundle of rank $r$ and degree $d$ on an elliptic curve $E$. Then $F$ is slope-stable if and only if $d$ and $r$ are relatively prime.
- Fix a point $x \in E$. For every rational number $\mu = \frac{d}{r}$, where $r > 0$ and $(d, r) = 1$, there exists a unique slope-stable vector bundle $V_{\mu}$ of rank $r$ and $\det(V_{\mu}) \cong \mathcal{O}_E(dx)$.

Hence the categorical systole is

$$\text{sys}(\sigma) = \min_{\substack{(d, r) = 1 \\ r > 0}} \{1, | - d + \tau r| \} = \min_{\substack{(d, r) \in \mathbb{Z}^2 \\ (d, r) \neq (0, 0)}} \{| - d + \tau r| \} = \lambda_1(L_\tau),$$

where $\lambda_1(L_\tau)$ denotes the least length of a nonzero element in the lattice $L_\tau = (1, \tau)$.

On the other hand, the categorical volume of $\sigma$ has been computed in Example 2.3.5, which equals to $2\omega$. Thus

$$\sup_{\tau \in \mathbb{H}} \frac{\text{sys}(\sigma) \cdot \omega}{\text{vol}(\sigma)} = \frac{1}{2} \cdot \sup_{\tau \in \mathbb{H}} \frac{\lambda_1(L_\lambda)^2}{\omega}.$$

Note that $\omega$ is the area of the parallelogram spanned by 1 and $\tau$. Hence the quantity $\sup_{\tau \in \mathbb{H}} \frac{\lambda_1(L_\lambda)^2}{\omega}$ is the so-called Hermite constant $\gamma_2$ of lattices in $\mathbb{R}^2$. It is classically known that the Hermite constant $\gamma_2 = \frac{2}{\sqrt{3}}$ (see for instance [19]). This concludes the proof. \qed

### 3.4 K3 surfaces case

In this section, we give an affirmative answer to Question 3.1.4 for K3 surfaces of Picard rank one. This is the main result of the present chapter. We start with recalling some standard notations.

Let $X$ be a smooth complex projective K3 surface and $\mathcal{D} = \mathcal{D}^b\text{Coh}(X)$ be its derived category. Sending an object $E \in \mathcal{D}$ to its Mukai vector $\nu(E) = \text{ch}(E) \sqrt{\text{td}(X)}$ identifies the
numerical Grothendieck group of $\mathcal{D}$ with the lattice

$$\mathcal{N}(\mathcal{D}) = H^0(X; \mathbb{Z}) \oplus \text{NS}(X) \oplus H^4(X; \mathbb{Z}).$$

The Mukai pairing on $\mathcal{N}(\mathcal{D})$ is given by

$$((r_1, D_1, s_1), (r_2, D_2, s_2)) = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1.$$

Note that the Mukai pairing on $\mathcal{N}(\mathcal{D})$ is non-degenerate, hence any group homomorphism $Z : \mathcal{N}(\mathcal{D}) \to \mathbb{C}$ can be written as

$$Z(\nu) = (\Omega, \nu) : \mathcal{N}(\mathcal{D}) \to \mathbb{C}$$

for some unique $\Omega \in \mathcal{N}(\mathcal{D}) \otimes \mathbb{C}$.

Define the period domains

$$Q^\pm(X) = \{ \Omega \in \mathcal{N}(\mathcal{D}) \otimes \mathbb{C} : (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0 \},$$

$$Q_0^\pm(X) = Q^\pm(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp.$$

Here $\Delta(X) = \{ \delta \in \mathcal{N}(\mathcal{D}) : \delta^2 = -2 \}$ denotes the root system. We let $Q^+(X)$ be the connected component containing classes $(1, i\omega, -\frac{1}{2} \omega^2)$ for $\omega \in \text{NS}(X)$ ample. These domains are invariant under the rescaling action of $\mathbb{C}^*$ on $\mathcal{N}(\mathcal{D}) \otimes \mathbb{C}$.

Define the tube domain

$$\mathcal{T}^\pm(X) = \{ \beta + i\omega : \beta, \omega \in \text{NS}(X) \otimes \mathbb{R}, (\omega, \omega) > 0 \},$$

where $\mathcal{T}^+(X)$ denotes the connected component containing $i\omega$ for ample classes $\omega$. One can check that the map

$$\mathcal{T}^+(X) \hookrightarrow Q^+(X), \quad \beta + i\omega \mapsto \Omega = \exp(\beta + i\omega)$$

is an embedding which gives a section of the $\mathbb{C}^*$-action on $Q^+(X)$.

We now briefly review some previous results on Bridgeland stability conditions on the derived categories $\mathcal{D} = \mathcal{D}^b\text{Coh}(X)$ of K3 surfaces. The existence of stability conditions on
$D$ was first proved by Bridgeland [15]. The main result of [15] is a complete description of the distinguished connected component $\text{Stab}^\dagger(D) \subset \text{Stab}(D)$ containing the set of geometric stability conditions, for which all skyscraper sheaves $O_x$ are stable and of the same phase.

Let $\text{Stab}^\Delta(D) \subset \text{Stab}(D)$ be the union of those connected components which are images of $\text{Stab}^\dagger(D)$ under the $\text{Aut}(D)$-actions. The complex submanifold $\text{Stab}^*(D) \subset \text{Stab}^\Delta(D)$ is defined to be the subset consisting of stability conditions $\sigma = (Z = (\Omega, -), P)$ such that the corresponding vector $\Omega \in \mathcal{N}(D) \otimes \mathbb{C}$ satisfies $(\Omega, \Omega) = 0$. Note that this submanifold is denoted by $\text{Stab}^*_\text{red}(X)$ in [8].

In the work of Bayer and Bridgeland [8], they define the stringy Kähler moduli space of the K3 surface $X$ as the quotient

$$M_{\text{Kah}}(X) := (\text{Stab}^*(D)/\mathbb{C})/(\text{Aut}_{\text{CY}}(D)/[2]),$$

where $\text{Aut}_{\text{CY}}(D) \subset \text{Aut}(D)$ is a certain subgroup of $\text{Aut}(D)$ consisting of Calabi–Yau autoequivalences. The mirror symmetry phenomenon in this context is that the complex orbifold $M_{\text{Kah}}(X)$ can be identified with the base of Dolgachev’s family of lattice-polarized K3 surfaces mirror to $X$ [30]. We refer to [8] for more details.

We now state the main theorem.

**Theorem 3.4.1.** Let $X$ be a K3 surface of Picard rank one, with $\text{NS}(X) = \mathbb{Z}H$ and $H^2 = 2n$. Then

$$\text{sys}(\sigma)^2 \leq (n + 1)\text{vol}(\sigma)$$

holds for any $\sigma \in \text{Stab}^*(D_{\text{bCoh}}(X))$.

The following proposition allows us to compute the categorical systole of a stability condition $\sigma = (Z_\sigma, P_\sigma) \in \text{Stab}^\Delta(D)$ using only its central charge $Z_\sigma$.

**Proposition 3.4.2.** Let $\sigma = (Z_\sigma, P_\sigma) \in \text{Stab}^\Delta(D)$. Then

$$\text{sys}(\sigma) = \min\{|Z_\sigma(v)| : 0 \neq v \in \mathcal{N}(D), v^2 = (v, v) \geq -2\}.$$

**Proof.** It suffices to prove the statement for $\sigma \in \text{Stab}^\dagger(D)$, since autoequivalences induce Hodge isometries on the numerical Grothendieck group $\mathcal{N}(D)$, and $\text{Stab}^\Delta(D)$ is the image
of the connected component \( \text{Stab}^\dagger(D) \) under the \( \text{Aut}(D) \)-actions.

Let \( v = mv_0 \in \mathcal{N}(D) \) be a Mukai vector, where \( m \in \mathbb{Z}_{>0} \) and \( v_0 \) is primitive. A result of Bayer and Macrì [9, Theorem 6.8], which is based on a previous result of Toda [98], says that if \( v_0^2 \geq -2 \), then there exists a \( \sigma \)-semistable object with Mukai vector \( v \) for any \( \sigma \in \text{Stab}^\dagger(D) \).

Hence

\[
\text{sys}(\sigma) = \min\{|Z_\sigma(E)| : E \text{ is } \sigma-\text{semistable} \} \\
\leq \min\{|Z_\sigma(v)| : 0 \neq v \in \mathcal{N}(D), \ v^2 \geq -2 \}.
\]

On the other hand, for any stable object \( E \),

\[
v(E)^2 = -\chi(E,E) = -\text{hom}^0(E,E) + \text{hom}^1(E,E) - \text{hom}^2(E,E) \\
= -2 + \text{hom}^1(E,E) \geq -2.
\]

Hence

\[
\text{sys}(\sigma) = \min\{|Z_\sigma(E)| : E \text{ is } \sigma-\text{stable} \} \\
\geq \min\{|Z_\sigma(v)| : 0 \neq v \in \mathcal{N}(D), \ v^2 \geq -2 \}.
\]

This concludes the proof. \( \Box \)

**Remark 3.4.3.** By Proposition 3.4.2, the set of Mukai vectors needed for computing the categorical systole is independent of stability condition \( \sigma \in \text{Stab}^\dagger(D) \). This is also the case for elliptic curves, where the categorical systole is the minimum among the absolute values of central charges of all the nonzero vectors \( (d,r) \in \mathbb{Z}^2 \). However, this is not true in general, see for instance Example 3.2.6.

We now prove the main theorem.

**Proof of Theorem 3.4.1.** Let \( X \) be a K3 surface of Picard rank one, with \( \text{NS}(X) = \mathbb{Z}H \) and \( H^2 = 2n \). The numerical Grothendieck group of \( D = \mathcal{D}^b\text{Coh}(X) \) is

\[
\mathcal{N}(D) = \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}.
\]

By Proposition 3.4.2 and the definition of categorical volume, in order to compute the
categorical systoles and volumes of stability conditions $\sigma \in \text{Stab}^*(D)$, it suffices to know their central charges.

By [8, 15], up to autoequivalences (which do not change the ratio $\text{sys}(\sigma)^2/\text{vol}(\sigma)$), $Z = (\Omega, -)$ is the central charge of a stability condition in the submanifold $\text{Stab}^*(D)$ if and only if $\Omega \in Q_0^+(X)$. Recall that $Q_0^+(X) \subset Q^+(X)$ admits a section of the $\mathbb{C}^*$-action given by the tube domain. Hence $\Omega$ can be written as $\Omega = c \exp((\beta + i\omega)H)$ for some nonzero complex number $c$, and $\beta \in \mathbb{R}$, $\omega > 0$.

By Proposition 3.4.2 and 2.4.6,
\[
\sup_{\sigma \in \text{Stab}^*(D)} \frac{\text{sys}(\sigma)^2}{\text{vol}(\sigma)} = \sup_{\Omega = \exp((\beta + i\omega)H) \in Q^+_0(X)} \frac{\min\{|(\Omega, v)|^2 : v^2 \geq -2, v \neq 0\}}{4n\omega^2} = \sup_{\Omega = \exp((\beta + i\omega)H) \in Q^+(X)} \frac{\min\{|(\Omega, v)|^2 : v^2 \geq -2, v \neq 0\}}{4n\omega^2}
\]

The second equality follows from the fact that the numerator vanishes if $\Omega \in \delta^\perp$.

Let $v = (r, dH, s) \in \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}$. One then can plug in the integers $r, d, s$ and the real numbers $\beta, \omega$ to the central charge $(\Omega, v)$, and rewrite the above quantity as
\[
\sup_{\beta \in \mathbb{R}} \frac{\min\{|s + 2n(\beta + i\omega)d + n(\beta + i\omega)^2r|^2 : (s, d, r) \in \mathbb{Z}^3 \setminus \{(0, 0, 0)\}, sr \leq nd^2 + 1\}}{4n\omega^2}
\]

In order to deal with the situation when $\omega$ is very close to zero, we need the following technical lemma.

**Lemma 3.4.4.** For any real number $\beta$ and any $0 < \omega < \frac{1}{\sqrt{n}}$, there exists integers $(s, d, r)$ such that:

\[
1 \leq r \leq \frac{1}{\sqrt{n}\omega},
\]
\[
|s + 2n\beta d + n\beta^2 r| < \sqrt{n}\omega,
\]
\[
0 \leq nd^2 - sr \leq n.
\]

**Proof of Lemma 3.4.4.** Let $l = \lfloor \frac{1}{\sqrt{n}\omega} \rfloor + 1$. For each $1 \leq j \leq l$, choose $d_j \in \mathbb{Z}$ such that
\[
-\frac{1}{2} < d_j + j\beta \leq \frac{1}{2}.
\]

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Consider the real numbers \( \{2n\beta d_j + n\beta^2 j\}_{1 \leq j \leq l} \) modulo 1. There is at least a pair 
\( (2n\beta d_j + n\beta^2 j, 2n\beta d_k + n\beta^2 k) \) has distance less than or equals to \( 1/l \) modulo 1. Say \( j > k \) without loss of generality. We choose \( r = j - k, d = d_j - d_k, \) and choose \( s \) to be the integer closest to \(-2n\beta d - n\beta^2 r\). Then

\[
1 \leq r \leq \left\lfloor \frac{1}{\sqrt{n}\omega} \right\rfloor
\]

and

\[
|s + 2n\beta d + n\beta^2 r| \leq \frac{1}{\left\lfloor \frac{1}{\sqrt{n}\omega} \right\rfloor + 1} < \sqrt{n}\omega.
\]

Let \( e = s + 2n\beta d + n\beta^2 r \). Then

\[
nd^2 - sr = nd^2 - \left( -2n\beta d - n\beta^2 r + e \right) r = n(d + r\beta)^2 - re.
\]

We have

\[
(d + r\beta)^2 = ((d_j + j\beta) - (d_k + k\beta))^2 < 1
\]

and

\[
|re| < \frac{1}{\sqrt{n}\omega} \cdot \sqrt{n}\omega = 1.
\]

Hence \(-1 < nd^2 - sr < n + 1\). Since it is an integer, thus \(0 \leq nd^2 - sr \leq n\). \(\Box\)

We are now ready to finish the proof Theorem 3.4.1. By previous discussion, it suffices to prove the following claim.

**Claim.** For any \( \beta \in \mathbb{R} \) and \( \omega > 0 \), there exists \( (s, d, r) \in \mathbb{Z}^3 \) not all zero such that 
\( sr \leq nd^2 + 1 \) and

\[
\frac{|s + 2n(\beta + i\omega)d + n(\beta + i\omega)^2 r|^2}{4n\omega^2} < n + 1.
\]

- If \( \omega \geq \frac{1}{\sqrt{n}} \), one can simply take the class of skyscraper sheaves \( (s, d, r) = (1, 0, 0) \) and check that it satisfies the two inequalities above.
• If $\omega < \frac{1}{\sqrt{n}}$, we choose $(s, d, r)$ as in Lemma 3.4.4. Then it satisfies $sr < nd^2 + 1$ and

$$\frac{|s + 2n(\beta + i\omega)d + n(\beta + i\omega)^2r|^2}{4n\omega^2} = \frac{1}{4n} \left( \frac{s + 2n\beta d + n\beta^2r}{\omega} + n\omega \right)^2 + (nd^2 - rs) \leq \frac{1}{4n} (\sqrt{n} + \sqrt{n})^2 + n = n + 1.$$ 

This concludes the proof of Theorem 3.4.1.

\[
\square
\]

**Remark 3.4.5 (Spherical systole).** The notion of spherical objects in a triangulated category was introduced by Seidel and Thomas [89]. These objects are the categorical analogue of Lagrangian spheres in derived Fukaya categories. An object $S \in \mathcal{D} \text{Coh} \mathcal{X}$ in the derived category of coherent sheaves on a Calabi–Yau $n$-fold $X$ is called spherical if $\text{Hom}_\mathcal{D}(S, S) = \mathbb{C} \oplus \mathbb{C}[-n]$.

One can define the spherical systole of a Bridgeland stability condition $\sigma$ on a triangulated category $\mathcal{D}$ as the minimum among masses of spherical objects:

$$\text{sys}_{\text{sph}}(\sigma) := \min \{ m_\sigma(S) : S \in \mathcal{D} \text{ spherical} \}.$$ 

Let $\mathcal{D} = \mathcal{D} \text{b Coh}(\mathcal{E})$ be the derived category of coherent sheaves on an elliptic curve $\mathcal{E}$. Then the categorical systole of any stability condition on $\mathcal{D}$ is achieved by some spherical object in $\mathcal{D}$, hence $\text{sys}(\sigma) = \text{sys}_{\text{sph}}(\sigma)$ for any $\sigma \in \text{Stab}(\mathcal{D})$. However, this is not true for the derived categories of K3 surfaces.

The following proposition shows that the systolic inequality does not hold for spherical systole on K3 surfaces.

**Proposition 3.4.6.** Let $X$ be a K3 surface of Picard rank one and $\mathcal{D} = \mathcal{D} \text{b Coh}(\mathcal{X})$ be its derived category of coherent sheaves. Then

$$\sup_{\sigma \in \text{Stab}^*(\mathcal{D})} \frac{\text{sys}_{\text{sph}}(\sigma)^2}{\text{vol}(\sigma)} = +\infty.$$ 

**Proof.** Let $\text{NS}(X) = \mathbb{Z}H$ and $H^2 = 2n$. Let $\omega H \in \text{NS}(X) \otimes \mathbb{R}$ be an ample class. Then $Z = (\exp(i\omega H), -)$ gives the central charge of a stability condition in $\text{Stab}^*(\mathcal{D})$ if $\omega > 1$.
By Proposition 3.4.2 and 2.4.6,
\[ \sup_{\sigma \in \text{Stab}^+(D)} \frac{\text{sys}_{\text{sph}}(\sigma)^2}{\text{vol}(\sigma)} \geq \sup_{\omega > 1} \min \left\{ \frac{|(\exp(i\omega H), v)|^2}{4n\omega^2} : v^2 = -2 \right\}. \]

Here we use the fact that the Mukai vector of a spherical object \( S \) satisfies \( v(S)^2 = -\text{hom}^0(S, S) + \text{hom}^1(S, S) - \text{hom}^2(S, S) = -2 \).

Let \( v = (r, dH, s) \in \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z} \) be a vector satisfying \( v^2 = 2nd^2 - 2rs = -2 \). Then \( rs \neq 0 \) and \( r, s \) are both positive or both negative. Hence
\[
\sup_{\sigma \in \text{Stab}^+(D)} \frac{\text{sys}_{\text{sph}}(\sigma)^2}{\text{vol}(\sigma)} \geq \sup_{\omega > 1} \min \left\{ \frac{|(\exp(i\omega H), v)|^2}{4n\omega^2} : v^2 = -2 \right\}
= \sup_{\omega > 1} \min \frac{1}{4n} \left( \frac{s}{\omega} + n\omega r \right)^2 - 1
\geq \sup_{\omega > 1} \frac{1}{4n} \left( \frac{1}{\omega} + n\omega \right)^2 - 1 = +\infty.
\]

\[ \square \]

**Remark 3.4.7.** In the proof of Theorem 3.4.1, we do not make use of the Mukai vectors of spherical objects. Recall that
\[ \text{sys}(\sigma) = \min \{|Z_\sigma(v)| : v^2 \geq -2, v \neq 0\}, \]
but we only use the Mukai vectors that satisfy \( 0 \leq v^2 \leq 2n \) to prove the systolic inequality (c.f. Lemma 3.4.4).

### 3.5 Further studies

#### 3.5.1 Optimal systolic ratio of K3 surfaces of Picard rank one

It is not clear whether the constant \( n + 1 \) in the categorical systolic inequality
\[ \text{sys}(\sigma)^2 \leq (n + 1)\text{vol}(\sigma) \]
is optimal. As discussed in the proof of Theorem 3.4.1, finding the optimal systolic ratio is equivalent to solving the following lattice-theoretic problem:

\[
\sup_{\beta \in \mathbb{R}} \min_{\omega > 0, \ (s, d, r) \in \mathbb{Z}^3} \frac{|s + 2n(\beta + i\omega)d + n(\beta + i\omega)^2r|^2}{4n\omega^2} = ?
\]

\[
\omega > 0, \ sr < nd^2 + 1, \ (s, d, r) \neq (0, 0, 0)
\]

One might expect the optimal is achieved by certain stability conditions on K3 surfaces with “extra symmetries”, like the optimal of Loewner’s torus systolic inequality is achieved by the flat equilateral torus.

### 3.5.2 Systolic inequality for K3 surfaces of higher Picard rank

Almost the whole argument of the proof of Theorem 3.4.1 can be applied to K3 surfaces of higher Picard rank. In particular, Bayer–Bridgeland’s description of stringy Kähler moduli space, Proposition 3.4.2 and 2.4.6 all work for general K3 surfaces. Hence the systolic problem is equivalent to the following:

\[
\sup_{\beta, \omega \in \text{NS}(X) \otimes \mathbb{R}} \min_{\omega^2 > 0, \ (s, D, r) \in \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}} \frac{|s + (\beta + i\omega)D + \frac{1}{2}(\beta + i\omega)^2r|^2}{2\omega^2} = ?
\]

\[
\omega^2 > 0, \ 2sr < D^2 + 2, \ (s, D, r) \neq (0, 0, 0)
\]

The problem becomes more complicated for K3 surfaces of higher Picard rank because of the indefiniteness of intersection pairing on \(\text{NS}(X)\).

### 3.5.3 Miscellaneous

- It is mentioned in Remark 3.4.3 that in the case of elliptic curves and K3 surfaces, the set of Mukai vectors needed for computing the categorical systole is independent of stability conditions. This is also true for the derived categories of coherent sheaves on smooth projective curves of genus \(\geq 1\), since there is no wall-crossings on the space of stability conditions [70]. It would be interesting to find other triangulated categories that also satisfy this property.

- Akrout [3] proves that the systole of Riemann surfaces is a topological Morse function on...
the Teichmüller space. It would be interesting to know whether the categorical systole is also a topological Morse function on the space of Bridgeland stability conditions.
Chapter 4

Entropy of autoequivalences\(^1\)

4.1 Introduction

In the pioneering work [29], Dimitrov–Haiden–Katzarkov–Kontsevich introduced the notion of categorical entropy of an endofunctor on a triangulated category with a split generator. A typical example of such triangulated category is given by the bounded derived category \(D^b(X)\) of coherent sheaves on a variety \(X\). When \(X\) is a smooth projective variety with ample (anti-)canonical bundle, the group of autoequivalences on \(D^b(X)\) is well-understood [12]. It is generated by tensoring line bundles, automorphisms on the variety, and degree shifts. The categorical entropy in this case was computed by Kikuta and Takahashi [63]. On the other hand, when the variety is Calabi–Yau, there are much more autoequivalences on \(D^b(X)\) because of the presence of spherical objects.

In this article, we show that the composite of the simplest spherical twist \(T_{\mathcal{O}_X}\) with \(- \otimes \mathcal{O}(-H)\) already gives an interesting categorical entropy.

**Theorem 4.1.1** (= Theorem 4.3.1). Let \(X\) be a strict Calabi–Yau manifold over \(\mathbb{C}\) of dimension \(d \geq 3\). Consider the autoequivalence \(\Phi := T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(-H))\) on \(D^b(X)\). The categorical entropy

\(^1\)References: [32, 33].
$h_t(\Phi)$ is a positive function in $t \in \mathbb{R}$. Moreover, for any $t \in \mathbb{R}$, $h_t(\Phi)$ is the unique $\lambda > 0$ satisfying
\[
\sum_{k \geq 1} \frac{\chi(O(kH))}{e^{k\lambda}} = e^{(d-1)t}.
\]

A simple argument on Hilbert polynomial shows that this equation defines an algebraic curve over $\mathbb{Q}$ in the coordinate $(e^t, e^\lambda)$. Thus the algebraicity conjecture in [29, Question 4.1] holds in this case.

The notion of categorical entropy is a categorical analogue of topological entropy of a continuous surjective self-map on a compact metric space. There is a fundamental theorem of topological entropy due to Gromov and Yomdin.

**Theorem 4.1.2** ([44, 45, 106]). Let $M$ be a compact Kähler manifold and let $f : M \to M$ be a surjective holomorphic map. Then
\[
h_{\text{top}}(f) = \log \rho(f^*),
\]
where $h_{\text{top}}(f)$ is the topological entropy of $f$, and $\rho(f^*)$ is the spectral radius of $f^* : H^*(M; \mathbb{C}) \to H^*(M; \mathbb{C})$.

Kikuta and Takahashi proposed the following analogous conjecture on categorical entropy.

**Conjecture 4.1.3** ([63]). Let $X$ be a smooth proper variety over $\mathbb{C}$ and let $\Phi$ be an autoequivalence on $D^b(X)$. Then
\[
h_0(\Phi) = \log \rho(\text{HH}_*(\Phi)),
\]
where $\text{HH}_*(\Phi) : \text{HH}_*(X) \to \text{HH}_*(X)$ is the induced $\mathbb{C}$-linear isomorphism on the Hochschild homology group of $X$, and $\rho(\text{HH}_*(\Phi))$ is its spectral radius.

Note that one can replace $\text{HH}_*(\Phi)$ by the induced Fourier-Mukai type action on the cohomology $\Phi_{H^*} : H^*(X; \mathbb{C}) \to H^*(X; \mathbb{C})$, because there is a commutative diagram
\[
\begin{array}{ccc}
\text{HH}_*(X) & \xrightarrow{\text{HH}_*(\Phi)} & \text{HH}_*(X) \\
\downarrow I_X^* & & \downarrow I_X^* \\
H^*(X; \mathbb{C}) & \xrightarrow{\Phi_{H^*}} & H^*(X; \mathbb{C}),
\end{array}
\]
where $I^X_K$ is the modified Hochschild–Kostant–Rosenberg isomorphism [71, Theorem 1.2]. Hence $\rho(HH_\bullet(\Phi)) = \rho(\Phi_{H^\bullet})$.

Conjecture 4.1.3 has been proved in several cases. See [61, 62, 63, 107].

Now we explain the motivation of the present work, namely why we should not expect Conjecture 4.1.3 to hold in general. We first note that Theorem 4.1.2 does not hold if $f$ is not holomorphic. For example, there is a construction by Thurston [95] of pseudo-Anosov maps that act trivially on the cohomology, and thus have zero spectral radius. Moreover, Dimitrov–Haiden–Katzarkov–Kontsevich [29, Theorem 2.18] showed that the categorical entropy of the induced autoequivalence on the derived Fukaya category is equal to $\log \lambda$, where $\lambda > 1$ is the stretch factor of the pseudo-Anosov map. Hence the analogous statement of Conjecture 4.1.3 is not true if $D^b(X)$ is replaced by the derived Fukaya categories of the symplectic manifolds considered in [29, 95].

Motivated by homological mirror symmetry, one may expect to find counterexamples of Conjecture 4.1.3 on the derived categories of coherent sheaves on Calabi–Yau manifolds. In other words, the discrepancy between complex and symplectic geometry should lead to the discrepancy between Theorem 4.1.2 and Conjecture 4.1.3.

Using Theorem 4.1.1, we construct counterexamples of Conjecture 4.1.3.

**Proposition 4.1.4** (= Proposition 4.4.1). For any even integer $d \geq 4$, let $X$ be a Calabi–Yau hypersurface in $\mathbb{CP}^{d+1}$ of degree $(d+2)$ and $\Phi = T_{\mathcal{O}_X} \circ (- \otimes \mathcal{O}(-1))$. Then

$$h_0(\Phi) > 0 = \log \rho(\Phi_{H^\bullet}).$$

In particular, Conjecture 4.1.3 fails in this case.

Interestingly, as pointed out to the author by Genki Ouchi, the same autoequivalence does not produce counterexamples of Conjecture 4.1.3 if $X$ is an odd dimensional Calabi–Yau manifold (see Remark 4.4.3).
4.2 Preliminaries

4.2.1 Categorical entropy

We recall the notion of categorical entropy introduced by Dimitrov-Haiden-Katzarkov-Kontsevich [29].

Let $D$ be a triangulated category. A triangulated subcategory is called thick if it is closed under taking direct summands. The split closure of an object $E \in D$ is the smallest thick triangulated subcategory containing $E$. An object $G \in D$ is called a split generator if its split closure is $D$.

**Definition 4.2.1** ([29]). Let $E$ and $F$ be non-zero objects in $D$. If $F$ is in the split closure of $E$, then the complexity of $F$ relative to $E$ is defined to be the function

$$
\delta_t(E, F) := \inf \left\{ \sum_{i=1}^{k} e^{n_i t} \right\}
$$

where $t$ is a real parameter that keeps track of the shiftings.

Note that when the category $D$ is $\mathbb{Z}_2$-graded in the sense that $[2] \equiv \text{id}_D$, only the value at $t$ equals to zero, $\delta_0(E, F)$, will be of any use.

**Definition 4.2.2** ([29]). Let $D$ be a triangulated category with a split generator $G$ and let $F : D \to D$ be an endofunctor. The categorical entropy of $F$ is defined to be the function $h_t(F) : \mathbb{R} \to (-\infty, \infty)$ given by

$$
h_t(F) := \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, F^n G).
$$

**Lemma 4.2.3** ([29]). The limit $\lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, F^n G)$ exists in $(-\infty, \infty)$ for every $t \in \mathbb{R}$, and is independent of the choice of the split generator $G$.

We will use the following proposition to compute categorical entropy.

**Proposition 4.2.4** ([29, 63]). Let $G$ and $G'$ be split generators of $D^b(X)$ and let $\Phi$ be an autoequivalence on $D^b(X)$. Then the categorical entropy equals to

$$
h_t(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{l \in \mathbb{Z}} \dim \text{Hom}_{D^b(X)}^l(G, \Phi^n G') e^{-lt}.
$$
4.2.2 Spherical objects and spherical twists

We recall the notions of spherical objects and spherical twists introduced by Seidel-Thomas [89]. They are the categorical analogue of Lagrangian spheres in symplectic manifolds and the Dehn twists along Lagrangian spheres.

**Definition 4.2.5 ([89]).** An object $S \in \mathcal{D}^b(X)$ is called spherical if $S \otimes \omega_X \cong S$ and $\operatorname{Hom}^\bullet_{\mathcal{D}^b(X)}(S, S) = \mathbb{C} \oplus \mathbb{C}[-\dim X]$.

**Definition 4.2.6 ([89]).** The spherical twist $T_S$ with respect to a spherical object $S$ is an autoequivalence on $\mathcal{D}^b(X)$ given by

$$E \mapsto T_S(E) := \operatorname{Cone}(\operatorname{Hom}^\bullet_{\mathcal{D}^b(X)}(S, E) \otimes S \to E).$$

We also recall the definition of strict Calabi–Yau manifolds.

**Definition 4.2.7.** A smooth projective variety $X$ is called strict Calabi–Yau if $\omega_X \cong \mathcal{O}_X$ and $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < \dim X$. This is equivalent to the condition that $\mathcal{O}_X$ is a spherical object.

4.3 Computation of categorical entropy

We fix the notations and assumptions that will be used throughout this section. We work over complex number field $\mathbb{C}$.

**Notations**

- $X$ is a strict Calabi–Yau manifold (Definition 4.2.7) with a very ample line bundle $H$.
- $d := \dim_\mathbb{C} X \geq 3$.
- $\mathcal{O} := \mathcal{O}_X$ and $\mathcal{O}(k) := \mathcal{O}_X(kH)$.
- $a_k := h^0(\mathcal{O}(k)) = \chi(\mathcal{O}(k))$ for $k > 0$. 
\[ G := \bigoplus_{i=1}^{d+1} \mathcal{O}(i) \text{ and } G' := \bigoplus_{i=1}^{d+1} \mathcal{O}(-i). \] By a result of Orlov [79], both \( G \) and \( G' \) are split generators of \( D^b(X) \).

The goal of this section is to prove the following theorem.

**Theorem 4.3.1.** Let \( X \) be a strict Calabi–Yau manifold over \( \mathbb{C} \) of dimension \( d \geq 3 \). Consider the autoequivalence \( \Phi := T_0 \circ (- \otimes \mathcal{O}(-1)) \) on \( D^b(X) \). The categorical entropy \( h_t(\Phi) \) is a positive function in \( t \in \mathbb{R} \). Moreover, for any \( t \in \mathbb{R} \), \( h_t(\Phi) \) is the unique \( \lambda > 0 \) satisfying

\[
\sum_{k \geq 1} \frac{\chi(\mathcal{O}(k))}{e^{k\lambda}} = e^{(d-1)t}.
\]

We begin with a lemma that will be crucial in the computation of categorical entropy.

**Lemma 4.3.2.** 1. For any integers \( n \geq 0 \) and \( k > 0 \), \( \text{Hom}^l(\mathcal{O}, \Phi^n(G') \otimes \mathcal{O}(-k)) \) is zero except for \( l = d, d + (d-1), \ldots, d + n(d-1) \).

2. For \( n \geq 0 \) and \( k > 0 \), define

\[
B_{n,k} := \sum_{m=0}^{n} \dim \text{Hom}^{d+m(d-1)}(\mathcal{O}, \Phi^n(G') \otimes \mathcal{O}(-k)) \cdot e^{-m(d-1)t}.
\]

In particular, \( B_{0,k} = \dim \text{Hom}^d(\mathcal{O}, \mathcal{O}(-k)) = a_k \). There is a recursive relation among \( B_{n,k} \)’s:

\[
B_{n,k} = B_{n-1,k+1} + a_k e^{-(d-1)t} B_{n-1,1}.
\]

**Proof.** We prove the first assertion by induction on \( n \). The statement is true when \( n = 0 \) by Kodaira vanishing theorem and Serre duality.

By the definition of \( \Phi \), there is an exact triangle:

\[
\Phi^{n-1}(G') \otimes \mathcal{O}(-1) \to \Phi^n(G') \to \text{Hom}^\bullet(\mathcal{O}, \Phi^{n-1}(G') \otimes \mathcal{O}(-1)) \otimes \mathcal{O}[1] \xrightarrow{+1}.
\]

By tensoring it with \( \mathcal{O}(-k) \) and applying \( \text{Hom}^\bullet(\mathcal{O}, -) \), we get a long exact sequence:

\[
\text{Hom}^\bullet(\mathcal{O}, \Phi^{n-1}(G') \otimes \mathcal{O}(-k-1)) \to \text{Hom}^\bullet(\mathcal{O}, \Phi^n(G') \otimes \mathcal{O}(-k))
\]

\[
\to \text{Hom}^\bullet(\mathcal{O}, \Phi^{n-1}(G') \otimes \mathcal{O}(-1)) \otimes \text{Hom}^\bullet(\mathcal{O}, \mathcal{O}(-k))[1] \xrightarrow{+1} \ldots
\]
Suppose the statement is true for \( n - 1 \). Then the first complex in the long exact sequence is non-zero only at degree \( d, d + (d - 1), \ldots, d + (n - 1)(d - 1) \), and the third complex is non-zero only at degree \( d + (d - 1), d + 2(d - 1), \ldots, d + n(d - 1) \). Since we assume the dimension \( d \geq 3 \), the long exact sequence splits into short exact sequences, and the proof follows.

The recursive relation in the second assertion also follows from the above long exact sequence.

Lemma 4.3.3. Define

\[
P_{s,k} := \sum_{i_1 + \cdots + i_q = s \atop i_1 \geq k} a_{i_1} a_{i_2} \cdots a_{i_q} e^{-q(d-1)t},
\]
where the summation runs over all ordered partitions of \( s \) with the first piece no less than \( k \). Then we have

\[
B_{n,k} = a_{n+k} + \sum_{s=1}^{n} a_s P_{n+k-s,k}.
\]

Proof. Induction on \( n \) and use the recursive relation (*).

Now we are ready to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. By Proposition 4.2.4, the categorical entropy can be written as

\[
h_t(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log \frac{1}{n} \sum_{l} \dim \text{Hom}^l(G, \Phi^n(G')) e^{-lt}
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{d+1} \sum_{l} \dim \text{Hom}^l(O, \Phi^n(G') \otimes O(-k)) e^{-lt}
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \log \sum_{k=1}^{d+1} B_{n,k}.
\]

By Lemma 4.3.3 and the fact that \( \{a_k\} \) is an increasing sequence, we have \( B_{n,1} \leq B_{n,k+1} \leq B_{n+k,1} \). Hence

\[
h_t(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log B_{n,1}.
\]
Define $C_n := B_{n,1} e^{-(d-1)t}$ and $a_k' := a_k e^{-(d-1)t}$. Then again by Lemma 4.3.3,

$$C_n = \left( a_{n+1} + \sum_{s=1}^{n} a_s P_{n+1-s,1} \right) e^{-(d-1)t}$$

$$= \sum_{i_1+\cdots+i_q=n+1} a'_{i_1} a'_{i_2} \cdots a'_{i_q}.$$

Thus

$$C_n = a_1' C_{n-1} + a_2' C_{n-2} \cdots + a_n' C_0 + a_{n+1}' .$$

Notice that $h_1(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log B_{n,1} = \lim_{n \to \infty} \frac{1}{n} \log C_n$ is a positive real number for any $t \in \mathbb{R}$, because there exists some $k > 0$ (depending on $d$ and $t$) such that $a_k' = a_k e^{-(d-1)t} > 1$, hence

$$\lim_{n \to \infty} \frac{1}{n} \log C_n \geq \lim_{n \to \infty} \frac{1}{n} \log(a_k')^{\frac{n+1}{k+1}} = \frac{1}{k} \log a_k' > 0 .$$

We claim that the categorical entropy $h_1(\Phi)$ is the unique $\lambda > 0$ satisfying

$$\sum_{k \geq 1} \frac{a_k'}{e^{k\lambda}} = 1 .$$

Or equivalently,

$$\sum_{k \geq 1} \frac{\chi(O(k))}{e^{k\lambda}} = e^{(d-1)t} .$$

This will conclude the proof of the theorem. We prove the claim in the following lemma, with a slight change in notations.

\[ \square \]

**Lemma 4.3.4.** Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers, and define $\{C_n\}_{n \geq 0}$ inductively by

$$C_n = a_1 C_{n-1} + a_2 C_{n-2} \cdots + a_n C_0 + a_{n+1} .$$

(\*)

Suppose that

$$\lim_{n \to \infty} \frac{1}{n} \log C_n = \lambda > 0 ,$$

then we have

$$\sum_{k \geq 1} \frac{a_k}{e^{k\lambda}} = 1 .$$
Proof. Assume that ∑_{k \geq 1} \frac{a_k}{e^{\lambda_1}} > 1. Then there exists some m > 0 so that ∑_{k=1}^{m} \frac{a_k}{e^{\lambda_1}} > 1. Moreover, there exists some λ_1 > λ such that ∑_{k=1}^{m} \frac{a_k}{e^{\lambda_1}} > ∑_{k=1}^{m} \frac{a_k}{e^{\lambda_1}} > 1.

Choose a constant D_1 > 0 such that C_k > D_1 e^{(k+1)\lambda_1} for all 0 ≤ k ≤ m. We can then prove by induction that C_n > D_1 e^{(n+1)\lambda_1} holds for all n ≥ 0. Indeed, by the induction hypothesis, (***) and ∑_{k=1}^{m} \frac{a_k}{e^{\lambda_1}} > 1, for n > m, we have

\[ C_n > a_1 C_{n-1} + \cdots + a_mC_{n-m} \]
\[ > D_1 (a_1 e^{n\lambda_1} + \cdots + a_m e^{(n-m+1)\lambda_1}) \]
\[ > D_1 e^{(n+1)\lambda_1}. \]

However, this contradicts with the assumption that lim_{n→∞} \frac{1}{n} \log C_n = λ since λ_1 > λ. Hence we have ∑_{k=1}^{m} \frac{a_k}{e^{\lambda_1}} ≤ 1.

On the other hand, assume that ∑_{k=1}^{m} \frac{a_k}{e^{\lambda_1}} > 1. Then there exists some 0 < λ_2 < λ such that ∑_{k=1}^{m} \frac{a_k}{e^{\lambda_2}} < 1.

Choose a constant D_2 > 1 such that C_0 < D_2 e^{λ_2}. We can prove by induction that C_n < D_2 e^{(n+1)λ_2} for all n ≥ 0. Indeed, by induction hypothesis, (***) and ∑_{k=1}^{m} \frac{a_k}{e^{λ_2}} < 1, we have

\[ C_n < D_2 (a_1 e^{nλ_2} + \cdots + a_n e^{λ_2} + a_{n+1}) < D_2 e^{(n+1)λ_2}. \]

This again contradicts with the assumption that lim_{n→∞} \frac{1}{n} \log C_n = λ since λ_2 < λ.

This concludes the proof of the lemma.

4.4 Counterexample of Kikuta-Takahashi

Using Theorem 4.3.1, we can now construct counterexamples of Conjecture 4.1.3.

**Proposition 4.4.1.** For any even integer d ≥ 4, let X be a Calabi–Yau hypersurface in CP^{d+1} of degree (d+2) and Φ = T_O o (− ⊗ O(−1)). Then

\[ h_0(Φ) > 0 = \log ρ(Φ_H^*). \]

In particular, Conjecture 4.1.3 fails in this case.
Proof. By Theorem 4.3.1, we only need to show that the spectral radius $\rho(\Phi_{H^*})$ equals to 1.

Consider another autoequivalence $\Phi' := T_{\mathcal{O}} \circ (- \otimes \mathcal{O}(1))$ on $D^b(X)$. By [7, Proposition 5.8], there is a commutative diagram

$$
\begin{array}{ccc}
D^b(X) & \xrightarrow{\Phi'} & D^b(X) \\
\downarrow{\Psi} & & \downarrow{\Psi} \\
\text{HMF}^{\text{gr}}(W) & \xrightarrow{\tau} & \text{HMF}^{\text{gr}}(W).
\end{array}
$$

Here $W$ is the defining polynomial of $X$, $\text{HMF}^{\text{gr}}(W)$ is the associated graded matrix factorization category, $\Psi$ is an equivalence introduced by Orlov [78], and $\tau$ is the grade shift functor on $\text{HMF}^{\text{gr}}(W)$ which satisfies $\tau^{d+2} = [2]$. Hence we have $(\Phi')^{d+2} = [2]$ and $(\Phi')^{d+2} = \text{id}_{H^*}$.

On the other hand, $(T_{\mathcal{O}})_{H^*}$ is an involution on $H^*(X; \mathbb{C})$ when $X$ is an even dimensional strict Calabi–Yau manifold ([51, Corollary 8.13]). Thus $(T_{\mathcal{O}})_{H^*} = (T_{\mathcal{O}})_{H^*}^{-1}$. Hence we also have $\Phi_{H^*}^{d+2} = \text{id}_{H^*}$, which implies that $\rho(\Phi_{H^*}) = 1$. \hfill $\Box$

Remark 4.4.2. The autoequivalence $\Phi'$ that we considered in the proof is the one that corresponds to the monodromy around the Gepner point ($\mathbb{Z}_{d+2}$-orbifold point) in the Kähler moduli of $X$.

Remark 4.4.3 (Ouchi). The functor $\Phi = T_{\mathcal{O}} \circ (- \otimes \mathcal{O}(-1))$ does not produce counterexamples of Conjecture 4.1.3 if $X$ is an odd dimensional Calabi–Yau manifold. When $X$ is of odd dimension, Lemma 4.3.2 implies

$$
h_0(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log \chi(G, \Phi^n(G')) \leq \log \rho([\Phi]).
$$

On the other hand, we have $h_0(\Phi) \geq \log \rho([\Phi])$ by Kikuta-Shiraishi-Takahashi [62, Theorem 2.13].

4.5 Categorical entropy of $\mathbb{P}$-twists

Despite the fact that we have disproved Conjecture 4.1.3, it still is an interesting problem to find a characterization of autoequivalences satisfying the conjecture. The conjecture holds true for autoequivalences on smooth projective curves [61], abelian surfaces [107], and smooth projective varieties with ample (anti)-canonical bundle [63]. Ouchi [80] also showed
that spherical twists satisfy the conjecture.

By slightly modifying Ouchi’s proof, we show that Conjecture 4.1.3 also holds for $\mathbb{P}$-twists. One can consider $\mathbb{P}$-twists as the categorical analogue of Dehn twists along Lagrangian complex projective space.

We start with an easy lemma on the complexity function.

**Lemma 4.5.1.**

1. $\delta_t(G, E \oplus F) \geq \delta_t(G, E)$.

2. For a distinguished triangle $A \to B \to C \to A[1]$, we have

$$\delta_t(G, B) \leq \delta_t(G, A) + \delta_t(G, C).$$

We recall the notion of $\mathbb{P}^d$-objects and $\mathbb{P}^d$-twists introduced by Huybrechts and Thomas [52]. They are the categorical analogue of the symplectomorphisms associated to a Lagrangian complex projective space [85].

**Definition 4.5.2** ([52]). An object $E \in \mathcal{D}^b(X)$ is called a $\mathbb{P}^d$-object if $E \otimes \omega_X \cong E$ and $\text{Hom}^*(E, E)$ is isomorphic as a graded ring to $H^*(\mathbb{P}^d, \mathbb{C})$.

For a $\mathbb{P}^d$-object $E$, a generator $h \in \text{Hom}^2(E, E)$ can be viewed as a morphism $h : E[-2] \to E$. The image of $h$ under the natural isomorphism $\text{Hom}^2(E, E) \cong \text{Hom}^2(E^\vee, E^\vee)$ will be denoted $h^\vee$.

**Definition 4.5.3** ([52]). The $\mathbb{P}^d$-twist $P_E$ with respect to a $\mathbb{P}^d$-object is an autoequivalence on $\mathcal{D}^b(X)$ given by the double cone construction

$$F \mapsto P_E(F) := \text{Cone}(\text{Cone}(\text{Hom}^{*-2}(E, F) \otimes E \to \text{Hom}^*(E, F) \otimes E) \to F),$$

where the first right arrow is given by $h^\vee \cdot \text{id} - \text{id} \cdot h$.

The following fact is crucial for computing the categorical entropy of $\mathbb{P}$-twists in the next section.

**Lemma 4.5.4** ([52]). Let $E \in \mathcal{D}^b(X)$ be a $\mathbb{P}^d$-object. Then

1. $P_E(E) \cong E[-2d]$. 

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2. \( P_\mathcal{E}(\mathcal{F}) \cong \mathcal{F} \) for any \( \mathcal{F} \in \mathcal{E}^\perp = \{ \mathcal{F} : \text{Hom}^*(\mathcal{E}, \mathcal{F}) = 0 \} \).

**Theorem 4.5.5.** Let \( X \) be a smooth projective variety of dimension \( 2d \) over \( \mathbb{C} \), and let \( P_\mathcal{E} \) be the \( \mathbb{P}^d \)-twist of a \( \mathbb{P}^d \)-object \( \mathcal{E} \in D^b(X) \). Then for \( t \leq 0 \), we have \( h_t(P_\mathcal{E}) = -2dt \). In particular, Conjecture 4.1.3 holds for \( \mathbb{P}^d \)-twists:

\[
h_0(P_\mathcal{E}) = 0 = \log \rho((P_\mathcal{E})_{\mathcal{H}^+}).
\]

Assume that \( \mathcal{E}^\perp \neq \phi \), then we have \( h_t(P_\mathcal{E}) = 0 \) for \( t > 0 \).

**Proof.** Fix a split generator \( G \in D^b(X) \), and let

\[
A := \text{Cone}(\text{Hom}^{*-2}(\mathcal{E}, G) \otimes \mathcal{E} \to \text{Hom}^*(\mathcal{E}, G) \otimes \mathcal{E}).
\]

By the definition of \( \mathbb{P}^d \)-twist, we have a distinguished triangle

\[
G \to P_\mathcal{E}(G) \to A[1] \to G[1].
\]

And by Lemma 4.5.4, we have \( P_\mathcal{E}(A) = A[-2d] \).

By applying \( P_\mathcal{E}^{n-1} \) to the distinguished triangle, we have

\[
\delta_t(G, P_\mathcal{E}^n(G)) \leq \delta_t(G, P_\mathcal{E}^{n-1}(G)) + \delta_t(G, A[-2d(n-1)+1])
= \delta_t(G, P_\mathcal{E}^{n-1}(G)) + \delta_t(G, A)e^{(-2d(n-1)+1)t}
\leq \cdots \text{(do this inductively)}
\leq 1 + \delta_t(G, A) \sum_{k=0}^{n-1} e^{(-2dk+1)t}
\]

For \( t \leq 0 \), we have \( \delta_t(G, P_\mathcal{E}^n(G)) \leq 1 + \delta_t(G, A) \cdot ne^{(-2d(n-1)+1)t} \). Hence

\[
h_t(P_\mathcal{E}) = \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, P_\mathcal{E}^n(G)) \leq -2dt.
\]

On the other hand, since \( G \oplus \mathcal{E} \) is also a split generator of \( D \), we can apply Lemma 4.5.1
and 4.5.4 to get

\[
\begin{align*}
h_t(P_E) &= \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, P^n_E(G \oplus \mathcal{E})) \\
&\geq \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, P^n_E(\mathcal{E})) \\
&= \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, \mathcal{E}[-2nd]) \\
&= \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, \mathcal{E}) e^{-2ndt} \\
&= -2dt.
\end{align*}
\]

Hence we have \( h_t(P_E) = -2dt \) for \( t \leq 0 \).

Note that the induced Fourier–Mukai type action \((P_E)_{H^*}\) is identity on the cohomology [52]. Thus Conjecture 4.1.3 holds for the \( \mathbb{P}^d \)-twist \( P_E \):

\[
h_0(P_E) = 0 = \log \rho((P_E)_{H^*}).
\]

For \( t > 0 \), we have \( \delta_t(G, P^n_E(G)) \leq 1 + \delta_t(G, A)(\epsilon^i + n - 1) \). Hence \( h_t(P_E) \leq 0 \). Assume that \( \mathcal{E}^\perp \neq \phi \) and take \( B \in \mathcal{E}^\perp \), then we can apply the same trick on the split generator \( G \oplus B \):

\[
\begin{align*}
\begin{align*}
h_t(P_E) &= \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, P^n_E(G \oplus B)) \\
&\geq \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, P^n_E(B)) \\
&= \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, B) \\
&= 0.
\end{align*}
\end{align*}
\]

This concludes the proof of the theorem.

\[ \square \]

**Remark 4.5.6.** When \( \dim(X) = 2 \), a \( \mathbb{P}^1 \)-object \( \mathcal{E} \in D^b(X) \) is also a spherical object \((S^2 \cong \mathbb{P}^1)\). For \( t \leq 0 \), we have \( h_t(P_E) = -2t \). On the other hand, \( h_t(T_E) = -t \) for the spherical twist \( T_E \) [80]. This matches with the fact that \( T_E^2 \cong P_E \) by Huybrechts and Thomas [52].
References


[83] W.-D. Ruan. Lagrangian torus fibrations and mirror symmetry of Calabi–Yau mani-
folds. In Symplectic geometry and mirror symmetry (Seoul, 2000), pages 385–427. World

1985.

2000.


[89] P. Seidel and R. Thomas. Braid group actions on derived categories of coherent

[90] N. Sheridan. Homological mirror symmetry for Calabi–Yau hypersurfaces in projective


2006.


[96] G. Tian. Smoothness of the universal deformation space of compact Calabi–Yau


