



# The Ultrapower Axiom

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# The Ultrapower Axiom

A dissertation presented

by

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to

The Harvard Department of Mathematics

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## The Ultrapower Axiom

### **Abstract**

The inner model problem for supercompact cardinals, one of the central open problems in modern set theory, asks whether there is a canonical model of set theory with a supercompact cardinal. The problem is closely related to the more precise question of the equiconsistency of strongly compact cardinals and supercompact cardinals. This dissertation approaches these two problems abstractly by introducing a principle called the Ultrapower Axiom which is expected to hold in all known canonical models of set theory. By investigating the consequences of the Ultrapower Axiom under the hypothesis that there is a supercompact cardinal, we provide evidence that the inner model problem can be solved. Moreover, we establish that under the Ultrapower Axiom, strong compactness and supercompactness are essentially equivalent.

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# Chapter 1

## Introduction

The goal of inner model theory is to construct and analyze canonical models of set theory. The simplest example of such a model is Gödel's constructible universe  $L$ , the smallest model of set theory that contains every ordinal number. One sense in which  $L$  is canonical is that seemingly every question about the internal structure of  $L$  can be answered. For example, Gödel proved that  $L$  satisfies the Generalized Continuum Hypothesis. This stands in stark contrast with the universe of sets  $V$ , many of whose most basic properties (for example, whether the Continuum Hypothesis holds) cannot be determined in any commonly accepted axiomatic system.

To what extent does  $L$  provide a good approximation to the universe of sets? On the one hand, the principle that every set belongs to  $L$  (or in other words,  $V = L$ ) cannot be refuted using the ZFC axioms, since  $L$  itself is a model of the theory  $\text{ZFC} + V = L$ . If  $V = L$ , then  $L$  approximates  $V$  *very* well. On the other hand, the model  $L$  fails to satisfy relatively weak large cardinal axioms. If one takes the stance that these large cardinal axioms are true in the universe of sets, one must conclude that  $V \neq L$ . Moreover, it follows from large cardinal axioms that  $L$  constitutes only a tiny fragment of the universe of sets. For example, assuming large cardinal axioms, the set of real numbers that lie in  $L$  is countable.

Are there canonical models generalizing  $L$  that yield better approximations to  $V$ ? It



turns out that there is a hierarchy of canonical models beyond  $L$ , satisfying stronger and stronger large cardinal axioms. The program of building such models has met striking success, reaching large cardinal axioms as strong as a Woodin limit of Woodin cardinals. Based on the pattern that has emerged so far, it is plausible every large cardinal axiom has a canonical model.

At present, however, a vast expanse of large cardinal axioms are not yet known to admit canonical models. A key target problem for inner model theory is the construction of a canonical model with a supercompact cardinal. Work of Woodin suggests that the solution to this problem alone will yield an *ultimate canonical model* that inherits essentially all large cardinals present in the universe. There is therefore hope that the goal of constructing inner models for all large cardinal axioms could be achieved in a single stroke. If this is possible, the resulting model would be of enormous set theoretic interest, since it would closely approximate the universe of sets and yet admit an analysis that is as detailed as that of Gödel's  $L$ .

This dissertation investigates whether there can be a canonical model with a supercompact cardinal. To do this, we develop an abstract approach to inner model theory. This is accomplished by introducing a combinatorial principle called the Ultrapower Axiom, which is expected to hold in all canonical models. If one could show that the Ultrapower Axiom is inconsistent with a supercompact cardinal, one would arguably have to conclude that there can be no canonical model with a supercompact cardinal.

Supplemented with large cardinal axioms, the Ultrapower Axiom turns out to have surprisingly strong and coherent consequences for the structure of the upper reaches of the universe of sets, particularly above the first supercompact cardinal. These consequences are entirely consistent with what one would expect to hold in a canonical model, yet are proved by methods that are completely different from the usual techniques of inner model theory. The coherence of this theory provides compelling evidence that the Ultrapower Axiom is consistent with a supercompact cardinal. If this is the case, it seems that the only possible

explanation is that the canonical model for a supercompact cardinal does indeed exist. Optimistically, studying the consequences of the Ultrapower Axiom will shed light on how this model should be constructed.

## Outline

We now describe the main results of this dissertation.

**Chapter 2.** In this introductory chapter, we introduce UA in the context of the problem of the linearity of the Mitchell order on normal ultrafilters. We show first that UA holds in all canonical inner models, a result that is philosophically central to this dissertation. More precisely, we prove that UA is a consequence of Woodin's Weak Comparison principle:

**Theorem 2.3.10.** *Assume that  $V = \text{HOD}$  and there is a  $\Sigma_2$ -correct worldly cardinal. If Weak Comparison holds, then the Ultrapower Axiom holds.*

We then show that UA implies the linearity of the Mitchell order:

**Theorem 2.3.11 (UA).** *The Mitchell order is linear.*

Two applications of this result to longstanding problems of Solovay-Reinhardt-Kanamori [1] are explained in the introduction to Chapter 2.

**Chapter 3.** This chapter introduces the Ketonen order, a generalization of the Mitchell order to all countably complete ultrafilters on ordinals. The restriction of this order to weakly normal ultrafilters was originally introduced by Ketonen. The first proof of the wellfoundedness of the generalization of this order to countably complete ultrafilters is due to the author:

**Theorem 3.3.8.** *The Ketonen order is wellfounded.*

The main theorem of this chapter explains the fundamental role of the Ketonen order in applications of the Ultrapower axiom:

**Theorem 3.6.1.** *The Ultrapower Axiom is equivalent to the linearity of the Ketonen order.*

In addition, we analyze the relationship between the Ketonen order and various well-known orders like the Rudin-Keisler order and the Mitchell order.

**Chapter 4.** The topic of this chapter is the generalized Mitchell order, which is defined in exactly the same way as the usual Mitchell order on normal ultrafilters but removing the requirement that the ultrafilters involved be normal. This order is not linear (assuming there is a measurable cardinal), and in fact it is quite pathological when considered on ultrafilters in general. The two main results of this chapter generalize the linearity of the Mitchell order to nice classes of ultrafilters:

**Theorem 4.3.29 (UA).** *The generalized Mitchell order is linear on Dodd sound ultrafilters.*

Dodd soundness is a generalization of normality that was first isolated in the context of inner model theory by Steel [2]. A uniform ultrafilter  $U$  on a cardinal  $\lambda$  is Dodd sound if the map  $h : P(\lambda) \rightarrow M_U$  defined by  $h(X) = j_U(X) \cap [\text{id}]_U$  belongs to  $M_U$ . The concept is discussed at great length in Section 4.3.

A better-known generalization of normality is the concept of a normal fine ultrafilter (Definition 4.4.7), introduced by Solovay, and underpinning the theory of supercompact cardinals. The second result of this chapter generalizes the linearity of the Mitchell order to this class of ultrafilters:

**Theorem 4.4.2 (UA).** *Suppose  $\lambda$  is a cardinal such that  $2^{<\lambda} = \lambda$ . Then the generalized Mitchell order is linear on normal fine ultrafilters on  $P_{\text{bd}}(\lambda)$ .*

Here  $P_{\text{bd}}(\lambda)$  denotes the set of bounded subsets of  $\lambda$ .

**Chapter 5.** We turn to another fundamental order on ultrafilters, the Rudin-Frolík order. The structure of the Rudin-Frolík order on countably complete ultrafilters is intimately related to the Ultrapower Axiom. For example, we point out the following simple connection:

**Corollary 5.2.9.** *The Ultrapower Axiom holds if and only if the Rudin-Frolík order is directed on countably complete ultrafilters.*

On the other hand, it is well-known that the Rudin-Frolík order is not directed on ultrafilters on  $\omega$ .

The chapter is devoted to deriving deeper structural features of the Rudin-Frolík order from UA. The most interesting one is that it is locally finite:

**Theorem 5.4.23 (UA).** *A countably complete ultrafilter has at most finitely many predecessors in the Rudin-Frolík order up to isomorphism.*

Given the finiteness of the Rudin-Frolík order, it turns out to be possible to represent every ultrafilter as a finite iterated ultrapower consisting of *irreducible ultrafilters*, ultrafilters whose ultrapowers cannot be factored as an iterated ultrapower (Theorem 5.3.16). We apply this to analyze ultrafilters on the least measurable cardinal under UA:

**Theorem 5.3.21 (UA).** *Every countably complete ultrafilter on the least measurable cardinal  $\kappa$  is isomorphic to  $U^n$  where  $U$  is the unique normal ultrafilter on  $\kappa$  and  $n$  is a natural number.*

This generalizes a classic theorem of Kunen [3].

**Chapter 6.** This chapter exposit two inner model principles that follow abstractly from UA in the presence of a supercompact cardinal:

**Theorem 6.2.8 (UA).** *Assume there is a supercompact cardinal. Then  $V$  is a generic extension of HOD.*

Thus UA almost implies  $V = \text{HOD}$ . This is best possible in the sense that it is consistent that UA holds and  $V$  is a *nontrivial* generic extension of HOD.

The main result of the chapter is that UA implies the Generalized Continuum Hypothesis:

**Theorem 6.3.12 (UA).** *Suppose  $\kappa$  is supercompact. Then for all  $\lambda \geq \kappa$ ,  $2^\lambda = \lambda^+$ .*

Thus UA almost implies the GCH. This is best possible in the sense that it is consistent that UA holds but CH fails.

**Chapter 7.** This chapter initiates an analysis of strongly compact and supercompact cardinals under UA. In this chapter, we investigate the structure of the least strongly compact cardinal, introducing the “least ultrafilters”  $\mathcal{K}_\lambda$ , and proving that they witness its supercompactness:

**Theorem 7.4.23 (UA).** *The least strongly compact cardinal is supercompact.*

**Chapter 8.** The main result of this chapter is the *Irreducibility Theorem* (Theorem 8.2.18, Corollary 8.2.20) that relates supercompactness and irreducibility (that is, Rudin-Frolík minimality) under UA. The original impetus for proving this theorem was to analyze all larger strongly compact cardinals:

**Theorem 8.3.9 (UA).** *A cardinal  $\kappa$  is strongly compact if and only if it is supercompact or a measurable limit of supercompact cardinals.*

We also analyze various other large cardinals using UA. For example, we consider huge cardinals (Theorem 8.4.5) and rank-into-rank cardinals (Theorem 8.4.12).

# Chapter 2

## The Linearity of the Mitchell Order

### 2.1 Introduction

#### Normal ultrafilters and the Mitchell Order

Normal ultrafilters are among the simplest objects that arise from modern large cardinal axioms, yet despite their apparent simplicity, and despite the past six decades of remarkable progress in the theory of large cardinals, the class of normal ultrafilters remains mysterious, its underlying structure inextricably bound up with some of the deepest and most difficult problems in set theory. The following questions, posed by Solovay-Reinhardt-Kanamori [1] in the 1970s, are among the most prominent open questions in this subject:

**Question 2.1.1.** Assume  $\kappa$  is  $2^\kappa$ -supercompact. Must there be more than one normal ultrafilter on  $\kappa$  concentrating on nonmeasurable cardinals?

**Question 2.1.2.** Assume  $\kappa$  is strongly compact. Must  $\kappa$  carry more than one normal ultrafilter?

These questions turn out to be merely the most concrete instances of a sequence of more and more general structural questions in the theory of large cardinals. Let us start down this path by stating a conjecture that would answer both questions at once:

**Conjecture 2.1.3.** *It is consistent with an extendible cardinal that every measurable cardinal carries a unique normal ultrafilter concentrating on nonmeasurable cardinals.*

This would obviously answer Question 2.1.1 negatively, but what bearing does it have on Question 2.1.2? Assume  $\kappa$  is extendible and every measurable cardinal carries a unique normal ultrafilter concentrating on nonmeasurable cardinals. Consider the least strongly compact cardinal  $\kappa$  that is a limit of strongly compact cardinals. By a theorem of Menas [4] (proved here as Theorem 8.1.1), the set of measurable cardinals below  $\kappa$  is nonstationary. It follows that every normal ultrafilter on  $\kappa$  concentrates on nonmeasurable cardinals. Since we assumed there is only one such ultrafilter,  $\kappa$  is a strongly compact cardinal that carries a unique ultrafilter. Conjecture 2.1.3 thus supplies a negative answer to Question 2.1.2 as well.

Why would someone make Conjecture 2.1.3? To answer this question, we must consider the broader question of the structure of the Mitchell order under large cardinal hypotheses. Recall that if  $U$  and  $W$  are normal ultrafilters, the *Mitchell order* is defined by setting  $U \triangleleft W$  if  $U$  belongs to the ultrapower of the universe by  $W$ . It is not hard to see that a normal ultrafilter  $U$  on a cardinal  $\kappa$  concentrates on nonmeasurable cardinals if and only if  $U$  is a minimal element in the Mitchell order on normal ultrafilters on  $\kappa$ . The following conjecture therefore generalizes Conjecture 2.1.3:

**Conjecture 2.1.4.** *It is consistent with an extendible cardinal that the Mitchell order is linear.*

How could one possibly prove Conjecture 2.1.4? The most general technique for proving consistency results, namely forcing, seems to be powerless in this instance. To force the linearity of the Mitchell order, one would in particular have to force that the least measurable cardinal carries a unique normal ultrafilter, but even this much more basic problem remains open.

Kunen [3] famously did prove that it is consistent for the least measurable cardinal to carry a unique normal ultrafilter, not by forcing but instead by inner model theory. In fact, he showed that if  $U$  is a normal ultrafilter on a cardinal  $\kappa$ , then the inner model  $L[U]$  satisfies that  $U \cap L[U]$  is the unique normal ultrafilter on  $\kappa$ . Mitchell [5] then isolated the Mitchell order in an attempt to generalize Kunen's results. He proceeded roughly as follows:

- Consider the model  $M = L[\langle U_\alpha : \alpha < \gamma \rangle]$  built from a coherent sequence  $\langle U_\alpha : \alpha < \gamma \rangle$  of normal ultrafilters.<sup>1</sup>
- Show that in  $M$ ,  $\langle U_\alpha \cap M : \alpha < \gamma \rangle$  is increasing in the Mitchell order and contains all normal ultrafilters.

In the decades since Mitchell's result, inner model theory has ascended much farther into the large cardinal hierarchy. Combining the results of many researchers (especially Neeman [6] and Schlutzenberg [7]), the following is the best partial result towards Conjecture 2.1.4 to date:

**Theorem.** *If it is consistent that there is a Woodin limit of Woodin cardinals, then the linearity of the Mitchell order is consistent with a Woodin limit of Woodin cardinals.*

The linearity proof, due to Schlutzenberg, is much harder, but the argument still roughly follows Mitchell's:

- Consider the model  $M = L[\langle E_\alpha : \alpha < \gamma \rangle]$  built from a coherent extender sequence  $\langle E_\alpha : \alpha < \gamma \rangle$ .
  - By the definition of a coherent extender sequence,  $\langle E_\alpha : \alpha < \gamma \rangle$  is linearly ordered by the Mitchell order in  $M$ .
- Show that in  $M$ , every normal ultrafilter appears on the sequence  $\langle E_\alpha : \alpha < \gamma \rangle$ .

---

<sup>1</sup>Coherence is a key technical definition that includes the assumption that  $\langle U_\alpha : \alpha < \gamma \rangle$  is increasing in the Mitchell order.



By now, it may appear that Conjecture 2.1.4 itself is subsumed by the far more important (but far less precise) Inner Model Problem:

**Conjecture 2.1.5.** *There is a canonical inner model with an extendible cardinal.*

The relationship between Conjecture 2.1.4 and Conjecture 2.1.5 is actually not as straightforward as one might expect, because if inner model theory can be extended to the level of extendible cardinals, the models must be significantly different from the current models. For example, the Woodin and Neeman-Steel models with long extenders are canonical inner models designed to accommodate large cardinals at the finite levels of supercompactness. It is not known whether the constructions actually succeed, but the following conjecture is plausible:

**Conjecture 2.1.6.** *If for all  $n < \omega$ , there is a cardinal  $\kappa$  that is  $\kappa^{+n}$ -supercompact, then for all  $n < \omega$ , there is an iterable Woodin model with a cardinal  $\kappa$  that is  $\kappa^{+n}$ -supercompact.*

Given the pattern described above, one might expect to generalize Mitchell and Schlutzenberg's results to the Woodin models, and therefore obtain for any  $n < \omega$ , the consistency of the linearity of the Mitchell order with a cardinal  $\kappa$  that is  $\kappa^{+n}$ -supercompact. But there is a catch: the proofs of these theorems cannot generalize verbatim to this level.

**Proposition 2.1.7.** *If  $L[\mathbb{E}]$  is an iterable Woodin model satisfying that  $\kappa$  is  $\kappa^{++}$ -supercompact, then in  $L[\mathbb{E}]$ , there is a normal ultrafilter on  $\kappa$  that does not lie on the coherent sequence  $\mathbb{E}$ .*

Therefore Mitchell's proof of the consistency of the linearity of the Mitchell order cannot extend to the level of a cardinal  $\kappa$  that is  $\kappa^{++}$ -supercompact. This might have been taken as a reason for skepticism about Conjecture 2.1.4.

## The Ultrapower Axiom

The problem of generalizing the linearity of the Mitchell order to canonical inner models at the finite levels of supercompactness was the original inspiration for all the work in this

dissertation. Our initial discovery was a new argument that shows that *any* canonical inner model built by the methodology of modern inner model theory must satisfy that the Mitchell order is linear. The argument is extremely simple and relies on a single fundamental property of the known canonical inner models: the *Comparison Lemma*. The Comparison Lemma roughly states that any two canonical inner models at the same large cardinal level can be embedded into a common model. The inner model constructions are perhaps best viewed as an attempt to build models satisfying the Comparison Lemma and accommodating large cardinals.

Upon further reflection, we realized that this argument relied solely on an abstract combinatorial principle that distills abstractly a consequence of the Comparison Lemma. This principle is called the *Ultrapower Axiom* and abbreviated by UA. (The definition appears in Section 2.3.) The Comparison Lemma implies that UA holds in all known canonical inner models. Since the Comparison Lemma is fundamental to the methodology of inner model theory, UA is expected to hold in any canonical inner model that will ever be built.

Our theorem on the linearity of the Mitchell order now reads:

**Theorem 2.3.11.** *Assume the Ultrapower Axiom. Then the Mitchell order is linear.*

Granting our contention that UA holds in every canonical inner model, we have reduced Conjecture 2.1.4 to the Inner Model Problem (for example, Conjecture 2.1.5). Moreover, we can state a perfectly precise test question that seems to capture the essence of the Inner Model Problem:

**Conjecture 2.1.8.** *The Ultrapower Axiom is consistent with an extendible cardinal.*

It is our expectation that neither this conjecture nor even Conjecture 2.1.4 will be proved without first solving the Inner Model Problem. But what makes Conjecture 2.1.8 much more interesting than Conjecture 2.1.4 is that UA turns out to have a host of structural consequences in the theory of large cardinals. By studying UA, one can hope to glean insight into the inner models that have not yet been built, or perhaps to refute their existence by

refuting UA from a large cardinal hypothesis. The latter has not happened. Instead a remarkable theory of large cardinals under UA has emerged which in our opinion provides evidence for Conjecture 2.1.8 and hence for the existence of inner models for very large cardinals.

## Outline of Chapter 2

We now briefly outline the contents of the rest of this chapter.

**Section 2.2.** This section contains preliminary definitions most of which are standard or self-explanatory. The topics we cover are ultrapowers, close embeddings, uniform ultrafilters, and normal ultrafilters.

**Section 2.3.** This section contains proofs of the linearity of the Mitchell order and motivation for the Ultrapower Axiom. We begin in Section 2.3 by introducing and motivating Woodin's Weak Comparison axiom. Then in Section 2.3, we give our original argument for the linearity of the Mitchell order under Weak Comparison (Theorem 2.3.4). In Section 2.3, we abstract from this argument the Ultrapower Axiom, the central principle in this dissertation and prove UA from Weak Comparison (Theorem 2.3.10). This proof is incomplete in the sense that several technical lemmas are deferred until the end of the chapter. In Section 2.3, we give the proof of the linearity of the Mitchell order from UA (Theorem 2.3.11), which is actually a simplification of the proof in Section 2.3. We also prove a sort of converse: UA restricted to normal ultrafilters is equivalent to the linearity of the Mitchell order. Finally, in Section 2.3, we prove the technical lemmas we had set aside in Section 2.3.

## 2.2 Preliminary definitions

### Ultrapowers

We briefly put down our conventions on ultrapowers. If  $U$  is an ultrafilter, we denote by

$$j_U : V \rightarrow M_U$$

the ultrapower of the universe by  $U$ . If  $M_U$  is wellfounded, or equivalently if  $U$  is countably complete, our convention is that  $M_U$  denotes the unique transitive class isomorphic to the ultrapower of the universe by  $U$ . The ultrafilters we consider will almost always be countably complete.

Many arguments in this dissertation proceed by applying an ultrafilter to a model to which it does not belong. This involves a taking *relativized ultrapower*. If  $N$  is a transitive model of ZFC and  $X \in N$ , an  $N$ -ultrafilter on  $X$  is a set  $U \subseteq P(X) \cap N$  such that  $(N, U) \models U$  is an ultrafilter. Equivalently,  $U$  is an ultrafilter on the Boolean algebra  $P(X) \cap N$ . One can form the ultrapower of  $N$  by  $U$ , denoted

$$j_U^N : N \rightarrow M_U^N$$

using a modified ultrapower construction that uses only functions that belong to  $N$ . For any function  $f \in N$  that is defined  $U$ -almost everywhere, we denote by  $[f]_U^N$  the point in  $M_U^N$  represented by  $f$ . Since the point  $[\text{id}]_U^N$  comes up so often, we introduce special notation for it:

**Definition 2.2.1.** If  $U$  is an  $N$ -ultrafilter,  $a_U^N$  denotes the point  $[\text{id}]_U^N$ .

We will drop the superscript  $N$  when it is convenient and unambiguous.

Derived ultrafilters allow us to extract combinatorial content from elementary embeddings:

**Definition 2.2.2.** Suppose  $N$  and  $M$  are transitive models of ZFC and  $j : N \rightarrow M$  is an elementary embedding. Suppose  $X \in N$  and  $a \in j(X)$ . The  $N$ -ultrafilter on  $X$  derived from  $j$  using  $a$  is the  $N$ -ultrafilter  $\{A \in P(X) \cap N : a \in j(A)\}$ .

What is the relationship between an elementary embedding and the ultrapowers by its derived ultrafilters? The answer is contained in the following lemma:

**Lemma 2.2.3.** *Suppose  $N$  and  $M$  are transitive models of ZFC and  $j : N \rightarrow M$  is an elementary embedding. Suppose  $X \in N$  and  $a \in j(X)$ . Let  $U$  be the  $N$ -ultrafilter on  $X$  derived from  $j$  using  $a$ . Then there is a unique embedding  $k : M_U^N \rightarrow M$  such that  $k \circ j_U^N = j$  and  $k(a_U) = a$ .  $\square$*

We refer to the embedding  $k$  as the *factor embedding* associated to the derived ultrafilter  $U$ .

Often we will wish to discuss an ultrapower embedding without the need to choose any particular ultrafilter giving rise to it, so we introduce the following terminology:

**Definition 2.2.4.** If  $N$  and  $M$  are transitive models of ZFC, an elementary embedding  $j : N \rightarrow M$  is an *ultrapower embedding* if there is an  $N$ -ultrafilter  $U$  such that  $M = M_U^N$  and  $j = j_U^N$ .

**Definition 2.2.5.** If  $N$  is a transitive model of ZFC, a *countably complete ultrafilter* of  $N$  is a point  $U \in N$  such that  $N$  satisfies that  $U$  is a countably complete ultrafilter.

An  $N$ -ultrafilter  $U$  is a countably complete ultrafilter of  $N$  if and only if its corresponding ultrapower  $j : N \rightarrow M$  is wellfounded and definable over  $N$ .

**Definition 2.2.6.** An ultrapower embedding  $j : N \rightarrow M$  is an *internal ultrapower embedding* of  $N$  if there is a countably complete ultrafilter  $U$  of  $N$  such that  $j = j_U^N$ .

An important point is that for our purposes, when we speak of ultrapower embeddings, we only mean ultrapower embeddings between wellfounded models. For example, if  $U$  is

an ultrafilter on  $\omega$ , then the embedding  $j_U : V \rightarrow M_U$  does not count as an ultrapower embedding.

There is a characterization of ultrapower embeddings that does not refer to ultrafilters at all.

**Definition 2.2.7.** Suppose  $N$  and  $M$  are transitive set models of ZFC. An elementary embedding  $j : N \rightarrow M$  is *cofinal* if for all  $a \in M$ , there is some  $X \in N$  such that  $a \in j(X)$ .

Equivalently,  $j$  is cofinal if  $j[\text{Ord} \cap N]$  is cofinal in  $\text{Ord} \cap M$ .

**Definition 2.2.8.** Suppose  $N$  and  $M$  are transitive set models of ZFC. An elementary embedding  $j : N \rightarrow M$  is a *weak ultrapower embedding* if there is some  $a \in M$  such that every element of  $M$  is definable in  $M$  from parameters in  $j[N] \cup \{a\}$ .

For metamathematical reasons (namely, the undefinability of definability), we cannot define the concept of a weak ultrapower embeddings when  $M$  is a proper class.

**Lemma 2.2.9.** *Suppose  $N$  and  $M$  are transitive set models of ZFC. An elementary embedding  $j : N \rightarrow M$  is an ultrapower embedding if and only if  $j$  is a cofinal weak ultrapower embedding.* □

The following notation will be extremely important in our analysis of elementary embeddings:

**Definition 2.2.10.** Suppose  $N$  and  $M$  are transitive models of ZFC. Suppose  $j : N \rightarrow M$  is a cofinal elementary embedding and  $S$  is a subclass of  $M$ . Then the *hull of  $S$  in  $M$  over  $j[N]$*  is the class  $H^M(j[N] \cup S) = \{j(f)(x_1, \dots, x_n) : x_1, \dots, x_n \in S\}$ .

The fundamental theorem about these hulls, which we use repeatedly and implicitly, is the following:

**Lemma 2.2.11.** *Suppose  $N$  and  $M$  are transitive models of ZFC. Suppose  $j : N \rightarrow M$  is a cofinal elementary embedding and  $S$  is a subclass of  $M$ . Then the hull of  $S$  in  $M$  over  $j[N]$  is the minimum elementary substructure of  $M$  containing  $j[N] \cup S$ .* □

For more on hulls, see [8] Chapter 1 Lemma 1.1.18. (Larson’s lemma should use a stronger theory than ZFC – Replacement;  $\Sigma_2$ -Replacement suffices.) Using hulls, we can give a metamathematically unproblematic model theoretic characterization of ultrapower embeddings between transitive models that are not assumed to be sets:

**Lemma 2.2.12.** *Suppose  $N$  and  $M$  are transitive models of ZFC. A cofinal elementary embedding  $j : N \rightarrow M$  is an ultrapower embedding if  $M = H^M(j[N] \cup \{a\})$  for some  $a \in M$ . □*

## Close embeddings

The property of being an internal ultrapower embedding is a very stringent requirement. *Closeness* is a natural weakening that originated in inner model theory:

**Definition 2.2.13.** Suppose  $N$  and  $M$  are transitive models of ZFC and  $j : N \rightarrow M$  is an elementary embedding. Then  $j$  is *close* to  $N$  if  $j$  is cofinal and for all  $X \in N$  and  $a \in j(X)$ , the  $N$ -ultrafilter on  $X$  derived from  $j$  using  $a$  belongs to  $N$ .

Close embeddings have a very natural model theoretic definition that makes no reference to ultrafilters:

**Lemma 2.2.14.** *Suppose  $N$  and  $P$  are transitive models of ZFC and  $j : N \rightarrow P$  is an elementary embedding. Then the following are equivalent:*

- (1)  $j$  is close to  $N$ .
- (2) For any  $a \in P$ ,  $j$  factors as  $N \xrightarrow{i} M \xrightarrow{k} P$  where  $i : N \rightarrow M$  is an internal ultrapower embedding,  $k : M \rightarrow P$  is an elementary embedding, and  $a \in k[M]$ .
- (3) For any set  $A \in P$ , the inverse image  $j^{-1}[A]$  belongs to  $N$ .

*Proof.* (1) implies (2): Immediate from the factor embedding construction Lemma 2.2.3.

(2) *implies* (3): Fix  $A \in P$ , and we will show  $j^{-1}[A] \in N$ . Factor  $j$  as  $N \xrightarrow{i} M \xrightarrow{k} P$  where  $i : N \rightarrow M$  is an internal ultrapower embedding,  $k : M \rightarrow P$  is an elementary embedding, and  $A \in k[M]$ . Fix  $B \in M$  such that  $k(B) = A$ . Now  $i^{-1}[B] \in N$  since  $i$  is an internal ultrapower embedding of  $N$ . We finish by showing  $i^{-1}[B] = j^{-1}[A]$ . First, by the elementarity of  $k$ ,  $B = k^{-1}[A]$ . Therefore  $i^{-1}[B] = i^{-1}[k^{-1}[A]] = (k \circ i)^{-1}[A] = j^{-1}[A]$ .

(3) *implies* (1): We first show that  $j$  is cofinal. Assume not, towards a contradiction. Then there is an ordinal  $\alpha \in P$  that lies above all ordinals in the range of  $j$ . Therefore  $j^{-1}[\alpha] = \text{Ord} \cap N \notin N$ , which is a contradiction.

Finally, fix  $X \in N$  and  $a \in P$  with  $a \in j(X)$ . We must show that the  $N$ -ultrafilter on  $X$  derived from  $j$  using  $a$  belongs to  $N$ . Let  $p_a^{j(X)}$  denote the principal  $N$ -ultrafilter on  $j(X)$  concentrated at  $a$ . Then the  $N$ -ultrafilter on  $X$  derived from  $j$  using  $a$  is precisely  $j^{-1}[p_a^{j(X)}]$ , which belongs to  $N$  by assumption.  $\square$

Most texts on inner model theory define *close extenders* rather than close embeddings, so we briefly describe the relationship between these two concepts. If  $N$  is a transitive model of ZFC and  $E$  is an  $N$ -extender of length  $\lambda$ , then  $E$  is *close to*  $N$  if  $E_a \in M$  for all  $a \in [\lambda]^{<\omega}$ .

**Lemma 2.2.15.** *An  $N$ -extender  $E$  is close to  $N$  if and only if the elementary embedding  $j_E^N$  is close to  $N$ .*  $\square$

The fact that the comparison process gives rise to close embeddings is less well-known than the fact that all extenders applied in a normal iteration tree are close, which for example is proved in [9]. Given that each of the individual extenders that are applied are close, the following fact shows that all the embeddings between models of ZFC in a normal iteration tree are close:

**Lemma 2.2.16.** (1) *If  $N \xrightarrow{i} M \xrightarrow{k} P$  are close embeddings, then the composition  $k \circ i$  is close to  $N$ .*

(2) *Suppose  $\mathcal{D} = \{M_p, j_{pq} : p \leq q \in I\}$  is a directed system of transitive models of ZFC and elementary embeddings. Suppose  $p \in I$  is an index such that for all  $q \geq p$  in  $I$ ,*



$j_{pq} : M_p \rightarrow M_q$  is close to  $M_p$ . Let  $N$  be the direct limit of  $\mathcal{D}$ , and assume  $N$  is transitive. Then the direct limit embedding  $j_{p\infty} : M_p \rightarrow N$  is close to  $M_p$ .

*Proof.* (1) is immediate from Lemma 2.2.14 (3). (2) is clear from Lemma 2.2.14 (2).  $\square$

An often useful trivial fact about close embeddings is that their right-factors are close:

**Lemma 2.2.17.** *If  $j : N \rightarrow P$  is a close embedding and  $j = k \circ i$  where  $N \xrightarrow{i} M \xrightarrow{k} P$  are elementary embeddings. Then  $i$  is close to  $N$ .*  $\square$

Another fact which is almost tautological is that an ultrapower embedding is internal if and only if it is close:

**Lemma 2.2.18.** *If  $j : N \rightarrow M$  is an ultrapower embedding, then  $j$  is an internal ultrapower embedding of  $N$  if and only if  $j$  is close to  $N$ .*  $\square$

## Uniform ultrafilters

One of the most basic notions from ultrafilter theory is that of a uniform ultrafilter:

**Definition 2.2.19.** An ultrafilter  $U$  is *uniform* if every set in  $U$  has the same cardinality. If  $U$  is an ultrafilter, the *size* of  $U$ , denoted  $\lambda_U$ , is the least cardinality of a set in  $U$ .

The cardinals  $\lambda_U$  for  $U$  a countably complete ultrafilter will become very important in Chapter 7.

Equivalently,  $U$  is a uniform ultrafilter on  $X$  if it extends the Fréchet filter on  $X$ , the collection of  $A \subseteq X$  such that  $|X \setminus A| < |X|$ . It will be important to distinguish between the notion of a uniform ultrafilter and the similar but distinct notion of a *tail uniform* ultrafilter on an ordinal, defined in Section 3.2. These notions are often confused in the literature.

**Definition 2.2.20.** Suppose  $U$  and  $W$  are ultrafilters. Then  $U$  and  $W$  are *isomorphic*, denoted  $U \cong W$ , if there exist  $X \in U$ ,  $Y \in W$ , and a bijection  $f : X \rightarrow Y$  such that for all  $A \subseteq X$ ,  $A \in U$  if and only if  $f[A] \in W$ .

Ultrafilter isomorphism is equivalent to the following model theoretic property:

**Definition 2.2.21.** Suppose  $j_0 : N \rightarrow M_0$  and  $j_1 : N \rightarrow M_1$  are elementary embeddings. We write  $(M_0, j_0) \cong (M_1, j_1)$  to denote that there is an isomorphism  $k : M_0 \rightarrow M_1$  such that  $k \circ j_0 = j_1$ .

The following lemma (due to Rudin-Keisler) is explained in Section 3.4:

**Lemma 2.2.22.** *If  $U$  and  $W$  are ultrafilters, then  $U$  and  $W$  are isomorphic if and only if  $(M_U, j_U) \cong (M_W, j_W)$ .*  $\square$

For countably complete ultrafilters, there is a much simpler model theoretic characterization of ultrafilter isomorphism (so we will not really need the notation from Definition 2.2.21):

**Corollary 2.2.23.** *If  $U$  and  $W$  are countably complete ultrafilters, then  $U$  and  $W$  are isomorphic if and only if  $j_U = j_W$ .*

*Proof.* Since  $M_U$  and  $M_W$  are transitive, the only possible isomorphism between  $M_U$  and  $M_W$  is the identity. Hence  $(M_U, j_U) \cong (M_W, j_W)$  if and only if  $j_U = j_W$ .  $\square$

Notice that if  $U \cong W$  then  $\lambda_U = \lambda_W$ . Since we are mostly interested in ultrapower embeddings and not ultrafilters themselves, the following lemma lets us focus our attention on uniform ultrafilters that lie on cardinals:

**Lemma 2.2.24.** *Any ultrafilter  $U$  is isomorphic to a uniform ultrafilter  $W$  on  $\lambda_U$ .*

*Proof.* Fix  $X \in U$  such that  $|X| = \lambda_U$ . Let  $f : X \rightarrow \lambda_U$  be a bijection. Let  $W = \{A \subseteq \lambda_U : f^{-1}[A] \in U\}$ . Then  $U \cong W$ . Moreover  $W$  is uniform since  $\lambda_W = \lambda_U$ .  $\square$

Let us also mention a basic generalization of uniformity to the relativized case:

**Definition 2.2.25.** Suppose  $M$  is a transitive model of ZFC and  $U$  is an  $M$ -ultrafilter. Then the *size* of  $U$  is the  $M$ -cardinal  $\lambda_U = \min\{|X|^M : X \in U\}$ .

## Normal ultrafilters and the Mitchell order

**Definition 2.2.26.** Suppose  $\langle X_\alpha : \alpha < \delta \rangle$  is a sequence of subsets of  $\delta$ . The *diagonal intersection* of  $\langle X_\alpha : \alpha < \delta \rangle$  is the set

$$\Delta_{\alpha < \delta} X_\alpha = \{ \alpha < \delta : \alpha \in \bigcap_{\beta < \alpha} X_\beta \}$$

**Definition 2.2.27.** A uniform ultrafilter on an infinite cardinal  $\kappa$  is *normal* if it is closed under diagonal intersections.

**Lemma 2.2.28.** *Suppose  $U$  is an ultrafilter on an ordinal  $\kappa$ . The following are equivalent:*

(1)  $U$  is normal.

(2)  $U$  is  $\kappa$ -complete and  $a_U = \kappa$ . □

The Mitchell order was introduced by Mitchell in [5].

**Definition 2.2.29.** Suppose  $U$  and  $W$  are normal ultrafilters. The *Mitchell order* is defined by setting  $U \triangleleft W$  if  $U \in M_W$ .

This definition makes sense by our convention that the ultrapower of the universe by a countably complete ultrafilter is taken to be transitive.

**Lemma 2.2.30.** *The Mitchell order is a wellfounded partial order.* □

Actually, the interested reader will find several generalizations of this fact scattered throughout this dissertation. For example, Theorem 3.3.8 and Theorem 4.2.47 come to mind.

**Definition 2.2.31.** If  $U$  is a normal ultrafilter on a cardinal  $\kappa$ , then  $o(U)$  denotes the rank of  $U$  in the restriction of the Mitchell order to normal ultrafilters on  $\kappa$ . For any ordinal  $\kappa$ ,  $o(\kappa)$  denotes the rank of the restriction of the Mitchell order to normal ultrafilters on  $\kappa$ .

## 2.3 The linearity of the Mitchell order

Our original proof of the linearity of the Mitchell order did not use the Ultrapower Axiom as a hypothesis. Instead, it used a principle called *Weak Comparison* that was introduced by Woodin [10] in his work on the Inner Model Problem for supercompact cardinals.

Weak Comparison is directly motivated by the Comparison Lemma of inner model theory, and it is immediately clear that Weak Comparison holds in all known canonical inner models. On the other hand, although the Ultrapower Axiom is a more elegant principle than Weak Comparison, the fact that the Ultrapower Axiom holds in all known canonical inner models is not nearly as obvious. But our proof of the linearity of the Mitchell order from Weak Comparison actually shows that the Ultrapower Axiom follows from Weak Comparison, and this is how the Ultrapower Axiom was originally isolated.

In this section, we will introduce Weak Comparison and then prove that Weak Comparison implies the linearity of the Mitchell order. We then isolate the Ultrapower Axiom by remarking that this proof breaks naturally into two implications: first, that Weak Comparison implies the Ultrapower Axiom, and second, that the Ultrapower Axiom implies the linearity of the Mitchell order. We hope that this “genetic approach” will help motivate the formulation of the Ultrapower Axiom. The reader who does not want to learn about Weak Comparison can skip ahead to Section 2.3. We emphasize, however, that the fact that Weak Comparison implies UA is central to the motivation for this work.

### Weak Comparison

Stating Weak Comparison requires a number of definitions. The following notational convention will make many of our arguments easier to read:

**Definition 2.3.1.** Suppose  $N_0, N_1, P$  are transitive models of ZFC. We write

$$(j_0, j_1) : (N_0, N_1) \rightarrow P$$

to mean that  $j_0 : N_0 \rightarrow P$  and  $j_1 : N_1 \rightarrow P$  are elementary embeddings.

Weak Comparison is a comparison principle for a class of structures. The next two definitions specify this class.

**Definition 2.3.2.** Suppose  $M$  is a model of ZFC. Then  $M$  is *finitely generated* if there is some  $a \in M$  such that every point in  $M$  is definable in  $M$  using  $a$  as a parameter.

**Definition 2.3.3.** Suppose  $M$  is a transitive set that satisfies ZFC. Then  $M$  is a  $\Sigma_2$ -*hull* if there is a  $\Sigma_2$ -elementary embedding  $\pi : M \rightarrow V$ .

We can now state Weak Comparison:

**Weak Comparison.** If  $M_0$  and  $M_1$  are finitely generated  $\Sigma_2$ -hulls such that  $P(\omega) \cap M_0 = P(\omega) \cap M_1$ , then there are close embeddings  $(k_0, k_1) : (M_0, M_1) \rightarrow N$ .

We conclude this section by sketching Woodin’s argument that Weak Comparison holds in all known canonical inner models. Assume that  $V$  itself is a canonical inner model, so that there is some sort of Comparison Lemma for countable sufficiently elementary substructures of  $V$ . Assume  $M_0$  and  $M_1$  are finitely generated  $\Sigma_2$ -hulls. We will show that there are close embeddings  $(k_0, k_1) : (M_0, M_1) \rightarrow N$ .

The fact that  $M_0$  and  $M_1$  are countable  $\Sigma_2$ -hulls implies that the Comparison Lemma applies to them. The comparison process therefore produces transitive structures  $N_0$  and  $N_1$  such that one of the following holds:

**Case 1.**  $N_0 = N_1$  and there are close embeddings  $(k_0, k_1) : (M_0, M_1) \rightarrow N_0$ .

**Case 2.**  $N_0 \in N_1$ ,  $P(\omega) \cap N_1 \subseteq M_1$ , and there is a close embedding  $k_0 : M_0 \rightarrow N_0$ .

**Case 3.**  $N_1 \in N_0$ ,  $P(\omega) \cap N_0 \subseteq M_0$ , and there is a close embedding  $k_1 : M_1 \rightarrow N_1$ .

Case 1 is the result of “coiteration,” while in Case 2 and Case 3, one of the models has “outiterated” the other. To obtain weak comparison for the pair  $M_0$  and  $M_1$ , it suffices to show that Case 1 holds. To do this we show that Case 2 and Case 3 cannot hold.

Assume towards a contradiction that Case 2 holds. Since  $M_0$  is finitely generated, there is some  $a \in M_0$  such that every point in  $M_0$  is definable in  $M_0$  from  $a$ . Therefore  $k_0[M_0]$  is equal to the set of points in  $N_0$  that are definable in  $N_0$  from  $k_0(a)$ . Since  $N_0 \in N_1$ , it follows that  $k_0[M_0] \in N_1$  and  $k_0[M_0]$  is countable in  $N_1$ . Therefore its transitive collapse, namely  $M_0$ , is in  $N_1$  and is countable in  $N_1$ . Let  $x \in P(\omega) \cap N_1$  code  $M_0$  in the sense that any transitive model  $H$  of ZFC with  $x \in H$  has  $M_0 \in H$ . Then  $x \in P(\omega) \cap N_1 \subseteq P(\omega) \cap M_1 = P(\omega) \cap M_0$ . It follows that  $x \in M_0$ . Since  $x$  codes  $M_0$ , this implies  $M_0 \in M_0$ , which is a contradiction.

A similar argument shows that Case 3 does not hold. Therefore Case 1 must hold.

This argument actually shows that a slight strengthening of Weak Comparison is true in all known canonical inner models:

**Weak Comparison (Strong Version).** If  $M_0$  and  $M_1$  are finitely generated  $\Sigma_2$ -hulls, either  $M_0 \in M_1$ ,  $M_1 \in M_0$ , or there are close embeddings  $(k_0, k_1) : (M_0, M_1) \rightarrow N$ .

The strong version of Weak Comparison implies the Continuum Hypothesis.<sup>2</sup> It is not clear if it has any other advantages.

## Weak Comparison and the Mitchell order

In the interest of full disclosure, we admit that we cannot actually prove the linearity of the Mitchell order from Weak Comparison. Rather we will need some auxiliary hypotheses:

**Theorem 2.3.4.** *Assume that  $V = \text{HOD}$  and there is a  $\Sigma_2$ -correct worldly cardinal. Assume Weak Comparison holds. Then the Mitchell order is linear.*

The need for these auxiliary hypotheses is one of the quirks of Weak Comparison, and it is part of the reason we think the Ultrapower Axiom is a more elegant principle.

Here a cardinal  $\kappa$  is *worldly* if  $V_\kappa$  satisfies ZFC, and  $\Sigma_2$ -*correct* if  $V_\kappa \prec_{\Sigma_2} V$ . This is a very weak large cardinal hypothesis. For example, if  $\kappa$  is inaccessible, then in  $V_\kappa$  there

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<sup>2</sup>Here one must assume in addition to the strong version of Weak Comparison that  $V = \text{HOD}$  and there is a  $\Sigma_2$ -correct worldly cardinal. In fact, these hypotheses are necessary for all our consequences of Weak Comparison.

is a  $\Sigma_2$ -correct worldly cardinal; indeed, Morse-Kelley set theory implies the existence of a  $\Sigma_2$ -correct worldly cardinal. If  $\kappa$  is a strong cardinal, then  $\kappa$  itself is a  $\Sigma_2$ -correct worldly cardinal. The hypothesis is motivated by the following lemma, which we defer to a later section:

**Lemma 2.3.19.** *The existence of a  $\Sigma_2$ -hull is equivalent to the existence of a  $\Sigma_2$ -correct worldly cardinal.*

If one wants to apply Weak Comparison at all, at the very least, one needs the existence of a  $\Sigma_2$ -hull, and therefore one needs a  $\Sigma_2$ -correct worldly cardinal. One also needs *finitely generated* models, and this is where we use the principle  $V = \text{HOD}$ :

**Definition 2.3.5.** Suppose  $M$  is a model of ZFC. Then  $M$  is *pointwise definable* if every point in  $M$  is definable in  $M$  without parameters.

**Lemma 2.3.6.** *Assume  $V = \text{HOD}$ . If there is a  $\Sigma_2$ -hull, then there is a pointwise definable  $\Sigma_2$ -hull.* □

The principle  $V = \text{HOD}$  arguably does not hold in all canonical inner models. (The standard counterexample is  $L[M_1^\#]$ , though one might instead argue that this is not a canonical inner model.) The proof that the Mitchell order is linear, however, really does work in any inner model. The fact that we must assume  $V = \text{HOD}$  is again a quirk of the formulation of Weak Comparison.

The key technical lemma of Theorem 2.3.4 is the following closure property:

**Lemma 2.3.17.** *The set of finitely generated  $\Sigma_2$ -hulls is closed under internal ultrapowers.*

We defer the proof to Section 2.3. We now proceed to the proof of Theorem 2.3.4 granting the lemmas.

*Proof of Theorem 2.3.4.* Since there is a  $\Sigma_2$ -correct worldly cardinal and  $V = \text{HOD}$ , we can fix a pointwise definable  $\Sigma_2$ -hull  $H$  (by Lemma 2.3.6). It suffices to show that the Mitchell order is linear in  $H$ , since this is a  $\Pi_2$ -statement and  $H \equiv_{\Pi_2} V$ .

Suppose  $U_0$  and  $U_1$  are normal ultrafilters of  $H$ . We must show that in  $H$ , either  $U_0 = U_1$ ,  $U_0 \triangleleft U_1$ , or  $U_0 \triangleright U_1$ . We might as well assume that  $U_0$  and  $U_1$  are normal ultrafilters on the same cardinal  $\kappa$ , since otherwise it is immediate that  $U_0 \triangleleft U_1$  or  $U_0 \triangleright U_1$ .

Let  $j_0 : H \rightarrow M_0$  be the ultrapower of  $H$  by  $U_0$  and let  $j_1 : H \rightarrow M_1$  be the ultrapower of  $H$  by  $U_1$ . By the closure of finitely generated  $\Sigma_2$ -hulls under internal ultrapowers (Lemma 2.3.17),  $M_0$  and  $M_1$  are finitely generated  $\Sigma_2$ -hulls. Since  $M_0$  and  $M_1$  are internal ultrapowers of  $H$ ,  $P(\omega) \cap M_0 = P(\omega) \cap M_1$ . Therefore, by Weak Comparison there are close embeddings

$$(k_0, k_1) : (M_0, M_1) \rightarrow N$$

Since  $H$  is pointwise definable,

$$k_0 \circ j_0 = k_1 \circ j_1 \tag{2.1}$$

This is because  $k_0 \circ j_0, k_1 \circ j_1$  are both elementary embeddings from  $H$  to  $N$ , and therefore must shift all parameter-free definable points in the same way.

The proof now splits into three cases.

**Case 1.**  $k_0(\kappa) = k_1(\kappa)$ .

**Case 2.**  $k_0(\kappa) < k_1(\kappa)$ .

**Case 3.**  $k_0(\kappa) > k_1(\kappa)$ .

In Case 1, we will show  $U_0 = U_1$ , in Case 2, we will show  $U_0 \triangleleft U_1$ , and in Case 3, we will show  $U_0 \triangleright U_1$ . This will complete the proof.



*Proof in Case 1.* Suppose  $A \subseteq \kappa$  and  $A \in H$ . We have

$$\begin{aligned}
A \in U_0 &\iff \kappa \in j_0(A) \\
&\iff k_0(\kappa) \in k_0(j_0(A)) \\
&\iff k_0(\kappa) \in k_1(j_1(A)) \tag{2.2}
\end{aligned}$$

$$\iff k_1(\kappa) \in k_1(j_1(A)) \tag{2.3}$$

$$\iff \kappa \in j_1(A)$$

$$\iff A \in U_1$$

To obtain (2.2), we use (2.1) above. To obtain (2.3), we use the case hypothesis that  $k_0(\kappa) = k_1(\kappa)$ . It follows that  $U_0 = U_1$ .  $\square$

*Proof in Case 2.* Suppose  $A \subseteq \kappa$  and  $A \in H$ . We have

$$\begin{aligned}
A \in U_0 &\iff \kappa \in j_0(A) \\
&\iff k_0(\kappa) \in k_0(j_0(A)) \\
&\iff k_0(\kappa) \in k_1(j_1(A)) \tag{2.4}
\end{aligned}$$

$$\iff k_0(\kappa) \in k_1(j_1(A)) \cap k_1(\kappa) \tag{2.5}$$

$$\iff k_0(\kappa) \in k_1(j_1(A) \cap \kappa)$$

$$\iff k_0(\kappa) \in k_1(A) \tag{2.6}$$

To obtain (2.4), we use (2.1) above. To obtain (2.5), we use the case hypothesis that  $k_0(\kappa) < k_1(\kappa)$ . To prove (2.6), we use that  $U_1$  is  $\kappa$ -complete, so  $\text{CRT}(j_1) = \kappa$  and hence  $j_1(A) \cap \kappa = A$  for any  $A \subseteq \kappa$ .

It follows from this calculation that  $U_0$  is the  $M_1$ -ultrafilter on  $\kappa$  derived from  $k_1$  using  $k_0(\kappa)$ . (Here we use that  $P(\kappa) \cap M_1 = P(\kappa) \cap H$ .) Since  $k_1$  is close to  $M_1$ , it follows that  $U_0 \in M_1$ . Since  $M_1 = M_{U_1}^H$ , this means that  $U_0 \triangleleft U_1$  in  $H$ .  $\square$

*Proof in Case 3.* The proof in this case is just like the proof in Case 2 but with  $U_0$  and  $U_1$  swapped.  $\square$

This completes the proof of Theorem 2.3.4. □

## Weak Comparison and the Ultrapower Axiom

We now define the Ultrapower Axiom, which arises naturally from the proof of Theorem 2.3.4. Notice that the first half of this proof, which justifies our application of Weak Comparison to the ultrapowers  $M_0$  and  $M_1$ , does not actually require that  $U_0$  and  $U_1$  are normal ultrafilters. Instead, it simply requires that they are countably complete.

In order to state UA succinctly, we make the following definitions.

**Definition 2.3.7.** Suppose  $N, M_0, M_1, P$  are transitive models of ZFC and  $j_0 : N \rightarrow M_0$ ,  $j_1 : N \rightarrow M_1$ , and  $(k_0, k_1) : (M_0, M_1) \rightarrow P$  are elementary embeddings.

- $(k_0, k_1)$  is a *comparison* of  $(j_0, j_1)$  if  $k_0 \circ j_0 = k_1 \circ j_1$ .
- $(k_0, k_1)$  is an *internal ultrapower comparison* if  $k_0$  is an internal ultrapower embedding of  $M_0$  and  $k_1$  is an internal ultrapower embedding of  $M_1$ .
- $(k_0, k_1)$  is a *close comparison* if  $k_0$  is close to  $M_0$  and  $k_1$  is close to  $M_1$ .

**Ultrapower Axiom.** Every pair of ultrapower embeddings of the universe of sets has an internal ultrapower comparison.

On the face of it, the statement that every pair of ultrapowers has a comparison by internal ultrapowers looks *much* stronger than the conclusion of Weak Comparison, which only supplies close comparisons. But this is an illusion.

**Lemma 2.3.8.** *Suppose  $N, M_0, M_1$  are transitive set models of ZFC and  $j_0 : N \rightarrow M_0$  and  $j_1 : N \rightarrow M_1$  are weak ultrapower embeddings. If  $(j_0, j_1)$  has a comparison by close embeddings, then  $(j_0, j_1)$  has a comparison by internal ultrapowers.*

*Proof.* Suppose  $(k_0, k_1) : (M_0, M_1) \rightarrow P$  is a comparison by close embeddings. Let  $H \prec P$  be defined by

$$H = H^P(k_0[M_0] \cup k_1[M_1])$$

Let  $Q$  be the transitive collapse of  $H$  and let  $h : Q \rightarrow P$  be the inverse of the transitive collapse embedding. Let  $i_0 = h^{-1} \circ k_0$  and  $i_1 = h^{-1} \circ k_1$ .

Obviously  $(i_0, i_1) : (M_0, M_1) \rightarrow Q$  is a comparison of  $(j_0, j_1)$  and

$$Q = H^Q(i_0[M_0] \cup i_1[M_1])$$

We need to show it is a comparison by internal ultrapowers, or in other words that  $i_0$  is an internal ultrapower embedding of  $M_0$  and  $i_1$  is an internal ultrapower embedding of  $M_1$ .

We first show that  $i_0$  is an ultrapower embedding of  $M_0$ . Since  $j_1 : N \rightarrow M_1$  is a weak ultrapower embedding, there is some  $a \in M_1$  such that every element of  $M_1$  is definable in  $M_1$  from parameters in  $j_1[N] \cup \{a\}$ . It follows easily that  $Q = H^Q(i_0[M_0] \cup \{i_1(a)\})$ . Therefore  $i_0$  is an ultrapower embedding by Lemma 2.2.12.

Next, we show that  $i_0$  is an *internal* ultrapower embedding. Since  $h \circ i_0 = k_0$  and  $k_0$  is close, in fact,  $i_0$  is close to  $M_0$  (Lemma 2.2.17). Since  $i_0$  is a close ultrapower embedding of  $M_0$ , in fact,  $i_0$  is an internal ultrapower embedding of  $M_0$  (Lemma 2.2.18).

A symmetric argument shows that  $i_1$  is an internal ultrapower embedding of  $M_1$ , completing the proof.  $\square$

This yields a strengthening of Weak Comparison:

**Theorem 2.3.9.** *Assume Weak Comparison and  $V = \text{HOD}$ . Suppose  $M_0$  and  $M_1$  are finitely generated  $\Sigma_2$ -hulls such that  $P(\omega) \cap M_0 = P(\omega) \cap M_1$ . Then there are internal ultrapower embeddings  $(i_0, i_1) : (M_0, M_1) \rightarrow Q$ .*

*Proof.* Applying Weak Comparison, fix close embeddings  $(k_0, k_1) : (M_0, M_1) \rightarrow P$ .

Since  $M_0$  and  $M_1$  are  $\Sigma_2$ -hulls, they satisfy any  $\Pi_3$  sentence true in  $V$ . Therefore they both satisfy  $V = \text{HOD}$ . Let  $H_0 \prec M_0$  be the set of points that are definable without parameters in  $M_0$ . Let  $H_1 \prec M_1$  be the set of points that are definable without parameters in  $M_1$ . Then  $k_0[H_0] = k_1[H_1]$  is the set of points that are definable without parameters in  $P$ . It follows that  $H_0 \cong H_1$ . Let  $N$  be the common transitive collapse of  $H_0$  and  $H_1$ , and let

$j_0 : N \rightarrow M_0$  and  $j_1 : N \rightarrow M_1$  be the inverses of the transitive collapse maps. Note that  $j_0$  and  $j_1$  are weak ultrapower embeddings, and since  $N$  is pointwise definable,  $k_0 \circ j_0 = k_1 \circ j_1$ .

The weak ultrapower embeddings  $(j_0, j_1)$  therefore have a comparison by close embeddings, namely  $(k_0, k_1)$ . It follows from Lemma 2.3.8 that they have a comparison by internal ultrapower embeddings.  $\square$

Lemma 2.3.8 also yields a proof of the Ultrapower Axiom from the same hypotheses as Theorem 2.3.4:

**Theorem 2.3.10.** *Assume that  $V = \text{HOD}$  and there is a  $\Sigma_2$ -correct worldly cardinal. If Weak Comparison holds, then the Ultrapower Axiom holds.*

*Proof.* Since there is a  $\Sigma_2$ -correct worldly cardinal and  $V = \text{HOD}$ , we can fix a pointwise definable  $\Sigma_2$ -hull  $H$  (by Lemma 2.3.6). Since UA is a  $\Pi_2$ -statement and  $H \equiv_{\Pi_2} V$ , it suffices to show that  $H$  satisfies UA.

Suppose  $j_0 : H \rightarrow M_0$  and  $j_1 : H \rightarrow M_1$  are internal ultrapower embeddings of  $H$ . We must show that  $H$  satisfies that  $(j_0, j_1)$  has an internal ultrapower comparison.

By the closure of finitely generated  $\Sigma_2$ -hulls under internal ultrapowers (Lemma 2.3.17),  $M_0$  and  $M_1$  are finitely generated  $\Sigma_2$ -hulls. Moreover, since  $M_0$  and  $M_1$  are internal ultrapowers of  $H$ ,  $P(\omega) \cap M_0 = P(\omega) \cap H = P(\omega) \cap M_1$ . Therefore by Theorem 2.3.9, there are internal ultrapower embeddings  $(i_0, i_1) : (M_0, M_1) \rightarrow Q$ . Moreover since  $H$  is finitely generated,  $i_0 \circ j_0 = i_1 \circ j_1$ . It follows that  $(i_0, i_1)$  is an internal ultrapower comparison of  $(j_0, j_1)$ . This is absolute to  $H$ , and therefore  $H$  satisfies that  $(j_0, j_1)$  has an internal ultrapower comparison, as desired.  $\square$

## The Ultrapower Axiom and the Mitchell order

In this subsection, we prove the linearity of the Mitchell order from the Ultrapower Axiom. We include this proof largely for the benefit of the reader who would prefer to skip over our discussions of Weak Comparison, since the proof is very similar to that of Theorem 2.3.4.

The reader who has followed Theorem 2.3.4 will no doubt notice that both the statement and proof of Theorem 2.3.11 below are much simpler and more elegant than those of Theorem 2.3.4. It is a general pattern that UA is easier to use than Weak Comparison. In fact, almost every known consequence of Weak Comparison is a consequence of UA.

**Theorem 2.3.11 (UA).** *The Mitchell order is linear.*

*Proof.* Suppose  $U_0$  and  $U_1$  are normal ultrafilters. We must show that either  $U_0 = U_1$ ,  $U_0 \triangleleft U_1$ , or  $U_0 \triangleright U_1$ . We may assume without loss of generality that  $U_0$  and  $U_1$  are normal ultrafilters on the same cardinal  $\kappa$ , since otherwise it is obvious that either  $U_0 \triangleleft U_1$  or  $U_0 \triangleright U_1$ .

Let  $j_0 : V \rightarrow M_0$  be the ultrapower of the universe by  $U_0$ . Let  $j_1 : V \rightarrow M_1$  be the ultrapower of the universe by  $U_1$ . Applying UA, there is an internal ultrapower comparison  $(i_0, i_1) : (M_0, M_1) \rightarrow P$  of  $(j_0, j_1)$ .

The proof now breaks into three cases.

**Case 1.**  $i_0(\kappa) = i_1(\kappa)$ .

**Case 2.**  $i_0(\kappa) < i_1(\kappa)$ .

**Case 3.**  $i_0(\kappa) > i_1(\kappa)$ .

In Case 1, we will prove  $U_0 = U_1$ . In Case 2, we will prove  $U_0 \triangleleft U_1$ . In Case 3, we will prove  $U_0 \triangleright U_1$ .

*Proof in Case 1.* Suppose  $A \subseteq \kappa$ . Then

$$\begin{aligned}
A \in U_0 &\iff \kappa \in j_0(A) \\
&\iff i_0(\kappa) \in i_0(j_0(A)) \\
&\iff i_0(\kappa) \in i_1(j_1(A)) \tag{2.7}
\end{aligned}$$

$$\iff i_1(\kappa) \in i_1(j_1(A)) \tag{2.8}$$

$$\iff \kappa \in j_1(A)$$

$$\iff A \in U_1$$

To obtain (2.7), we use the fact that  $(i_0, i_1)$  is a comparison, and in particular that  $i_1 \circ j_1 = i_0 \circ j_0$ . To obtain (2.8), we use the case hypothesis that  $i_0(\kappa) = i_1(\kappa)$ . It follows that  $U_0 = U_1$ .  $\square$

*Proof in Case 2.* Suppose  $A \subseteq \kappa$ . Then

$$\begin{aligned}
A \in U_0 &\iff \kappa \in j_0(A) \\
&\iff i_0(\kappa) \in i_0(j_0(A)) \\
&\iff i_0(\kappa) \in i_1(j_1(A)) \\
&\iff i_0(\kappa) \in i_1(j_1(A)) \cap i_1(\kappa) \tag{2.9}
\end{aligned}$$

$$\begin{aligned}
&\iff i_0(\kappa) \in i_1(j_1(A) \cap \kappa) \\
&\iff i_0(\kappa) \in i_1(A) \tag{2.10}
\end{aligned}$$

To obtain (2.9), we use the case hypothesis that  $i_0(\kappa) < i_1(\kappa)$ . To obtain (2.10), we use that  $U_1$  is  $\kappa$ -complete; therefore  $\text{CRT}(j_1) = \kappa$  so  $j_1(A) \cap \kappa = A$  for any  $A \subseteq \kappa$ .

It follows that  $U_0$  is the  $M_1$ -ultrafilter derived from  $i_1$  using  $i_0(\kappa)$ . (Here we use that  $P(\kappa) \subseteq M_1$ .) Since  $i_1$  is an internal ultrapower embedding of  $M_1$ ,  $i_1$  is definable over  $M_1$ , and therefore  $U_0$  is definable over  $M_1$  from  $i_1$  and  $i_0(\kappa)$ . It follows that  $U_0 \in M_1$ . Since  $M_1 = M_{U_1}$ , this means  $U_0 \triangleleft U_1$ , as desired.  $\square$

*Proof in Case 3.* The proof in this case is identical to the proof in Case 2 but with  $U_0$  and  $U_1$  swapped.  $\square$

Thus no matter which of the cases hold, either  $U_0 = U_1$ ,  $U_0 \triangleleft U_1$ , or  $U_0 \triangleright U_1$ . This completes the proof.  $\square$

There is a partial converse to Theorem 2.3.11 that helps explain the motivation for the proof of Theorem 2.3.11. To state this converse, we first defines a restricted version of the Ultrapower Axiom for ultrapower embeddings coming from normal ultrafilters:

**Definition 2.3.12.** The *Normal Ultrapower Axiom* is the statement that any pair of ultrapower embeddings of the universe of sets associated to normal ultrafilters have a comparison by internal ultrapowers.

**Proposition 2.3.13.** *The Normal Ultrapower Axiom is equivalent to the linearity of the Mitchell order.*

*Proof.* The proof that the Normal Ultrapower Axiom implies the linearity of the Mitchell order is immediate from the proof of Theorem 2.3.11.

Conversely, assume the Mitchell order is linear. Suppose  $U_0$  and  $U_1$  are normal ultrafilters, and let  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  be their ultrapowers. We will show  $(j_0, j_1)$  has a comparison by internal ultrapowers. Assume without loss of generality that  $U_0 \triangleleft U_1$ . Let  $i_0 : M_0 \rightarrow P_0$  be the ultrapower of  $M_0$  by  $j_0(U_1)$ . Let  $i_1 : M_1 \rightarrow P_1$  be the ultrapower of  $M_1$  by  $U_0$ . Then  $i_0$  and  $i_1$  are internal ultrapowers of  $M_0$  and  $M_1$  respectively. Moreover  $i_0 = j_0(j_1)$  and  $i_1 = j_0 \upharpoonright M_1$ ,<sup>3</sup> so

$$i_0 \circ j_0 = j_0(j_1) \circ j_0 = j_0 \circ j_1 = i_1 \circ j_1$$

It follows that  $(i_0, i_1)$  is a comparison of  $(j_0, j_1)$  by internal ultrapowers.  $\square$

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<sup>3</sup>Suppose  $M, N$ , and  $P$  are transitive models of ZFC. Suppose  $j : M \rightarrow N$  and  $i : M \rightarrow P$  are elementary embeddings. Assume  $j \upharpoonright x \in M$  for all  $x \in M$ . Assume  $i$  is a cofinal embedding. Then  $i(j) = \bigcup_{X \in M} i(j \upharpoonright X)$ .

The proof of Proposition 2.3.13 is local in the sense that it shows that the comparability of two normal ultrafilters in the Mitchell order is equivalent to their comparability by internal ultrapowers. This is a special feature of the Mitchell order on normal ultrafilters. For the generalized Mitchell order (defined in Chapter 4), neither implication is provable. Motivated by this issue, we develop in Section 5.5 a variant of the generalized Mitchell order called the *internal relation*.

## Technical lemmas related to Weak Comparison

In this section, we prove several lemmas promised in Section 2.3.

**Lemma 2.3.14.** *Suppose  $N$  is a finitely generated model of ZFC and  $U$  is an  $N$ -ultrafilter. Then  $M_U^N$  is finitely generated.*

*Proof.* Fix  $b \in N$  such that every element of  $N$  is definable in  $N$  using  $b$  as a parameter. Obviously every element of  $j_U[N]$  is definable in  $M_U^N$  using  $j_U(b)$  as a parameter. But  $M_U^N = \{j_U(f)(a_U) : f \in N\} = \{g(a_U) : g \in j_U[N]\}$ . Therefore every element of  $M_U^N$  is definable using  $j_U(b)$  and  $a_U$  as parameters.  $\square$

The next lemma, standard in the case of fully elementary embeddings, is the key to our analysis of  $\Sigma_2$ -hulls:

**Lemma 2.3.15.** *Suppose  $j : N \rightarrow M$  is a  $\Sigma_2$ -elementary embedding between transitive models of ZFC. Suppose  $X \in N$ , and  $a \in j(X)$ . Let  $U$  be the  $N$ -ultrafilter on  $X$  derived from  $j$  using  $a$ . Then there is a unique  $\Sigma_2$ -elementary embedding  $k : M_U^N \rightarrow M$  such that  $k \circ j_U^N = j$  and  $k(a_U) = a$ .*

*Proof.* We begin with a simple remark. Suppose  $\varphi(v_1, \dots, v_n)$  is a  $\Sigma_2$ -formula and  $f_1, \dots, f_n$  are functions in  $N$  that are defined  $U$ -almost everywhere. The statement  $S = \{x \in X : N \models \varphi(f_1(x), \dots, f_n(x))\}$  can be written as a Boolean combination of  $\Sigma_2$  formulas in the variables



$S$  and  $f_1, \dots, f_n$ . It follows that

$$j(\{x \in X : N \models \varphi(f_1(x), \dots, f_n(x))\}) = \{x \in j(X) : M \models \varphi(j(f_1)(x), \dots, j(f_n)(x))\}$$

For any function  $f \in N$  defined  $U$ -almost everywhere, set

$$k([f]_U) = j(f)(a)$$

Fix a  $\Sigma_2$ -formula  $\varphi(x_1, \dots, x_n)$ . The following calculation shows that  $k$  is a well-defined  $\Sigma_2$ -elementary embedding from  $M_U^N$  to  $M$ :

$$\begin{aligned} M_U^N \models \varphi([f_1]_U, \dots, [f_n]_U) &\iff \{x \in X : N \models \varphi(f_1(x), \dots, f_n(x))\} \in U \\ &\iff M \models a \in j(\{x \in X : N \models \varphi(f_1(x), \dots, f_n(x))\}) \\ &\iff M \models a \in \{x \in j(X) : M \models \varphi(j(f_1)(x), \dots, j(f_n)(x))\} \\ &\iff M \models \varphi(j(f_1)(a), \dots, j(f_n)(a)) \\ &\iff M \models \varphi(k([f_1]_U), \dots, k([f_n]_U)) \quad \square \end{aligned}$$

Lemma 2.3.15 yields a  $\Sigma_2$ -elementary generalization of the standard Realizability Lemma:

**Lemma 2.3.16.** *Suppose  $N$  is a countable  $\Sigma_2$ -hull and  $N \models U$  is a countably complete ultrafilter. Then  $M_U^N$  is a  $\Sigma_2$ -hull.*

*Proof.* Let  $\pi : N \rightarrow V$  be a  $\Sigma_2$ -elementary embedding. Let  $U' = \pi(U)$ , so  $U'$  is a countably complete ultrafilter. Since  $\pi[U] \subseteq U'$  is countable, there is some  $a \in \bigcap \pi[U]$ . Note that  $U = \pi^{-1}\{a\}$ . Therefore by Lemma 2.3.15, there is a  $\Sigma_2$ -elementary embedding  $k : M_U^N \rightarrow V$ , so  $M_U^N$  is a  $\Sigma_2$ -hull.  $\square$

**Lemma 2.3.17.** *The set of finitely generated  $\Sigma_2$ -hulls is closed under internal ultrapowers.*

*Proof.* Immediate from the conjunction of Lemma 2.3.14 and Lemma 2.3.16.  $\square$

Lemma 2.3.15 can also be used to prove the following fact:

**Lemma 2.3.18.** *Suppose  $N$  is a set model of ZFC and  $j : N \rightarrow M$  is a  $\Sigma_2$ -elementary embedding. Then  $j$  factors as a cofinal elementary embedding followed by a  $\Sigma_2$ -elementary end extension.  $\square$*

**Proposition 2.3.19.** *There is a  $\Sigma_2$ -hull if and only if there is a  $\Sigma_2$ -correct worldly cardinal.*

*Proof.* Suppose  $N$  is a  $\Sigma_2$ -hull. Let  $\pi : N \rightarrow V$  be a  $\Sigma_2$ -elementary embedding. By Lemma 2.3.15,  $\pi$  factors as a cofinal elementary embedding  $\pi : N \rightarrow H$  followed by a  $\Sigma_2$ -elementary end extension  $H \prec_{\Sigma_2} V$ . Since  $H \prec_{\Sigma_2} V$ ,  $H = V_\kappa$  for some cardinal  $\kappa$ . Since  $\pi : N \rightarrow V_\kappa$  is fully elementary,  $V_\kappa$  satisfies ZFC. Thus  $\kappa$  is a  $\Sigma_2$ -correct worldly cardinal.  $\square$

# Chapter 3

## The Ketonen Order

### 3.1 Introduction

#### Ketonen's order

Central to Chapter 2 was an argument that the Mitchell order is linear in all known canonical inner models. In Section 2.3, we delved deeper into the first half of this proof, extracting from it a general inner model principle called the Ultrapower Axiom. It turns out that a closer look at the second half of the proof also yields more information: it shows that the Ultrapower Axiom implies not only the linearity of the Mitchell order, but also the linearity of a much more general order on countably complete ultrafilters.

This order dates back to the early 1970s. A remarkable theorem of Ketonen [11] from this period states that if every regular cardinal  $\lambda \geq \kappa$  carries a  $\kappa$ -complete uniform ultrafilter, then  $\kappa$  is strongly compact. Ketonen gave two proofs of this theorem. The first is an induction. The second is not as well-known, but is of much greater interest here. Ketonen introduced a wellfounded order on countably complete weakly normal ultrafilters, and showed that certain minimal elements in this order witness the strong compactness of  $\kappa$ . (We give this proof in Theorem 7.2.15 since generalizations of the proof form a key component of our analysis of strong compactness and supercompactness under UA.)

Independently of Ketonen’s work, and a rather long time after, we extracted from the proof of the linearity of the Mitchell order under UA (Theorem 2.3.11) a more general version of Ketonen’s order, which we now call the *Ketonen order*. The Ketonen order is a wellfounded partial order on countably complete ultrafilters concentrating on ordinals. The key realization, which distinguishes our work from Ketonen’s, is that the Ketonen order can be *linear*. In fact, the totality of the Ketonen order is an immediate consequence of UA (Theorem 3.3.6). In fact, the linearity of the Ketonen order is equivalent to the Ultrapower Axiom. This equivalence is Theorem 3.6.1, which is probably the hardest theorem of this chapter. The Ketonen order will be our main tools in the investigation of the structure of countably complete ultrafilters under UA.

### Outline of Chapter 3

Let us outline the rest of Chapter 3.

**Section 3.2.** We introduce some more preliminary definitions that will be used throughout the rest of this dissertation. Especially important are limits of ultrafilters, which we introduce both in the traditional ultrafilter theoretic sense and in a generalized setting in terms of inverse images.

**Section 3.3.** We introduce the main object of study of this chapter, a fundamental tool in the theory of the Ultrapower Axiom: a wellfounded partial order on countably complete ultrafilters called the Ketonen order. In Section 3.3, we define the Ketonen order and give various alternate characterizations. The most important characterization is given by Lemma 3.3.4, which shows that the Ketonen order can be reformulated in terms of comparisons. This immediately leads to the observation that the Ketonen order is linear under the Ultrapower Axiom. In Section 3.3, we establish the basic order-theoretic properties of the Ketonen order: it is a preorder on the class of countably complete ultrafilters concentrating on ordinals. Restricted to tail uniform ultrafilters, it is a partial order. Lemma 3.3.15 shows that the Ketonen order is *graded* in the sense that if  $\alpha < \beta$ , then the tail uniform ultrafil-

ters on  $\alpha$  all lie below those on  $\beta$ . In particular, the Ketonen order is setlike. We finally prove the wellfoundedness of the Ketonen order (Theorem 3.3.8). The general proof of the wellfoundedness of the Ketonen order is due to the author.

**Section 3.4.** We explore the relationship between the Ketonen order and two well-known orders on ultrafilters. Section 3.4 shows that the restriction of the Ketonen order to normal ultrafilters is precisely the Mitchell order. In this sense the Ketonen order is a generalized Mitchell order. In Section 3.4 we turn to perhaps the best-known order on ultrafilters: the Rudin-Keisler order. We take this opportunity to set down some basic facts about this order, sometimes with proofs. The Ketonen order is not isomorphism invariant, so it cannot extend the Rudin-Keisler order. To explain these orders' relationship better, we define an auxiliary order called the *revised Rudin-Keisler order* which is contained in the intersection of the Rudin-Keisler order and the Ketonen order. Moreover we introduce the concept of an *incompressible ultrafilter*, an ultrafilter  $U$  whose generator  $a_U$  is as small as possible (see Lemma 3.4.18). An argument due to Solovay shows that the strict Rudin-Keisler order and the revised Rudin-Keisler order coincide on incompressible ultrafilters. Thus the Ketonen order extends the strict Rudin-Keisler order on countably complete incompressible ultrafilters.

**Section 3.5.** We study several variants of the Ketonen order. In Section 3.5, we investigate the relationship between the Ketonen order and notions from inner model theory. We introduce a model theoretic generalization of the Ketonen order whose domain is the class of pointed models of ZFC, structures  $(M, \xi)$  where  $M$  is a transitive model of ZFC and  $\xi$  is an ordinal of  $M$ . This defines a coarse analog of the Dodd-Jensen order, the canonical prewellorder on mice. We give a generalized wellfoundedness proof for this order (Theorem 3.5.8) that is closely related to the proof of the wellfoundedness of the Dodd-Jensen order. We use this to prove an often useful lemma that is a coarse analog of the Dodd-Jensen lemma: Theorem 3.5.10 shows that definable embeddings are pointwise minimal on the ordinals.

In the next subsection, Section 3.5, we introduce the *seed order*. In early versions of this work, we mainly used the seed order where we now use the Ketonen order. There is no substantive difference between these approaches since under UA the two orders coincide. In ZFC, however, one cannot prove that the seed order is transitive: indeed, we show by a silly argument that the transitivity of the seed order implies the Ultrapower Axiom. We also introduce an extension of the seed order to pointed ultrapowers. The next subsection Section 3.5 is spent relating this order to the structure of the direct limit of all ultrapowers under UA.

The next two subsections are devoted to combinatorial generalizations of the Ketonen order. One does not need to read them to understand the rest of this dissertation. In Section 3.5, we introduce a generalized version of the Lipschitz order, and show that this order extends the Ketonen order on countably complete ultrafilters. Therefore under UA, the two orders coincide, which gives a strange analog of the linearity of the Lipschitz order in determinacy theory. Section 3.5 introduces a combinatorial generalization of the Ketonen order to filters, which demonstrates a relationship between the Ketonen order and the canonical order on stationary sets due to Jech [12].

**Section 3.6.** This section contains Theorem 3.6.1, the most substantive result of the chapter: the linearity of the Ketonen order is equivalent to the Ultrapower Axiom. The fact that UA implies the linearity of the Ketonen order is immediate. (The proof appears in Section 3.3.) The converse, however, is subtle. Since we will mostly work under the assumption of UA, this equivalence is itself not that important (although it does show that all of our results can be proved from an a priori weaker premise). More important is the proof, which identifies a canonical way to compare a pair of ultrafilters assuming the linearity of the Ketonen order.

## 3.2 Preliminary definitions

### Tail uniform ultrafilters

A common notational issue we will encounter in this dissertation is that two ultrafilters may differ only in the sense that they have different underlying sets. The change-of-space relation, defined below, articulates our tendency to identify such ultrafilters.

**Definition 3.2.1.** Suppose  $U$  is an ultrafilter on  $X$  and  $C$  is a class. We say  $U$  *concentrates on  $C$*  if  $C \cap X \in U$ . If  $C$  is a set and  $U$  concentrates on  $C$ , the *projection of  $U$  on  $C$*  is the ultrafilter  $U \upharpoonright C = \{A \subseteq C : A \cap X \in U\}$ .

**Definition 3.2.2.** The *change-of-space relation* is defined on ultrafilters  $U$  and  $W$  by setting  $U =_E W$  if  $U = W \upharpoonright X$  where  $X$  is the underlying set of  $U$ .

**Lemma 3.2.3.** *Suppose  $U$  and  $W$  are ultrafilters. Then the following are equivalent:*

- (1)  $U =_E W$ .
- (2) For some set  $S \in U \cap W$ ,  $U \cap P(S) = W \cap P(S)$ .
- (3) For all sets  $A$ ,  $a_U \in j_U(A)$  if and only if  $a_W \in j_W(A)$ .
- (4) There is a comparison  $(k, h)$  of  $(j_U, j_W)$  such that  $k(a_U) = h(a_W)$ . □

The change-of-space relation is therefore an equivalence relation on ultrafilters.

The Ketonen order will be a partial order on the class of ultrafilters on ordinals. On such general ultrafilters, however, the (nonstrict) Ketonen order is only a preorder, due to the existence of  $=_E$ -equivalent ultrafilters (see Lemma 3.3.16). Thus we sometimes restrict further to those ultrafilters that are uniform in a slightly nonstandard sense:

**Definition 3.2.4.** A filter  $F$  on an ordinal  $\delta$  is *tail uniform* if it contains  $\delta \setminus \alpha$  for every  $\alpha < \delta$ .

For any ordinal  $\delta$ , the *tail filter* on  $\delta$  is the filter generated by sets of the form  $\delta \setminus \alpha$  for  $\alpha < \delta$ . A filter is therefore tail uniform if it extends the tail filter. Equivalently,  $F$  is tail uniform if every element of  $F^+$  is cofinal in  $\alpha$ .

For example, the principal ultrafilter on  $\alpha + 1$  concentrated at  $\alpha$  is uniform.

**Definition 3.2.5.** If  $U$  is an ultrafilter that concentrates on ordinals, then  $\delta_U$  denotes the least ordinal  $\delta$  on which  $U$  concentrates.

**Lemma 3.2.6.** *If  $U$  is an ultrafilter that concentrates on ordinals, then  $U$  is tail uniform if and only if  $\delta_U$  is the underlying set of  $U$ .* □

The key property of tail uniform ultrafilters, which is quite obvious, is that they yield canonical representatives of  $=_E$  equivalence classes of ultrafilters concentrating on ordinals.

**Lemma 3.2.7.** *For any ultrafilter  $U$  that concentrates on ordinals, then  $U \upharpoonright \delta_U$  is the unique tail uniform ultrafilter  $W$  such that  $U =_E W$ . In particular, if  $U$  and  $W$  are tail uniform ultrafilters such that  $U =_E W$ , then  $U = W$ .* □

There is an obvious but useful characterization of  $\delta_U$  in terms of elementary embeddings:

**Lemma 3.2.8.** *If  $U$  is an ultrafilter that concentrates on ordinals, then  $\delta_U$  is the least ordinal  $\delta$  such that  $a_U < j_U(\delta)$ .* □

**Definition 3.2.9.** The class of countably complete tail uniform ultrafilters is denoted by  $\text{Un}$ .

Let us just point out that tail uniformity and uniformity are not the same concept, and moreover neither is a strengthening of the other. The simplest way to separate these concepts is by considering the Fréchet and tail filters themselves. For any set  $X$ , let  $F_X$  denote the Fréchet filter on  $X$ . For any ordinal  $\alpha$ , let  $T_\alpha$  denote the tail filter on  $\alpha$ .

**Lemma 3.2.10.** *Suppose  $\lambda$  is an ordinal.*

- $T_\lambda \subseteq F_\lambda$  if and only if  $\lambda$  is a cardinal.



- $F_\lambda \subseteq T_\lambda$  if and only if  $|\lambda| = \text{cf}(\lambda)$ .

Thus  $T_\lambda = F_\lambda$  if and only if  $\lambda$  is a regular cardinal. If  $\lambda$  is a singular cardinal,  $T_\lambda$  is tail uniform but not uniform. If  $\lambda$  is not a cardinal, then  $F_\lambda$  is uniform but not tail uniform.  $\square$

One can easily obtain ultrafilters that are counterexamples to the equivalence of tail uniformity and true uniformity by combining the previous lemma with the Ultrafilter Lemma.

## Limits of ultrafilters

The following definition comes from classical ultrafilter theory:

**Definition 3.2.11.** Suppose  $W$  is an ultrafilter,  $I$  is a set in  $W$ , and  $\langle U_i : i \in I \rangle$  is a sequence of ultrafilters on a set  $X$ . The  $W$ -limit of  $\langle U_i : i \in I \rangle$  is the ultrafilter

$$W\text{-}\lim_{i \in I} U_i = \{A \subseteq X : \{i \in I : A \in U_i\} \in W\}$$

It is often easier to think about limits in terms of elementary embeddings:

**Lemma 3.2.12.** Suppose  $W$  is an ultrafilter,  $I$  is a set in  $W$ , and  $\langle U_i : i \in I \rangle$  is a sequence of ultrafilters on a fixed set  $X$ . Then

$$W\text{-}\lim_{i \in I} U_i = j_W^{-1}[Z]$$

where  $Z = [\langle U_i : i \in I \rangle]_W$ .

*Proof.* Suppose  $A \subseteq X$ . Then

$$\begin{aligned} A \in W\text{-}\lim_{i \in I} U_i &\iff A \in U_i \text{ for } W\text{-almost all } i \in I \\ &\iff j_W(A) \in [\langle U_i : i \in I \rangle]_W \\ &\iff A \in j_W^{-1}[Z] \end{aligned}$$

where the middle equivalence follows from Łoś's Theorem.  $\square$

Limits generalize the usual derived ultrafilter and pushforward constructions:

**Definition 3.2.13.** Suppose  $X$  is a set and  $a \in X$ . The *principal ultrafilter on  $X$  concentrated at  $a$*  is the ultrafilter  $p_a^X = \{A \subseteq X : a \in A\}$ .

**Definition 3.2.14.** Suppose  $W$  is an ultrafilter,  $I$  is a set in  $W$ , and  $f : I \rightarrow X$  is a function. Then the *pushforward* of  $W$  by  $f$  is the ultrafilter  $f_*(W) = \{A \subseteq X : f^{-1}[A] \in W\}$ .

The following lemmas relate the derived ultrafilter construction to inverse images, limits, and pushforwards.

**Lemma 3.2.15.** *Suppose  $N$  and  $P$  are transitive models of ZFC,  $X$  is a set in  $N$ ,  $i : N \rightarrow P$  is an elementary embedding, and  $a \in i(X)$ . Then the  $N$ -ultrafilter on  $X$  derived from  $i$  using  $a$  is simply  $i^{-1}[p_a^{i(X)}]$ .  $\square$*

**Lemma 3.2.16.** *Suppose  $W$  is an ultrafilter,  $I$  is a set in  $W$ , and  $f : I \rightarrow X$  is a function. Then*

$$f_*(W) = W\text{-}\lim_{i \in I} p_{f(i)}^X = j_W^{-1}[p_{[f]_W}^{j_W(X)}]$$

*In other words,  $f_*(W)$  is the ultrafilter on  $X$  derived from  $j_W$  using  $[f]_W$ .  $\square$*

These lemmas are trivial, but it turns out that many calculations are significantly simpler when one treats limits and derived ultrafilters uniformly as inverse images.

To be really pedantic, the reader might point out that for example in Lemma 3.2.16,  $p_{[f]_W}^{j_W(X)}$  is not an  $M_W$ -ultrafilter but a  $V$ -ultrafilter. Moreover if  $M_W$  is not wellfounded, then the statement  $[f]_W \in j_W(X)$  so  $p_{[f]_W}^{j_W(X)}$  is not well-defined. Of course,  $p_{[f]_W}^{j_W(X)}$  really denotes  $(p_{[f]_W}^{j_W(X)})^{M_W}$ . For the reader's own sake, we will try to omit all these superscripts in our notation for principal ultrafilters when they can be guessed from context. For example, in Lemma 3.2.16, we would usually write:

$$f_*(W) = W\text{-}\lim_{i \in I} p_{f(i)} = j_W^{-1}[p_{[f]_W}]$$

The key to understanding derived ultrafilters is to consider the natural factor embeddings associated to them. There is a generalization of the factor embedding construction to the

case of limits. In fact, this works somewhat more generally for arbitrary inverse images of ultrafilters:

**Lemma 3.2.17.** *Suppose  $N$  and  $P$  are transitive models of ZFC,  $X$  is a set in  $N$ ,  $i : N \rightarrow P$  is an elementary embedding, and  $D$  is a  $P$ -ultrafilter on  $i(X)$ . Let  $U = i^{-1}[D]$ . There is a unique elementary embedding  $k : M_U^N \rightarrow M_D^P$  such that  $k(a_U) = a_D$  and  $k \circ j_U^N = j_D^P \circ i$ .*

*Proof.* For any function  $f \in N$  defined on a set in  $U$ , set

$$k([f]_U^N) = [i(f)]_D^P$$

It is immediate from this definition that  $k(a_U) = a_D$  and  $k \circ j_U^N = j_D^P \circ i$ . We must show that  $k$  is well-defined and elementary. This follows from the usual calculation:

$$\begin{aligned} M_U^N \models \varphi([f_1]_U^N, \dots, [f_n]_U^N) &\iff N \models \varphi(f_1(x), \dots, f_n(x)) \text{ for } U\text{-almost all } x \\ &\iff P \models \varphi(i(f_1)(x), \dots, i(f_n)(x)) \text{ for } D\text{-almost all } x \\ &\iff M_D^P \models \varphi([i(f_1)]_D^P, \dots, [i(f_n)]_D^P) \quad \square \end{aligned}$$

### 3.3 The Ketonen order

#### Characterizations of the Ketonen order

Let us begin our investigation of the Ketonen order with a purely combinatorial definition.

**Definition 3.3.1.** Suppose  $X$  is a set and  $A$  is a class. Then  $\mathcal{B}(X)$  denotes the set of countably complete ultrafilters on  $X$ , and  $\mathcal{B}(X, A)$  denotes the set of countably complete ultrafilters on  $X$  that concentrate on  $A$ .

**Definition 3.3.2.** Suppose  $\delta$  is an ordinal. The *Ketonen order* is defined on  $\mathcal{B}(\delta)$  as follows. For  $U, W \in \mathcal{B}(\delta)$ :

- $U <_k W$  if there is a set  $I \in W$  and a sequence  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\delta, \alpha)$  such that  $U = W\text{-}\lim_{i \in I} U_\alpha$ .

- $U \leq_{\mathbb{k}} W$  if there is a set  $I \in W$  and a sequence  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\delta, \alpha + 1)$  such that  $U = W\text{-}\lim_{i \in I} U_\alpha$ .

We refer to  $<_{\mathbb{k}}$  and  $\leq_{\mathbb{k}}$  as the *strict* and *non-strict* Ketonen orders.

Of course one could take  $I = \delta \setminus \{0\}$  in the first bullet-point and  $I = \delta$  in the second, but the combinatorics are typically clearer if one does not make this demand.

There is perhaps a potential ambiguity in our notation, since the order  $<_{\mathbb{k}}$  depends on the ordinal  $\delta$ , which we suppress in our notation. This dependence is always immaterial, however, since there are canonical embeddings between the various Ketonen orders. These embeddings allow us to spin all these orders together into one (Definition 3.3.13).

Let us first explain the straightforward relationship between the strict and nonstrict Ketonen orders.

**Proposition 3.3.3.** *Suppose  $\delta$  is an ordinal and  $U, W \in \mathcal{B}(\delta)$ . Then  $U \leq_{\mathbb{k}} W$  if and only if  $U <_{\mathbb{k}} W$  or  $U = W$ .*

*Proof.* Let  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\delta, \alpha + 1)$  witness  $U \leq_{\mathbb{k}} W$ . Let

$$J = \{i \in I : U_\alpha \in \mathcal{B}(\delta, \alpha)\}$$

If  $J \in W$ , then  $\langle U_\alpha : \alpha \in J \rangle \in \prod_{\alpha \in J} \mathcal{B}(\delta, \alpha)$  witnesses  $U <_{\mathbb{k}} W$ .

Assume therefore that  $J \notin W$ . For all  $\alpha \in I \setminus J$ ,  $U_\alpha \in \mathcal{B}(\delta, \alpha + 1) \setminus \mathcal{B}(\delta, \alpha)$ . Note that  $\mathcal{B}(\delta, \alpha + 1) \setminus \mathcal{B}(\delta, \alpha)$  contains only the principal ultrafilter  $p_\alpha^\delta$ , and hence  $U_\alpha = p_\alpha^\delta$  for  $\alpha \in I \setminus J$ . Thus

$$U = W\text{-}\lim_{\alpha \in I} U_\alpha = W\text{-}\lim_{\alpha \in I \setminus J} p_\alpha^\delta = W$$

where the final equality follows easily from the definitions (or from Lemma 3.2.16).  $\square$

We therefore focus our attention on the strict Ketonen order  $<_{\mathbb{k}}$  for now. Before establishing its basic order-theoretic properties, let us give some fairly obvious alternate characterizations of it. We think the characterization Lemma 3.3.4 (2) is quite elegant in that it demonstrates a basic relationship between the Ketonen order, the covering properties of

ultrapowers, and extensions of filter bases to countably complete ultrafilters, foreshadowing the powerful interactions between strong compactness and the Ultrapower Axiom that we will see in Chapter 7 and Chapter 8. Lemma 3.3.4 (3) and (4) are more useful, though, linking the Ketonen order and the Ultrapower Axiom through the concept of a comparison (Definition 2.3.7).

**Lemma 3.3.4.** *Suppose  $\delta$  is an ordinal and  $U, W \in \mathcal{B}(\delta)$ . The following are equivalent:*

(1)  $U <_{\mathbf{k}} W$ .

(2)  $j_W[U]$  extends to an  $M_W$ -ultrafilter  $Z \in \mathcal{B}^{M_W}(j_W(\delta), a_W)$ .

(3) There is a comparison  $(k, h) : (M_U, M_W) \rightarrow P$  of  $(j_U, j_W)$  such that  $h$  is an internal ultrapower embedding of  $M_W$  and  $k(a_U) < h(a_W)$ .

(4) There is a comparison  $(k, h) : (M_U, M_W) \rightarrow P$  of  $(j_U, j_W)$  such that  $h$  is close to  $M_W$  and  $k(a_U) < h(a_W)$ .

*Proof.* (1) implies (2): Fix  $I \in W$  and  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\delta, \alpha)$  witnessing  $U <_{\mathbf{k}} W$ . Let  $Z = [\langle U_\alpha : \alpha \in I \rangle]_W$ . By Łoś's Theorem,  $Z \in \mathcal{B}^{M_W}(j_W(\delta), a_W)$ , and by Lemma 3.2.12,  $j_W^{-1}[Z] = W\text{-}\lim_{i \in I} U_i = U$ . This implies  $j_W[U] \subseteq Z$ .

(2) implies (1): Similar.

(2) implies (3): Fix  $Z \in \mathcal{B}^{M_W}(j_W(\delta), a_W)$  such that  $j_W[U] \subseteq Z$ . Because of the basic structure of ultrafilters, the fact that  $j_W[U] \subseteq Z$  implies that  $j_W^{-1}[Z] = U$ . Let  $h : M_W \rightarrow N$  be the ultrapower of  $M_W$  by  $Z$ . Since  $Z$  concentrates on  $a_W$ ,  $a_Z < h(a_W)$ . By Lemma 3.2.17, there is a unique elementary embedding  $k : M_U \rightarrow N$  such that  $k(a_U) = a_Z$  and  $k \circ j_U = h \circ j_W$ . The former equation implies  $k(a_U) < h(a_W)$ , while the latter equation says that  $(k, h)$  is a comparison of  $(j_U, j_W)$ . Therefore (3) holds.

(3) implies (4): Internal ultrapower embeddings are close.

(4) implies (2): Let  $Z$  be the  $M_W$ -ultrafilter on  $j_W(\delta)$  derived from  $h$  using  $k(a_U)$ . Thus  $Z = h^{-1}[p_{k(a_U)}]$ . (Here  $p_{k(a_U)}$  denotes the principal ultrafilter on  $k(j_U(\delta))$  concentrated at

$k(a_U)$ ; see Definition 3.2.13 and the ensuing discussion.) Since  $h$  is close,  $Z$  belongs to  $M_W$ , and since  $k(a_U) < h(a_W)$ ,  $Z$  concentrates on  $a_W$ . Thus  $Z \in \mathcal{B}^{M_W}(j_W(\delta), a_W)$ . Moreover,

$$j_W^{-1}[Z] = j_W^{-1}[h^{-1}[p_{k(a_U)}]] = j_U^{-1}[k^{-1}[p_{k(a_U)}]] = j_U^{-1}[p_{a_U}] = U$$

In particular,  $j_W[U] \subseteq Z$ , which shows (2).  $\square$

Of course, there are identical characterizations for the nonstrict Ketonen order as well:

**Lemma 3.3.5.** *Suppose  $\delta$  is an ordinal and  $U, W \in \mathcal{B}(\delta)$ . The following are equivalent:*

- (1)  $U \leq_{\mathbf{k}} W$ .
- (2)  $j_W[U]$  extends to an  $M_W$ -ultrafilter  $Z \in \mathcal{B}^{M_W}(j_W(\delta), a_W + 1)$ .
- (3) There is a comparison  $(k, h) : (M_U, M_W) \rightarrow P$  of  $(j_U, j_W)$  such that  $h$  is an internal ultrapower embedding of  $M_W$  and  $k(a_U) \leq h(a_W)$ .
- (4) There is a comparison  $(k, h) : (M_U, M_W) \rightarrow P$  of  $(j_U, j_W)$  such that  $h$  is close to  $M_W$  and  $k(a_U) \leq h(a_W)$ .  $\square$

Lemma 3.3.4 and Lemma 3.3.5 lead to the central linearity theorem for the Ketonen order under UA:

**Theorem 3.3.6 (UA).** *Suppose  $\delta$  is an ordinal and  $U, W \in \mathcal{B}(\delta)$ . Either  $U <_{\mathbf{k}} W$  or  $W \leq_{\mathbf{k}} U$ .*

*Proof.* Let  $(k, h) : (M_U, M_W) \rightarrow N$  be an internal ultrapower comparison of  $(j_U, j_W)$ . If  $k(a_U) < h(a_W)$ , then Lemma 3.3.4 (3) implies  $U <_{\mathbf{k}} W$ . Otherwise,  $h(a_W) \leq k(a_U)$  and so  $W \leq_{\mathbf{k}} U$  by Lemma 3.3.5.  $\square$

This linearity theorem is only interesting, of course, if we know that the Ketonen order is “well defined”: if  $U <_{\mathbf{k}} W$  and  $W <_{\mathbf{k}} U$  held for all  $U, W \in \mathcal{B}(\delta)$ , it wouldn’t be very useful. We now show that in fact the Ketonen order is a wellfounded partial order.

## Basic properties of the Ketonen order

We state the main theorem of this section, which we will prove in pieces:

**Theorem.** *For any ordinal  $\delta$ ,  $(\mathcal{B}(\delta), <_{\mathbf{k}})$  is a strict wellfounded partial order.*

Thus we must show the following facts:

**Proposition 3.3.7.** *For any ordinal  $\delta$ ,  $<_{\mathbf{k}}$  is a transitive relation on  $\mathcal{B}(\delta)$ .*

**Theorem 3.3.8.** *For any ordinal  $\delta$ ,  $<_{\mathbf{k}}$  is a wellfounded relation on  $\mathcal{B}(\delta)$ .*

Let us warm up to this by proving irreflexivity:

**Proposition 3.3.9.** *For any ordinal  $\delta$ ,  $<_{\mathbf{k}}$  is an irreflexive relation on  $\mathcal{B}(\delta)$ .*

*Proof.* Suppose towards a contradiction that  $U \in \mathcal{B}(\delta)$  satisfies  $U <_{\mathbf{k}} U$ . Fix  $I \in U$ , and  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\delta, \alpha)$  such that

$$U = U\text{-}\lim_{\alpha \in I} U_\alpha$$

Define  $A \subseteq \delta$  by induction: put  $\alpha \in A$  if and only if  $A \cap \alpha \notin U_\alpha$ . Then

$$\begin{aligned} A \in U &\iff \{\alpha \in I : A \in U_\alpha\} \in U \\ &\iff \{\alpha \in I : A \cap \alpha \in U_\alpha\} \in U \\ &\iff \{\alpha \in I : \alpha \notin A\} \in U \\ &\iff I \setminus A \in U \end{aligned}$$

Since  $I \in U$  and  $U$  is an ultrafilter, either  $A$  or  $I \setminus A$  must belong to  $U$ . Thus both belong to  $U$ , contradicting that  $U$  is closed under intersections.  $\square$

Notice that the proof does not use the wellfoundedness of  $U$ . We now give two proofs of the transitivity of the Ketonen order.

*Proof of Proposition 3.3.7.* Suppose  $U <_{\mathbf{k}} W \leq_{\mathbf{k}} Z$ . We will show that  $U <_{\mathbf{k}} Z$ . Fix the following objects:

- A set  $I \in W$  and a sequence  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\delta, \alpha)$  such that  $U = W\text{-}\lim_{\alpha \in I} U_\alpha$ .
- A set  $J \in Z$  and a sequence  $\langle W_\beta : \beta \in J \rangle \in \prod_{\beta \in J} \mathcal{B}(\delta, \beta)$  such that  $W = Z\text{-}\lim_{\beta \in J} W_\beta$ .

Since  $I \in W = Z\text{-}\lim_{\beta \in J} W_\beta$ , the set  $J' = \{\beta \in J : I \in W_\beta\}$  belongs to  $Z$ . For  $\beta \in J'$ , we can define  $U'_\beta = W_\beta\text{-}\lim_{\alpha \in I} U_\alpha$ . Thus:

$$\begin{aligned}
U &= W\text{-}\lim_{\alpha \in I} U_\alpha \\
&= (Z\text{-}\lim_{\beta \in J} W_\beta)\text{-}\lim_{\alpha \in I} U_\alpha \\
&= Z\text{-}\lim_{\beta \in J'} (W_\beta\text{-}\lim_{\alpha \in I} U_\alpha) \\
&= Z\text{-}\lim_{\beta \in J'} U'_\beta
\end{aligned}$$

Finally, if  $\beta \in J'$ , then  $\{\alpha \in I : U_\alpha \in \mathcal{B}(\delta, \beta)\} \supseteq I \cap (\beta + 1) \in W_\beta$ , so

$$\langle U'_\beta : \beta \in J' \rangle \in \prod_{\beta \in J'} \mathcal{B}(\delta, \beta)$$

Therefore  $\langle U'_\beta : \beta \in J' \rangle$  witnesses  $U <_{\mathbf{k}} Z$ . □

We are still just warming up, so let us give another proof of the transitivity of the Ketonen order that is more diagrammatic:

*Alternate Proof of Proposition 3.3.7.* Using Lemma 3.3.4, fix the following objects:

- A comparison  $(k_0, h_0) : (M_U, M_W) \rightarrow N_0$  of  $(j_U, j_W)$  such that  $h_0$  is an internal ultra-power embedding of  $M_W$  and  $k_0(a_U) < h_0(a_W)$ .
- A comparison  $(k_1, h_1) : (M_W, M_Z) \rightarrow N_1$  of  $(j_W, j_Z)$  such that  $h_1$  is an internal ultra-power embedding of  $M_Z$  and  $k_1(a_W) \leq h_1(a_Z)$ .

The rest of the proof is contained in Fig. 3.1. Consider the embeddings  $h_0 : M_W \rightarrow N_0$  and  $k_1 : M_W \rightarrow N_1$ . There is a very general construction that yields a comparison of  $(h_0, k_1)$ . Since  $h_0$  is amenable to  $M_W$ , one can define  $k_1(h_0) : N_1 \rightarrow k_1(N_0)$  by shifting the



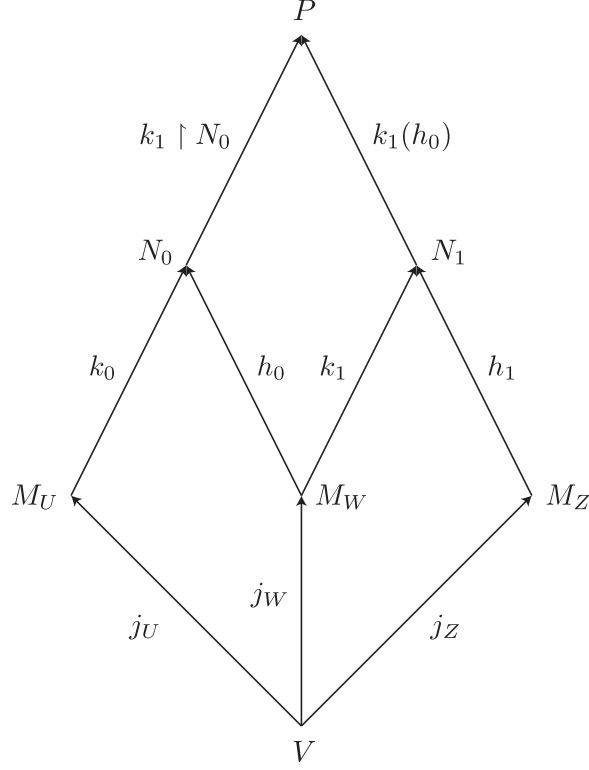


Figure 3.1: The transitivity of the Ketonen order

fragments of  $h_0$  using  $k_1$ . The well-known identity  $(k_1 \upharpoonright N_0) \circ h_0 = k_1(h_0) \circ k_1$  implies that  $(k_1 \upharpoonright N_0, k_1(h_0)) : (N_0, N_1) \rightarrow k_1(N_0)$  is a comparison of  $(h_0, k_1)$ .

It follows easily that  $((k_1 \upharpoonright N_0) \circ k_0, k_1(h_0) \circ h_1)$  is a comparison of  $(j_U, j_Z)$ . Easily  $k_1(h_0) \circ h_0$  is an internal ultrapower embedding of  $M_Z$ . Finally

$$(k_1 \upharpoonright N_0) \circ k_0(a_U) < (k_1 \upharpoonright N_0) \circ h_0(a_W) = k_1(h_0) \circ k_1(a_W) \leq k_1(h_0) \circ h_1(a_Z)$$

Thus  $U <_{\mathbf{k}} Z$  by Lemma 3.3.4. □

We finally turn to wellfoundedness. We will give a combinatorial proof, but the reader can consult Section 3.5 for a diagrammatic approach in a more general context. The proof proceeds by iterating the following *strong transitivity lemma* for the Ketonen order, abstracted from the proof of Proposition 3.3.7:

**Lemma 3.3.10.** *Suppose  $\delta$  is an ordinal,  $U, W \in \mathcal{B}(\delta)$ , and  $U <_{\mathbf{k}} W$ . Suppose  $Z$  is an ultrafilter,  $J$  is a set in  $Z$ , and  $\{W_x : x \in J\} \subseteq \mathcal{B}(\delta)$  is a sequence such that*

$$W = Z\text{-}\lim_{x \in J} W_x$$

*Then there is a set  $J' \subseteq J$  in  $Z$  and  $\{U_x : x \in J'\} \subseteq \mathcal{B}(\delta)$  with  $U_x <_{\mathbf{k}} W_x$  for all  $x \in J'$  such that*

$$U = Z\text{-}\lim_{x \in J'} U_x$$

*Sketch.* Fix  $I \in W$  and  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\delta, \alpha)$  such that  $U = W\text{-}\lim_{\alpha \in I} U_\alpha$ . Let  $J' = \{x \in J : I \in W_x\}$ . For  $x \in J'$ , let  $U_x = W_x\text{-}\lim_{\alpha \in I} U_\alpha$ . Then  $\langle U_\alpha : \alpha \in I \rangle$  witnesses that  $U_x <_{\mathbf{k}} W_x$ . Moreover the calculation in Proposition 3.3.7 shows that  $U = Z\text{-}\lim_{x \in J'} U_x$ .  $\square$

This is more elegantly stated using elementary embeddings:

**Lemma 3.3.11.** *Suppose  $\delta$  is an ordinal,  $U, W \in \mathcal{B}(\delta)$ , and  $U <_{\mathbf{k}} W$ . Suppose  $j : V \rightarrow M$  is an elementary embedding and  $W_* \in j(\mathcal{B}(\delta))$  extends  $j[W]$ . Then there is some  $U_* \in j(\mathcal{B}(\delta))$  extending  $j[U]$  such that  $M \models U_* <_{\mathbf{k}} W_*$ .  $\square$*

Recall now the notation  $U \upharpoonright C$  from Definition 3.2.1, denoting the projection of an ultrafilter  $U$  to a set  $C$  on which it concentrates. We will need the following trivial lemma, which is also implicit in the proof of Proposition 3.3.7:

**Lemma 3.3.12.** *Suppose  $\epsilon$  and  $\delta$  are ordinals,  $U \in \mathcal{B}(\delta)$ , and  $W \in \mathcal{B}(\delta, \epsilon)$ . If  $U \leq_{\mathbf{k}} W$ , then  $U \in \mathcal{B}(\delta, \epsilon)$  and  $U \upharpoonright \epsilon \leq_{\mathbf{k}} W \upharpoonright \epsilon$  in the Ketonen order on  $\mathcal{B}(\epsilon)$ .*

*Proof.* Fix  $I \in W$  and  $\langle U_\alpha : \alpha \in I \rangle \in \prod \mathcal{B}(\delta, \alpha)$  such that  $U = W\text{-}\lim_{\alpha \in I} U_\alpha$ . Then since  $U = W\text{-}\lim_{\alpha \in I} U_\alpha = W\text{-}\lim_{\alpha \in I \cap \epsilon} U_\alpha$  is a limit of ultrafilters concentrating on  $\epsilon$ , so  $U$  itself concentrates on  $\epsilon$ . Moreover  $\langle U_\alpha \upharpoonright \epsilon : \alpha \in I \cap \epsilon \rangle$  witnesses that  $U \upharpoonright \epsilon <_{\mathbf{k}} W \upharpoonright \epsilon$  in the Ketonen order on  $\mathcal{B}(\epsilon)$ .  $\square$

As we prove Theorem 3.3.8, the reader may profit from the observation that the proof consists of the combinatorial core of the proof of the wellfoundedness of the Mitchell order on normal ultrafilters, stripped of all applications of normality and Loś's Theorem.

*Proof of Theorem 3.3.8.* Assume towards a contradiction that there is an ordinal  $\delta$  such that  $<_{\mathbf{k}}$  is illfounded on  $\mathcal{B}(\delta)$ . Fix the least such  $\delta$ . Choose a sequence  $\{U_n : n < \omega\} \subseteq \mathcal{B}(\delta)$  that is  $<_{\mathbf{k}}$ -descending:

$$U_0 >_{\mathbf{k}} U_1 >_{\mathbf{k}} U_2 >_{\mathbf{k}} \cdots$$

For each positive number  $n$ , we will define by recursion a set  $J_n \in U_0$  and a sequence of ultrafilters  $\langle U_\alpha^n : \alpha \in J_n \rangle \in \prod_{\alpha \in J_n} \mathcal{B}(\delta, \alpha)$  such that for all  $n < \omega$ , the following hold:

- $U_n = U\text{-}\lim_{\alpha \in J_n} U_\alpha^n$ .
- If  $n > 1$ , then  $J_n \subseteq J_{n-1}$  and for all  $\alpha \in J_n$ ,  $U_\alpha^n <_{\mathbf{k}} U_\alpha^{n-1}$ .

To start, fix  $J_1 \in U_0$  and  $\langle U_\alpha^1 : \alpha \in J_1 \rangle \in \prod_{\alpha \in J_1} \mathcal{B}(\delta, \alpha)$  witnessing that  $U_1 <_{\mathbf{k}} U_0$ ; that is,

$$U_1 = U_0\text{-}\lim_{\alpha \in J_1} U_\alpha^1$$

Suppose  $n > 1$  and  $J_{n-1} \in U_0$  and  $\langle U_\alpha^{n-1} : \alpha \in J_{n-1} \rangle \in \prod_{\alpha \in J_{n-1}} \mathcal{B}(\delta, \alpha)$  have been defined so that  $U_{n-1} = U\text{-}\lim_{\alpha \in J_{n-1}} U_\alpha^{n-1}$ . Lemma 3.3.10 (with  $U = U_n$ ,  $W = U_{n-1}$ , and  $Z = U_0$ ) yields  $J_n \subseteq J_{n-1}$  and  $\{U_\alpha^n : \alpha \in J_n\} \subseteq \mathcal{B}(\delta)$  such that the two bullet points above are satisfied. We must verify that  $\langle U_\alpha^n : \alpha \in J_n \rangle \in \prod_{\alpha \in J_n} \mathcal{B}(\delta, \alpha)$ . But for any  $\alpha \in J_n$ ,  $U_\alpha^n <_{\mathbf{k}} U_\alpha^{n-1} \in \mathcal{B}(\delta, \alpha)$ , and therefore  $U_\alpha^n \in \mathcal{B}(\delta, \alpha)$  by Lemma 3.3.12, as desired. This completes the recursive definition.

Now let  $J = \bigcap_{n < \omega} J_n$ . For any  $\alpha \in J$ , we have

$$U_\alpha^1 >_{\mathbf{k}} U_\alpha^2 >_{\mathbf{k}} U_\alpha^3 >_{\mathbf{k}} \cdots$$

by the second bullet point above. Since  $U_\alpha^n \in \mathcal{B}(\delta, \alpha)$  for all  $n < \omega$ , Lemma 3.3.12 implies

$$U_\alpha^1 \mid \alpha >_{\mathbf{k}} U_\alpha^2 \mid \alpha >_{\mathbf{k}} U_\alpha^3 \mid \alpha >_{\mathbf{k}} \cdots$$

Thus the restriction of  $<_{\mathbf{k}}$  to  $\mathcal{B}(\alpha)$  is illfounded. This contradicts the minimality of  $\delta$ .  $\square$

Observe that the proof of Theorem 3.3.8 goes through in ZF + DC. The structure of countably complete ultrafilters on ordinals is of great interest in the context of the Axiom

of Determinacy, and so the existence of a combinatorial analog of the Mitchell order in that context raises a number of very interesting structural questions that we will not pursue seriously in this dissertation.

## The global Ketonen order

Lemma 3.3.12 above suggests extending the Ketonen order to an order on ultrafilters that is agnostic about the underlying sets of the ultrafilters involved:

**Definition 3.3.13.** Suppose  $U$  and  $W$  are countably complete ultrafilters on ordinals. The (global) *Ketonen order* is defined as follows:

- $U <_{\mathbf{k}} W$  if  $U \upharpoonright \delta <_{\mathbf{k}} W \upharpoonright \delta$ .
- $U \leq_{\mathbf{k}} W$  if  $U \upharpoonright \delta \leq_{\mathbf{k}} W \upharpoonright \delta$ .

where  $\delta$  is any ordinal such that  $U$  and  $W$  both concentrate on  $\delta$ .

By Lemma 3.3.12, this definition does not conflict with our original definition of the Ketonen order on  $\mathcal{B}(\delta)$ . In fact, various characterizations of the Ketonen order from Lemma 3.3.4 translate smoothly to this context:

**Lemma 3.3.14.** Suppose  $\epsilon$  and  $\delta$  are ordinals,  $U \in \mathcal{B}(\epsilon)$ , and  $W \in \mathcal{B}(\delta)$ . Then the following are equivalent:

- (1)  $U <_{\mathbf{k}} W$ .
- (2) There exist  $I \in W$  and  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\epsilon, \alpha)$  such that  $U = W\text{-}\lim_{\alpha \in I} U_\alpha$ .
- (3)  $j_W[U] \subseteq Z$  extends to an  $M_W$ -ultrafilter  $Z \in \mathcal{B}^{M_W}(j_W(\epsilon), a_W)$ .
- (4) There is a comparison  $(k, h) : (M_U, M_W) \rightarrow P$  of  $(j_U, j_W)$  such that  $h$  is an internal ultrapower embedding of  $M_W$  and  $k(a_U) < h(a_W)$ .

(5) There is a comparison  $(k, h) : (M_U, M_W) \rightarrow P$  of  $(j_U, j_W)$  such that  $h$  is close to  $M_W$  and  $k(a_U) < h(a_W)$ .  $\square$

We have the following simple relationship between the space of an ultrafilter and its position in the Ketonen order:

**Lemma 3.3.15.** *Suppose  $U$  and  $W$  are countably complete ultrafilters on ordinals.*

- If  $\delta_U < \delta_W$ , then  $U <_{\mathbf{k}} W$ .
- If  $U \leq_{\mathbf{k}} W$ , then  $\delta_U \leq \delta_W$ .

*Proof.* To see the first bullet point, note that for any  $\alpha \in [\delta_U, \delta_W)$ ,  $\alpha \geq \delta_U$  and hence  $U$  concentrates on  $\alpha$ . Thus the constant sequence  $\langle U : \alpha \in [\delta_U, \delta_W) \rangle$  belongs to  $\prod_{\alpha \in [\delta_U, \delta_W)} \mathcal{B}(\epsilon, \alpha)$ , and clearly  $U = W\text{-}\lim_{\alpha \in [\delta_U, \delta_W)} U$ . By Lemma 3.3.14,  $U <_{\mathbf{k}} W$ .

The second bullet point is immediate from Lemma 3.3.12.  $\square$

The one issue with the global Ketonen order, which presents only notational difficulties, is that in this generalized context,  $\leq_{\mathbf{k}}$  is no longer the irreflexive part of  $<_{\mathbf{k}}$ . Instead we have the following fact, where  $=_E$  is the change-of-space relation defined in Definition 3.2.2:

**Lemma 3.3.16.** *Suppose  $U$  and  $W$  are countably complete ultrafilters on ordinals. Then  $U \leq_{\mathbf{k}} W$  if and only if  $U <_{\mathbf{k}} W$  or  $U =_E W$ .*  $\square$

Since the  $=_E$ -relation convenient to restrict the global Ketonen order to the class of tail uniform ultrafilters  $\text{Un}$ :

**Lemma 3.3.17.** *Suppose  $U, W \in \text{Un}$ . Then  $U \leq_{\mathbf{k}} W$  if and only if  $U <_{\mathbf{k}} W$  or  $U = W$ .*

**Definition 3.3.18.** For any ordinal  $\delta$ , let  $\text{Un}(\delta)$  denote the set of tail uniform ultrafilters  $U$  such that  $\delta_U \leq \delta$ .

**Lemma 3.3.19.** *For all ordinals  $\delta$ , the map  $\phi : \mathcal{B}(\delta) \rightarrow \text{Un}(\delta)$  defined by  $\phi(U) = U \upharpoonright \delta_U$  is an isomorphism from  $(\mathcal{B}(\delta), <_{\mathbf{k}}, \leq_{\mathbf{k}})$  to  $(\text{Un}(\delta), <_{\mathbf{k}}, \leq_{\mathbf{k}})$ . Thus the Ketonen order is a set-like wellfounded partial order on  $\text{Un}$ .*  $\square$

The following easy lemma generalizes our work in this section, showing that not only do the various Ketonen orders on  $\mathcal{B}(\delta)$  cohere, but in fact, order-preserving maps between ordinals induce order-preserving maps on their associated Ketonen orders:

**Lemma 3.3.20.** *Suppose  $\epsilon \leq \delta$  are ordinals and  $f : \epsilon \rightarrow \delta$  is an increasing function. For any  $U, W \in \mathcal{B}(\epsilon)$ ,  $U <_{\mathbf{k}} W$  in the Ketonen order on  $\mathcal{B}(\epsilon)$  if and only if  $f_*(U) <_{\mathbf{k}} f_*(W)$  in the Ketonen order on  $\mathcal{B}(\delta)$ .*

*Sketch.* Fix  $I \in W$  and  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\epsilon, \alpha)$  such that  $U = W\text{-}\lim_{\alpha \in I} U_\alpha$ . Let  $J = f[I]$ , and for  $\alpha \in I$ , let  $Z_{f(\alpha)} = f_*(W_\alpha)$ . Thus  $J \in f_*(W)$ . Moreover, for all  $\alpha \in I$ ,  $f(\alpha) \supseteq f[\alpha] \in Z_{f(\alpha)}$  since  $f$  is increasing, so  $Z_{f(\alpha)} \in \mathcal{B}(\delta, f(\alpha))$ . Thus  $\langle Z_\beta : \beta \in J \rangle \in \prod_{\beta \in J} \mathcal{B}(\delta, \beta)$ . Finally

$$f_*(U) = W\text{-}\lim_{\alpha \in I} f_*(W_\alpha) = f_*(W)\text{-}\lim_{\beta \in f[I]} Z_\alpha$$

It follows that  $f_*(U) <_{\mathbf{k}} f_*(W)$ . The other direction is similar. □

### 3.4 Orders on ultrafilters

In this section, we discuss some generalizations of the Ketonen order and compare the Ketonen order with other well-known orders.

#### The Mitchell order

The Ketonen order can be seen as a combinatorial generalization of the Mitchell order on normal ultrafilters. We will discuss the relationship between the Ketonen order and the generalization of the Mitchell order to arbitrary countably complete ultrafilters at length in Chapter 4, but for now, we satisfy ourselves by proving that the Ketonen and Mitchell orders coincide on normal ultrafilters.

**Theorem 3.4.1.** *Suppose  $U$  and  $W$  are normal ultrafilters. Then  $U \triangleleft W$  if and only if  $U <_{\mathbf{k}} W$ .*

*Proof.* Suppose first that  $U$  and  $W$  are normal ultrafilters on distinct cardinals  $\kappa$  and  $\lambda$ . Clearly  $U \triangleleft W$  if and only if  $\kappa < \lambda$ . Moreover by Lemma 3.3.15,  $U <_{\mathbf{k}} W$  if and only if  $\kappa < \lambda$ . Thus  $U \triangleleft W$  if and only if  $U <_{\mathbf{k}} W$ .

Assume instead that  $U$  and  $W$  lie on the same cardinal  $\kappa$ . By Lemma 2.2.28,  $U$  and  $W$  are  $\kappa$ -complete and  $\kappa = a_U = a_W$ . The key fact we use is that since  $\text{CRT}(j_W) = \kappa$ ,  $j_W(A) \cap \kappa = A$  for all  $A \subseteq \kappa$ .

Suppose first that  $U \triangleleft W$ . Then  $U \in M_W$ . Working in  $M_W$ , consider the projection  $Z = U \upharpoonright j_W(\kappa) \in \mathcal{B}^{M_W}(j_W(\kappa), \kappa)$ . For any  $A \subseteq \kappa$ ,  $j_W(A) \cap \kappa = A \in U$ , or in other words,  $j_W(A) \in Z$ . In other words,  $j_W[U] \subseteq Z$ , so by Lemma 3.3.4,  $U \triangleleft W$ .

Conversely, suppose  $U <_{\mathbf{k}} W$ . Fix  $Z \in \mathcal{B}^{M_W}(j_W(\kappa), \kappa)$  such that  $U = j_W^{-1}[Z]$ . Suppose  $A \subseteq \kappa$ . Then  $A \in U$  if and only if  $j_W(A) \cap \kappa \in Z$  if and only if  $A \in Z$ . Therefore  $U = Z \upharpoonright \kappa$ , so  $U \in M_W$ . This implies  $U \triangleleft W$ .  $\square$

Thus the wellfoundedness of the Mitchell order follows from the wellfoundedness of the Ketonen order. Notice that this theorem gives another proof of the linearity of the Mitchell order on normal ultrafilters under UA. Finally, the proof has the following consequence:

**Corollary 3.4.2.** *Suppose  $\kappa$  is a cardinal,  $U \in \mathcal{B}(\kappa)$ , and  $W$  is a normal ultrafilter on  $\kappa$ . Then  $U <_{\mathbf{k}} W$  if and only if  $U \triangleleft W$ .*  $\square$

Thus the Ketonen predecessors of a normal ultrafilter  $W$  on  $\kappa$  are precisely  $\mathcal{B}(\kappa) \cap M_W$ . We will see various nontrivial generalizations of this fact to more general types of ultrafilters than normal ones.

## The Rudin-Keisler order

In this section, we briefly recall the theory of the Rudin-Keisler order and explain its relationship with the Ketonen order. We also introduce the notion of an *incompressible ultrafilter*, which will be a useful technical tool.

The Rudin-Keisler order is defined in terms of pushforward ultrafilters (Definition 3.2.14).

**Definition 3.4.3.** Suppose  $U$  and  $W$  are ultrafilters. The *Rudin-Keisler order* is defined by setting  $U \leq_{\text{RK}} W$  if there is a function  $f : I \rightarrow X$  such that  $f_*(W) = U$  where  $I \in W$  and  $X$  is the underlying set of  $U$ .

We could of course take  $I$  to be the underlying set of  $W$  above. The Rudin-Keisler order is a (nonstrict) preorder on the class of ultrafilters. For us, the most important characterization of the Rudin-Keisler order uses elementary embeddings:

**Lemma 3.4.4.** *Suppose  $U$  and  $W$  are ultrafilters. Then  $U \leq_{\text{RK}} W$  if and only if there is an elementary embedding  $k : M_U \rightarrow M_W$  such that  $k \circ j_U = j_W$ .*

*Proof.* Let  $X$  be the underlying set of  $U$ .

First assume  $U \leq_{\text{RK}} W$ . Fix  $I \in W$  and  $f : I \rightarrow X$  such that  $f_*(W) = U$ . Let  $a = [f]_W$ , so by Lemma 3.2.16,  $U$  is the ultrafilter on  $X$  derived from  $j_W$  using  $a$ . Let  $k : M_U \rightarrow M_W$  be the factor embedding, so  $k(a_U) = a$  and  $k \circ j_U = j_W$ . Then  $k$  witnesses the conclusion of the lemma.

Conversely, assume there is an elementary embedding  $k : M_U \rightarrow M_W$  such that  $k \circ j_U = j_W$ . Let  $b = k(a_U)$ . Then  $b \in j_W(X)$ . On the one hand,  $U$  is equal to the ultrafilter on  $X$  derived from  $j_W$  using  $b$ . (Explicitly:  $U = j_U^{-1}[p_{a_U}] = j_U^{-1}[k^{-1}[k(p_{a_U})]] = j_W^{-1}[p_b]$ .) Fix  $I \in W$  and  $f : I \rightarrow X$  such that  $[f]_W = b$ . Then by Lemma 3.2.16,  $f_*(W)$  is the ultrafilter on  $X$  derived from  $j_W$  using  $b$ , or in other words  $f_*(W) = U$ . Thus  $U \leq_{\text{RK}} W$  as desired.  $\square$

A second combinatorial formulation of the Rudin-Keisler order is in terms of partitions which will become relevant when we study indecomposability (especially in Theorem 7.5.24):

**Lemma 3.4.5.** *Suppose  $U$  and  $W$  are ultrafilters. Let  $X$  be the underlying set of  $U$ . Then  $U \leq_{\text{RK}} W$  if and only if there is a sequence of pairwise disjoint sets  $\langle Y_x : x \in X \rangle$  such that  $U = \{A \subseteq X : \bigcup_{x \in A} Y_x \in W\}$ .*  $\square$

The following is the fundamental theorem of the Rudin-Keisler order:



**Theorem 3.4.6.** *Suppose  $U$  and  $W$  are ultrafilters. Then  $U \cong W$  if and only if  $U \leq_{\text{RK}} W$  and  $W \leq_{\text{RK}} U$ .*

We sketch the proof even though we do not need it in what follows. This involves a very interesting rigidity theorem for pushforwards:

**Lemma 3.4.7.** *Suppose  $U$  is an ultrafilter on  $X$  and  $f : X \rightarrow X$  is a function. If  $f_*(U) = U$  then  $f(x) = x$  for  $U$ -almost all  $x \in X$ .*

*Proof.* Assume  $f : X \rightarrow X$  is such that  $f(x) \neq x$  for all  $x \in X$ . We will show that  $f_*(U) \neq U$ .

**Claim.** *There is a partition  $X = A_0 \cup A_1 \cup A_2$  such that  $f[A_n] \subseteq X \setminus A_n$  for  $n = 0, 1, 2$ .*

*Sketch.* Consider the directed graph  $G$  with vertices  $X$  and a directed edge from  $x$  to  $f(x)$  for each  $x \in X$ . Our claim above amounts to the fact that  $G$  is 3-colorable. It suffices to show that each connected subgraph  $H \subseteq G$  is 3-colorable. Therefore suppose  $H$  is a connected subgraph of  $G$ . The key point is that  $H$  contains at most one cycle (since  $G$  is a “functional graph”), so one obtains an acyclic graph  $H'$  by removing an edge of  $H$  if necessary. Since  $H'$  is acyclic,  $H'$  is 2-colorable. By changing the color of at most one vertex in the coloring of  $H'$ , one obtains a 3-coloring of  $H$ .  $\square$

Since  $A_0 \cup A_1 \cup A_2 = X \in U$ , either  $A_0, A_1$ , or  $A_2$  belongs to  $U$ . Assume without loss of generality that  $A_0 \in U$ . Then  $X \setminus A_0 \supseteq f[A_0] \in f_*(U)$ , so  $X \setminus A_0 \in f_*(U)$  as desired.  $\square$

Let us reformulate this in terms of ultrapowers:

**Theorem 3.4.8.** *Suppose  $U$  is an ultrafilter and  $k : M_U \rightarrow M_U$  is an elementary embedding such that  $k \circ j_U = j_U$ . Then  $k$  is the identity.*

*Proof.* Let  $X$  be the underlying set of  $U$ . Fix  $X \in U$  and a function  $f : X \rightarrow X$  such that  $[f]_U = k(a_U)$ . Then by Lemma 3.2.16,  $f_*(U)$  is the ultrafilter on  $X$  derived from  $j_U$  using  $k(a_U)$ , which is easily seen to equal  $U$ . (Yet another inverse image calculation:  $j_U^{-1}[p_{k(a_U)}] = (k \circ j_U)^{-1}[p_{k(a_U)}] = j_U^{-1}[k^{-1}[p_{k(a_U)}]] = j_U^{-1}[p_{a_U}] = U$ .) Therefore by Lemma 3.4.7,  $f \upharpoonright X = \text{id}$

for some  $I \in U$ . Thus  $k(a_U) = [f]_U = a_U$ . It follows that  $k \upharpoonright j_U[V] \cup \{a_U\}$  is the identity, so  $k \upharpoonright M_U$  is the identity since  $M_U = H^{M_U}(j_U[V] \cup \{a_U\})$ .  $\square$

Lemma 3.4.7 immediately implies Theorem 3.4.6:

*Proof of Theorem 3.4.6.* Let  $X$  be the underlying set of  $U$  and  $Y$  be the underlying set of  $W$ . The trivial direction is to prove that  $U \cong W$  implies  $U \leq_{\text{RK}} W$  and  $W \leq_{\text{RK}} U$ . Fix  $I \in U$ ,  $J \in W$ , and a bijection  $f : I \rightarrow J$  such that for all  $A \subseteq I$ ,  $A \in U$  if and only if  $f[A] \in W$ . Viewing  $f$  as a function  $p : I \rightarrow Y$ , we have  $W = p_*(U)$ . Viewing  $f^{-1}$  as a function  $p : J \rightarrow X$ , we have  $U = p_*(W)$ . This implies  $W \leq_{\text{RK}} U$  and  $U \leq_{\text{RK}} W$ .

Conversely assume  $U \leq_{\text{RK}} W$  and  $W \leq_{\text{RK}} U$ . Fix  $I \in U$  and  $f : I \rightarrow Y$  such that  $f_*(U) = W$ . Fix  $J \in W$  and  $g : J \rightarrow X$  such that  $g_*(W) = U$ . We claim there is a set  $I' \subseteq I$  such that  $I' \in U$  and  $g \circ f \upharpoonright I'$  is the identity. To see this, note that  $(g \circ f)_*(U) = g_*(f_*(U)) = g_*(W) = U$ . Therefore by Lemma 3.4.7, there is a set  $I' \subseteq I$  such that  $I' \in U$  and  $g \circ f$  is the identity.  $\square$

Theorem 3.4.6 motivates the following definition:

**Definition 3.4.9.** The *strict Rudin-Keisler order* is defined on ultrafilters  $U$  and  $W$  by setting  $U <_{\text{RK}} W$  if  $U \leq_{\text{RK}} W$  and  $W \not\cong U$ .

We now discuss the structure of the Rudin-Keisler order on countably complete ultrafilters and its relationship to the Ketonen order. To facilitate this discussion, we introduce a revised version of the Rudin-Keisler order. Recall that a function  $f$  defined on a set of ordinals  $I$  is *regressive* if  $f(\alpha) < \alpha$  for all  $\alpha \in I$ .

**Definition 3.4.10.** Suppose  $U$  and  $W$  are ultrafilters on ordinals. Let  $X$  be the underlying set of  $U$ . The *revised Rudin-Keisler order* is defined by setting  $U <_{\text{rk}} W$  if there is a set  $I \in W$  and a regressive function  $f : I \rightarrow X$  such that  $f_*(W) = U$ .

**Lemma 3.4.11.** *If  $U$  and  $W$  are ultrafilters on ordinals, then  $U <_{\text{rk}} W$  if and only if there is an elementary embedding  $k : M_U \rightarrow M_W$  such that  $k \circ j_U = j_W$  and  $k(a_U) < a_W$ .*  $\square$

**Corollary 3.4.12.** *The Ketonen order and the Rudin-Keisler order extend the revised Rudin-Keisler order.*  $\square$

**Lemma 3.4.13.** *For any ultrafilter  $U$ , the collection of tail uniform ultrafilters isomorphic to  $U$  is linearly ordered by the revised Rudin-Keisler order.*

*Proof.* Suppose  $W_0 \cong U \cong W_1$  are tail uniform ultrafilters. Then  $W_0 <_{\text{rk}} W_1$  if and only if  $M_U \models a_{W_0} < a_{W_1}$ .  $\square$

We now introduce a concept that is very useful in the study of countably complete ultrafilters. (The same concept was considered by Ketonen [13], who called them normalized ultrafilters.)

**Definition 3.4.14.** A tail uniform ultrafilter  $U$  on an ordinal  $\lambda$  is *incompressible* if for any set  $I \in U$ , no regressive function on  $I$  is one-to-one.

**Lemma 3.4.15.** *Suppose  $U$  is tail uniform. The following are equivalent:*

(1)  $U$  is incompressible.

(2) If  $W <_{\text{rk}} U$ , then  $W <_{\text{RK}} U$ .  $\square$

**Lemma 3.4.16.** *A tail uniform ultrafilter  $U$  is incompressible if and only if it is the  $<_{\text{rk}}$ -minimum element of  $C = \{U' \in \text{Un} : U' \cong U\}$ .*

*Proof.* By Lemma 3.4.15,  $U$  is an  $<_{\text{rk}}$ -minimal element of  $C$ . Since  $<_{\text{rk}}$  linearly orders  $C$  by Lemma 3.4.13,  $U$  is the  $<_{\text{rk}}$ -minimum element of  $C$ .  $\square$

**Corollary 3.4.17.** *An ultrafilter is isomorphic to at most one incompressible ultrafilter.*  $\square$

**Lemma 3.4.18.** *Suppose  $U$  is tail uniform ultrafilter on  $\delta$ . Then the following are equivalent:*

(1)  $U$  is incompressible.

(2)  $a_U$  is the least ordinal  $a$  of  $M_U$  such that  $M_U = H^{M_U}(j_U[V] \cup \{a\})$ .

(3)  $a_U$  is the largest ordinal  $a$  of  $M_U$  such that  $a \neq j_U(f)(b)$  for any function  $f : \delta \rightarrow \delta$  and  $b < a$ . □

If  $U$  is countably complete, then the collection of tail uniform ultrafilters isomorphic to  $U$  is *wellordered* by  $<_{\text{rk}}$ , and therefore it has a minimum element. The following is the key existence theorem for incompressible ultrafilters:

**Lemma 3.4.19.** *Any countably complete ultrafilter  $U$  is isomorphic to a unique incompressible ultrafilter  $W$  which can be obtained in any of the following ways:*

- $W$  is the  $<_{\text{rk}}$ -minimum element of the isomorphism class of  $U$ .
- $W = f_*(U)$  where  $f : \delta_U \rightarrow \delta_U$  is the least one-to-one function modulo  $U$ .
- $W$  is the tail uniform ultrafilter derived from  $j_U$  using  $\alpha$  where  $\alpha$  is the ordinal defined in either of the following ways:
  - $\alpha$  is least such that  $M_U = H^{M_U}(j_U[V] \cup \{\alpha\})$ .
  - $\alpha$  is largest such that  $\alpha \neq j_U(f)(\beta)$  for any  $\beta < \alpha$ . □

What makes incompressible ultrafilters useful is the following dual to Lemma 3.4.15:

**Proposition 3.4.20.** *Suppose  $U$  is incompressible and  $W$  is an ultrafilter on an ordinal. If  $U <_{\text{RK}} W$  then  $U <_{\text{rk}} W$ .*

*Proof.* Assume  $U <_{\text{RK}} W$ . Fix  $k : M_U \rightarrow M_W$  such that  $k \circ j_U = j_W$ . Since  $U \not\cong W$ ,  $k$  is not an isomorphism. It follows that  $a_W \notin k[M_U]$ : otherwise  $j_W[V] \cup \{a_W\} \subseteq k[M_U]$  and so  $M_W = H^{M_W}(j_W[V] \cup \{a_W\}) \subseteq k[M_U]$ , and therefore  $k$  is surjective and hence an isomorphism.

To show that  $U <_{\text{rk}} W$ , it suffices by Lemma 3.4.11 to show that  $k(a_U) < a_W$ . Suppose not. Then  $a_W \leq k(a_U)$ , and since  $a_W \notin k[M_U]$ , in fact  $a_W < k(a_U)$ . Since  $M_W =$

$H^{M_W}(j_W[V] \cup \{a_W\})$  we can fix a function  $f : \delta_W \rightarrow \delta_W$  such that  $j_W(f)(a_W) = k(a_U)$ . Since  $a_W < k(a_U)$ ,

$$M_W \models \exists \xi < k(a_U) \ j_W(f)(\xi) = k(a_U)$$

Since  $j_W(f) = k(j_U(f))$ , the elementarity of  $k : M_U \rightarrow M_W$  implies

$$M_U \models \exists \xi < a_U \ j_U(f)(\xi) = a_U$$

This contradicts Lemma 3.4.18 (3), which in particular states that  $a_U \neq j_U(f)(\xi)$  for any  $\xi < a_U$ . □

**Corollary 3.4.21.** *The strict Rudin-Keisler order and the revised Rudin-Keisler order coincide on incompressible ultrafilters.* □

**Corollary 3.4.22.** *The Ketonen order extends the strict Rudin-Keisler order on countably complete incompressible ultrafilters.* □

We remark that given Corollary 3.4.22, one might guess that  $<_{\text{rk}} = \leq_{\text{RK}} \cap <_{\mathbf{k}}$ , but it is not hard to construct a counterexample under weak large cardinal assumptions.

**Corollary 3.4.23** (Solovay). *The strict Rudin-Keisler order is wellfounded on countably complete ultrafilters.*

*Proof.* Suppose towards a contradiction that

$$U_0 >_{\text{RK}} U_1 >_{\text{RK}} U_2 >_{\text{RK}} \cdots$$

is a descending sequence of countably complete ultrafilters in the strict Rudin-Keisler order. For each  $n$ , let  $W_n$  be the unique incompressible ultrafilter isomorphic to  $U_n$ . Then

$$W_0 >_{\text{RK}} W_1 >_{\text{RK}} W_2 >_{\text{RK}} \cdots$$

since the strict Rudin-Keisler order is isomorphism invariant. But by Corollary 3.4.22, the Ketonen order extends the strict Rudin-Keisler order on countably complete incompressible ultrafilters, and therefore

$$W_0 >_{\mathbf{k}} W_1 >_{\mathbf{k}} W_2 >_{\mathbf{k}} \cdots$$

This contradicts the wellfoundedness of the Ketonen order (Lemma 3.3.19). □

Note that this yields another proof of Theorem 3.4.6 in the case that  $U$  and  $W$  are countably complete.

## 3.5 Variants of the Ketonen order

### Minimality of internal embeddings

In this subsection, we study an extension of the Ketonen order that provides some insight into arbitrary extender embeddings (as opposed to just ultrapower embeddings). This order also clarifies the connection between the Ketonen order and the mouse order from inner model theory (which is also known as the Dodd-Jensen order). Using these ideas, we prove a lemma (Theorem 3.5.10) that states that if  $N$  and  $M$  are transitive models and  $j : N \rightarrow M$  is an elementary embedding that is definable over  $N$  from parameters, then  $j(\alpha) \leq k(\alpha)$  for any other elementary embedding  $k : N \rightarrow M$ .

**Definition 3.5.1.** A *pointed model* is a structure  $(M, \xi)$  such that  $M$  is a transitive model of ZFC and  $\xi \in \text{Ord}^M$ . If  $\mathcal{M} = (M, \xi)$  is a pointed model, then  $\xi_{\mathcal{M}} = \xi$ .

We allow pointed models  $(M, \xi)$  where  $M$  is a proper class. We abuse notation by confusing a pointed model  $\mathcal{M} = (M, \xi)$  with its underlying set  $M$ . We therefore sometimes denote  $\xi$  by  $\xi_M$  instead of  $\xi_{\mathcal{M}}$ .

When we discuss elementary embeddings of pointed models, we never impose elementarity in the language of a pointed model (i.e., with a distinguished constant for  $\xi$ ), only elementarity in the language of set theory.

**Definition 3.5.2.** Suppose  $N$  and  $M$  are transitive models of ZFC. An elementary embedding  $j : N \rightarrow M$  is:

- an *extender embedding* if  $j$  is cofinal and  $M = H^M(j[M] \cup S)$  for some  $S \in M$ .

- an *internal extender embedding* if it is furthermore definable over  $N$ .

**Definition 3.5.3.** Suppose  $M$  and  $N$  are pointed models.

The *Ketonen order on models* is defined on  $M$  and  $N$  by setting  $M <_{\mathbf{k}} N$  if there are embeddings  $(k, h) : (M, N) \rightarrow P$  such that  $k(\xi_M) < h(\xi_N)$  and  $h$  is an internal extender embedding of  $N$ .

The *nonstrict Ketonen order* is defined on  $M$  and  $N$  by setting  $M \leq_{\mathbf{k}} N$  if there are embeddings  $(k, h) : (M, N) \rightarrow P$  such that  $k(\xi_M) \leq h(\xi_N)$  and  $h$  is an internal extender embedding of  $N$ .

*Ketonen equivalence* is defined on  $M$  and  $N$  by setting  $M =_E N$  if  $M \leq_{\mathbf{k}} N$  and  $N \leq_{\mathbf{k}} M$ .

The transitivity for the Ketonen order on pointed models uses a trivial “comparison lemma” that is provable in ZFC.

**Lemma 3.5.4.** *Suppose  $M$ ,  $N_0$ , and  $N_1$  are transitive models of ZFC. Suppose  $h : M \rightarrow N_0$  is an internal extender embedding and  $k : M \rightarrow N_1$  is a cofinal elementary embedding. Then there is a comparison  $(\ell, i) : (N_0, N_1) \rightarrow P$  of  $(h, k)$  such that  $i$  is an internal extender embedding of  $N_1$ .*

*Proof.* Let  $\ell = k \upharpoonright N_0$  and  $i = k(h)$ . Then  $i$  is an internal extender embedding and  $\ell \circ h = k \circ h = k(h) \circ k = i \circ k$ , so  $(\ell, i)$  is a comparison of  $(h, k)$ .  $\square$

**Lemma 3.5.5.** *If  $M_0 <_{\mathbf{k}} M_1 \leq_{\mathbf{k}} M_2$ , then  $M_0 <_{\mathbf{k}} M_2$ .*

*Proof.* Suppose  $M_0 <_{\mathbf{k}} M_1 \leq_{\mathbf{k}} M_2$ . Let  $(k_0, h_0) : (M_0, M_1) \rightarrow N_0$  witness  $M_0 <_{\mathbf{k}} M_1$ . Let  $(k_1, h_1) : (M_1, M_2) \rightarrow N_1$  witness  $M_1 \leq_{\mathbf{k}} M_2$ . Applying Lemma 3.5.4, let

$$(\ell, i) : (N_0, N_1) \rightarrow P$$

be a comparison of  $(h_0, k_1)$  such that  $i$  is an internal extender embedding of  $N_1$ . Let  $k = \ell \circ k_0$  and let  $h = i \circ h_1$ . Then  $(k, h) : (M_0, M_2) \rightarrow P$  and  $h$  is an internal extender embedding of  $M_2$  since it is the composition of the internal extender embeddings  $i$  and  $h_1$ . Finally,

$$k(\xi_{M_0}) = \ell \circ k_0(\xi_{M_0}) < \ell \circ h_0(\xi_{M_1}) = i \circ k_1(\xi_{M_1}) \leq i \circ h_1(\xi_{M_2}) = h(\xi_{M_2}) \quad \square$$

We will prove the wellfoundedness of the Ketonen order on pointed models that satisfy a very weak form of iterability.

**Definition 3.5.6.** A transitive model  $M$  of ZFC is  $\omega$ -linearly iterable if the following holds.

Suppose

$$M = M_0 \xrightarrow{h_0} M_1 \xrightarrow{h_1} M_2 \xrightarrow{h_2} \dots$$

is such that for all  $i < \omega$ ,  $h_i : M_i \rightarrow M_{i+1}$  is an internal extender embedding, then its direct limit is wellfounded.

The following is a well-known fact, versions of which are due to Gaifman, Kunen, and Mitchell (see [14]):

**Lemma 3.5.7.** *Suppose  $M$  is a model of ZFC such that  $\text{Ord}^M$  has uncountable cofinality. Then  $M$  is  $\omega$ -linearly iterable. Similarly, any inner model is  $\omega$ -linearly iterable.  $\square$*

The proof of the wellfoundedness of the Ketonen order on pointed models is based on the proof of the wellfoundedness of the Dodd-Jensen order.

**Theorem 3.5.8.** *The Ketonen order is wellfounded on  $\omega$ -linearly iterable pointed models.*

*Proof.* To simplify notation, we isolate the main step of the proof as a lemma:

**Lemma 3.5.9.** *Suppose that  $M_0 >_{\mathbf{k}} M_1 >_{\mathbf{k}} M_2 >_{\mathbf{k}} \dots$  is a descending sequence of pointed models. Then there is a descending sequence  $N_0 >_{\mathbf{k}} N_1 >_{\mathbf{k}} N_2 >_{\mathbf{k}} \dots$  of pointed models and an internal extender embedding  $h : M_0 \rightarrow N_0$  with  $\xi_{N_0} < h(\xi_{M_0})$ .*

*Proof.* The proof is illustrated by Fig. 3.2. Let  $(h_i, k_i) : (M_i, M_{i+1}) \rightarrow N_i$  witness  $M_i >_{\mathbf{k}} M_{i+1}$ . We endow  $N_i$  with the structure of a pointed model by letting  $\xi_{N_i} = k_i(\xi_{M_{i+1}})$ .

Setting  $h = h_0$ , particular,  $h$  is an internal extender embedding and  $\xi_{N_0} < h(\xi_{M_0})$ . It remains to verify that  $N_0 >_{\mathbf{k}} N_1 >_{\mathbf{k}} N_2 >_{\mathbf{k}} \dots$ . Fix  $i < \omega$ . By Lemma 3.5.4, there is a comparison  $(h'_i, k'_i) : (N_i, N_{i+1}) \rightarrow P_i$  of  $(k_i, h_{i+1})$  such that  $h'_i$  is an internal extender



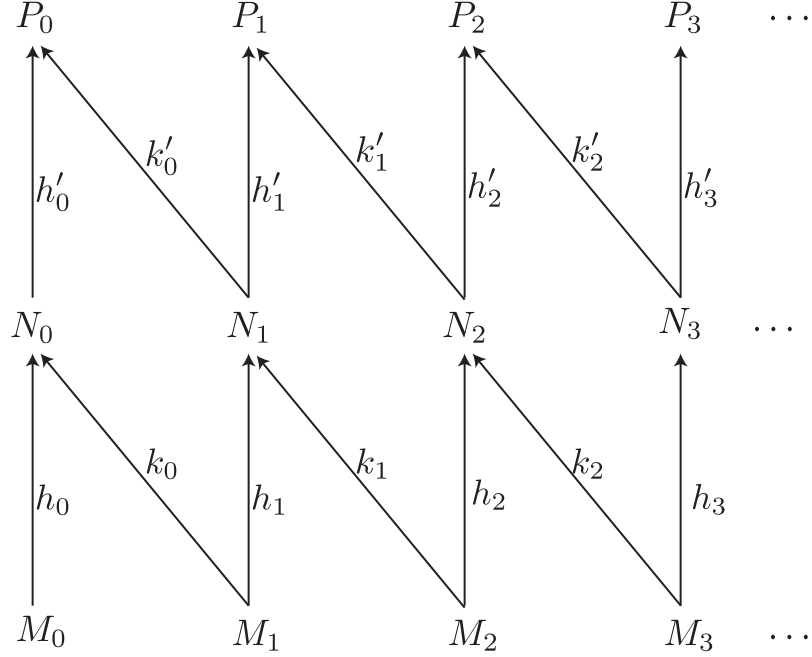


Figure 3.2: The proof of Lemma 3.5.9

embedding of  $N_i$ . As in Lemma 3.5.5,

$$h'_i(\xi_{N_i}) = h'_i(k_i(\xi_{M_{i+1}})) = k'_i(h_{i+1}(\xi_{M_{i+1}})) > k'_i(k_{i+1}(\xi_{M_{i+1}})) = k'_i(\xi_{N_{i+1}})$$

and hence  $N_i >_{\mathbb{k}} N_{i+1}$ . □

Now suppose towards a contradiction that  $M_0^0 >_{\mathbb{k}} M_1^0 >_{\mathbb{k}} M_2^0 >_{\mathbb{k}} \dots$  is a descending sequence of  $\omega$ -linearly iterable pointed models. By recursion, using the lemma, one obtains sequences  $M_0^i >_{\mathbb{k}} M_1^i >_{\mathbb{k}} M_2^i >_{\mathbb{k}} \dots$  and internal extender embeddings  $h^i : M_0^i \rightarrow M_0^{i+1}$  with  $\xi_{M_0^{i+1}} < h^i(\xi_{M_0^i})$  for all  $i < \omega$ . But then the iteration

$$M_0^0 \xrightarrow{h^0} M_0^1 \xrightarrow{h^1} M_0^2 \xrightarrow{h^2} \dots$$

has an illfounded direct limit, which contradicts that  $M_0^0$  is  $\omega$ -linearly iterable. □

The wellfoundedness of the Ketonen order on pointed models has some useful consequences. Of course, it provides an alternate proof of the wellfoundedness of the Ketonen order:

*Alternate Proof of Theorem 3.3.8.* For  $U \in \text{Un}$ , let  $\Phi(U) = (M_U, a_U)$ . Then for any  $U \in \text{Un}$ ,  $\Phi(U)$  is an  $\omega$ -linearly iterable pointed model. Moreover, if  $U <_{\mathbb{k}} W$ , then  $\Phi(U) <_{\mathbb{k}} \Phi(W)$  since internal ultrapower embeddings are internal extender embeddings. Thus the Ketonen order is wellfounded on  $\text{Un}$  since by Theorem 3.5.8, the Ketonen order is wellfounded on  $\omega$ -linearly iterable models.  $\square$

More interestingly, Theorem 3.5.8 implies a coarse version of the Dodd-Jensen Lemma (proved for example, [15]):

**Theorem 3.5.10.** *Suppose  $M$  is an  $\omega$ -linearly iterable model. Suppose  $h, k : M \rightarrow N$  are elementary embeddings and  $h$  is an internal extender embedding of  $M$ . Then for all  $\alpha \in \text{Ord}^M$ ,  $h(\alpha) \leq k(\alpha)$ .*

*Proof.* Suppose towards a contradiction that  $k(\alpha) < h(\alpha)$ . Then  $(k, h) : (M, M) \rightarrow N$  witnesses  $(M, \alpha) <_{\mathbb{k}} (M, \alpha)$ , contradicting Theorem 3.5.8.  $\square$

The idea of generalizing arguments from inner model theory to prove results like Theorem 3.5.10 is due to Woodin [10], who proved the similar theorem that if  $M$  and  $N$  are models of ZFC and  $M$  is finitely generated, then there is at most one close embedding from  $M$  to  $N$ . Woodin's theorem actually follows from the restriction of Theorem 3.5.10 to ultrapower embeddings.

By tracing through the proof of this theorem, one can prove the following fact, which is really a theorem scheme:

**Theorem 3.5.11.** *Suppose  $M$  and  $N$  are inner models,  $h, k : M \rightarrow N$  are elementary embeddings. If  $h$  is definable over  $M$ , then  $h(\alpha) \leq k(\alpha)$  for all ordinals  $\alpha$ .*  $\square$

There are some metamathematical difficulties involving the linear iterability of an inner model  $M$  by an  $\omega$ -sequence of definable embeddings: it is not in general clear that this is first-order expressible in the language of set theory with a predicate for  $M$ . The iterability

required for the proof of Theorem 3.5.10, however, can be stated and proved. We omit the proof since we have no applications of this more general theorem.

## The seed order

We now define the seed order, a variant of the Ketonen order that uses fully internal ultrapower comparisons.

**Definition 3.5.12.** Suppose  $U$  and  $W$  are countably complete ultrafilters on ordinals.

The *seed order* is defined by setting  $U <_S W$  if there is an internal ultrapower comparison  $(k, h)$  of  $(j_U, j_W)$  such that  $k(a_U) < h(a_W)$ .

The *nonstrict seed order* is defined by setting  $U \leq_S W$  if there is an internal ultrapower comparison  $(k, h)$  of  $(j_U, j_W)$  such that  $k(a_U) \leq h(a_W)$ .

*Seed equivalence* is defined by setting  $U =_S W$  if there is an internal ultrapower comparison  $(k, h)$  of  $(j_U, j_W)$  such that  $k(a_U) = h(a_W)$ .

**Lemma 3.5.13.** *If  $U$  and  $W$  are countably complete ultrafilters, then  $U =_S W$  if and only if  $U =_E W$ .* □

**Lemma 3.5.14.** *Suppose  $U_0$  and  $U_1$  are countably complete ultrafilters concentrating on ordinals. Then  $U_0 \leq_S U_1$  if and only if  $U_0 <_S U_1$  or  $U_0 =_S U_1$ .* □

By the characterization of the Ketonen order in terms of comparisons (Lemma 3.3.4) we have the following fact:

**Lemma 3.5.15.** *The Ketonen order extends the seed order.* □

It follows that the seed order is a strict wellfounded set-like relation. (Transitivity is another story; see Proposition 3.5.18 below.)

**Corollary 3.5.16.** *Suppose  $U$  and  $W$  are countably complete ultrafilters on ordinals. Then  $U =_S W$  if and only if  $U \leq_S W$  and  $W \leq_S U$ .* □

**Proposition 3.5.17** (UA). *The seed order linearly orders  $Un$ .*

*Proof.* By the definition of the Ultrapower Axiom, the nonstrict seed order is a total relation on  $Un$ . By Corollary 3.5.16 and the fact that  $=_S$  restricts to equality on  $Un$ , the seed order is antisymmetric on  $Un$ . Thus the seed order linearly orders  $Un$ .  $\square$

The seed order, unlike the Ketonen order, is not provably transitive in ZFC (for mundane reasons):

**Proposition 3.5.18.** *The seed order is transitive if and only if the Ultrapower Axiom holds.*

The proof uses the following trivial variant of Lemma 3.3.15.

**Lemma 3.5.19.** *Suppose  $\alpha$  is an ordinal and  $U$  is a countably complete ultrafilter that concentrates on ordinals. Then  $U$  and the principal ultrafilter  $p_\alpha$  are comparable in the seed order:*

- $U <_S p_\alpha$  if and only if  $\delta_U \leq \alpha$ .
- $U =_E p_\alpha$  if and only if  $\delta_U = \alpha + 1$ .
- $U >_S p_\alpha$  if and only if  $\alpha + 1 < \delta_U$ .  $\square$

*Proof of Proposition 3.5.18.* Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  are ultrapower embeddings. We will show they can be compared. For  $i = 0, 1$ , fix ordinals  $\alpha_i \in M_i$  such that  $M_i = H^{M_i}(j_i[V] \cup \{\alpha_i\})$  with the further property that letting  $U_i$  be the tail uniform ultrafilter derived from  $j_i$  using  $\alpha_i$ ,  $\delta_{U_0} < \delta_{U_1}$ .

By Lemma 3.5.19,

$$U_0 <_S p_{\delta_{U_0}} \leq_S U_1$$

Thus if the seed order is transitive,  $U_0 \leq_S U_1$ . Since  $j_0 = j_{U_0}$  and  $j_1 = j_{U_1}$  the fact that  $U_0 <_S U_1$  implies in particular that there is an internal ultrapower comparison of  $(j_0, j_1)$ . This verifies the Ultrapower Axiom for the pair  $(j_0, j_1)$ .  $\square$

We now consider the seed order on pointed models.

**Definition 3.5.20.** A *pointed ultrapower* is a pointed model  $\mathcal{M}$  whose underlying class  $M$  is an ultrapower of the universe  $V$ . A *pointed ultrapower embedding* is a pair  $(j, \xi)$  where  $j$  is an ultrapower embedding and  $\xi$  is an ordinal.

There is a natural identification of countably complete ultrafilters with a certain class of pointed ultrapower embeddings:

**Definition 3.5.21.** Suppose  $U$  is a countably complete ultrafilter on an ordinal. Then the *pointed ultrapower embedding representing  $U$*  is  $(j_U, a_U)$ . A pointed ultrapower embedding  $(j, \xi)$  *represents an ultrafilter* if it is the pointed ultrapower embedding representing some ultrafilter.

We apologize for bombarding the reader with definitions. The following definitions extend the seed order and Ketonen order to pointed ultrapowers and embeddings.

**Definition 3.5.22.** Suppose  $M$  and  $N$  are pointed ultrapowers.

The *seed order (nonstrict seed order)* is defined by setting  $M <_S N$  ( $M \leq_S N$ ) if there is an internal ultrapower comparison  $(k, h) : (M, N) \rightarrow P$  such that  $k(\xi_M) < h(\xi_N)$  ( $k(\xi_M) \leq h(\xi_N)$ ).

*Seed equivalence* is defined by setting  $M \leq_S N$  if there is an internal ultrapower comparison  $(k, h) : (M, N) \rightarrow P$  such that  $k(\xi_M) = h(\xi_N)$ .

**Definition 3.5.23.** Suppose  $(i, \nu)$  and  $(j, \xi)$  are pointed embeddings.

The *seed order (nonstrict seed order)* is defined by setting  $(i, \nu) <_S (j, \xi)$  ( $(i, \nu) \leq_S (j, \xi)$ ) if there is an internal ultrapower comparison  $(k, h)$  of  $(i, j)$  such that  $k(\nu) < h(\xi)$  ( $k(\nu) \leq h(\xi)$ ).

*Seed equivalence* is defined on by setting  $(i, \nu) =_S (j, \xi)$  if there is an internal ultrapower comparison  $(k, h)$  of  $(i, j)$  such that  $k(\nu) = h(\xi)$ .

The *Ketonen order* (*nonstrict Ketonen order*) is defined on by setting  $(i, \nu) <_{\mathbf{k}} (j, \xi)$  ( $(i, \nu) \leq_{\mathbf{k}} (j, \xi)$ ) if there is a comparison  $(k, h)$  of  $(i, j)$  such that  $h$  is an internal ultrapower embedding and  $k(\nu) < h(\xi)$  ( $k(\nu) \leq h(\xi)$ ).

*Ketonen equivalence* is defined by setting  $(i, \nu) =_{\mathbf{k}} (j, \xi)$  if  $(i, \nu) \leq_{\mathbf{k}} (j, \xi)$  and  $(j, \xi) \leq_{\mathbf{k}} (i, \nu)$

The following is in a sense the strongest consequence of UA for these orders:

**Proposition 3.5.24** (UA). *Suppose  $(i, \nu)$  and  $(j, \xi)$  are pointed ultrapower embeddings. Then either  $(i, \nu) <_S (j, \xi)$ ,  $(i, \nu) =_S (j, \xi)$ , or  $(i, \nu) >_S (j, \xi)$ .  $\square$*

**Definition 3.5.25.** If  $(j, \xi)$  is a pointed ultrapower embedding, then  $M(j)$  denotes the target model of  $j$  and  $M(j, \xi)$  denotes the pointed ultrapower  $(M(j), \xi)$ .

**Lemma 3.5.26** (UA). *Suppose  $(i, \nu)$  and  $(j, \xi)$  are pointed ultrapower embeddings. Then the following are equivalent:*

- (1)  $(i, \nu) \leq_S (j, \xi)$ .
- (2)  $(i, \nu) \leq_{\mathbf{k}} (j, \xi)$ .
- (3)  $M(i, \nu) \leq_S M(j, \xi)$ .
- (4)  $M(i, \nu) \leq_{\mathbf{k}} M(j, \xi)$ .

*Proof.* The implications from (1) to (2) to (3) to (4) are trivial, so to prove the lemma, it suffices to show that (4) implies (1). Therefore assume  $M(i, \nu) \leq_{\mathbf{k}} M(j, \xi)$ . Assume (4) fails, towards a contradiction, so that by Proposition 3.5.24,  $(j, \xi) <_S (i, \nu)$ . Therefore  $M(j, \xi) <_S M(i, \nu)$  and hence  $M(j, \xi) <_{\mathbf{k}} M(i, \nu)$ . By Lemma 3.5.5,  $M(i, \nu) <_{\mathbf{k}} M(i, \nu)$ , contradicting Theorem 3.5.8.  $\square$

Unlike their restrictions to ultrafilters, the relations  $=_E$  and  $=_S$  are far from trivial on pointed ultrapowers and embeddings. When one of the pointed ultrapower embeddings involved represents an ultrafilter,  $=_S$  is closely related to the Rudin-Frolík order (Chapter 5):

**Lemma 3.5.27.** *Suppose  $(i, \nu)$  and  $(j, \xi)$  are pointed ultrapower embeddings such that  $(i, \nu)$  represents an ultrafilter. Then  $(i, \nu) =_S (j, \xi)$  if and only if there is an internal ultrapower embedding  $e : M(i) \rightarrow M(j)$  such that  $e \circ i = j$  and  $e(\nu) = \xi$ .*

*Proof.* We prove the forwards direction, since the converse is trivial. Let  $M = M(i)$  and  $N = M(j)$ . Fix an internal ultrapower comparison  $(k, h) : (M, N) \rightarrow P$  of  $(i, j)$  with  $k(\nu) = h(\xi)$ .

We claim that  $k[M] \subseteq h[N]$ . Since  $i$  represents an ultrafilter,  $M = H^M(i[V] \cup \{\nu\})$ , and hence  $k[M] = H^P(k[i[V] \cup \{\nu\}])$ . But  $k(\nu) = h(\xi) \in h[N]$  and  $k[i[V]] = h[j[V]] \subseteq h[N]$ . Thus  $k[i[V] \cup \{\nu\}] \subseteq h[N]$ , so that  $k[M] = H^P(k \circ i[V] \cup \{k(\nu)\}) \subseteq h[N]$ , as claimed.

Let  $e = h^{-1} \circ k$ . Then  $e : M \rightarrow N$  is an elementary embedding and since  $e \circ i = h^{-1} \circ k \circ i = h^{-1} \circ h \circ j$ , we have  $e \circ i = j$ . It follows that  $e$  is an ultrapower embedding of  $M$ . Since  $h \circ e = k$  and  $k$  is close to  $M$ ,  $e$  is an internal ultrapower embedding of  $M$  by Lemma 2.2.17. Finally,  $e(\nu) = h^{-1}(k(\nu)) = h^{-1}(h(\xi)) = \xi$ .  $\square$

**Corollary 3.5.28 (UA).** *Suppose  $i : V \rightarrow M$  and  $j : V \rightarrow M$  are ultrapower embeddings with the same target model. Then  $i = j$ .*

*Proof.* Fix  $\xi$  such that  $M = H^M(i[V] \cup \{\xi\})$ . Since  $M(i, \xi) = (M, \xi) = M(j, \xi)$ , we must have  $(i, \xi) =_S (j, \xi)$  by Lemma 3.5.26. By Lemma 3.5.27, it follows that there is an internal ultrapower embedding  $k : M \rightarrow M$  such that  $k \circ i = j$  and  $k(\xi) = \xi$ . Since  $k$  is internal to  $M$ ,  $k$  is the identity, and therefore  $i = j$ .  $\square$

The structure of the equivalence relation  $=_S$  on pointed models under UA seems quite interesting. For example, for all we know, if  $M =_S N$  then there some  $H$  of which both  $M$  and  $N$  are ultrapowers such that  $M =_S H =_S N$ .

## The width of an embedding

As a brief digression, we make some general remarks about the size of ultrafilters necessary to realize compositions of ultrapower embeddings. It turns out to be easier to work in a bit

more generality, using a definition due to Cummings [16]:

**Definition 3.5.29.** Suppose  $M$  and  $N$  are transitive models of ZFC and  $j : M \rightarrow N$  is an extender embedding. The *width* of  $j$ , denoted  $\text{WIDTH}(j)$ , is the least  $M$ -ordinal  $\iota$  such that  $N = H^N(j[M] \cup \sup j[\iota])$ .

Note that an embedding is an extender embedding if and only if its width is well-defined. For ultrapower embeddings, there is a simple relationship between width and size. Recall from Definition 2.2.25, which generalizes the notion of size (i.e.,  $\lambda_U$ ) to  $M$ -ultrafilters  $U$ .

**Proposition 3.5.30.** *If  $j : M \rightarrow N$  is the ultrapower embedding associated to an  $M$ -ultrafilter  $U$ , then  $\text{WIDTH}(j) = \lambda_U + 1$ .  $\square$*

There are really two key facts about width, both of which are generalized by the theory of generators (Lemma 5.4.25). The first can be summarized that narrow embeddings are continuous at large regular cardinals  $\lambda$  (i.e.,  $j(\lambda) = \sup j[\lambda]$ ):

**Lemma 3.5.31.** *Suppose  $j : M \rightarrow N$  is an extender embedding and  $\lambda$  is an ordinal of  $M$ -cofinality at least  $\text{WIDTH}(j)$ . Then  $j(\lambda) = \sup j[\lambda]$ .*

*Proof.* Suppose  $\alpha \in \text{Ord}^N$  and  $\alpha < j(\lambda)$ . We will show  $\alpha \leq j(\gamma)$  for some  $\gamma < \lambda$ . Since  $N = H^N(j[M] \cup \sup j[\lambda])$ , we can find a function  $f \in M$  and an ordinal  $\nu < \lambda$  such that  $\alpha = j(f)(\xi)$  for some  $\xi < j(\nu)$ . Since the  $M$ -cofinality of  $\lambda$  is above  $\nu$ ,  $f[\nu] \cap \lambda$  is bounded by some  $\gamma < \lambda$ . Hence  $\alpha = j(f)(\xi) \leq \sup j(f)[j(\nu)] \cap j(\lambda) = j(\sup f[\nu] \cap \lambda) = j(\gamma)$ , as desired.  $\square$

This has a useful consequence for ultrapower embeddings (which is essentially equivalent):

**Lemma 3.5.32.** *Suppose  $M$  is a transitive model of ZFC and  $U$  is an  $M$ -ultrafilter on a set  $X \in M$ . Then for any ordinal  $\delta$  such that  $\text{cf}^M(\delta) > \lambda_U$ ,  $j_U^M$  is continuous at  $\delta$ .  $\square$*

It is worth mentioning a related fact here:



**Lemma 3.5.33.** *If  $U$  is an ultrafilter on  $X$ , then for any cardinal  $\gamma$ ,  $|j_U(\gamma)| \leq \gamma^{|X|}$ . Thus if  $\lambda$  is a strong limit cardinal above  $|X|$ ,  $j_U[\lambda] \subseteq \lambda$ . If moreover  $\text{cf}(\lambda) > |X|$ , then  $j_U(\lambda) = \lambda$ .  $\square$*

The second provides a computation of the width of a composition in terms of the width of the factors:

**Lemma 3.5.34.** *Suppose  $M \xrightarrow{i} N \xrightarrow{j} P$  are elementary embeddings. Then*

$$\text{WIDTH}(j \circ i) = \max\{\text{WIDTH}(i), \gamma\}$$

where  $\gamma$  is the least ordinal such that  $\text{WIDTH}(j) \leq \sup i[\gamma]$ .

*Proof.* Let  $\iota = \max\{\text{WIDTH}(i), \gamma\}$ .

We first show  $\text{WIDTH}(j \circ i) \leq \iota$ . Since  $\text{WIDTH}(i) \leq \iota$ ,  $N = H^N(i[M] \cup \sup i[\iota])$ . Since  $\text{WIDTH}(j) \leq \sup i[\iota]$ ,  $P = H^P(j[N] \cup \sup j[\sup i[\iota]]) = H^P(j[N] \cup \sup j \circ i[\iota])$ . Putting these calculations together,

$$P = H^P(j[i[M] \cup \sup i[\iota]] \cup \sup j \circ i[\iota]) = H^P(j \circ i[M] \cup \sup j \circ i[\iota])$$

It follows that  $\text{WIDTH}(j \circ i) \leq \iota$ .

We now show  $\iota \leq \text{WIDTH}(j \circ i)$ . First, we show  $\gamma \leq \text{WIDTH}(j \circ i)$ . Fix  $\eta < \gamma$ , and we will show  $\eta < \text{WIDTH}(j \circ i)$ . This follows from the fact that

$$H^P(j \circ i[M] \cup \sup j \circ i[\eta]) \subseteq H^P(j[N] \cup \sup j[\sup i[\eta]]) \subsetneq P$$

The final inequality uses that  $\sup i[\eta] < \text{WIDTH}(j)$ .

We finish by showing  $\text{WIDTH}(i) \leq \text{WIDTH}(j \circ i)$ . This uses the argument from Proposition 3.4.20. Suppose  $\eta < \text{WIDTH}(i)$  is an  $M$ -cardinal, and we will show  $\eta < \text{WIDTH}(j \circ i)$ . Fix  $a \in N$  such that  $a \notin H^N(i[M] \cup \sup i[\eta])$ . Suppose towards a contradiction that  $j(a) \in H^P(j \circ i[M] \cup \sup j \circ i[\eta])$ . Fix  $\xi < \eta$  and  $f \in M$  such that  $j(a) = j(i(f))(\alpha)$  for some  $\alpha \leq j(i(\xi))$ . Then by the elementarity of  $j$ ,  $N$  satisfies that  $a = i(f)(\alpha)$  for some  $\alpha \leq i(\xi)$ . This contradicts our assumption that  $a \notin H^N(i[M] \cup \sup i[\eta])$ . Therefore  $j(a) \notin H^P(j \circ i[M] \cup \sup j \circ i[\eta])$ , so  $\eta < \text{WIDTH}(j \circ i)$ , as desired.  $\square$

## The direct limit of all ultrapowers

Under the Ultrapower Axiom, it is possible to take the direct limit of all ultrapower embeddings. The properties of this structure, which is denoted  $M_\infty$ , turn out to be closely related both to the Ketonen order on pointed ultrapowers and to the theory of supercompact cardinals.

To save ink, it is convenient to let  $\infty$  be a formal symbol such that (by definition) every ordinal  $\lambda$  satisfies  $\lambda < \infty$ .

**Definition 3.5.35.** Suppose  $M$  is an ultrapower of the universe and  $\lambda$  is an ordinal. Then  $\nu(M, \lambda) = \sup i[\lambda]$  where  $i : V \rightarrow M$  is any ultrapower embedding.

By Theorem 3.5.10,  $\nu(M, \lambda)$  does not depend on the choice of  $i$ . We also set  $\nu(M, \infty) = \infty$ .

**Definition 3.5.36.** If  $\lambda$  is a cardinal or  $\lambda = \infty$ , then  $\mathcal{D}_\lambda$  denotes the following category:

- An inner model  $M$  is an object of  $\mathcal{D}_\lambda$  if there is an ultrapower embedding  $i : V \rightarrow M$  with  $\text{WIDTH}(i) \leq \lambda$ .
- If  $M, N \in \mathcal{D}_\lambda$ , an internal ultrapower embedding  $j : M \rightarrow N$  is a morphism of  $\mathcal{D}_\lambda$  if  $\text{WIDTH}(j) \leq \nu(M, \lambda)$ .

Thus  $\mathcal{D}_\infty$  is the category of all ultrapowers of the universe equipped with their internal ultrapower embeddings. As an immediate consequence of Corollary 3.5.28, UA implies that  $\mathcal{D}_\lambda$  is a full subcategory of  $\mathcal{D}_\infty$ ; that is, it contains all morphisms between the objects it sees. It is not clear whether this is the case in ZFC (and it is not even clear whether this subcategory is locally small in the natural sense).<sup>1</sup>

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<sup>1</sup>This raises an interesting question in the general theory of elementary embeddings:

**Question 3.5.37** (ZFC). Can two ultrapower embeddings of the universe have the same target model but different widths?

The question has something to do with uniform ultrafilters on singular cardinals.

**Definition 3.5.38.** A category  $\mathcal{C}$  is a *partial order* if every pair of objects  $a, b \in \mathcal{C}$  there is at most one morphism from  $a$  to  $b$  in  $\mathcal{C}$ . A category  $\mathcal{C}$  is *directed* if for every pair of objects  $a, b \in \mathcal{C}$ , there is a further object  $c \in \mathcal{C}$  admitting morphisms  $a \rightarrow c$  and  $b \rightarrow c$ .

We have the following equivalences:

**Lemma 3.5.39.** *The following are equivalent:*

- (1) *The Ultrapower Axiom.*
- (2) *For all cardinals  $\lambda$ ,  $\mathcal{D}_\lambda$  is a directed partial order.*
- (3)  *$\mathcal{D}_\infty$  is a directed partial order.*

*Proof.* (1) *implies* (2): The fact that  $\mathcal{D}_\lambda$  is a partial order follows immediately from Corollary 3.5.28. The directedness of  $\mathcal{D}_\lambda$  follows from an easy localization of UA (Proposition 5.4.16), which states that if  $U$  and  $W$  are countably complete ultrafilters on a cardinal  $\gamma$ , then there is a countably complete ultrafilter  $Z$  on  $\gamma$  such that there are internal ultrapower embeddings  $k : M_U \rightarrow M_Z$  and  $h : M_W \rightarrow M_Z$ .

(2) *implies* (3): Immediate.

(3) *implies* (1): Immediate. □

**Definition 3.5.40 (UA).** If  $\lambda$  is a cardinal or  $\lambda = \infty$ , let

$$M_\lambda = \lim \mathcal{D}_\lambda$$

For all  $M \in \mathcal{D}_\lambda$ ,

$$j_{M,\lambda} : M \rightarrow M_\lambda$$

denotes the direct limit embedding.

The models  $M_\lambda$  are wellfounded by a standard application of the linear iterability of the universe (Lemma 3.5.7). We will see that  $M_\infty$  need not be set-like. By convention we identify its set-like part with an inner model.

The following lemma is the key to the analysis of the models  $M_\lambda$  for  $\lambda$  a regular cardinal:

**Lemma 3.5.41** (UA). *Suppose  $\lambda$  is a regular cardinal or  $\lambda = \infty$ . For any ultrapower embedding  $i : V \rightarrow N$  with  $i \in \mathcal{D}_\lambda$ ,  $i(M_\lambda) = M_\lambda$  and  $i(j_{V,\lambda}) = j_{N,\lambda}$ .*

*Proof.* The key point is that since  $\lambda$  is regular and  $\text{WIDTH}(i) \leq \lambda$ ,  $i(\lambda) = \sup i[\lambda] = \nu(N, \lambda)$ . Thus  $i(\mathcal{D}_\lambda) = \mathcal{D}_{i(\lambda)}^N = \mathcal{D}_{\nu(N,\lambda)}^N$ , which is equal to the cone above  $N$  in  $\mathcal{D}_\lambda$ . Since  $\mathcal{D}_\lambda$  is a directed partial order, this cone is cofinal in  $\mathcal{D}_\lambda$ , and thus its direct limit is equal to that of  $\mathcal{D}_\lambda$ . In other words,  $i(M_\lambda) = \lim i(\mathcal{D}_\lambda) = \lim \mathcal{D}_\lambda = M_\lambda$ , and similarly  $i(j_{V,\lambda}) = j_{N,\lambda}$ .  $\square$

In the case  $\lambda = \infty$  above, we are heavily abusing notation: when  $M_\infty$  is not set-like (and so cannot be identified with a transitive class), a more careful statement would involve isomorphism rather than equality.

We now further explore the set-likeness of  $M_\infty$ .

**Definition 3.5.42.** We say an ordinal  $\kappa$  *can be mapped arbitrarily high by ultrapower embeddings* if for all  $\alpha > \kappa$ , there is an ultrapower embedding  $j : V \rightarrow M$  such that  $j(\kappa) > \alpha$ . The *ultrapower threshold* is the least ordinal  $\kappa$  that can be mapped arbitrarily high by ultrapower embeddings.

The existence of the ultrapower threshold is a large cardinal principle closely related to two recently popularized weakenings of strong compactness: the strongly tall cardinals of Hamkins [17] and the  $\omega_1$ -strongly compact cardinals of Bagaria-Magidor [18]. Certainly a strongly tall cardinal or an  $\omega_1$ -strongly compact cardinal is greater than or equal to the ultrapower threshold. (By theorems of Gitik [19], it is consistent with ZFC that these inequalities are strict.) The ultrapower threshold is in a sense a hybrid of these notions in the sense that it weakens strong compactness in the Hamkins and Bagaria-Magidor directions simultaneously, producing a super-weakening of strong compactness.

If it exists, the ultrapower threshold is a Beth fixed point, but by the arguments of Bagaria-Magidor [18], it cannot be proved to be inaccessible in ZFC. The nonexistence of the ultrapower threshold has the following structural consequence for ultrapower embeddings:

**Lemma 3.5.43.** *If the ultrapower threshold does not exist, then unboundedly many ordinals are fixed by all ultrapower embeddings.*

*Proof.* Fix an ordinal  $\xi$ . Let

$$T_\xi = \{j(\xi) : j \text{ is an ultrapower embedding of } V\}$$

and let  $C$  be the class of ordinals fixed by all ultrapower embeddings. Note that if  $\xi$  is an ordinal and  $i : V \rightarrow N$  is an ultrapower embedding,

$$i(T_\xi) = \{j(i(\xi)) : j \text{ is an ultrapower embedding of } N\} \subseteq T_\xi$$

This is because the composition of ultrapower embeddings is an ultrapower embedding.

Since the ultrapower threshold does not exist,  $T_\xi$  is a set for all ordinals  $\xi$ . So assume  $T_\xi$  is a set, and we will show  $C \setminus \xi$  is nonempty. Let  $\alpha = \sup(T_\xi)$ . Obviously  $\alpha \geq \xi$ . Suppose  $i$  is an ultrapower embedding. Then

$$i(\alpha) = \sup i(T_\xi) \leq \sup(T_\xi) = \alpha$$

Thus  $i(\alpha) = \alpha$ . It follows that  $\alpha \in C \setminus \xi$ , as desired. Thus if  $T_\xi$  is a set, then there is an ordinal above  $\xi$  fixed by all ultrapower embeddings. It follows that if  $T_\xi$  is a set for all  $\xi$ , then  $C$  is a proper class.  $\square$

The embedding  $j_{V,\infty}$  actually encodes the class of common fixed points of ultrapower embeddings by a standard argument:

**Lemma 3.5.44.** *An ordinal belongs to the range of  $j_{V,\infty}$  if and only if it is fixed by all ultrapower embeddings.*

*Proof.* Suppose first that  $\beta$  is an ordinal in the range of  $j_{V,\infty}$ . Fix an ordinal  $\alpha$  such that  $\beta = j_{V,\infty}(\alpha)$ . Suppose  $i : V \rightarrow N$  is an ultrapower embedding. Then  $i(\beta) = i(j_{V,\infty}(\alpha)) = i(j_{V,\infty})(i(\alpha)) = j_{N,\infty} \circ i(\alpha) = j_{V,\infty}(\alpha) = \beta$ . Thus  $\beta$  is fixed by all ultrapower embeddings.

Conversely, suppose  $\beta$  is fixed by all ultrapower embeddings. Let  $\alpha$  be the least ordinal such that  $j_{V,\infty}(\alpha) \geq \beta$ . Suppose towards a contradiction that  $j_{V,\infty}(\alpha) > \beta$ . Then there is an

ultrapower embedding  $i : V \rightarrow N$  and some ordinal  $\xi < i(\alpha)$  and  $j_{N,\infty}(\xi) \geq \beta$ . But by the elementarity of  $i$ ,  $i(\alpha)$  is the least ordinal  $\alpha'$  such that  $i(j_{V,\infty})(\alpha') \geq i(\beta)$ . Since  $i(\beta) = \beta$ , this means that  $i(\alpha)$  is the least ordinal  $\alpha'$  such that  $j_{N,\infty}(\alpha') \geq \beta$ . This contradicts the existence of  $\xi < i(\alpha)$  such that  $j_{N,\infty}(\xi) \geq \beta$ .  $\square$

**Theorem 3.5.45 (UA).** *Exactly one of the following holds:*

(1) *The ultrapower threshold exists.*

(2)  *$M_\infty$  is set-like.*

*Proof.* Suppose (1) holds. For any ordinal  $\xi$ , the images of  $\xi$  under ultrapower embeddings are bounded above by  $j_{V,\infty}(\xi)$ , and so  $\xi$  is not the ultrapower threshold. Thus (2) fails.

Suppose (2) fails. By Lemma 3.5.43, the class  $C$  of ordinals fixed by all ultrapower embeddings is unbounded in the ordinals. By Lemma 3.5.44,  $C = j_{V,\infty}[\text{Ord}] \cap \text{Ord}$ . The function  $j_{V,\infty} \upharpoonright \text{Ord}$  is therefore equal to the increasing enumeration of  $C$ . It follows that for every ordinal  $\alpha$ ,  $j_{V,\infty}(\alpha)$  is an ordinal. In other words,  $M_\infty$  is set-like. Thus (1) holds.  $\square$

The analysis of supercompactness under UA has the following surprising consequence (Theorem 7.4.26): the ultrapower threshold is supercompact.

**Theorem 3.5.46 (UA).** *Exactly one of the following holds:*

(1)  *$M_\infty$  is set-like.*

(2) *There is a supercompact cardinal.*

*Proof given Theorem 7.4.26.* Suppose  $M_\infty$  is not set-like. By Theorem 3.5.45, the ultrapower threshold exists. By Theorem 7.4.26, the ultrapower threshold is supercompact.  $\square$

In fact, if the ultrapower threshold  $\kappa$  is supercompact, then  $j_{V,\infty}(\kappa)$  is isomorphic to  $\text{Ord}$  while  $j_{V,\infty} \upharpoonright V_\kappa = (j_{V,\infty})^{V_\kappa}$ .

We now explain the connection between the models  $M_\lambda$  and the Ketonen order on pointed ultrapower embeddings.

**Definition 3.5.47.** Let  $\mathcal{P}_\lambda$  denote the collection of pointed ultrapowers  $(M, \xi)$  such that  $M \in \mathcal{D}_\lambda$ . For any  $M \in \mathcal{P}_\lambda$ ,  $o_\lambda(M)$  denotes the rank of  $M$  in the Ketonen order restricted to  $\mathcal{P}_\lambda$ . (If  $\lambda = \infty$ , this rank may not exist.) If  $W$  is a countably complete ultrafilter then  $o_\lambda(W) = o_\lambda(M_W, a_W)$ .

The following theorem shows that the ordinals  $o_\lambda(M)$  are highly structured under UA:

**Theorem 3.5.48 (UA).** *Assume  $\lambda$  is regular or  $\lambda = \infty$ . For any  $M \in \mathcal{P}_\lambda$ ,  $o_\lambda(M) = j_{M,\lambda}(\xi_M)$ .*

*Proof.* Consider the partial function  $\Phi_\lambda : \mathcal{P}_\lambda \rightarrow \text{Ord}$  defined by

$$\Phi_\lambda(M) = j_{M,\lambda}(\xi_M)$$

If  $\lambda = \infty$ , we leave  $\Phi_\lambda(M)$  undefined if  $j_{M,\lambda}(\xi_M)$  is not in the set-like part of  $M_\infty$ .

For  $M, N \in \mathcal{P}_\lambda$ ,  $M <_{\mathbb{k}} N$  implies  $\Phi_\lambda(M) <_{\mathbb{k}} \Phi_\lambda(N)$  and  $M =_E N$  implies  $\Phi_\lambda(M) = \Phi_\lambda(N)$ . Moreover the image of  $\Phi_\lambda$  is the set-like initial segment of  $\text{Ord}^{M_\lambda}$ . Therefore  $\Phi_\lambda$  is equal to the rank function of  $(\mathcal{P}_\lambda, <_{\mathbb{k}})$ . That is, for all  $M \in \mathcal{P}_\lambda$ ,  $o_\lambda(M) = \Phi_\lambda(M) = j_{M,\lambda}(\xi_M)$ .  $\square$

Combining this with Theorem 3.5.46, we obtain:

**Corollary 3.5.49.** *Exactly one of the following holds:*

(1) *There is a supercompact cardinal.*

(2) *For any pointed ultrapower  $M$ ,  $o_\infty(M)$  exists.*  $\square$

In conclusion, the Ketonen order, which is in a sense the simplest structure associated with the Ultrapower Axiom, bears a deep relationship to supercompactness under UA. This relationship is one of the topics of Chapter 7 and Chapter 8.

## The Lipschitz order

In this short subsection, we describe a generalization of the Ketonen order that raises an interesting philosophical question. Throughout the section, we fix an infinite ordinal  $\delta$ .

**Definition 3.5.50.** Suppose  $f : P(\delta) \rightarrow P(\delta)$ . Then  $f$  is:

- a *reduction* if for  $A \subseteq \delta$  and  $\alpha < \delta$ ,  $f(A) \cap \alpha$  depends only on  $A \cap \alpha$ .
- a *contraction* if for  $A \subseteq \delta$  and  $\alpha < \delta$ ,  $f(A) \cap (\alpha + 1)$  depends only on  $A \cap \alpha$ .

We say  $X$  *reduces* (*contracts*) to  $Y$  if there is a reduction (contraction)  $f : P(\delta) \rightarrow P(\delta)$  such that  $f^{-1}[Y] = X$ . In this case we say  $f$  is a reduction (contraction) from  $X$  to  $Y$ .

These concepts can be formulated in terms of long games:

**Definition 3.5.51.** In the *Lipschitz game of length  $\delta$*  associated to sets  $X, Y \subseteq P(\delta)$ , denoted  $G_\delta(X, Y)$ , two players I and II alternate playing 0s or 1s. I plays at limit stages. The play lasts for  $\delta \cdot 2$  moves, so that I and II produce sequences  $x_I, x_{II} \in {}^\delta 2$ . Let  $A_I = \{\alpha < \delta : x_I(\alpha) = 1\}$  and  $A_{II} = \{\alpha < \delta : x_{II}(\alpha) = 1\}$ . Then II wins if  $A_I \in X \iff A_{II} \in Y$ .

Player II has a winning strategy if and only if  $X$  reduces to  $Y$ , and Player I has a winning strategy if and only if  $Y$  contracts to  $P(\delta) \setminus X$ .

**Definition 3.5.52.** The *Lipschitz order* is defined on  $X, Y \subseteq P(\delta)$  by setting  $X <_L Y$  if  $X$  and  $P(\delta) \setminus X$  contract to  $Y$ . The *nonstrict Lipschitz order* is defined on  $X, Y \subseteq P(\delta)$  by setting  $X \leq_L Y$  if  $X$  reduces to  $Y$ .

This notation is perhaps misleading since it might suggest that  $X <_L Y$  if and only if  $X \leq_L Y$  and  $Y \not\leq_L X$ . Under the Axiom of Determinacy, this is true when  $X$  and  $Y$  are contained in  $P(\omega)$ .

The Lipschitz order is transitive in the following strong sense:



**Lemma 3.5.53.** *The composition of a contraction and a reduction is a contraction. Therefore if  $X$  contracts to  $Y$  and  $Y$  reduces to  $Z$ , then  $X$  contracts to  $Z$ . In particular, if  $X <_L Y \leq_L Z$  then  $X <_L Z$ .  $\square$*

A generalization of the proof of Proposition 3.3.9 shows that the Lipschitz order is irreflexive:

**Lemma 3.5.54.** *Suppose  $X \subseteq P(\delta)$ . Then  $X$  does not contract to  $P(\delta) \setminus X$ .*

*Proof.* It suffices to show that every contraction  $f : P(\delta) \rightarrow P(\delta)$  has a fixed point  $A$ : then  $A \in X$  if and only if  $f(A) \in X$  so  $f$  is not a contraction from  $X$  to  $P(\delta) \setminus X$ .

We define  $A$  by recursion. Suppose  $\alpha < \delta$  and we have defined  $A \cap \alpha$ . We then put  $\alpha \in A$  if and only if  $\alpha \in f(A \cap \alpha)$ . Then for any  $\alpha < \delta$ ,

$$\begin{aligned} \alpha \in A &\iff \alpha \in f(A \cap \alpha) \\ &\iff \alpha \in f(A) \end{aligned}$$

The final equivalence follows from the fact that  $f$  is a contraction. Thus  $f(A) = A$ , as desired.  $\square$

**Corollary 3.5.55.** *The Lipschitz order is a strict partial order.  $\square$*

By the proof of the Martin-Monk theorem (see [20]) descending sequences in the Lipschitz order give rise to pathological subsets of Cantor space:

**Theorem 3.5.56** (ZF + DC). *The following are equivalent:*

- (1) *There is a flip set.*
- (2) *The Lipschitz order on  $P(\omega)$  is illfounded.*
- (3) *The Lipschitz order on  $P(\delta)$  is illfounded.*

*Proof.* To see (1) implies (2), suppose  $F \subseteq 2^\omega$  is a flip set. Define  $(E_n)_{n < \omega}$  by recursion, setting  $E_0 = F$  and  $E_{n+1} = \{s \in 2^\omega : 0s \in E_n\}$ . It is easy to see that  $E_{n+1}$  and  $2^\omega \setminus E_{n+1}$  both contract to  $E_n$ , via the contractions  $s \mapsto 0s$  and  $s \mapsto 1s$  respectively.

(2) trivially implies (3).

We finally show (3) implies (1). Fix  $X_0 >_L X_1 >_L X_2 >_L \cdots$  a descending sequence of subsets of  $P(\delta)$ . For  $n < \omega$ , fix contractions  $f_n^0$  from  $X_{n+1}$  to  $X_n$  and  $f_n^1$  from  $X_{n+1}$  to  $P(\delta) \setminus X_n$ . For each  $s \in 2^\omega$ , we define sets  $A_n^s \subseteq \delta$  such that

$$A_n^s = f_n^{s(n)}(A_{n+1}^s)$$

Suppose  $A_n^s \cap \alpha$  has been defined for all  $n < \omega$ . Then

$$A_n^s \cap (\alpha + 1) = f_n^{s(n)}(A_{n+1}^s \cap \alpha) \cap (\alpha + 1)$$

Since  $f_n^i$  is a contraction for all  $n < \omega$  and  $i \in \{0, 1\}$ ,  $A_n^s$  is well-defined and  $A_n^s = f_n^{s(n)}(A_{n+1}^s)$ .

Define  $F_n \subseteq 2^\omega$  by putting  $s \in F_n$  if and only if  $A_n^s \in X_n$ . Whether  $s \in F_n$  depends only on  $s \upharpoonright (\omega \setminus n)$ . Moreover, if  $s \in F_{n+1}$  then  $s \in F_n$  if and only if  $s(n) = 0$ . It is easy to show by induction that if  $s$  and  $s'$  agree on  $\omega \setminus n$  and  $\sum_{k < n} s(k) = \sum_{k < n} s'(k) \pmod{2}$ , then  $s \in F_0$  if and only if  $s' \in F_0$ . Similarly, if  $s$  and  $s'$  agree on  $\omega \setminus n$  and  $\sum_{k < n} s(k) \neq \sum_{k < n} s'(k) \pmod{2}$ , then  $s \in F_0$  if and only if  $s' \notin F_0$ . It follows that  $F_0$  is a flip set.  $\square$

Of course, (1), (2), and (3) are all provable in ZFC. In the choiceless context of ZF + DC, however, there may be no flip sets (for example, if every subset of Cantor space has the Baire property or is Lebesgue measurable). In this case, Theorem 3.5.56 shows that the Lipschitz order is wellfounded not only on subsets of Cantor space but also on subsets of  $P(\delta)$ .<sup>2</sup> The proof also shows that the wellfounded part of the Lipschitz order is equal to the collection of sets that do not lie above a flip set.

We turn now to the relationship between the Lipschitz order and the Ketonen order.

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<sup>2</sup>Under the same hypotheses, one can show that the Lipschitz order on  ${}^\delta S$  is wellfounded for any set  $S$  after generalizing the definition of the Lipschitz order in the natural way.

**Definition 3.5.57.** A set  $Z \subseteq P(\delta)$  *concentrates* on a set  $S$  if for all  $A, B \subseteq \delta$  with  $A \cap S = B \cap S$ ,  $A \in Z$  if and only if  $B \in Z$ .

Note that if  $Z$  is an ultrafilter that concentrates on a class  $S$  in the sense of Definition 3.2.1, then  $Z$  concentrates on  $S$  in the sense of Definition 3.5.57.

**Lemma 3.5.58.** *Suppose  $X \subseteq P(\delta)$  and  $W$  is an ultrafilter on  $\delta$ . Then the following are equivalent:*

- (1)  $X <_L W$ .
- (2)  $X$  contracts to  $W$ .
- (3) For some  $Z \in M_W$  that concentrates on  $a_W$ ,  $X = j_W^{-1}[Z]$ .

*Proof.* (1) implies (2): Trivial.

(2) implies (1): Assume  $X$  contracts to  $W$ . Since  $W$  is an ultrafilter,  $W$  reduces to  $P(\delta) \setminus W$ . Since  $X$  contracts to  $W$  and  $W$  reduces to  $P(\delta) \setminus W$ ,  $X$  contracts to  $P(\delta) \setminus W$ . Therefore  $X <_L W$ .

(1) implies (3): Let  $f : P(\delta) \rightarrow P(\delta)$  be a contraction from  $X$  to  $W$ . For each  $\alpha$ , let  $X_\alpha = \{A \subseteq \delta : \alpha \in f(A)\}$ . Since  $f$  is a contraction,  $X_\alpha$  concentrates on  $\alpha$ . Let  $Z = [\langle X_\alpha : \alpha < \delta \rangle]_W$ . By Łoś's Theorem,  $Z$  concentrates on  $a_W$ . Then

$$\begin{aligned} A \in X &\iff f(A) \in W \\ &\iff \{\alpha < \delta : A \in X_\alpha\} \in W \\ &\iff j_W(A) \in Z \end{aligned}$$

Thus  $j_W^{-1}[Z] = X$ .

(3) implies (1): Fix  $Z \in M_W$  concentrating on  $a_W$  such that  $X = j_W^{-1}[Z]$ . Let  $\langle X_\alpha : \alpha \in I \rangle$  be such that  $Z = [\langle X_\alpha : \alpha \in I \rangle]_W$  and  $X_\alpha$  concentrates on  $\alpha$  for all  $\alpha \in I$ . Define  $f : P(\delta) \rightarrow P(\delta)$  by setting  $f(X) = \{\alpha \in I : X \in X_\alpha\}$ . Then  $f$  is a contraction since  $X_\alpha$

concentrates on  $\alpha$  for all  $\alpha \in I$ . Moreover,

$$\begin{aligned}
A \in X &\iff j_W(A) \in Z \\
&\iff \{\alpha < \delta : A \in X_\alpha\} \in W \\
&\iff f(X) \in W \quad \square
\end{aligned}$$

Using Lemma 3.3.4, this has the following corollary:

**Corollary 3.5.59.** *The Lipschitz order extends the Ketonen order on  $\mathcal{B}(\delta)$ .*  $\square$

Under UA, it follows that the two orders coincide:

**Corollary 3.5.60 (UA).** *The Lipschitz order and the Ketonen order coincide on  $\mathcal{B}(\delta)$ . In particular, the Lipschitz order linearly orders  $\mathcal{B}(\delta)$ .*

*Proof.* Since  $<_L$  is a strict partial order extending the total relation  $<_{\mathbf{k}}$  (Theorem 3.3.6), the two orders must be equal.  $\square$

Another way to state this is as a determinacy consequence of UA:

**Corollary 3.5.61 (UA).** *For all ordinals  $\delta$ , for any  $U, W \in \mathcal{B}(\delta)$ , the game  $G_\delta(U, W)$  is determined.*

We conclude this section with a question that is perhaps of some philosophical significance:

**Question 3.5.62.** Assume that for any ordinal  $\delta$ , for any  $U, W \in \mathcal{B}(\delta)$ , the game  $G_\delta(U, W)$  is determined. Does the Ultrapower Axiom hold?

If this were true then the Ultrapower Axiom would be a long determinacy principle. In Section 3.6, we give partial positive answer.

## The Ketonen order on filters

We briefly discuss a generalization of the Ketonen order to a wellfounded partial order on arbitrary countably complete filters that is suggested by the proof of Theorem 3.3.8. This order will not appear elsewhere in this dissertation, but it seems potentially quite interesting since it identifies a connection between the Ketonen order and stationary reflection.

**Definition 3.5.63.** Suppose  $F$  is a filter,  $I \in F$ , and  $\langle G_i : i \in I \rangle$  is a sequence of filters on a fixed set  $Y$ . The  $F$ -limit of  $\langle G_i : i \in I \rangle$  is the filter

$$F\text{-}\lim_{i \in I} G_i = \{A \subseteq Y : \{i \in I : A \in G_i\} \in F\}$$

**Definition 3.5.64.** If  $F$  is a filter on a set  $X$  and  $C$  is a class, then  $F$  *concentrates* on  $C$  if  $C \cap X \in F$ .

**Definition 3.5.65.** Suppose  $X$  is a set and  $C$  is a class. Let  $\mathcal{F}(X)$  denote the set of countably complete filters on  $X$  and let  $\mathcal{F}(X, C)$  denote the set of filters on  $X$  that concentrate on  $C$ .

**Definition 3.5.66.** Suppose  $\epsilon$  and  $\delta$  are ordinals,  $F \in \mathcal{F}(\epsilon)$ , and  $G \in \mathcal{F}(\delta)$ . The *Ketonen order on filters* is defined on by setting  $F <_{\mathbb{k}} G$  if there is a set  $I \in G$  and a sequence  $\langle F_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{F}(\epsilon, \alpha)$  such that  $F \subseteq G\text{-}\lim_{\alpha \in I} F_\alpha$ .

Under the Ultrapower Axiom, the restriction to ultrafilters of the Ketonen order on filters coincides with the Ketonen order as it is defined in Section 3.3. We do not know whether this is provable in ZFC.

Note that the proof of Proposition 3.3.9 breaks down when we consider filters instead of ultrafilters. In fact, in a sense this simple proof cannot be remedied, since irreflexivity fails if we allow filters that are countably incomplete, and it is not clear how countable completeness could come in to the argument of Proposition 3.3.9. It is somewhat surprising that one can in fact prove the irreflexivity of the Ketonen order by instead using countable completeness and the argument of Theorem 3.3.8:

**Theorem 3.5.67.** *The Ketonen order on filters is wellfounded.*

We include the proof, which is closely analogous to that of Theorem 3.3.8.

**Lemma 3.5.68.** *Suppose  $H$  is a filter and  $F <_{\mathbb{k}} G$  are countably complete filters on ordinals  $\epsilon$  and  $\delta$ . Suppose  $J \in H$  and  $\langle G_x : x \in J \rangle$  is a sequence of countably complete filters such that  $G \subseteq H\text{-}\lim_{x \in J} G_x$ . Then there is a set  $J' \subseteq J$  in  $H$  and a sequence  $\langle F_x : x \in J' \rangle$  of countably complete filters such that  $F_x <_{\mathbb{k}} G_x$  for all  $x \in K$  and  $F \subseteq H\text{-}\lim_{x \in K} F_x$ .*

*Proof.* Since  $F <_{\mathbb{k}} G$ , we can fix  $I \in G$  and countably complete filters  $\langle D_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{F}(\epsilon, \alpha)$  such that  $F \subseteq G\text{-}\lim_{\alpha \in I} D_\alpha$ .

Let  $J' = \{b \in J : I \in G_b\}$ . Since  $I \in G \subseteq H\text{-}\lim_{x \in J} G_x$ , we have  $J' \in H$  by the definition of a limit. For each  $x \in J'$ , let

$$F_x = G_x\text{-}\lim_{\alpha \in I} D_\alpha$$

Then  $F_x \in \mathcal{B}(\epsilon)$ , and the sequence  $\langle D_\alpha : \alpha \in I \rangle$  witnesses  $F_b <_{\mathbb{k}} G_b$ .

Finally,

$$\begin{aligned} F &\subseteq G\text{-}\lim_{\alpha \in I} D_\alpha \\ &\subseteq (H\text{-}\lim_{x \in J} G_x)\text{-}\lim_{\alpha \in I} D_\alpha \\ &= H\text{-}\lim_{x \in J'} (G_x\text{-}\lim_{\alpha \in I} D_\alpha) \\ &= H\text{-}\lim_{x \in J'} F_x \end{aligned}$$

Thus  $F \subseteq H\text{-}\lim_{x \in K} F_x$ , as desired. □

*Proof of Theorem 3.5.67.* Suppose towards a contradiction that  $\delta$  is the least ordinal such that the Ketonen order is illfounded below a countably complete filter that concentrates on  $\delta$ . Fix a descending sequence  $F_0 >_{\mathbb{k}} F_1 >_{\mathbb{k}} F_2 >_{\mathbb{k}} \cdots$  such that  $F_0$  concentrates on  $\delta$ .

We will define sets of ordinals  $I_1 \supseteq I_2 \supseteq \cdots$  in  $F$  and sequences  $\langle F_\alpha^m : \alpha \in I_m \rangle$  of countably complete filters such that

$$F_m \subseteq F\text{-}\lim_{\alpha \in I_m} F_\alpha^m$$

for all  $1 \leq m < \omega$ . We will have:

- For all  $\alpha \in I_1$ ,  $F_\alpha^1$  concentrates on  $\alpha$ .
- For all  $1 \leq m < \omega$ , for all  $\alpha \in I_{m+1}$ ,  $F_\alpha^{m+1} <_{\mathbb{k}} F_\alpha^m$ .

Since  $F_1 <_{\mathbb{k}} F$ , there is a set of ordinals  $I_1 \in F$  and a sequence  $\langle F_\alpha^1 : \alpha \in I_1 \rangle$  of countably complete ultrafilters such that  $F_1 \subseteq F\text{-}\lim_{\alpha \in I_1} F_\alpha^1$  and  $F_\alpha^1$  concentrates on  $\alpha$  for all  $\alpha \in I_1$ .

Suppose  $1 \leq m < \omega$  and  $\langle F_\alpha^m : \alpha \in I_m \rangle$  has been defined. We now apply Lemma 3.5.68 with  $H = F$ ,  $F = F_{m+1}$ , and  $G = F_m$ . This yields a set  $I_{m+1} \subseteq I_m$  in  $F$  and a sequence  $\langle F_\alpha^{m+1} : \alpha \in I_m \rangle$  of countably complete filters on  $\delta$  such that  $F_\alpha^{m+1} <_{\mathbb{k}} F_\alpha^m$  for all  $\alpha \in I_{m+1}$  and

$$F_{m+1} \subseteq F\text{-}\lim_{\alpha \in I_{m+1}} F_\alpha^{m+1}$$

This completes the definition of the sets  $I_1 \supseteq I_2 \supseteq \dots$  and sequences  $\langle F_\alpha^m : \alpha \in I_m \rangle$  for  $1 \leq m < \omega$ .

Now let  $I = \bigcap_{1 \leq m < \omega} I_m$ . Since  $F_0$  is countably complete,  $I$  is nonempty, so we can fix an ordinal  $\alpha \in I$ . Then since  $\alpha \in I_m$  for all  $1 \leq m < \omega$ ,

$$F_\alpha^1 >_{\mathbb{k}} F_\alpha^2 >_{\mathbb{k}} F_\alpha^3 >_{\mathbb{k}} \dots$$

Since  $F_\alpha^1$  concentrates on  $\alpha < \delta$ , this contradicts the minimality of  $\delta$ . □

Recall the following definition, due to Jech [12]:

**Definition 3.5.69.** Assume  $\delta$  is a regular cardinal. The *canonical order on stationary sets* is defined on stationary sets  $S, T \subseteq \delta$  by setting  $S < T$  if there is a closed unbounded set  $C \subseteq \delta$  such that  $S \cap \alpha$  is stationary in  $\alpha$  for all  $\alpha \in C \cap T$ .

**Definition 3.5.70.** For any ordinal  $\alpha$ , let  $\mathcal{C}_\alpha$  denote the filter of closed cofinal subsets of  $\alpha$ .

**Definition 3.5.71.** Suppose  $F$  is a filter on a set  $X$  and  $S$  is a set such that  $F$  does not concentrate on the complement of  $S$ . The *projection of  $F$  on  $S$*  is the filter defined by

$$F \upharpoonright S = \{A \cap S : A \in F\}$$

The following proposition connects the canonical order on stationary sets and the Ketonen order on filters:

**Proposition 3.5.72.** *Suppose  $\delta$  is a regular cardinal and  $S$  and  $T$  are stationary subsets of  $\delta$ . Then  $S < T$  implies  $\mathcal{C}_\delta \upharpoonright S <_{\mathbb{k}} \mathcal{C}_\delta \upharpoonright T$ .*

*Proof.* Fix a closed unbounded set  $C \subseteq \delta$  such that  $S \cap \alpha$  is stationary in  $\alpha$  for all  $\alpha \in C \cap T$ . Note that  $C \cap T \in \mathcal{C}_\delta \upharpoonright T$ , and for all  $\alpha \in C \cap T$ ,  $\mathcal{C}_\alpha \upharpoonright S$  is a countably complete filter concentrating on ordinals less than  $\alpha$ .

**Claim 1.**  $\mathcal{C}_\delta \upharpoonright S \subseteq (\mathcal{C}_\delta \upharpoonright T)\text{-}\lim_{\alpha \in C \cap T} \mathcal{C}_\alpha \upharpoonright S$ .

*Proof.* Suppose  $A \in \mathcal{C}_\delta \upharpoonright S$ . We will show that  $A \in (\mathcal{C}_\delta \upharpoonright T)\text{-}\lim_{\alpha \in C \cap T} \mathcal{C}_\alpha \upharpoonright S$ . Fix  $E \in \mathcal{C}_\delta$  such that  $S \cap E \subseteq A$ . Let  $E'$  be the set of accumulation points of  $E$ . Then for any  $\alpha \in E'$ ,  $S \cap (E \cap \alpha) \subseteq A$  and  $E \cap \alpha \in \mathcal{C}_\alpha$ , so  $A \in \mathcal{C}_\alpha \upharpoonright S$ . Thus

$$E' \cap C \cap T \subseteq \{\alpha \in C \cap T : A \in \mathcal{C}_\alpha \upharpoonright S\}$$

Since  $E' \cap C \in \mathcal{C}_\delta$ ,  $E' \cap C \cap T \in \mathcal{C}_\delta \upharpoonright T$ , and therefore  $\{\alpha \in C \cap T : A \in \mathcal{C}_\alpha \upharpoonright S\} \in \mathcal{C}_\delta \upharpoonright T$ . It follows that  $A \in (\mathcal{C}_\delta \upharpoonright T)\text{-}\lim_{\alpha \in C \cap T} \mathcal{C}_\alpha \upharpoonright S$ , as desired.  $\square$

The claim implies  $\mathcal{C}_\delta \upharpoonright S <_{\mathbb{k}} \mathcal{C}_\delta \upharpoonright T$ , as desired.  $\square$

As a corollary of Theorem 3.5.67 and Proposition 3.5.72, we have the following theorem of Jech:

**Corollary 3.5.73.** *The canonical order on stationary sets is wellfounded.*  $\square$

## 3.6 The linearity of the Ketonen order

In this final section, we prove a converse to Proposition 3.5.17, which can also be seen as a partial positive answer to Question 3.5.62. We say that *the Ketonen order is linear* if for all ordinals  $\delta$ , the Ketonen order on  $\mathcal{B}(\delta)$  is a linear order. The Ketonen order is linear if and only if its restriction to  $\text{Un}$  is a linear order.



**Theorem 3.6.1.** *The Ketonen order is linear if and only if the Ultrapower Axiom holds.*

Given Lemma 3.3.4, the linearity of the Ketonen order would appear to be a much weaker assumption than UA: the linearity of the Ketonen order only guarantees comparisons  $(k, h) : (M_U, M_W) \rightarrow N$  such that  $h$  is an internal ultrapower embedding of  $M_W$ , while UA asserts the existence of comparisons with both  $k$  and  $h$  internal. How can one transform partially internal comparisons into the fully internal comparisons required by UA?

To properly describe it, let us make some definitions:

**Definition 3.6.2.** Suppose  $M_0$ ,  $M_1$ , and  $N$  are transitive models of ZFC and

$$(k_0, k_1) : (M_0, M_1) \rightarrow N$$

are elementary embeddings.

- $(k_0, k_1)$  is *0-internal* if  $k_0$  is definable over  $M_0$ .
- $(k_0, k_1)$  is *1-internal* if  $k_1$  is definable over  $M_1$ .
- $(k_0, k_1)$  is *internal* if it is both 0-internal and 1-internal.

We indicated above that the difficulty in proving Theorem 3.6.1 is that it is not clear how to transform the 1-internal comparisons given by the linearity of the Ketonen order into the internal comparisons required to witness UA. In fact, it is simply impossible to do this in general, since as a consequence of the proof of Lemma 3.5.4, 1-internal comparisons can be proved to exist in ZFC alone.

**Proposition 3.6.3.** *Any pair of ultrapower embeddings has a 0-internal ultrapower comparison and a 1-internal ultrapower comparison. □*

Thus the true power of the linearity of the Ketonen order lies not in the mere existence of 1-internal comparisons  $(k, h)$  but rather in the existence of  $(k, h)$  witnessing  $U <_{\mathbf{k}} W$  (or  $W \leq_{\mathbf{k}} U$ ); that is, with the additional property  $k(a_U) < h(a_W)$ .

Theorem 3.6.1 is an immediate consequence of our next theorem, which shows how to explicitly define a comparison of a pair of ultrafilters:

**Theorem 3.6.4.** *Assume the Ketonen order is linear. Suppose  $\epsilon$  and  $\delta$  are ordinals. Suppose  $U \in \mathcal{B}(\epsilon)$  and  $W \in \mathcal{B}(\delta)$ .*

- *Let  $W_*$  be the least element of  $j_U(\mathcal{B}(\delta), <_{\mathbb{k}})$  extending  $j_U[W]$ .*
- *Let  $U_*$  be the least element of  $j_W(\mathcal{B}(\epsilon), <_{\mathbb{k}})$  extending  $j_W[U]$ .*

*Then  $(j_{W_*}^{M_U}, j_{U_*}^{M_W})$  is a comparison of  $(j_U, j_W)$ .*

The definitions of  $W_*$  and  $U_*$  rely on the fact that  $j_U(\mathcal{B}(\delta), <_{\mathbb{k}})$  and  $j_W(\mathcal{B}(\epsilon), <_{\mathbb{k}})$  are wellorders, not only in  $M_U$  and  $M_W$  but also by absoluteness in the true universe  $V$ . This, however, is not the main use of the linearity of the Ketonen order in the proof. Indeed, it is consistent that there is a pair of countably complete ultrafilters  $U$  and  $W$  such that the minimum extensions  $W_*$  and  $U_*$  are well-defined yet  $(j_U, j_W)$  admits no comparison.<sup>3</sup> Instead we will use the linearity of the Ketonen order to compare  $(j_{W_*}^{M_U} \circ j_U, j_{U_*}^{M_W} \circ j_W)$ :

**Lemma 3.6.5.** *Suppose  $\epsilon$  and  $\delta$  are ordinals. Suppose  $U \in \mathcal{B}(\epsilon)$  and  $W \in \mathcal{B}(\delta)$ .*

- *Let  $W_*$  be an element of  $j_U(\mathcal{B}(\delta))$  extending  $j_U[W]$ .*
- *Let  $U_*$  be a minimal element of  $j_W(\mathcal{B}(\epsilon), <_{\mathbb{k}})$  extending  $j_W[U]$ .*

*For any 1-internal ultrapower comparison*

$$(k, h) : (M_{W_*}^{M_U}, M_{U_*}^{M_W}) \rightarrow P$$

*of  $(j_{W_*}^{M_U} \circ j_U, j_{U_*}^{M_W} \circ j_W)$ , the following hold:*

$$h(j_{U_*}^{M_W}(a_W)) \leq k(a_{W_*}) \tag{3.1}$$

$$h(a_{U_*}) \leq k(j_{W_*}^{M_U}(a_U)) \tag{3.2}$$

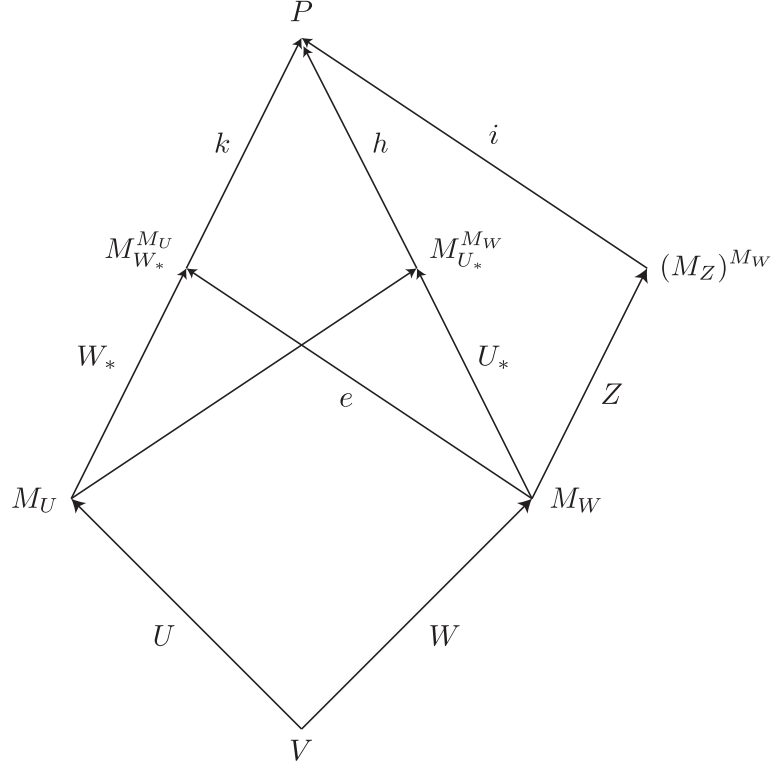


Figure 3.3: The proof of Lemma 3.6.5.

*Proof.* Let us direct the reader's attention to the key diagram, Fig. 3.3.

We first prove (3.1). By Lemma 3.2.17, there is an elementary embedding  $e : M_W \rightarrow M_{W_*}^{M_U}$  such that  $e \circ j_W = j_{W_*}^{M_U} \circ j_U$  and  $e(a_W) = a_{W_*}$ . We now apply the minimality of internal embeddings (Theorem 3.5.10). Note that  $k \circ e$  and  $h \circ j_{U_*}^{M_W}$  are both elementary embeddings from  $M_W$  to  $P$ , but  $h \circ j_{U_*}^{M_W}$  is an internal ultrapower embedding. Thus by Theorem 3.5.10,  $h(j_{U_*}^{M_W}(\alpha)) \leq k(e(\alpha))$  for all ordinals  $\alpha$ . It follows in particular that  $h(j_{U_*}^{M_W}(a_W)) \leq k(e(a_W)) = k(a_{W_*})$ , proving (3.1).

We now prove (3.2). To reduce subscripts, we define:

$$\alpha = j_{W_*}^{M_U}(a_U)$$

Let  $Z$  be the  $M_W$ -ultrafilter on  $j_W(\epsilon)$  derived from  $h \circ j_{U_*}^{M_W}$  using  $k(\alpha)$ , so

$$Z = (h \circ j_{U_*}^{M_W})^{-1} [p_{k(\alpha)}]$$

---

<sup>3</sup>Take  $U$  and  $W$  to be Mitchell incomparable normal ultrafilters. Apply Theorem 3.4.1 and Lemma 8.2.11 to see that  $j_U(W)$  and  $j_W(U)$  are the *only* extensions of  $j_U[W]$  and  $j_W[U]$  in  $M_U$  and  $M_W$  respectively.

Since  $h \circ j_{U_*}^{M_W}$  is an internal ultrapower embedding of  $M_W$ ,  $Z$  is a countably complete ultrafilter of  $M_W$ ; in other words,  $Z \in j_W(\mathcal{B}(\epsilon))$ . Moreover, it is not hard to compute that  $Z$  extends  $j_W[U]$ , or equivalently  $j_W^{-1}[Z] = U$ :

$$\begin{aligned}
j_W^{-1}[Z] &= j_W^{-1}[(h \circ j_{U_*}^{M_W})^{-1}[\mathbf{p}_{k(\alpha)}]] \\
&= (h \circ j_{U_*}^{M_W} \circ j_W)^{-1}[\mathbf{p}_{k(\alpha)}] \\
&= (k \circ j_{W_*}^{M_U} \circ j_U)^{-1}[\mathbf{p}_{k(\alpha)}] \\
&= (j_{W_*}^{M_U} \circ j_U)^{-1}[k^{-1}[\mathbf{p}_{k(\alpha)}]] \\
&= (j_{W_*}^{M_U} \circ j_U)^{-1}[\mathbf{p}_\alpha] \\
&= j_U^{-1}[(j_{W_*}^{M_U})^{-1}[\mathbf{p}_{j_{W_*}^{M_U}(a_U)}]] \\
&= j_U^{-1}[\mathbf{p}_{a_U}] = U
\end{aligned}$$

Since  $U_*$  is a minimal element of  $j_W(\mathcal{B}(\epsilon), <_{\mathbb{k}})$  extending  $j_W[U]$ ,  $M_W$  satisfies  $Z \not\prec_{\mathbb{k}} U_*$ .

Since  $Z$  is derived from  $h \circ j_{U_*}^{M_W}$  using  $k(\alpha)$ , there is a factor embedding  $i : (M_Z)^{M_W} \rightarrow P$  specified by the following properties:

$$i \circ j_Z^{M_W} = h \circ j_{U_*}^{M_W} \tag{3.3}$$

$$i(a_Z) = k(\alpha) \tag{3.4}$$

Note that these properties define  $i$  over  $M_W$ . Therefore by (3.3),  $(i, h)$  is a 1-internal ultrapower comparison of  $(j_Z^{M_W}, j_{U_*}^{M_W})$  in  $M_W$ . The fact that  $Z \not\prec_{\mathbb{k}} U_*$  in  $M_W$  implies

$$h(a_{U_*}) \leq i(a_Z) = k(\alpha) = k(j_{W_*}^{M_U}(a_U))$$

proving (3.2). □

Lemma 3.6.5 can be read as asserting that the natural ultrafilter representing the embedding  $j_{U_*}^{M_W} \circ j_W$  is not strictly above the one representing  $j_{W_*}^{M_U} \circ j_U$  in the Ketonen order. To make this precise, we need to define what these natural ultrafilters. This is related to the well-known notion of an ultrafilter sum:

**Definition 3.6.6.** Suppose  $U$  is an ultrafilter on  $X$ ,  $I$  is a set in  $U$ , and  $\langle W_i : i \in I \rangle$  is a sequence of ultrafilters on  $Y$ . The  $U$ -sum of  $\langle W_i : i \in I \rangle$  is the ultrafilter defined by

$$U\text{-}\sum_{i \in I} W_i = \{A \subseteq X \times Y : \{i \in I : A_i \in W_i\} \in U\}$$

In the definition above, if  $A \subseteq X \times Y$  and  $i \in X$ , then  $A_i = \{j \in Y : (i, j) \in A\}$ .

There is an obvious connection between sums and limits: the projection of a sum of ultrafilters onto its second coordinate is precisely equal to the limit of those ultrafilters.

**Lemma 3.6.7.** *Suppose  $U$  is an ultrafilter,  $I$  is a set in  $U$ , and  $\langle W_i : i \in I \rangle$  is a sequence of ultrafilters on  $Y$ . Let  $Z = [\langle W_i : i \in I \rangle]_U$  and let  $D = U\text{-}\sum_{i \in I} W_i$ . Then  $M_D = M_Z^{M_U}$ ,  $j_D = j_Z^{M_U} \circ j_U$ , and  $a_D = (j_Z^{M_U}(a_U), a_Z)$ .  $\square$*

Motivated this lemma, we introduce the following nonstandard notation.

**Definition 3.6.8.** Suppose  $U$  is an ultrafilter on  $X$ , and  $W_*$  is an  $M_U$ -ultrafilter on  $j_U(Y)$ . Then  $U\text{-}\sum W_*$  denotes the ultrafilter on  $X \times Y$  derived from  $j_{W_*}^{M_U} \circ j_U$  using  $(j_{W_*}^{M_U}(a_U), a_{W_*})$ .

In this section we will only require sums of ultrafilters where  $W_* \in M_U$ , but it is just more convenient not to choose a representative for  $W_*$ .

**Lemma 3.6.9.** *Suppose  $U$  is an ultrafilter and  $W_*$  is an  $M_U$ -ultrafilter on  $j_U(Y)$ . Then  $j_{U\text{-}\sum W_*} = j_{W_*}^{M_U} \circ j_U$ , and  $a_{U\text{-}\sum W_*} = (j_{W_*}^{M_U}(a_U), a_{W_*})$ .  $\square$*

In the context of Theorem 3.6.4, we would like to use Lemma 3.6.5 to conclude that the ultrafilters  $U\text{-}\sum W_*$  and  $W\text{-}\sum U_*$  are either equal or incomparable in the Ketonen order, and thus conclude by the linearity of the Ketonen order that  $U\text{-}\sum W_* = W\text{-}\sum U_*$ . The only remaining problem is that  $U\text{-}\sum W_*$  and  $W\text{-}\sum U_*$  are not ultrafilters on ordinals. But obviously we can associate Ketonen orders to an arbitrary wellorder:

**Definition 3.6.10.** Suppose  $(X, \prec)$  is a wellorder. The *Ketonen order associated to  $(X, \prec)$*  is the order  $(\mathcal{B}(X), \prec^k)$  defined on  $U, W \in \mathcal{B}(X)$  by setting  $U \prec^k W$  if there exist  $I \in W$  and  $\langle U_x : x \in I \rangle \in \prod_{x \in I} \mathcal{B}(X, X_{\prec x})$  such that  $U = W\text{-}\lim_{x \in I} U_x$ .

If  $(X, \prec)$  and  $(X', \prec')$  are isomorphic wellorders, then the associated Ketonen orders are also isomorphic, so in particular all the characterizations of the Ketonen order generalize to arbitrary wellorders:

**Lemma 3.6.11.** *Suppose  $(X, \prec)$  is a wellorder and  $U, W \in \mathcal{B}(X)$ . Then the following are equivalent:*

(1)  $U \prec^{\mathbb{k}} W$ .

(2) *There is a 1-internal ultrapower comparison  $(k, h) : (M_U, M_W) \rightarrow N$  of  $(j_U, j_W)$  such that  $k(a_U) \prec^* h(a_W)$  where  $\prec^* = k(j_U(\prec)) = h(j_W(\prec))$ .  $\square$*

It is convenient to introduce some notation for the statement of Lemma 3.6.13:

**Definition 3.6.12.** Let  $\text{flip} : \text{Ord} \times \text{Ord} \rightarrow \text{Ord} \times \text{Ord}$  be defined by  $\text{flip}(\alpha, \beta) = (\beta, \alpha)$ . Let  $\prec$  denote the Gödel order on  $\text{Ord} \times \text{Ord}$ .

The only property of the Gödel order that we need is that  $(\alpha_0, \beta_0) \prec (\alpha_1, \beta_1)$  implies that either  $\alpha_0 < \alpha_1$  or  $\beta_0 < \beta_1$ .

**Lemma 3.6.13.** *Suppose  $\epsilon$  and  $\delta$  are ordinals. Suppose  $U \in \mathcal{B}(\epsilon)$  and  $W \in \mathcal{B}(\delta)$ . Assume the Ketonen order  $(\mathcal{B}(\epsilon \times \delta), \prec^{\mathbb{k}})$  is linear.*

- *Let  $W_*$  be the least element of  $j_U(\mathcal{B}(\delta), \prec_{\mathbb{k}})$  extending  $j_U[W]$ .*
- *Let  $U_*$  be the least element of  $j_W(\mathcal{B}(\epsilon), \prec_{\mathbb{k}})$  extending  $j_W[U]$ .*

*Then  $U\text{-}\sum W_* = \text{flip}_*(W\text{-}\sum U_*)$ .*

*Proof.* Assume towards a contradiction that  $U\text{-}\sum W_* \prec^{\mathbb{k}} \text{flip}_*(W\text{-}\sum U_*)$ . The following identities are easily verified using Lemma 3.6.9:

$$\begin{aligned} j_{U\text{-}\sum W_*} &= j_{W_*}^{M_U} \circ j_U & j_{\text{flip}_*(W\text{-}\sum U_*)} &= j_{U_*}^{M_W} \circ j_W \\ a_{U\text{-}\sum W_*} &= (j_{W_*}^{M_U}(a_U), a_{W_*}) & a_{\text{flip}_*(W\text{-}\sum U_*)} &= (a_{U_*}, j_{U_*}^{M_W}(a_W)) \end{aligned}$$

By Lemma 3.6.11, the assumption that  $U\text{-}\sum W_* \prec^k \text{flip}_*(W\text{-}\sum U_*)$  is equivalent to the existence of a 1-internal comparison

$$(k, h) : (M_{W_*}^{M_U}, M_{U_*}^{M_W}) \rightarrow N$$

of  $(j_{W_*}^{M_U} \circ j_U, j_{U_*}^{M_W} \circ j_W)$  such that

$$k(j_{W_*}^{M_U}(a_U), a_{W_*}) \prec h(a_{U_*}, j_{U_*}^{M_W}(a_W))$$

Therefore either  $k(j_{W_*}^{M_U}(a_U)) < h(a_{U_*})$  or  $k(a_{W_*}) < h(j_{U_*}^{M_W}(a_W))$ , contradicting Lemma 3.6.5.

A symmetric argument shows that we cannot have  $\text{flip}_*(W\text{-}\sum U_*) \prec^k U\text{-}\sum W_*$  either. Thus by the linearity of  $(\mathcal{B}(\epsilon \times \delta), \prec^k)$ , we must have  $U\text{-}\sum W_* = \text{flip}_*(W\text{-}\sum U_*)$ , which proves the theorem.  $\square$

As an immediate consequence, we can prove Theorem 3.6.4:

*Proof of Theorem 3.6.4.* Let  $\alpha$  be the ordertype of the Gödel order on  $\epsilon \times \delta$ . Since the Ketonen order is linear on  $\mathcal{B}(\alpha)$ , the isomorphic order  $(\mathcal{B}(\epsilon \times \delta), \prec^k)$  is also linear. Thus we can apply Lemma 3.6.13 to conclude that  $U\text{-}\sum W_* = \text{flip}_*(W\text{-}\sum U_*)$ . In particular,  $U\text{-}\sum W_* \cong W\text{-}\sum U_*$ , so applying Lemma 3.6.9,

$$j_{W_*}^{M_U} \circ j_U = j_{U\text{-}\sum W_*} = j_{W\text{-}\sum U_*} = j_{U_*}^{M_W} \circ j_W$$

Thus  $(j_{W_*}^{M_U}, j_{U_*}^{M_W})$  is a comparison of  $(j_U, j_W)$ , as desired.  $\square$

Let us make some comments on this theorem. It is not immediately obvious from the definition that the linearity of the Ketonen order on  $\mathcal{B}(\lambda)$  implies the linearity of the Ketonen order on  $\mathcal{B}(\delta)$  for all ordinals  $\delta < \lambda^+$ .<sup>4</sup>

**Definition 3.6.14.** Suppose  $\lambda$  is a cardinal.

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<sup>4</sup>Note that if  $\kappa$  is regular, then for any  $n < \omega$ , the collection of subsets of  $\kappa^n$  of ordertype less than  $\kappa^n$  forms a  $\kappa$ -complete ideal; this is closely related to the Milner-Rado Paradox. Therefore for example if  $\kappa$  is  $2^\kappa$ -strongly compact, there is a  $\kappa$ -complete ultrafilter on  $\kappa^n$  that does not concentrate on a set of ordertype less than  $\kappa^n$ . (It suffices that  $\kappa$  is measurable.) This suggests it may be nontrivial to reduce the linearity of  $(\mathcal{B}(\kappa^2), \prec_k)$  to that of  $(\mathcal{B}(\kappa), \prec_k)$  by a direct combinatorial argument.

- $\text{UA}_{<\lambda}$  is the assertion that any pair of ultrapower embeddings of width less than  $\lambda$  have an internal ultrapower comparison.
- $\text{UA}_{\leq\lambda}$  is another way of writing  $\text{UA}_{<\lambda^+}$ .

**Corollary 3.6.15.** *Suppose  $\lambda$  is an infinite cardinal and the Ketonen order is linear on  $\mathcal{B}(\lambda)$ . Then  $\text{UA}_{\leq\lambda}$  holds. In particular, the Ketonen order is linear on  $\mathcal{B}(\delta)$  for all  $\delta < \lambda^+$ .*

*Proof.* Suppose  $U$  and  $W$  are ultrafilters on  $\lambda$ . To see  $\text{UA}_{\leq\lambda}$ , it suffices to show that  $(j_U, j_W)$  has a comparison. Since the Ketonen order  $(\mathcal{B}(\lambda), <_{\mathbf{k}})$  is linear, so is  $(\mathcal{B}(X), \prec^{\mathbf{k}})$  whenever  $(X, \prec)$  is a wellorder of ordertype  $\lambda$ . Since  $\lambda$  is an infinite cardinal, the Gödel order on  $\lambda \times \lambda$  has ordertype  $\lambda$ . Thus  $(\mathcal{B}(\lambda \times \lambda), \prec^{\mathbf{k}})$  is linear, and so we can apply Lemma 3.6.13 and the proof of Theorem 3.6.4 to conclude that  $(j_U, j_W)$  has a comparison.  $\square$

Surely with some extra work one can prove the following conjecture:

**Conjecture 3.6.16.** *If the Ketonen order is linear on countably complete incompressible ultrafilters, then the Ultrapower Axiom holds.*

The proof of Theorem 3.6.1 that we have given here uses Łoś's Theorem, which makes significant use of the Axiom of Choice. With care, however, the combinatorial content of Theorem 3.6.1, namely Lemma 3.6.13, can actually be established in  $\text{ZF} + \text{DC}$  alone. This makes the following question seem interesting:

**Question 3.6.17.** Assume  $\text{AD} + V = L(\mathbb{R})$ . Is the Ketonen order linear?



# Chapter 4

## The Generalized Mitchell Order

### 4.1 Introduction

#### The linearity of the generalized Mitchell order

The topic of this section is the generalized Mitchell order, which is defined by extending the definition of the Mitchell order to a broader class of objects:

**Definition 4.1.1.** The *generalized Mitchell order* is defined on countably complete ultrafilters  $U$  and  $W$  by setting  $U \triangleleft W$  if  $U \in M_W$ .

The main question we investigate here is to what extent this generalized order is linear assuming the Ultrapower Axiom. Recall that UA implies the linearity of the Mitchell order on normal ultrafilters (Theorem 2.3.11). On the other hand the generalized Mitchell order is obviously not a linear order on arbitrary countably complete ultrafilters (Section 4.2). The main theorem of this chapter is the generalization of the linearity of the Mitchell order on normal ultrafilters to normal fine ultrafilters:

**Definition 4.1.2.** For any ordinal  $\lambda$ , the *bounded powerset of  $\lambda$*  is the set  $P_{\text{bd}}(\lambda) = \bigcup_{\alpha < \lambda} P(\alpha)$ .

**Theorem 4.4.2 (UA).** *Suppose  $\lambda$  is a cardinal such that  $2^{<\lambda} = \lambda$ . Then the Mitchell order is linear on normal fine ultrafilters on  $P_{\text{bd}}(\lambda)$ .*

This amounts to the most general form of the linearity of the Mitchell order on normal fine ultrafilters that one could hope for (Proposition 4.4.12), except for the cardinal arithmetic assumption on  $\lambda$  (which we dispense with much later in Theorem 7.5.39).

## Outline of Chapter 4

We now outline the rest of this chapter.

**Section 4.2.** This contains various folklore facts about large cardinals and the generalized Mitchell order. None of the results here are due to the author. We give a brief exposition of the theory of strong embeddings (Section 4.2) and supercompact embeddings (Section 4.2) centered around the relationship between these concepts and the generalized Mitchell order. We also exposit the Kunen Inconsistency Theorem, which is closely related to the wellfoundedness properties of the Mitchell order. Finally we establish the basic order theoretic properties of the generalized Mitchell order, especially its transitivity, wellfoundedness (Theorem 4.2.47), and nonlinearity (Section 4.2).

**Section 4.3.** This section introduces the notion of Dodd soundness. This concept first arose in inner model theory, and our exposition is the first to put it into a general context. We begin by giving a very simple definition of Dodd soundness that will hopefully help the reader view it as a natural refinement of supercompactness. We then prove the equivalence of this notion with the definition of Dodd soundness from fine structure theory (Theorem 4.3.22).

A theorem of Schlutzenberg [7] (stated as Theorem 4.3.1 below) shows that the Mitchell order is linear on Dodd sound ultrafilters in the canonical inner models. We prove this theorem (Theorem 4.3.29) here under the much weaker assumption of UA and by a completely different and much simpler argument directly generalizing the proof of the linearity of the Mitchell order on normal ultrafilters.

**Section 4.4.** We finally turn to the Mitchell order on normal fine ultrafilters, the natural generalization of normal ultrafilters associated with supercompact cardinals. Our analysis proceeds by showing that normal fine ultrafilters are isomorphic to Dodd sound

ultrafilters, and then citing the linearity of the Mitchell order on Dodd sound ultrafilters. To do this, we introduce the notion of an *isonormal ultrafilter* and prove that every normal fine ultrafilter is isomorphic to an isonormal ultrafilter (Theorem 4.4.37). The main difficulty is the “singular case” (Section 4.4) which amounts to generalizing Solovay’s Lemma [21] (proved as Theorem 4.4.27) to singular cardinals. Theorem 4.4.25 states that if  $2^{<\lambda} = \lambda$ , then isonormal ultrafilters on  $\lambda$  are Dodd sound. Putting these theorems together, we obtain that under the Generalized Continuum Hypothesis, normal fine ultrafilters are isomorphic to Dodd sound ultrafilters, yielding the main theorem of the chapter (Theorem 4.4.2), the linearity of the Mitchell order on normal fine ultrafilters.

## 4.2 Folklore of the generalized Mitchell order

### Strength and the Mitchell order

The generalized Mitchell order is often viewed as a more finely calibrated generalization of the concept of the strength of an elementary embedding. In this subsection, we set down the basic theory of strength and discuss its relationship with the Mitchell order.

**Definition 4.2.1.** Suppose  $M$  is a transitive class and  $\lambda$  is a cardinal.

- An elementary embedding  $j : V \rightarrow M$  is  $\lambda$ -strong if  $P(\lambda) \subseteq M$ .
- An elementary embedding is  $<\lambda$ -strong if  $P_{\text{bd}}(\lambda) \subseteq M$ .

Notice that the property of being  $\lambda$ -strong depends only on  $M$ . The basic lemmas we prove about  $\lambda$ -strong embeddings almost all apply to arbitrary inner models containing  $P(\lambda)$ . (The embedding  $j$  just comes along for the ride.)

Most authors define  $j$  to be  $\alpha$ -strong if  $V_\alpha \subseteq M$ . The definition used here is arguably preferable (if one is assuming the Axiom of Choice and not assuming the Generalized Continuum Hypothesis). This is because it is more expressive:

**Lemma 4.2.2.** *Suppose  $j : V \rightarrow M$  is an elementary embedding.*

- *If  $\alpha$  is an ordinal, then  $V_{\alpha+1} \subseteq M$  if and only if  $j$  is  $\beth_\alpha$ -strong.*
- *If  $\gamma$  is a limit ordinal, then  $V_\gamma \subseteq M$  if and only if  $j$  is  $<\beth_\gamma$ -strong.* □

It would be strange to define strong embeddings without defining strong cardinals, so let us include the definition even though we will have little to say about the concept:

**Definition 4.2.3.** Suppose  $\kappa \leq \lambda$  are cardinals. Then  $\kappa$  is  $\lambda$ -strong if there is an inner model  $M$  and a  $\lambda$ -strong elementary embedding  $j : V \rightarrow M$  such that  $\text{CRT}(j) = \kappa$  and  $j(\kappa) > \lambda$ .  $\kappa$  is *strong* if  $\kappa$  is  $\lambda$ -strong for all  $\lambda$ .

The requirement that  $j(\kappa) > \lambda$  above is not actually necessary as a consequence of Theorem 4.2.37. We use standard notation for hereditary cardinality:

**Definition 4.2.4.** If  $x$  is a set,  $\text{tc}(x)$  denotes the smallest transitive set  $y$  with  $x \subseteq y$ . The *hereditary cardinality* of  $x$  is the cardinality of  $\text{tc}(x)$ . For any cardinal  $\lambda$ ,  $H(\lambda)$  denotes the collection of sets of hereditary cardinality less than  $\lambda$ .

**Lemma 4.2.5.** *For any infinite cardinal  $\lambda$ ,*

- *$H(\lambda^+)$  is a transitive set.*
- *$H(\lambda^+)$  is bi-interpretable with  $P(\lambda)$ .*
- *$H(\lambda)$  is bi-interpretable with  $P_{\text{bd}}(\lambda)$ .* □

The bi-interpretability of  $H(\lambda^+)$  and  $P(\lambda)$  yields the following lemma:

**Lemma 4.2.6.** *An embedding  $j : V \rightarrow M$  is  $<\lambda$ -strong if and only if  $H(\lambda) \subseteq M$  and  $\lambda$ -strong if and only if  $H(\lambda^+) \subseteq M$ .* □

**Definition 4.2.7.** The *strength* of an elementary embedding  $j : V \rightarrow M$ , denoted  $\text{STR}(j)$ , is the largest cardinal  $\lambda$  such that  $j$  is  $<\lambda$ -strong.

The following fact specifies exactly which powersets are contained in the target model of an elementary embedding in terms of its strength:

**Lemma 4.2.8.** *Suppose  $j : V \rightarrow M$  is an elementary embedding and  $\lambda$  is a cardinal. Then the following are equivalent:*

(1)  $\text{STR}(j) = \lambda$ .

(2) For all  $X \in M$ ,  $P(X) \subseteq M$  if and only if  $|X|^M < \lambda$ . □

The main limitation on the strength of an elementary embedding is known as the *Kunen Inconsistency Theorem* [22]:

**Theorem 4.2.9** (Kunen). *Suppose  $j : V \rightarrow M$  is a nontrivial elementary embedding and  $\lambda$  is the first fixed point of  $j$  above  $\text{CRT}(j)$ . Then  $\text{STR}(j) \leq \lambda$ .* □

We prove this and other related facts in Section 4.2.

The basic relationship between strength and the Mitchell order is given by the following two lemmas:

**Lemma 4.2.10.** *Suppose  $U$  and  $W$  are countably complete ultrafilters and  $U \triangleleft W$ . Then  $M_W$  is  $\lambda$ -strong where  $\lambda$  is the cardinality of the underlying set  $X$  of  $U$ . In fact,  $P(X) \subseteq M_W$ .*

*Proof.* Clearly  $X \in M_W$  since  $X \in U \in M_W$ . It suffices to show that  $P(X) \subseteq M_W$ . Fix  $A \subseteq X$ , and we will show  $A \in M_W$ . Since  $U$  is an ultrafilter, either  $A \in U$  or  $X \setminus A \in U$ . If  $A \in U$ , then  $A \in U \in M_W$ , so  $A \in M_W$ . If  $X \setminus A \in U$ , then similarly  $X \setminus A \in M_W$ , and since  $X \in M_W$ , it follows that  $A = X \setminus (X \setminus A) \in M_W$ . Therefore in either case,  $A \in M_W$ . □

**Lemma 4.2.11.** *Suppose  $W$  is a countably complete ultrafilter and  $j_W$  is  $2^\lambda$ -strong. Then for any countably complete ultrafilter  $U$  on  $\lambda$ ,  $U \triangleleft W$ .*

*Proof.* Since  $U \subseteq P(\lambda)$ ,  $U \in H((2^\lambda)^+) \subseteq M_W$ . □

This strength requirement implicit in the definition of the generalized Mitchell order may seem somewhat unnatural. What if one modified the Mitchell order, considering for example the *amenability relation* defined on countably complete ultrafilters by setting  $U \triangleleft W$  if and only if  $U$  concentrates on  $M_W$  and  $U \cap M_W \in M_W$ ? Such modified Mitchell orders are the subject of Section 5.5.

For the time being, we must point out some irritating properties of the generalized Mitchell order that suggest that in some sense it may be a little bit *too* general. The issue is that the definition of  $U \triangleleft W$  above has a strong dependence on the choice of the underlying set of  $U$ . For example, if  $W$  is nonprincipal, then the following hold:

- There is a principal ultrafilter  $D$  on an ordinal such that  $D \not\triangleleft W$ .
- There is a set  $x$  such that the principal ultrafilter  $\{\{x\}\} \not\triangleleft W$ .

For the first bullet point, let  $\lambda$  be the strength of  $j_W$ , and let  $D$  be any principal ultrafilter on  $\lambda$ . For the second bullet point, let  $x$  be any set that does not belong to  $M_W$ .

These silly counterexamples suggest that the generalized Mitchell order is only a well-behaved relation on a restricted class of ultrafilters. Recall that for any ultrafilter  $U$  on a set  $X$ ,  $\lambda_U$  is defined to be the least cardinality of a set in  $U$ , and  $U$  is said to be uniform if  $|X| = \lambda_U$ . Hereditary uniformity is a strengthening of uniformity:

**Definition 4.2.12.** An ultrafilter  $U$  on a set  $X$  is *hereditarily uniform* if  $\lambda_U$  is the hereditary cardinality of  $X$ .

Any ultrafilter  $U$  is isomorphic to a hereditarily uniform ultrafilter since in fact  $U$  is isomorphic to an ultrafilter on  $\lambda_U$  (Lemma 2.2.24). The following lemma argues that the generalized Mitchell order is a reasonable relation on the class of hereditarily uniform ultrafilters:

**Lemma 4.2.13.** *Suppose  $U' \leq_{\text{RK}} U \triangleleft W$  are countably complete ultrafilters. Let  $X$  and  $X'$  be the underlying sets of  $U$  and  $U'$ , and assume  $X' \in M_W$  and  $M_W$  satisfies  $|X'| \leq |X|$ . Then  $U' \triangleleft W$  and  $M_W$  satisfies  $U' \leq_{\text{RK}} U$ . If  $U' \cong U$ , then  $M_W$  satisfies  $U' \cong U$ .*

**Lemma 4.2.14.** *Suppose  $U' \leq_{\text{RK}} U \triangleleft W$  are countably complete ultrafilters and  $U'$  is hereditarily uniform. Then  $U' \triangleleft W$  and  $M_W$  satisfies  $U' \leq_{\text{RK}} U$ . If  $U' \cong U$ , then  $M_W$  satisfies  $U' \cong U$ . In particular, the restriction of the generalized Mitchell order to hereditarily uniform ultrafilters is isomorphism invariant.*

Lemma 4.2.13 and Lemma 4.2.14 follow from a fact that is both more general and easier to prove:

**Lemma 4.2.15.** *Suppose  $M$  is an inner model of ZFC,  $\lambda$  is a cardinal, and  $X \in M$  is a set of cardinality  $\lambda$  such that  $P(X) \subseteq M$ .*

- *For any set  $Y \in M$  such that  $M \models |Y| \leq |X|$ ,  $P(Y) \subseteq M$ .*
- *For any set  $Y \in M$  such that  $M \models |Y| \leq |X|$ ,  $P(X \times Y) \subseteq M$ .*
- *For any set  $Y \in M$  such that  $M \models |Y| \leq |X|$ , every function from  $X$  to  $P(Y)$  belongs to  $M$ .*
- *$P(\lambda) \subseteq M$ .*
- *Every set of hereditary cardinality at most  $\lambda$  belongs to  $M$  and has hereditary cardinality at most  $\lambda$  in  $M$ .* □

The bullet points are arranged in such a way that the reader should have no trouble proving each one in turn.<sup>1</sup>

*Proof of Lemma 4.2.13.* Fix  $f : X \rightarrow X'$  such that  $f_*(U) = U'$ . By Lemma 4.2.15,  $f \in M_W$ , and hence  $U' = f_*(U) \in M_W$ . Moreover  $f$  witnesses  $U' \leq_{\text{RK}} U$  in  $M_W$ . Finally if  $U' \cong U$ , then this is also witnessed by some  $g \in M_W$ . □

<sup>1</sup>It is likely, however, that the second bullet-point cannot be established if  $M$  is not assumed to satisfy the Axiom of Choice.

*Proof of Lemma 4.2.14.* By Lemma 4.2.15, the underlying set of  $U'$  belongs to  $M_W$  and has hereditary cardinality at most  $\lambda_{U'} \leq \lambda_U \leq |X|$  in  $M_W$ , so the lemma follows from Lemma 4.2.13.  $\square$

## Supercompactness and the Mitchell order

We now turn to a concept that is more pertinent to this dissertation than strength: supercompactness.

**Definition 4.2.16.** Suppose  $M$  is a transitive class and  $X$  is a set. An elementary embedding  $j : V \rightarrow M$  is  *$X$ -supercompact* if  $j[X] \in M$ .

The following lemma allows us to focus solely on the case of  $\lambda$ -supercompact embeddings for  $\lambda$  a cardinal:

**Lemma 4.2.17.** *Suppose  $X$  and  $Y$  are sets such that  $|X| = |Y|$ . Then an elementary embedding  $j : V \rightarrow M$  is  $X$ -supercompact if and only if it is  $Y$ -supercompact. In particular,  $j$  is  $X$ -supercompact if and only if  $j$  is  $|X|$ -supercompact.*

*Proof.* Suppose  $j$  is  $X$ -supercompact and  $f : X \rightarrow Y$  is a surjection. Then

$$j(f)[j[X]] = j[Y]$$

so  $j$  is  $Y$ -supercompact.  $\square$

**Definition 4.2.18.** Suppose  $\kappa \leq \lambda$  are cardinals. Then  $\kappa$  is  *$\lambda$ -supercompact* if there is a  $\lambda$ -supercompact embedding  $j : V \rightarrow M$  such that  $\text{CRT}(j) = \kappa$  and  $j(\kappa) > \lambda$ ;  $\kappa$  is *supercompact* if  $\kappa$  is  $\lambda$ -supercompact for all cardinals  $\lambda \geq \kappa$ .

The results of this dissertation (Section 8.4) single out a class of ultrapower embeddings that are just shy of  $\lambda$ -supercompact, so the following is an important definition:

**Definition 4.2.19.** Suppose  $\lambda$  is a cardinal. An elementary embedding  $j : V \rightarrow M$  is  *$<\lambda$ -supercompact* if  $j$  is  $\delta$ -supercompact for all cardinals  $\delta < \lambda$ .



The definition of supercompactness is motivated by its relationship with the closure of  $M$  under  $\lambda$ -sequences:

**Lemma 4.2.20.** *Suppose  $j : V \rightarrow M$  is an elementary embedding and  $\lambda$  is a cardinal.*

- (1)  *$j$  is  $\lambda$ -supercompact if and only if  $j \upharpoonright \lambda \in M$ .*
- (2) *If  $j$  is  $\lambda$ -supercompact, then  $j$  is  $\lambda$ -strong.*
- (3) *If  $j$  is  $\lambda$ -supercompact, then  $j[X] \in M$  for all  $X$  of cardinality  $\lambda$ .*
- (4) *If  $j$  is  $\lambda$ -supercompact and  $M = H^M(j[V] \cup S)$  for some  $S \subseteq M$  such that  $S^\lambda \subseteq M$ , then  $M^\lambda \subseteq M$ .*

*Proof.* For (1), note that  $j \upharpoonright \lambda$  is the inverse of the transitive collapse of  $j[\lambda]$ .

For (2), suppose  $A \subseteq \lambda$ . Then  $A = (j \upharpoonright \lambda)^{-1}[j(A)]$ , so since  $j \upharpoonright \lambda$  and  $j(A)$  both belong to  $M$ , so does  $A$ .

(3) is immediate from Lemma 4.2.17.

For (4), fix  $\langle x_\alpha : \alpha < \lambda \rangle \in M^\lambda$ . Fix  $\langle f_\alpha : \alpha < \lambda \rangle$  and  $\langle a_\alpha : \alpha < \lambda \rangle \in S^\lambda$  such that  $x_\alpha = j(f_\alpha)(a_\alpha)$  for all  $\alpha < \lambda$ . The function  $G : j[\lambda] \rightarrow M$  defined by  $G(j(\alpha)) = j(f_\alpha)$  belongs to  $M$  by (3), since

$$G = j[\{(\alpha, f_\alpha) : \alpha < \lambda\}]$$

Therefore the sequence  $\langle j(f_\alpha) : \alpha < \lambda \rangle$  can be computed from  $G$  and  $j \upharpoonright \lambda$ :

$$j(f_\alpha) = G \circ (j \upharpoonright \lambda)(\alpha)$$

Since both  $G$  and  $j \upharpoonright \lambda$  belong to  $M$  by (1),  $\langle j(f_\alpha) : \alpha < \lambda \rangle \in M$ . Finally,

$$\langle x_\alpha : \alpha < \lambda \rangle = \langle j(f_\alpha)(a_\alpha) : \alpha < \lambda \rangle$$

can be computed from  $\langle j(f_\alpha) : \alpha < \lambda \rangle$  and  $\langle a_\alpha : \alpha < \lambda \rangle$ . Both these sequences belong to  $M$ , since  $\langle a_\alpha : \alpha < \lambda \rangle \in S^\lambda \subseteq M$ , so  $\langle x_\alpha : \alpha < \lambda \rangle \in M$ , as desired.  $\square$

For the purposes of this dissertation, the most relevant corollary of Lemma 4.2.20 is its application to ultrapower embeddings:

**Corollary 4.2.21.** *An ultrapower embedding  $j : V \rightarrow M$  is  $\lambda$ -supercompact if and only if  $M^\lambda \subseteq M$ .*

*Proof.* Fix  $a \in M$  such that  $M = H^M(j[V] \cup \{a\})$ . The corollary follows from applying Lemma 4.2.20 (4) in the case  $S = \{a\}$ .  $\square$

We can make good use of Corollary 4.2.21 since it is always possible to derive a  $\lambda$ -supercompact ultrapower embeddings from a  $\lambda$ -supercompact embedding:

**Lemma 4.2.22.** *Suppose  $j : V \rightarrow M$  is an  $X$ -supercompact embedding,  $V \xrightarrow{i} N \xrightarrow{k} M$  are elementary embeddings,  $k \circ i = j$ , and  $j[X] \in k[N]$ . Then  $i$  is  $X$ -supercompact and  $k(i[X]) = j[X]$ . In particular, letting  $\lambda = |X|$ ,  $k \upharpoonright \lambda + 1$  is the identity.*

*Proof.* Fix  $S \in M$  such that  $k(S) = j[X]$ . Then

$$S = k^{-1}[k(S)] = k^{-1}[j[X]] = k^{-1} \circ j[X] = i[X]$$

Thus  $i[X] = S \in M$ , so  $i$  is  $X$ -supercompact, and moreover,  $k(i[X]) = k(S) = j[X]$ .

Since  $k(i[X]) = j[X]$ , the argument of Lemma 4.2.17 shows  $k(i[\lambda]) = j[\lambda]$ . But then if  $\alpha \leq \lambda$ ,  $k(\alpha) = k(\text{ot}(i[\lambda] \cap i(\alpha))) = \text{ot}(k(i[\lambda]) \cap k(i(\alpha))) = \text{ot}(j[\lambda] \cap j(\alpha)) = \alpha$ .  $\square$

**Definition 4.2.23.** The *supercompactness* of an elementary embedding  $j : V \rightarrow M$  is the least cardinal  $\lambda$  such that  $j$  is not  $\lambda$ -supercompact.

Which cardinals are the supercompactness of an elementary embedding? Which are the supercompactness of an ultrapower embedding? This turns out to be a major distinction:

**Proposition 4.2.24.** *Suppose  $\lambda$  is a singular cardinal and  $j : V \rightarrow M$  is an elementary embedding such that  $M^{<\lambda} \subseteq M$ . Then  $M^\lambda \subseteq M$ .*  $\square$

Thus the supercompactness of an ultrapower embedding is always regular, while it is easy to see this must fail for arbitrary embeddings if there is a  $\kappa^+\omega$ -supercompact cardinal. An important point is that if the cofinality of  $\lambda$  is small,  $\lambda$ -supercompactness is equivalent to  $\lambda^+$ -supercompactness:

**Lemma 4.2.25.** *Suppose  $\lambda$  is a cardinal,  $j : V \rightarrow M$  is elementary embedding, and  $\kappa = \text{CRT}(j)$ . If  $j$  is  $\lambda$ -supercompact, then  $j$  is  $\lambda^{<\kappa}$ -supercompact.*

*Proof.* Assume  $j[\lambda] \in M$ , and we will show that  $j[P_\kappa(\lambda)] \in M$ . Note that for  $\sigma \in P_\kappa(\lambda)$ ,  $j(\sigma) = j[\sigma]$ . Thus

$$j[P_\kappa(\lambda)] = \{j[\sigma] : \sigma \in P_\kappa(\lambda)\} = P_\kappa(j[\lambda])$$

One consequence of this is that  $P_\kappa(j[\lambda]) \subseteq M$ , since  $j[P_\kappa(\lambda)] \subseteq M$ , and therefore  $P_\kappa(j[\lambda]) = (P_\kappa(j[\lambda]))^M \in M$ . It follows that  $j[P_\kappa(\lambda)] \in M$ , as desired.  $\square$

It follows for example that the supercompactness of an elementary embedding is never the successor of a singular cardinal  $\gamma$  of countable cofinality, since  $\gamma^\omega = \gamma^+$ . This is an important component in the proof of Kunen's Inconsistency Theorem (Theorem 4.2.37).

We now begin to examine the relationship between supercompactness and the Mitchell order, which turns out to be central to the rest of this dissertation. The key point is that if  $U \triangleleft W$ , then the supercompactness of  $M_W$  determines the extent to which the ultrapower of  $M_W$  by  $U$  is correctly computed by  $M_W$ .

**Lemma 4.2.26.** *Suppose  $U \triangleleft W$  are countably complete ultrafilters. Then there is a unique elementary embedding  $k : (M_U)^{M_W} \rightarrow j_U(M_W)$  such that  $k \circ (j_U)^{M_W} = j_U \upharpoonright M_W$  and  $k(a_U^{M_W}) = a_U$ . Let  $X$  be the underlying set of  $U$ . Then  $k \upharpoonright j_U((2^\lambda)^{M_W}) + 1$  where  $\lambda = |X|$ .*

*Proof.* Since  $P(X) \subseteq M_W$ ,  $U$  is the ultrafilter derived from  $j_U \upharpoonright M_W$  using  $a_U$ . Thus there is a unique factor embedding  $k : (M_U)^{M_W} \rightarrow j_U(M_W)$  such that  $k \circ (j_U)^{M_W} = j_U \upharpoonright M_W$  and  $k(a_U^{M_W}) = a_U$ . This establishes the first part of the lemma.

As for the second part, since  $U \triangleleft W$ , we have  $P(X) \subseteq M_W$  and hence by Lemma 4.2.15,  $P(\lambda) \subseteq M_W$  and every function from  $X$  to  $P(\lambda)$  belongs to  $M_W$ . It follows that  $j_U(P(\lambda)) \subseteq \text{ran}(k)$ : if  $A \in j_U(P(\lambda))$ , then  $A = j_U(f)(a_U)$  for some  $f : X \rightarrow P(\lambda)$ , and therefore

$$A = k(j_U^{M_W}(f)(a_U^{M_W})) \in \text{ran}(k)$$

Since there is a surjection  $g : P(\lambda) \rightarrow (2^\lambda)^{M_W}$  in  $M_W$ ,

$$j_U(g)[j_U(P(\lambda))] = j_U((2^\lambda)^{M_W}) \subseteq \text{ran}(k)$$

Moreover  $j_U((2^\lambda)^{M_W}) \in j_U[M_W] \subseteq \text{ran}(k)$ . Thus  $j_U((2^\lambda)^{M_W}) + 1 \subseteq \text{ran}(k)$ , or in other words,  $k \upharpoonright j_U((2^\lambda)^{M_W}) + 1 = \text{id}$ .  $\square$

We will refer to the embedding of Lemma 4.2.26 as a *factor embedding*.

**Lemma 4.2.27.** *Suppose  $U$  and  $W$  are countably complete ultrafilters with  $U \triangleleft W$ . Let  $X$  be the underlying set of  $U$ , let  $\lambda = |X|$  and let  $\delta = ((2^\lambda)^+)^{M_W}$ . Then*

$$j_U^{M_W} \upharpoonright H^{M_W}(\delta) = j_U \upharpoonright H^{M_W}(\delta)$$

*Proof.* Let  $k : (M_U)^{M_W} \rightarrow j_U(M_W)$  be the factor embedding with  $k \circ (j_U)^{M_W} = j_U \upharpoonright M_W$  and  $k(a_U^{M_W}) = a_U$ . Then Lemma 4.2.26 implies  $k \upharpoonright j_U^{M_W}(\delta)$  is the identity, and therefore  $k \upharpoonright j_U^{M_W}(H^{M_W}(\delta))$  is the identity. Now

$$j_U^{M_W} \upharpoonright H^{M_W}(\delta) = (k \upharpoonright j_U^{M_W}(H^{M_W}(\delta))) \circ (j_U^{M_W} \upharpoonright H^{M_W}(\delta)) = j_U \upharpoonright H^{M_W}(\delta) \quad \square$$

Our next proposition, Proposition 4.2.28, suggests that the Mitchell order on ultrafilters be seen as a generalization of supercompactness that asks for one ultrapower  $M_W$  how much it can see of another embedding  $j_U$ . (On this view supercompactness is the special case in which we ask how much of  $j_U$  is seen by  $M_U$  itself.)

**Proposition 4.2.28.** *Suppose  $U$  and  $W$  are countably complete ultrafilters. Let  $X$  be the underlying set of  $U$ , let  $\lambda = |X|$  and let  $\delta = ((2^\lambda)^+)^{M_W}$ . Then the following are equivalent:*

(1)  $U \triangleleft W$ .

(2)  $j_U \upharpoonright H^{M_W}(\delta) \in M_W$ .

(3)  $j_U \upharpoonright P(\lambda) \in M_W$ .

(4)  $j_U \upharpoonright P(X) \in M_W$ .

*Proof.* (1) implies (2). Immediate from Lemma 4.2.27.

(2) implies (3). Immediate since  $P(\lambda) \subseteq H^{M_W}(\delta)$ .

(2) implies (3). This is probably clear enough (and in any case, (1) implies (4) is easy), but let us just make sure. By Lemma 4.2.15,  $|X|^M = \lambda$ . Let  $\rho : \lambda \rightarrow X$  be a surjection in  $M_W$ . For  $A \in P(X)$ ,

$$j_U(A) = j_U(\rho)[j_U(\rho^{-1}[A])]$$

(3) implies (1). If  $j_U \upharpoonright P(X)$  belongs to  $M_W$ , then  $U = \{A \subseteq X : a_U \in j_U(A)\}$  belongs to  $M_W$  as well. □

Given Lemma 4.2.27, it is reasonable to wonder whether the whole embedding  $j_U \upharpoonright M_W$  might be correctly computed by  $M_W$  as well; that is, perhaps the factor embedding  $k$  is always trivial. We provide a counterexample in Proposition 5.5.5.<sup>2</sup> This is equivalent to the supercompactness of  $j_W$ , a phenomenon we exploit later:

**Proposition 4.2.29.** *Suppose  $U \triangleleft W$  are countably complete ultrafilters. Then the following are equivalent:*

(1)  $(j_U)^{M_W} = j_U \upharpoonright M_W$ .

(2)  $j_W$  is  $\lambda_U$ -supercompact.

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<sup>2</sup>This counterexample also shows that in the context of Lemma 4.2.26, the lower bound given there on  $\text{CRT}(k)$  can be tight in the sense that (consistently) one can have

$$\text{CRT}(k) = j_U \left( (2^\lambda)^{M_W} \right)^{+(M_U)^{M_W}}$$

*Proof.* (1) implies (2): Let  $k : (M_U)^{M_W} \rightarrow j_U(M_W)$  be the factor embedding of Lemma 4.2.26, with  $k \circ j_U^{M_W} = j_U \upharpoonright M_W$  and  $k(a_U^{M_W}) = a_U$ . Since  $(j_U)^{M_W} = j_U \upharpoonright M_W$ , we have that  $k : j_U(M_W) \rightarrow j_U(M_W)$  and  $k \circ j_U \circ j_W = j_U \circ j_W$ . Hence by the basic theory of the Rudin-Keisler order (Theorem 3.4.8),  $k$  is the identity.

It follows in particular that  $j_U(j_W)(a_U) \in \text{ran}(k)$ . Fix  $f : X \rightarrow M_W$  in  $M_W$  such that

$$k(j_U^{M_W}(f)(a_U^{M_W})) = j_U(j_W)(a_U)$$

Thus  $j_U(f)(a_U) = j_U(j_W)(a_U)$ , so by Łoś's Theorem, there is a set  $A \in U$  such that  $f \upharpoonright A = j_W \upharpoonright A$ . Since  $P(X) \subseteq M_W$ ,  $A \in M_W$ , and hence  $j_W \upharpoonright A = f \upharpoonright A \in M_W$ . In particular,  $j_W[A] \in M_W$ , so  $j_W$  is  $A$ -supercompact. By Lemma 4.2.17,  $j_W$  is  $|A|$ -supercompact, and since  $\lambda_U \leq |A|$ , it follows that  $j_W$  is  $\lambda_U$ -supercompact.

(2) implies (1): Obvious. □

Down the line (Theorem 8.3.26) we will show that under UA, whenever  $U \triangleleft W$ , in fact  $j_W$  is  $\lambda_U$ -supercompact (and in fact it suffices that  $P(\lambda_U) \subseteq M_W$ ), and thus  $j_W^{M_U} = j_W \upharpoonright M_U$ . For now, let us mention a generalization of Proposition 4.2.29, whose proof we omit:

**Proposition 4.2.30.** *Suppose  $U$  and  $W$  are countably complete ultrafilters such that  $U$  concentrates on a set in  $M_W$ . The following are equivalent:*

- (1)  $j_{U \cap M_W}^{M_W} = j_U \upharpoonright M_W$
- (2) *There is a function  $f \in M_W$  such that  $f \upharpoonright A = j_W \upharpoonright A$  for some  $A \in U$ .*
- (3) *For all  $f : I \rightarrow M_W$  where  $I \in U$ , there is some  $g \in M_W$  such that  $g \upharpoonright A = f \upharpoonright A$  for some  $A \in U$ .* □

We finish this section with a restriction on the supercompactness of an ultrafilter:

**Proposition 4.2.31.** *Suppose  $U$  is an ultrafilter and  $j_U$  is  $\lambda_U^+$ -supercompact. Then  $U$  is principal.*

We use the following lemma:

**Lemma 4.2.32.** *Suppose  $j : V \rightarrow M$  is an elementary embedding that is discontinuous at the infinite cardinal  $\lambda$ . Let  $\lambda_* = \sup j[\lambda]$ . Then*

$$\lambda^+ \leq \lambda_*^{+M} < j(\lambda)^{+M} = j(\lambda^+)$$

*If  $j$  is continuous at  $\lambda^+$ , then  $j(\lambda^+)$  is a singular ordinal of cofinality  $\lambda^+$ , so  $j(\lambda^+) < j(\lambda)^+$ .*

*Proof.* We first show that  $\lambda^+ \leq \lambda_*^{+M}$ . Suppose  $\alpha < \lambda^+$ . Let  $\prec$  be a wellorder of  $\lambda$  such that  $\text{ot}(\prec) = \alpha$ . Then  $\prec_* = j(\prec) \upharpoonright \lambda_*$  is a wellorder of  $\lambda_*$  and  $j$  restricts to an order-preserving embedding from  $(\lambda, \prec)$  into  $(\lambda_*, \prec_*)$ . Therefore

$$\alpha \leq \text{ot}(\lambda_*, \prec_*) < \lambda_*^{+M}$$

The final inequality follows from the fact that  $(\lambda_*, \prec_*)$  belongs to  $M$ . Since  $\alpha < \lambda^+$  was arbitrary, it follows that  $\lambda^+ \leq \lambda_*^{+M}$ .

To prove  $\lambda_*^{+M} < j(\lambda)^{+M}$ , it is of course enough to show  $\lambda_*^{+M} \leq j(\lambda)$ . But  $j(\lambda)$  is a cardinal of  $M$  that is greater than  $\lambda_*$ , and hence  $\lambda_*^{+M} \leq j(\lambda)$ .

Finally, assume that  $j$  is continuous at  $\lambda^+$ . Obviously  $j(\lambda^+)$  has cofinality  $\lambda^+$ , but the point is that this implies  $j(\lambda^+)$  is *singular*, since the inequalities above show  $\lambda^+ < j(\lambda^+)$ . We can therefore conclude  $j(\lambda)^{+M} < j(\lambda)^+$ : obviously  $j(\lambda)^{+M} \leq j(\lambda)^+$ , but the point is that equality cannot hold since  $j(\lambda^+)$  is singular and  $j(\lambda)^+$  is regular.  $\square$

*Proof of Proposition 4.2.31.* Let  $\lambda = \lambda_U$ . Without loss of generality, we may assume that  $U$  is a uniform ultrafilter on  $\lambda$  and  $\lambda$  is infinite. Thus  $j_U$  is discontinuous at  $\lambda$ . Assume towards a contradiction that  $j_U[\lambda^+] \in M_U$ . By Lemma 4.2.32,  $j_U(\lambda^+) > \lambda^+$ . But by Lemma 3.5.32,  $j_U$  is continuous at  $\lambda^+$ , and therefore  $j_U[\lambda^+] \in M_U$  is a cofinal subset of  $j_U(\lambda^+)$  of ordertype  $\lambda^+$ . Hence  $\text{cf}^{M_U}(j_U(\lambda)^{+M_U}) = \lambda^+ < j_U(\lambda^+)$ , and this contradicts that  $j_U(\lambda^+)$  is regular in  $M_U$ .  $\square$

## The Kunen Inconsistency

The story of the Kunen Inconsistency Theorem is often cast as a cautionary tale with the moral that a large cardinal hypothesis may turn out to be false for nontrivial combinatorial reasons:

**Theorem 4.2.33** (Kunen). *There is no nontrivial elementary embedding from the universe to itself.* □

A more pragmatic perspective is to view the Kunen Inconsistency as a proof technique, providing at least some constraint on the elementary embeddings a large cardinal theorist is bound to analyze. Examples pervade this work, but to pick the closest one, the Kunen Inconsistency will form a key component of the proof of the wellfoundedness of the Mitchell order in Section 4.2. Since our applications of Kunen's theorem will require the basic concepts from the proof (especially the notion of a critical sequence), we devote this subsection for a brief exposition of this topic.

We first give a proof of a version of Kunen's inconsistency Theorem that is due to Harada. (Another writeup of this proof appears in Kanamori's textbook [23].) The methods are purely ultrafilter-theoretic methods and very much in the spirit of this dissertation:

**Proposition 4.2.34** (Kunen). *Suppose  $j : V \rightarrow M$  is an elementary embedding and  $\eta$  is a strong limit cardinal such that  $j$  is  $\eta$ -supercompact and  $j[\eta] \subseteq \eta$ . Then  $j \upharpoonright \eta = \text{id}$ .*

*Proof.* Assume the proposition holds for all  $\bar{\eta} < \eta$ .

If  $\eta$  has uncountable cofinality, then there is an  $\omega$ -closed unbounded set of  $\bar{\eta} < \eta$  such that  $j[\bar{\eta}] \subseteq \bar{\eta}$ . Therefore  $j \upharpoonright \bar{\eta} = \text{id}$  for unboundedly many  $\bar{\eta} < \eta$ , so  $j \upharpoonright \eta = \text{id}$ .

Assume instead that  $\eta$  has countable cofinality. Then  $j$  is continuous at  $\eta$ , so since  $j[\eta] \subseteq \eta$ , we have  $j(\eta) = \eta$ . We essentially reduce to the case that  $j$  is the ultrapower of the universe by an ultrafilter  $U$  on  $P(\eta)$ . Let  $U$  be the ultrafilter on  $P(\eta)$  derived from  $j$  using  $j[\eta]$ . Let  $k : M_U \rightarrow M$  be the factor embedding. Then  $j[\eta] \in k[M_U]$ , so by our analysis



of the supercompactness of derived embeddings (Lemma 4.2.22),  $j_U$  is  $\eta$ -supercompact and  $k \upharpoonright \eta + 1 = \text{id}$ . It follows that  $j_U(\eta) = \eta$ . Moreover, if we show  $j_U \upharpoonright \eta = \text{id}$  then we can conclude  $j \upharpoonright \eta = \text{id}$ . In fact, we will show that  $U$  is principal.

By Lemma 4.2.25,  $j_U$  is actually  $\eta^\omega$ -supercompact. Since  $\eta$  is a strong limit cardinal of countable cofinality,  $\eta^\omega = 2^\eta$ . Thus  $j_U$  is  $2^\eta$ -supercompact. Recall that Proposition 4.2.31 states that if  $j_U$  is  $\lambda_U^+$ -supercompact, then  $U$  is principal. Thus to show  $U$  is principal, it suffices to show that  $\lambda_U < 2^\eta$ .

Since  $U$  is an ultrafilter on  $P(\eta)$ ,  $\lambda_U \leq |P(\eta)| \leq 2^\eta$ , so in fact, we need only show  $\lambda_U \neq 2^\eta$ . Since  $U$  is isomorphic to a uniform ultrafilter on  $\lambda_U$ ,  $j_U$  is discontinuous at  $\lambda_U$ , and in particular  $j_U(\lambda_U) \neq \lambda_U$ . On the other hand, since  $M_U$  is closed under  $2^\eta$ -sequences,  $(2^\eta)^{M_U} = 2^\eta$ , and hence

$$j_U(2^\eta) = (2^{j_U(\eta)})^{M_U} = (2^\eta)^{M_U} = 2^\eta$$

Since  $\lambda_U$  is moved by  $j_U$  while  $2^\eta$  is fixed,  $\lambda_U \neq 2^\eta$ , and hence  $\lambda_U < 2^\eta$ , as desired.  $\square$

Suppose  $j : V \rightarrow M$  is an elementary embedding with critical point  $\kappa$ . Let  $\lambda$  be the first ordinal above  $\kappa$  such that  $j[\lambda] \subseteq \lambda$ , the first ordinal at which one might be able to apply the Kunen argument. Proposition 4.2.34 tells us that  $j[\lambda] \notin M$  if  $\lambda$  is a strong limit; we would like to see that in fact  $j[\lambda]$  never belongs to  $M$ . This follows from the critical sequence analysis of  $\lambda$ :

**Definition 4.2.35.** Suppose  $N$  and  $P$  are transitive models of ZFC with the same ordinals and  $j : N \rightarrow P$  is a nontrivial elementary embedding. The *critical sequence* of  $j$  is the sequence  $\langle \kappa_n : n < \omega \rangle$  defined by recursion:  $\kappa_0 = \text{CRT}(j)$  and for all  $n < \omega$ ,  $\kappa_{n+1} = j(\kappa_n)$ .

In the context of Definition 4.2.35, let  $\lambda = \sup_{n < \omega} \kappa_n$ . Clearly  $\lambda$  is the least ordinal such that  $j[\lambda] \subseteq \lambda$ . If  $\text{cf}^M(\lambda) = \omega$ ,  $j$  is continuous at  $\lambda$ , so  $j(\lambda) = \lambda$ . In particular, if  $N = V$ , which is the case of interest in this section, then  $\lambda$  is the first fixed point of  $j$  above  $\text{CRT}(j)$ .

In the case  $n > 1$ , the conclusion of the following lemma is a considerable understatement:

**Lemma 4.2.36.** *Suppose  $j : V \rightarrow M$  is a nontrivial elementary embedding and  $\langle \kappa_n : n < \omega \rangle$  is its critical sequence. For any  $n < \omega$ , if  $j$  is  $\kappa_n$ -strong then  $\kappa_n$  is measurable.*

*Proof.* The proof is by induction on  $n$ . Certainly  $\kappa_0 = \text{CRT}(j)$  is measurable. Assume the lemma is true for  $n = m$ , and we will show it is true for  $n = m + 1$ . Therefore assume  $j$  is  $\kappa_{m+1}$ -strong. In particular,  $j$  is  $\kappa_m$ -strong, so by our induction hypothesis,  $\kappa_m$  is measurable. By elementarity,  $\kappa_{m+1} = j(\kappa_m)$  is measurable in  $M$ . Since  $j$  is  $\kappa_{m+1}$ -strong,  $P(\kappa_{m+1}) \subseteq M$ . Thus the measurability of  $\kappa_{m+1}$  in  $M$  is upwards absolute to  $V$ , so  $\kappa_{m+1}$  is measurable.  $\square$

**Theorem 4.2.37** (Kunen). *Suppose  $j : V \rightarrow M$  is a nontrivial elementary embedding and  $\lambda$  is the least ordinal above  $\text{CRT}(j)$  with  $j[\lambda] \subseteq \lambda$ . Then  $j[\lambda] \notin M$ .*

*Proof.* Assume towards a contradiction that  $j[\lambda] \in M$ . By Lemma 4.2.20,  $j$  is  $\lambda$ -strong. Therefore  $\kappa_n$  is measurable for all  $n < \omega$  by Lemma 4.2.36, and so  $\lambda$  is a strong limit cardinal. Since  $j[\lambda] \subseteq \lambda$  and  $P(\lambda) \subseteq M$ ,  $j[\lambda] \in M$ . Thus  $\lambda$  is a strong limit cardinal,  $j[\lambda] \subseteq \lambda$ , and  $j$  is  $\lambda$ -supercompact. From Proposition 4.2.34 we can therefore conclude that  $\text{CRT}(j) \geq \lambda$ , contradicting that  $\text{CRT}(j) < \lambda$ .  $\square$

A useful structural consequence of Kunen's Inconsistency Theorem is the following lemma:

**Lemma 4.2.38.** *Suppose  $\gamma$  is a cardinal,  $j : V \rightarrow M$  is a nontrivial elementary embedding,  $\text{CRT}(j) \leq \gamma$ , and  $P(\gamma) \subseteq M$ . Then there is a measurable cardinal  $\kappa \leq \gamma$  such that  $j(\kappa) > \gamma$ .*

*Proof.* Let  $\langle \kappa_n : n < \omega \rangle$  be the critical sequence of  $j$  and  $\lambda = \sup_{n < \omega} \kappa_n$ . Thus  $\lambda$  is the least ordinal with  $j[\lambda] \subseteq \lambda$ . By Theorem 4.2.37,  $P(\lambda) \not\subseteq M$ , so since  $P(\gamma) \subseteq M$ , we have  $\gamma < \lambda$ . Let  $n < \omega$  be least such that  $\kappa_n \leq \gamma < \kappa_{n+1}$ . Lemma 4.2.36 implies  $\kappa_n$  is measurable, and  $j(\kappa_n) = \kappa_{n+1} > \gamma$ . Thus taking  $\kappa = \kappa_n$  proves the lemma.  $\square$

In one instance (Theorem 4.4.36), we will need a strengthening of Lemma 4.2.38 which has essentially the same proof:

**Lemma 4.2.39.** *Suppose  $\gamma \leq \lambda$  are cardinals and  $j : V \rightarrow M$  is a nontrivial  $\lambda$ -supercompact elementary embedding with  $\text{CRT}(j) \leq \gamma$ . Then there is a  $\lambda$ -supercompact cardinal  $\kappa \leq \gamma$  such that  $j(\kappa) > \gamma$ .  $\square$*

## The wellfoundedness of the generalized Mitchell order

The main theorem of this subsection states that the generalized Mitchell order is a wellfounded partial order when restricted to a reasonable class of countably complete ultrafilters. In fact, the wellfoundedness of the generalized Mitchell order on countably complete ultrafilters is a special case of Steel's wellfoundedness theorem for the Mitchell order on extenders [24], since countably complete ultrafilters are amenable extenders in the sense of [24], but we will give a much simpler proof here.

We start with the fundamental fact that the Mitchell order is irreflexive:

**Lemma 4.2.40.** *Suppose  $U$  is a countably complete nonprincipal ultrafilter. Then  $U \not\triangleleft U$ .*

*Proof.* Suppose towards a contradiction that  $U \triangleleft U$ . By Lemma 4.2.14, if  $U' \cong U$  is a uniform ultrafilter on a cardinal (as given by Lemma 2.2.24) then  $U' \triangleleft U'$  as well. We can therefore assume without loss of generality that  $U$  is a uniform ultrafilter on a cardinal  $\lambda$ . By Proposition 4.2.28,  $j_U \upharpoonright P(\lambda) \in M_U$ . In particular,  $j_U \upharpoonright \lambda \in M_U$ , so  $M_U^\lambda \subseteq M_U$  by Lemma 4.2.20. Therefore  $j_U^{M_U} = j_U \upharpoonright M_U$ , for example as a consequence of Proposition 4.2.29. Thus  $j_U$  is  $\delta$ -supercompact for all cardinals  $\delta$ . This contradicts Proposition 4.2.31.  $\square$

We now turn to the transitivity and wellfoundedness of the generalized Mitchell order. The following lemma (which in the language of [24] states that countably complete ultrafilters are *amenable*), is the key to the proof.

**Lemma 4.2.41.** *Suppose  $U$  is a nonprincipal countably complete ultrafilter on a set  $X$ . Suppose  $\lambda$  is a cardinal such that  $P(\lambda) \subseteq M_U$ . Then  $M_U \models 2^\lambda < j_U(|X|)$ .*

*Proof.* The proof proceeds by finding a measurable cardinal  $\kappa \leq |X|$  such that  $2^\lambda < j_U(\kappa)$ .

If  $\lambda < \text{CRT}(j_U)$ , then  $\kappa = \text{CRT}(j_U)$  works. Therefore assume  $\text{CRT}(j_U) \leq \lambda$ . By Lemma 4.2.38, there is an measurable cardinal  $\kappa \leq \lambda$  such that  $j_U(\kappa) > \lambda$ . We claim that  $\kappa \leq |X|$ , which completes the proof. Assume not. Then  $|X|$  is smaller than the inaccessible cardinal  $\kappa$ , and hence  $j_U(\kappa) = \kappa \leq \lambda$  (Lemma 3.5.33), a contradiction.  $\square$

We really only use the following consequence of Lemma 4.2.41:

**Corollary 4.2.42.** *Suppose  $U_0 \triangleleft U_1$  are countably complete nonprincipal hereditarily uniform ultrafilters. Then  $M_{U_1} \models 2^{\lambda_{U_0}} < j_{U_1}(\lambda_{U_1})$ .*

*Proof.* This is immediate from Lemma 4.2.41, using the fact (Lemma 4.2.10) that if  $U_0 \triangleleft U_1$  then  $P(\lambda_{U_0}) \subseteq M_{U_1}$ .  $\square$

**Corollary 4.2.43.** *Suppose  $U_0 \triangleleft U_1$  are countably complete nonprincipal hereditarily uniform ultrafilters. Let  $\lambda = \lambda_{U_1}$ . Then  $U_0 \in j_{U_1}(H(\lambda))$ .*

*Proof.* Since  $U_0$  is hereditarily uniform,  $M_{U_1} \models |\text{tc}(U_0)| = 2^{\lambda_{U_0}}$ . By Corollary 4.2.42,  $M_{U_1} \models 2^{\lambda_{U_0}} < j_{U_1}(\lambda)$ . Therefore  $U_0 \in H^{M_{U_1}}(j_{U_1}(\lambda)) = j_{U_1}(H(\lambda))$ .  $\square$

**Proposition 4.2.44.** *Suppose  $U_0 \triangleleft U_1 \triangleleft U_2$  are countably complete nonprincipal hereditarily uniform ultrafilters. Then  $U_0 \triangleleft U_2$  and  $M_{U_2} \models U_0 \triangleleft U_1$ .*

*Proof.* Let  $\lambda = \lambda_{U_1}$ . Then  $U_0 \in j_{U_1}(H(\lambda))$ . By Lemma 4.2.27,  $M_{U_2}$  contains  $j_{U_1}(H(\lambda))$ , so  $U_0 \in M_{U_2}$ , which yields  $U_0 \triangleleft U_2$ . In fact, by Lemma 4.2.27,  $j_{U_1}(H(\lambda)) = j_{U_1}^{M_{U_2}}(H(\lambda))$ , and so  $U_0 \in j_{U_1}^{M_{U_2}}(H(\lambda)) \subseteq M_{U_1}^{M_{U_2}}$ . Thus  $U_0 \in M_{U_1}^{M_{U_2}}$ , or other words,  $M_{U_2} \models U_0 \triangleleft U_1$ .  $\square$

**Corollary 4.2.45.** *The generalized Mitchell order is transitive on countably complete nonprincipal hereditarily uniform ultrafilters.*  $\square$

It is worth pointing out that the generalized Mitchell order on extenders is *not* transitive if there is a cardinal that is  $\kappa$ -strong where  $\kappa$  is a measurable cardinal. The counterexample

is described in [24]. (The generalized Mitchell order is not transitive on arbitrary countably complete ultrafilters either as a consequence of the silly counterexamples in Section 4.2.)

**Proposition 4.2.46.** *The generalized Mitchell order is wellfounded on countably complete nonprincipal hereditarily uniform ultrafilters.*

*Proof.* Suppose not, and let  $\lambda$  be the least cardinal such that there is a descending sequence

$$U_0 \triangleright U_1 \triangleright U_2 \triangleright \cdots$$

of countably complete hereditarily uniform ultrafilters with  $\lambda_{U_0} = \lambda$ .

By Proposition 4.2.44 and the closure of  $M_{U_0}$  under countable sequences, the sequence  $\langle U_n : 1 \leq n < \omega \rangle$  belongs to  $M_{U_0}$  and

$$M_{U_0} \models U_1 \triangleright U_2 \triangleright \cdots$$

Note that in  $M_{U_0}$ ,  $U_1$  is a countably complete nonprincipal hereditarily uniform ultrafilter, and by Corollary 4.2.42,  $\lambda_{U_1} < j_{U_0}(\lambda)$ .

On the other hand, by the elementarity of  $j_{U_0}$ , from the perspective of  $M_{U_0}$ ,  $j_{U_0}(\lambda)$  is the least cardinal  $\lambda'$  such that there is a descending sequence

$$W_0 \triangleright W_1 \triangleright W_2 \triangleright \cdots$$

of countably complete hereditarily uniform ultrafilters such that  $\lambda_{W_0} = \lambda'$ . This is a contradiction. □

One can prove a slightly more general result than Proposition 4.2.46 although this generality is never useful.

**Theorem 4.2.47.** *The generalized Mitchell order is wellfounded on nonprincipal countably complete ultrafilters.*

*Proof.* Assume towards a contradiction that  $U_0 \triangleright U_1 \triangleright \cdots$  are nonprincipal countably complete ultrafilters. For each  $n < \omega$ , let  $U'_n$  be a hereditarily uniform ultrafilter such that  $U'_n \cong U_n$ . Then by Lemma 4.2.14,  $U'_0 \triangleright U'_1 \triangleright \cdots$ . This contradicts Proposition 4.2.46. □

## The nonlinearity of the generalized Mitchell order

Before we discuss the extent to which the generalized Mitchell order is linear under UA, it is worth pointing out:

- the obvious counterexamples to linearity
- the maximal amount of linearity one can reasonably hope for.

The fact is that if there is a measurable cardinal, then the generalized Mitchell order is not linear, even restricting to uniform countably complete ultrafilters on cardinals. The known counterexamples to the linearity of the generalized Mitchell order are closely related to the *Rudin-Frolík order* (the subject of Chapter 5):

**Definition 4.2.48.** The *Rudin-Frolík order* is defined on countably complete ultrafilters  $U$  and  $W$  by setting  $U \leq_{\text{RF}} W$  if there is an internal ultrapower embedding  $i : M_D \rightarrow M_W$  such that  $i \circ j_D = j_W$ .

By Lemma 3.4.4, the Rudin-Keisler order can be defined in exactly the same way except omitting the requirement that  $i$  be internal.

**Proposition 4.2.49.** *If  $U \leq_{\text{RF}} W$  are nonprincipal countably complete ultrafilters, then  $U$  and  $W$  are incomparable in the generalized Mitchell order.*

*Proof.* We first show  $U \not\triangleleft W$ . Since  $U \leq_{\text{RF}} W$ ,  $M_W \subseteq M_U$ . Therefore the fact that  $U \notin M_U$  implies that  $U \notin M_W$ , and hence  $U \not\triangleleft W$ .

We now show  $W \not\triangleleft U$ . Assume towards a contradiction that  $W \triangleleft U$ . Assume without loss of generality that  $U$  is a uniform ultrafilter on a cardinal  $\lambda$ . (Since the Mitchell order is isomorphism invariant in its second argument, this does not change our situation.) Since  $U \leq_{\text{RF}} W$ , we have  $U \leq_{\text{RK}} W$  by Lemma 3.4.4. Since  $U$  is hereditarily uniform and  $U \leq_{\text{RK}} W \triangleleft U$ , our lemma on the invariance of the Mitchell order (Lemma 4.2.14) yields that  $U \triangleleft U$ . This contradicts Lemma 4.2.40.  $\square$

A similar argument shows the following:

**Proposition 4.2.50.** *Suppose  $U$  and  $W$  are countably complete ultrafilters and there is a nonprincipal  $D \leq_{\text{RF}} U, W$ . Then  $U$  and  $W$  are incomparable in the generalized Mitchell order.*  $\square$

Even this does not exhaust the known counterexamples to the linearity of the generalized Mitchell order:

**Proposition 4.2.51.** *Suppose  $U_0 \triangleleft U_1 \triangleleft U_2$ . Suppose  $U_0, U_2 \leq_{\text{RF}} W$ . Then  $U_1$  and  $W$  are incomparable in the Mitchell order.*  $\square$

We omit the proof. The hypotheses of the proposition are satisfied if  $U_0, U_1, U_2$  are normal ultrafilters on measurable cardinals  $\kappa_0 < \kappa_1 < \kappa_2$  respectively and  $W = U_0 \times U_2$ .

All known examples of nonlinearity in the generalized Mitchell order are accompanied by nontrivial relations in the Rudin-Frolík order. A driving question in this work is whether assuming UA, these are the only counterexamples.

**Definition 4.2.52.** A nonprincipal countably complete ultrafilter  $W$  is *irreducible* if for all  $U \leq_{\text{RF}} W$ , either  $U$  is principal or  $U$  is isomorphic to  $W$ .

The Irreducible Ultrafilter Hypothesis (IUH) essentially states that the sort of counterexamples to the linearity of the Mitchell order that we have described are the only ones.

**Irreducible Ultrafilter Hypothesis.** Suppose  $U$  and  $W$  are hereditarily uniform irreducible ultrafilters. Either  $U \cong W$ ,  $U \triangleleft W$ , or  $W \triangleleft U$ .

We can now make precise the question of the extent of the linearity of the Mitchell order under UA:

**Question 4.2.53.** Does UA imply IUH?

With this in mind, let us turn to the positive results on linearity.

## 4.3 Dodd soundness

### Introduction

*Dodd soundness* is a fine-structural generalization of supercompactness, introduced by Steel [2] in the context of inner model theory as a strengthening of the initial segment condition. The following remarkable theorem is due to Schlutzenberg [7]:

**Theorem 4.3.1** (Schlutzenberg). *Suppose  $L[\mathbb{E}]$  is an iterable Mitchell-Steel model and  $U$  is a countably complete ultrafilter of  $L[\mathbb{E}]$ . Then the following are equivalent:*

(1)  *$U$  is irreducible.*

(2)  *$U$  is isomorphic to a Dodd sound ultrafilter.*

(3)  *$U$  is isomorphic to an extender on the sequence  $\mathbb{E}$ .* □

Since the total extenders on  $\mathbb{E}$  are linearly ordered by the Mitchell order, this has the following consequence:

**Theorem 4.3.2** (Schlutzenberg). *Suppose  $L[\mathbb{E}]$  is an iterable Mitchell-Steel model. Then  $L[\mathbb{E}]$  satisfies the Irreducible Ultrafilter Hypothesis.* □

It is open whether this theorem can be extended to the Woodin models at the finite levels of supercompactness. The main result of this section (Theorem 4.3.29) states that UA alone suffices to prove the linearity of the generalized Mitchell order on Dodd sound ultrafilters.

### Dodd sound embeddings, extenders, and ultrafilters

In this subsection, we present a definition of Dodd soundness due to the author that is much simpler than the one given in [2] and [7], and that is easier to use in certain contexts. (The other definition is also useful.) We then show that the two definitions are equivalent.



**Definition 4.3.3.** Suppose  $M$  is a transitive class,  $j : V \rightarrow M$  is an elementary embedding, and  $\alpha$  is an ordinal. Let  $\delta$  be the least ordinal such that  $j(\delta) \geq \alpha$ . Then

$$j^\alpha : P(\delta) \rightarrow M$$

is the function defined by  $j^\alpha(X) = j(X) \cap \alpha$ . The embedding  $j$  is said to be  $\alpha$ -*sound* if  $j^\alpha$  belongs to  $M$ .

Recall that the bounded powerset of an ordinal  $\delta$  is defined by  $P_{\text{bd}}(\delta) = \bigcup_{\xi < \delta} P(\xi)$ . In the context of Definition 4.3.3, if  $\alpha = \sup j[\delta]$  it have been natural to define  $j^\alpha = j \upharpoonright P_{\text{bd}}(\delta)$ . With this alternate definition,  $j^\alpha \in M$  is an a priori weaker requirement. The next lemma shows that this does not actually make a difference:

**Lemma 4.3.4.** *Suppose  $M$  is a transitive class,  $j : V \rightarrow M$  is an elementary embedding, and  $\delta$  is an ordinal. Let  $\delta_* = \sup j[\delta]$ . Then the following are equivalent:*

- (1)  $j$  is  $\delta_*$ -*sound*.
- (2)  $j[P_{\text{bd}}(\delta)] \in M$  or equivalently  $j$  is  $2^{<\delta}$ -*supercompact*.
- (3)  $j \upharpoonright P_{\text{bd}}(\delta) \in M$ .
- (4)  $j \upharpoonright P_{\text{bd}}^M(\delta) \in M$ .

*Proof.* (1) *implies* (2): Trivial. (The equivalence of  $j[P_{\text{bd}}(\delta)] \in M$  with  $2^{<\delta}$ -supercompactness is immediate from Lemma 4.2.17.)

(2) *implies* (3):  $j \upharpoonright P_{\text{bd}}(\delta)$  is the inverse of the transitive collapse of  $j[P_{\text{bd}}(\delta)]$ .

(3) *implies* (4): Trivial.

(4) *implies* (1): Assume  $j \upharpoonright P_{\text{bd}}^M(\delta) \in M$ . Since  $\delta \subseteq P_{\text{bd}}^M(\delta)$ ,

$$j \upharpoonright \delta = (j \upharpoonright P_{\text{bd}}^M(\delta)) \upharpoonright \delta \in M$$

Therefore  $j$  is  $\delta$ -supercompact. Since supercompactness implies strength (Lemma 4.2.20),  $P(\delta) \subseteq M$ . In particular  $j \upharpoonright P_{\text{bd}}^M(\delta) = j \upharpoonright P_{\text{bd}}(\delta)$ . Finally for  $X \subseteq \delta$ ,  $j^{\delta_*}(X) = \bigcup_{\xi < \delta} j(X \cap \xi)$ , so  $j^{\delta_*}$  is definable from  $j \upharpoonright P_{\text{bd}}(\delta)$  and hence  $j^{\delta_*} \in M$ , which shows (1).  $\square$

**Lemma 4.3.5.** *Suppose  $M$  is a transitive class,  $j : V \rightarrow M$  is an elementary embedding, and  $\alpha$  is an ordinal. Then  $j$  is  $\alpha$ -sound if and only if  $\{j(X) \cap \alpha : X \in V\} \in M$ .*

*Proof.* The forward direction is immediate since  $\{j(X) \cap \alpha : X \in V\} = \text{ran}(j^\alpha)$ . The reverse direction follows from the fact that  $j^\alpha$  is the inverse of the transitive collapse of  $\{j(X) \cap \alpha : X \in V\}$ .  $\square$

Our next lemma states that the fragments  $j^\alpha$  “pull back” under elementary embeddings.

**Lemma 4.3.6.** *Suppose  $V \xrightarrow{i} N \xrightarrow{k} M$  are elementary embeddings and  $j = k \circ i$ . Suppose  $j^\alpha \in \text{ran}(k)$ . Then  $k^{-1}(j^\alpha) = i^{k^{-1}(\alpha)}$ .*

*Proof.* Let  $\delta$  be the least ordinal such that  $j(\delta) \geq \alpha$ . Note that  $j^\alpha[\text{Ord}] = j[\delta] \in \text{ran}(k)$ , so by our analysis of derived embeddings (Lemma 4.2.22),  $k \upharpoonright \delta + 1$  is the identity and  $i$  is  $\delta$ -supercompact. In particular,  $P_{\text{bd}}(\delta) \subseteq M$  and  $k(P_{\text{bd}}(\delta)) = P_{\text{bd}}(\delta)$ .

Let  $h = k^{-1}(j^\alpha)$ . Then  $\text{dom}(h) = k^{-1}(P_{\text{bd}}(\delta)) = P_{\text{bd}}(\delta)$ . Thus for  $X \in \text{dom}(h)$ ,  $k(X) = X$ , and hence

$$k(h(X)) = k(h)(k(X)) = k(h)(X) = j^\alpha(X) = j(X) \cap \alpha = k(i(X)) \cap \alpha$$

By the elementarity of  $k$ , this implies that  $h(X) = i(X) \cap k^{-1}(\alpha)$ , or in other words  $k^{-1}(j^\alpha) = h = i^{k^{-1}(\alpha)}$ , as desired.  $\square$

We now turn to Dodd soundness.

**Definition 4.3.7.** If  $j : V \rightarrow M$  is an extender embedding, the *Dodd length* of  $j$ , denoted  $\alpha(j)$ , is the least ordinal  $\alpha$  such that every element of  $M$  is of the form  $j(f)(\xi)$  for some  $\xi < \alpha$ .

On first glance, one might believe that the Dodd length of an elementary embedding  $j$  is the same as its *natural length*, denoted  $\nu(j)$ , the least  $\nu$  such that  $M = H^M(j[V] \cup \nu)$ . In fact, equality may fail: the issue is that  $\nu(j)$  is the least ordinal such that every element of  $M$  is of the form  $j(f)(p)$  for a finite set  $p \subseteq \nu$ , whereas in the definition of  $\alpha(j)$ , one must

write every element of  $M$  in the form  $j(f)(\xi)$  where  $\xi$  is not a finite set but a single ordinal below  $\nu$ .

Our main focus, of course, is on ultrafilters, and in this case the Dodd length has an obvious characterization:<sup>3</sup>

**Lemma 4.3.8.** *If  $j : V \rightarrow M$  is an ultrapower embedding, then  $\alpha(j) = \xi + 1$  where  $\xi$  is the least ordinal such that  $M = H^M(j[V] \cup \{\xi\})$ . Therefore  $U$  is incompressible if and only if  $U$  is tail uniform and  $\alpha(j_U) = a_U + 1$ .  $\square$*

Our next lemma establishes a limit on the solidity of an extender embedding. (It is equivalent to the statement that no extender belongs to its own ultrapower.)

**Lemma 4.3.9.** *Suppose  $j : V \rightarrow M$  is an extender embedding and  $\alpha = \alpha(j)$ . Then  $j$  is not  $\alpha$ -sound.*

*Proof.* Let us first show that if  $U$  is a countably complete tail uniform ultrafilter on an ordinal  $\delta$ , then  $j_U$  is not  $a_U + 1$ -sound. Note that

$$U = \{A \subseteq \delta : a_U \in j(A)\} = \{A \subseteq \delta : a_U \in j_U^{a_U+1}(A)\}$$

so since  $U \notin M_U$ ,  $j_U^{a_U+1} \notin M$ . Thus  $j_U$  is not  $a_U + 1$ -sound, as claimed.

We now handle the case where  $j$  is an arbitrary extender embedding. By the definition of Dodd length, there is some  $\xi < \alpha$  and some function  $f \in V$  such that  $j^\alpha = j(f)(\xi)$ . Let  $U$  be the tail uniform ultrafilter derived from  $j$  using  $\xi$ , and let  $k : M_U \rightarrow M$  be the factor embedding. Then  $\xi \in \text{ran}(k)$  and so  $j^\alpha \in \text{ran}(k)$ . Applying our lemma on pullbacks of the fragments  $j^\alpha$  (Lemma 4.3.6),  $k^{-1}(j^\alpha) = j_U^{k^{-1}(\alpha)}$ . Therefore  $j_U$  is  $k^{-1}(\alpha)$ -sound. But note that  $a_U = k^{-1}(\xi) < k^{-1}(\alpha)$ . Hence  $j_U$  is  $a_U + 1$ -sound, and this contradicts the first paragraph.  $\square$

An embedding is *Dodd sound* if it is as sound as it can possibly be:

---

<sup>3</sup>This gives us a counterexample to the equality of Dodd length and natural length. Suppose  $U$  is a normal ultrafilter on  $\kappa$ . Let  $W = U^2$ . Then  $\nu(j_W) = j_U(\kappa) + 1$  but  $\alpha(j_W) = j_U(\kappa) + \kappa + 1$ .

**Definition 4.3.10.** Suppose  $M$  is a transitive class and  $j : V \rightarrow M$  is an elementary embedding. Then  $j$  is said to be *Dodd sound* if  $j$  is  $\beta$ -sound for all  $\beta < \alpha(j)$ .

We now prove the equivalence between the Dodd soundness of an extender  $E$  as it is defined in [2] and the Dodd soundness of its associated embedding  $j_E$  as it is defined in Definition 4.3.3.

**Definition 4.3.11.** • A *parameter* is a finite set of ordinals.

- The *parameter order* is defined on parameters  $p$  and  $q$  by

$$p < q \iff \max(p \triangle q) \in q$$

- If  $p$  is a parameter, then  $\langle p_i : i < |p| \rangle$  denotes the *descending* enumeration of  $p$ .
- For any  $k \leq |p|$ ,  $p \upharpoonright k$  denotes the parameter  $\{p_i : i < k\}$ .

The point of enumerating parameters in descending order is that the parameter order is then transformed into the lexicographic order:

**Lemma 4.3.12.** *Suppose  $p$  and  $q$  are parameters of length  $n$  and  $m$  respectively. Then  $p < q$  if and only if  $\langle p_0, \dots, p_{n-1} \rangle <_{\text{lex}} \langle q_0, \dots, q_{m-1} \rangle$ . □*

**Lemma 4.3.13.** *The parameter order is a set-like wellorder.* □

**Definition 4.3.14.** If  $j : V \rightarrow M$  is an elementary embedding and  $p$  is a parameter, then  $\mu_j(p)$  is the least ordinal  $\mu$  such that  $p \subseteq j(\mu)$ .

**Definition 4.3.15.** Suppose  $j : V \rightarrow M$  is an elementary embedding,  $p$  is a parameter, and  $\nu < \min(p)$  is an ordinal. Let  $\delta = \mu_j(p)$ . Then the *extender of  $j$  below  $(p, \nu)$*  is the set

$$E^j \upharpoonright p \cup \nu = \{(q, A) : q \in [\nu]^{<\omega}, A \subseteq [\delta]^{<\omega}, \text{ and } p \cup q \in j(A)\}$$

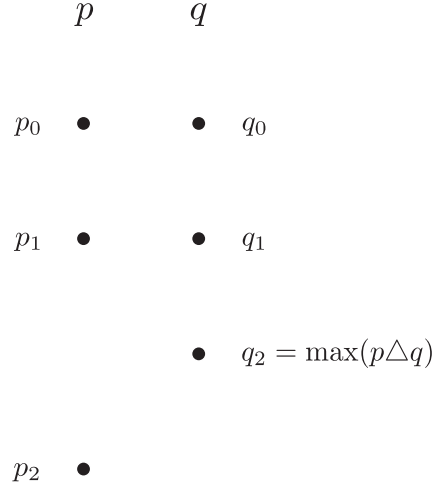


Figure 4.1: The parameter order

The restriction  $E^j \upharpoonright p \cup \nu$  can be thought of as an extender *relativized* to the parameter  $p$ . It is possible to axiomatize relativized extenders as directed systems of ultrafilters and associate to them ultrapower embeddings, namely the direct limit of these systems. Instead we make the following definition:

**Definition 4.3.16.** A *relativized extender* is a set of the form  $E^j \upharpoonright p \cup \nu$  for some elementary embedding  $j$ . The *extender embedding* associated to a relativized extender  $E$ , denoted

$$j_E : V \rightarrow M_E$$

is the unique  $j : V \rightarrow M$  such that  $E = E^j \upharpoonright p \cup \nu$  for some  $p, \nu$  and  $M = H^M(j[V] \cup p \cup \nu)$ .

If  $E$  is a relativized extender,  $\nu$  is an ordinal, and  $p$  is a parameter, then

$$E \upharpoonright p \cup \nu = E^j \upharpoonright p \cup \nu$$

where  $j = j_E$ .

The *Dodd parameter* of an extender is the key to the fine-structural proofs of Dodd soundness, which are motivated by the fundamental solidity proofs from fine structure theory.

**Definition 4.3.17.** Suppose  $j : V \rightarrow M$  is an extender embedding. Then  $\eta(j)$  is the least ordinal  $\eta$  such that for some parameter  $p$ ,

$$M = H^M(j[V] \cup p \cup \eta)$$

The *Dodd parameter* of  $j$ , denoted  $p(j)$ , is the least parameter  $p$  such that

$$M = H^M(j[V] \cup p \cup \eta(j))$$

Thus if  $j$  is an ultrapower embedding, as it always will be in our applications, then  $\eta = 0$ . More generally,  $\eta$  is obviously always a limit ordinal.

The Dodd parameter can also be defined recursively using the concept of an  $x$ -generator of an elementary embedding:

**Definition 4.3.18.** Suppose  $M$  and  $N$  are transitive models of ZFC,  $j : M \rightarrow N$  is an elementary embedding, and  $x \in N$ . Then an ordinal  $\xi \in N$  is an  $x$ -generator of  $j$  if  $\xi \notin H^N(j[M] \cup \xi \cup \{x\})$ .

**Lemma 4.3.19.** *Suppose  $j : V \rightarrow M$  is an extender embedding. Let  $q$  be the  $\subseteq$ -maximum parameter with the property that  $q_k$  is the largest  $q \upharpoonright k$ -generator of  $j$  for all  $k < |q|$ . Then  $p(j) = q$  and  $\eta(j)$  is the strict supremum of the  $q$ -generators of  $j$ .*

*Proof.* Let  $p = p(j)$ ,  $n = |p|$ , and  $\eta = \eta(j)$ . Fix  $k < n$ . We will show  $p_k$  is the largest  $p \upharpoonright k$ -generator.

Since  $M = H^M(j[V] \cup p \cup \eta) \subseteq H^M(j[V] \cup p \upharpoonright k \cup (p_k + 1))$ , there are no  $p \upharpoonright k$ -generators strictly above  $p_k$ . It therefore suffices to show that  $p_k$  is a  $p \upharpoonright k$ -generator. Assume not. Then  $p_k \in H^M(j[V] \cup p \upharpoonright k \cup p_k)$ . Fix  $u \subseteq p_k$  such that  $p_k = j(f)(p \cup r)$  for some function  $f \in V$ . Let  $r = p \setminus \{p_k\} \cup u$ . Then  $r < p$  in the parameter order, but  $p \subseteq H^M(j[V] \cup r)$ , and hence  $M = H^M(j[V] \cup r \cup \eta)$ , contrary to the minimality of the Dodd parameter  $p$ .

By the maximality of  $q$ , this shows that  $p = q \upharpoonright n$ . We now show that  $\eta$  is the strict supremum of the  $p$ -generators of  $j$ . Since  $M = H^M(j[V] \cup p \cup \eta)$ , there are no  $p$ -generators

greater than or equal to  $\eta$ . It therefore suffices to show that for any  $\alpha < \eta$ , there is a  $p$ -generator of  $j$  above  $\alpha$ . Suppose  $\alpha < \eta$ . By the minimality of  $\eta$ ,  $M \neq H^M(j[V] \cup p \cup \alpha)$ , and so there is a  $p$ -generator of  $j$  above  $\alpha$ , as desired.

Since  $\eta$  is a limit ordinal, there is no largest  $p$ -generator, and hence  $p = q$ .  $\square$

**Corollary 4.3.20.** *Suppose  $j : V \rightarrow M$  is an extender embedding and  $p = p(j)$ . Then for all  $i < |p|$ ,  $p_i$  is a  $\{p_0, \dots, p_{i-1}\}$ -generator.*  $\square$

The following is Steel's definition of the Dodd soundness of an extender:

**Definition 4.3.21.** Suppose  $E$  is an extender,  $p = p(j_E)$ , and  $\eta = \eta(j_E)$ .

- $E$  is *Dodd solid* if

$$E \upharpoonright \{p_0, \dots, p_{i-1}\} \cup p_i \in M_E$$

for all  $i < |p|$ .

- $E$  is *Dodd sound* if  $E$  is Dodd solid and

$$E \upharpoonright p \cup \nu \in M_E$$

for all  $\nu < \eta$ .

If  $E$  is an extender such that  $j_E$  is an ultrapower embedding, then  $E$  is Dodd solid if and only if  $E$  is Dodd sound, simply because  $\eta(j_E) = 0$  (so the extra requirement for Dodd soundness holds vacuously).

The following fact is essentially a matter of rearranging definitions:

**Theorem 4.3.22.** *Suppose  $E$  is an extender. Then  $E$  is Dodd sound in the sense of Definition 4.3.21 if and only if  $j_E$  is Dodd sound in the sense of Definition 4.3.10.*

*Proof.* Before we prove the equivalence, we prove three preliminary claims.

Let  $j = j_E$  and  $M = M_E$ . Let  $\eta = \eta(j)$  and let  $p = p(j)$  be the Dodd parameter of  $j$ .

**Claim 1.**  $p \cup \{\eta\}$  is the least parameter  $s$  such that every element of  $M$  is of the form  $j(f)(q)$  for some  $q < s$ .

*Proof.* Suppose not. Then fix  $s < p \cup \{\eta\}$  such that every element of  $M$  is of the form  $j(f)(q)$  for some  $q < s$ . Fix  $q < s$  such that  $p = j(f)(q)$  for some  $f$ . Then  $M = H^M(j[V] \cup q \cup \eta)$ . Since  $p$  is the least parameter with this property (by the definition of the Dodd parameter), it follows that  $p \leq q$ . In particular  $p < s$ . Since  $p < s < p \cup \{\eta\}$ ,  $s = p \cup r$  for some  $r \in [\eta]^{<\omega}$ . Now let  $\xi < \eta$  be a  $p$ -generator such that  $r \subseteq \xi$ . Then  $p \cup \{\xi\} = j(f)(u)$  for some  $u < s$ . Since  $u$  generates  $p$ , we must have  $p \leq u$ . Since  $p \leq u \leq p \cup r$ ,  $u = p \cup t$  for some  $t < r$ . In particular, since  $r \subseteq \xi$ ,  $t \subseteq \xi$ . Now  $\xi = j(f)(p \cup r)$  where  $r \in [\xi]^{<\omega}$ , contradicting that  $\xi$  is not a  $p$ -generator.  $\square$

Let  $\varphi$  be the function that sends a parameter to its rank in the parameter order.

**Claim 2.** Suppose  $x \in M$  and  $q$  is a parameter. Then  $x = j(f)(q)$  for some function  $f \in V$  if and only if  $x = j(g)(\varphi(q))$  for some function  $g \in V$ .

*Proof.* For the forwards direction, let  $g = f \circ \varphi^{-1}$ , and for the reverse direction, let  $f = g \circ \varphi$ .  $\square$

From Claim 1 and Claim 2, we obtain the following key identity:

$$\varphi(p \cup \{\eta\}) = \alpha(j) \tag{4.1}$$

(Recall that  $\alpha(j)$  denotes the Dodd length of  $j$ , the least ordinal  $\alpha$  such that every element of  $M$  is of the form  $j(f)(\xi)$  for some  $\xi < \alpha$ .)

**Claim 3.** Suppose  $q$  is a parameter and  $m = |q|$ . For  $i < m$ , let

$$F_i = E \upharpoonright \{q_0, \dots, q_{i-1}\} \cup q_i$$

Then for any transitive model  $N$  of ZFC, the following are equivalent:

(1)  $F_0, \dots, F_{m-1} \in N$ .



(2)  $j^{\varphi(q)} \in N$ .

*Sketch.* (1) implies (2): Let  $\mu = \mu_j(q) = \mu_j(\{q_0\})$ . If  $F_0, \dots, F_{m-1} \in N$ , then so is the function  $e : P([\mu]^{<\omega}) \rightarrow M$  defined by  $e(X) = \{r < q : r \in j(X)\}$ . ( $e$  is the parameter version of  $j^q$ .) This is because  $r \in e(X)$  if and only if  $(r, X) \in F_i$  where  $i$  is such that  $\max(q \Delta r) = q_i$ .

Let  $\delta$  be least such that  $j(\delta) \geq \varphi(q)$ . Then  $\varphi[\delta] \subseteq \mu$  and for  $A \subseteq \delta$ ,  $j(A) \cap \varphi(q) = \varphi^{-1}[e(\varphi[A])]$ . This shows  $j^{\varphi(q)} \in N$ .

(2) implies (1): Similar. □

Having proved the three claims, we finally turn to the equivalence of the two notions of Dodd soundness. (We will leave some of the parameter order theoretic details to the reader.)

Assume first that  $E$  is Dodd sound in the sense of Definition 4.3.21. Suppose  $\beta < \alpha(j)$ , and we will show that  $j$  is  $\beta$ -sound. It suffices to show that  $j$  is  $\beta'$ -sound for some  $\beta' \geq \beta$ , which allows us to increase  $\beta$  throughout the argument if necessary. By (4.1), by increasing  $\beta$ , we may assume  $\varphi(p) \leq \beta$ . Thus  $p \leq \varphi^{-1}(\beta) < \varphi^{-1}(\alpha(j)) = p \cup \{\eta\}$ , as a consequence of (4.1). Let  $q = \varphi^{-1}(\beta)$ . Then  $p \leq q < p \cup \{\eta\}$ , so  $q = p \cup r$  for some  $r \subseteq \eta$ . Since  $\eta$  is a limit ordinal, by increasing  $\beta$  if necessary, we may assume  $|r| \leq 1$ . By the Dodd soundness of  $E$ , for all  $i < |q|$ ,

$$E \upharpoonright \{q_0, \dots, q_{i-1}\} \cup q_i \in M$$

This is because either  $\{q_0, \dots, q_{i-1}\} \cup q_i = \{p_0, \dots, p_{i-1}\} \cup p_i$  or  $\{q_0, \dots, q_{i-1}\} \cup q_i = p \cup \xi$  for some  $\xi < \eta$ . Therefore by Claim 3,  $j^\beta \in M$  so  $j$  is  $\beta$ -sound.

Conversely, assume that  $j$  is Dodd sound as an elementary embedding. Let  $\beta = \varphi(p)$ . Since  $p < p \cup \{\eta\}$ , by (4.1),  $\beta < \alpha$ . Therefore  $j^\beta \in M$  by the Dodd soundness of  $j$ . By Claim 3, it follows that  $E \upharpoonright \{p_0, \dots, p_{i-1}\} \cup p_i$  for all  $i < |p|$ , so  $E$  is Dodd solid. If  $\eta = 0$ , it follows that  $E$  is Dodd sound. Assume instead that  $\eta > 0$ . Fix  $\xi < \eta$ , and we will show  $E \upharpoonright p \cup \xi \in M$ . Let  $q = p \cup \{\xi\}$ . Then  $q < p \cup \{\eta\}$ , so  $\varphi(q) < \alpha$ . Therefore by the Dodd soundness of  $j$ ,  $j^{\varphi(q)} \in M$ . Applying Claim 3, it follows that  $E \upharpoonright p \cup \xi \in M$ . □

It is worth remarking that the proof shows that an extender  $E$  is Dodd solid if and only if  $j_E$  is  $\beta$ -solid where  $\beta$  is the rank of  $p(j_E)$  in the canonical wellorder on parameters.

We now define Dodd sound ultrafilters. One could define an ultrafilter to be Dodd sound if its ultrapower embedding is Dodd sound, but then there would be many isomorphic Dodd sound ultrafilters all with the same associated embedding, which complicates the statements of our theorems and adds no real generality. Instead, we ensure that a Dodd sound ultrafilter is the canonical element of its isomorphism class:

**Definition 4.3.23.** A countably complete ultrafilter is *Dodd sound* if it is incompressible and its ultrapower embedding is Dodd sound.

The following alternate characterization of Dodd soundness for ultrafilters is immediate from Lemma 4.3.8 and Lemma 4.3.9:

**Lemma 4.3.24.** *A tail uniform ultrafilter  $U$  on an ordinal  $\delta$  is Dodd sound if and only if  $j_U$  is  $a_U$ -sound. That is,  $U$  is Dodd sound if and only if the function  $h : P(\delta) \rightarrow M_U$  defined by  $h(X) = j_U(X) \cap a_U$  belongs to  $M_U$ .  $\square$*

We finally provide a combinatorial characterization of Dodd soundness for ultrafilters:

**Definition 4.3.25.** Suppose  $U$  is an ultrafilter on an ordinal  $\delta$ .

- A sequence  $\langle S_\alpha : \alpha < \delta \rangle$  is  *$U$ -threadable* if there is a set  $S \subseteq \delta$  such that  $S \cap \alpha = S_\alpha$  for  $U$ -almost all  $\alpha < \delta$ .
- A *soundness sequence* for  $U$  is a sequence  $\langle \mathcal{A}_\alpha : \alpha < \delta \rangle$  such that for any sequence  $\langle S_\alpha : \alpha < \delta \rangle$ , the following are equivalent:
  - (1)  $\langle S_\alpha : \alpha < \delta \rangle$  is  $U$ -threadable.
  - (2)  $S_\alpha \in \mathcal{A}_\alpha$  for  $U$ -almost all  $\alpha$ .

**Theorem 4.3.26.** *A tail uniform ultrafilter  $U$  is Dodd sound if and only if it has a soundness sequence.*

*Proof.* Note that a sequence  $\langle S_\alpha : \alpha < \delta \rangle$  is  $U$ -threadable if and only if

$$[\langle S_\alpha : \alpha < \delta \rangle] = j_U(S) \cap a_U$$

some  $S \subseteq \delta$ . Thus  $\langle \mathcal{A}_\alpha : \alpha < \delta \rangle$  is a soundness sequence for  $U$  if and only if

$$[\langle \mathcal{A}_\alpha : \alpha < \delta \rangle]_U = \{j_U(S) \cap a_U : S \subseteq \delta\}$$

By Lemma 4.3.5, it follows that  $U$  has a soundness sequence if and only if  $j_U$  is  $a_U$ -sound, or in other words (applying Lemma 4.3.24)  $U$  is Dodd sound.  $\square$

## The generalized Mitchell order on Dodd sound ultrafilters

In this short section, we prove the linearity of the Mitchell order on Dodd sound ultrafilters. We first prove a stronger statement that characterizes  $P(P(\lambda)) \cap M_W$  when  $W$  is Dodd solid in terms of the Lipschitz order on subsets of  $P(\lambda)$ .

**Proposition 4.3.27.** *Suppose  $W$  is a Dodd sound ultrafilter on a cardinal  $\lambda$ . Then*

$$P(P(\lambda)) \cap M_W = \{X \subseteq P(\lambda) : X <_L W\}$$

*Proof.* Suppose  $X \subseteq P(\lambda)$ .

Assume first that  $X <_L W$ . By our characterization of the Lipschitz order where the second argument is an ultrafilter (Lemma 3.5.58), this means that there is a set  $Z \in M_W$  such that for all  $A \subseteq \delta$ ,  $A \in X$  if and only if  $j_W(A) \cap a_W \in Z$ . But then  $X = (j^{a_W})^{-1}[Z]$ , so  $X \in M_W$ .

Conversely, suppose  $X \in M_W$ . Let  $Z = j^{a_W}[X]$ . Then  $Z \in M_W$  and for all  $A \subseteq \delta$ ,  $A \in X$  if and only if  $j_W(A) \cap a_W = j^{a_W}(A) \in Z$ . It follows that  $X <_L W$ .  $\square$

**Corollary 4.3.28.** *Suppose  $U$  and  $W$  are countably complete ultrafilters on  $\lambda$  and  $W$  is Dodd sound. Then  $U <_L W$  if and only if  $U \triangleleft W$ . In particular, if  $U \leq_{\mathbf{k}} W$  then  $U \triangleleft W$ .  $\square$*

In particular, the Lipschitz order is wellfounded on Dodd sound ultrafilters.

**Theorem 4.3.29 (UA).** *The generalized Mitchell order is linear on Dodd sound ultrafilters.*

*Proof.* Suppose  $U$  and  $W$  are Dodd sound ultrafilters. By the linearity of the Lipschitz order on  $\text{Un}$ , either  $U <_L W$ ,  $U = W$ , or  $U >_L W$ . Therefore by Proposition 4.3.27, either  $U \triangleleft W$ ,  $U = W$ , or  $U \triangleright W$ , as desired.  $\square$

Notice that the linearity of the Mitchell order on Dodd sound ultrafilters actually follows from the linearity of the Lipschitz order, which perhaps is weaker than UA.

As a consequence of Corollary 4.3.28, if  $W$  is Dodd sound and  $U <_{\mathbb{k}} W$ , then  $U \triangleleft W$ . We now prove a strong converse, which is closely related to Proposition 4.2.29:

**Proposition 4.3.30.** *Suppose  $U$  is a countably complete ultrafilter on a cardinal  $\lambda$  and  $W$  is a nonprincipal uniform ultrafilter on a cardinal  $\delta$  such that  $j_W$  is  $\lambda$ -supercompact. If  $U \triangleleft W$ , then  $U <_S W$ .*

*Proof.* Note that  $(j_U(j_W), j_U \upharpoonright M_W)$  is a 0-internal comparison of  $(j_U, j_W)$  by the standard identity:

$$j_U(j_W) \circ j_U = j_U \circ j_W$$

Since  $j_W$  is  $\lambda$ -supercompact,  $j_U \upharpoonright M_W = j_U^{M_W}$ , which is definable over  $M_W$  since  $U \triangleleft W$ .

Since  $j_W$  is  $\lambda$ -supercompact,  $\lambda \leq \delta$  by Proposition 4.2.31. Therefore for all  $\alpha < \lambda$ ,  $j_W(\alpha) < a_W$ . Applying Łoś's Theorem,

$$j_U(j_W)(a_U) = [j_W \upharpoonright \lambda]_U < j_U(a_W)$$

Thus  $(j_U(j_W), j_U \upharpoonright M_W)$  witnesses that  $U <_S W$ .  $\square$

This raises the question of whether the Ketonen order extends the generalized Mitchell order. One should restrict attention here to countably complete uniform ultrafilters on cardinals, or else there are silly counterexamples. If this were true, it would complete the picture in which the wellfoundedness of the Ketonen order explains that of all the other known wellfounded orders. It is consistently false, however (Proposition 5.5.5):

**Proposition 4.3.31.** *Suppose  $\kappa$  is  $2^\kappa$ -supercompact and  $2^\kappa = 2^{\kappa^+}$ . Then there are  $\kappa$ -complete uniform ultrafilters  $U$  and  $W$  on  $\kappa$  and  $\kappa^+$  respectively such that  $W \triangleleft U$ .  $\square$*

Thus  $W \triangleleft U$  but  $U \not<_{\mathbb{k}} W$  simply because  $\delta_U < \delta_W$ . (This is a consequence of Lemma 3.3.15.) By Proposition 4.3.30, if  $U$  and  $W$  are uniform ultrafilters on the same cardinal  $\lambda$  and both  $j_U$  and  $j_W$  are  $\lambda$ -supercompact, then  $U \triangleleft W$  implies  $U <_{\mathbb{k}} W$ .

**Lemma 4.3.32.** *Suppose  $\lambda$  is a cardinal,  $W$  is a countably complete ultrafilter on  $\lambda$ , and  $Z$  is a countably complete ultrafilter such that  $W \triangleleft Z$ . Assume that for all  $\alpha < \lambda$ ,  $\mathcal{B}(\lambda, \alpha) \subseteq M_Z$  and  $M_Z \models \mathcal{B}(\lambda, \alpha) \leq 2^\lambda$ . Then for any  $U <_{\mathbb{k}} W$ ,  $U \triangleleft Z$ .*

*Proof.* Since  $W \triangleleft Z$ ,  $P(\lambda) \subseteq M_Z$  and in fact  $P(\lambda)^\lambda \subseteq M_Z$ . Moreover

$$M_Z \models \left| \bigcup_{\alpha < \lambda} \mathcal{B}(\lambda, \alpha) \right| \leq 2^\lambda = |P(\lambda)|$$

Hence  $(\bigcup_{\alpha < \lambda} \mathcal{B}(\lambda, \alpha))^\lambda \subseteq M_Z$ , so  $\prod_{\alpha \in I} \mathcal{B}(\lambda, \alpha) \in M_Z$  for any set  $I \subseteq \lambda$ .

Now suppose  $U <_{\mathbb{k}} W$ . Fix  $I \in W$  and  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\lambda, \alpha)$  such that  $U = W\text{-}\lim_{\alpha \in I} U_\alpha$ . Then the sequence  $\langle U_\alpha : \alpha \in I \rangle \in M_Z$ , so  $U \in M_Z$ , so  $U \triangleleft Z$ , as desired.  $\square$

In fact, this lemma yields the somewhat stronger result that for any  $I \in W$  and sequence  $\langle U_\alpha : \alpha \in I \rangle$  of ultrafilters with  $\delta_{U_\alpha} < \lambda$ ,  $W\text{-}\lim_{\alpha \in I} U_\alpha \triangleleft Z$ .

**Corollary 4.3.33 (UA).** *Assume  $\lambda$  is a cardinal such that  $2^{<\lambda} = \lambda$ . If  $W$  and  $Z$  are countably complete ultrafilters on  $\lambda$  such that  $W \triangleleft Z$ , then  $W <_{\mathbb{k}} Z$ .*

*Proof.* Given the assumption that  $2^{<\lambda} = \lambda$  and the fact that  $P(\lambda) \subseteq M_Z$ , it is not hard to show that  $\text{Un}_\alpha \in M_Z$  and  $M_Z \models |\text{Un}_\alpha| \leq 2^\lambda$  for all  $\alpha < \lambda$ . Therefore we are in a position to apply Lemma 4.3.32 to any ultrafilter  $U <_{\mathbb{k}} W$ . Assume towards a contradiction that  $W \not<_{\mathbb{k}} Z$ . By the linearity of the Ketonen order,  $Z <_{\mathbb{k}} W$ . Now  $Z <_{\mathbb{k}} W \triangleleft Z$ , so by Lemma 4.3.32,  $Z \triangleleft Z$ . This contradicts the strictness of the Mitchell order (Lemma 4.2.40).  $\square$

**Corollary 4.3.34** (UA + GCH). *The Ketonen order extends the generalized Mitchell order on countably complete uniform ultrafilters on infinite cardinals.*  $\square$

Corollary 8.3.27 shows that the same conclusion can be deduced from UA alone. This will be achieved by proving from UA that if  $W \triangleleft Z$ , then  $Z$  is  $\lambda_W$ -supercompact. The result then follows from Proposition 4.3.30.

## 4.4 Generalizations of normality

In this section, we develop the theory of normal fine ultrafilters, the natural combinatorial generalization of normal ultrafilters, and a central component of the classical theory of supercompact cardinals. The main result of the section (Theorem 4.4.2) states roughly that UA + GCH implies that all these ultrafilters are linearly ordered by the Mitchell order.

**Definition 4.4.1.** For any infinite cardinal  $\lambda$ , let  $\mathcal{N}_\lambda$  be set of normal fine ultrafilters on  $P_{\text{bd}}(\lambda)$ . Let  $\mathcal{N} = \bigcup_\lambda \mathcal{N}_\lambda$ .

We provide the definitions of normality and fineness in Section 4.4.

**Theorem 4.4.2** (UA). *Suppose  $\lambda$  is a cardinal such that  $2^{<\lambda} = \lambda$ . Then  $\mathcal{N}_\lambda$  is wellordered by the Mitchell order. Therefore assuming the Generalized Continuum Hypothesis,  $\mathcal{N}$  is linearly ordered by the Mitchell order.*

We asserted that UA + GCH would roughly imply that the Mitchell order is linear on the class of all normal fine ultrafilters, but our theorem only mentions the subclass  $\mathcal{N}$ . In fact, the class of all normal fine ultrafilters is not literally linearly ordered by the Mitchell order for a number of reasons: one reason is that distinct normal fine ultrafilters can be isomorphic and hence Mitchell incomparable. Proposition 4.4.12 below, however, shows that every normal fine ultrafilter is isomorphic to an element of  $\mathcal{N}$ , so Theorem 4.4.2 essentially covers all the bases.

A key concept in the proof of Theorem 4.4.2, introduced here for the first time, is that of an isonormal ultrafilter.

**Definition 4.4.3.** Suppose  $\lambda$  is a cardinal. An ultrafilter  $U$  on  $\lambda$  is *isonormal* if  $U$  is weakly normal and  $j_U$  is a  $\lambda$ -supercompact embedding.

We define weak normality in Section 4.4. The concept dates back to Solovay and Ketonen [13]. The other main theorem of this section explains how isonormal ultrafilters get their name:

**Theorem 4.4.37.** *Suppose  $U$  is a nonprincipal ultrafilter. Then  $U$  is isonormal if and only if  $U$  is the incompressible ultrafilter isomorphic to a normal fine ultrafilter. In particular, every normal fine ultrafilter is isomorphic to a unique isonormal ultrafilter.*

The proof appears in Section 4.4. The forwards direction is quite easy, but the reverse implication requires quite a bit of work amounting to a generalization of the theorem of [21] known as *Solovay's Lemma* to singular cardinals. This generalization constitutes a fundamental and (apparently) new fact about supercompactness whose proof requires some basic notions from PCF theory.

The investigation of isonormal ultrafilters is related back to the linearity of the Mitchell order by the following proposition:

**Theorem 4.4.25.** *Suppose  $2^{<\lambda} = \lambda$ . Then every isonormal ultrafilter  $U$  on  $\lambda$  is Dodd sound.*

This is basically just a matter of defining weakly normal ultrafilters on singular cardinals. We actually prove our main theorem (Theorem 4.4.2) right now. But we will need to assume Theorem 4.4.25 and Theorem 4.4.37. We also need a lemma that shows  $\mathcal{N}$  is well-behaved under the Mitchell order assuming GCH:

**Lemma 4.4.4.** *If  $2^{<\lambda} = \lambda$ , then any  $\mathcal{U} \in \mathcal{N}_\lambda$  is hereditarily uniform and satisfies  $\lambda_{\mathcal{U}} = \lambda$ .*

*Proof.* Since  $P_{\text{bd}}(\lambda)$  is transitive,  $|\text{tc}(P_{\text{bd}}(\lambda))| = |P_{\text{bd}}(\lambda)| = 2^{<\lambda} = \lambda$ . On the other hand, since  $j_{\mathcal{U}}$  is  $\lambda$ -supercompact, Proposition 4.2.31 implies  $\lambda_{\mathcal{U}} \geq \lambda$ . Thus  $|\text{tc}(P_{\text{bd}}(\lambda))| = \lambda_{\mathcal{U}}$ , so  $\mathcal{U}$  is hereditarily uniform.  $\square$

We finally prove Theorem 4.4.2 assuming Theorem 4.4.25 and Theorem 4.4.37.

*Proof of Theorem 4.4.2.* Suppose  $\mathcal{U}$  and  $\mathcal{W}$  are elements of  $\mathcal{N}_\lambda$ . We show that either  $\mathcal{U} \triangleleft \mathcal{W}$ ,  $\mathcal{U} = \mathcal{W}$ , or  $\mathcal{U} \triangleright \mathcal{W}$ . Applying Theorem 4.4.37, let  $U$  be the isonormal ultrafilter isomorphic to  $\mathcal{U}$  and let  $W$  be the isonormal ultrafilter isomorphic to  $\mathcal{W}$ . Note that  $U$  and  $W$  are uniform ultrafilters on the cardinal  $\lambda_{\mathcal{U}} = \lambda_{\mathcal{W}} = \lambda$  (Lemma 4.4.4). We have  $2^{<\lambda} = \lambda$  by assumption, so Theorem 4.4.25 yields that  $U$  and  $W$  are Dodd sound. By the linearity of the Mitchell order on Dodd sound ultrafilters (Theorem 4.3.29), we are in one of the following cases:

**Case 1.**  $U = W$ .

*Proof in Case 1.* Since  $\mathcal{U} \cong U = W \cong \mathcal{W}$ , Lemma 4.4.11 below implies  $\mathcal{U} = \mathcal{W}$ .  $\square$

**Case 2.**  $U \triangleleft W$ .

*Proof in Case 2.* Since  $W \cong \mathcal{W}$ , we have  $U \triangleleft \mathcal{W}$ . Since  $\mathcal{U}$  is hereditarily uniform (Lemma 4.4.4) and isomorphic to  $U$ , the isomorphism invariance of the generalized Mitchell order on hereditarily uniform ultrafilters (Lemma 4.2.14) implies  $\mathcal{U} \triangleleft \mathcal{W}$ .  $\square$

**Case 3.**  $U \triangleright W$ .

*Proof in Case 3.* Proceeding as in Case 2, we obtain  $\mathcal{U} \triangleright \mathcal{W}$ .  $\square$

This shows that either  $\mathcal{U} \triangleleft \mathcal{W}$ ,  $\mathcal{U} = \mathcal{W}$ , or  $\mathcal{U} \triangleright \mathcal{W}$ , as desired.

We finally sketch the proof that  $\mathcal{N}$  is linearly ordered by the Mitchell order assuming  $\text{UA} + \text{GCH}$ . It suffices to show the following: suppose  $\mathcal{U} \in \mathcal{N}_\gamma$ ,  $\mathcal{W} \in \mathcal{N}_\lambda$ , and  $2^{<\lambda} = \lambda$ . Then  $\mathcal{U} \triangleleft \mathcal{W}$ . Let  $U$  be the isonormal ultrafilter of  $\mathcal{U}$ , so by the proof of Lemma 4.4.4,  $U$  is



an ultrafilter on  $\gamma$ . Since  $2^\gamma \leq 2^{<\lambda} = \lambda$ ,  $U \in H_{(2^\gamma)^+} \subseteq H_{\lambda^+} \subseteq M_{\mathcal{W}}$ . Since  $P(P_{bd}(\gamma)) \subseteq M_{\mathcal{W}}$ , this easily implies that  $\mathcal{U} \triangleleft \mathcal{W}$ .  $\square$

## Normal fine ultrafilters

In this section, we give the general definition of a normal fine ultrafilter, which is the natural combinatorial generalization of the notion of a normal ultrafilter on a cardinal. This begins with the generalized diagonal intersection operation:

**Definition 4.4.5.** Suppose  $X$  is a set and  $\langle A_x : x \in X \rangle$  is a sequence with  $A_x \subseteq P(X)$  for all  $x \in X$ . The *diagonal intersection* of  $\langle A_x : x \in X \rangle$  is the set

$$\Delta_{x \in X} A_x = \{ \sigma \in P(X) : \sigma \in \bigcap_{x \in \sigma} A_x \}$$

**Definition 4.4.6.** If  $X$  is a set, a *family over  $X$*  is a family  $Y$  of subsets of  $X$  such that every element of  $X$  belongs to some element of  $Y$ .

Thus any set  $Y$  is a family on a unique set (namely  $X = \bigcup Y$ ).

**Definition 4.4.7.** Suppose  $Y$  is a family over  $X$ . A filter  $\mathcal{F}$  on  $Y$  is:

- *fine* if for any  $x \in X$ ,  $\mathcal{F}$  concentrates on  $\{ \sigma : x \in \sigma \}$ .
- *normal* if for any  $\{ A_x : x \in X \} \subseteq \mathcal{F}$ ,  $\Delta_{x \in X} A_x \in \mathcal{F}$ .

**Remark 4.4.8.** Let us make some remarks regarding this definition.

- (1) It makes sense to discuss normal fine filters on  $Y$  without mention of  $X$ , since  $X = \bigcup Y$  is determined from  $Y$ .
- (2) The structure of the underlying set  $Y$  is usually not that important since a normal fine ultrafilter  $\mathcal{U}$  on  $Y$  can always be lifted to a normal fine ultrafilter on  $P(X)$  where  $X = \bigcup_{\sigma \in Y} \sigma$ . Therefore it is tempting to restrict consideration to normal fine ultrafilters on  $P(X)$  for some  $X$ . It is often important for technical reasons, however, that the underlying set  $Y$  be small; usually we want  $|Y| = |\bigcup Y|$ .

- (3) The structure of the set  $X$  is also usually irrelevant, but sometimes it is useful that  $X$  be transitive or that  $X$  be a cardinal. Suppose  $X$  and  $X'$  are sets and  $f : X \rightarrow X'$  is a surjection. If  $Y$  is a family over  $X$ , then  $Y' = \{f[\sigma] : \sigma \in Y\}$  is a family over  $X'$  and  $g(\sigma) = f[\sigma]$  defines a surjection from  $Y$  to  $Y'$ . If  $\mathcal{U}$  is an ultrafilter on  $Y$ , then  $g_*(\mathcal{U})$  is an isomorphic ultrafilter on  $Y'$  and moreover  $\mathcal{U}'$  is normal (fine) if and only if  $\mathcal{U}$  is normal (fine). (This is the ultrafilter theoretic analog of Lemma 4.2.17.)
- (4) An ultrafilter on an ordinal is fine if and only if it is tail uniform. Thus a normal fine ultrafilter on  $\kappa$  is the same thing as a normal ultrafilter on  $\kappa$ .

The connection between normality and supercompactness is clear from the following lemma:

**Lemma 4.4.9.** *Suppose  $Y$  is a family over  $X$  and  $\mathcal{U}$  is an ultrafilter on  $Y$ .*

(1)  $\mathcal{U}$  is fine if and only if  $j_{\mathcal{U}}[X] \subseteq a_{\mathcal{U}}$ .

(2)  $\mathcal{U}$  is normal if and only if  $a_{\mathcal{U}} \subseteq j_{\mathcal{U}}[X]$ . □

Thus  $\mathcal{U}$  is a normal fine ultrafilter on  $Y$  over  $X$  if and only if  $a_{\mathcal{U}} = j_{\mathcal{U}}[X]$ , or in other words,  $a_{\mathcal{U}}$  witnesses that  $j_{\mathcal{U}}$  is  $X$ -supercompact.

Lemma 4.4.9 yields the main source of normal fine ultrafilters.

**Lemma 4.4.10.** *Suppose  $j : V \rightarrow M$  is an  $X$ -supercompact elementary embedding and  $Y \subseteq P(X)$  is such that  $j[X] \in j(Y)$ .*

- $Y$  is a family over  $X$ .
- The ultrafilter  $\mathcal{U}$  on  $Y$  derived from  $j$  using  $j[X]$  is a normal fine ultrafilter on  $Y$ .
- Let  $k : M_{\mathcal{U}} \rightarrow M$  be the factor embedding. Then  $k(\alpha) = \alpha$  for all  $\alpha \leq |X|$ .

*Proof.* Immediate from Lemma 4.2.22 and Lemma 4.4.9. □

Another consequence of Lemma 4.4.9 is the following fact, which does not seem to have a simple combinatorial proof:

**Lemma 4.4.11.** *Suppose  $\mathcal{U}$  and  $\mathcal{W}$  are normal fine ultrafilters on  $Y$ . If  $\mathcal{U} \cong \mathcal{W}$  then  $\mathcal{U} = \mathcal{W}$ .*

*Proof.* Let  $X = \bigcup Y$ . Since  $\mathcal{U} \cong \mathcal{W}$ ,  $j_{\mathcal{U}} = j_{\mathcal{W}}$ . By Lemma 4.4.9,  $a_{\mathcal{U}} = j_{\mathcal{U}}[X] = j_{\mathcal{W}}[X] = a_{\mathcal{W}}$ . Thus  $\mathcal{U} = \{A \subseteq Y : a_{\mathcal{U}} \in j_{\mathcal{U}}(A)\} = \{A \subseteq Y : a_{\mathcal{W}} \in j_{\mathcal{W}}(A)\} = \mathcal{W}$ .  $\square$

It also follows that any normal fine ultrafilter is countably complete. This is because the proof that an  $\omega$ -supercompact ultrapower embedding  $j : V \rightarrow M$  has the property that  $M^\omega \subseteq M$  does not really require that  $M$  is wellfounded. (The reader will lose nothing by simply appending countable completeness to the definition of normality, rather than proving it from the definition we have given.)

Recall the class  $\mathcal{N}$  defined in the previous section. We finish this section by proving that every normal fine ultrafilter is isomorphic to a unique element of  $\mathcal{N}$ .

**Proposition 4.4.12.** *Any nonprincipal normal fine ultrafilter  $\mathcal{D}$  is isomorphic to a unique ultrafilter  $\mathcal{U} \in \mathcal{N}$ .*

For this we will use a basic lemma about supercompactness:

**Lemma 4.4.13.** *Suppose  $j : V \rightarrow M$  is  $\lambda$ -supercompact and  $\sup j[\lambda] = j(\lambda)$ . Then  $j$  is  $\lambda^\iota$ -supercompact where  $\iota = \text{cf}(\lambda)$ . In particular,  $j$  is  $\lambda^+$ -supercompact.*

*Proof.* Let  $\kappa = \text{CRT}(j)$ . Lemma 4.2.25 states that  $j$  is  $\lambda^{<\kappa}$ -supercompact. It suffices to show that  $\iota < \kappa$ : then since  $j$  is  $\lambda^{<\kappa}$ -supercompact,  $j$  is  $\lambda^\iota$ -supercompact, and so since  $\lambda^\iota > \lambda$ ,  $j$  is  $\lambda^+$ -supercompact.

We now show  $\iota < \kappa$ . Since  $\sup j[\lambda] = j(\lambda)$  and  $j[\lambda] \in M$ ,  $\text{cf}^M(j(\lambda)) = \text{cf}(\lambda) = \iota$ . On the other hand, by elementarity  $\text{cf}^M(j(\lambda)) = j(\text{cf}(\lambda)) = j(\iota)$ . It follows that  $j(\iota) = \iota$ . Since  $j$  is  $\iota$ -supercompact, the Kunen Inconsistency Theorem (Theorem 4.2.37) implies  $\iota < \kappa$  where  $\kappa = \text{CRT}(j)$ .  $\square$

Actually, we always have  $\lambda^+ = \lambda^{<\kappa}$  in the context of Lemma 4.4.13, and this is how SCH above a supercompact is proved.

*Proof of Proposition 4.4.12.* Obviously, any normal fine ultrafilter is isomorphic to a normal fine ultrafilter on  $P(\lambda)$  for some cardinal  $\lambda$ . Therefore assume  $\mathcal{D}$  is a normal ultrafilter on  $P(\lambda)$ , and we will show that  $\mathcal{D}$  is isomorphic to a normal fine ultrafilter on  $P_{\text{bd}}(\lambda')$  for some cardinal  $\lambda'$ .

If  $\mathcal{D}$  concentrates on  $P_{\text{bd}}(\lambda)$ , we are done, since  $\mathcal{D}$  is then isomorphic to  $\mathcal{D} \upharpoonright P_{\text{bd}}(\lambda)$ . So assume  $\mathcal{D}$  does not concentrate on  $P_{\text{bd}}(\lambda)$ . By Łoś's Theorem,  $a_{\mathcal{D}} = j_{\mathcal{D}}[\lambda]$  is unbounded in  $j_{\mathcal{D}}(\lambda)$ . In other words,  $j_{\mathcal{D}}$  is continuous at  $\lambda$ . Therefore by Lemma 4.4.13,  $j_{\mathcal{D}}$  is  $\lambda^+$ -supercompact. Note that  $j_{\mathcal{D}}[\lambda^+]$  is not cofinal in  $j_{\mathcal{D}}(\lambda^+)$ : otherwise  $j_{\mathcal{D}}(\lambda^+) = \text{cf}^{M_{\mathcal{D}}}(j_{\mathcal{D}}(\lambda^+)) = \lambda^+$ , so  $\text{CRT}(j_{\mathcal{D}}) > \lambda^+$  by Theorem 4.4.32, which implies that  $\mathcal{D}$  is principal. Therefore let  $\mathcal{U}$  be the normal fine ultrafilter on  $P_{\text{bd}}(\lambda^+)$  derived from  $j_{\mathcal{D}}$  using  $j_{\mathcal{D}}[\lambda^+]$ . Then  $\mathcal{U}$  is isomorphic to  $\mathcal{D}$ : by construction  $\mathcal{U} \leq_{\text{RK}} \mathcal{D}$ , and on the other hand, the map  $f: P_{\text{bd}}(\lambda^+) \rightarrow Y$  defined by  $f(\sigma) = \sigma \cap \lambda$  pushes  $\mathcal{U}$  forward to  $\mathcal{D}$  so  $\mathcal{D} \leq_{\text{RK}} \mathcal{U}$ .  $\square$

## Weakly normal ultrafilters

Another combinatorial generalization of the notion of a normal ultrafilter, due to Solovay and Ketonen [13], is the notion of a weakly normal ultrafilter.

**Definition 4.4.14.** A uniform ultrafilter  $U$  on a cardinal  $\lambda$  is *weakly normal* if for any set  $A \in U$ , if  $f: A \rightarrow \lambda$  is regressive, then there is some  $B \subseteq A$  such that  $B \in U$  and  $f[B]$  has cardinality less than  $\lambda$ .

Solovay's definition of a weakly normal ultrafilter applied only to regular cardinals  $\lambda$ , asserting that every regressive function on  $\lambda$  is *bounded* on a set of full measure. The generalization of the concept of weak normality to singular cardinals is due to Ketonen.

**Lemma 4.4.15.** *Suppose  $U$  is a uniform ultrafilter on a cardinal  $\lambda$ . Then the following are equivalent:*

(1)  $U$  is weakly normal.

(2) Suppose  $\langle A_\alpha : \alpha < \lambda \rangle$  is a sequence of subsets of  $\lambda$  such that  $\bigcap_{\alpha \in \sigma} A_\alpha \in U$  for all nonempty  $\sigma \in P_\lambda(\lambda)$ . Then  $\Delta_{\alpha < \lambda} A_\alpha \in U$ .  $\square$

**Corollary 4.4.16.** *A uniform ultrafilter on a regular cardinal is weakly normal if and only if it is closed under decreasing diagonal intersections.*  $\square$

Weakly normal ultrafilters on regular cardinals have a simple characterization in terms of their ultrapowers:

**Lemma 4.4.17.** *Suppose  $\lambda$  is a regular cardinal. An ultrafilter  $U$  on  $\lambda$  is weakly normal if and only if  $a_U = \sup j_U[\lambda]$ .*

*Proof.* Suppose  $U$  is weakly normal. Since  $U$  is a tail uniform ultrafilter on  $\lambda$ ,  $a_U > j_U(\alpha)$  for all  $\alpha < \lambda$ . We will show that  $j_U[\lambda]$  is cofinal in  $a_U$ , which proves  $a_U = \sup j_U[\lambda]$ . Suppose  $\xi < a_U$ . Then  $\xi = [f]_U$  for some  $f : \lambda \rightarrow \lambda$  that is regressive on a set in  $U$ . Since  $U$  is weakly normal, there is a set  $A \in U$  such that  $|f[A]| < \lambda$ . Since  $\lambda$  is regular,  $f[A]$  is bounded below  $\lambda$ . Fix  $\alpha < \lambda$  such that  $f(\xi) < \alpha$  for all  $\xi \in A$ . Then  $[f]_U < j_U(\alpha)$ .

Conversely suppose  $a_U = \sup j_U[\lambda]$ . Since  $a_U > j_U(\alpha)$  for all  $\alpha < \lambda$ ,  $\delta_U \geq \lambda$ , and hence  $U$  is tail uniform. Since  $\lambda$  is regular, it follows that  $\lambda$  is uniform. Next, suppose  $A \in U$  and  $f : A \rightarrow \lambda$  is regressive. Then  $[f]_U < a_U$ . Since  $j_U[\lambda]$  is cofinal in  $a_U$ , fix  $\alpha < \lambda$  with  $[f]_U < j_U(\alpha)$ . Then for a set  $B \in U$  with  $B \subseteq A$ ,  $f(\beta) < \alpha$  for all  $\beta \in B$ . In particular,  $f$  takes fewer than  $\lambda$  values on  $B$ .  $\square$

Lemma 4.4.17 yields the main source of weakly normal ultrafilters on regular cardinals:

**Corollary 4.4.18.** *Suppose  $j : V \rightarrow M$  is an elementary embedding and  $\lambda$  is a regular cardinal such that  $\sup j[\lambda] < j(\lambda)$ . Then the ultrafilter on  $\lambda$  derived from  $j$  using  $\sup j[\lambda]$  is weakly normal.*

To help motivate the concept of weak normality on singular cardinals, let us explain its relationship to an isomorphism invariant notion:

**Definition 4.4.19.** Suppose  $\lambda$  is an infinite cardinal. An ultrafilter  $U$  is  $\lambda$ -minimal if  $\lambda_U = \lambda$  and for any  $W <_{\text{RK}} U$ ,  $\lambda_W < \lambda$ .

If  $2^\lambda = \lambda^+$ , there is a  $\lambda$ -minimal (countably incomplete) ultrafilter on  $\lambda$ , according to a result of Comfort-Negrepointis [25]. On the other hand, the existence of a weakly normal ultrafilter (with no completeness assumptions) implies the existence of an inner model with a measurable cardinal [26]. Weakly normal ultrafilters, however, are the revised Rudin-Keisler analog (Definition 3.4.10) of  $\lambda$ -minimal ones:

**Lemma 4.4.20.** *An ultrafilter  $U$  on a cardinal  $\lambda$  is weakly normal if and only if  $\lambda_U = \lambda$  and for all  $W <_{\text{rk}} U$ ,  $\lambda_W < \lambda$ .* □

Lemma 4.4.20 yields a generalization of Scott's theorem that every countably complete ultrafilter has a derived normal ultrafilter:

**Corollary 4.4.21.** *If  $Z$  is a countably complete uniform ultrafilter on  $\lambda$ , there is a weakly normal ultrafilter  $U$  on  $\lambda$  such that  $U \leq_{\text{RK}} Z$ .*

*Proof.* Since  $<_{\text{rk}}$  is wellfounded on countably complete ultrafilters, there is a countably complete ultrafilter  $U$  that is  $<_{\text{rk}}$ -minimal with the property that  $\lambda_U = \lambda$  and  $U \leq_{\text{RK}} Z$ . Then  $U$  satisfies the conditions of Lemma 4.4.20: if  $W <_{\text{rk}} U$ , then  $W \leq_{\text{RK}} Z$ , so by the  $<_{\text{rk}}$ -minimality of  $U$ , it must be the case that  $\lambda_W < \lambda$  □

The following theorem shows that every countably complete  $\lambda$ -minimal ultrafilter is isomorphic to a weakly normal ultrafilter.

**Proposition 4.4.22.** *A countably complete uniform ultrafilter  $U$  on a cardinal  $\lambda$  is weakly normal if and only if it is  $\lambda$ -minimal and incompressible.*

*Proof.* Suppose  $U$  is weakly normal. To see  $U$  is incompressible, note that any function that is regressive on a set in  $U$  takes less than  $\lambda$ -many values on a set in  $U$ , and hence is not one-to-one. To see  $U$  is  $\lambda$ -minimal, suppose  $W <_{\text{RK}} U$  and we will show that  $\lambda_W < \lambda$ .

Since  $W \leq_{\text{RK}} U$ ,  $W$  is countably complete, and hence  $W$  is isomorphic to an incompressible ultrafilter. We can therefore assume without loss of generality that  $W$  is incompressible. Then by the key lemma about the strict Rudin-Keisler order on incompressible ultrafilters (Proposition 3.4.20) the fact that  $W <_{\text{RK}} U$  implies  $W <_{\text{rk}} U$ . Now by Lemma 4.4.20,  $\lambda_W < \lambda$ .

Conversely suppose  $U$  is  $\lambda$ -minimal and incompressible. Suppose  $W <_{\text{rk}} U$ , and we will show  $\lambda_W < \lambda$ . We can then conclude that  $U$  is weakly normal using Lemma 4.4.20. Since  $U$  is incompressible,  $W <_{\text{rk}} U$  implies  $W <_{\text{RK}} U$  (Lemma 3.4.15, essentially the definition of incompressibility). Therefore by the definition of  $\lambda$ -minimality,  $\lambda_W < \lambda$ , as desired.  $\square$

It is not clear to us whether Proposition 4.4.22 can be proved without the assumption of countable completeness, though of course countable completeness is not required if  $\lambda$  is regular.

The following characterization of weak normality is the one that is most relevant to our investigations of supercompactness.

**Proposition 4.4.23.** *Suppose  $\lambda$  is an infinite cardinal. A countably complete ultrafilter  $U$  on  $\lambda$  is weakly normal if and only if  $a_U$  is the unique generator of  $j_U$  that lies above  $j(\delta)$  for all  $\delta < \lambda$ .*

For the proof, we will need an obvious lemma:

**Lemma 4.4.24.** *Suppose  $\lambda$  is an infinite cardinal. An ultrafilter  $U$  on  $\lambda$  is uniform if and only if  $a_U \notin H^{M_U}(j_U[V] \cup j_U(\delta))$  for any  $\delta < \lambda$ .*  $\square$

*Proof of Proposition 4.4.23.* We begin with the forwards direction. Suppose  $U$  is weakly normal.

We first show that for any ordinal  $\xi$  such that  $\xi < a_U$ ,  $\xi \in H^{M_U}(j_U[V] \cup j_U(\delta))$  for some  $\delta < \lambda$ . Assume not, towards a contradiction. Let  $W$  be the tail uniform ultrafilter derived from  $j_U$  using  $\xi$ . Then  $W <_{\text{rk}} U$ , as witnessed by the factor embedding  $k : M_W \rightarrow$

$M_U$ . By Lemma 4.4.20, it follows that  $W$  is not a uniform ultrafilter on  $\lambda$ , and so by Lemma 4.4.24, there is some  $\delta < \lambda$  such that  $\xi \in H^{M_W}(j_W[V] \cup j_W(\delta))$ . It follows that  $\xi \in H^{M_U}(j_U[V] \cup j_U(\delta))$ .

Next we show that  $a_U$  is a generator of  $j_U$ . Since  $U$  is uniform, Lemma 4.4.24 implies  $a_U \notin H^{M_U}(j_U[V] \cup j_U(\delta))$  for any  $\delta < \lambda$ . But by the previous paragraph, for all  $\xi < a_U$ ,  $\xi \in H^{M_U}(j_U[V] \cup j_U(\delta))$  for some  $\delta < \lambda$ . Thus  $a_U \notin H^{M_U}(j_U[V] \cup \vec{\xi})$  for any  $\vec{\xi} \in [a_U]^{<\omega}$ . In other words,  $a_U$  is a generator of  $j_U$ . By the previous paragraph,  $a_U$  is clearly the unique generator above  $j_U(\delta)$  for all  $\delta < \lambda$ .

We now turn to the converse. Assume  $a_U$  is the unique generator of  $j_U$  that lies above  $j(\delta)$  for all  $\delta < \lambda$ . We will show  $U$  is weakly normal by verifying the conditions of Proposition 4.4.22. Since  $a_U$  is a generator,  $U$  is incompressible. Since  $M_U$  is wellfounded, there is a least ordinal that does not belong to  $H^{M_U}(j_U[V] \cup j_U(\delta))$  for any  $\delta$ , and clearly this ordinal is a generator of  $j_U$  that lies above  $j_U(\delta)$  for all  $\delta < \lambda$ . Thus it must equal  $a_U$ . In other words, for any  $\xi < a_U$ ,  $\xi \in H^{M_U}(j_U[V] \cup j_U(\delta))$  for some  $\delta < \lambda$ .

Fix an ultrafilter  $W$  on  $\lambda$  such that  $W <_{\text{rk}} U$ . We will show  $\lambda_W < \lambda$ , verifying the second condition of Proposition 4.4.22. Let  $k : M_W \rightarrow M_U$  be an elementary embedding with  $k \circ j_W = j_U$  and  $k(a_W) < a_U$ . Then by the previous paragraph,  $k(a_W) \in H^{M_U}(j_U[V] \cup j_U(\delta))$  for some  $\delta < \lambda$ . It follows that  $a_W \in H^{M_W}(j_W[V] \cup j_W(\delta))$  (by the proof of Lemma 3.5.34). By Lemma 4.4.24, this implies  $W$  is not uniform on  $\lambda$ , or in other words,  $\lambda_W < \lambda$ .  $\square$

Using Proposition 4.4.23, we can prove the Dodd soundness of isonormal ultrafilters on  $\lambda = 2^{<\lambda}$ .

**Theorem 4.4.25.** *Suppose  $2^{<\lambda} = \lambda$ . Then every isonormal ultrafilter  $U$  on  $\lambda$  is Dodd sound.*

*Proof.* Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $U$ . Since  $j$  is  $\lambda$ -supercompact,  $j$  is  $2^{<\lambda}$ -supercompact. By Lemma 4.3.4,  $j$  is  $\lambda_*$ -sound where  $\lambda_* = \sup j[\lambda]$ .

We now show that  $j$  is  $\xi$ -sound where  $\xi$  is the least generator of  $j$  such that  $\xi \geq \lambda_*$ . Since



$\lambda_*$  is closed under pairing, the  $\lambda_*$ -soundness of  $j$  implies that the extender

$$E = E_j \upharpoonright \lambda_* = \{(p, X) : p \in [\lambda_*]^{<\omega}, X \subseteq [\lambda]^{<\omega}, \text{ and } p \in j(X)\}$$

belongs to  $M_U$ . Let  $j_E : V \rightarrow M_E$  be the associated extender embedding and let  $k : M_E \rightarrow M$  be the factor embedding. Then

$$\text{CRT}(k) = \min\{\alpha : \alpha \notin H^M(j[V] \cup \lambda_*)\} = \xi$$

by the definition of a generator. Therefore  $j_E^\xi = j^\xi$ . Moreover since  $M$  is closed under  $\lambda$ -sequences by Corollary 4.2.21,  $j_E^M = j_E \upharpoonright M$ . Therefore  $j^\xi = j_E^\xi = (j_E^M)^\xi \in M$ , so  $j$  is  $\xi$ -sound.

By Proposition 4.4.23,  $\xi = a_U$ . Therefore  $j$  is  $a_U$ -sound, which implies that  $U$  is Dodd sound. □

We should point out that the assumption  $\lambda = 2^{<\lambda}$  is *necessary*:

**Lemma 4.4.26.** *Suppose  $\lambda$  is a cardinal that carries a Dodd sound ultrafilter  $U$ . Then  $2^{<\lambda} = \lambda$ .*

*Proof.* Since  $U$  is Dodd sound,  $j_U$  is  $a_U$ -sound. In particular,  $j_U$  is  $\sup j_U[\lambda]$ -sound since  $\sup j_U[\lambda] \leq a_U$ . Therefore by Lemma 4.3.4,  $j_U$  is  $2^{<\lambda}$ -supercompact. By Proposition 4.2.31,  $j_U$  is not  $\lambda^+$ -supercompact. It follows that  $2^{<\lambda} < \lambda^+$ , or in other words  $2^{<\lambda} = \lambda$ . □

## Solovay's Lemma

A special case of our main theorem, Theorem 4.4.37, was known long before our work.

**Theorem 4.4.27** (Solovay's Lemma). *Suppose  $\lambda$  is a regular cardinal. Then there is a set  $B \subseteq P(\lambda)$  such that the following hold:*

- *For any family  $Y$  over  $\lambda$ , any normal fine ultrafilter  $\mathcal{U}$  on  $Y$  concentrates on  $B$ .*
- *If  $\sigma$  and  $\tau$  are elements of  $B$  with the same supremum, then  $\sigma = \tau$ .*

Before proving Solovay’s Lemma, let us explain its relevance to isonormal ultrafilters. Essentially, Solovay’s Lemma yields the “regular case” of the key isomorphism theorem for isonormal ultrafilters (Theorem 4.4.37):

**Corollary 4.4.28.** *Suppose  $\lambda$  is a regular cardinal,  $Y$  is a family over  $\lambda$ , and  $\mathcal{U}$  is a non-principal normal fine ultrafilter on  $Y$ . Then  $\mathcal{U}$  is isomorphic to the ultrafilter*

$$U = \{A \subseteq \lambda : \{\sigma \in Y : \sup \sigma \in A\} \in \mathcal{U}\}$$

Moreover,  $U$  is an isonormal ultrafilter.

*Proof.* To see  $\mathcal{U} \cong U$ , let  $f : P(\lambda) \rightarrow \lambda + 1$  be the function  $f(\sigma) = \sup \sigma$ . Then  $f_*(\mathcal{U}) = U$  and by Theorem 4.4.27,  $f$  is one-to-one on a set in  $\mathcal{U}$ .

To see  $U$  is isonormal, we must verify that  $U$  is weakly normal and  $j_U$  is  $\lambda$ -supercompact. The latter is trivial:  $j_{\mathcal{U}}$  is  $\lambda$ -supercompact by Lemma 4.4.9, and  $j_{\mathcal{U}} = j_U$  since  $\mathcal{U}$  and  $U$  are isomorphic. As for weak normality, by Lemma 3.2.16,  $U = f_*(\mathcal{U})$  is the ultrafilter on  $\lambda$  derived from  $j_{\mathcal{U}}$  using  $[f]_{\mathcal{U}}$  so  $U$  is weakly normal by Corollary 4.4.18.  $\square$

The proof of Solovay’s lemma uses the observation that if  $j : V \rightarrow M$  is an elementary embedding,  $j[\lambda]$  is definable from the action of  $j$  on a stationary partition:<sup>4</sup>

**Lemma 4.4.29.** *Suppose  $\lambda$  is a cardinal,  $j : V \rightarrow M$  is an elementary embedding, and  $\mathcal{P} \subseteq P(\lambda)$  is a partition of  $S_{\omega}^{\lambda} = \{\alpha < \lambda : \text{cf}(\alpha) = \omega\}$  into stationary sets. Then*

$$j[\mathcal{P}] = \{T \in j(\mathcal{P}) : T \text{ is stationary in } \sup j[\lambda]\}$$

It is worth noting that Lemma 4.4.29 is perfectly general; we really do allow  $j$  to be an arbitrary elementary embedding of  $V$ .

*Proof.* Let  $\lambda_* = \sup j[\lambda]$ .

**Claim 1.**  $j[\mathcal{P}] \subseteq \{T \in j(\mathcal{P}) : T \text{ is stationary in } \lambda_*\}$ .

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<sup>4</sup>Solovay’s published proof [21] uses the combinatorics of  $\omega$ -Jonsson algebras instead of stationary sets. Woodin rediscovered the proof using stationary sets, which was already known to Solovay.

*Proof.* Fix  $S \in \mathcal{P}$ . We will show that  $j(S)$  intersects every closed cofinal subset of  $\lambda_*$ . Suppose  $C \subseteq \lambda_*$  is closed cofinal in  $\lambda_*$ . Then  $j^{-1}[C]$  is  $\omega$ -closed cofinal in  $\lambda$ . Since  $S$  is a stationary subset of  $S_\omega^\lambda$ ,  $S \cap j^{-1}[C] \neq \emptyset$ . But  $j(S) \cap C = j(S) \cap C \supseteq j[S \cap j^{-1}[C]] \neq \emptyset$ . So  $j(S) \cap C \neq \emptyset$ , as desired.  $\square$

**Claim 2.**  $\{T \in j(\mathcal{P}) : T \text{ is stationary in } \lambda_*\} \subseteq j[\mathcal{P}]$ .

*Proof.* Fix  $T \in j(\mathcal{P})$  such that  $T$  is stationary in  $\lambda_*$ . We will show that there is some  $S \in \mathcal{P}$  such that  $j(S) = T$ . Since  $j[\lambda]$  is  $\omega$ -closed cofinal in  $\lambda_*$ ,  $T \cap j[\lambda] \neq \emptyset$ . Take  $\xi < \lambda$  such that  $j(\xi) \in T$ . Since  $j(\xi) \in T \subseteq j(S_\omega^\lambda)$ ,  $\xi \in S_\omega^\lambda$ . Therefore  $\xi \in S$  for some  $S \in \mathcal{P}$ , since  $\bigcup \mathcal{P} = S_\omega^\lambda$ . Now  $j(\xi) \in j(S) \cap T$ . Therefore  $j(S)$  and  $T$  are not disjoint, so since  $j(\mathcal{P})$  is a partition,  $j(S) = T$ , as desired.  $\square$

Combining the two claims yields the lemma.  $\square$

Lemma 4.4.29 leads to a characterization of supercompactness that looks surprisingly weak:

**Corollary 4.4.30.** *Suppose  $j : V \rightarrow M$  is an elementary embedding and  $\lambda$  is a regular cardinal. The following are equivalent:*

- (1)  $j$  is  $\lambda$ -supercompact.
- (2)  $M$  is correct about stationary subsets of  $\lambda_* = \sup j[\lambda]$ .

*Proof.* (1) implies (2): Assume  $j$  is  $\lambda$ -supercompact. Suppose  $M$  satisfies that  $S$  is stationary in  $\lambda_*$ , and we will show that  $S$  is truly stationary in  $\lambda_*$ . Fix a closed cofinal set  $C \subseteq \lambda_*$ . We will show  $S \cap C \neq \emptyset$ . Note that  $C \cap j[\lambda] \in M$  by Lemma 4.2.20 (3). Let  $E$  be the closure of  $C \cap j[\lambda]$  in  $\lambda_*$ . Then  $E \in M$ ,  $E \subseteq C$ , and  $E$  is closed cofinal in  $\lambda_*$ . Since  $E \in M$  and  $S$  is stationary from the perspective of  $M$ ,  $S \cap E \neq \emptyset$ . In particular,  $S \cap C \neq \emptyset$ .

(2) implies (1): Since  $\lambda$  is regular, there is a partition  $\mathcal{P}$  of  $S_\omega^\lambda$  into stationary sets such that  $|\mathcal{P}| = \lambda$ . By Lemma 4.4.29,  $j[\mathcal{P}] = \{T \in j(\mathcal{P}) : T \text{ is stationary in } \lambda_*\}$ , which is defin-

able over  $M$  since  $M$  is correct about stationary subsets of  $\lambda_*$ . Thus  $j$  is  $\mathcal{P}$ -supercompact, so by Lemma 4.2.17,  $j$  is  $\lambda$ -supercompact, as desired.  $\square$

Of course the implication from (1) to (2) is not very surprising, but it allows us to restate Lemma 4.4.29 in a useful way:

**Corollary 4.4.31.** *Suppose  $\lambda$  is a regular cardinal,  $j : V \rightarrow M$  is a  $\lambda$ -supercompact elementary embedding, and  $\langle S_\alpha : \alpha < \lambda \rangle$  is a partition of  $S_\omega^\lambda$  into stationary sets. Let  $\langle T_\beta : \beta < j(\lambda) \rangle = j(\langle S_\alpha : \alpha < \lambda \rangle)$ . Then  $j[\lambda] = \{\beta < j(\lambda) : M \models T_\beta \text{ is stationary in } \lambda_*\}$ .  $\square$*

We now prove Solovay's Lemma.

*Proof of Theorem 4.4.27.* Let  $\langle S_\alpha : \alpha < \lambda \rangle$  be a partition of  $S_\omega^\lambda = \{\alpha < \lambda : \text{cf}(\alpha) = \omega\}$  into stationary sets. Let

$$B = \{\sigma \subseteq \lambda : \sigma = \{\beta < \lambda : S_\beta \text{ is stationary in } \sup(\sigma)\}\}$$

By construction, any two elements of  $B$  with the same supremum are equal.

To finish, suppose  $Y$  is a family over  $\lambda$  and  $\mathcal{U}$  is a normal fine on  $Y$ . We must show that  $\mathcal{U}$  concentrates on  $B$ , or equivalently, that  $a_{\mathcal{U}} \in j_{\mathcal{U}}(B)$ . Since  $a_{\mathcal{U}} = j_{\mathcal{U}}[\lambda]$  (Lemma 4.4.9), this amounts to showing

$$j_{\mathcal{U}}[\lambda] = \{\beta < j_{\mathcal{U}}(\lambda) : M_{\mathcal{U}} \models j_{\mathcal{U}}(S)_\beta \text{ is stationary in } \sup j_{\mathcal{U}}[\lambda]\}$$

which is of course a consequence of Corollary 4.4.31.  $\square$

Another corollary of Solovay's Lemma is Woodin's proof of the Kunen Inconsistency Theorem:

**Theorem 4.4.32.** *Suppose  $j : V \rightarrow M$  is an elementary embedding,  $\iota$  is a regular cardinal,  $j$  is  $\iota$ -supercompact, and  $j(\iota) = \sup j[\iota]$ . Then  $j \upharpoonright \iota + 1$  is the identity.*

*Proof.* Let  $\langle S_\alpha : \alpha < \iota \rangle$  be a partition of  $S_\omega^\iota$  into stationary sets. By Corollary 4.4.31, and using the fact that  $j(\iota) = \sup j[\iota]$ ,

$$j[\iota] = \{\beta < j(\iota) : M \models j(S)_\beta \text{ is stationary in } j(\iota)\} = j(\iota)$$

But this means  $j \upharpoonright \iota + 1$  is the identity, as desired.  $\square$

Applying Theorem 4.4.32 at  $\iota = \lambda^+$  where  $\lambda$  is the first fixed point of  $j$  above  $\text{CRT}(j)$  yields a proof of the Kunen Inconsistency (Theorem 4.2.37).

## Supercompactness and singular cardinals

In this section, we finish the proof of Theorem 4.4.37. We do this by proving an analog of Solovay's Lemma at singular cardinals. One basic issue, however, is that Theorem 4.4.27 itself cannot generalize: in fact, if  $\lambda$  is a singular cardinal,  $Y$  is a family over  $\lambda$ , and  $\mathcal{U}$  is a normal fine ultrafilter on  $Y$ , then the supremum function is *not* one-to-one on any set in  $\mathcal{U}$ .

**Proposition 4.4.33.** *Suppose  $\lambda$  is a cardinal of cofinality  $\iota$ ,  $Y$  is a family over  $\lambda$ , and  $\mathcal{U}$  is a normal fine ultrafilter on  $Y$ . Define  $f : Y \rightarrow \lambda + 1$  by*

$$f(\sigma) = \sup \sigma$$

*Define  $g : Y \rightarrow \iota + 1$  by*

$$g(\sigma) = \sup(\sigma \cap \iota)$$

*Then  $f_*(\mathcal{U}) \cong g_*(\mathcal{U})$ .*

It is a bit easier to prove the following equivalent statement first (which in any case turns out to be more useful):

**Proposition 4.4.34.** *Suppose  $j : V \rightarrow M$  is an elementary embedding and  $\lambda$  is a cardinal of cofinality  $\iota$ . Then  $\sup j[\lambda]$  and  $\sup j[\iota]$  are interdefinable in  $M$  from parameters in  $j[V]$ .*

*Proof.* Let  $h : \iota \rightarrow \lambda$  be an increasing cofinal function. Then

$$\sup j[\lambda] = \sup j[h[\iota]] = \sup j(h) \circ j[\iota] = \sup j(h)[\sup j[\iota]]$$

Therefore  $\sup j[\lambda]$  is definable in  $M$  from  $j(h)$  and  $\sup j[\iota]$ . Moreover,

$$\sup j[\iota] = \sup j(h)^{-1}[\sup j[\lambda]]$$

so  $\sup j[\iota]$  is definable in  $M$  from  $j(h)$  and  $\sup j[\lambda]$ .  $\square$

*Proof of Proposition 4.4.33.* Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{U}$ . Then (by Lemma 3.2.16)  $f_*(\mathcal{U})$  is the ultrafilter on  $\lambda + 1$  derived from  $j$  using  $[f]_{\mathcal{U}} = \sup j[\lambda]$  and  $g_*(\mathcal{U})$  is the ultrafilter on  $\iota + 1$  derived from  $j$  using  $[g]_{\mathcal{U}} = \sup j[\iota]$ . By Proposition 4.4.34,

$$H^M(j[V] \cup \{\sup j[\lambda]\}) = H^M(j[V] \cup \{\sup j[\iota]\})$$

But

$$(M_{f_*(\mathcal{U})}, j_{f_*(\mathcal{U})}) \cong (H^M(j[V] \cup \{\sup j[\lambda]\}), j) \cong (M_{g_*(\mathcal{U})}, j_{g_*(\mathcal{U})})$$

It follows that  $f_*(\mathcal{U}) \cong g_*(\mathcal{U})$ . □

**Corollary 4.4.35.** *Suppose  $\lambda$  is a cardinal of cofinality  $\iota$ ,  $Y$  is a family over  $\lambda$ , and  $\mathcal{U}$  is a normal fine ultrafilter on  $Y$ . Then there is a set  $B \in \mathcal{U}$  on which the supremum function takes at most  $\iota$ -many values.*

*Proof.* Let  $f : Y \rightarrow \lambda$  be the supremum function. Since  $f_*(\mathcal{U})$  is isomorphic to an ultrafilter on  $\iota + 1$ ,  $f$  takes at most  $\iota$ -many values on a set in  $\mathcal{U}$ . □

What we show instead is that an analog of Lemma 4.4.29 holds:

**Theorem 4.4.36.** *Suppose  $\lambda$  is a cardinal and  $j : V \rightarrow M$  is a  $\lambda$ -supercompact elementary embedding. Let  $\theta$  be the least generator of  $j$  with  $\theta \geq \sup j[\lambda]$ . Then*

$$j[\lambda] \in H^M(j[V] \cup \{\theta\})$$

*Moreover if  $\sup j[\lambda] < j(\lambda)$ , then  $\theta < j(\lambda)$ .*

As a corollary, we prove the second of the main theorems of this section:

**Theorem 4.4.37.** *Suppose  $U$  is a nonprincipal ultrafilter. Then  $U$  is isonormal if and only if  $U$  is the incompressible ultrafilter isomorphic to a normal fine ultrafilter. In particular, every normal fine ultrafilter is isomorphic to a unique isonormal ultrafilter.*

*Proof.* We begin with the forward direction, which turns out to follow from Proposition 4.4.22. Suppose  $U$  is an isonormal ultrafilter on a cardinal  $\lambda$ . We will show that  $U$  is incompressible

and isomorphic to a normal fine ultrafilter on  $P_{\text{bd}}(\lambda)$ . Since  $U$  is weakly normal, Proposition 4.4.22 implies  $U$  is incompressible.

Since  $U$  is uniform on  $\lambda$ ,  $\sup j_U[\lambda] < j_U(\lambda)$  and thus  $j_U \upharpoonright \lambda \in j_U(P_{\text{bd}}(\lambda))$ . Let  $\mathcal{U}$  be the ultrafilter on  $P_{\text{bd}}(\lambda)$  derived from  $j_U$  using  $j_U \upharpoonright \lambda$ . Then  $\mathcal{U} \leq_{\text{RK}} U$  and  $\mathcal{U}$  is a normal fine ultrafilter on  $P_{\text{bd}}(\lambda)$  by Lemma 4.4.9. It follows that  $j_{\mathcal{U}}$  is  $\lambda$ -supercompact, and therefore  $\lambda_{\mathcal{U}} \geq \lambda$  by Proposition 4.2.31. Since  $U$  is weakly normal, Proposition 4.4.22 implies  $U$  is  $\lambda$ -minimal and therefore  $\mathcal{U} \not\leq_{\text{RK}} U$ . Since  $\mathcal{U} \leq_{\text{RK}} U$  and  $\mathcal{U} \not\leq_{\text{RK}} U$ , we must have  $\mathcal{U} \cong U$  (by definition).

Conversely, suppose  $U$  is incompressible and isomorphic to a normal fine ultrafilter, and we will show that  $U$  is isonormal. Since every normal fine ultrafilter is isomorphic to an element of  $\mathcal{N}$  (Proposition 4.4.12), for some cardinal  $\lambda$ ,  $U$  is isomorphic to a normal fine ultrafilter  $\mathcal{U}$  on  $P_{\text{bd}}(\lambda)$ . In particular  $j_U = j_{\mathcal{U}}$  is  $\lambda$ -supercompact. To show that  $U$  is isonormal, it therefore suffices to show that  $U$  is a weakly normal ultrafilter on  $\lambda$ .

Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $U$ . Let  $\theta$  be the least generator of  $j$  with  $\theta \geq \sup j[\lambda]$ . Since  $P_{\text{bd}}(\lambda) \in \mathcal{U}$ ,  $\sup j[\lambda] < j(\lambda)$ , and so by Theorem 4.4.36,  $\theta < j(\lambda)$ . Since  $\theta$  is a generator of  $j = j_U$ ,  $\theta \leq a_U$ . In fact, we claim  $a_U = \theta$ . On the other hand, by Theorem 4.4.36,

$$M = H^M(j[V] \cup \{j[\lambda]\}) = H^M(j[V] \cup \{\theta\})$$

The ultrapower theoretic characterization of incompressibility (Lemma 3.4.18) implies that  $a_U$  is the least ordinal  $\alpha$  such that  $M = H^M(j[V] \cup \{\alpha\})$ . Thus  $a_U \leq \theta$ . Hence  $a_U = \theta$ , as desired.

Since  $U$  is tail uniform (by the definition of incompressibility) and  $a_U < j_U(\lambda)$ ,  $U$  is an ultrafilter on  $\lambda$ . Since  $a_U$  is the least generator of  $j$  above  $\sup j[\lambda]$ , the characterization of weakly normal ultrafilters in terms of generators (Proposition 4.4.23) implies that  $U$  is a weakly normal ultrafilter on  $\lambda$ . □

We conclude this chapter by proving Theorem 4.4.36. The proof relies on some basic notions from PCF theory.

**Definition 4.4.38.** Suppose  $\iota$  is an ordinal. We denote by  $J_{\text{bd}}^\iota$  the ideal of bounded subsets of  $\iota$ , omitting the superscript  $\iota$  when it is clear from context. If  $f$  and  $g$  are functions from  $\iota$  to  $\text{Ord}$ ,

- $f <_{\text{bd}} g$  if  $\{\alpha < \iota : f(\alpha) \geq g(\alpha)\} \in J_{\text{bd}}$ .
- $f =_{\text{bd}} g$  if  $\{\alpha < \iota : f(\alpha) \neq g(\alpha)\} \in J_{\text{bd}}$ .

**Definition 4.4.39.** Suppose  $C$  is a set of functions from  $\iota$  to  $\text{Ord}$ . A function  $s : \iota \rightarrow \text{Ord}$  is an *exact upper bound* of  $C$  if the following hold:

- For all  $f \in C$ ,  $f <_{\text{bd}} s$ .
- For all  $g <_{\text{bd}} s$ , for some  $f \in C$ ,  $g <_{\text{bd}} f$ .

The following trivial fact plays a key role in the proof of Theorem 4.4.36:

**Lemma 4.4.40.** *Suppose  $C$  is a set of functions from  $\iota$  to  $\text{Ord}$  and  $s$  and  $t$  are exact upper bounds of  $C$ . Then  $s =_{\text{bd}} t$ .*

*Proof.* Suppose  $s$  and  $t$  are exact upper bounds of  $C$ . Suppose towards a contradiction that  $s \neq_{\text{bd}} t$ . Without loss of generality, we can assume that there is an unbounded set  $A \subseteq \iota$  such that  $s(\alpha) < t(\alpha)$  for all  $\alpha \in A$ . Define  $g : \iota \rightarrow \text{Ord}$  by setting

$$g(\alpha) = \begin{cases} s(\alpha) & \text{if } \alpha \in A \\ 0 & \text{otherwise} \end{cases}$$

Then  $g < t$ , so since  $t$  is an exact upper bound of  $C$ , there is some  $f \in C$  such that  $g <_{\text{bd}} f$ . Since  $s$  is an upper bound of  $C$ ,  $f <_{\text{bd}} s$ . Therefore  $g <_{\text{bd}} s$ . This contradicts that  $A = \{\alpha < \iota : g(\alpha) = s(\alpha)\}$  is unbounded in  $\iota$ . □

**Definition 4.4.41.** If  $s : \iota \rightarrow \text{Ord}$  is a function and  $\delta$  is an ordinal, a *scale of length  $\delta$*  in  $\prod_{\alpha < \iota} s(\alpha)$  is a  $<_{\text{bd}}$ -increasing cofinal sequence  $\langle f_\alpha : \alpha < \delta \rangle \subseteq \prod_{\alpha < \iota} s(\alpha)$ .



Shelah's Representation Theorem [27] states that if  $\lambda$  is a singular cardinal of cofinality  $\iota$ , then there is a cofinal continuous sequence  $u : \iota \rightarrow \lambda$  such that  $\prod_{\alpha < \iota} u(\alpha)^+$  has a scale of length  $\lambda^+$ . This is a deep theorem in the context of ZFC, but since we are assuming large cardinals, we will have enough SCH to get away with using only the following trivial version of Shelah's theorem:

**Lemma 4.4.42.** *Suppose  $\lambda$  is a singular cardinal of cofinality  $\iota$  such that  $\lambda^\iota = \lambda^+$ . Suppose  $\langle \delta_\alpha : \alpha < \iota \rangle$  is a sequence of regular cardinals cofinal in  $\lambda$ . Then there is a scale of length  $\lambda^+$  in  $\prod_{\alpha < \iota} \delta_\alpha$ .*

*Proof.* We start by proving the standard fact that  $\mathbb{P} = (\prod_{\alpha < \iota} \delta_\alpha, <_{\text{bd}})$  is a  $\leq \lambda$ -directed partial order. The proof proceeds in two steps.

First, we prove that  $\mathbb{P}$  is  $< \lambda$ -directed. Suppose  $\gamma < \lambda$  and  $\{g_i : i < \gamma\} \subseteq \mathbb{P}$ . We will find a  $<_{\text{bd}}$ -upper bound  $g$  of  $\{g_i : i < \gamma\}$ . Fix  $\alpha_0$  such that  $\gamma < \delta_{\alpha_0}$ . For  $\alpha < \iota$ , define

$$g(\alpha) = \begin{cases} \sup_{i < \gamma} g_i(\alpha) + 1 & \text{if } \alpha_0 \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

If  $\alpha_0 \leq \alpha$ , then  $\delta_\alpha$  is a regular cardinal greater than  $\gamma$ , so  $\sup_{i < \gamma} g_i(\alpha) < \delta_\alpha$ . Hence  $g \in \prod_{\alpha < \iota} \delta_\alpha$  and  $g$  is a  $<_{\text{bd}}$ -upper bound of  $\{g_i : i < \gamma\}$ .

Second, we prove that  $\mathbb{P}$  is  $\lambda$ -directed. Fix  $\{g_i : i < \lambda\} \subseteq \mathbb{P}$ . For  $\alpha < \iota$ , let  $h_\alpha \in \mathbb{P}$  be a  $<_{\text{bd}}$ -upper bound of  $\{g_i : i < \delta_\alpha\}$ . Finally let  $g \in \mathbb{P}$  be a  $<_{\text{bd}}$ -upper bound of  $\{g_i : i < \iota\}$ . Then  $g$  is a  $<_{\text{bd}}$ -upper bound of  $\{g_i : i < \lambda\}$ , as desired.

Enumerate  $\prod_{\alpha < \iota} \delta_\alpha$  as  $\{g_\xi : \xi < \lambda^+\}$ . We define  $\langle f_\xi : \xi < \lambda^+ \rangle$  recursively. If  $\langle f_\xi : \xi < \theta \rangle$  has been defined, choose a  $<_{\text{bd}}$ -upper bound  $f_\theta \in \mathbb{P}$  of  $\{f_\xi : \xi < \theta\} \cup \{g_\xi\}$ . (Such a function exists by the  $\lambda$ -directedness of  $\mathbb{P}$ .) By construction  $\langle f_\xi : \xi < \lambda^+ \rangle$  is a scale in  $\prod_{\alpha < \iota} \delta_\alpha$ .  $\square$

This concludes our summary of the basic notions from PCF theory used in the proof of Theorem 4.4.36, which we now commence.

*Proof of Theorem 4.4.36.* For the purposes of the proof, let us say that  $x$  is *weakly definable from  $y$*  (in  $M$ ) if  $x$  is definable in  $M$  from parameters in  $j[V] \cup \{y\}$ , or in other words,  $x \in H^M(j[V] \cup \{y\})$ . Note that weak definability is a transitive relation.

By Lemma 4.4.29, we may assume  $\lambda$  is a singular cardinal. Let  $\iota$  be the cofinality of  $\lambda$ . Let  $\lambda_* = \sup j[\lambda]$ .

**Claim 1.** *Suppose  $\langle \delta_\alpha : \alpha < \iota \rangle$  is an increasing sequence of regular cardinals cofinal in  $\lambda$ . Let  $e$  be the equivalence class of  $\langle \sup j[\delta_\alpha] : \alpha < \iota \rangle$  modulo  $J_{\text{bd}}$ . Then  $j[\lambda]$  is weakly definable from  $e$  and  $j \upharpoonright \iota$  in  $M$ .*

*Proof of Claim 1.* Fix a sequence  $\langle S^\alpha : \alpha < \iota \rangle$  such that  $S^\alpha = \{S_\beta^\alpha : \beta < \delta_\alpha\}$  is a partition of  $S_\omega^{\delta_\alpha}$  into stationary sets. Note that  $\langle j(S^\alpha) : \alpha < \iota \rangle = j(\langle S^\alpha : \alpha < \iota \rangle) \circ j \upharpoonright \iota$  is weakly definable from  $j \upharpoonright \iota$ .

Solovay's Lemma (Corollary 4.4.31) implies that for all  $\alpha < \iota$ ,  $j[\delta_\alpha]$  is equal to the set  $\{\beta < j(\delta_\alpha) : M \models j(S^\alpha)_\beta \text{ is stationary in } \sup j[\delta_\alpha]\}$ . It follows that

$$\begin{aligned} \beta \in j[\lambda] &\iff \{\alpha < \iota : M \models j(S^\alpha)_\beta \text{ is stationary in } \sup j[\delta_\alpha]\} \notin J_{\text{bd}} \\ &\iff \exists s \in e \{\alpha < \iota : M \models j(S^\alpha)_\beta \text{ is stationary in } s(\alpha)\} \notin J_{\text{bd}} \end{aligned}$$

Thus  $j[\lambda]$  is weakly definable from  $e$  and  $\langle j(S^\alpha) : \alpha < \iota \rangle$ . Since  $\langle j(S^\alpha) : \alpha < \iota \rangle$  is weakly definable from  $j \upharpoonright \iota$ , this proves the claim.  $\square$

It is not hard to see that  $j \upharpoonright \iota$  is itself weakly definable from  $e$ , but we will not need this. The following observation, however, will be crucial:

**Observation 1.**  *$j \upharpoonright \iota$  and  $j[\iota]$  are weakly definable from  $\sup j[\iota]$ .*

This is an immediate consequence of Solovay's Lemma (Corollary 4.4.31).

Let  $\mathcal{D}$  be the normal fine ultrafilter on  $P(\iota)$  derived from  $j$  using  $j[\iota]$  and let  $k : M_{\mathcal{D}} \rightarrow M$  be the factor embedding. Let  $\lambda_{\mathcal{D}} = \sup j_{\mathcal{D}}[\lambda]$ .<sup>5</sup> By Lemma 4.4.10,  $\text{CRT}(k) > \iota$  and hence  $k(\lambda_{\mathcal{D}}) = \sup k[\lambda_{\mathcal{D}}] = \lambda_*$ .

<sup>5</sup>If  $\eta^t < \lambda$  for all  $\eta < \lambda$ , then  $\lambda_{\mathcal{D}} = \lambda$ , but we do not assume this.

**Observation 2.**  $k[M_{\mathcal{D}}]$  consists of all  $x \in M$  that are weakly definable from  $\sup j[\iota]$ .

Observation 2 follows from the fact that  $k[M_{\mathcal{D}}] = H^M(j[V] \cup \{j[\iota]\})$  combined with the fact (Observation 1) that  $j[\iota]$  and  $\sup j[\iota]$  are weakly definable from each other.

Let

$$\theta = \sup k[\lambda_{\mathcal{D}}^{+M_{\mathcal{D}}}]$$

The ordinal  $\theta$  will turn out to be the least generator of  $j$  above  $\lambda_*$ . For now, let us just show that there is no smaller generator:

**Claim 2.**  $\theta \subseteq H^M(j[V] \cup \lambda_*)$ .

*Proof.* Suppose  $\alpha < \theta$ . The claim amounts to showing that  $\alpha$  is weakly definable from a finite set of ordinals below  $\lambda_*$ . By the definition of  $\theta$ ,  $\alpha < k(\xi)$  for some  $\xi < \lambda_{\mathcal{D}}^{+M_{\mathcal{D}}}$ . Fix a surjection  $p : \lambda_{\mathcal{D}} \rightarrow \xi$  with  $p \in M_{\mathcal{D}}$ . Observation 2 implies  $k(p)$  is weakly definable from  $\sup j[\iota]$ . Since  $k(p)$  is a surjection from  $\lambda_*$  onto  $k(\xi)$ , for some  $\nu < \lambda_*$ ,  $\alpha = k(p)(\nu)$ . Thus  $\alpha$  is weakly definable from  $\sup j[\iota]$  and  $\nu$ , which both lie below  $\lambda_*$ , proving the claim.  $\square$

Fix a sequence  $\langle \delta_\alpha : \alpha < \iota \rangle$  of regular cardinals greater than  $\iota$  that is increasing and cofinal in  $\lambda$ .

**Claim 3.** In  $M_{\mathcal{D}}$ , there is a scale  $\vec{f} = \langle f_\alpha : \alpha < \lambda_{\mathcal{D}}^{+M_{\mathcal{D}}} \rangle$  in  $\prod_{\alpha < \iota} j_{\mathcal{D}}(\delta_\alpha)$ .

*Proof.* Applying Lemma 4.4.42 in  $M_{\mathcal{D}}$ , it suffices to show that  $M_{\mathcal{D}}$  satisfies  $\lambda_{\mathcal{D}}^{\iota} = \lambda_{\mathcal{D}}^{+\mathcal{D}}$ . By the critical sequence analysis given by the Kunen Inconsistency Theorem (Lemma 4.2.39), there is a  $\lambda$ -supercompact cardinal  $\kappa \leq \iota$  such that  $j_{\mathcal{D}}(\kappa) > \iota$ . Thus  $\iota < j_{\mathcal{D}}(\kappa) < \lambda_*$  and  $j_{\mathcal{D}}(\kappa)$  is  $\lambda_{\mathcal{D}}^{+\mathcal{D}}$ -supercompact in  $M_{\mathcal{D}}$ . By the local version of Solovay's theorem [21] (which appears as Corollary 6.3.2) applied in  $M_{\mathcal{D}}$ , it follows that in  $M_{\mathcal{D}}$ ,  $\lambda_{\mathcal{D}}^{\iota} \leq (\lambda_{\mathcal{D}}^+)^{< j_{\mathcal{D}}(\kappa)} = \lambda_{\mathcal{D}}^{+M_{\mathcal{D}}}$ , as desired.  $\square$

**Claim 4.**  $\langle \sup j[\delta_\alpha] : \alpha < \iota \rangle$  is an exact upper bound of  $k(\vec{f}) \upharpoonright \theta$ .

Before proving Claim 4, let us show how it implies the theorem.

Let  $e$  be the equivalence class of  $\langle \sup j[\delta_\alpha] : \alpha < \iota \rangle$  modulo the bounded ideal on  $\iota$ . Then Claim 4 and Lemma 4.4.40 imply that  $e$  is definable in  $M$  from the parameters  $\theta$  and  $k(\vec{f})$ . Thus by Claim 1,  $j[\lambda]$  is weakly definable from  $\theta$ ,  $k(\vec{f})$ , and  $j \upharpoonright \iota$ .

Note that  $\lambda_*$  is definable in  $M$  from  $\theta$ :  $\lambda_*$  is the largest  $M$ -cardinal below  $\theta$ . By Proposition 4.4.34,  $\sup j[\iota]$  is weakly definable from  $\lambda_*$  and hence from  $\theta$ . Thus by Observation 1  $j \upharpoonright \iota$  is weakly definable from  $\theta$ , and by Observation 2,  $k(\vec{f})$  is weakly definable from  $\theta$ . Combining this with the previous paragraph,  $j[\lambda]$  is weakly definable from  $\theta$  alone. This yields:

$$j[\lambda] \in H^M(j[V] \cup \{\theta\})$$

We now show  $\theta$  is the least generator of  $j$  above  $\lambda_*$ . It suffices by Claim 2 to show that  $\theta$  is a generator of  $j$ . Assume towards a contradiction that this fails. Then  $\theta \in H^M(j[V] \cup \theta) = H^M(j[V] \cup \lambda_*)$  by Claim 2. Thus  $j[\lambda] \in H^M(j[V] \cup \lambda_*)$ . Fix  $\xi < \lambda_*$  such that  $j[\lambda] \in H^M(j[V] \cup \{\xi\})$ . Let  $W$  be the ultrafilter derived from  $j$  using  $\xi$ . Then by Lemma 4.2.22,  $j_W$  is  $\lambda$ -supercompact, yet  $\lambda_W < \lambda$ , and this contradicts Proposition 4.2.31. Thus our assumption was false, and in fact  $\theta$  is a generator of  $j$ .

Thus  $j[\lambda] \in H^M(j[V] \cup \{\theta\})$  where  $\theta$  is the least generator of  $j$  greater than or equal to  $\lambda_*$ . To finish, we must show that if  $\lambda_* < j(\lambda)$  then  $\theta < j(\lambda)$ . But  $\theta \leq \lambda_*^{+M}$  while  $j(\lambda)$  is a limit cardinal of  $M$  above  $\lambda_*$ . Hence  $\lambda_*^{+M} < j(\lambda)$ , as desired.

We now turn to the proof of Claim 4. It will be important here that for any  $s : \iota \rightarrow \text{Ord}$ ,  $k(s) = k \circ s$  since  $\text{CRT}(k) > \iota$ .

*Proof of Claim 4.* We first show that for all  $\nu < \theta$ ,

$$k(\vec{f})_\nu <_{\text{bd}} \langle \sup j[\delta_\alpha] : \alpha < \iota \rangle$$

For any  $\nu < \theta$ , there is some  $\xi < \lambda_D^{+M}$  such that  $\nu < k(\xi)$ . Therefore

$$k(\vec{f})_\nu <_{\text{bd}} k(\vec{f})_{k(\xi)} = k(f_\xi)$$

Hence it suffices to show that for any  $\xi < \lambda_{\mathcal{D}}^{+M_{\mathcal{D}}}$ ,  $k(f_{\xi}) < \langle \sup j[\delta_{\alpha}] : \alpha < \iota \rangle$ . For all  $\alpha < \iota$ , we have that  $\delta_{\alpha}$  is a regular cardinal above  $\iota$ . By Corollary 4.4.28,  $\lambda_{\mathcal{D}} = \iota$ , so since ultrapower embeddings are continuous at regular cardinals above their size (Lemma 3.5.32),

$$j_{\mathcal{D}}(\delta_{\alpha}) = \sup j_{\mathcal{D}}[\delta_{\alpha}]$$

Since  $f_{\xi} \in \prod_{\alpha < \iota} j_{\mathcal{D}}(\delta_{\alpha})$ , we therefore have  $f_{\xi}(\alpha) < \sup j_{\mathcal{D}}[\delta_{\alpha}]$  and hence  $k(f_{\xi})(\alpha) = k(f_{\xi}(\alpha)) < \sup j[\delta_{\alpha}]$  for all  $\alpha < \iota$ , as desired.

We finish by showing that for any  $g : \iota \rightarrow \text{Ord}$  such that  $g <_{\text{bd}} \langle \sup j[\delta_{\alpha}] : \alpha < \iota \rangle$ , there is some  $\xi < \lambda_{\mathcal{D}}^{+M_{\mathcal{D}}}$  such that  $g <_{\text{bd}} k(f_{\xi})$ . For  $\alpha < \iota$ , let  $h(\alpha) < \delta_{\alpha}$  be least such that  $g(\alpha) \leq j(h(\alpha))$ . Then  $j_{\mathcal{D}} \circ h \in M_{\mathcal{D}}$  (since  $M_{\mathcal{D}}$  is closed under  $\iota$ -sequences by Corollary 4.2.21). Since  $\langle f_{\xi} : \xi < \lambda^+ \rangle$  is cofinal, in  $\prod_{\alpha < \iota} j_{\mathcal{D}}(\delta_{\alpha})$ , there is some  $\xi < \lambda_{\mathcal{D}}^{+M_{\mathcal{D}}}$  such that  $j_{\mathcal{D}} \circ h <_{\text{bd}} f_{\xi}$ . It follows that

$$g \leq j \circ h = k \circ j_{\mathcal{D}} \circ h = k(j_{\mathcal{D}} \circ h) <_{\text{bd}} k(f_{\xi})$$

as desired. □

This completes the proof of Theorem 4.4.36. □

# Chapter 5

## The Rudin-Frolík Order

### 5.1 Introduction

#### Ultrafilters on the least measurable cardinal

This chapter is motivated by a single simple question. Chapter 2 established the linearity of the Mitchell order on normal ultrafilters assuming UA. As a consequence, the least measurable cardinal  $\kappa$  carries a unique normal ultrafilter. But what are the other countably complete ultrafilters on  $\kappa$ ? The following theorem of Kunen [22] answers this question under a hypothesis that is much more restrictive than UA:

**Theorem 5.1.1** (Kunen). *Suppose  $U$  is a normal ultrafilter on  $\kappa$  and  $V = L[U]$ . Then every countably complete ultrafilter is isomorphic to  $U^n$  for some  $n < \omega$ .*

Here  $U^n$  is the ultrafilter on  $[\kappa]^n$  generated by sets of the form  $[A]^n$  where  $A \in U$ . An even stronger theorem of Kunen characterizes every elementary embedding of the universe when  $V = L[U]$ :

**Theorem 5.1.2** (Kunen). *Suppose  $V = L[U]$  for some normal ultrafilter  $U$ . Then any elementary embedding  $j : V \rightarrow M$  is an iterated ultrapower of  $U$ .*

Kunen's proofs of these theorems rely heavily on the structure of  $L[U]$ , so much so that it might seem unlikely UA alone could imply analogous results. The results of this chapter, however, show that UA does just as well:

**Theorem 5.3.21** (UA). *Let  $\kappa$  be the least measurable cardinal. Then there is a unique normal ultrafilter  $U$  on  $\kappa$ , and every countably complete ultrafilter is isomorphic to  $U^n$  for some  $n < \omega$ .*

**Theorem 5.3.23** (UA). *Let  $\kappa$  be the least measurable cardinal. Let  $U$  be the unique normal ultrafilter on  $\kappa$ . Then any elementary embedding  $j : V \rightarrow M$  such that  $M = H^M(j[V] \cup j(\kappa))$  is an iterated ultrapower of  $U$ .*

The requirement that  $M = H^M(j[V] \cup j(\kappa))$  is necessary because for example there could be two measurable cardinals. (Actually one could make do with the requirement that  $M = H^M(j[V] \cup j(\alpha))$  and there are no measurable cardinals in the interval  $(\kappa, \alpha]$ .)

Thus there is an abstract generalization of Kunen's analysis of  $L[U]$  to arbitrary models of UA. Far more interesting, however, is that this generalization leads to the discovery of new structure high above the least measurable cardinal.

**Definition 5.1.3.** A nonprincipal countably complete ultrafilter  $U$  is *irreducible* if its ultrapower embedding cannot be written nontrivially as a linear iterated ultrapower.

Irreducible ultrafilters arise in the generalization of Kunen's theorem, which really factors into the following two theorems:

**Theorem 5.3.14** (UA). *Every irreducible ultrafilter on the least measurable cardinal  $\kappa$  is isomorphic to the unique normal ultrafilter on  $\kappa$ .*

**Theorem 5.3.16** (UA). *Every ultrapower embedding can be written as a finite linear iterated ultrapower of irreducible ultrafilters.*

The first of these theorems is highly specific to the least measurable cardinal, but the second is a perfectly general fact: under UA, the structure of countably complete ultrafilters

in general can be reduced to the structure of irreducible ultrafilters. The nature of irreducible ultrafilters in general is arguably the most interesting problem raised by this dissertation, intimately related to the theory of supercompactness and strong compactness under UA.

## Outline of Chapter 5

We now outline the rest of this chapter.

**Section 5.2.** We introduce the fundamental Rudin-Frolík order, which measures how an ultrapower embedding can be factored as a finite iterated ultrapower. We explain how the topological definition of the Rudin-Frolík order is related to the concept of an internal ultrapower embedding (Corollary 5.2.7). We show that the Ultrapower Axiom is equivalent to the directedness of the Rudin-Frolík order on countably complete ultrafilters, and we show that the Rudin-Frolík order is not directed on ultrafilters on  $\omega$ .

**Section 5.3.** In this section, we answer the basic question, characterizing the ultrafilters on the least measurable cardinal up to isomorphism. It turns out that this can be done for all ultrafilters below the least  $\mu$ -measurable cardinal. (In fact, the analysis extends quite a bit further, but we have omitted this work from this dissertation.) Towards this, in Section 5.3, we introduce irreducible ultrafilters and analyze the irreducible ultrafilters up to isomorphism. We then prove that every ultrafilter can be factored into finitely many irreducible ultrafilters in Section 5.3.

**Section 5.4.** In this section, we investigate the deeper structural properties of the Rudin-Frolík order assuming UA. We show that the Rudin-Frolík order satisfies the local ascending chain condition (Theorem 5.3.17), which was actually required as a step in the irreducible factorization theorem. We show that the Rudin-Frolík order induces a lattice on the isomorphism types of countably complete ultrafilters. This involves showing that every pair of ultrapower embeddings has a minimum comparison, which we call a *pushout*. In Section 5.4, we use pushouts to prove the local finiteness of the Rudin-Frolík order: a countably complete ultrafilter has at most finitely many Rudin-Frolík predecessors assuming



UA. Finally, in Section 5.4, we study the structure of pushouts and their relationship to the minimal covers of Section 3.6. This involves the key notion of a *translation* of ultrafilters.

**Section 5.5.** In this section, we use the theory of comparisons developed in Section 5.4 to investigate a variant of the generalized Mitchell order called the internal relation.

## 5.2 The Rudin-Frolík order

Irreducible ultrafilters are most naturally studied in the setting of the *Rudin-Frolík order*, an order on ultrafilters due to Rudin and Frolík [28] that dates back to the study of ultrafilters by Mary Ellen Rudin’s school in the late 1960s. The structure of the Rudin-Frolík order on countably complete ultrafilters turns out to encapsulate many of the phenomena we have been studying so far. For example, the Ultrapower Axiom is equivalent to the statement that the Rudin-Frolík order is directed, while irreducible ultrafilters are simply the minimal elements of the Rudin-Frolík order. The deeper properties of this order developed in this chapter (especially the existence of least upper bounds) will provide some of the key tools of the supercompactness analysis.

In this section, we discuss the theory of the Rudin-Frolík order without yet restricting to countably complete ultrafilters. For this reason, this subsection is a bit out of step with the rest of this dissertation, and the only fact that will be truly essential going forward is the characterization of the Rudin-Frolík order on countably complete ultrafilters given by Corollary 5.2.8, which the reader who is not interested in ultrafilter combinatorics can take as the definition of the Rudin-Frolík order on countably complete ultrafilters.

**Definition 5.2.1.** A sequence of ultrafilters  $\langle W_i : i \in I \rangle$  is *discrete* if there is a sequence of pairwise disjoint sets  $\langle Y_i : i \in I \rangle$  such that  $Y_i \in W_i$  for all  $i \in I$ .

Typically (for example, in Definition 5.2.2) we will consider discrete sequences of ultrafilters that all lie on the same set  $X$ . Then discreteness says these ultrafilters can be simultaneously separated from each other.

**Definition 5.2.2.** Suppose  $U$  is an ultrafilter on  $X$  and  $W$  is an ultrafilter on  $Y$ . The *Rudin-Frolík order* is defined by setting  $U \leq_{\text{RF}} W$  if there is a set  $I \in U$  and a discrete sequence of ultrafilters  $\langle W_i : i \in I \rangle$  on  $Y$  such that  $W = U\text{-}\lim_{i \in I} W_i$ .

Recall that if  $U$  is an ultrafilter on  $X$ ,  $I$  is a set in  $U$ , and  $\langle W_i : i \in I \rangle$  is a sequence of ultrafilters on  $Y$ , then the  $U$ -sum of  $\langle W_i : i \in I \rangle$  is defined by

$$U\text{-}\sum_{i \in I} W_i = \{A \subseteq X \times Y : \{i \in I : A_i \in W_i\} \in U\}$$

The projection  $\pi^0 : X \times Y \rightarrow X$  defined by  $\pi^0(i, j) = i$  satisfies  $\pi_*^0(U\text{-}\sum_{i \in I} W_i) = U$ , and the projection  $\pi^1 : X \times Y \rightarrow Y$  defined by  $\pi^1(i, j) = j$  satisfies

$$\pi_*^1\left(U\text{-}\sum_{i \in I} W_i\right) = U\text{-}\lim_{i \in I} W_i$$

The model-theoretic characterization of the Rudin-Frolík order uses the following lemma:

**Lemma 5.2.3.** *Suppose  $U$  is an ultrafilter,  $I \in U$ , and  $\langle W_i : i \in I \rangle$  is a sequence of ultrafilters on  $Y$ . Then the following are equivalent:*

- (1) *There is a  $U$ -large set  $J \subseteq I$  such that  $\langle W_i : i \in J \rangle$  is discrete.*
- (2)  *$\pi^1$  is one-to-one on a set in  $U\text{-}\sum_{i \in I} W_i$ .*
- (3)  *$U\text{-}\sum_{i \in I} W_i \cong U\text{-}\lim_{i \in I} W_i$ .*

*Proof.* (1) implies (2): Fix  $J \in U$  contained in  $I$  and pairwise disjoint sets  $\langle Y_i : i \in J \rangle$  with  $Y_i \in W_i$  for all  $i \in J$ . We will show  $\pi^1$  is one-to-one on a set in  $U\text{-}\sum_{i \in I} W_i$ . Let

$$A = \{(i, j) : i \in J \text{ and } j \in Y_i\}$$

Then  $A \in U\text{-}\sum_{i \in I} W_i$  and  $\pi^1$  is one-to-one on  $A$ .

(2) implies (1): Fix  $A \in U\text{-}\sum_{i \in I} W_i$  on which  $\pi^1$  is one-to-one. For each  $i \in I$ , let  $Y_i = \{j \in Y : (i, j) \in A\}$ . Since  $\pi^1$  is one-to-one on  $A$ , the sets  $Y_i$  are disjoint. Since  $A \in U\text{-}\sum_{i \in I} W_i$ , the set  $J = \{i \in I : Y_i \in W_i\}$  belongs to  $U$ . Thus  $J \in U$ ,  $J \subseteq I$ , and  $\langle W_i : i \in J \rangle$  is witnessed to be discrete by  $\langle Y_i : i \in J \rangle$ , as desired.

(2) *implies* (3): Trivial.

(3) *implies* (2): By Theorem 3.4.8, if  $Z \cong Z'$  and  $f$  is such that  $f_*(Z) = Z'$ , then  $f$  is one-to-one on a set in  $Z$ . Therefore since  $\pi_*^1(U\text{-}\sum_{i \in I} W_i) = U\text{-}\lim_{i \in I} W_i$ ,  $\pi^1$  is one-to-one on a set in  $U\text{-}\sum_{i \in I} W_i$ .  $\square$

**Corollary 5.2.4.** *If  $U$  and  $W$  are ultrafilters, the following are equivalent:*

(1)  $U \leq_{\text{RF}} W$ .

(2) *There exist  $I \in U$  and ultrafilters  $\langle W_i : i \in I \rangle$  on a set  $Y$  such that  $W \cong U\text{-}\sum_{i \in I} W_i$ .*

*Proof.* (1) *implies* (2): Obvious from Lemma 5.2.3.

(2) *implies* (1): The proof uses the fact that the Rudin-Frolík order is isomorphism invariant, which should be easy enough to see from the definition.

Let  $Y' = I \times Y$ . Let  $f^i : Y \rightarrow Y'$  be the embedding defined by  $f^i(y) = (i, y)$ , and let  $W'_i = f_*^i(W_i)$ . Then  $W'_i \cong W_i$  and  $\langle W'_i : i \in I \rangle$  is discrete. We have

$$W \cong U\text{-}\sum_{i \in I} W_i \cong U\text{-}\sum_{i \in I} W'_i \cong U\text{-}\lim_{i \in I} W'_i$$

where the last isomorphism follows from Lemma 5.2.3. By the definition of the Rudin-Frolík order  $U \leq_{\text{RF}} U\text{-}\lim_{i \in I} W'_i$ , so by the isomorphism invariance of the Rudin-Frolík order,  $U \leq_{\text{RF}} W$ .  $\square$

The following generalization of closeness to possibly illfounded models in our view simplifies the theory of the Rudin-Frolík order on countably incomplete ultrafilters:

**Definition 5.2.5.** Suppose  $N$  and  $M$  are models of ZFC. A cofinal elementary embedding  $h : N \rightarrow M$  is *close* to  $N$  if for all  $X \in N$  and all  $a \in M$  such that  $M \models a \in h(X)$ , the  $N$ -ultrafilter on  $X$  derived from  $h$  using  $a$  belongs to  $N$ .

It is really not quite accurate to say that this  $N$ -ultrafilter belongs to  $N$ ; we really mean that it is the extension of a point in  $N$ .

**Lemma 5.2.6.** *If  $h : N \rightarrow M$  is close and  $M = H^M(h[N] \cup \{a\})$  for some  $a \in M$ , then there is an ultrafilter  $Z$  of  $N$  and an isomorphism  $k : M_Z^N \rightarrow M$  such that  $k \circ j_Z^N = h$ .  $\square$*

**Corollary 5.2.7.** *If  $U$  and  $W$  are ultrafilters, the following are equivalent:*

(1)  $U \leq_{\text{RF}} W$ .

(2) *There is a close embedding  $h : M_U \rightarrow M_W$  such that  $h \circ j_U = j_W$ .*

*Sketch.* (1) implies (2): By Corollary 5.2.4, fix  $I \in U$  and a sequence of ultrafilters  $\langle W_i : i \in I \rangle$  such that  $W \cong U\text{-}\sum_{i \in I} W_i$ . Let  $D = U\text{-}\sum_{i \in I} W_i$  and let  $Z = [\langle W_i : i \in I \rangle]_U$ . We have  $(M_Z^{M_U}, j_Z^{M_U} \circ j_U) \cong (M_D, j_D) \cong (M_W, j_W)$ , so fix an isomorphism  $k : M_Z^{M_U} \rightarrow M_W$  such that  $k \circ j_Z^{M_U} \circ j_U = j_W$ . It is easy to see that  $k \circ j_Z^{M_U}$  is close to  $M_U$ .

(2) implies (1): Let  $Y$  be the underlying set of  $W$  and let  $Z$  be the  $M_U$ -ultrafilter on  $j_U(Y)$  derived from  $h$  using  $a_W$ . Let  $k : M_Z^{M_U} \rightarrow M_W$  be the factor embedding. It is easy to see that  $k$  is surjective. Thus  $k$  is an isomorphism. It follows that  $U\text{-}\sum Z \cong W$ , so by Corollary 5.2.4,  $U \leq_{\text{RF}} W$ .  $\square$

Note that the close embedding given by Corollary 5.2.7 is “isomorphic to” a (possibly illfounded) internal ultrapower embedding of  $M_U$ . But the language of close embeddings makes it possible to work with the Rudin-Frolík order in fairly simple model theoretic terms while keeping our language precise.

In the countably complete case, Corollary 5.2.7 really does imply that there is an internal ultrapower embedding from  $M_U$  to  $M_W$ :

**Corollary 5.2.8.** *If  $U$  and  $W$  are countably complete ultrafilters, then  $U \leq_{\text{RF}} W$  if and only if there is an internal ultrapower embedding  $h : M_U \rightarrow M_W$  such that  $h \circ j_U = j_W$ .  $\square$*

**Corollary 5.2.9.** *The Ultrapower Axiom holds if and only if the Rudin-Frolík order is directed on countably complete ultrafilters.*

*Proof.* Assume the Ultrapower Axiom. Suppose  $U$  and  $W$  are countably complete ultrafilters. Let  $j : V \rightarrow M$  and  $i : V \rightarrow N$  be their respective ultrapower embeddings. Using

UA, fix an internal ultrapower comparison  $(k, h) : (M, N) \rightarrow P$ . Then the composition  $k \circ j = h \circ i$  is an ultrapower embedding of  $V$ , associated say to the countably complete ultrafilter  $D$ . Then  $U \leq_{\text{RF}} D$  since  $k : M_U \rightarrow M_D$  is an internal ultrapower embedding such that  $k \circ j_U = k \circ j = j_D$ . Similarly,  $W \leq_{\text{RF}} D$ . Thus the Rudin-Frolík order is directed on countably complete ultrafilters. The converse is similar.  $\square$

Corollary 5.2.7 makes the relationship between the Rudin-Frolík order and the Rudin-Keisler order clear:

**Corollary 5.2.10.** *The Rudin-Keisler order extends the Rudin-Frolík order.*

*Proof.* Suppose  $U \leq_{\text{RF}} W$ . Then by Corollary 5.2.7, there is an elementary embedding  $h : M_U \rightarrow M_W$  such that  $h \circ j_U = j_W$ . By Lemma 3.4.4,  $U \leq_{\text{RK}} W$ .  $\square$

Thus by Theorem 3.4.6, if  $U \leq_{\text{RF}} W$  and  $W \leq_{\text{RF}} U$ , then  $U \cong W$ . This motivates the following definition:

**Definition 5.2.11.** The *strict Rudin-Frolík order* is defined on ultrafilters  $U$  and  $W$  by setting  $U <_{\text{RF}} W$  if  $U \leq_{\text{RF}} W$  but  $U \not\cong W$ .

**Lemma 5.2.12.** *The strict Rudin-Frolík order is wellfounded on countably complete ultrafilters.*

*Proof.* This follows from the fact that the strict Rudin-Keisler order extends the strict Rudin-Frolík order (Corollary 5.2.10) and is wellfounded on countably complete ultrafilters (Corollary 3.4.23).  $\square$

The Rudin-Frolík order is *not* directed on arbitrary ultrafilters. In fact, the Rudin-Frolík order restricted to ultrafilters on  $\omega$  already fails to be directed. We sketch a proof of this fact that bears a striking resemblance to many of the comparison arguments used throughout this dissertation, especially Theorem 5.3.11 below. We hope it demonstrates that the close embedding approach to the Rudin-Frolík order really yields some simplifications.

**Theorem 5.2.13** (Rudin). *If  $U$  and  $W$  are ultrafilters on  $\omega$  that have an upper bound in the Rudin-Frolík order, then either  $U \leq_{\text{RF}} W$  or  $W \leq_{\text{RF}} U$ .*

*Sketch.* By Corollary 5.2.7 (3), the existence of an  $\leq_{\text{RF}}$ -upper bound of  $U$  and  $W$  implies the existence of close embeddings  $(k, h) : (M_U, M_W) \rightarrow N$  such that  $k \circ j_U = h \circ j_W$ . Assume without loss of generality that  $k(a_U) < h(a_W)$ . Let  $Z$  be the  $M_W$ -ultrafilter on  $j_W(\omega)$  derived from  $h$  using  $k(a_U)$ . Then  $Z$  concentrates on  $a_W < j_W(\omega)$ . Since  $Z$  belongs to  $M_W$  and concentrates on an  $M_W$ -finite set,  $Z$  is principal. Since  $Z$  is derived from  $h$  using  $k(a_U)$ , we must in fact have  $h(a_Z) = k(a_U)$ .

We now follow the argument of Lemma 3.5.27. Since  $k(a_U) \in h[M_W]$ , it is easy to see that  $k[M_U] = H^N(k \circ j_U[V] \cup \{k(a_U)\}) \subseteq h[M_W]$ . Define  $e : M_U \rightarrow M_W$  by  $e = h^{-1} \circ k$ . Then  $e$  is an elementary embedding and  $h \circ e = k$ , so since  $k$  is close to  $M_U$ ,  $e$  is close to  $M_U$ . Thus there is a close embedding  $e : M_U \rightarrow M_W$ , and it follows that  $U \leq_{\text{RF}} W$ .  $\square$

This theorem is often summarized by the statement that “the Rudin-Frolík order forms a tree,” but this is only true of the Rudin-Frolík order on  $\omega$ . The reader should note that this proof is very similar to the proof of the linearity of the Mitchell order from UA. The argument shows that natural generalization of the seed order to  $\beta(\omega)$  is equal to the Rudin-Frolík order, while the natural generalization of the Ketonen order is equal to the revised Rudin-Keisler order.

**Corollary 5.2.14.** *The Rudin-Frolík order on  $\beta(\omega)$  is not directed.*

*Proof.* Assume towards a contradiction that the Rudin-Frolík order on  $\beta(\omega)$  is directed. Then it is linear. This contradicts the well-known theorem of Kunen [29] that the Rudin-Keisler order is not linear on ultrafilters on  $\omega$ .  $\square$

Thus, unsurprisingly, the analog of the Ultrapower Axiom for countably incomplete ultrafilters is false.

We conclude this section with a basic rigidity lemma for the Rudin-Frolík order on countably complete ultrafilters that apparently had not been noticed:

**Theorem 5.2.15.** *Suppose  $U$  is a countably complete ultrafilter. Suppose  $I \in U$  and  $\langle W_i : i \in I \rangle$  and  $\langle W'_i : i \in I \rangle$  are discrete sequences of countably complete ultrafilters such that*

$$U\text{-}\lim_{i \in I} W_i = U\text{-}\lim_{i \in I} W'_i$$

*Then for  $U$ -almost all  $i \in I$ ,  $W_i = W'_i$ .*

In other words, there is at most one way to realize one countably complete ultrafilter as a discrete limit with respect to another.

**Lemma 5.2.16.** *Suppose  $U$  and  $W$  are countably complete ultrafilters. Then there is at most one internal ultrapower embedding  $h : M_U \rightarrow M_W$  such that  $h \circ j_U = j_W$ .*

*Proof.* Suppose  $h, k : M_U \rightarrow M_W$  are internal ultrapower embeddings such that  $h \circ j_U = k \circ j_U$ . In other words,  $h \upharpoonright j_U[V] = k \upharpoonright j_U[V]$ . Moreover  $h \upharpoonright \text{Ord} = k \upharpoonright \text{Ord}$  by Theorem 3.5.10. Since  $M_U = H^{M_U}(j_U[V] \cup \text{Ord})$ , it follows that  $h = k$ .  $\square$

*Proof of Theorem 5.2.15.* Let  $Z = [\langle W_i : i \in I \rangle]_U$  and let  $Z' = [\langle W'_i : i \in I \rangle]_U$ . By Lemma 5.2.3,  $U\text{-}\sum_{i \in I} Z_i \cong U\text{-}\sum_{i \in I} Z'_i$  and their projections to the second coordinate are equal. Using the ultrapower theoretic characterization of sums (Lemma 3.6.9), this means:

$$j_Z^{M_U} \circ j_U = j_{Z'}^{M_U} \circ j_U$$

$$a_Z = a_{Z'}$$

Lemma 5.2.16 now implies  $j_Z^{M_U} = j_{Z'}^{M_U}$ . But  $Z$  and  $Z'$  are derived from  $j_Z^{M_U} = j_{Z'}^{M_U}$  using  $a_Z = a_{Z'}$ . Thus  $Z = Z'$ . Finally, by Łoś's Theorem we have that  $W_i = W'_i$  for  $U$ -almost all  $i \in I$ .  $\square$

The author's intuition is that Theorem 5.2.15 should be true for countably incomplete ultrafilters as well, and the fact that the proof does not just generalize is a bit of a subtle point.

## 5.3 Below the first $\mu$ -measurable cardinal

### Introduction

In a sense, the first large cardinal axiom that is significantly beyond any “normal ultrafilter axiom” is the existence of a  $\mu$ -measurable cardinal:

**Definition 5.3.1.** A cardinal  $\kappa$  is said to be  $\mu$ -measurable if there is an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that the ultrafilter on  $\kappa$  derived from  $j$  using  $\kappa$  belongs to  $M$ .

The existence of a  $\mu$ -measurable cardinal is a large cardinal axiom that is stronger than the existence of a measurable cardinal  $\kappa$  such that  $o(\kappa) = 2^{2^\kappa}$ , but weaker than the existence of a cardinal  $\kappa$  that is  $2^\kappa$ -strong.

As an example of the strength of  $\mu$ -measurable cardinals, let us show the following fact:

**Proposition 5.3.2.** *Suppose  $\kappa$  is a  $\mu$ -measurable cardinal. Then there is a normal ultrafilter on  $\kappa$  that concentrates on cardinals  $\delta$  such that for any  $A \subseteq P(\delta)$ , there is a normal ultrafilter  $D$  on  $\delta$  such that  $A \in M_D$ .*

*Proof.* Let  $j : V \rightarrow M$  witness that  $\kappa$  is  $\mu$ -measurable and let  $U$  be the normal ultrafilter on  $\kappa$  derived from  $j$  using  $\kappa$ . Thus  $U \in M$ .

**Claim 1.**  *$M_U$  satisfies the statement that for all  $A \subseteq P(\kappa)$ , there is a normal ultrafilter  $D$  on  $\kappa$  such that  $A \in M_D$ .*

*Proof.* Suppose not, and fix  $A \subseteq P(\kappa)$  such that  $M_U$  satisfies that there is no normal ultrafilter  $D$  on  $\kappa$  with  $A \in (M_D)^{M_U}$ . Let  $k : M_U \rightarrow M$  be the factor embedding. By Lemma 4.4.10,  $\text{CRT}(k) > \kappa$  and  $P(\kappa) \cap M_U = P(\kappa) = P(\kappa) \cap M$ , so  $k(A) = A$ . Therefore since  $k$  is elementary,  $M$  satisfies that there is no normal ultrafilter  $D$  on  $\kappa$  with  $A \in (M_D)^M$ . But  $A \in j_U(V_\kappa) \subseteq (M_U)^M$ , and  $U \in M$  is a normal ultrafilter. This is a contradiction.  $\square$



By Łoś's Theorem,  $U$  concentrates on cardinals  $\delta$  such that for all  $A \subseteq P(\delta)$ , there is a normal ultrafilter  $D$  on  $\delta$  such that  $A \in M_D$ .  $\square$

Thus a  $\mu$ -measurable cardinal is a limit of cardinals  $\delta$  such that  $o(\delta) = 2^{2^\delta}$ .

## Irreducible ultrafilters and $\mu$ -measurability

The goal of the next few subsections is to analyze the countably complete ultrafilters in  $V_\kappa$  where  $\kappa$  is the least  $\mu$ -measurable cardinal. We first analyze simpler ultrafilters called *irreducible ultrafilters* and then we reduce the general case to the irreducible case.

**Definition 5.3.3.** An a nonprincipal countably complete ultrafilter  $U$  is *irreducible* if every ultrafilter  $D <_{\text{RF}} U$  is principal.

Let us give some examples of irreducible ultrafilters.

**Proposition 5.3.4.** *If  $U$  is a normal ultrafilter on a cardinal  $\kappa$ , then  $U$  is irreducible.*

*Proof.* Suppose  $D <_{\text{RF}} U$ . By Corollary 5.2.10,  $D <_{\text{RK}} U$ , and therefore by Proposition 4.4.22,  $\lambda_D < \kappa$ . But since  $D \leq_{\text{RK}} U$ ,  $D$  is  $\kappa$ -complete. Since  $D$  is  $\kappa$ -complete and  $\lambda_D < \kappa$ ,  $D$  is principal.  $\square$

A direct generalization of this yields:

**Proposition 5.3.5.** *Normal fine ultrafilters are irreducible.*

*Proof.* Suppose  $\mathcal{U}$  is a normal fine ultrafilter. By Theorem 4.4.37,  $\mathcal{U}$  is isomorphic to an isonormal ultrafilter  $U$  on a cardinal  $\lambda$ . It suffices to show that  $U$  is irreducible. Suppose  $D <_{\text{RF}} U$ , and we will show  $D$  is principal. By Corollary 5.2.10,  $D <_{\text{RK}} U$ , and therefore by Proposition 4.4.22,  $\lambda_D < \kappa$ . Since  $D \leq_{\text{RF}} U$ ,  $M_U$  is contained in  $M_D$ , and so because  $j_U$  is  $\lambda$ -supercompact, using Corollary 4.2.21,  $\text{Ord}^\lambda \subseteq M_U \subseteq M_D$ . In particular,  $j_D \upharpoonright \lambda \in M_D$ , so  $j_D$  is  $\lambda$ -supercompact. Since  $\lambda_D < \lambda$  and  $j_D$  is  $\lambda$ -supercompact,  $D$  is principal by Proposition 4.2.31.  $\square$

Dodd sound ultrafilters are also irreducible:

**Proposition 5.3.6.** *If  $U$  is a Dodd sound ultrafilter, then  $U$  is irreducible.*

*Proof.* Suppose  $D <_{\text{RF}} U$ , and we will show  $D$  is principal. We may assume without loss of generality that  $D$  is incompressible. Then since  $D <_{\text{RK}} U$ , in fact  $D <_{\mathbf{k}} U$  by Corollary 3.4.22. Since the Lipschitz order extends the Ketonen order,  $D <_L U$ , so by Corollary 4.3.28,  $D \triangleleft U$ . But then  $D \in M_U \subseteq M_D$ , so  $D \triangleleft D$ , which implies  $D$  is principal by Lemma 4.2.40.  $\square$

Finally returning to  $\mu$ -measurable cardinals, we have the following fact:

**Proposition 5.3.7.** *Suppose  $j : V \rightarrow M$  is such that  $\text{CRT}(j) = \kappa$  and  $U_0 \in M$  where  $U_0$  is the normal ultrafilter on  $\kappa$  derived from  $j$ . Let  $U_1$  be the ultrafilter on  $V_\kappa$  derived from  $j$  using  $U_0$ . Then  $U_1$  is irreducible and  $U_1$  is not isomorphic to a normal ultrafilter.*

*Proof.* Let  $j_1 : V \rightarrow M_1$  be the ultrapower of  $V$  by  $U_1$ . The key point, which is easily verified, is that  $a_{U_1} = U_0$ . Also note that

$$M_1 = H^{M_1}(j_1[V] \cup (2^{2^\kappa})^{M_1})$$

since  $a_{U_1} = U_0 \in H^{M_1}(j_1[V] \cup (2^{2^\kappa})^{M_1})$ ,  $U_0$  being a subset of  $P(\kappa)$ .

We now show that  $U_1$  is irreducible. Suppose  $D \leq_{\text{RF}} U_1$  and  $D$  is nonprincipal. We must show  $D \cong U_1$ . Since  $\lambda_D = \kappa$ , we have  $\text{CRT}(j_D) = \kappa$ . Let  $k : M_D \rightarrow M_1$  be the unique internal ultrapower embedding with  $k \circ j_D = j_1$ . We claim  $k(\kappa) = \kappa$ . Supposing the contrary, we have that  $k(\kappa) > \kappa$  is an inaccessible cardinal that is a generator of  $j_1$ , contradicting that  $M_1 = H^{M_1}(j_1[V] \cup (2^{2^\kappa})^{M_1})$ . Thus  $k(\kappa) = \kappa$ . Since  $M_1 \subseteq M_D$ ,  $U_0 \in M_D$ , and since  $k(\kappa) = \kappa$ ,  $k(U_0) = U_0$ . Since  $U_0 = a_{U_1}$ , it follows that  $k$  is surjective. Thus  $k$  is an isomorphism, and it follows that  $D \cong U_1$ .

Finally we show that  $U_1$  is not isomorphic to a normal ultrafilter. Suppose towards a contradiction that it is. Then in fact,  $U_1$  is isomorphic to the ultrafilter on  $\kappa$  derived from

$j_{U_1}$  using  $\kappa$ , namely  $U_0$ . In particular,  $M_{U_0} = M_{U_1}$ , so since  $U_0 \in M_{U_1}$ , in fact  $U_0 \in M_{U_0}$ . This contradicts the fact that the Mitchell order is irreflexive (Lemma 4.2.40).  $\square$

Under UA, Proposition 5.3.7 has a converse:

**Theorem 5.3.8 (UA).** *Suppose  $\kappa$  is a measurable cardinal. Exactly one of the following holds:*

- (1)  $\kappa$  is  $\mu$ -measurable.
- (2) Every irreducible ultrafilter  $U$  of completeness  $\kappa$  is isomorphic to a normal ultrafilter.

The proof will use some of the machinery from Chapter 3. Recall that a *pointed ultrapower embedding* is a pair  $(j, \alpha)$  such that  $j : V \rightarrow M$  is an ultrapower embedding and  $\alpha$  is an ordinal. For the reader's convenience we restate here the definition of the Ketonen order and the seed order to pointed ultrapower embeddings:

**Definition 5.3.9.** Suppose  $(j, \alpha)$  and  $(i, \beta)$  are pointed ultrapower embeddings.

- $(j, \alpha) \leq_{\mathbf{k}} (i, \beta)$  (resp.  $(j, \alpha) <_{\mathbf{k}} (i, \beta)$ ) if there is a 1-internal comparison  $(k, h)$  of  $(j, i)$  such that  $k(\alpha) \leq h(\beta)$  (resp.  $k(\alpha) < h(\beta)$ ).
- $(j, \alpha) =_E (i, \beta)$  if  $(j, \alpha) \leq_{\mathbf{k}} (i, \beta)$  and  $(i, \beta) \leq_{\mathbf{k}} (j, \alpha)$ .
- $(j, \alpha) \leq_S (i, \beta)$  (resp.  $(j, \alpha) <_S (i, \beta)$ ) if there is an internal comparison  $(k, h)$  of  $(j, i)$  such that  $k(\alpha) \leq h(\beta)$  (resp.  $k(\alpha) < h(\beta)$ ).
- $(j, \alpha) =_S (i, \beta)$  if  $(j, \alpha) \leq_S (i, \beta)$  and  $(i, \beta) \leq_S (j, \alpha)$ .

Equivalently  $(j, \alpha) =_S (i, \beta)$  if there is an internal comparison  $(k, h)$  of  $(j, i)$  such that  $k(\alpha) = h(\beta)$ . Two fundamental consequences of UA are that  $\leq_{\mathbf{k}}$  and  $\leq_S$  coincide on pointed ultrapower embeddings (Lemma 3.5.26) and are prewellorders (Proposition 3.5.24).

The following lemma is an immediate consequence of Lemma 3.5.27:

**Lemma 5.3.10.** *Suppose  $U$  and  $W$  are countably complete ultrafilters concentrating on ordinals. Then  $U \leq_{\text{RF}} W$  if and only if for some ordinal  $\alpha$ ,  $(j_U, a_U) =_S (j_W, \alpha)$ .*

The following theorem can be viewed as yet another generalization of the proof that the Mitchell order is linear under UA.

**Theorem 5.3.11 (UA).** *Suppose  $U$  is a countably complete ultrafilter. Let  $D$  be the normal ultrafilter on  $\kappa = \text{CRT}(j_U)$  derived from  $j_U$  using  $\kappa$ . Then either  $D \leq_{\text{RF}} U$  or  $D \triangleleft U$ .*

*Proof.* Let  $i : M_D \rightarrow M_U$  be the factor embedding. Then  $(i, \text{id}) : (M_D, M_U) \rightarrow M_U$  witnesses that  $(j_D, \kappa) \leq_k (j_U, \kappa)$ . Thus  $(j_D, \kappa) \leq_S (j_U, \kappa)$ , so let  $(k, h) : (M_D, M_U) \rightarrow N$  be an internal ultrapower comparison of  $(j_D, j_U)$  witnessing this. In other words,  $k(\kappa) \leq h(\kappa)$ . The proof now breaks into two cases:

**Case 1.**  $k(\kappa) = h(\kappa)$

*Proof in Case 1.* Then  $(k, h)$  witnesses  $(j_D, \kappa) =_S (j_U, \kappa)$ . Lemma 5.3.10 therefore implies that  $D \leq_{\text{RF}} U$ . □

**Case 2.**  $k(\kappa) < h(\kappa)$

*Proof in Case 2.* We will show that  $D \in M_U$ . The key point is that for any  $A \subseteq \kappa$ ,

$$h(j_U(A)) \cap h(\kappa) = h(A) \cap h(\kappa)$$

and therefore

$$\begin{aligned} A \in D &\iff a_D \in j_D(A) \\ &\iff k(a_D) \in k(j_D(A)) \\ &\iff k(a_D) \in h(j_U(A)) \\ &\iff k(a_D) \in h(A) \end{aligned}$$

Since  $h$  is definable over  $M_U$  and  $P(\kappa) \subseteq M_U$ , it follows that  $D$  is definable over  $M_U$ , and hence  $D \in M_U$ . □

Thus in Case 1,  $D \leq_{\text{RF}} U$ , and in Case 2,  $D \triangleleft U$ . This proves the theorem.  $\square$

A more abstract perspective on this argument is that it generalizes the linearity of the Mitchell order on Dodd sound ultrafilters (Theorem 4.3.29):

**Proposition 5.3.12.** *Suppose  $\alpha$  is an ordinal and  $i : V \rightarrow N$  is an  $\alpha$ -sound elementary embedding. Suppose  $U$  is a countably complete tail uniform ultrafilter on an ordinal  $\eta$  such that*

$$(j_U, a_U) <_{\mathbb{k}} (i, \alpha)$$

Then  $U \in N$ .

*Sketch.* Recall that  $i^\alpha : P(\delta) \rightarrow N$  is the function  $i^\alpha(A) = i(A) \cap \alpha$  where  $\delta$  is least such that  $i(\delta) \geq \alpha$ . The  $\alpha$ -soundness of  $i$  amounts to the fact that  $i^\alpha \in N$ .

Fix a 1-internal comparison  $(k, h) : (M_U, N) \rightarrow P$ . We have  $U = i^{-1}[h^{-1}[\mathbb{P}_{k(a_U)}^{h(i(\eta))}]]$  by the usual argument, so since  $k(a_U) < h(\alpha)$ ,

$$U = (i^\alpha)^{-1}[h^{-1}[\mathbb{P}_{k(a_U)}^{h(\alpha)}]] \cap P(\eta)$$

Since  $i^\alpha \in N$  and  $h$  is definable over  $N$ , it follows that  $U \in N$ .  $\square$

Theorem 5.3.11 leads to the proof of Theorem 5.3.8.

*Proof of Theorem 5.3.8.* Assume (1) fails, and we will show (2) holds. Let  $U$  be an irreducible ultrafilter such that  $\text{CRT}(j_U) = \kappa$  but  $U$  is not isomorphic to a normal ultrafilter. Let  $D$  be the normal ultrafilter on  $\kappa$  derived from  $j_U$  using  $\kappa$ . By Theorem 5.3.11, either  $D \leq_{\text{RF}} U$  or  $D \triangleleft U$ . If  $D \leq_{\text{RF}} U$  then since  $D$  is nonprincipal and  $U$  is irreducible,  $D \cong U$ , contrary to our hypothesis that  $U$  is not isomorphic to a normal ultrafilter. Therefore  $D \triangleleft U$ . Then  $j_U : V \rightarrow M_U$  has critical point  $\kappa$  and the normal ultrafilter on  $\kappa$  derived from  $j_U$  using  $\kappa$  belongs to  $M_U$ , so  $\kappa$  is a  $\mu$ -measurable cardinal. Therefore (2) holds.

If (2) holds, then (1) fails as a consequence of Proposition 5.3.7.  $\square$

**Corollary 5.3.13 (UA).** *Suppose  $\kappa$  is the least  $\mu$ -measurable cardinal. Then every irreducible ultrafilter in  $V_\kappa$  is isomorphic to a normal ultrafilter.*

*Proof.* This follows from Theorem 5.3.8 applied in  $V_\kappa$ , which is a model of ZFC + UA that also satisfies the statement that there are no  $\mu$ -measurable cardinals.  $\square$

**Corollary 5.3.14 (UA).** *Let  $\kappa$  be the least measurable cardinal. Then  $\kappa$  carries a unique irreducible ultrafilter up to isomorphism.*  $\square$

## Factorization into irreducibles

The results of the previous section motivate understanding how arbitrary countably complete ultrafilters relate to irreducible ultrafilters. The main theorem of this subsection answers the question in complete generality: every ultrapower embedding can be written as a finite iterated ultrapower of irreducible ultrafilters. To be perfectly precise, let us introduce some notation for iterated ultrapowers.

**Definition 5.3.15.** Suppose  $\nu$  is an ordinal. An *iterated ultrapower of length  $\nu$*  is a sequence  $\langle M_\beta, U_\alpha, j_{\alpha,\beta} : \alpha < \beta < \nu \rangle$  such that the following hold:

- For all  $\alpha$  with  $\alpha + 1 < \nu$ ,  $U_\alpha$  is a countably complete ultrafilter of  $M_\alpha$  and  $j_{\alpha,\alpha+1} : M_\alpha \rightarrow M_{\alpha+1}$  is the ultrapower of  $M_\alpha$  by  $U_\alpha$ .
- For all  $\alpha < \beta < \gamma < \nu$ ,  $j_{\alpha,\gamma} = j_{\beta,\gamma} \circ j_{\alpha,\beta}$ .
- For all limit ordinals  $\gamma < \nu$ ,  $M_\gamma$  is the direct limit of  $\langle M_\alpha, j_{\alpha,\beta} : \alpha < \beta < \gamma \rangle$  and for all  $\alpha < \gamma$ ,  $j_{\alpha,\gamma} : M_\alpha \rightarrow M_\gamma$  is the direct limit embedding.

Note that the iterated ultrapower  $\langle M_\beta, U_\alpha, j_{\alpha,\beta} : \alpha < \beta < \nu \rangle$  is actually completely determined by the sequence  $\langle U_\alpha : \alpha + 1 < \nu \rangle$ . We make the convention that for  $\beta < \nu$ ,  $j_{\beta,\beta}$  is the identity function on  $M_\beta$ .

**Theorem 5.3.16** (UA). *Suppose  $W$  is a countably complete ultrafilter. Then there is a finite linear iterated ultrapower  $\langle M_n, U_m, j_{m,n} : m < n \leq \ell \rangle$  such that  $M_0 = V$ ,  $M_\ell = M_W$ ,  $U_m$  is an irreducible ultrafilter of  $M_m$  for all  $m < \ell$ , and  $j_W = j_{0,\ell}$ .*

The proof of this theorem relies on a stronger structural property of the Rudin-Frolík order:

**Theorem 5.3.17** (UA). *Suppose  $W$  is a countably complete ultrafilter. Then there is no ascending chain  $D_0 <_{\text{RF}} D_1 <_{\text{RF}} D_2 <_{\text{RF}} \cdots$  such that  $D_n \leq_{\text{RF}} W$  for all  $n < \omega$ .*

More succinctly, the Rudin-Frolík order satisfies the *local ascending chain condition*. Later we will give a deeper explanation of why this is true (Theorem 5.4.23): a countably complete ultrafilter has only finitely many Rudin-Frolík predecessors up to isomorphism.

We defer the proof of Theorem 5.3.17 to the next section. In this section we will derive Theorem 5.3.16 from Theorem 5.3.17, and show how this can be used to analyze ultrafilters on the least measurable cardinal.

Before we can proceed, we need a simple lemma about the pervasiveness of irreducible ultrafilters:

**Lemma 5.3.18.** *Suppose  $D <_{\text{RF}} W$  are countably complete ultrafilters. Then there is a countably complete ultrafilter  $F$  with  $D <_{\text{RF}} F \leq_{\text{RF}} W$  and an irreducible ultrafilter  $U$  of  $M_D$  such that  $j_F = j_U^{M_D} \circ j_D$ .*

*Proof.* By the wellfoundedness of the Rudin-Frolík order on countably complete ultrafilters (Lemma 5.2.12), let  $F$  be  $<_{\text{RF}}$ -minimal among ultrafilters  $Z$  such that  $D <_{\text{RF}} Z \leq_{\text{RF}} W$ . By Corollary 5.2.8, fix a countably complete ultrafilter  $U$  of  $M_Z$  such that  $j_F = j_U^{M_Z} \circ j_D$ .

We claim  $U$  is an irreducible ultrafilter of  $M_D$ . Suppose  $\bar{U} <_{\text{RF}} U$  in  $M_D$ , and we will show that  $\bar{U}$  is principal in  $M_D$ . Let  $\bar{F}$  be a countably complete ultrafilter such that  $j_{\bar{F}} = j_{\bar{U}}^{M_D} \circ j_D$ . One easily computes:

$$D \leq_{\text{RF}} \bar{F} <_{\text{RF}} F \leq_{\text{RF}} W$$

Assume towards a contradiction  $D <_{\text{RF}} \bar{F}$ ; then  $D <_{\text{RF}} \bar{F} \leq_{\text{RF}} W$  and  $\bar{F} <_{\text{RF}} F$ , contradicting that  $F$  is  $<_{\text{RF}}$ -minimal among ultrafilters  $Z$  such that  $D <_{\text{RF}} Z \leq_{\text{RF}} W$ . Therefore  $D \not<_{\text{RF}} \bar{F}$ , or in other words  $D \cong \bar{F}$ . Now

$$M_D = M_{\bar{F}} = M_{\bar{U}}^{M_D}$$

It follows that  $\bar{U}$  is principal in  $M_D$ . □

We now deduce Theorem 5.3.16 from Theorem 5.3.17.

*Proof of Theorem 5.3.16 assuming Theorem 5.3.17.* By recursion, we construct a finite sequence of countably complete ultrafilters  $D_0 <_{\text{RF}} D_1 <_{\text{RF}} \cdots <_{\text{RF}} D_\ell \cong W$  and an iterated ultrapower  $\langle M_n, U_m, j_{m,n} : m < n \leq \ell \rangle$  such that  $M_0 = V$ ,  $U_m$  is an irreducible ultrafilter of  $M_m$  for all  $m < \ell$ , and  $j_{0,n} = j_{D_n}$  for all  $n \leq \ell$ .

To begin, let  $M_0 = V$  and let  $D_0$  be principal.

Suppose  $D_0 \leq_{\text{RF}} D_1 \leq_{\text{RF}} \cdots \leq_{\text{RF}} D_k \leq_{\text{RF}} W$  and  $\langle M_n, U_m, j_{m,n} : m < n \leq k \rangle$  have been constructed. If  $D_k \cong W$ , we set  $\ell = k$  and terminate the construction. Otherwise,  $D_k <_{\text{RF}} W$ . Using Lemma 5.3.18, fix  $D_{k+1}$  with  $D_k <_{\text{RF}} D_{k+1} \leq_{\text{RF}} W$  and an irreducible ultrafilter  $U_k$  of  $M_{D_k} = M_k$  such that  $j_{D_{k+1}} = j_{U_k}^{M_k} \circ j_{D_k}$ . Let  $\langle M_n, U_m, j_{m,n} : m < n \leq k+1 \rangle$  be the iterated ultrapower given by the sequence  $\langle U_m : m \leq k \rangle$ .

This recursion must terminate in finitely many steps, since otherwise we will produce  $D_0 <_{\text{RF}} D_1 <_{\text{RF}} \cdots$  with  $D_n \leq_{\text{RF}} W$  for all  $n < \omega$ , contradicting the local ascending chain condition (Theorem 5.3.17). When the process terminates, we have  $D_\ell \cong W$ . This yields the objects promised in the first paragraph.

In particular, we have produced an iterated ultrapower  $\langle M_n, U_m, j_{m,n} : m < n \leq \ell \rangle$  such that  $U_m$  is an irreducible ultrafilter of  $M_m$  for all  $m < \ell$  and  $j_{0,\ell} = j_{D_\ell} = j_W$ , as desired. □

We now turn our sights back to the countably complete ultrafilters below the least  $\mu$ -measurable cardinal.



**Theorem 5.3.19 (UA).** *Assume that there are no  $\mu$ -measurable cardinals. Suppose  $W$  is a countably complete ultrafilter. Then there is a finite iterated ultrapower  $\langle M_n, U_m, j_{m,n} : m < n \leq \ell \rangle$  such that  $M_0 = V$ ,  $M_\ell = M_W$ ,  $U_m$  is a normal ultrafilter of  $M_m$  for all  $m < \ell$ , and  $j_W = j_{0,\ell}$ .*

*Proof.* This is immediate from Theorem 5.3.8 and Theorem 5.3.16. □

Stated more succinctly, if there are no  $\mu$ -measurable cardinals and the Ultrapower Axiom holds, then every ultrapower embedding is given by a finite iteration of normal ultrafilters. Combining this with the linearity of the Mitchell order on normal ultrafilters, Theorem 5.3.19 comes very close to a complete analysis of all countably complete ultrafilters below the least  $\mu$ -measurable cardinal on the assumption of the Ultrapower Axiom alone. In any case, it gives as complete an analysis as the Ultrapower Axiom ever will:

**Proposition 5.3.20.** *The following are equivalent:*

- (1) *The Mitchell order is linear and every ultrapower embedding is given by a finite iteration of normal ultrafilters.*
- (2) *The Ultrapower Axiom holds and there are no  $\mu$ -measurable cardinals.* □

The proof is as obvious as it is tedious, and it is omitted.

We now derive the analog of Kunen's theorem (Theorem 5.1.1 above):

**Theorem 5.3.21 (UA).** *Suppose  $\kappa$  is the least measurable cardinal. Let  $U$  be the unique normal ultrafilter on  $\kappa$ . Then every countably complete ultrafilter on  $\kappa$  is isomorphic to  $U^n$  for some  $n < \omega$ .*

*Proof.* We first prove the theorem assuming  $\kappa$  is the only measurable cardinal. Then  $U$  is the only normal ultrafilter. Thus by Theorem 5.3.16, every ultrapower embedding is given by a finite iterated ultrapower of  $U$ . In other words, every countably complete ultrafilter is isomorphic to  $U^n$  for some  $n < \omega$ .

We now prove the theorem assuming there are two measurable cardinals. Let  $\delta$  be the second one. Since  $V_\delta$  is a model of UA and satisfies that  $\kappa$  is the only measurable cardinal, by the previous paragraph  $V_\delta$  satisfies that every countably complete ultrafilter is isomorphic to  $U^n$  for some  $n < \omega$ . Since every countably complete ultrafilter on  $\kappa$  belongs to  $V_\delta$ , it follows that (in  $V$ ) every countably complete ultrafilter on  $\kappa$  is isomorphic to  $U^n$  for some  $n < \omega$ .  $\square$

We sketch how this implies the transfinite version of Kunen's theorem.

**Definition 5.3.22.** Suppose  $U$  is a countably complete ultrafilter and  $\nu$  is an ordinal. Then  $j_{U^\nu} : V \rightarrow M_{U^\nu}$  denotes the elementary embedding  $j_{0,\nu} : V \rightarrow M_\nu$  where  $\langle M_\beta, U_\alpha, j_{\alpha,\beta} : \alpha < \beta \leq \nu \rangle$  is the iterated ultrapower given by setting  $U_\alpha = j_{0,\alpha}(U)$  for all  $\alpha < \nu$ .

**Theorem 5.3.23 (UA).** *Let  $\kappa$  be the least measurable cardinal. Let  $U$  be the unique normal ultrafilter on  $\kappa$ . Suppose  $M$  is an inner model and  $j : V \rightarrow M$  is an elementary embedding such that  $M = H^M(j[V] \cup j(\kappa))$ . Then  $j = j_{U^\nu}$  for some ordinal  $\nu$ .*

**Lemma 5.3.24.** *Suppose  $M$  is an inner model,  $j : V \rightarrow M$  is an elementary embedding, and  $\langle \xi_\alpha : \alpha < \nu \rangle$  is the increasing enumeration of the generators of  $j$ . For any  $p \in [\nu]^{<\omega}$ , let  $U_p$  be the ultrafilter on  $[\mu_j(p)]^{|p|}$  derived from  $j$  using  $\{\xi_\alpha : \alpha \in p\}$ . Then  $j$  is uniquely determined by the sequence  $\langle U_p : p \in [\nu]^{<\omega} \rangle$ .*

*Sketch.* This follows from the usual extender ultrapower construction. This proof is not intended as an exposition of this construction; we are merely checking, for the sake of the reader already familiar with this construction, that a slightly modified version (i.e., using only generators) works just as well.

For  $p \in [\nu]^{<\omega}$ , let  $j_p : V \rightarrow M_p$  be the ultrapower of the universe by  $U_p$  and let  $k_p : M_p \rightarrow M$  be the unique elementary embedding such that  $k_p \circ j_p = j$  and  $k_p(a_{U_p}) = \{\xi_\alpha : \alpha \in p\}$ .

For  $p \subseteq q \in [\nu]^{<\omega}$ , define  $k_{p,q} : M_p \rightarrow M_q$  by setting

$$k_{p,q}([f]_{U_p}) = [f']_{U_q}$$

where, letting  $e : |p| \rightarrow |q|$  be the unique function such that  $q_{e(n)} = p_n$ ,  $f'$  is defined for  $U_q$ -almost every  $s$  by

$$f'(s) = f(\{s_{e(n)} : n < |p|\})$$

Then

$$\langle M_{U_q}, k_{p,q} : p \subseteq q \in [\nu]^{<\omega} \rangle$$

is a directed system. Let  $N$  be its direct limit and let  $j_{p,\infty} : M_p \rightarrow N$  be the direct limit map.

For any  $p \subseteq q \in [\nu]^{<\omega}$ , it is easy to check that  $k_q \circ k_{p,q} = k_p$ . Therefore by the universal property of the direct limit, there is a map  $k : N \rightarrow M$  such that  $k \circ j_{p,\infty}$  is equal to  $k_p : M_p \rightarrow M$ .

We claim  $k$  is the identity. Towards a contradiction, suppose not. Let  $\xi = \text{CRT}(k)$ . Then  $\xi$  is a generator of  $j$ , so  $\xi = \xi_\alpha$  for some  $\alpha < \nu$ . But then letting  $a = a_{U_{\{\alpha\}}}$ , we have  $\{\xi\} = k_{\{\alpha\}}(a) = k \circ j_{p,\infty}(a) \in \text{ran}(k)$ , so  $\xi \in \text{ran}(k)$ , contradicting that  $\xi = \text{CRT}(k)$ .

Since  $k$  is the identity,  $j_{0,\infty} = j$ . Since the directed system  $\langle M_{U_q}, k_{p,q} : p \subseteq q \in [\nu]^{<\omega} \rangle$ , and thus the embedding  $j_{0,\infty}$ , were defined only with reference to the sequence  $\langle U_p : p \in [\nu]^{<\omega} \rangle$ , the lemma follows.  $\square$

**Lemma 5.3.25.** *Suppose  $U$  is a normal ultrafilter,  $M$  is an inner model, and  $j : V \rightarrow M$  is an elementary embedding such that for any  $a \in M$ , the ultrafilter derived from  $j$  using  $a$  is isomorphic to  $U^n$  for some  $n < \omega$ . Then  $M = M_{U^\nu}$  and  $j = j_{U^\nu}$  for some ordinal  $\nu$ .*

*Sketch.* For all  $m < \omega$ , let  $\kappa_m = j_{U^m}(\kappa)$ , so  $\kappa_m$  is the  $m$ -th generator of  $j_{U^m}$  for any  $n > m$ . Let  $W_n$  be the ultrafilter on  $[\kappa]^n$  derived from  $j_{U^n}$  using  $\{\kappa_{n-1}, \dots, \kappa_0\}$ . Thus  $W_n$  is the unique ultrafilter with the following properties:

- $W_n \cong U^m$  for some  $m < \omega$ .
- The underlying set of  $W_n$  is  $[\kappa]^n$ .
- Every element of  $a_{W_m}$  is a generator of  $j_{W_m}$ .

Since every ultrafilter  $Z$  derived from  $j$  is isomorphic to an ultrafilter on  $\kappa$ , the class of generators of  $j$  is contained in  $j(\kappa)$ , and in particular it forms a set. Let  $\langle \xi_\alpha : \alpha < \nu \rangle$  enumerate this set in increasing order. For any finite set  $p \subseteq \nu$ , the ultrafilter on  $[\kappa]^n$  derived from  $j$  using  $\{\xi_\alpha : \alpha \in p\}$  has the properties enumerated above, and hence is equal to  $W_n$ .

Let  $\langle \xi'_\alpha : \alpha < \nu \rangle$  denote the sequence of generators of  $j_{U^\nu}$ . Then for any finite set  $p \subseteq \nu$ , the ultrafilter on  $[\kappa]^n$  derived from  $j_{U^\nu}$  using  $\{\xi'_\alpha : \alpha \in p\}$  is equal to  $W_n$ .

By Lemma 5.3.24, it follows that  $j = j_{U^\nu}$ . □

*Proof of Theorem 5.3.23.* The assumption that  $M = H^M(j[V] \cup j(\kappa))$  implies that every ultrafilter derived from  $j$  is isomorphic to an ultrafilter on  $\kappa$ . By Theorem 5.1.1, it follows that every ultrafilter derived from  $j$  is isomorphic to  $U^n$  for some  $n$ . By Lemma 5.3.25,  $j = j_{U^\nu}$  for some ordinal  $\nu$ . □

## 5.4 The structure of the Rudin-Frolík order

### The local ascending chain condition

The goal of this subsection is to prove Theorem 5.3.17, the local ascending chain condition for the Rudin-Frolík order. This uses two lemmas, the first of which is often useful in the context of UA. The approach taken here uses the following concept:

**Definition 5.4.1.** Suppose  $Y$  is a set,  $W \in \mathcal{B}(Y)$ , and  $U \leq_{\text{RF}} W$ . Then the *translation of  $U$  by  $W$* , denoted  $t_U(W)$ , is the unique  $M_U$ -ultrafilter  $Z \in j_U(\mathcal{B}(Y))$  such that  $j_Z^{M_U} \circ j_U = j_W$  and  $a_Z = a_W$ .

The uniqueness of  $Z$  follows from the fact (Lemma 5.2.16) that there is at most one internal ultrapower embedding  $k : M_U \rightarrow M_W$  such that  $k \circ j_U = j_W$ . Then  $t_U(W)$  must be the  $M_U$ -ultrafilter on  $j_U(Y)$  derived from  $k$  using  $a_W$ . We view  $t_U(W)$  as a version of  $W$  inside  $M_U$ .

A more elegant, less comprehensible characterization of  $t_U(W)$  is immediate from the proof of Theorem 5.2.15:

**Lemma 5.4.2.** *Suppose  $U$  and  $W$  are countably complete ultrafilters. Suppose  $I$  is a set in  $U$  and  $\langle W_i : i \in I \rangle$  is a discrete sequence of ultrafilters such that  $W = U\text{-}\lim_{i \in I} W_i$ . Then  $t_U(W) = [\langle W_i : i \in I \rangle]_U$ .  $\square$*

The following lemma links translations to the minimal covers from the proof of Theorem 3.6.1:

**Lemma 5.4.3.** *Suppose  $\delta$  is an ordinal,  $W \in \mathcal{B}(\delta)$ , and  $U \leq_{\text{RF}} W$  is a countably complete ultrafilter. Then  $t_U(W)$  is  $<_{\mathbb{k}}^{M_U}$ -minimal among all  $Z \in j_U(\mathcal{B}(\delta))$  such that  $j_U[W] \subseteq Z$ .*

*Proof.* Fix  $Z \in j_U(\mathcal{B}(\delta))$  with  $j_U[W] \subseteq Z$ . For ease of notation, let  $N = M_Z^{M_U}$ . Then by Lemma 3.2.17, there is a unique embedding  $e : M_W \rightarrow N$  such that  $e \circ j_W = j_Z^{M_U} \circ j_U$  and  $e(a_W) = a_Z$ . Thus the 1-internal comparison  $(e, \text{id}) : (M_W, N) \rightarrow N$  witnesses

$$(M_W, a_W) \leq_{\mathbb{k}} (N, a_Z)$$

Suppose now towards a contradiction that  $Z <_{\mathbb{k}} t_U(W)$  in  $M_U$ . Let  $(k, h)$  be a 1-internal comparison of  $(j_Z^{M_U}, j_{t_U(W)}^{M_U})$  such that  $k(a_Z) < h(a_{t_U(W)})$ . Since  $M_Z^{M_U} = N$ ,  $M_{t_U(W)}^{M_U} = M_W$ , and  $a_{t_U(W)} = a_W$ ,  $(k, h) : (N, M_W) \rightarrow P$  is a 1-internal comparison witnessing

$$(N, a_Z) <_{\mathbb{k}} (M_W, a_W)$$

This contradicts the wellfoundedness of the Ketonen order on pointed ultrapowers (Theorem 3.5.8).  $\square$

**Lemma 5.4.4.** *Suppose  $U \leq_{\text{RF}} W$  are countably complete ultrafilters. If  $U$  is nonprincipal, then  $t_U(W) \neq j_U(W)$ .*

*Proof.* Assume  $t_U(W) = j_U(W)$ , and we will show that  $U$  is principal. By Lemma 5.4.2, fix a set  $I \in U$  and a discrete sequence  $\langle W_i : i \in I \rangle$  such that  $[\langle W_i : i \in I \rangle]_U = t_U(W)$ . Since

$\langle W_i : i \in I \rangle$  is discrete, in particular the  $W_i$  are pairwise distinct. Since  $t_U(W) = j_U(W)$ , Loś's Theorem implies that there is a  $U$ -large set  $J \subseteq I$  such that  $W_i = W$  for all  $i \in J$ . Since the  $W_i$  are pairwise distinct, it follows that  $|J| = 1$ . Thus  $U$  contains a set of size 1, so  $U$  is principal.  $\square$

**Proposition 5.4.5 (UA).** *Suppose  $\delta$  is an ordinal,  $W \in \mathcal{B}(\delta)$ , and  $U \leq_{\text{RF}} W$  is a nonprincipal ultrafilter. Then  $t_U(W) <_{\mathbb{k}} j_U(W)$  in  $M_U$ .*

*Proof.* By Lemma 5.4.3 and the linearity of the Ketonen order,  $t_U(W) \leq_{\mathbb{k}} j_U(W)$ . By Lemma 5.4.4,  $t_U(W) \neq j_U(W)$ . It follows that  $t_U(W) <_{\mathbb{k}} j_U(W)$ .  $\square$

The following simple lemma on the preservation of the Rudin-Frolík order under translation functions will be used in the proof of Theorem 5.3.17:

**Lemma 5.4.6.** *Suppose  $U, W$ , and  $Z$  are countably complete ultrafilters with  $U \leq_{\text{RF}} W, Z$ .*

- $W \leq_{\text{RF}} Z$  if and only if  $t_U(W) \leq_{\text{RF}} t_U(Z)$  in  $M_U$ .
- $W <_{\text{RF}} Z$  if and only if  $t_U(W) <_{\text{RF}} t_U(Z)$  in  $M_U$ .  $\square$

We finally prove the local ascending chain condition.

*Proof of Theorem 5.3.17.* Assume towards a contradiction that the theorem is false. Let  $C$  be the class of countably complete tail uniform ultrafilters  $Z$  such that there is an infinite  $<_{\text{RF}}$ -ascending sequence  $\langle U_n : n < \omega \rangle$  sequence  $\leq_{\text{RF}}$ -bounded above by  $Z$ . Let  $W$  be a  $<_{\mathbb{k}}$ -minimal element of  $C$ , and fix  $U_0 <_{\text{RF}} U_1 <_{\text{RF}} \dots$  such that  $U_n \leq_{\text{RF}} W$  for all  $n < \omega$ . We may assume without loss of generality that  $U_0$  is nonprincipal. By elementarity,  $j_{U_0}(W)$  is a  $<_{\mathbb{k}}^{M_{U_0}}$ -minimal element of  $j_{U_0}(C)$ .

Since translation functions preserve the Rudin-Frolík order (Lemma 5.4.6),  $M_{U_0}$  satisfies  $t_{U_0}(U_0) <_{\text{RF}} t_{U_0}(U_1) <_{\text{RF}} t_{U_0}(U_2) <_{\text{RF}} \dots$  and  $t_{U_0}(U_n) \leq_{\text{RF}} t_{U_0}(W)$  for all  $n \leq \omega$ . Since  $M_{U_0}$  is closed under countable sequences, it follows that  $t_{U_0}(W) \in j_{U_0}(C)$ . But by Proposition 5.4.5,  $t_{U_0}(W) <_{\mathbb{k}} j_{U_0}(W)$ . This contradicts that  $j_{U_0}(W)$  is a  $<_{\mathbb{k}}^{M_{U_0}}$ -minimal element of  $j_{U_0}(C)$ .  $\square$

## Pushouts and the Rudin-Frolík lattice

In this section we establish that the Rudin-Frolík order is a lattice:

**Theorem (UA).** *The Rudin-Frolík order is a lattice in the following sense:*

- *If  $U_0$  and  $U_1$  are countably complete ultrafilters, there is an  $\leq_{\text{RF}}$ -minimum countably complete ultrafilter  $W \geq_{\text{RF}} U_0, U_1$ .*
- *If  $U_0$  and  $U_1$  are countably complete ultrafilters, there is an  $\leq_{\text{RF}}$ -maximum countably complete ultrafilter  $D \leq_{\text{RF}} U_0, U_1$ .*

The Rudin-Frolík order is not literally a lattice because it is only a preorder, but the theorem above shows that it induces a lattice structure on isomorphism types of countably complete ultrafilters. The two parts will be proved as Corollary 5.4.15 and Proposition 5.4.17 below.

We begin by establishing the existence of least upper bounds in the Rudin-Frolík order, which is by far the most important part of the theorem. Here it is cleaner to work with the elementary embeddings rather than the ultrafilters:

**Definition 5.4.7.** Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  are ultrapower embeddings. An internal ultrapower comparison  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  is a *pushout* of  $(j_0, j_1)$  if for any internal ultrapower comparison  $(k_0, k_1) : (M_0, M_1) \rightarrow P$ , there is a unique internal ultrapower embedding  $h : N \rightarrow P$  such that  $h \circ i_0 = k_0$  and  $h \circ i_1 = k_1$ .

Pushout comparisons are simply the model theoretic manifestation of least upper bounds in the Rudin-Frolík order. We will prove:

**Theorem 5.4.8 (UA).** *Every pair of ultrapower embeddings has a unique pushout.*

The uniqueness of pushouts is a standard category theoretic fact: the pushout of a pair of embeddings is what a category theorist would call the pushout of these morphisms in the category  $\mathcal{D}_\infty$  of all ultrapower embeddings. In general, if two morphisms in a category have

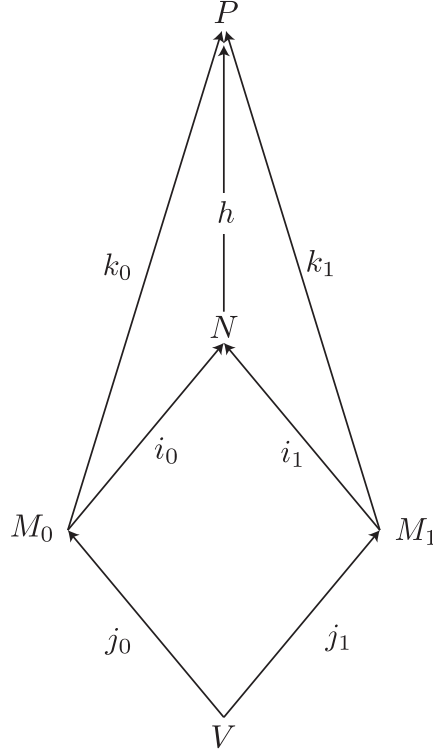


Figure 5.1: The pushout of  $(j_0, j_1)$ .

a pushout, it is unique up to isomorphism. Since the only isomorphisms in  $\mathcal{D}_\infty$  are identity functions, this implies the uniqueness of ultrapower pushouts up to equality.

We now begin the proof of Theorem 5.4.8. The proof involves the following auxiliary concept:

**Definition 5.4.9.** Suppose  $M_0$  and  $M_1$  are transitive models of ZFC. A pair of elementary embeddings  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  to a transitive model  $N$  is *minimal* if  $N = H^N(i_0[M_0] \cup i_1[M_1])$ .

In the context of ultrapower embeddings, minimality has the following alternate characterization:

**Lemma 5.4.10.** Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  are elementary embeddings and  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  is a comparison of  $(j_0, j_1)$ . Suppose  $a \in M_1$  is such that  $M_1 = H^{M_1}(j_1[V] \cup \{a\})$ . Then  $(i_0, i_1)$  is minimal if and only if  $N = H^N(i_0[M_0] \cup \{i_1(a)\})$ .  $\square$



Embedded in any pair  $(k_0, k_1) : (M_0, M_1) \rightarrow P$ , there is a unique minimal pair  $(i_0, i_1) : (M_0, M_1) \rightarrow N$ . This follows from a trivial hull argument:

**Lemma 5.4.11.** *Suppose  $(k_0, k_1) : (M_0, M_1) \rightarrow P$  is a pair of elementary embeddings. Then there exists a unique minimal  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  admitting an elementary embedding  $h : N \rightarrow P$  such that  $h \circ i_0 = k_0$  and  $h \circ i_1 = k_1$ .*

*Proof.* Let  $H = H^P(k_0[M_0] \cup k_1[M_1])$ . Let  $N$  be the transitive collapse of  $H$ . Let  $h : N \rightarrow P$  be the inverse of the transitive collapse. Let  $i_0 = h^{-1} \circ k_0$  and  $i_1 = h^{-1} \circ k_1$ . Then  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  and  $h \circ i_0 = k_0$  and  $h \circ i_1 = k_1$ . Moreover

$$h[H^N(i_0[M_0] \cup i_1[M_1])] = H^P(k_0[M_0] \cup k_1[M_1]) = h[N]$$

which implies  $H^N(i_0[M_0] \cup i_1[M_1]) = N$  since  $h$  is injective. Thus  $(i_0, i_1)$  is minimal.

Uniqueness is obvious; we omit the proof. □

**Corollary 5.4.12** (UA). *Every pair of ultrapower embeddings of  $V$  has a minimal internal ultrapower comparison.*

*Proof.* Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  are ultrapower embeddings. Fix an internal ultrapower comparison  $(k_0, k_1) : (M_0, M_1) \rightarrow P$  of  $(j_0, j_1)$ . By Lemma 5.4.11, there is a minimal pair  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  and an elementary  $h : N \rightarrow P$  with  $h \circ i_0 = k_0$  and  $h \circ i_1 = k_1$ . It follows immediately that  $(i_0, i_1)$  is a comparison of  $(j_0, j_1)$ . By Lemma 5.4.10,  $i_0$  is an ultrapower embedding of  $M_0$ . Since  $k_0$  is close to  $M_0$  and  $h \circ i_0 = k_0$ ,  $i_0$  is close to  $M_0$ . Thus  $i_0$  is a close ultrapower embedding of  $M_0$ , so  $i_0$  is an internal ultrapower embedding of  $M_0$ . Similarly  $i_1$  is an internal ultrapower embedding of  $M_1$ . Thus  $(i_0, i_1)$  is a minimal internal ultrapower comparison of  $(j_0, j_1)$ . □

**Lemma 5.4.13.** *Suppose  $(k_0, k_1) : (M_0, M_1) \rightarrow P$  is a pair of elementary embeddings and  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  is a minimal pair. Then there is at most one elementary embedding  $h : N \rightarrow P$  such that  $h \circ i_0 = k_0$  and  $h \circ i_1 = k_1$ .*

*Proof.* Suppose  $h, h' : N \rightarrow P$  satisfy  $h \circ i_0 = h' \circ i_0 = k_0$  and  $h \circ i_1 = h' \circ i_1 = k_1$ . Then  $h \upharpoonright i_0[M_0] = h' \upharpoonright i_0[M_0]$  and  $h \upharpoonright i_1[M_1] = h' \upharpoonright i_1[M_1]$ . Since  $N = H^N(i_0[M_0] \cup i_1[M_1])$ , it follows that  $h = h'$ .  $\square$

**Lemma 5.4.14** (UA). *Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  are ultrapower embeddings and  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  is a minimal comparison of  $(j_0, j_1)$ . Then  $(i_0, i_1)$  is the pushout of  $(j_0, j_1)$ .*

*Proof.* Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  are ultrapower embeddings. Suppose  $(k_0, k_1) : (M_0, M_1) \rightarrow P$  is a comparison of  $(j_0, j_1)$ . It suffices to find an internal ultrapower embedding  $h : N \rightarrow P$  such that  $h \circ i_0 = k_0$  and  $h \circ i_1 = k_1$ ; uniqueness is then immediate from Lemma 5.4.13.

Fix  $a \in M_1$  such that  $M_1 = H^{M_1}(j_1[V] \cup \{a\})$ . By Lemma 5.4.10,

$$N = H^N(i_0[M_0] \cup i_1(a))$$

By the definition of  $=_S$ , we have:

$$(N, i_1(a)) =_S (M_1, a) =_S (P, k_1(a))$$

Thus by the transitivity of the seed order,  $(N, i_1(a)) =_S (P, k_1(a))$ . Since the objects witnessing  $(N, i_1(a)) =_S (P, k_1(a))$  are internal ultrapower embeddings of  $N$  and  $P$ , which are themselves internal ultrapowers of  $M_0$ , it follows that  $M_0$  satisfies  $(N, i_1(a)) =_S (P, k_1(a))$ . By the equivalence between the seed order on models and the seed order on embeddings (Lemma 3.5.26),  $M_0$  satisfies  $(i_0, i_1(a)) =_S (k_0, k_1(a))$ . Applying Lemma 5.3.10 in  $M_0$ , it follows that there is an internal ultrapower embedding  $h : N \rightarrow P$  such that  $h \circ i_0 = k_0$  and  $h(i_1(a)) = k_1(a)$ .

We claim  $h \circ i_1 = k_1$ . Note that  $h \circ i_1 \circ j_1 = h \circ i_0 \circ j_0 = k_0 \circ j_0 = k_1 \circ j_1$ , so  $h \circ i_1 \upharpoonright j_1[V] = k_1 \upharpoonright j_1[V]$ . Moreover  $h(i_1(a)) = k_1(a)$ . Thus

$$h \circ i_1 \upharpoonright j_1[V] \cup \{a\} = k_1 \upharpoonright j_1[V] \cup \{a\}$$

Since  $M_1 = H^{M_1}(j_1[V] \cup \{a\})$ , it follows that  $h \circ i_1 = k_1$ , as desired.

Thus  $h : N \rightarrow P$  is an internal ultrapower embedding with  $h \circ i_0 = k_0$  and  $h \circ i_1 = k_1$ .  $\square$

*Proof of Theorem 5.4.8.* The existence of pushouts is an immediate consequence of Corollary 5.4.12 and Lemma 5.4.14.  $\square$

The existence of least upper bounds in the Rudin-Frolík order is a trivial restatement of Theorem 5.4.8:

**Corollary 5.4.15.** *Suppose  $U_0$  and  $U_1$  are countably complete ultrafilters. Suppose  $(i_0, i_1) : (M_{U_0}, M_{U_1}) \rightarrow N$  is the pushout of  $(j_{U_0}, j_{U_1})$ . Suppose  $W$  is a countably complete ultrafilter such that  $j_W = i_0 \circ j_0 = i_1 \circ j_1$ . Then  $W$  is the  $\leq_{\text{RF}}$ -minimum countably complete ultrafilter  $W \geq_{\text{RF}} U_0, U_1$ .*

*Proof.* The internal ultrapower embeddings  $i_0$  and  $i_1$  witness that  $U_0 \leq_{\text{RF}} W$  and  $U_1 \leq_{\text{RF}} W$ . Suppose  $U_0 \leq_{\text{RF}} Z$  and  $U_1 \leq_{\text{RF}} Z$ . We will show  $W \leq_{\text{RF}} Z$ . Let  $k_0 : M_{U_0} \rightarrow M_Z$  and  $k_1 : M_{U_1} \rightarrow M_Z$  witness  $U_0 \leq_{\text{RF}} Z$  and  $U_1 \leq_{\text{RF}} Z$ . Then since  $(i_0, i_1)$  is a pushout and  $(k_0, k_1) : (M_{U_0}, M_{U_1}) \rightarrow M_Z$ , there is an internal ultrapower embedding  $h : M_W \rightarrow M_Z$  such that  $h \circ i_0 = k_0$  and  $h \circ i_1 = k_1$ . In particular  $h \circ j_W = h \circ i_0 \circ j_{U_0} = k_0 \circ j_{U_0} = j_Z$ , so  $h$  witnesses that  $W \leq_{\text{RF}} Z$ .  $\square$

It is worth noting the following bound here:

**Proposition 5.4.16.** *Suppose  $U_0$  and  $U_1$  are countably complete ultrafilters. If  $W$  is a minimal upper bound of  $U_0$  and  $U_1$  in the Rudin-Frolík order, then  $\lambda_W = \max\{\lambda_{U_0}, \lambda_{U_1}\}$ .*

*Proof.* Let  $\lambda = \max\{\lambda_{U_0}, \lambda_{U_1}\}$ . Let  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  be the ultrapowers by  $U_0$  and  $U_1$  respectively. There is a minimal comparison  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  of  $(j_0, j_1)$  such that  $i_0 \circ j_0 = i_1 \circ j_1 = j_W$ . Fix  $\alpha < j_0(\lambda)$  such that  $M_0 = H^{M_0}(j_0[V] \cup \{\alpha\})$ . By Lemma 5.4.10,  $N = H^N(i_1[M_1] \cup \{i_0(\alpha)\}) \subseteq H^N(i_1[M_1] \cup i_1(j_1(\lambda)))$ . It follows that  $\text{WIDTH}(i_1) \leq j_1(\lambda) + 1$ . Therefore by our lemma on the width of the composition of two elementary embeddings (Lemma 3.5.34),  $\text{WIDTH}(j_W) = \text{WIDTH}(i_1 \circ j_1) = \lambda + 1$ . In other words,  $\lambda_W = \lambda$ .  $\square$

We now show the existence of greatest lower bounds in the Rudin-Frolík order. In fact we do a bit better:

**Proposition 5.4.17.** *Suppose  $A$  is a nonempty class of countably complete ultrafilters. Then  $A$  has a greatest lower bound in the Rudin-Frolík order.*

This follows purely abstractly from what we have proved so far. Recall that a partial order  $(P, \leq)$  has the local ascending chain condition if for any  $p \in P$ , there is no ascending sequence  $a_0 < a_1 < \dots$  in  $P$  with  $a_n \leq p$  for all  $n < \omega$ .

**Lemma 5.4.18.** *Suppose  $(P, \leq)$  is a join semi-lattice with a minimum element that satisfies the local ascending chain condition. For any nonempty set  $A \subseteq P$ ,  $A$  has a greatest lower bound in  $P$ .*

*Proof.* Consider the set  $B \subseteq P$  of lower bounds of  $A$ . In other words,

$$B = \{b \in P : \forall a \in A \ b \leq a\}$$

Since  $P$  has a minimum element,  $B$  is nonempty. Since  $A$  is nonempty, fixing  $p \in A$ , every element of  $B$  lies below  $p$ . Therefore by the local ascending chain condition,  $B$  has a maximal element  $b_0$ . (The ascending chain condition says that the relation  $>$  is wellfounded on  $\{c \in P : c \leq p\}$ , so the nonempty set  $B$  has a  $>$ -minimal element, or equivalently a  $<$ -maximal element.)

We claim  $B$  is a directed subset of  $(P, \leq)$ . Suppose  $b, c \in B$ . For any  $a \in A$ , by the definition of  $B$ ,  $b, c \leq a$ , and therefore their least upper bound  $b \vee c \leq a$ . In other words,  $b \vee c \leq a$  for all  $a \in A$ , so  $b \vee c \in B$ . This shows that  $B$  is directed.

Finally since  $b_0$  is a maximal element of the directed set  $B$ , in fact  $b_0$  is its maximum element. □

*Proof of Proposition 5.4.17.* The Rudin-Frolík order induces a partial order on the isomorphism types of countably complete ultrafilters. This partial order is a join semi-lattice by Corollary 5.4.15, and it has the local ascending chain condition by Theorem 5.3.17. It has a

minimum element, namely the isomorphism type of the principal ultrafilters. Therefore the conditions of Lemma 5.4.18 are met (except that we are considering a set-like partial order instead of a set, which makes no difference). This implies the proposition.  $\square$

Let us give another application of pushouts to the Rudin-Frolík order. The following characterization of the internal ultrapower embeddings of a pushout is remarkably easy to prove:

**Theorem 5.4.19.** *Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  are ultrapower embeddings and  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  is their pushout. Suppose  $h : N \rightarrow P$  is an ultrapower embedding. Then the following are equivalent:*

- (1)  $h$  is amenable to both  $M_0$  and  $M_1$ .
- (2)  $h$  is an internal ultrapower embedding of  $N$ .

*Proof.* (1) implies (2): Let  $k_0 = h \circ i_0$  and  $k_1 = h \circ i_1$ . Since  $h$  is an ultrapower embedding of  $N$ ,  $k_0$  is an ultrapower embedding of  $M_0$ . Since  $h$  is amenable to  $M_0$ ,  $k_0$  is amenable to  $M_0$ , and hence  $k_0$  is close to  $M_0$ . Since  $k_0$  is a close ultrapower embedding of  $M_0$ , in fact  $k_0$  is an internal ultrapower embedding of  $M_0$ . Similarly  $k_1$  is an internal ultrapower embedding of  $M_1$ . Thus  $(k_0, k_1)$  is an internal ultrapower comparison of  $(j_0, j_1)$ . Since  $(i_0, i_1)$  is a pushout, there is an internal ultrapower embedding  $h' : N \rightarrow P$  such that  $h' \circ i_0 = k_0$  and  $h' \circ i_1 = k_1$ . By Lemma 5.4.13, however,  $h$  is the unique elementary embedding from  $N$  to  $P$  such that  $h \circ i_0 = k_0$  and  $h \circ i_1 = k_1$ . Thus  $h = h'$ , so  $h$  is an internal ultrapower embedding, as desired.

(2) implies (1): Trivial.  $\square$

An elegant way to restate this is in terms of the ultrafilters amenable to a pushout:

**Corollary 5.4.20.** *Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  are ultrapower embeddings and  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  is their pushout. Suppose  $W$  is a countably complete  $N$ -ultrafilter. Then  $W \in N$  if and only if  $W \in M_0 \cap M_1$ .  $\square$*

Theorem 5.4.19 permits an interesting generalization of the uniqueness of ultrapower embeddings:

**Corollary 5.4.21** (UA). *Suppose  $U$  and  $W$  are countably complete ultrafilters. Then the following are equivalent:*

(1)  $U \leq_{\text{RF}} W$ .

(2)  $M_W \subseteq M_U$ .

*Proof.* (1) implies (2): Trivial.

(2) implies (1): Let  $(h.k) : (M_U, M_W) \rightarrow N$  be the pushout of  $(j_U, j_W)$ . Since  $M_W \subseteq M_U$  and  $k$  is an internal ultrapower of  $M_U$ ,  $k$  is amenable to  $M_U$ . In particular,  $k \upharpoonright N$  is amenable to both  $M_U$  and  $M_W$ . Therefore  $k \upharpoonright N$  is an internal ultrapower of  $N$ . Thus  $k$  is  $\gamma$ -supercompact for all ordinals  $\gamma$ . It follows from Proposition 4.2.31 that  $k$  is the identity. Hence  $h : M_U \rightarrow M_W$  is an internal ultrapower embedding with  $h \circ j_U = j_W$ , so  $U \leq_{\text{RF}} W$ .  $\square$

## The finiteness of the Rudin-Frolík order

The goal of this subsection is to prove the central structural fact about the Rudin-Frolík order under UA: any countably complete ultrafilter has at most finitely many predecessors in the Rudin-Frolík order up to isomorphism. The following terminology allows us to state this more precisely:

**Definition 5.4.22.** The *type* of an ultrafilter  $U$  is the class  $\{U' : U' \cong U\}$ .

**Theorem 5.4.23** (UA). *If  $W$  is a countably complete ultrafilter, then  $\{U : U \leq_{\text{RF}} W\}$  is the union of finitely many types.*

The proof heavily uses the concept of a Dodd parameter, introduced in Section 4.3 in a slightly more general context. Let us just remind the reader what this is in the special

case of ultrapower embeddings. We identify finite sets of ordinals with their *decreasing* enumerations: if  $p \subseteq \text{Ord}$  and  $|p| = \ell$ , then  $\langle p_n : n < \ell \rangle$  denotes the unique decreasing sequence such that  $p = \{p_0, \dots, p_{\ell-1}\}$ . The canonical order on finite sets of ordinals is then the lexicographic order on their decreasing enumerations.

**Definition 5.4.24.** Suppose  $j : V \rightarrow M$  is an ultrapower embedding. The *Dodd parameter* of  $j$ , denoted  $p(j)$ , is the least finite set of ordinals  $p$  such that  $H^M(j[V] \cup p) = M$ .

Note that since  $j$  is an ultrapower embedding,  $M = H^M(j[V] \cup \{\alpha\})$  for some ordinal  $\alpha$ , so  $p(j)$  certainly exists.

Recall the notion of an  $x$ -generator of an elementary embedding: if  $j : M \rightarrow N$  is an elementary embedding between transitive models of ZFC and  $x \in N$ , then an ordinal  $\xi \in N$  is an  $x$ -generator of  $j$  if  $\xi \notin H^N(j[V] \cup \xi \cup \{x\})$ . We need a basic but not completely trivial lemma about  $x$ -generators:

**Lemma 5.4.25.** *Suppose  $M \xrightarrow{j} N \xrightarrow{i} P$  are elementary embeddings between transitive models and  $\xi$  is an  $x$ -generator of  $j$ . Then  $i(\xi)$  is an  $i(x)$ -generator of  $i \circ j$ .*

*Proof.* Suppose not, and fix a function  $f$  and a finite set  $p \subseteq i(\xi)$  such that

$$i(\xi) = i(j(f))(p, i(x))$$

Then  $P$  satisfies the statement that for some finite set  $q \subseteq i(\xi)$ ,  $i(\xi) = i(j(f))(q, i(x))$ . Since  $i$  is elementary,  $N$  satisfies that for some finite set  $q \subseteq \xi$ ,  $\xi = j(f)(q, x)$ , and this contradicts that  $\xi$  is an  $x$ -generator of  $j$ . □

The key lemma regarding the Dodd parameter is that each of its elements is a generator in a strong sense:

**Lemma 5.4.26.** *Suppose  $j : V \rightarrow M$  is an ultrapower embedding. Let  $p = p(j)$ . Let  $\ell = |p|$ . Then for any  $n < \ell$ ,  $p_n$  is the largest  $p \upharpoonright n$ -generator of  $j$ .*

*Proof.* We first show that  $p_n$  is a  $p \upharpoonright n$ -generator of  $j$ . Suppose not, towards a contradiction. Fix a finite set  $q \subseteq p_n$  such that  $p_n \in H^M(j[V] \cup p \upharpoonright n \cup q)$ . Let  $r = (p \setminus \{p_n\}) \cup q$ . Then  $r < p$  but  $p \in H^M(j[V] \cup r)$ . Therefore

$$M = H^M(j[V] \cup p) \subseteq H^M(j[V] \cup r)$$

so  $H^M(j[V] \cup r) = M$ , contrary to the minimality of the Dodd parameter  $p$ .

Now let  $\xi$  be the largest  $p \upharpoonright n$ -generator of  $j$ . Suppose towards a contradiction that  $p_n < \xi$ . Then  $p \subseteq \xi \cup \{p_0, \dots, p_{n-1}\}$ , so since  $\xi \notin H^M(j[V] \cup \xi \cup p \upharpoonright n)$ , in fact  $\xi \notin H^M(j[V] \cup p)$ . This contradicts the definition of  $p(j)$ .  $\square$

The key to the proof of the finiteness of the Rudin-Frolík order is to partition the Rudin-Frolík predecessors of a countably complete ultrafilter according to their relationship with its Dodd parameter.

**Definition 5.4.27.** Suppose  $U <_{\text{RF}} W$  are countably complete ultrafilters. Let  $p = p(j_W)$ . Let  $i : M_U \rightarrow M_W$  be the unique internal ultrapower embedding such that  $i \circ j_U = j_W$ . Then  $n(U, W)$  is the least number  $n$  such that  $p_n \notin i[M_U]$ .

Note that  $n(U, W)$  depends only on the types of  $U$  and  $W$ . Note moreover that  $n(U, W)$  exists whenever  $U <_{\text{RF}} W$ : otherwise  $p \subseteq i[M_U]$ , so  $M_W = H^{M_W}(j_W[V] \cup p) \subseteq i[M_U]$ , which implies that  $i$  is surjective; thus  $i$  is an isomorphism, so  $U \cong W$ , contrary to the assumption that  $U <_{\text{RF}} W$ .

**Lemma 5.4.28.** *Suppose  $U <_{\text{RF}} W$  are countably complete ultrafilters. Let  $p = p(j_W)$ . Let  $i : M_U \rightarrow M_W$  be the unique internal ultrapower embedding such that  $i \circ j_U = j_W$ . Let  $n = n(U, W)$ . Then*

$$i[M_U] \subseteq H^{M_W}(j_W[V] \cup p \upharpoonright n \cup p_n)$$

*Proof.* Suppose towards a contradiction that the lemma fails. Let  $\xi$  be the least ordinal such that  $i(\xi) \notin H^{M_W}(j_W[V] \cup p \upharpoonright n \cup p_n)$ . Then  $i[\xi] \subseteq H^{M_W}(j_W[V] \cup p \upharpoonright n \cup p_n)$ .



By the minimality of  $n$ ,  $p \upharpoonright n \in i[M_U]$ . Therefore let  $q \in M_U$  be such that  $i(q) = p \upharpoonright n$ . We claim  $\xi$  is a  $q$ -generator of  $j_U$ . Supposing the contrary, we have  $\xi \in H^{M_U}(j_U[V] \cup \xi \cup q)$ , so

$$i(\xi) \in i[H^{M_U}(j_U[V] \cup \xi \cup q)] \subseteq H^{M_W}(j_W[V] \cup p \upharpoonright n \cup p_n)$$

which contradicts the definition of  $\xi$ .

Since  $\xi$  is a  $q$  generator of  $j_U$ ,  $i(\xi)$  is an  $i(q)$ -generator of  $i \circ j_U$  by Lemma 5.4.25. In other words,  $i(\xi)$  is a  $p \upharpoonright n$ -generator of  $j_W$ . By Lemma 5.4.26,  $p_n$  is the largest  $p \upharpoonright n$ -generator of  $j_W$ , so  $i(\xi) \leq p_n$ . This contradicts that  $i(\xi) \notin H^{M_W}(j_W[V] \cup p \upharpoonright n \cup p_n)$ .  $\square$

**Definition 5.4.29.** Suppose  $W$  is a countably complete ultrafilter and  $p = p(j_W)$ . For any  $n < |p|$ ,  $\mathcal{D}_n(W) = \{U <_{\text{RF}} W : n(U, W) = n\}$ .

**Lemma 5.4.30.** For any countably complete ultrafilter  $W$ ,

$$\{U : U <_{\text{RF}} W\} = \bigcup_{n < |p(j_W)|} \mathcal{D}_n(W)$$

*Proof.* See the remarks following Definition 5.4.27.  $\square$

The following fact is the key to the proof of the finiteness of the Rudin-Frolík order:

**Lemma 5.4.31.** Suppose  $U_0, U_1 \in \mathcal{D}_n(W)$  and  $D$  is the  $\leq_{\text{RF}}$ -minimum countably complete ultrafilter such that  $U_0, U_1 \leq_{\text{RF}} D$ . Then  $D \in \mathcal{D}_n(W)$ .

*Proof.* Let  $M_0 = M_{U_0}$  and let  $M_1 = M_{U_1}$ . Let  $(i_0, i_1) : (M_0, M_1) \rightarrow M_D$  be internal ultrapower embeddings witnessing that  $U_0, U_1 \leq_{\text{RF}} D$  and let  $(k_0, k_1) : (M_0, M_1) \rightarrow M_W$  be internal ultrapower embeddings witnessing that  $U_0, U_1 \leq_{\text{RF}} W$ .

Since  $D$  is the  $\leq_{\text{RF}}$ -minimum countably complete ultrafilter with  $U_0, U_1 \leq_{\text{RF}} D$ , in fact  $D \leq_{\text{RF}} W$ . Let  $h : M_D \rightarrow M_W$  be the unique internal ultrapower embedding such that  $h \circ j_D = j_W$ . Notice that

$$h \circ i_0 = k_0$$

$$h \circ i_1 = k_1$$

by Lemma 5.2.16.

Since  $D$  is the  $\leq_{\text{RF}}$ -minimum ultrafilter with  $U_0, U_1 \leq_{\text{RF}} D$ ,  $(i_0, i_1) : (M_0, M_1) \rightarrow M_D$  must be minimal in the sense of Definition 5.4.9:

$$M_D = H^{M_D}(i_0[M_0] \cup i_1[M_1])$$

(The proof is a trivial diagram chase. Let  $(\bar{i}_0, \bar{i}_1) : (M_0, M_1) \rightarrow N$  be the unique minimal pair admitting  $e : N \rightarrow M_D$  such that  $e \circ \bar{i}_0 = i_0$  and  $e \circ \bar{i}_1 = i_1$ . By the proof of Corollary 5.4.12,  $N$  is an internal ultrapower of  $M_0$  and  $M_1$ , so since  $D$  is a least upper bound of  $U_0, U_1$ , there is an internal ultrapower embedding  $d : M_D \rightarrow N$  such that  $d \circ i_0 = \bar{i}_0$  and  $d \circ i_1 = \bar{i}_1$ . Then  $d \circ e : N \rightarrow N$  satisfies  $d \circ e \circ \bar{i}_0 = \bar{i}_0$  and  $d \circ e \circ \bar{i}_1 = \bar{i}_1$ , and hence by Lemma 5.4.13,  $d \circ e$  must be the identity map. Hence  $d$  and  $e$  are inverses, so by transitivity  $N = M_D$  and  $e$  is the identity. Now  $\bar{i}_0 = e \circ \bar{i}_0 = i_0$  and  $\bar{i}_1 = e \circ \bar{i}_1 = i_1$  so  $(\bar{i}_0, \bar{i}_1) = (i_0, i_1)$ . Since  $(\bar{i}_0, \bar{i}_1)$  is minimal, so is  $(i_0, i_1)$ .) Therefore

$$h[M_D] = h[H^{M_D}(i_0[M_0] \cup i_1[M_1])] = H^{M_W}(k_0[M_0] \cup k_1[M_1])$$

Let  $p = p(j_W)$ . Since  $U_0 \in \mathcal{D}_n(W)$ ,  $k_0[M_0] \subseteq H^{M_W}(j_W[V] \cup p \upharpoonright n \cup p_n)$  by Lemma 5.4.28. Similarly,  $k_1[M_1] \subseteq H^{M_W}(j_W[V] \cup p \upharpoonright n \cup p_n)$ . Thus

$$k_0[M_0] \cup k_1[M_1] \subseteq H^{M_W}(j_W[V] \cup p \upharpoonright n \cup p_n)$$

It follows that  $h[M_D] = H^{M_W}(k_0[M_0] \cup k_1[M_1]) \subseteq H^{M_W}(j_W[V] \cup p \upharpoonright n \cup p_n)$ . In particular, since  $p_n$  is a  $p \upharpoonright n$ -generator of  $j_W$  by Lemma 5.4.26,  $p_n \notin h[M_D]$ . Clearly

$$p \upharpoonright n \in k_0[M_0] \subseteq h[M_D]$$

so  $n$  is the least number such that  $p_n \notin h[M_D]$ . It follows that  $n(D, W) = n$ . In other words,  $D \in \mathcal{D}_n(W)$ . □

The point now is that by Theorem 5.3.17 and Corollary 5.4.15, we can find a maximum element of  $\mathcal{D}_n$ :

**Proposition 5.4.32** (UA). *Suppose  $W$  is a countably complete ultrafilter and  $n < |p(j_W)|$ . If  $\mathcal{D}_n(W)$  is nonempty, then  $\mathcal{D}_n(W)$  has a  $\leq_{\text{RF}}$ -maximum element.*

*Proof.* By Corollary 5.4.15, every pair of countably complete ultrafilters has a least upper bound in the Rudin-Frolík order. Combining this with Lemma 5.4.31, the class  $\mathcal{D}_n(W)$  is directed under  $\leq_{\text{RF}}$ . Moreover it is bounded below  $W$  in  $\leq_{\text{RF}}$ . Therefore by Theorem 5.3.17, it has a maximal element  $U$ . By the  $\leq_{\text{RF}}$ -directedness of  $\mathcal{D}_n(W)$ , this maximal element is a maximum element.  $\square$

We finally prove Theorem 5.4.23.

*Proof of Theorem 5.4.23.* The proof is by induction on the wellfounded relation  $<_{\text{RF}}$ . (See Lemma 5.2.12.) Assume  $W$  is a countably complete ultrafilter. Our induction hypothesis is that for all  $U <_{\text{RF}} W$ ,  $\{D : D \leq_{\text{RF}} U\}$  is the union of finitely many types. We aim to show that  $\{U : U \leq_{\text{RF}} W\}$  is the union of finitely many types.

Let  $p = p(j_W)$  and let  $\ell = |p(j_W)|$ . By Lemma 5.4.30,

$$\{U : U <_{\text{RF}} W\} = \bigcup_{n < \ell} \mathcal{D}_n(W)$$

We claim that for any  $n < \ell$ ,  $\mathcal{D}_n(W)$  is the union of finitely many types. If  $\mathcal{D}_n(W)$  is empty, this is certainly true. If  $\mathcal{D}_n(W)$  is nonempty, then by Proposition 5.4.32, there is a  $\leq_{\text{RF}}$ -maximum element  $U$  of  $\mathcal{D}_n(W)$ . Since  $U \in \mathcal{D}_n(W)$ ,  $U <_{\text{RF}} W$  so by our induction hypothesis  $\{D : D \leq_{\text{RF}} U\}$  is the union of finitely many types. But since  $U$  is a  $\leq_{\text{RF}}$ -maximum element of  $\mathcal{D}_n(W)$ ,  $\mathcal{D}_n(W) \subseteq \{D : D \leq_{\text{RF}} U\}$ . Thus  $\mathcal{D}_n(W)$  is the union of finitely many types.

Since  $\{U : U <_{\text{RF}} W\} = \bigcup_{n < \ell} \mathcal{D}_n(W)$  is a finite union of classes  $\mathcal{D}_n(W)$  each of which is itself the union of finitely many types,  $\{U : U <_{\text{RF}} W\}$  is the union of finitely many types. The collection  $\{U : U \leq_{\text{RF}} W\}$  contains just one more type than  $\{U : U <_{\text{RF}} W\}$ , namely that of  $W$ . So  $\{U : U \leq_{\text{RF}} W\}$  is the union of finitely many types, completing the induction.  $\square$

## Translations and limits

In this section we explain the relationship between pushouts, ultrafilter translations, and the minimal covers defined for the proof of UA from the linearity of the Ketonen order in Section 3.6.

Recall Definition 5.4.1, which defined for any countably complete ultrafilters  $U \leq_{\text{RF}} W$  the translation of  $W$  by  $U$ , the canonical countably complete ultrafilter of  $M_U$  that leads from  $M_U$  into  $M_W$ . It turns out that there is a natural generalization of  $t_U(W)$  for any ultrafilters that admit a pushout:

**Definition 5.4.33.** Suppose  $U$  and  $W \in \mathcal{B}(Y)$  are countably complete ultrafilters. Suppose  $(k, h) : (M_U, M_W) \rightarrow N$  is the pushout of  $(j_U, j_W)$ . Then  $t_U(W)$  denotes the  $M_U$ -ultrafilter on  $j_U(Y)$  derived from  $k$  using  $h(a_W)$ .

The point of this definition is that  $t_U(W)$  is the canonical ultrafilter of  $M_U$  giving rise to the  $M_U$ -side of the pushout of  $(j_U, j_W)$ :

**Lemma 5.4.34.** *Suppose  $U$  and  $W \in \mathcal{B}(Y)$  are countably complete ultrafilters. Suppose  $(k, h) : (M_U, M_W) \rightarrow N$  is the pushout of  $(j_U, j_W)$ . Then  $t_U(W)$  is the unique ultrafilter  $Z \in j_U(\mathcal{B}(Y))$  such that  $j_Z^{M_U} = k$  and  $a_Z = h(a_W)$ .  $\square$*

We will try to omit superscripts when we can:

**Corollary 5.4.35.** *If  $U$  and  $W$  are countably complete ultrafilters, then  $(j_{t_U(W)}, j_{t_W(U)})$  is the pushout of  $(j_U, j_W)$  if it exists.*

The notation  $t_U(W)$  generalizes the notation  $t_U(W)$  introduced in Definition 5.4.1 when  $U \leq_{\text{RF}} W$ . To see this, assume  $U \leq_{\text{RF}} W$  and let  $k : M_U \rightarrow M_W$  be the unique internal ultrapower embedding of  $M_U$  such that  $k \circ j_U = j_W$ . Then  $(k, \text{id}) : (M_U, M_W) \rightarrow M_W$  is the pushout of  $(j_U, j_W)$ , and hence  $t_U(W)$  as we have defined it here is just the  $M_U$ -ultrafilter derived from  $k$  using  $a_W$ , which is precisely  $t_U(W)$  as defined in Definition 5.4.1.

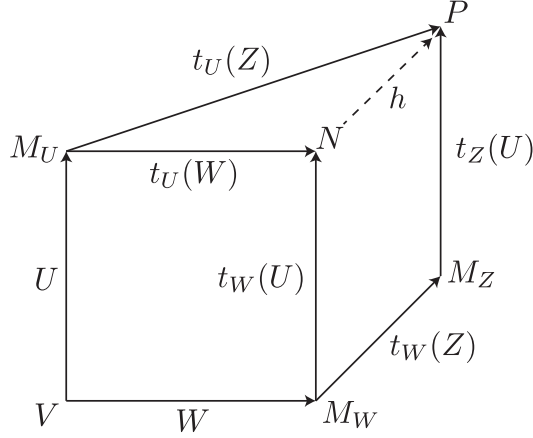


Figure 5.2: The proof of Lemma 5.4.38.

It turns out that in the definition of a translation, one does not need to use the pushout (as long as the pushout exists):

**Lemma 5.4.36.** *Suppose  $U$  and  $W \in \mathcal{B}(Y)$  are countably complete ultrafilters such that the pair  $(j_U, j_W)$  has a pushout. Let  $(k, h) : (M_U, M_W) \rightarrow P$  be a close comparison of  $(j_U, j_W)$ . Then  $t_U(W)$  is the  $M_U$ -ultrafilter on  $j_U(Y)$  derived from  $k$  using  $h(a_W)$ .  $\square$*

It is not hard to see that translations are isomorphism invariant:

**Lemma 5.4.37.** *Suppose  $U \cong U'$  and  $W \cong W'$ . Then  $t_U(W) \cong t_{U'}(W')$  in  $M_U$ .*

In fact, we can do quite a bit better than this: translation functions preserve the Rudin-Frolík order.

**Lemma 5.4.38.** *Suppose  $U$ ,  $W$ , and  $Z$  are countably complete ultrafilters. If  $W \leq_{\text{RF}} Z$ , then  $t_U(W) \leq_{\text{RF}} t_U(Z)$  in  $M_U$ .*

*Proof.* Let  $N = M_{t_U(W)}^{M_U}$  and let  $P = M_{t_U(Z)}^{M_U}$ . The proof is contained in Fig. 5.2. By Corollary 5.4.35:

- $(j_{t_U(W)}, j_{t_U(Z)}) : (M_U, M_W) \rightarrow N$  is the pushout of  $(j_U, j_W)$ .

- $(j_{t_U(Z)}, j_{t_Z(U)} \circ j_{t_W(Z)}) : (M_U, M_W) \rightarrow P$  is an internal ultrapower comparison of  $(j_U, j_W)$ .

Since  $(j_{t_U(W)}, j_{t_W(U)})$  is a pushout, there is an internal ultrapower embedding  $h : N \rightarrow P$  such that  $h \circ j_{t_U(W)} = j_{t_U(Z)}$  and  $h \circ j_{t_W(U)} = j_{t_Z(U)} \circ j_{t_W(Z)}$ . In particular, the first of these equations says that  $h$  witnesses  $t_U(W) \leq_{\text{RF}} t_U(Z)$  in  $M_U$ .  $\square$

We occasionally use the following fact, which is an immediate consequence of Lemma 5.4.34:

**Lemma 5.4.39.** *Suppose  $U$  and  $W$  are countably complete ultrafilters on  $X$  and  $Y$ . Then the following are equivalent:*

- (1)  $U \leq_{\text{RF}} W$ .
- (2) For some  $I \in U$  and some discrete sequence  $\langle W_i : i \in I \rangle$  of countably complete ultrafilters on  $Y$ ,  $t_U(W) = [\langle W_i : i \in I \rangle]_U$ .
- (3)  $j_{t_U(W)} \circ j_U = j_W$ .
- (4)  $t_W(U)$  is a principal ultrafilter of  $M_W$ .
- (5)  $t_W(U) = \mathfrak{p}_{h(a_U)}^{j_W(X)}$  where  $h : M_U \rightarrow M_W$  is the unique internal ultrapower embedding such that  $h \circ j_U = j_W$ .  $\square$

The following fundamental fact connects translations back to the minimal covers of Section 3.6:

**Theorem 5.4.40 (UA).** *Suppose  $\delta$  is an ordinal,  $U$  is a countably complete ultrafilter, and  $W \in \mathcal{B}(\delta)$ . Then  $t_U(W)$  is the least element of  $j_U(\mathcal{B}(\delta), <_{\mathbb{k}})$  that extends  $j_U[W]$ .*

*Proof.* By replacing  $U$  with an isomorphic ultrafilter, we may assume that for some ordinal  $\epsilon$ ,  $U \in \mathcal{B}(\epsilon)$ , putting us in a position to apply the results of Section 3.6.

Let  $W_*$  be the least element of  $j_U(\mathcal{B}(\delta), <_{\mathbb{k}})$  that extends  $j_U[W]$  and let  $U_*$  be the least element of  $j_W(\mathcal{B}(\epsilon), <_{\mathbb{k}})$  that extends  $j_W[U]$ . By Theorem 3.6.4,

$$(j_{W_*}^{M_U}, j_{U_*}^{M_W}) : (M_U, M_W) \rightarrow N$$

is a comparison of  $(j_U, j_W)$ . Moreover, as a consequence of Lemma 3.6.13,  $a_{W_*} = j_{U_*}^{M_W}(a_W)$ .

In particular,

$$N = H^N(j_{W_*}^{M_U}[M_U] \cup \{a_{W_*}\}) = H^N(j_{W_*}^{M_U}[M_U] \cup \{j_{U_*}^{M_W}(a_W)\})$$

It follows from Lemma 5.4.10 that  $(j_{W_*}^{M_U}, j_{U_*}^{M_W})$  is minimal. Therefore by Lemma 5.4.14,  $(j_{W_*}^{M_U}, j_{U_*}^{M_W})$  is the pushout of  $(j_U, j_W)$ . Since  $W_*$  is the  $M_U$ -ultrafilter on  $j_U(\delta)$  derived from  $j_{W_*}^{M_U}$  using  $j_{U_*}^{M_W}(a_W)$ , by definition  $W_* = t_U(W)$ .  $\square$

This yields the following bound on  $t_U(W)$  that is not a priori obvious:

**Corollary 5.4.41** (UA). *Suppose  $U$  is a countably complete ultrafilter and  $W$  is a countably complete ultrafilter on an ordinal. Then  $t_U(W) \leq_{\mathbb{k}} j_U(W)$  in  $M_U$ .*

*Proof.* Let  $\delta$  be the underlying ordinal of  $W$ . Then  $j_U(W) \in j_U(\mathcal{B}(\delta))$  and  $j_U[W] \subseteq j_U(W)$ . Thus  $t_U(W) \leq_{\mathbb{k}} j_U(W)$  in  $M_U$  by Theorem 5.4.40.  $\square$

We finally show that translation functions preserve the Ketonen order:

**Theorem 5.4.42** (UA). *Translation functions preserve the Ketonen order. More precisely, suppose  $Z$  is a countably complete ultrafilter and  $U$  and  $W$  are countably complete ultrafilters on ordinals. Then  $U <_{\mathbb{k}} W$  if and only if  $M_Z \models t_Z(U) <_{\mathbb{k}} t_Z(W)$ .*

For this we need the strong transitivity of the Ketonen order (Lemma 3.3.10). We actually use the following immediate corollary of Lemma 3.3.10 and the characterization of limits in terms of inverse images (Lemma 3.2.12):

**Lemma 5.4.43.** *Suppose  $Z$  is an ultrafilter,  $\delta$  is an ordinal, and  $U, W \in \mathcal{B}(\delta)$  satisfy  $U <_{\mathbb{k}} W$ . For any  $W_* \in j_Z(\mathcal{B}(\delta))$  with  $j_Z[W] \subseteq W_*$ , there is some  $U_* \in j_Z(\mathcal{B}(\delta))$  with  $U_* <_{\mathbb{k}}^{M_Z} W_*$  and  $j_Z[U] \subseteq U_*$ .*  $\square$

With Theorem 5.4.40 and Lemma 5.4.43 in hand, we can prove Theorem 5.4.42.

*Proof of Theorem 5.4.42.* Assume that  $U <_{\mathbb{k}} W$  are countably complete ultrafilters on ordinals. We will show  $t_Z(U) <_{\mathbb{k}}^{M_Z} t_Z(W)$ . For ease of notation, we will assume (without real loss of generality) that  $U, W \in \mathcal{B}(\delta)$  for a fixed ordinal  $\delta$ .

Let  $W_* = t_Z(W)$ . Theorem 5.4.40 implies that  $j_Z[W] \subseteq W_*$ . (This is actually much easier to prove than Theorem 5.4.40.) By Lemma 5.4.43, it follows that there is some  $U_* \in j_Z(\mathcal{B}(\delta))$  with

$$U_* <_{\mathbb{k}}^{M_Z} W_*$$

and  $j_Z[U] \subseteq U_*$ . Since  $t_Z(U)$  is the minimal extension of  $j_Z[U]$  by Theorem 5.4.40, we have

$$t_Z(U) \leq_{\mathbb{k}}^{M_Z} U_*$$

By the transitivity of the Ketonen order,  $t_Z(U) \leq_{\mathbb{k}}^{M_Z} t_Z(W)$ , as desired.  $\square$

## 5.5 The internal relation

### A generalized Mitchell order

In this section, we introduce a variant of the generalized Mitchell order that will serve as a powerful tool in the theory of countably complete ultrafilters. The trouble with using the Mitchell order itself to prove general theorems about countably complete ultrafilters is that the Mitchell order is only meaningful for ultrafilters that have a certain amount of strength: a precondition for  $U \triangleleft W$  is that  $P(\lambda_U) \subseteq M_W$ . In order to analyze a more general class of ultrafilters, we need a way to talk about the Mitchell order on ultrafilters that are not assumed to be strong.

There are a number of possible approaches, but the one that has proved most successful is called the *internal relation*:

**Definition 5.5.1.** The *internal relation* is defined on countably complete ultrafilters  $U$  and  $W$  by setting  $U \sqsubset W$  if  $j_U \upharpoonright M_W$  is an internal ultrapower embedding of  $M_W$ .



The topic of this section is the theory of the internal relation under UA. The reason that we have saved it for this chapter is that it is closely related to the theory of pushouts from Section 5.4.

Before we proceed through the basic theory below, let us mention that the supercompactness analysis of Chapter 7 and Chapter 8 yields a set theoretically simpler description of the internal relation on a very large class of ultrafilters. In fact, the internal relation and the Mitchell order are essentially one and the same:

**Theorem 8.3.30 (UA).** *Suppose  $U$  and  $W$  are hereditarily uniform irreducible ultrafilters. Then the following are equivalent:*

- (1)  $U \sqsubset W$ .
- (2) Either  $U \triangleleft W$  or  $W \in V_\kappa$  where  $\kappa = \text{CRT}(j_U)$ .

The second part of condition (2) should be compared with Kunen's commuting ultrapowers lemma (Theorem 5.5.20).

## The Mitchell order versus the internal relation

To understand the nature of the internal relation, it helps to consider its relationship with the Mitchell order.

**Proposition 5.5.2.** *Suppose  $U$  is a countably complete ultrafilter on a set  $X$  and  $W$  is a countably complete ultrafilter such that  $X \in M_W$  and  $U \sqsubset W$ . Then the  $M_W$ -ultrafilter  $U \cap M_W$  belongs to  $M_W$ . In particular, if  $P(X) \subseteq M_W$ , then  $U \triangleleft W$ .  $\square$*

In general, however,  $U \sqsubset W$  does not imply  $U \triangleleft W$ . This is a consequence of Kunen's commuting ultrapowers lemma (Theorem 5.5.20):

**Proposition 5.5.3.** *Suppose  $\kappa$  is a measurable cardinal,  $U \in V_\kappa$  is a countably complete ultrafilter and  $W$  is a  $\kappa$ -complete ultrafilter. Then  $W \sqsubset U$ .  $\square$*

Note that in the situation above, if  $W$  is nonprincipal, then  $\lambda_W \geq \kappa$ , and in particular  $W \not\triangleleft U$  since  $P(\kappa) \not\subseteq M_U$ .

Whether  $U \triangleleft W$  always implies  $U \sqsubset W$  is a considerably subtler question. This implication is consistently false. (This is closely related to Proposition 4.2.29.) We begin with the following fact:

**Proposition 5.5.4.** *Suppose  $\kappa$  is  $2^\kappa$ -supercompact and  $2^\kappa = 2^{(\kappa^+)}$ . Then there is a normal ultrafilter  $D$  on  $\kappa$  and a  $\kappa$ -complete normal fine ultrafilter  $\mathcal{U}$  on  $P_\kappa(\kappa^+)$  such that  $\mathcal{U} \triangleleft D$ .*

*Sketch.* Since  $\kappa$  is  $\kappa^+$ -supercompact, there is a  $\kappa$ -complete normal fine ultrafilter  $\mathcal{U}$  on  $P_\kappa(\kappa^+)$ . By Solovay's theorem on SCH above a strongly compact cardinal (Theorem 7.2.16),  $|P_\kappa(\kappa^+)| = \kappa^+$ . By Solovay's ultrafilter-capturing theorem (Theorem 6.3.3), for any set  $A$  of hereditary cardinality at most  $2^\kappa$ , there is a normal ultrafilter  $D$  on  $\kappa$  such that  $A \in M_D$ . But  $\mathcal{U} \subseteq P(P_\kappa(\kappa^+))$  has hereditary cardinality  $2^{\kappa^+} = 2^\kappa$ . Thus there is a normal ultrafilter  $D$  on  $\kappa$  such that  $\mathcal{U} \in M_D$ , or in other words,  $\mathcal{U} \triangleleft D$ .  $\square$

Thus given a failure of the weak GCH at a supercompact, one must have a rather unusual situation in which  $\mathcal{U} \triangleleft D$  even though  $\lambda_{\mathcal{U}} > \lambda_D$ . On the other hand, the internal relation does not hold between these ultrafilters:

**Proposition 5.5.5.** *Assume  $D$  is a  $\kappa$ -complete uniform ultrafilter on  $\kappa$  and  $\mathcal{U}$  is a  $\kappa$ -complete normal fine ultrafilter on  $P_\kappa(\kappa^+)$ .<sup>1</sup> Then  $\mathcal{U} \not\sqsubset D$ .*

*Proof.* Suppose towards a contradiction that  $\mathcal{U} \sqsubset D$ . Then  $j_{\mathcal{U}}(M_D) \subseteq M_D$  since  $j_{\mathcal{U}} \upharpoonright M_D$  is an internal ultrapower embedding of  $M_D$ . But  $j_{\mathcal{U}}(M_D) = (M_{j_{\mathcal{U}}(D)})^{M_{\mathcal{U}}}$ . Since  $j_{\mathcal{U}}(D)$  is  $j_{\mathcal{U}}(\kappa)$ -complete in  $M_{\mathcal{U}}$ ,

$$\text{Ord}^{(\kappa^+)} \subseteq \text{Ord}^{j_D(\kappa)} \cap M_{\mathcal{U}} \subseteq M_{j_{\mathcal{U}}(D)}^{M_{\mathcal{U}}} = j_{\mathcal{U}}(M_D) \subseteq M_D$$

It follows that  $j_D$  is  $\kappa^+$ -supercompact, and this contradicts the bound on the supercompactness of the ultrapower by an ultrafilter on  $\kappa$  (Proposition 4.2.31).  $\square$

<sup>1</sup>The proof only requires that  $\mathcal{U}$  is a  $\kappa$ -complete fine ultrafilter on  $P_\kappa(\kappa^+)$ .

We have not checked that the implication from  $U \triangleleft W$  to  $U \sqsubset W$  can fail under the Generalized Continuum Hypothesis, but we are confident that it can. Under UA, however, this implication is a theorem:

**Theorem 8.3.26** (UA). *Suppose  $U$  and  $W$  are countably complete ultrafilters. If  $U \triangleleft W$ , then  $U \sqsubset W$ . □*

## Basic theory of the internal relation

The true motivation for the definition of the internal relation comes from the theory of ultrapower comparisons:

**Lemma 5.5.6.** *Suppose  $U$  and  $W$  are countably complete ultrafilters. Then*

$$(j_U(j_W), j_U \upharpoonright M_W) : (M_U, M_W) \rightarrow j_U(M_W)$$

*is a 0-internal minimal comparison of  $(j_U, j_W)$ . It is an internal ultrapower comparison if and only if  $U \sqsubset W$ .*

*Proof.* The fact that  $(j_U(j_W), j_U \upharpoonright M_W)$  is a comparison of  $(j_U, j_W)$  is immediate from the standard application-composition identity:

$$j_U(j_W) \circ j_U = (j_U \upharpoonright M_W) \circ j_W$$

Since  $j_W$  is an internal ultrapower embedding of  $V$ ,  $j_U(j_W)$  is an internal ultrapower embedding of  $M_U$  by the elementarity of  $j_U$ . In particular,  $(j_U(j_W), j_U \upharpoonright M_W)$  is 0-internal. Moreover, if  $U \sqsubset W$  then  $j_U \upharpoonright M_W$  is an internal ultrapower embedding of  $M_W$ , and hence  $(j_U(j_W), j_U \upharpoonright M_W)$  is an internal ultrapower comparison.

Let us finally show that  $(j_U(j_W), j_U \upharpoonright M_W)$  is a minimal comparison of  $(j_U, j_W)$ , or in other words that

$$j_U(M_W) = H^{j_U(M_W)}(j_U(j_W)[M_U] \cup j_U[M_W])$$

The proof begins with the standard fact that  $M_W = H^{M_W}(j_W[V] \cup \{a_W\})$ . Applying  $j_U$  to both sides of the equation, we obtain:

$$j_U(M_W) = H^{j_U(M_W)}(j_U(j_W)[M_U] \cup \{j_U(a_W)\})$$

Since  $j_U(a_W) \in j_U[M_W]$ ,

$$H^{j_U(M_W)}(j_U(j_W)[M_U] \cup \{j_U(a_W)\}) \subseteq H^{j_U(M_W)}(j_U(j_W)[M_U] \cup j_U[M_W])$$

This yields that  $j_U(M_W) \subseteq H^{j_U(M_W)}(j_U(j_W)[M_U] \cup j_U[M_W])$ , which of course implies that equality holds, as desired.  $\square$

Combining Lemma 5.5.6 with the fact that minimal comparisons of ultrapowers are ultrapower comparisons (Lemma 5.4.10), we obtain the following lemma:

**Lemma 5.5.7.** *Suppose  $U$  and  $W$  are countably complete ultrafilters. Then  $j_U \upharpoonright M_W$  is an ultrapower embedding of  $M_W$ .*

Of course, we do not mean that  $j_U \upharpoonright M_W$  is necessarily an *internal* ultrapower embedding of  $M_W$ , just that there is a point  $a \in j_U(M_W)$  such that  $j_U(M_W) = H^{j_U(M_W)}(j_U[M_W] \cup \{a\})$ . An important point is that this point  $a$  need not be  $a_U$  itself.

Applying the proof of Lemma 5.4.10 in to the minimal comparison  $(j_U(j_W), j_U \upharpoonright M_W)$  identifies a specific  $M_W$ -ultrafilter giving rise to the embedding  $j_U \upharpoonright M_W$ :

**Definition 5.5.8.** Suppose  $U$  and  $W$  are countably complete ultrafilters. Let  $X$  be the underlying set of  $U$ . Then the *pushforward of  $U$  into  $M_W$*  is the  $M_W$ -ultrafilter  $s_W(U)$  on  $j_W(X)$  defined as follows: if  $A \subseteq j_W(X)$  and  $A \in M_W$ ,

$$A \in s_W(U) \iff j_W^{-1}[A] \in U$$

The reason we call  $s_W(U)$  a pushforward is that it is literally equal to the pushforward  $f_*(U) \cap M_W$  where  $f : X \rightarrow j_W(X)$  is the restriction  $f = j_W \upharpoonright X$ .

For the reader's convenience, let us chase through all the lemmas and prove that  $s_W(U)$  behaves as it should:

**Lemma 5.5.9.** *Suppose  $U$  and  $W$  are countably complete ultrafilters on  $X$  and  $Y$ . Then  $s_W(U)$  is the  $M_W$ -ultrafilter on  $j_W(X)$  derived from  $j_U \upharpoonright M_W$  using  $j_U(j_W)(a_U)$ . Moreover,*

$$j_{s_W(U)}^{M_W} = j_U \upharpoonright M_W$$

*Thus  $U \sqsubset W$  if and only if  $s_W(U) \in M_W$ .*

*Proof.* Let  $f = j_W \upharpoonright X$ . Then  $f_*(U)$  is the ultrafilter derived from  $j_U$  using  $j_U(f)(a_U)$  by the basic theory of the Rudin-Keisler order (Lemma 3.2.16). Thus  $f_*(U) \cap M_W$  is the  $M_W$ -ultrafilter derived from  $j_U \upharpoonright M_W$  using  $j_U(f)(a_U) = j_U(j_W)(a_U)$ . But  $f_*(U) \cap M_W = s_W(U)$ , so  $s_W(U)$  is the  $M_W$ -ultrafilter on  $j_W(X)$  derived from  $j_U \upharpoonright M_W$  using  $j_U(j_W)(a_U)$ .

We finish by proving  $j_{s_W(U)}^{M_W} = j_U \upharpoonright M_W$ . Since  $s_U(W)$  is derived from  $j_U \upharpoonright M_W$  using  $j_U(j_W)(a_U)$ , there is a factor embedding  $k : M_{s_W(U)}^{M_W} \rightarrow j_U(M_W)$  with  $k \circ j_{s_W(U)}^{M_W} = j_U \upharpoonright M_W$  and  $k(a_{s_W(U)}) = j_U(j_W)(a_U)$ . Since  $(j_U(j_W), j_U \upharpoonright M_W) : (M_U, M_W) \rightarrow j_U(M_W)$  is a minimal comparison of  $(j_U, j_W)$ , Lemma 5.4.10 yields:

$$j_U(M_W) = H^{j_U(M_W)}(j_U[M_W] \cup \{j_U(j_W)(a_U)\})$$

But  $H^{j_U(M_W)}(j_U[M_W] \cup \{j_U(j_W)(a_U)\}) \subseteq k[M_{s_W(U)}^{M_W}]$ . In other words,  $k$  is a surjection. It follows that  $M_{s_W(U)}^{M_W} = j_U(M_W)$  and  $k$  is the identity. Therefore  $j_{s_W(U)}^{M_W} = k \circ j_{s_W(U)}^{M_W} = j_U \upharpoonright M_W$  as desired.  $\square$

As a corollary, one can characterize the internal relation in terms of amenability of ultrafilters.

**Lemma 5.5.10.** *Suppose  $U$  and  $W$  are countably complete ultrafilters. Then the following are equivalent:*

- (1)  $U \sqsubset W$ .
- (2) For all  $U' \leq_{\text{RK}} U$ ,  $U' \cap M_W \in M_W$ .
- (3) For all  $U' \cong U$ ,  $U' \cap M_W \in M_W$ .

*Proof.* (1) implies (2): Suppose  $U' \leq_{\text{RK}} U \sqsubset W$ . Fix a set  $X$  and a point  $a \in M_U$  such that  $U'$  is the ultrafilter on  $X$  derived from  $j_U$  using  $a$ . If  $X \cap M_W \notin U'$ , then  $U' \cap M_W = \emptyset$ , and so  $U' \cap M_W \in M_W$  vacuously. Therefore assume  $X \cap M_W \in U'$ . In other words,  $a \in j_U(X \cap M_W)$ , so  $a \in j_U(M_W)$ . Then  $U' \cap M_W$  is the ultrafilter derived from  $j_U \upharpoonright M_W$  using  $a$ , so since  $j_U \upharpoonright M_W$  is an internal ultrapower embedding of  $M_W$ ,  $U' \cap M_W \in M_W$ .

(2) implies (3): Trivial.

(3) implies (1): Let  $X$  be the underlying set of  $U$ . Let  $f : X \rightarrow j_W(X)$  be the restriction  $f = j_W \upharpoonright X$ . Since  $j_W$  is injective,  $f_*(U) \cong U$ . Moreover  $f_*(U) \cap M_W = s_W(U)$ , so if  $f_*(U) \cap M_W \in M_W$ , then  $U \sqsubset W$  by Lemma 5.5.9.  $\square$

This has the following corollary, which is perhaps not immediately obvious:

**Corollary 5.5.11.** *Suppose  $U$ ,  $W$ , and  $Z$  are countably complete ultrafilters and*

$$Z \leq_{\text{RK}} U \sqsubset W$$

*Then  $Z \sqsubset W$ .*

*Proof.* By Lemma 5.5.10, for all  $U' \leq_{\text{RK}} U$ ,  $U' \sqsubset W$ . In particular (by the transitivity of the Rudin-Keisler order), for all  $U' \leq_{\text{RK}} Z$ ,  $U' \sqsubset W$ . Applying Lemma 5.5.10 again,  $Z \sqsubset W$ , as desired.  $\square$

There is also an obvious relationship in the other direction between the Rudin-Frolík order and the internal relation:

**Proposition 5.5.12.** *Suppose  $U$ ,  $W$ , and  $Z$  are countably complete ultrafilters and*

$$U \leq_{\text{RF}} W \sqsupset Z$$

*Then  $Z \sqsubset U$ .*

*Proof.* Since  $Z \sqsubset W$ , Lemma 5.5.9 implies  $s_W(Z) \in M_W$ . Since  $U \leq_{\text{RF}} W$ , there is an internal ultrapower embedding  $h : M_U \rightarrow M_W$ . We claim that  $h^{-1}[s_W(Z)] = s_U(Z)$ . Let  $X$

be the underlying set of  $Z$ . If  $A \in j_U(P(X))$ ,

$$\begin{aligned}
A \in h^{-1}[s_W(Z)] &\iff h(A) \in s_W(Z) \\
&\iff j_W^{-1}[h(A)] \in Z \\
&\iff (h \circ j_U)^{-1}[h(A)] \in Z \\
&\iff j_U^{-1}[A] \in Z \\
&\iff A \in s_U(Z)
\end{aligned}$$

Since  $h$  is definable over  $M_U$  and  $s_W(Z) \in M_W \subseteq M_U$ ,  $s_U(Z) = h^{-1}[s_W(Z)] \in M_U$ . Hence  $Z \sqsubset U$  by Lemma 5.5.9, as desired.  $\square$

The key to understanding the internal relation under UA is the following theorem, which takes advantage of the theory of pushouts and translations (Section 5.4 and Section 5.4):

**Lemma 5.5.13** (UA). *Suppose  $U$  and  $W$  are countably complete ultrafilters. Then the following are equivalent:*

- (1)  $U \sqsubset W$ .
- (2)  $(j_U(j_W), j_U \upharpoonright M_W)$  is the pushout of  $(j_W, j_U)$ .
- (3)  $t_U(W) = j_U(W)$ .
- (4)  $t_W(U) = s_W(U)$ .

If the underlying set of  $W$  is an ordinal, we can add to the list:

- (5)  $M_U \vDash j_U(W) \leq_k t_U(W)$ .

*Proof.* (1) implies (2): Since  $U \sqsubset W$ ,  $(j_U(j_W), j_U \upharpoonright M_W)$  is a minimal internal ultrapower comparison of  $(j_U, j_W)$ . Therefore by Lemma 5.4.14,  $(j_U(j_W), j_U \upharpoonright M_W)$  is the pushout of  $(j_U, j_W)$ , so (2) holds.

(2) implies (3): Let  $X$  be the underlying set of  $W$ . By the definition of  $t_U(W)$ ,  $t_U(W)$  is the  $M_U$ -ultrafilter on  $j_U(X)$  derived from  $k$  using  $h(a_W)$  where  $(k, h) : (M_U, M_W) \rightarrow N$

is the pushout of  $(j_U, j_W)$ . By (2),  $(k, h) = (j_U(j_W), j_U \upharpoonright M_W)$ , and hence  $t_U(W)$  is the  $M_U$ -ultrafilter on  $j_U(X)$  derived from  $j_U(j_W)$  using  $j_U(a_W)$ . Since  $W$  is the ultrafilter on  $X$  derived from  $j_W$  using  $a_W$ , by the elementarity of  $j_U$ ,  $j_U(W)$  is the ultrafilter on  $j_U(X)$  derived from  $j_U(j_W)$  using  $j_U(a_W)$ . This yields that  $t_U(W) = j_U(W)$ , so (3) holds.

(3) implies (4): Let  $(k, h) : (M_U, M_W) \rightarrow N$  be the pushout of  $(j_U, j_W)$ . Since  $t_U(W) = j_U(W)$ , Lemma 5.4.34 implies  $k = j_U(j_W)$  and  $h(a_W) = a_{j_U(W)} = j_U(a_W)$ .

We claim that  $h = j_U \upharpoonright M_W$ . Note that  $h \upharpoonright j_W[V] = j_U \upharpoonright j_W[V]$  since  $h \circ j_W = k \circ j_U = j_U(j_W) \circ j_U = j_U \circ j_W$ . Moreover  $h(a_W) = j_U(a_W)$ , so

$$h \upharpoonright j_W[V] \cup \{a_W\} = j_U \upharpoonright j_W[V] \cup \{a_W\}$$

Since  $M_W = H^{M_W}(j_W[V] \cup \{a_W\})$  it follows that  $h = j_U \upharpoonright M_W$ , as claimed.

Now  $t_W(U)$  is the  $M_W$ -ultrafilter derived from  $h = j_U \upharpoonright M_W$  using  $k(a_U) = j_U(j_W)(a_U)$ . By Lemma 5.5.9,  $t_W(U) = s_W(U)$ .

(4) implies (1): Since  $t_W(U) = s_W(U)$ ,  $s_W(U) \in M_W$ . By Lemma 5.5.9,  $U \sqsubset W$ .

Finally, assume that the underlying set of  $W$  is an ordinal  $\delta$ , and we will show the equivalence of (3) and (5). Clearly (3) implies (5), so let us prove the converse. Assume (5) holds. By Corollary 5.4.41,  $t_U(W) \leq_{\mathbb{k}} j_U(W)$  in  $M_U$ . Thus  $t_U(W) \leq_{\mathbb{k}} j_U(W)$  and  $j_U(W) \leq_{\mathbb{k}} t_U(W)$  in  $M_U$ , so  $j_U(W) = t_U(W)$  since the Ketonen order is antisymmetric.  $\square$

## Commuting ultrapowers and wellfoundedness

The comparison characterization of the internal relation (Lemma 5.5.6) leads to a connection between the internal relation and the seed order on pointed ultrapower embeddings, which will give us some insight into the wellfoundedness of the internal relation:

**Lemma 5.5.14.** *Suppose  $\delta$  is a limit ordinal and  $U \sqsubset W$  are countably complete ultrafilters. Then for any  $\alpha < j_U(\delta)$ ,  $(j_U, \alpha) <_S (j_W, \sup j_W[\delta])$ .*

*Proof.* Since  $U \sqsubset W$ ,  $(j_U(j_W), j_U \upharpoonright M_W)$  is an internal ultrapower comparison of  $(j_U, j_W)$  by Lemma 5.5.6. To show that  $(j_U, \alpha) <_S (j_W, \sup j_W[\delta])$ , it therefore suffices to show that



$j_U(j_W)(\alpha) < (j_U \upharpoonright M_W)(\sup j_W[\delta])$ . Note however that

$$(j_U \upharpoonright M_W)(\sup j_W[\delta]) = j_U(\sup j_W[\delta]) = \sup j_U(j_W)[j_U(\delta)] > j_U(j_W)(\alpha)$$

since  $\alpha < j_U(\delta)$ . □

As an immediate corollary, we have that the seed order extends the internal relation in many cases:

**Theorem 5.5.15.** *Suppose  $U \sqsubset W$  are ultrafilters concentrating on ordinals. Then  $U <_S W$  if and only if  $\delta_U \leq \delta_W$ .* □

We also obtain a wellfoundedness theorem for the internal relation, which becomes more interesting when one realizes that the internal relation is not in fact wellfounded.

**Theorem 5.5.16.** *Suppose  $\delta$  is an ordinal. Then the internal relation is wellfounded on the class of countably complete ultrafilters whose ultrapowers are discontinuous at  $\delta$ .*

*Proof.* Suppose towards a contradiction  $U_0 \sqsupset U_1 \sqsupset U_2 \sqsupset \dots$  are all discontinuous at  $\delta$ . For  $n < \omega$ , let  $j_n : V \rightarrow M_n$  denote the ultrapower of the universe by  $U_n$ , and let  $\delta_n = \sup j_n[\delta]$ . Since  $\delta_{n+1} < j_{n+1}(\delta)$  and  $U_{n+1} \sqsubset U_n$ , Lemma 5.5.14 implies  $(j_{n+1}, \delta_{n+1}) <_S (j_n, \delta_n)$ . Writing this a different way, we have:

$$(j_0, \delta_0) >_S (j_1, \delta_1) >_S (j_2, \delta_2) >_S \dots$$

This immediately contradicts the wellfoundedness of the Ketonen order on pointed models (Theorem 3.5.8). □

**Corollary 5.5.17.** *If  $U$  is a nonprincipal countably complete ultrafilter, then  $U \not\sqsubset U$ .* □

Unlike the Mitchell order, the internal relation is not strict. In fact, it has 2-cycles, which typically come from the phenomenon of *commuting ultrafilters*:

**Definition 5.5.18.** Suppose  $U$  and  $W$  are countably complete ultrafilters. Then  $U$  and  $W$  *commute* if  $j_U(j_W) = j_W \upharpoonright M_U$  and  $j_W(j_U) = j_U \upharpoonright M_W$ .

Clearly if  $U$  and  $W$  commute, then  $U \sqsubset W$  and  $W \sqsubset U$ . Let us provide some obvious combinatorial characterizations of commuting ultrafilters:

**Lemma 5.5.19.** *Suppose  $U$  and  $W$  are countably complete ultrafilters on sets  $X$  and  $Y$ . The following are equivalent:*

(1)  $U$  and  $W$  commute.

(2) For all  $A \subseteq X \times Y$ ,  $\forall^U x \forall^W y (x, y) \in A \iff \forall^W y \forall^U x (x, y) \in A$ .

(3) The function  $\text{flip}(x, y) = (y, x)$  satisfies  $\text{flip}_*(U \times W) = W \times U$ . □

Somewhat surprisingly, there are nontrivial examples of commuting ultrafilters:

**Theorem 5.5.20** (Kunen). *Suppose  $U$  and  $W$  are countably complete ultrafilters and  $U \in V_\kappa$  where  $\kappa = \text{CRT}(j_W)$ . Then  $j_W(j_U) = j_U \upharpoonright M_W$  and  $j_U(j_W) = j_W \upharpoonright M_U$ .*

Let us give our pet proof of Theorem 5.5.20, which uses the following somewhat surprising reformulation of commutativity:

**Proposition 5.5.21.** *Suppose  $U$  and  $W$  are countably complete ultrafilters such that  $j_W(j_U) = j_U \upharpoonright M_W$ . Then  $U$  and  $W$  commute.*

*Proof.* To show  $U$  and  $W$  commute, we must show that  $j_W \upharpoonright M_U = j_U(j_W)$ . By Lemma 5.5.6,  $(j_W \upharpoonright M_U, j_W(j_U))$  and  $(j_U(j_W), j_U \upharpoonright M_W)$  are 0-internal and 1-internal minimal comparisons of  $(j_U, j_W)$ . Since  $j_W(j_U) = j_U \upharpoonright M_W$ , we can conclude that

$$(j_W \upharpoonright M_U) \circ j_U = j_U(j_W) \circ j_U$$

In particular,  $j_W \upharpoonright M_U$  and  $j_U(j_W)$  are elementary embeddings of  $M_U$  with the same target model, which we will denote by

$$N = j_W(M_U) = j_U(j_W)(M_U) = j_U(M_W) = j_W(j_U)(M_W)$$

so since  $j_U(j_W)$  is an internal ultrapower embedding of  $M_U$ ,  $j_U(j_W)(\alpha) \leq j_W(\alpha)$  for all ordinals  $\alpha$ .

Let  $\xi$  be the least ordinal such that  $M_U = H^{M_U}(j_U[V] \cup \{\xi\})$ . We claim that

$$j_W(\xi) = j_U(j_W)(\xi)$$

By the previous paragraph, we have  $j_U(j_W)(\xi) \leq j_W(\xi)$ , so it suffices to prove the reverse inequality.

By elementarity,  $j_W(\xi)$  is the least ordinal  $\alpha$  with  $N = H^N(j_W(j_U)[M_W] \cup \{\alpha\})$ . On the other hand, since  $(j_U(j_W), j_U \upharpoonright M_W)$  is a minimal comparison of  $(j_U, j_W)$  (Lemma 5.5.6),  $N = H^N(j_U[M_W] \cup \{j_U(j_W)(\xi)\})$  (Lemma 5.4.10). Since  $j_U \upharpoonright M_W = j_W(j_U) \upharpoonright M_W$ , this yields

$$N = H^N(j_W(j_U)[M_W] \cup \{j_U(j_W)(\xi)\})$$

By the minimality of  $j_W(\xi)$ ,  $j_W(\xi) \leq j_U(j_W)(\xi)$ , as desired.

Thus  $j_U(j_W)$  and  $j_W \upharpoonright M_U$  coincide on  $j_U[V] \cup \{\xi\}$ . Since  $M_U = H^{M_U}(j_U[V] \cup \{\xi\})$ , it follows that  $j_U(j_W) = j_W \upharpoonright M_U$ , as desired.  $\square$

*Proof of Theorem 5.5.20.* It is trivial to see that  $j_W(j_U) = j_U \upharpoonright M_W$ . Hence by Proposition 5.5.21,  $U$  and  $W$  commute.  $\square$

Under UA, the only counterexamples to the strictness of the internal relation are commuting ultrafilters:

**Theorem 5.5.22** (UA). *Suppose  $U \sqsubset W$  and  $W \sqsubset U$ . Then  $U$  and  $W$  commute.*

*Proof.* Since  $U \sqsubset W$ ,  $t_U(W) = j_U(W)$ . Since  $W \sqsubset U$ ,  $t_U(W) = s_U(W)$ . Therefore  $j_U(W) = s_U(W)$ . It follows that  $j_U(j_W) = j_{j_U(W)}^{M_U} = j_{s_U(W)}^{M_U} = j_W \upharpoonright M_U$  by Lemma 5.5.9. Similarly,  $j_W(j_U) = j_U \upharpoonright M_W$ . In other words,  $U$  and  $W$  commute, as desired.  $\square$

This raises an interesting technical question:

**Question 5.5.23** (ZFC). *Suppose  $U$  and  $W$  are countably complete ultrafilters such that  $U \sqsubset W$  and  $W \sqsubset U$ . Do  $U$  and  $W$  commute?*

Theorem 5.5.16 gives some information regarding this question:

**Proposition 5.5.24.** *If  $U \sqsubset W$  and  $W \sqsubset U$ , then  $U$  and  $W$  have no common points of discontinuity.  $\square$*

The supercompactness analysis of Chapter 7 occasionally requires a partial converse to Theorem 5.5.20: the only way certain nice pairs of ultrafilters can commute is if one lies below the completeness of the other.

**Definition 5.5.25.** Suppose  $\lambda$  is a cardinal. A countably complete ultrafilter  $W$  is  $\lambda$ -*internal* if  $U \sqsubset W$  for all  $U$  such that  $\lambda_U < \lambda$ .

**Proposition 5.5.26.** *Suppose  $U$  and  $W$  are countably complete hereditarily uniform ultrafilters such that  $U$  is  $\lambda_U$ -internal and  $W$  is  $\lambda_W$ -internal. Let  $\kappa_U = \text{CRT}(j_U)$  and  $\kappa_W = \text{CRT}(j_W)$ . Then the following are equivalent:*

- (1)  $U$  and  $W$  commute.
- (2) Either  $U \in V_{\kappa_W}$  or  $W \in V_{\kappa_U}$ .

One can also state Proposition 5.5.26 avoiding the notion of hereditary uniformity: if  $U$  is  $\lambda_U$ -internal and  $W$  is  $\lambda_W$ -internal, then  $U$  and  $W$  commute if and only if  $\lambda_U < \kappa_W$  or  $\lambda_W < \kappa_U$ .

The proof of Proposition 5.5.26 requires a number of lemmas. The first allows us to approximate an arbitrary ultrapower embedding by a small ultrafilter:

**Lemma 5.5.27.** *Suppose  $j : V \rightarrow M$  is an ultrapower embedding. Then for any cardinal  $\lambda$ , there is a countably complete ultrafilter  $D$  with  $\lambda_D \leq 2^\lambda$  such that there is an elementary embedding  $k : M_D \rightarrow M$  with  $k \circ j_D = j$  and  $\text{CRT}(k) > \lambda$ .*

*Proof.* Suppose  $\gamma$  is an ordinal. We will find an ultrafilter  $D$  on  $\gamma^\gamma$  such that there is an elementary embedding  $k : M_D \rightarrow M$  with  $k \circ j_D = j$  and  $\text{CRT}(k) \geq \gamma$ . Taking  $\gamma = \lambda + 1$  proves the lemma.

Fix  $a \in M$  such that  $M = H^M(j[V] \cup \{a\})$  and  $X$  such that  $a \in j(X)$ . Fix functions  $\langle f_\alpha : \alpha < \gamma \rangle$  on  $X$  such that  $\alpha = j(f_\alpha)(a)$ . Define a function  $g : X \rightarrow \gamma^\gamma$  by letting  $g(x)$  be the function with  $g(x)(\alpha) = f_\alpha(x)$  for all  $\alpha < \gamma$ .

Let  $D$  be the ultrafilter on  $\gamma^\gamma$  derived from  $j$  using  $j(g)(a)$ . Let  $k : M_D \rightarrow M$  be the factor embedding such that  $k \circ j_D = j$  and  $k(a_D) = j(g)(a)$ .

We claim that  $\text{CRT}(k) \geq \gamma$ . It suffices to show that  $\gamma \subseteq k[M_D] = H^M(j[V] \cup \{j(g)(a)\})$ . Fix  $\alpha < \gamma$ . Then

$$\alpha = j(f_\alpha)(a) = j(f)_{j(\alpha)}(a) = j(g)(a)(j(\alpha))$$

Thus  $\alpha$  is definable in  $M$  from  $j(g)(a)$  and  $j(\alpha)$ . Thus  $\alpha \in H^M(j[V] \cup \{j(g)(a)\})$ , as desired.  $\square$

The coarseness of the bound  $2^\lambda$  actually causes a number of problems down the line. An argument due to Silver (which appears as Theorem 7.5.24) provides a major improvement in a special case, and is instrumental in our analysis of the linearity of the Mitchell order on normal fine ultrafilters under UA without GCH assumptions. Further improvements could potentially solve the problems concerning so-called isolated cardinals discussed in Section 7.5.

Using Lemma 5.5.27, we prove the following lemma, which can be seen as a version of the Kunen Inconsistency Theorem (Theorem 4.2.37) that replaces the strength requirement of that theorem with a requirement involving the internal relation:

**Lemma 5.5.28.** *Suppose  $U$  is a countably complete ultrafilter and  $\kappa$  is a strong limit cardinal. Then the following are equivalent:*

(1)  *$U$  is  $\kappa$ -internal and  $\sup j_U[\kappa] \subseteq \kappa$ .*

(2)  *$U$  is  $\kappa$ -complete.*

*Proof.* (1) implies (2). Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $U$ . We first show that  $j$  is  $<\kappa$ -supercompact. Fix  $\gamma < \kappa$ , and we will prove that  $j \upharpoonright \gamma \in M$ . Let  $\lambda = j(\gamma)$ , so  $\lambda < \kappa$  by the assumption that  $j[\kappa] \subseteq \kappa$ . By Lemma 5.5.27, one can find a countably

complete ultrafilter  $D$  with  $\lambda_D \leq 2^\lambda < \kappa$  and an elementary embedding  $k : M_D \rightarrow M$  with  $k \circ j_D = j$  and  $\text{CRT}(k) > \lambda = j(\gamma)$ . In particular  $j_D \upharpoonright \gamma = j \upharpoonright \gamma$ . Moreover since  $\lambda_D < \kappa$ ,  $D \sqsubset U$ . Therefore  $j \upharpoonright \gamma = j_D \upharpoonright \gamma \in M$ , as desired.

Now  $j$  is  $<\kappa$ -supercompact and  $j[\kappa] \subseteq \kappa$ . Since  $j$  is an ultrapower embedding, if  $\kappa$  is singular, then  $j$  is  $\kappa$ -supercompact. Therefore the Kunen Inconsistency Theorem (Theorem 4.2.37 or Theorem 4.4.32) implies  $\text{CRT}(j) \geq \kappa$ , so  $U$  is  $\kappa$ -complete.

(2) *implies* (1). Trivial. □

**Lemma 5.5.29.** *Suppose  $U$  and  $W$  are nonprincipal countably complete ultrafilters. Let  $\kappa_U = \text{CRT}(j_U)$  and  $\kappa_W = \text{CRT}(j_W)$ . Assume  $U$  is  $\kappa_W$ -internal and  $W$  is  $\kappa_U$ -internal. Then either  $j_U(\kappa_W) > \kappa_W$  or  $j_W(\kappa_U) > \kappa_U$ .*

*Proof.* Assume towards a contradiction that  $j_U(\kappa_W) = \kappa_W$  and  $j_W(\kappa_U) = \kappa_U$ . Since  $U$  is  $\kappa_W$ -internal and  $j_U[\kappa_W] \subseteq \kappa_W$ ,  $U$  is  $\kappa_W$ -complete. Therefore  $\kappa_U \geq \kappa_W$ . By symmetry,  $\kappa_W \geq \kappa_U$ . Thus  $\kappa_U = \kappa_W$ . This contradicts that  $j_U(\kappa_W) = \kappa_W$  while  $j_U(\kappa_U) > \kappa_U$  by the definition of a critical point. □

We can finally prove Proposition 5.5.26:

*Proof of Proposition 5.5.26.* (1) *implies* (2): Since  $U$  and  $W$  commute,  $j_U(\kappa_W) = \kappa_W$  and  $j_W(\kappa_U) = \kappa_U$ . By Lemma 5.5.29, either  $U$  is not  $\kappa_W$ -internal or  $W$  is not  $\kappa_U$ -internal. Therefore either  $\lambda_U < \kappa_W$  or  $\lambda_W < \kappa_U$ .

Assume first that  $\lambda_U < \kappa_W$ . Since  $U$  is hereditarily uniform, the underlying set of  $U$  has hereditary cardinality  $\lambda_U$ , and hence  $U \in V_{\kappa_W}$  since  $\kappa_W$  is inaccessible.

If instead  $\lambda_W < \kappa_U$ , then  $W \in V_{\kappa_U}$  by a similar argument.

(2) *implies* (1): Immediate from Theorem 5.5.20. □

## **$j$ on the ordinals**

In this section, we briefly survey some results that tie the structure of the internal relation under UA to the behavior of elementary embeddings on the ordinals. We only sketch most

of the proofs since the material is a bit of a detour from the main line of this dissertation.

Recall the notion of the rank of a pointed ultrapower in the Ketonen order (Definition 3.5.47): if  $\lambda$  is a cardinal, then  $\mathcal{P}_\lambda$  denotes the collection of pointed ultrapowers  $(M, \xi)$  such that  $M$  is the ultrapower by an ultrafilter  $U$  with  $\lambda_U < \lambda$  and  $\xi$  is an ordinal;  $o_\lambda(M, \xi)$  denotes the rank of  $(M, \xi)$  in the prewellorder  $(\mathcal{P}_\lambda, <_{\mathbf{k}})$ .

**Lemma 5.5.30** (UA). *Suppose  $\lambda$  is a regular cardinal,  $j : V \rightarrow M$  is an ultrapower of width less than  $\lambda$ ,  $i : M \rightarrow N$  is an ultrapower embedding such that  $i \circ j$  has width less than  $\lambda$ , and  $\xi$  is an ordinal such that  $M = H^M(j[V] \cup \{\xi\})$ . Then the following are equivalent:*

(1)  *$i$  is an internal ultrapower embedding.*

(2)  $(M, \xi) =_S (N, i(\xi))$ .

(3)  $i(o_\lambda(M, \xi)) = o_\lambda(N, i(\xi))$ .

*Proof.* The equivalence of (1) and (2) is an immediate consequence of Lemma 3.5.27. The equivalence of (2) and (3) follows from the fact that  $(M, \xi), (N, i(\xi)) \in \mathcal{P}_\lambda$  (and does not require the assumption that  $M = H^M(j[V] \cup \{\xi\})$ ).  $\square$

A surprising consequence of this is that under UA, the ultrafilters to which a countably complete ultrafilter  $U$  is internal are determined solely by the class of fixed points of  $j_U$ . Recall here that if  $W$  is a countably complete ultrafilter on an ordinal with  $\lambda_W < \lambda$ , then  $o_\lambda(W) = o_\lambda(M_W, a_W)$ .

**Theorem 5.5.31** (UA). *Suppose  $U$  is a countably complete ultrafilter and  $W$  is a countably complete ultrafilter on an ordinal. Then the following are equivalent:*

(1)  $U \sqsubset W$

(2)  $j_U(o_\lambda(W)) = o_\lambda(W)$  for all regular cardinals  $\lambda > \lambda_U, \lambda_W$ .

(3)  $j_U(o_\lambda(W)) = o_\lambda(W)$  for some regular cardinal  $\lambda > \lambda_U, \lambda_W$ .  $\square$

In particular, if  $j_U \upharpoonright \text{Ord} = j_{U'} \upharpoonright \text{Ord}$ , then  $U$  and  $U'$  are internal to exactly the same ultrafilters. The following observation shows that this is not vacuous, in that there are many nonisomorphic ultrafilters that have the same action on the ordinals:

**Theorem 5.5.32.** *Suppose  $U$  is a countably complete ultrafilter and  $Z$  is a countably complete ultrafilter of  $M_U$  such that  $j_U(Z)$  and  $j_U(j_U)(Z)$  commute in  $j_U(M_U)$ . Then  $j_Z^{M_U}$  fixes every ordinal in the range of  $j_U$ .*

The proof uses a lemma due to Kunen (which should be compared with Theorem 5.5.16):

**Lemma 5.5.33** (Kunen). *Suppose  $\alpha$  is an ordinal and  $S$  is a set of pairwise commuting countably complete ultrafilters such that  $j_W(\alpha) > \alpha$  for all  $W \in S$ . Then  $S$  is finite.*

*Proof.* Suppose towards a contradiction that  $\alpha$  is the least ordinal such that there is an infinite set  $S$  of pairwise commuting countably complete ultrafilters such that for all  $W \in S$ ,  $j_W(\alpha) > \alpha$ . Fix  $W \in S$ . Let  $T$  be a countably infinite subset of  $S$  such that  $W \notin T$ . Then in  $M_W$ ,  $j_W(T)$  is an infinite set of pairwise commuting countably complete ultrafilters. For any  $U \in T$ , since  $U$  and  $W$  commute,

$$j_W(j_U)(\alpha) = j_U(\alpha) > \alpha$$

Thus for any  $Z \in j_W(T)$ ,  $j_Z^{M_W}(\alpha) > \alpha$ . Here we use that  $T$  is countable so  $j_W(T) = j_W[T]$ .

In particular, in  $M_W$  there is an infinite set of pairwise commuting ultrafilters all of whose associated embeddings move  $\alpha$ . But by the elementarity of  $j_W$  and the definition of  $\alpha$ ,  $M_W$  satisfies that  $j_W(\alpha)$  is the least ordinal  $\xi$  such that there is an infinite set of pairwise commuting ultrafilters all of whose associated embeddings move  $\xi$ . Since  $j_W(\alpha) > \alpha$ , this is a contradiction.  $\square$

*Proof of Theorem 5.5.32.* Let  $X$  be the underlying set of  $U$ . Choose countably complete ultrafilters  $\langle Z_x : x \in X \rangle$  such that

$$Z = [\langle Z_x : x \in X \rangle]_U$$



The assumption that  $j_U(Z)$  and  $j_U(j_U)(Z)$  commute can be reformulated as follows:

$$\{(x, y) \in X \times X : Z_x \text{ and } Z_y \text{ commute}\} \in U \times U \quad (5.1)$$

Fix an ordinal  $\alpha$ . We must show that  $j_Z^{M_U}(j_U(\alpha)) = j_U(\alpha)$ . By Łoś's Theorem, it suffices to show that for almost all  $x \in X$ ,  $\{x \in X : j_{Z_x}(\alpha) = \alpha\} \in U$ . Let

$$A = \{x \in X : j_{Z_x}(\alpha) > \alpha\}$$

and assume towards a contradiction that  $A \in U$ .

Fix  $i < \omega$ . If  $n < m < i$ , the function  $f : X^i \rightarrow X^2$  defined by  $f(x_0, \dots, x_{i-1}) = (x_n, x_m)$  pushes  $U^i$  forward onto  $U^2$ , so by (5.1),

$$\{(x_0, \dots, x_{i-1}) : Z_{x_n} \text{ and } Z_{x_m} \text{ commute}\} \in U^i$$

Thus  $C_i \in U^i$  where

$$C_i = \{(x_0, \dots, x_{i-1}) \in X^i : \text{for all } n, m < i, Z_{x_n} \text{ and } Z_{x_m} \text{ commute}\}$$

Letting  $B_i = A^i \cap C_i$ , since  $A \in U$  by assumption, we have  $B_i \in U^i$ .

Since  $\langle U^i : i < \omega \rangle$  is a countably complete tower of ultrafilters with  $B_i \in U^i$  for all  $i < \omega$ , there is a sequence  $\langle x_n : n < \omega \rangle$  such that  $(x_0, \dots, x_{i-1}) \in B_i$  for all  $i < \omega$ . Now  $\{Z_{x_i} : i < \omega\}$  is an infinite set of pairwise commuting ultrafilters whose associated embeddings all move  $\alpha$ , contradicting Lemma 5.5.33. Thus our assumption was false, so in fact  $j_Z^{M_U}(j_U(\alpha)) = j_U(\alpha)$ , as desired.  $\square$

A corollary of the proof of Corollary 5.5.35 is the following more general fact about extenders.

**Definition 5.5.34.** A pair of extenders  $E$  and  $F$  *commute* if  $j_E(j_F) = j_F \upharpoonright M_E$  and  $j_F(j_E) = j_E \upharpoonright M_F$ .

**Corollary 5.5.35.** *Suppose  $E$  is an extender. Suppose  $F$  is an  $M_E$ -extender such that  $j_E(F)$  and  $j_E(j_E)(F)$  commute in  $j_E(M_E)$ . Then  $j_F^{M_E}$  fixes every ordinal in the range of  $j_E$ .*

*Sketch.* The first step is to reduce to the case that  $E$  is an ultrafilter. Let  $U$  be the ultrafilter derived from  $j_E$  using  $F$ . Let  $\bar{F} = a_U$ . A simple diagram chase shows that  $U^2$  is the ultrafilter derived from  $j_E(j_E) \circ j_E$  using  $(j_E(j_E)(F), j_E(F))$ . As a consequence of this,  $j_U(\bar{F})$  and  $j_U(j_U)(\bar{F})$  commute in  $j_U(M_U)$ . It suffices to show that  $j_{\bar{F}}^{M_U}$  fixes every ordinal in the range of  $j_U$ , since then by the elementarity of the factor embedding  $k : M_U \rightarrow M_E$ ,  $j_F^{M_E}$  fixes every ordinal in the range of  $j_E$ .

Now one generalizes the proof of Theorem 5.5.32 to the case where  $Z$  is an extender  $F$  rather than an ultrafilter. This presents no real difficulties.  $\square$

Another interesting corollary regards the relationship between the Mitchell order and pointwise domination of elementary embeddings on the ordinals.

**Theorem 5.5.36.** *Suppose  $F \triangleleft E$  are extenders. Let  $\kappa = \text{CRT}(F)$  and  $\iota = \text{WIDTH}(F)$ . Assume that the following hold:*

- $(M_F)^{<\kappa} \subseteq M_F$  and  $(M_E)^{<\iota} \subseteq M_E$ .
- $j_E(j_E)(F) \in j_E(V_\kappa)$ .

*Then for all ordinals  $\alpha$ ,  $j_F(\alpha) \leq j_E(\alpha)$  with equality if and only if  $j_E(\alpha) = \alpha$ .*

*Sketch.* Since  $(M_E)^{<\iota} \subseteq M_E$ , we have  $j_F^{M_E} = j_F \upharpoonright M_E$ . Since  $j_E(j_E)(F) \in j_E(V_\kappa)$  and  $M_F^{<\kappa} \subseteq M_F$ ,  $j_E(j_E)(F)$  and  $j_E(F)$  commute in  $j_E(M_E)$  by a generalization of the proof of Theorem 5.5.20. Therefore applying Corollary 5.5.35 yields that  $j_F(j_E(\alpha)) = j_E(\alpha)$  for all ordinals  $\alpha$ , which easily implies the conclusion of the theorem.  $\square$

The requirement that  $j_E(j_E)(F) \in j_E(V_\kappa)$  may seem ad hoc, but in fact it is necessary. For example, suppose  $\kappa < \lambda$  are cardinals,  $F$  is a  $(\kappa, \lambda)$ -extender that witnesses that  $\kappa$  is  $\lambda$ -strong, and  $U$  is a normal ultrafilter on  $\lambda$ . Trivially  $F \triangleleft U$ ,  $(M_U)^{<\kappa} \subseteq M_U$ ,  $(M_F)^{<\kappa} \subseteq M_F$ , and yet  $j_F(\kappa) > j_U(\kappa)$ .

Theorem 5.5.31 above implies that under UA, the question of whether  $U \sqsubset W$  depends only on  $j_U \upharpoonright \text{Ord}$  and  $M_W$ . The following theorem explains why:

**Theorem 5.5.37 (UA).** *Suppose  $U$  and  $W$  are countably complete ultrafilters such that  $j_U \upharpoonright \lambda \in M_W$  for all cardinals  $\lambda$ . Then  $U \sqsubset W$ .*

*Sketch.* For the proof, we need the following weak consequence of the analysis of directed systems of internal ultrapower embeddings (Lemma 3.5.41): there is an inner model  $N$  such that the following hold:<sup>2</sup>

- $j_U(N) \subseteq M_W$ .
- There is an elementary embedding  $k : M_W \rightarrow N$  that is amenable to  $M_W$ .

We claim that the embedding  $j_U \upharpoonright N$  is amenable to  $M_W$ . To see this, suppose  $X \in N$  is a transitive set. We will show  $j_U \upharpoonright X \in M_W$ . Since  $X$  is transitive, it suffices to show that  $j_U[X] \in M_W$ . Let  $\lambda$  be a cardinal and  $p : \lambda \rightarrow X$  be a surjection with  $p \in N$ . Then  $j_U(p) \in j_U(N) \subseteq M_W$ , so  $j_U[X] = j_U(p)[j_U[\lambda]] \in M_W$ , as desired.

Let  $U_* = (k \circ j_W)_*(U) \cap N$ . In other words, by the basic theory of the Rudin-Keisler order (Lemma 3.2.16),  $U_*$  is the  $N$ -ultrafilter derived from  $j_U \upharpoonright N$  using  $j_U(k \circ j_W)(a_U)$ . Since  $j_U \upharpoonright N$  is amenable to  $M_W$ , it follows that  $U_* \in M_W$ . Since  $k$  is amenable to  $M_W$ ,  $k^{-1}[U_*] \in M_W$ , but

$$k^{-1}[U_*] = k^{-1}[(k \circ j_W)_*(U) \cap N] = (j_W)_*(U) \cap M_W = s_W(U)$$

Thus  $s_W(U) \in M_W$ , so  $U \sqsubset W$  by Lemma 5.5.9. □

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<sup>2</sup>The inner model  $N$  can be taken to be the direct limit  $M_\lambda$  of all ultrapowers of width less than  $\lambda$  for some sufficiently large regular cardinal  $\lambda$ . Then by Lemma 3.5.41,  $N = j_U(N) = j_W(N) \subseteq M_W$ , as desired. The embedding  $k$  is obtained by setting  $k = j_W(j_\lambda) = j_{M_W, \lambda}$  (by Lemma 3.5.41 again). Then  $k$  is an elementary embedding from  $j_W(V) = M_W$  to  $j_W(N) = N$ , as desired. Note that we cannot assume that  $k$  is an ultrapower embedding.

# Chapter 6

## Ordinal Definability and Cardinal Arithmetic under UA

### 6.1 Introduction

#### The universe above a supercompact cardinal

In this short chapter, we exposit two results that show that something remarkable happens when UA is combined with very large cardinal hypotheses: instead of simply proving structural results for countably complete ultrafilters, the axiom now begins to resolve major questions independent from the usual axioms of set theory.

Let us describe the main results of this section. Since UA is preserved by forcing to add a Cohen real, UA does not imply  $V = \text{HOD}$ , no matter what large cardinals one assumes in addition to UA. But it turns out it is possible to prove that forcing is the only obstruction:

**Theorem 6.2.8** (UA). *Assume there is a supercompact cardinal. Then  $V$  is a generic extension of  $\text{HOD}$ .*

Similarly, UA is preserving by forcing to change the value of the continuum, so UA does not imply the Continuum Hypothesis. But UA for sufficiently large cardinals  $\lambda$ , UA implies

$$2^\lambda = \lambda^+:$$

**Theorem 6.1.1 (UA).** *Assume  $\kappa$  is supercompact. Then for all cardinals  $\lambda \geq \kappa$ ,  $2^\lambda = \lambda^+$ .*

It seems that above a supercompact cardinal, UA imposes incredible structure on the universe of sets. This is explored further in Chapter 7 and Chapter 8.

## Outline of Chapter 6

We now outline the rest of the chapter.

**Section 6.2.** We prove the results on ordinal definability under UA and large cardinals. This is quite straightforward, but many open questions remain. For example, we prove that if  $\kappa$  is supercompact and UA holds, then  $V$  is a generic extension of HOD. How small is the forcing? The best upper bound we know is  $\kappa^{++}$ , which comes from Section 6.3 below.

**Section 6.3.** We prove the results on GCH under UA and large cardinals. We begin by discussing related results in ZFC, especially Solovay's theorem on SCH above a supercompact cardinal. In Section 6.3, we prove a result regarding the Mitchell order and supercompactness that shows that under UA, if  $D$  and  $U$  are ultrafilters with  $\lambda_D$  below the supercompactness of  $U$ , then  $D \triangleleft U$ . This is immediate given GCH, but proving this using UA alone is a little bit subtle. In Section 6.3, we use this result to conclude that GCH holds above a supercompact.

## 6.2 Ordinal definability

Theorem 6.2.8 is quite easy given what we have shown so far, ultimately relying on the following simple fact:

**Proposition 6.2.1 (UA).** *Every countably complete ultrafilter on an ordinal is ordinal definable.*

*Proof.* Suppose  $\delta$  is an ordinal. Then the set  $\mathcal{B}(\delta)$  of all countably complete ultrafilters on  $\delta$  is wellordered by the Ketonen order. Thus every element of  $\mathcal{B}(\delta)$  is ordinal definable from its rank in the Ketonen order.  $\square$

**Corollary 6.2.2 (UA).** *For any set of ordinals  $X$  and any ultrapower embedding  $j : V \rightarrow M$ :*

(1)  $j(\text{OD}_X) \subseteq \text{OD}_X$ .

(2)  $j(\text{HOD}_X) \subseteq \text{HOD}_X$ .

(3) *For any  $Y \in \text{HOD}_X$ ,  $j \upharpoonright Y \in \text{HOD}_X$ .*

*Proof.* We first prove (1). We have  $j(\text{OD}_X) = \text{OD}_{j(X)}^M$ . Fix a countably complete ultrafilter  $U$  on an ordinal such that  $j = j_U$ . Then since  $M$  is definable from  $U$  and  $U \in \text{OD}$  by Proposition 6.2.1,  $\text{OD}_{j(X)}^M \subseteq \text{OD}_{j(X)}$ . Moreover  $j(X) = j_U(X)$  is definable from  $X$  and  $U$ , so  $j(X) \in \text{OD}_X$ . Hence  $\text{OD}_{j(X)}^M \subseteq \text{OD}_{j(X)} \subseteq \text{OD}_X$ .

For (2), note that  $j(\text{HOD}_X)$  is the class of sets that are hereditarily  $j(\text{OD}_X)$ , and this is contained in the class of sets that are hereditarily  $\text{OD}_X$  by (1).

For (3), clearly  $j \upharpoonright Y \in \text{OD}_Y \subseteq \text{OD}_X$ . But moreover by (2),  $j \upharpoonright Y \subseteq \text{HOD}_X$ . Therefore  $j \upharpoonright Y \in \text{HOD}_X$ .  $\square$

The following lemma should be compared with the theorem of Shelah that if  $\lambda$  is a singular strong limit cardinal of uncountable cofinality, then for any  $X$  such that  $P(\alpha) \subseteq \text{HOD}_X$  for all  $\alpha < \lambda$ , in fact  $P(\lambda) \subseteq \text{HOD}_X$ .

**Lemma 6.2.3 (UA).** *Suppose  $\kappa$  is  $\lambda$ -supercompact and  $X \subseteq \kappa$  is such that  $V_\kappa \subseteq \text{HOD}_X$ . Then  $P(\lambda) \subseteq \text{HOD}_X$ .*

*Proof.* Fix a  $\lambda$ -supercompact ultrapower embedding  $j : V \rightarrow M$  such that  $\text{CRT}(j) = \kappa$  and  $j(\kappa) > \lambda$ . Then

$$P(\lambda) \subseteq j(V_\kappa) \subseteq j(\text{HOD}_X) \subseteq \text{HOD}_X$$

The final inclusion follows from Corollary 6.2.2.  $\square$

**Theorem 6.2.4 (UA).** *Suppose  $\kappa$  is supercompact. Then  $V = \text{HOD}_X$  for some  $X \subseteq \kappa$ .*

*Proof.* Fix  $X \subseteq \kappa$  such that  $V_\kappa \subseteq \text{HOD}_X$ .<sup>1</sup> Since  $\kappa$  is supercompact, Lemma 6.2.3 implies that for all  $\lambda \geq \kappa$ ,  $P(\lambda) \subseteq \text{HOD}_X$ , and therefore  $V = \text{HOD}_X$ .  $\square$

To connect this to generic extensions of HOD, we use Vopěnka's Theorem.

**Definition 6.2.5.** Suppose  $X$  is a set such that  $X \cup \{X\} \subseteq \text{OD}$ . The *OD-cardinality* of  $X$ , denoted  $|X|^{\text{OD}}$ , is the least ordinal  $\lambda$  such that there is an OD bijection between  $\lambda$  and  $X$ .

The OD cardinality of  $X$  is defined for all  $X$  with  $X \cup \{X\} \subseteq \text{OD}$ . It is always a HOD-cardinal. In fact OD cardinality satisfies all the usual properties of cardinality; for example,  $|X|^{\text{OD}}$  is the least ordinal that ordinal definably surjects onto  $X$  and the least ordinal into which  $X$  ordinal definably injects.

**Definition 6.2.6.** Suppose  $\kappa$  is an ordinal. Let  $\mathbb{A}_\kappa$  be the Boolean algebra  $P(P(\kappa)) \cap \text{OD}$  and let  $\lambda = |\mathbb{A}_\kappa|^{\text{OD}}$ . Fix an OD bijection  $\pi : \lambda \rightarrow \mathbb{A}$ . Then  $\mathbb{V}_\kappa$  is the Boolean algebra on  $\lambda$  given by pulling back the operations on  $\mathbb{A}$  under  $\pi$ .

Note that  $\mathbb{V}_\kappa \in \text{HOD}$ . The Boolean algebra  $\mathbb{V}_\kappa$  is called the *Vopěnka algebra* at  $\kappa$ .

**Theorem 6.2.7 (Vopěnka).** *If  $\kappa$  is an ordinal, then  $\mathbb{V}_\kappa$  is a complete Boolean algebra and for any  $X \subseteq \kappa$ , there is a HOD-generic ultrafilter  $G \subseteq \mathbb{V}_\kappa$  such that  $\text{HOD}_X \subseteq \text{HOD}[G]$ .*  $\square$

This yields a proof of our main theorem on HOD:

**Theorem 6.2.8 (UA).** *Assume there is a supercompact cardinal. Then  $V$  is a generic extension of HOD.*

*Proof.* Let  $\kappa$  be the least supercompact cardinal. By Theorem 6.2.4,  $V = \text{HOD}_X$  for some  $X \subseteq \kappa$ , so by Theorem 6.2.7,  $V = \text{HOD}[G]$  for some generic  $G \subseteq \mathbb{V}_\kappa$ .  $\square$

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<sup>1</sup>To obtain such a set  $X$ , let  $E$  be a binary relation on  $\kappa$  such that  $(V_\kappa, \in) \cong (\kappa, E)$  using the fact that  $|V_\kappa| = \kappa$ . Code  $E$  as a subset of  $\kappa$  using a pairing function  $\kappa \rightarrow \kappa \times \kappa$ .

**Question 6.2.9 (UA).** Let  $\kappa$  be the least supercompact cardinal.

- Is  $V = \text{HOD}[X]$  for some  $X \subseteq \kappa$ ?
- Is  $V = \text{HOD}[G]$  for  $G \subseteq \kappa$  generic for a partial order  $\mathbb{P} \in \text{HOD}$  such that  $|\mathbb{P}| \leq \kappa$ ?  
What about a  $\kappa$ -cc Boolean algebra?
- Is  $V = \text{HOD}_{V_\kappa}$ ?

Assuming UA, one can actually calculate the cardinality of  $V_\kappa$  precisely. For example, in the next section, we will obtain:

**Theorem 6.3.24 (UA).** *If  $\kappa$  is  $\kappa^{++}$ -supercompact then  $|V_\kappa|^{\text{HOD}} = \kappa^{++}$ .*

Thus if  $\kappa$  is supercompact, then  $V = \text{HOD}[A]$  for some  $A \subseteq \kappa^{++}$ . As an immediate consequence, we have that HOD is very close to  $V$ :

**Corollary 6.2.10 (UA).** *Let  $\kappa$  be the least supercompact cardinal. Then for all cardinals  $\lambda \geq \kappa^{++}$ :*

(1)  $\lambda^{+\text{HOD}} = \lambda^+$ .

(2)  $(2^\lambda)^{\text{HOD}} = 2^\lambda$ .

Moreover if  $\delta > \kappa^{++}$  is regular, then HOD is correct about stationary subsets of  $\delta$ . □

Of course by the Levy-Solovay Theorem [30], HOD is also close to  $V$  in the sense that it absorbs large cardinals above  $\kappa$ . The structure of HOD at  $\kappa$  itself becomes a key question:

**Question 6.2.11 (UA).** Assume  $\kappa$  is supercompact. Is  $\kappa^{+\text{HOD}} = \kappa^+$ ?

Another question in this vein is whether  $\kappa$  is supercompact in HOD. Here the answer turns out to be yes:

**Definition 6.2.12.** If  $N$  is an inner model and  $S$  is a set, we say  $S$  is *amenable to  $N$*  if  $S \cap N \in N$ .



**Definition 6.2.13.** Suppose  $\kappa$  is supercompact. An inner model  $N$  is a *weak extender model at  $\kappa$*  if for all ordinals  $\lambda \geq \kappa$ , there is a normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\lambda)$  that concentrates on  $N$  and is amenable to  $N$ .

**Lemma 6.2.14.** *Suppose  $N$  is an inner model and  $\kappa$  is supercompact. Then the following are equivalent:*

- (1)  $N$  is a weak extender model at  $\kappa$ .
- (2) For arbitrarily large  $\delta \geq \kappa$ , there is a normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\delta)$  that concentrates on  $N$  and is amenable to  $N$ .

*Proof.* (1) implies (2): Trivial.

(2) implies (1): Fix  $\lambda \geq \kappa$ . We will show that there is a normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\lambda)$  that concentrates on  $N$  and is amenable to  $N$ . By (2), there is some  $\delta \geq \lambda$  such that there is a normal fine  $\kappa$ -complete ultrafilter  $\mathcal{U}$  on  $P_\kappa(\delta)$  that concentrates on  $N$  and is amenable to  $N$ . Let  $\mathcal{W} = f_*(\mathcal{U})$  where  $f : P_\kappa(\delta) \rightarrow P_\kappa(\lambda)$  is defined by  $f(\sigma) = \sigma \cap \lambda$ . Easily  $\mathcal{W}$  is a normal fine ultrafilter. Moreover  $f^{-1}[P_\kappa(\lambda) \cap M] = P_\kappa(\delta) \cap M \in \mathcal{U}$ , so  $P_\kappa(\lambda) \cap M \in \mathcal{W}$ . Thus  $\mathcal{W}$  concentrates on  $M$ . Finally, letting  $g = f \upharpoonright M$ , clearly  $g \in M$  and hence  $\mathcal{W} \cap M = f_*(\mathcal{U}) \cap M = g_*(\mathcal{U} \cap M) \in M$  since  $\mathcal{U} \cap M \in M$ . Thus  $\mathcal{W}$  is amenable to  $M$ . □

**Theorem 6.2.15 (UA).** *Let  $\kappa$  be the least supercompact cardinal. Then HOD is a weak extender model at  $\kappa$ .*

*Proof.* First note that every normal fine ultrafilter on an ordinal definable set is ordinal definable. We will prove this using the fact that isomorphic normal fine ultrafilters on the same set are equal (Lemma 4.4.11). Suppose  $\mathcal{U}$  is a normal fine ultrafilter on  $Y \in \text{OD}$ , and let  $U$  be a countably complete ultrafilter on an ordinal isomorphic to  $\mathcal{U}$ ; then by Lemma 4.4.11,  $\mathcal{U}$  is the *unique* normal fine ultrafilter on  $Y$  isomorphic to  $U$ , and hence  $\mathcal{U} \in \text{OD}_{Z,U} = \text{OD}_U = \text{OD}$ , with the final equality coming from Proposition 6.2.1.

In particular, for all  $\lambda \geq \kappa$ , every normal fine ultrafilter  $\mathcal{U}$  on  $P_\kappa(\lambda)$  is amenable to HOD. The issue is to show that there are such  $\mathcal{U}$  concentrating on HOD.

Fix a regular cardinal  $\delta > \kappa^{++}$ . Then by Corollary 6.2.10, HOD is correct about stationary subsets of  $\delta$ . Let  $\langle S_\alpha : \alpha < \delta \rangle \in \text{HOD}$  be a partition of  $S_\omega^\delta$  into stationary subsets. Let  $j : V \rightarrow M$  be an elementary embedding with critical point  $\kappa$  such that  $j(\kappa) > \delta$  and  $j[\delta] \in M$ . We claim that  $j[\delta] \in \text{HOD}^M$ .

By Corollary 4.4.31,

$$j[\delta] = \{\alpha < j(\delta) : M \models j(S)_\alpha \text{ is stationary in } \sup j[\delta]\}$$

Thus

$$j[\delta] \in \text{HOD}_{j(\langle S_\alpha : \alpha < \delta \rangle)}^M$$

But since  $\langle S_\alpha : \alpha < \delta \rangle$  is in HOD,  $j(\langle S_\alpha : \alpha < \delta \rangle) \in \text{HOD}^M$ . Thus  $j[\delta] \in \text{HOD}^M$ .

Let  $\mathcal{U}$  be the ultrafilter on  $P_\kappa(\delta)$  derived from  $j$  using  $j[\delta]$ . Since  $j[\delta] \in \text{HOD}^M = j(\text{HOD})$ ,  $\mathcal{U}$  concentrates on HOD by Łoś's Theorem. Thus  $\mathcal{U}$  is a normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\delta)$  that concentrates on HOD and is amenable to HOD.

This shows that for unboundedly many cardinals  $\delta$ , there is a normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\delta)$  that concentrates on HOD and is amenable to HOD. Therefore by Lemma 6.2.14, HOD is a weak extender model at  $\kappa$ .  $\square$

As a consequence of theorems of Woodin [10], this implies that a version of Jensen's Covering Lemma is true for HOD:

**Corollary 6.2.16 (UA).** *Any set  $A \subseteq \text{HOD}$  is contained in a set  $B \in \text{HOD}$  such that  $|B| \leq |A| + \gamma$  for some  $\gamma < \kappa$ .*  $\square$

We omit the proof. Of course one has a much stronger covering results above  $\kappa^{+++}$  as a consequence of Theorem 6.3.24.

## 6.3 The Generalized Continuum Hypothesis

### Introduction

In this section, we prove that GCH holds above the least supercompact assuming UA. We actually prove a more local version of this theorem. We stress that proving this local version requires some far from obvious tricks that are not actually necessary for the global result. We need the local result at various points in Chapter 7 and Chapter 8.

### Two theorems of Solovay

Let us begin by explaining the intuition that led to the expectation that UA might imply the eventual Generalized Continuum Hypothesis. This begins with two remarkable theorems of Solovay. First, of course, is his theorem on the Singular Cardinals Hypothesis:

**Theorem 6.3.1** (Solovay). *Suppose  $\kappa$  and  $\delta$  are cardinals with  $\text{cf}(\delta) \geq \kappa$ . Suppose  $\kappa$  is  $\delta$ -strongly compact. Then  $\delta^{<\kappa} = \delta$ . □*

We will give a proof in Corollary 6.3.2. As a corollary, the Singular Cardinals Hypothesis holds above a strongly compact cardinal:

**Corollary 6.3.2.** *Suppose  $\kappa \leq \lambda$  are cardinals,  $\lambda$  is a strong limit singular cardinal, and  $\kappa$  is  $\lambda$ -strongly compact. Then  $2^\lambda = \lambda^+$ .*

*Proof.* Note that  $2^\lambda = \lambda^{\text{cf}(\lambda)}$  since  $\lambda$  is a strong limit cardinal and in general  $2^\lambda = (2^{<\lambda})^{\text{cf}(\lambda)}$ .

First assume  $\text{cf}(\lambda) < \kappa$ . Then

$$2^\lambda = \lambda^{\text{cf}(\lambda)} \leq \lambda^{<\kappa} \leq (\lambda^+)^{<\kappa} = \lambda^+$$

by Theorem 6.3.1.

Assume instead that  $\kappa \leq \text{cf}(\lambda)$ . Assume by induction that  $2^\gamma = \gamma^+$  for all strong limit singular cardinals in the interval  $(\kappa, \lambda)$ . Let  $\iota = \text{cf}(\lambda)$ . Let  $j : V \rightarrow M$  be an elementary

embedding such that  $\text{cf}^M(\sup j[\iota]) < j(\kappa)$ , which exists since  $\kappa$  is  $\text{cf}(\lambda)$ -strongly compact. Then  $\lambda_* = \sup j[\lambda]$  is a strong limit singular cardinal of  $M$  and

$$\text{cf}^M(\lambda_*) = \text{cf}^M(\sup j[\iota]) < j(\kappa)$$

Therefore  $(2^{\lambda_*})^M = \lambda_*^{+M}$ . But there is an injection from  $P(\lambda)$  to  $P^M(\lambda_*)$ , namely the map  $A \mapsto j(A) \cap \lambda_*$ . Therefore

$$2^\lambda \leq |\lambda_*^{+M}|$$

By the usual computations of cardinalities of ultrapowers (Lemma 3.5.33), the fact that  $\lambda_*$  is a strong limit implies  $\lambda_* = \lambda$ . Thus  $2^\lambda \leq \lambda_*^{+M} = \lambda^{+M} \leq \lambda^+$ , as desired.  $\square$

The second of Solovay's theorems regards the number of normal fine ultrafilters generated by supercompactness assumptions:

**Theorem 6.3.3** (Solovay). *Suppose  $\kappa$  and  $\delta$  are cardinals,  $\kappa \leq \text{cf}(\delta)$ , and  $\kappa$  is  $2^\delta$ -supercompact. Then for all  $A \subseteq P(\delta)$ , there is a normal fine  $\kappa$ -complete ultrafilter  $\mathcal{U}$  on  $P_\kappa(\delta)$  such that  $A \in M_{\mathcal{U}}$ .*

Since this argument will be used repeatedly, it is worth working in a slightly more general context.

**Lemma 6.3.4.** *Suppose  $\lambda$  is a cardinal and  $j : V \rightarrow M$  is a  $\lambda$ -supercompact elementary embedding. Suppose  $Y \subseteq P(\lambda)$  is a set such that  $j[\lambda] \in j(Y)$ . Let  $\mathcal{U}$  be the normal fine ultrafilter on  $Y$  derived from  $j$  using  $j[\lambda]$ . Assume  $Y \in M_{\mathcal{U}}$  and  $\mathcal{U} \in M$ . Then both  $M_{\mathcal{U}}$  and  $M$  satisfy the following statement: for any  $A \subseteq P(\lambda)$ , there is a normal fine ultrafilter  $\mathcal{W}$  on  $Y$  such that  $A \in M_{\mathcal{W}}$ .*

*Proof.* Let  $k : M_{\mathcal{U}} \rightarrow M$  be the factor embedding with  $k(a_{\mathcal{U}}) = j[\lambda]$  and  $k \circ j_{\mathcal{U}} = j$ . Then by Lemma 4.4.10,  $\text{CRT}(k) > \lambda$  if  $k$  is nontrivial. In particular,  $k(\lambda) = \lambda$  and  $k(Y) = Y$ . Therefore by the elementarity of  $k$ , if  $M_{\mathcal{U}}$  satisfies the statement that for any  $A \subseteq P(\lambda)$ , there is a normal fine ultrafilter  $\mathcal{W}$  on  $Y$  such that  $A \in M_{\mathcal{W}}$ , then so does  $M$ . Therefore it suffices to show that this statement is true in  $M_{\mathcal{U}}$ .

Let  $D$  be the ultrafilter derived from  $j$  using  $\langle \mathcal{U}, j[\lambda] \rangle$ . Note that  $j_D : V \rightarrow M_D$  is a  $\lambda$ -supercompact elementary embedding,  $\mathcal{U}$  is the normal fine ultrafilter on  $Y$  derived from  $j_D$  using  $j[\lambda]$ , and  $\mathcal{U} \in M_D$ . By replacing  $j$  with  $j_D$ , we may therefore assume that  $j$  is an ultrapower embedding. In particular, by Corollary 4.2.21,  $M^\lambda \subseteq M$ .

We claim that  $P(P(\lambda)) \cap (M_{\mathcal{U}})^M = P(P(\lambda)) \cap M_{\mathcal{U}}$ . (This is a consequence of Lemma 4.2.27, but we give a proof here.) Since  $M$  is closed under  $\lambda$ -sequences,  $(M_{\mathcal{U}})^M = j_{\mathcal{U}}(M)$  by Proposition 4.2.29. In particular,  $(M_{\mathcal{U}})^M \subseteq M_{\mathcal{U}}$ , so  $P(P(\lambda)) \cap (M_{\mathcal{U}})^M \subseteq P(P(\lambda)) \cap M_{\mathcal{U}}$ . We now show the reverse inclusion. By Lemma 4.2.38, there is an inaccessible cardinal  $\kappa \leq \lambda$  such that  $j(\kappa) > \lambda$ . Since  $j$  is  $\lambda$ -supercompact,  $j$  is  $\lambda$ -strong, and so in particular  $V_\kappa \subseteq M$ . Therefore

$$P(P(\lambda)) \cap (M_{\mathcal{U}})^M \subseteq V_{j_{\mathcal{U}}(\kappa)} \cap M_{\mathcal{U}} = j_{\mathcal{U}}(V_\kappa) \subseteq j_{\mathcal{U}}(M) = (M_{\mathcal{U}})^M$$

as desired.

Suppose  $A \subseteq P(\lambda)$  and  $A \in M_{\mathcal{U}}$ . Note that  $k(A) = A \in P(P(\lambda)) \cap M_{\mathcal{U}} \subseteq (M_{\mathcal{U}})^M$ . Thus  $M$  satisfies that  $k(A)$  belongs to  $M_{\mathcal{W}}$  for some normal fine ultrafilter on  $k(Y)$ . (Namely, take  $\mathcal{W} = \mathcal{U}$ .) By the elementarity of  $k$ , it follows that  $M_{\mathcal{U}}$  satisfies that  $A$  belongs to  $M_{\mathcal{W}}$  for some normal fine ultrafilter on  $Y$ .

This shows that  $M_{\mathcal{U}}$  satisfies the statement that for any  $A \subseteq P(\lambda)$ , there is a normal fine ultrafilter  $\mathcal{W}$  on  $Y$  such that  $A \in M_{\mathcal{W}}$ , completing the proof.  $\square$

*Proof of Theorem 6.3.3.* By our large cardinal assumption, there is an elementary embedding  $j : V \rightarrow M$  such that the following hold:

- $\text{CRT}(j) = \kappa$  and  $j(\kappa) > \delta$ .
- $j$  is  $\delta$ -supercompact.
- $j$  is  $2^\delta$ -strong.

Let  $\mathcal{D}$  be the normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\delta)$  derived from  $j$  using  $j[\delta]$ , and let  $k : M_{\mathcal{D}} \rightarrow M$  be the factor embedding.

**Claim 1.**  $\mathcal{D} \in M$ .

*Proof.* Since  $j$  is  $2^\delta$ -strong,  $H_{2^\delta} \subseteq M$ . Since  $|P_\kappa(\delta)| = \delta$  by Corollary 6.3.2,  $P(P_\kappa(\delta)) \in H_{2^\delta}$ . Therefore since  $\mathcal{D} \in P(P_\kappa(\delta))$ ,  $\mathcal{D} \in M$ .  $\square$

Therefore applying Lemma 6.3.4 yields that  $M$  satisfies that for all  $A \subseteq P(\delta)$ , there is a normal fine ultrafilter  $\mathcal{W}$  on  $P_\kappa(\delta)$  such that  $A \in M_{\mathcal{W}}$ . But every  $A \subseteq P(\delta)$  belongs to  $M$ . Moreover, if  $M$  satisfies that  $\mathcal{W}$  is a normal fine ultrafilter on  $P_\kappa(\delta)$ , then  $\mathcal{W}$  actually is a normal fine ultrafilter on  $P_\kappa(\delta)$  and moreover  $P(P(\delta)) \cap (M_{\mathcal{W}})^M = P(P(\delta)) \cap M_{\mathcal{W}}$ . Thus we can conclude that  $V$  satisfies that every  $A \subseteq P(\delta)$  belongs to  $M_{\mathcal{W}}$  for some normal fine ultrafilter  $\mathcal{W}$  on  $P_\kappa(\delta)$ . This proves the theorem.  $\square$

As a corollary, Solovay observed that instances of GCH follow from the linearity of the Mitchell order:

**Theorem 6.3.5** (Solovay). *Suppose  $\kappa$  is  $2^\kappa$ -supercompact and the set of normal ultrafilters on  $\kappa$  is linearly ordered by the Mitchell order. Then  $2^{2^\kappa} = (2^\kappa)^+$ .*

*Proof.* First note that for any normal ultrafilter  $U$  on  $\kappa$ ,

$$|P(P(\kappa)) \cap M_U| \leq |j_U(V_\kappa)| \leq |(V_\kappa)^\kappa| = 2^\kappa$$

The first inequality follows from the inclusion  $P(P(\kappa)) \cap M_U \subseteq j_U(V_\kappa)$ , and the second from the existence of a surjection  $\pi : (V_\kappa)^\kappa \rightarrow j_U(V_\kappa)$ , defined by  $\pi(f) = j_U(f)(\kappa)$ .

Let  $\mathcal{N}$  be the set of normal ultrafilters on  $\kappa$ . The result will follow from counting  $\mathcal{N}$  in two different ways.

First, note that a normal ultrafilter  $U$  has at most  $2^\kappa$  predecessors in the Mitchell order. We are assuming the Mitchell order on  $\mathcal{N}$  is a wellorder, and therefore the ordertype of  $(\mathcal{N}, \triangleleft)$  is at most  $(2^\kappa)^+$ : any proper initial segment of  $(\mathcal{N}, \triangleleft)$  has cardinality  $2^\kappa$ . In particular  $|\mathcal{N}| \leq (2^\kappa)^+$ .

Second, note that

$$P(P(\kappa)) = \bigcup_{U \in \mathcal{N}} P(P(\kappa)) \cap M_U$$

Thus

$$2^{2^\kappa} = |P(P(\kappa))| = |\mathcal{N}| \cdot \sup_{U \in \mathcal{N}} |P(P(\kappa)) \cap M_U| = |\mathcal{N}| \cdot 2^\kappa$$

It follows that  $|\mathcal{N}| = 2^{2^\kappa}$ .

Thus we have shown  $2^{2^\kappa} = |\mathcal{N}| \leq (2^\kappa)^+$ . It follows that  $2^{2^\kappa} = (2^\kappa)^+$ , as desired.  $\square$

More generally, and by exactly the same argument, one can show:

**Proposition 6.3.6** (UA). *Assume  $\lambda$  is a cardinal such that  $2^{<\lambda} = \lambda$  and every  $A \subseteq P(\lambda)$  belongs to  $M_{\mathcal{W}}$  for some normal fine ultrafilter  $\mathcal{W}$  on  $P_{\text{bd}}(\lambda)$ . Then  $2^{2^\lambda} = (2^\lambda)^+$ .*

*Proof.* Recall that  $\mathcal{N}_\lambda$  denotes the set of normal fine ultrafilters on  $P_{\text{bd}}(\lambda)$ .

We claim that for any  $\mathcal{W} \in \mathcal{N}_\lambda$ ,  $P(P(\lambda)) \cap M_{\mathcal{W}}$  has cardinality at most  $2^\lambda$ . By Theorem 4.4.37,  $\lambda_{\mathcal{W}} = \lambda$ . By Lemma 4.2.38, there is an inaccessible cardinal  $\kappa \leq \lambda$  such that  $j_{\mathcal{W}}(\kappa) > \lambda$ . Thus  $P(P(\lambda)) \cap M_{\mathcal{W}} \subseteq j_{\mathcal{W}}(V_\kappa)$ . But  $|j_{\mathcal{W}}(V_\kappa)| \leq |V_\kappa|^{\lambda_{\mathcal{W}}} = \kappa^\lambda = 2^\lambda$ . In particular  $|P(P(\lambda)) \cap M_{\mathcal{W}}| \leq 2^\lambda$ .

This bound has two consequences.

First, it follows that any  $\mathcal{W} \in \mathcal{N}_\lambda$  has at most  $2^\lambda$  predecessors in the Mitchell order. This is because if  $\mathcal{U} \triangleleft \mathcal{W}$ , then  $\mathcal{U} \in P(P(P_{\text{bd}}(\lambda))) \cap M_{\mathcal{W}}$ , and

$$|P(P(P_{\text{bd}}(\lambda))) \cap M_{\mathcal{W}}| = |P(P(\lambda)) \cap M_{\mathcal{W}}|$$

since  $|P_{\text{bd}}(\lambda)|^{M_{\mathcal{W}}} = (2^{<\lambda})^{M_{\mathcal{W}}} = \lambda$ . (One does not actually need to use  $2^{<\lambda} = \lambda$  here, but it is convenient.) Hence  $(\mathcal{N}_\lambda, \triangleleft)$  is a wellfounded partial order of rank at most  $(2^\lambda)^+$ . Since  $2^{<\lambda} = \lambda$ , Theorem 4.4.2 implies that  $(\mathcal{N}_\lambda, \triangleleft)$  is a wellorder, and hence  $|\mathcal{N}_\lambda| \leq (2^\lambda)^+$ .

Second, it follows that  $|\mathcal{N}_\lambda| = 2^{2^\lambda}$ : by our assumption that every  $A \subseteq P(\lambda)$  belongs to  $M_{\mathcal{W}}$  for some  $\mathcal{W} \in \mathcal{N}_\lambda$ ,

$$P(P(\lambda)) = \bigcup_{\mathcal{W} \in \mathcal{N}_\lambda} P(P(\lambda)) \cap M_{\mathcal{W}}$$

Thus

$$2^{2^\lambda} = |P(P(\lambda))| = |\mathcal{N}_\lambda| \cdot \sup_{\mathcal{W} \in \mathcal{N}_\lambda} |P(P(\lambda)) \cap M_{\mathcal{W}}| = |\mathcal{N}_\lambda| \cdot 2^\lambda = |\mathcal{N}_\lambda|$$

Putting everything together,  $2^{2^\lambda} = |\mathcal{N}_\lambda| \leq (2^\lambda)^+$ , which proves the theorem.  $\square$

Regarding this lemma, a much more complicated argument (Theorem 7.4.28) shows that under UA, a set  $X$  carries at most  $(2^{|X|})^+$  countably complete ultrafilters.

Let us mention a little fact, proved very early on in this work, that gave the first indication that GCH above a supercompact cardinal might be provable from UA:

**Proposition 6.3.7** (UA). *Suppose  $\kappa$  is supercompact. Let  $\lambda = \beth_\kappa(\kappa)$ . Then  $2^\delta = \delta^+$  for all cardinals  $\delta \in [\lambda, \lambda^{+\omega}]$ .*

*Proof.* Since  $\lambda$  is a singular strong limit cardinal, Corollary 6.3.2 implies  $2^\lambda = \lambda^+$ . Since  $2^{<\lambda} = \lambda$ , Proposition 6.3.6 implies  $2^{2^\lambda} = (2^\lambda)^+$ . In other words,  $2^{(\lambda^+)} = \lambda^{++}$ . Since  $2^{<\lambda^+} = 2^\lambda = \lambda^+$ , Proposition 6.3.6 implies  $2^{2^{(\lambda^+)}} = (2^{(\lambda^+)})^+$ . In other words,  $2^{(\lambda^{++})} = \lambda^{+++}$ . Continuing this way yields the result for cardinals  $\delta$  such that  $\lambda \leq \delta < \lambda^{+\omega}$ . Then  $\lambda^{+\omega}$  is a strong limit cardinal, so  $2^{(\lambda^{+\omega})} = \lambda^{+\omega+1}$  by Corollary 6.3.2.  $\square$

This proof breaks down completely at  $\lambda^{+\omega+1}$ , and it gives no hint of whether  $2^\kappa = \kappa^+$  should hold when  $\kappa$  is supercompact. But the fact that one gets GCH at  $\omega + 1$  cardinals in a row strongly suggests that one should be able to prove the eventual GCH. To handle the case  $\delta = \lambda^{+\omega+1}$  and the case  $\delta = \kappa$  turns out to require a completely different argument, which we turn to now.

## More on the Mitchell order

**Definition 6.3.8.** A countably complete ultrafilter  $U$  is  $\lambda$ -Mitchell if for all hereditarily uniform ultrafilters  $D$  such that  $\lambda_D < \lambda$ ,  $D \triangleleft U$ .

If  $\lambda = 2^{<\lambda}$  and  $U$  is a countably complete ultrafilter such that  $j_U$  is  $\lambda$ -strong, then  $U$  is  $\lambda$ -Mitchell. The first step in the proof of GCH we will give is to prove the same result without assuming that  $2^{<\lambda} = \lambda$ , and instead using UA and a supercompactness hypothesis.

**Proposition 6.3.9** (UA). *Suppose  $\lambda$  is an infinite cardinal and  $U$  is a countably complete ultrafilter such that  $j_U$  is  $\lambda$ -supercompact. Then  $U$  is  $\lambda$ -Mitchell.*



In order to prove Proposition 6.3.9, we need two preliminary lemmas. The first is the obvious attempt to extend the proof of the linearity of the Mitchell order on normal ultrafilters from Chapter 2 to normal fine ultrafilters. (This was in fact the first proof we attempted in the very early days of UA, before realizing that the generalization was related to cardinal arithmetic.)

**Lemma 6.3.10.** *Suppose  $\lambda$  is a cardinal,  $U$  is a countably complete ultrafilter, and  $j_U$  is  $\lambda$ -supercompact. Suppose  $D$  is a countably complete ultrafilter on an ordinal  $\gamma \leq \lambda$ . Suppose  $(k, i) : (M_D, M_U) \rightarrow N$  is a 1-internal comparison of  $(j_D, j_U)$  such that  $k([\text{id}]_D) \in i(j_U[\lambda])$ . Then  $D \triangleleft U$ .*

*Proof.* Note that for any  $A \subseteq \gamma$ ,

$$\begin{aligned}
A \in D &\iff [\text{id}]_D \in j_D(A) \\
&\iff k([\text{id}]_D) \in k(j_D(A)) \\
&\iff k([\text{id}]_D) \in i(j_U(A)) \\
&\iff k([\text{id}]_D) \in i(j_U(A)) \cap i(j_U[\lambda]) \\
&\iff k([\text{id}]_D) \in i(j_U(A) \cap j_U[\lambda]) \\
&\iff k([\text{id}]_D) \in i(j_U[A])
\end{aligned}$$

Therefore

$$D = \{A \subseteq \gamma : k([\text{id}]_D) \in i(j_U[A])\} \tag{6.1}$$

Since  $j \upharpoonright \gamma \in M_U$ , the function defined on  $P(\gamma)$  by  $A \mapsto j_U[A]$  belongs to  $M_U$ . Moreover  $i$  is an internal ultrapower embedding of  $M_U$ . Therefore (6.1) shows that  $D$  is definable over  $M_U$  from parameters in  $M_U$ , and hence  $D \triangleleft U$ .  $\square$

Incidentally, this lemma suggests considering the following generalized Ketonen order: for  $D \in \mathcal{B}(X)$  and  $\mathcal{U} \in \mathcal{B}(Y)$ , set  $D \in^k \mathcal{U}$  if there exist  $I \in \mathcal{U}$  and  $\langle D_\sigma : \sigma \in I \rangle \in \prod_{\sigma \in I} \mathcal{B}(X, \sigma)$  such that  $D = \mathcal{U}\text{-}\lim_{\sigma \in I} D_\sigma$ . Lemma 6.3.10 can be restated as follows: if  $\lambda$  is a cardinal,

$D$  is a countably complete ultrafilter on  $\lambda$ , and  $\mathcal{U}$  is a normal fine ultrafilter on  $P(\lambda)$ , then  $D \in^k \mathcal{U}$  if and only if  $D \triangleleft \mathcal{U}$ .

Our next lemma puts us in a position to apply Lemma 6.3.10. For the proof of Proposition 6.3.9, we will only need the case  $A = j_U[\lambda]$ , but the general statement is used in the proof of level-by-level equivalence at singular cardinals (Theorem 8.3.22).

**Lemma 6.3.11.** *Suppose  $\lambda$  is a cardinal,  $U$  is a countably complete ultrafilter, and  $A \subseteq j_U(\lambda)$  is a nonempty set that is closed under  $j_U(f)$  for every  $f : \lambda \rightarrow \lambda$ . Suppose  $D$  is a countably complete ultrafilter on an ordinal  $\gamma < \lambda$ . Suppose  $(k, i) : (M_D, M_U) \rightarrow N$  is a 0-internal comparison of  $(j_D, j_U)$ . Then  $k([\text{id}]_D) \in i(A)$ .*

*Proof.* Let  $B = k^{-1}[i(A)]$ . By the definition of a 1-internal comparison,  $k : M_D \rightarrow N$  is an internal ultrapower embedding, and therefore  $B \in M_D$ . We must show that  $[\text{id}]_D \in B$ .

We first show that  $j_D[\lambda] \subseteq B$ . Note that  $j_U[\lambda] \subseteq A$  since  $A$  is nonempty and closed under  $j(c_\alpha)$  for any  $\alpha < \lambda$ , where  $c_\alpha : \lambda \rightarrow \lambda$  is the constant function with value  $\alpha$ . Thus  $i \circ j_U[\lambda] \subseteq i(A)$ . Since  $(i, k)$  is a 1-internal comparison,  $k \circ j_D[\lambda] = i \circ j_U[\lambda] \subseteq i(A)$ . So  $j_D[\lambda] \subseteq k^{-1}[i(A)] = B$ .

We now show that  $B$  is closed under  $j_D(f)$  for any  $f : \lambda \rightarrow \lambda$ . Fix  $\xi \in B$  and  $f : \lambda \rightarrow \lambda$ ; we will show  $j_D(f)(\xi) \in B$ . By assumption  $A$  is closed under  $j_U(f)$ , and so by elementarity  $i(A)$  is closed under  $i(j_U(f))$ . In particular, since  $k(\xi) \in i(A)$ ,  $i(j_U(f))(k(\xi)) \in i(A)$ . But  $i(j_U(f))(k(\xi)) = k(j_D(f)(\xi))$ . Now  $k(j_D(f)(\xi)) \in i(A)$  so  $j_D(f)(\xi) \in k^{-1}(i(A)) = B$ , as desired.

Since  $\gamma < \lambda$  and  $j_D[\lambda] \subseteq B$ , in particular  $j_D[\gamma^+] \subseteq B$ . Thus  $B$  is cofinal in the  $M_D$ -regular cardinal  $j_D(\gamma^+) = \sup j_D[\gamma^+]$ . In particular,  $|B|^{M_D} \geq j_D(\gamma^+)$ . Fix  $\langle B_\xi : \xi < \gamma \rangle$  with  $B = j_D(\langle B_\xi : \xi < \gamma \rangle)_{[\text{id}]_D}$ . By Łoś's Theorem, we may assume without loss of generality that  $B_\xi \subseteq \lambda$  and  $|B_\xi| \geq \gamma^+$  for all  $\xi < \gamma$ . Therefore there is an injective function  $g : \gamma \rightarrow \lambda$  such that  $g(\xi) \in B_\xi$  for all  $\xi < \gamma$ . By Łoś's Theorem,  $j_D(g)([\text{id}]_D) \in B$ . Since  $g$  is injective, there is a function  $f : \lambda \rightarrow \lambda$  be a function satisfying  $f(g(\xi)) = \xi$  for all  $\xi < \gamma$ . But  $B$  is closed under  $j_D(f)$ , and  $j_D(f)(j_D(g)([\text{id}]_D)) = [\text{id}]_D$ , so  $[\text{id}]_D \in B$ , as desired.  $\square$

Proposition 6.3.9 now follows easily.

*Proof of Proposition 6.3.9.* Fix a countably complete hereditarily uniform ultrafilter  $D$  with  $\lambda_D < \lambda$ . We must show  $D \triangleleft U$ . By the isomorphism invariance of the Mitchell order on hereditarily uniform ultrafilters (Lemma 4.2.14), we may assume  $D$  lies on an ordinal  $\gamma < \lambda$ . By UA, there is an internal ultrapower comparison  $(k, i) : (M_D, M_U) \rightarrow N$  of  $(j_D, j_U)$ . By Lemma 6.3.11 with  $A = j[\lambda]$ ,  $k([\text{id}]_D) \in i(j[\lambda])$ . Therefore the hypotheses of Lemma 6.3.10 are satisfied, so  $D \triangleleft U$ , as desired.  $\square$

It is natural to hope that the proof of Proposition 6.3.9 can be generalized to show the linearity of the Mitchell order on normal fine ultrafilters without assuming GCH. The trouble of course is removing the assumption  $\gamma < \lambda$  in Lemma 6.3.11. If  $\lambda$  is regular (which turns out to be the hard case), the proof of Lemma 6.3.11 goes through under the assumption that  $k[j_D(\lambda)] = \sup i \circ j_U[\lambda]$ . We know how to prove the lower bound  $k[j_D(\lambda)] \leq \sup i \circ j_U[\lambda]$  (using Lemma 8.2.11), but we do not know how to prove  $k[j_D(\lambda)] \geq \sup i \circ j_U[\lambda]$  directly. The only proof we know of the linearity of the Mitchell order that does not require a GCH assumption (Theorem 7.5.39) requires a good deal of the supercompactness analysis of Chapter 7.

## The proof of GCH

Theorem 6.1.1 above follows immediately from the following statement, which is much more local (and much harder to prove):

**Theorem 6.3.12** (UA). *Suppose  $\kappa \leq \delta$  are cardinals with  $\kappa \leq \text{cf}(\delta)$ . If  $\kappa$  is  $\delta^{++}$ -supercompact, then for any cardinal  $\lambda$  with  $\kappa \leq \lambda \leq \delta^{++}$ ,  $2^\lambda = \lambda^+$ .*

Combining Theorem 6.3.12 with the results of Chapter 7, the hypothesis that  $\kappa$  is  $\delta^{++}$ -supercompact can be weakened to the assumption that  $\kappa$  is  $\delta^{++}$ -strongly compact.

The hard part of the proof is contained in the following theorem:

**Theorem 6.3.13 (UA).** *Suppose  $\kappa$  and  $\delta$  are cardinals such that  $\text{cf}(\delta) \geq \kappa$ . Suppose  $\kappa$  is  $\delta^{++}$ -supercompact. Then  $2^\delta = \delta^+$ .*

*Proof.* Assume towards a contradiction that  $2^\delta > \delta^+$ . We use this assumption to prove the following claim, following the proof of Theorem 6.3.3:

**Claim 1.** *Every subset of  $\delta^{++}$  belongs to the ultrapower of the universe by a normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\delta)$ .*

*Proof.* Let  $j : V \rightarrow M$  be a  $\delta^{++}$ -supercompact ultrapower embedding with  $\text{CRT}(j) = \kappa$  and  $j(\kappa) > \delta^{++}$ . Since  $P(\delta^{++}) \subseteq M$ , it suffices to show the claim is true in  $M$ . Let  $\mathcal{U}$  be the normal fine ultrafilter on  $P_\kappa(\delta)$  derived from  $j$  using  $j[\delta]$ . By Proposition 6.3.9,  $\mathcal{U} \in M$  (since in fact every countably complete ultrafilter on  $P_\kappa(\delta)$  is in  $M$ ). Therefore by Lemma 6.3.4,  $M$  satisfies that every subset of  $P(\delta)$  belongs to the ultrapower of the universe by a normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\delta)$ . Since  $\delta^{++} \leq (2^\delta)^M$ , it follows that  $M$  satisfies that every subset of  $\delta^{++}$  belongs to the ultrapower of the universe by a normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\delta)$ , as desired.  $\square$

Let  $W$  be a  $\delta^+$ -supercompact ultrafilter on  $\delta^+$  with  $j_W(\kappa) > \delta^+$ . We claim  $P(\delta^{++}) \subseteq M_W$ . Suppose  $A \subseteq \delta^{++}$ . For some normal fine  $\kappa$ -complete ultrafilter  $U$  on  $P_\kappa(\delta)$ ,  $A \in M_U$ . But since  $|P_\kappa(\delta)| = \delta$  (by Solovay's Theorem on SCH above a strongly compact cardinal, Theorem 6.3.1), Proposition 6.3.9 implies  $U \in M_W$ . It is easy to see that this implies  $A \in M_W$ .

Let  $Z$  be a  $\delta^{++}$ -supercompact ultrafilter on  $\delta^{++}$  with  $j_Z(\kappa) > \delta^{++}$ . Let  $k : M_W \rightarrow N$  be the ultrapower of  $M_W$  by  $Z$  using functions in  $M_W$ . We have  $\text{WIDTH}(j_W) = \delta^{++}$  and  $\text{WIDTH}(k) = \delta^{++} < j_W(\delta^{++})$ , so by the lemma on the width of compositions (Lemma 3.5.34),  $\text{WIDTH}(k \circ j_W) = \delta^{++}$ . In other words, there is an ultrafilter  $D$  on  $\delta^+$  such that  $M_D = N$  and  $j_D = k \circ j_W$ .

Since  $P(\delta^{++}) \subseteq M_W$ ,  $(V_\kappa)^{\delta^{++}} \subseteq M_W$ . Therefore letting  $\kappa' = j_Z(\kappa) = k(\kappa)$ , we have  $V_{\kappa'} \cap N = V_{\kappa'} \cap M_Z$ . By Proposition 6.3.9,  $D \triangleleft Z$ , and so since  $\kappa' > \delta^+$ ,  $D \in V_{\kappa'} \cap M_Z \subseteq$

$N$ . It follows that  $D \in N = M_D$ , contradicting the irreflexivity of the Mitchell order (Lemma 4.2.40).  $\square$

*Proof of Theorem 6.3.12.* Suppose  $\lambda$  is a cardinal with  $\kappa \leq \lambda \leq \delta^{++}$ .

**Case 1.**  $\lambda \leq \delta$

If  $\lambda$  is regular then by Theorem 6.3.13,  $2^\lambda = \lambda^+$ . If  $\lambda$  is singular then  $2^{<\lambda} = \lambda$  by Theorem 6.3.13, so  $2^\lambda = \lambda^+$  by the local version of Solovay's theorem [21].

**Case 2.**  $\lambda = \delta^+$ .

Since  $\kappa$  is  $\delta^+$ -supercompact and  $2^\delta = \delta^+$ ,  $\kappa$  is  $2^\delta$ -supercompact. Therefore by Proposition 6.3.6,  $2^{2^\delta} = (2^\delta)^+$ . In other words,  $2^{(\delta^+)} = \delta^{++}$ .

**Case 3.**  $\lambda = \delta^{++}$

Given that  $2^{(\delta^+)} = \delta^{++}$  by Case 2, the case that  $\lambda = \delta^{++}$  can be handled in the same way as Case 2.  $\square$

**Corollary 6.3.14 (UA).** *Suppose  $\kappa \leq \delta$  and  $\kappa$  is  $2^\delta$ -supercompact. Then  $2^\delta = \delta^+$ .*

*Proof.* Assume first that  $\delta$  is singular. Since  $\kappa$  is  $<\delta$ -supercompact,  $2^{<\delta} = \delta$  by Theorem 6.3.12. Now  $2^\delta = \delta^+$  by Corollary 6.3.2.

Assume instead that  $\delta$  is regular. Assume towards a contradiction that  $2^\delta \geq \delta^{++}$ . Then  $\kappa$  is  $\delta^{++}$ -supercompact, so by Theorem 6.3.12,  $2^\delta = \delta^+$ , a contradiction.  $\square$

Let us point out another consequence that one can obtain using a result in Chapter 7:

**Theorem 6.3.15 (UA).** *Suppose  $\nu$  is a cardinal and  $\nu^+$  carries a countably complete uniform ultrafilter. Then  $2^{<\nu} = \nu$ .*

*Proof.* By Corollary 7.4.10 below, some cardinal  $\kappa \leq \nu$  is  $\nu^+$ -supercompact. If  $\kappa = \nu$  then obviously  $2^{<\nu} = \nu$ . So assume  $\kappa < \nu$ . If  $\nu$  is a limit cardinal, then the hypotheses of Theorem 6.3.12 hold for all sufficiently large  $\lambda < \nu$  and hence GCH holds on a tail below  $\nu$ ,

so  $2^{<\nu} = \nu$ . So assume  $\nu = \lambda^+$  is a successor cardinal. If  $\lambda$  is singular, then  $\lambda$  is a strong limit singular cardinal by Theorem 6.3.12, so  $2^\lambda = \lambda^+$  by Solovay's theorem Corollary 6.3.2, and hence  $2^{<\nu} = \nu$ . Finally if  $\lambda$  is regular, we can apply Theorem 6.3.12 directly to conclude that  $2^\lambda = \lambda^+$ , so again  $2^{<\nu} = \nu$ .  $\square$

This leaves open some questions about further localizations of the GCH proof.

**Question 6.3.16 (UA).** Suppose  $\kappa$  is  $\delta$ -supercompact. Must  $2^\delta = \delta^+$ ?

We conjecture that it is consistent with UA that  $\kappa$  is measurable but  $2^\kappa > \kappa^+$ , which would give a negative answer in the case  $\kappa = \delta$ . In certain cases, the question has a positive answer as an essentially immediate consequence of our main theorem:

**Proposition 6.3.17 (UA).** *Suppose  $\kappa \leq \lambda$ ,  $\text{cf}(\lambda) = \omega$ , and  $\kappa$  is  $\lambda$ -supercompact. Then  $2^\lambda = \lambda^+$ .*

*Suppose  $\kappa \leq \lambda$ ,  $\omega_1 \leq \text{cf}(\lambda) < \lambda$ , and  $\kappa$  is  $<\lambda$ -supercompact. Then  $2^\lambda = \lambda^+$ .*

*Suppose  $\kappa \leq \lambda$ ,  $\lambda$  is the double successor of a cardinal of cofinality at least  $\kappa$ , and  $\kappa$  is  $\lambda$ -supercompact. Then  $2^\lambda = \lambda^+$ .*  $\square$

Another interesting localization question is the following:

**Question 6.3.18 (UA).** Suppose  $\kappa$  is the least ordinal  $\alpha$  such that there is an ultrapower embedding  $j : V \rightarrow M$  with  $j(\alpha) > (2^\kappa)^+$ . Must  $2^\kappa = \kappa^+$ ?

### $\diamond$ on the critical cofinality

We conclude with the observation that stronger combinatorial principles than GCH follow from UA.

**Theorem 6.3.19 (UA).** *Suppose  $\kappa$  is  $\delta^{++}$ -supercompact where  $\text{cf}(\delta) \geq \kappa$ . Then  $\diamond(S_{\delta^+}^{\delta^{++}})$  holds.*

For the proof, we need a theorem of Kunen.

**Definition 6.3.20.** Suppose  $\lambda$  is a regular uncountable cardinal and  $S \subseteq \lambda$  is a stationary set. Suppose  $\langle \mathcal{A}_\alpha : \alpha \in S \rangle$  is a sequence of sets with  $\mathcal{A}_\alpha \subseteq P(\alpha)$  and  $|\mathcal{A}_\alpha| \leq \alpha$  for all  $\alpha < \lambda$ . Then  $\langle \mathcal{A}_\alpha : \alpha \in S \rangle$  is a  $\diamond^-(S)$ -sequence if for all  $X \subseteq \lambda$ ,  $\{\alpha \in S : X \cap \alpha \in \mathcal{A}_\alpha\}$  is stationary.

**Definition 6.3.21.**  $\diamond^-(S)$  is the assertion that there is a  $\diamond^-(S)$ -sequence.

**Theorem 6.3.22** (Kunen, [31]). *Suppose  $\lambda$  is a regular uncountable cardinal and  $S \subseteq \lambda$  is a stationary set. Then  $\diamond^-(S)$  is equivalent to  $\diamond(S)$ .*  $\square$

*Proof of Theorem 6.3.19.* By Theorem 6.3.12, GCH holds on the interval  $[\kappa, \delta^{++}]$ , and we will use this without further comment.

For each  $\alpha < \delta^{++}$ , let  $\mathcal{U}_\alpha$  be the unique ultrafilter of rank  $\alpha$  in the Mitchell order on normal fine  $\kappa$ -complete ultrafilters on  $P_\kappa(\delta)$ . The uniqueness of  $\mathcal{U}_\alpha$  follows from the linearity of the Mitchell order on normal fine ultrafilters on  $P_\kappa(\delta)$ , a consequence of Theorem 4.4.2 which applies in this context since  $2^{<\delta} = \delta$ . Let  $\mathcal{A}_\alpha = P(\alpha) \cap M_{\mathcal{U}_\alpha}$ . Note that  $|\mathcal{A}_\alpha| \leq \kappa^\delta = \delta^+$ .

Let

$$\vec{\mathcal{A}} = \langle \mathcal{A}_\alpha : \alpha < \delta^{++} \rangle$$

Note that  $\vec{\mathcal{A}}$  is definable in  $H_{\delta^{++}}$  without parameters.

**Claim 1.**  $\vec{\mathcal{A}}$  is a  $\diamond^-(S_{\delta^+}^{\delta^{++}})$ -sequence.

*Proof.* Suppose towards a contradiction that  $\vec{\mathcal{A}}$  is not a  $\diamond^-(S_{\delta^+}^{\delta^{++}})$ -sequence. Let  $\mathcal{W}$  be a  $\kappa$ -complete normal fine ultrafilter on  $P_\kappa(\delta^{++})$ . Then in  $M_{\mathcal{W}}$ ,  $\vec{\mathcal{A}}$  is not a  $\diamond^-(S_{\delta^+}^{\delta^{++}})$ -sequence. Let  $\mathcal{U}$  be the  $\kappa$ -complete normal fine ultrafilter on  $\delta$  derived from  $\mathcal{W}$  and let  $k : M_{\mathcal{U}} \rightarrow M_{\mathcal{W}}$  be the factor embedding. Let  $\gamma = \text{CRT}(k) = \delta^{++M_{\mathcal{U}}}$ .

Since  $\vec{\mathcal{A}}$  is definable in  $H_{\delta^{++}}$  without parameters,  $\vec{\mathcal{A}} \in \text{ran}(k)$ . Therefore  $k^{-1}(\vec{\mathcal{A}}) = \vec{\mathcal{A}} \upharpoonright \gamma$  is not a  $\diamond^-(S_{\delta^+}^\gamma)$ -sequence in  $M_{\mathcal{U}}$ . Fix a witness  $A \in P(\gamma) \cap M_{\mathcal{U}}$  and a closed unbounded set  $C \in P(\gamma) \cap M_{\mathcal{U}}$  such that for all  $\alpha \in C \cap S_{\delta^+}^\gamma$ ,  $A \cap \alpha \notin \mathcal{A}_\alpha$ . By elementarity, for all  $\alpha \in k(C) \cap S_{\delta^+}^{\delta^{++}}$ ,  $k(A) \cap \alpha \notin \mathcal{A}_\alpha$ . Since  $\mathcal{U}$  is  $\delta$ -supercompact,  $\text{cf}(\gamma) = \delta^+$ , and so in particular  $k(A) \cap \gamma \notin \mathcal{A}_\gamma$ . Since  $\gamma = \text{CRT}(k)$ , this means  $A \notin \mathcal{A}_\gamma$ .

Note however that  $\mathcal{U}$  has Mitchell rank  $\delta^{++M_{\mathcal{U}}} = \gamma$ , so  $\mathcal{U} = \mathcal{U}_{\gamma}$ . Therefore  $\mathcal{A}_{\gamma} = P(\gamma) \cap M_{\mathcal{U}}$ , so  $A \in \mathcal{A}_{\gamma}$  by choice of  $A$ . This is a contradiction.  $\square$

By Theorem 6.3.22, this completes the proof.  $\square$

## The size of the Vopěnka algebra

**Theorem 6.3.23** (UA). *Suppose  $\kappa$  is an inaccessible cardinal such that every  $A \subseteq P(\kappa)$  belongs to  $M_U$  for some countably complete ultrafilter  $U$  on  $\kappa$ . Then  $|\mathbb{V}_{\kappa}|^{\text{HOD}} = (2^{\kappa})^+$ .*

*Proof.* Let  $\lambda = |\mathbb{V}_{\kappa}|^{\text{HOD}}$ . Note that  $\lambda = |P(P(\kappa)) \cap \text{OD}|^{\text{OD}}$ .

Recall that  $\mathcal{B}(\kappa)$  denotes the set of countably complete ultrafilters on  $\kappa$ . As in Theorem 6.3.5,  $|\mathcal{B}(\kappa)| = 2^{2^{\kappa}}$ .

We claim that in fact  $|\mathcal{B}(\kappa)| = (2^{\kappa})^+$ . It suffices to show the upper bound  $|\mathcal{B}(\kappa)| \leq 2^{2^{\kappa}}$ . For this, we show that every initial segment of the Ketonen order has cardinality  $2^{\kappa}$ .

Since  $\kappa$  is inaccessible, for any  $\alpha < \kappa$ , the set  $\mathcal{B}(\kappa, \alpha)$  of countably complete ultrafilters on  $\kappa$  that concentrate on  $\alpha$  has cardinality less than  $\kappa$ . Thus for any  $U \in \mathcal{B}(\kappa)$ ,  $U$  has at most  $2^{\kappa} \cdot \prod_{\alpha < \kappa} |S_{\alpha}| = 2^{\kappa}$  predecessors in the Ketonen order, since if  $W <_{\mathbb{k}} U$ , then

$$W = U\text{-}\lim_{\alpha \in I} W_{\alpha}$$

for some  $I \in U$  and some sequence  $\langle W_{\alpha} : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\kappa, \alpha)$ .

Therefore let  $\langle U_{\alpha} : \alpha < (2^{\kappa})^+ \rangle$  be the  $<_{\mathbb{k}}$ -increasing enumeration of  $\mathcal{B}(\kappa)$ .

For the lower bound  $(2^{\kappa})^+ \leq \lambda$ , we apply the fact that every countably complete ultrafilter on an ordinal is OD (Proposition 6.2.1) to obtain  $\mathcal{B}(\kappa) \subseteq P(P(\kappa)) \cap \text{OD}$ , so in fact  $\lambda \geq |\mathcal{B}(\kappa)| = 2^{2^{\kappa}} = (2^{\kappa})^+$ .

We now turn to the upper bound.

Suppose  $U \in \mathcal{B}(\kappa)$ . Then  $|P(P(\kappa)) \cap M_U| \leq |j_U(V_{\kappa})| \leq |(V_{\kappa})^{\kappa}| = 2^{\kappa}$ . Let

$$\mathcal{A}_U = P(P(\kappa)) \cap M_U \cap \text{OD}$$



Note that  $P(P(\kappa) \cap M_U \cap \text{OD})$  is an ordinal definable subset of OD, so let  $\gamma_U = |\mathcal{A}_U|^{\text{OD}}$  and let  $\pi_U : \gamma_U \rightarrow \mathcal{A}_U$  be the OD-least bijection. Note that  $|\mathcal{A}_U| \leq 2^\kappa$  so  $\gamma_U < (2^\kappa)^+$ .

Let  $\lambda_0 = \sup\{\gamma_U : U \in S\}$ , so  $\lambda_0 \leq (2^\kappa)^+$ . Define  $\pi : (2^\kappa)^+ \times \lambda_0 \rightarrow P(P(\kappa)) \cap \text{OD}$  by

$$\pi(\alpha, \beta) = \pi_{f(\alpha)}(\beta)$$

Then our large cardinal assumption on  $\kappa$  implies that  $\pi$  is a surjection and  $\pi$  is ordinal definable, so  $\lambda \leq (2^\kappa)^+ \cdot \lambda_0 = (2^\kappa)^+$ .  $\square$

We finally prove Theorem 6.3.24, the fact that under UA, if  $\kappa$  is supercompact then  $V$  is a generic extension of HOD for a forcing of size  $\kappa^{++}$ .

**Theorem 6.3.24 (UA).** *If  $\kappa$  is  $\kappa^{++}$ -supercompact then  $|\mathbb{V}_\kappa|^{\text{HOD}} = \kappa^{++}$ .*

*Proof.* Note that since  $\kappa$  is  $\kappa^{++}$ -supercompact, by Theorem 6.3.12,  $(2^\kappa)^+ = \kappa^{++}$ . In particular,  $\kappa$  is  $2^\kappa$ -supercompact, so the hypotheses of Theorem 6.3.23 hold by Theorem 6.3.3. Thus  $|\mathbb{V}_\kappa|^{\text{HOD}} = (2^\kappa)^+ = \kappa^{++}$ .  $\square$

# Chapter 7

## The Least Supercompact Cardinal

### 7.1 Introduction

#### The identity crisis

How large is the least strongly compact cardinal? This question was first posed by Tarski in a precise form shortly after his discovery of strong compactness: *is the least strongly compact cardinal larger than the least measurable cardinal?* About a decade later, Solovay mounted the first serious attack on this problem. He fused the Scott's elementary embedding analysis of measurability with the combinatorial properties of strongly compact cardinals to produce what has become the central large cardinal concept: supercompactness. He then conjectured that every strongly compact cardinal is supercompact. This is certainly a natural conjecture to make since supercompact cardinals and strongly compact cardinals share some rather deep structural similarities. (See Section 7.2 and especially Section 7.2.) But unlike the least strongly compact cardinal, the size of the least supercompact cardinal is no mystery at all: it is upon first glance a staggeringly large object, much larger than the least measurable cardinal. Thus Solovay's conjecture implies a positive answer to Tarski's question.

Telis Menas, then a graduate student under Solovay at UC Berkeley, was the first to realize that Solovay's conjecture is false. Menas climbed up far beyond the least strongly compact

cardinal, and up there he discovered a strongly compact cardinal that is not supercompact.

**Theorem 8.1.1** (Menas). *The least strongly compact limit of strongly compact cardinals is not supercompact.*

This theorem closed off Solovay’s approach to Tarski’s question while leaving the question itself wide open. The fundamental breakthrough occurred mere months after Menas’s discovery on the other side of the world, with Magidor’s landmark independence result [32]:

**Theorem** (Magidor). *Suppose  $\kappa$  is a cardinal.*

- *If  $\kappa$  is strongly compact, then there is a forcing extension in which  $\kappa$  remains strongly compact but becomes the least measurable cardinal.*
- *If  $\kappa$  is supercompact, then there is a forcing extension in which  $\kappa$  remains supercompact but becomes the least strongly compact cardinal.  $\square$*

Thus the ZFC axioms are insufficient to answer Tarski’s question. Magidor described this peculiar situation as an “identity crisis” for the least strongly compact cardinal. The main result of this chapter is that the Ultrapower Axiom resolves this crisis:

**Theorem 7.4.23** (UA). *The least strongly compact cardinal is supercompact.*

We will prove much stronger results than this that explain exactly why the least strongly compact cardinal is supercompact, and that identify much weaker properties that are sufficient (under UA) for supercompactness. We defer until the final chapter the analysis of larger strongly compact cardinals.

## Outline of Chapter 7

We now outline the rest of the chapter.

**Section 7.2.** We exposit the basic theory of strong compactness. We use the theory of the Ketonen order to prove Ketonen’s Theorem [11] that  $\kappa$  is strongly compact if and only

if every regular cardinal carries a  $\kappa$ -complete ultrafilter (Theorem 7.2.15). This argument is the basis for much of the theory of this chapter. We use Ketonen’s Theorem to prove the local version of Solovay’s Theorem [21] on SCH above a strongly compact that we have by now cited several times (Theorem 7.2.16).

**Section 7.3.** In this section, we introduce the notion of a Fréchet cardinal and its associated Ketonen ultrafilters. Under UA, each Fréchet cardinal  $\lambda$  carries a unique Ketonen ultrafilter  $\mathcal{K}_\lambda$ . For regular  $\lambda$ , we analyze  $\mathcal{K}_\lambda$  under the assumption that some  $\kappa \leq \lambda$  is  $\lambda$ -strongly compact, showing that its associated embedding is  $<\lambda$ -supercompact and  $\lambda$ -tight (Proposition 7.4.11).

**Section 7.4.** Given the analysis of  $\mathcal{K}_\lambda$  in the previous section, we would like to show that if  $\lambda$  is a regular Fréchet cardinal, then some cardinal  $\kappa \leq \lambda$  is  $\lambda$ -strongly compact. In this section, we prove our best result towards this, showing that this is true unless  $\lambda$  is *isolated* (Theorem 7.4.9). Isolated cardinals are rare enough that this implies the supercompactness of the least strongly compact cardinal (Theorem 7.4.23).

**Section 7.5.** In this section we study the structure of isolated cardinals, which arose in Section 7.4 as a pathological case in our analysis of Fréchet cardinals. We rule out pathological isolated cardinals assuming GCH (Proposition 7.5.4). Without GCH, assuming just UA, we are still able to fully analyze ultrafilters on an isolated cardinal (Section 7.5), which turn out to look just like ultrafilters on the least measurable cardinal. We prove that nonmeasurable isolated cardinals are associated with serious failures of GCH (Theorem 7.5.21 and Theorem 7.5.23). We leverage these results to prove the linearity of the Mitchell order on normal fine ultrafilters without assuming GCH (Theorem 7.5.39).

## 7.2 Strong compactness

### Some characterizations of strong compactness

After almost 70 years of research, strong compactness remains one of the most important and mysterious large cardinal notions. Strongly compact cardinals were first isolated by Tarski in the context of infinitary logic:  $\kappa$  is strongly compact if the logic  $\mathcal{L}_{\kappa,\kappa}$  satisfies a generalized version of the Compactness Theorem. In keeping with modern large cardinal theory, we will introduce strongly compact cardinals in terms of elementary embeddings of the universe of sets into inner models with closure properties. The closure property we have in mind is a two-cardinal version of the covering property:

**Definition 7.2.1.** Suppose  $M$  is an inner model,  $\lambda$  is a cardinal, and  $\delta$  is an  $M$ -cardinal. Then  $M$  has the  $(\lambda, \delta)$ -covering property if every set  $A \subseteq M$  such that  $|A| < \lambda$  is contained in a set  $B \in M$  such that  $|B|^M < \delta$ .

**Definition 7.2.2.** A cardinal  $\kappa$  is strongly compact if for any cardinal  $\lambda \geq \kappa$ , there is an elementary embedding  $j : V \rightarrow M$  such that  $\text{CRT}(j) = \kappa$  and  $M$  has the  $(\lambda, j(\kappa))$ -covering property.

**Definition 7.2.3.** We make the following abbreviations:

- The  $(\leq \lambda, \delta)$ -covering property is the  $(\lambda^+, \delta)$ -covering property.
- The  $(\lambda, \leq \delta)$ -covering property is the  $(\lambda, \delta^{+M})$ -covering property.
- The  $(\leq \lambda, \leq \delta)$ -covering property is the  $(\lambda^+, \delta^{+M})$ -covering property.
- The  $\lambda$ -covering property is the  $(\lambda, \lambda)$ -covering property.
- The  $\leq \lambda$ -covering property is the  $(\leq \lambda, \leq \lambda)$ -covering property.

This notation is chosen so that, for example, an inner model  $M$  has the  $(\leq\lambda, \leq\delta)$ -covering property if every subset  $A \subseteq M$  such that  $|A| \leq \lambda$  is contained in a set  $B \in M$  such that  $|B|^M \leq \delta$ .

We will be particularly interested in the following local version of strong compactness (especially when  $\lambda$  is regular):

**Definition 7.2.4.** Suppose  $\kappa \leq \lambda$  are cardinals. Then  $\kappa$  is  $\lambda$ -strongly compact if there is an inner model  $M$  and an elementary embedding  $j : V \rightarrow M$  with  $\text{CRT}(j) = \kappa$  such that  $M$  has the  $(\leq\lambda, j(\kappa))$ -covering property.

Note that if  $j : V \rightarrow M$  and  $M$  has the  $(\leq\lambda, j(\kappa))$ -covering property, then  $j(\kappa) > \lambda$ .

Theorem 7.2.10 puts down several equivalent reformulations of strong compactness. These involve the notions of *tightness* and *filter bases*, which we now define.

The concept of *tightness* had not been given a name before this dissertation, but it plays a role analogous to that of supercompactness in the theory of supercompact cardinals:

**Definition 7.2.5.** Suppose  $M$  is an inner model,  $\lambda$  is a cardinal, and  $\delta$  is an  $M$ -cardinal. An elementary embedding  $j : V \rightarrow M$  is  $(\lambda, \delta)$ -tight if there is a set  $A \in M$  with  $|A|^M \leq \delta$  such that  $j[\lambda] \subseteq A$ . An elementary embedding is said to be  $\lambda$ -tight if it is  $(\lambda, \lambda)$ -tight.

Thus  $(\lambda, \delta)$ -tightness is a weakening of  $\lambda$ -supercompactness. Any  $j : V \rightarrow M$  such that  $M$  has the  $(\leq\lambda, j(\kappa))$ -covering property is  $(\leq\lambda, j(\kappa))$ -tight. Moreover, many of the general theorems about supercompact embeddings generalize to the context of  $(\lambda, \delta)$ -tight ones. For example, Lemma 4.2.17 generalizes:

**Lemma 7.2.6.** *Suppose  $j : V \rightarrow M$  is an elementary embedding. The following are equivalent:*

- (1)  $j$  is  $(\lambda, \delta)$ -tight.
- (2) For some  $X$  with  $|X| = \lambda$ , there is some  $Y \in M$  with  $|Y|^M \leq \delta$  such that  $j[X] \subseteq Y$ .

(3) For any  $A$  such that  $|A| \leq \lambda$ , there is some  $B \in M$  with  $|B|^M \leq \delta$  such that  $j[A] \subseteq B$ .

*Proof.* (1) implies (2): Trivial.

(2) implies (3): Suppose  $|A| \leq \lambda$ . We will find  $B \in M$  with  $|B|^M \leq \delta$  such that  $j[A] \subseteq B$ . Using (2), fix  $X$  with  $|X| = \lambda$  such that for some  $Y \in M$  with  $|Y|^M \leq \delta$  such that  $j[X] \subseteq Y$ . Let  $p : X \rightarrow A$  be a surjection. Then

$$j[A] = j(p)[j[X]] \subseteq j(p)[Y]$$

Let  $B = j(p)[Y]$ . Then  $j[A] \subseteq B$ ,  $B \in M$ , and  $|B|^M \leq |Y|^M \leq \delta$ .

(3) implies (1): Trivial. □

The relationship between the  $\lambda$ -supercompactness of an embedding and the closure of its target model under  $\lambda$ -sequences is analogous to the relationship between the  $(\lambda, \delta)$ -tightness of an elementary embedding and the  $(\leq \lambda, \leq \delta)$ -covering property of its target model. For example, there is an analog of Corollary 4.2.21:

**Lemma 7.2.7.** *Suppose  $j : V \rightarrow M$  is a  $(\lambda, \delta)$ -tight ultrapower embedding. Then  $M$  has the  $(\leq \lambda, \leq \delta)$ -covering property.*

*Proof.* Suppose  $A \subseteq M$  with  $|A| \leq \lambda$ , and we will find  $B \in M$  such that  $|B|^M \leq \delta$  and  $A \subseteq B$ . Fix  $a \in M$  such that  $M = H^M(j[V] \cup \{a\})$ . Fix a set of functions  $F$  of cardinality  $\lambda$  such that  $A = \{j(f)(a) : f \in F\}$ . By Lemma 7.2.6, fix  $G \in M$  with  $|G|^M \leq \delta$  and such that  $j[F] \subseteq G$ . Let  $B = \{g(a) : g \in G\}$ . Then  $B \in M$ ,  $A \subseteq B$ , and  $|B|^M \leq |G|^M \leq \delta$ , as desired. □

Many filters are most naturally presented in terms of a smaller family of sets that “generates” the filter. The notion of a filter base makes this precise:

**Definition 7.2.8.** A *filter base* on  $X$  is a family  $\mathcal{B}$  of subsets of  $X$  with the finite intersection property: for all  $A_0, A_1 \in \mathcal{B}$ ,  $A_0 \cap A_1 \neq \emptyset$ . If  $\kappa$  is a cardinal, a filter base  $\mathcal{B}$  is said to be  $\kappa$ -*complete* if for all  $\nu < \kappa$ , for all  $\{A_\alpha : \alpha < \nu\} \subseteq \mathcal{B}$ ,  $\bigcap_{\alpha < \nu} A_\alpha \neq \emptyset$ .

The term “filter base” is motivated by the fact that every filter base  $\mathcal{B}$  on  $X$  generates a filter.

**Definition 7.2.9.** Suppose  $\mathcal{B}$  is a filter base. The *filter generated by  $\mathcal{B}$*  is the filter  $F(\mathcal{B}) = \{A \subseteq X : \exists A_0, \dots, A_{n-1} \in \mathcal{B} \ A_0 \cap \dots \cap A_{n-1} \subseteq A\}$ . If  $\mathcal{B}$  is a  $\kappa$ -complete filter base, the  *$\kappa$ -complete filter generated by  $\mathcal{B}$*  is the filter

$$F_\kappa(\mathcal{B}) = \{A \subseteq X : \exists S \in P_\kappa(\mathcal{B}) \ \bigcap S \subseteq A\}$$

We finally prove our equivalences with strong compactness:

**Theorem 7.2.10.** *Suppose  $\kappa \leq \lambda$  are uncountable cardinals. Then the following are equivalent:*

- (1)  $\kappa$  is  $\lambda$ -strongly compact.
- (2) There is an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  that is  $(\lambda, \delta)$ -tight for some  $M$ -cardinal  $\delta < j(\kappa)$ .
- (3) Every  $\kappa$ -complete filter base of cardinality  $\lambda$  extends to a  $\kappa$ -complete ultrafilter.
- (4) There is a  $\kappa$ -complete fine ultrafilter on  $P_\kappa(\lambda)$ .
- (5) There is an ultrapower embedding  $j : V \rightarrow M$  with critical point  $\kappa$  that is  $(\lambda, \delta)$ -tight for some  $M$ -cardinal  $\delta < j(\kappa)$ .
- (6) There is an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $M$  has the  $(\lambda, \delta)$ -covering property for some  $M$ -cardinal  $\delta < j(\kappa)$ .

*Proof.* (1) implies (2): Trivial.

(2) implies (3): Let  $j : V \rightarrow M$  be an elementary embedding such that  $\text{CRT}(j) = \kappa$  and  $j$  is  $(\lambda, \delta)$ -tight for some  $M$ -cardinal  $\delta < j(\kappa)$ . Suppose  $\mathcal{B}$  is a  $\kappa$ -complete filter base on  $X$  of cardinality  $\lambda$ . By Lemma 7.2.6, there is a set  $S \in M$  such that  $j[\mathcal{B}] \subseteq S$  and  $|S|^M < j(\kappa)$ . By replacing  $S$  with  $S \cap j(\mathcal{B})$ , we may assume without loss of generality that  $S \subseteq j(\mathcal{B})$ . By



the elementarity of  $j$ , since  $j(\mathcal{B})$  is  $j(\kappa)$ -complete, the intersection  $\bigcap j(S)$  is nonempty. Fix  $a \in \bigcap j(S)$ . Since  $j[\mathcal{B}] \subseteq S$ , it follows that  $a \in j(A)$  for all  $A \in \mathcal{B}$ . Let  $U$  be the ultrafilter on  $X$  derived from  $j$  using  $a$ . Then  $U$  extends  $\mathcal{B}$  and  $U$  is  $\kappa$ -complete since  $\text{CRT}(j) = \kappa$ .

(3) *implies* (4): For any  $\alpha < \lambda$ , let  $A_\alpha = \{\sigma \in P_\kappa(\lambda) : \alpha \in \sigma\}$ , and let  $\mathcal{B} = \{A_\alpha : \alpha < \lambda\}$ . Then  $\mathcal{B}$  a  $\kappa$ -complete filter base on  $P_\kappa(\lambda)$ , and any filter on  $P_\kappa(\lambda)$  that extends  $\mathcal{B}$  is fine. By (3), there is a  $\kappa$ -complete ultrafilter extending  $\mathcal{B}$ . Thus there is a  $\kappa$ -complete fine ultrafilter on  $P_\kappa(\lambda)$ , as desired.

(4) *implies* (5): Suppose  $\mathcal{U}$  is a  $\kappa$ -complete fine ultrafilter on  $P_\kappa(\lambda)$ . Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{U}$ . The  $\kappa$ -completeness of  $\mathcal{U}$  implies that  $\text{CRT}(j) \geq \kappa$ . By Lemma 4.4.9,  $j[\lambda] \subseteq a_{\mathcal{U}}$ . Moreover  $a_{\mathcal{U}} \in j(P_\kappa(\lambda))$ , so letting  $\delta = |a_{\mathcal{U}}|^M$ ,  $\delta < j(\kappa)$ . Therefore  $j$  is an ultrapower embedding that is  $(\lambda, \delta)$ -tight for some  $\delta < j(\kappa)$ . Since  $\kappa \leq \lambda$  and  $\lambda \leq \text{ot}(j[\lambda]) \leq \delta^{+M} < j(\kappa)$ , it follows that  $j(\kappa) > \lambda$ . In particular,  $\text{CRT}(j) = \kappa$ .

(5) *implies* (6): This is an immediate consequence of the fact that tight ultrapowers have the covering property (Lemma 7.2.7).

(6) *implies* (1): Trivial. □

## Ketonen's Theorem

The main theorem of this subsection is a famous theorem of Ketonen [11] that amounts to a deeper ultrafilter theoretic characterization of strong compactness:

**Theorem 7.2.11** (Ketonen). *A cardinal  $\kappa$  is strongly compact if and only if every regular cardinal  $\lambda \geq \kappa$  carries a uniform  $\kappa$ -complete ultrafilter.*

Part of what is surprising about this theorem is that it does not even require that the ultrafilters in the hypothesis be  $\kappa^+$ -incomplete. Beyond this, it is not even obvious at the outset that the existence of  $\kappa$ -complete ultrafilters on, say,  $\kappa$  and  $\kappa^+$  implies that  $\kappa$  is  $\kappa^+$ -strongly compact.

We begin, however, with a less famous but no less important theorem of Ketonen, which is also a key step in the proof of Theorem 7.2.11. This theorem is in a sense the strongly compact generalization of Solovay’s Lemma [21]. Suppose  $j : V \rightarrow M$  is an elementary embedding. For regular cardinals  $\lambda$ , Solovay’s Lemma (or more specifically Corollary 4.4.30) yields a simple criterion for whether  $j$  is  $\lambda$ -supercompact solely in terms of the inner model  $M$  and the ordinal  $\sup j[\lambda]$ :

**Theorem 4.4.30** (Solovay). *Suppose  $j : V \rightarrow M$  is an elementary embedding and  $\lambda$  is a regular cardinal. Then  $j$  is  $\lambda$ -supercompact if and only if  $M$  is correct about stationary subsets of  $\sup j[\lambda]$ .  $\square$*

Ketonen proved a remarkable analog of this theorem for strongly compact embeddings:

**Theorem 7.2.12** (Ketonen). *Suppose  $j : V \rightarrow M$  is an elementary embedding,  $\lambda$  is a regular uncountable cardinal, and  $\delta$  is an  $M$ -cardinal. Then  $j$  is  $(\lambda, \delta)$ -tight if and only if  $\text{cf}^M(\sup j[\lambda]) \leq \delta$ .*

For example, suppose  $j : V \rightarrow M$  is an ultrapower embedding. Theorem 7.2.12 implies that all that is required for  $M$  to have the  $\leq \lambda$ -covering property is that  $M$  correctly compute the cofinality of  $\sup j[\lambda]$ .

The proof of Theorem 7.2.12 we give is due to Woodin, and is a bit different from Ketonen’s original proof. The trick is to choose the cover first, and then choose the set whose image is being covered:

*Proof of Theorem 7.2.12.* First assume  $j$  is  $(\lambda, \delta)$ -tight. Fix  $A \in M$  with  $j[\lambda] \subseteq A$  such that  $|A|^M \leq \delta$ . Then  $A \cap \sup j[\lambda]$  is cofinal in  $\sup j[\lambda]$ , so  $\sup j[\lambda]$  has cofinality at most  $|A|^M$  in  $M$ .

Now we prove the converse. Assume  $\text{cf}^M(\sup j[\lambda]) \leq \delta$ . Let  $Y \in M$  be an  $\omega$ -closed cofinal subset of  $\sup j[\lambda]$  of order type at most  $\delta$ . Note that  $j[\lambda]$  is itself an  $\omega$ -closed cofinal subset of  $\sup j[\lambda]$ , so since  $\sup j[\lambda]$  has uncountable cofinality,  $Y \cap j[\lambda]$  is an  $\omega$ -closed cofinal subset

of  $\lambda$ . In particular, since  $\text{cf}(\sup j[\lambda]) = \lambda$ ,  $Y \cap j[\lambda]$  has order type at least  $\lambda$ . Let  $X = j^{-1}[Y]$ . Then  $j[X] = Y \cap j[\lambda]$ , so

$$\text{ot}(X) = \text{ot}(j[X]) = \text{ot}(Y \cap j[\lambda]) \geq \lambda$$

Thus  $|X| = \lambda$ . Since  $|X| = \lambda$ ,  $Y \in M$ ,  $j[X] \subseteq Y$ , and  $|Y|^M \leq \delta$ , Lemma 7.2.6 implies that  $j$  is  $(\lambda, \delta)$ -tight.  $\square$

With Theorem 7.2.12 in hand, we turn to the proof of Ketonen's characterization of strong compactness. The key point is that the strong compactness of an elementary embedding is equivalent to an ultrafilter theoretic property:

**Proposition 7.2.13.** *Suppose  $\kappa \leq \lambda$  are uncountable cardinals and  $\lambda$  is regular. Suppose  $M$  is an inner model and  $j : V \rightarrow M$  is an elementary embedding. Suppose every regular cardinal in the interval  $[\kappa, \lambda]$  carries a uniform  $\kappa$ -complete ultrafilter. Then the following are equivalent:*

- (1)  $j$  is  $(\lambda, \delta)$ -tight for some  $M$ -cardinal  $\delta < j(\kappa)$ .
- (2)  $\sup j[\lambda]$  carries no  $j(\kappa)$ -complete tail uniform ultrafilter.

*Proof.* (1) implies (2): Assume (1). By Theorem 7.2.12,  $\text{cf}^M(\sup j[\lambda]) < j(\kappa)$ . Therefore the tail filter on  $\sup j[\lambda]$  is not  $j(\kappa)$ -complete in  $M$ , so  $\sup j[\lambda]$  does not carry a  $j(\kappa)$ -complete tail uniform ultrafilter in  $M$ .

(2) implies (1): Assume (2). Then in particular  $\text{cf}^M(\sup j[\lambda])$  carries no uniform  $j(\kappa)$ -complete ultrafilter in  $M$ . By elementarity, every  $M$ -regular cardinal in the interval  $j([\kappa, \lambda])$  carries a uniform  $\kappa$ -complete ultrafilter. Therefore  $\text{cf}^M(\sup j[\lambda])$  does not lie in the interval  $j([\kappa, \lambda])$ . Clearly  $\text{cf}^M(\sup j[\lambda]) \leq j(\lambda)$ , so it follows that  $\text{cf}^M(\sup j[\lambda]) < j(\kappa)$ .  $\square$

Ketonen introduced the Ketonen order as a tool to prove the following theorem, generalizing a theorem of Solovay that states that any measurable cardinal carries a normal ultrafilter that concentrates on nonmeasurable cardinals.

**Theorem 7.2.14** (Ketonen). *Suppose  $\lambda$  is a regular cardinal. If  $\lambda$  carries a  $\kappa$ -complete uniform ultrafilter, then  $\lambda$  carries a  $\kappa$ -complete uniform ultrafilter  $U$  such that  $\sup j_U[\lambda]$  carries no tail uniform  $\kappa$ -complete ultrafilter in  $M_U$ . Indeed, any  $<_{\mathbb{k}}$ -minimal  $\kappa$ -complete uniform ultrafilter on  $\lambda$  has this property.*

*Proof.* Let  $U$  be a  $<_{\mathbb{k}}$ -minimal element of the set of uniform  $\kappa$ -complete ultrafilters on  $\lambda$ . Suppose towards a contradiction that in  $M_U$ ,  $\sup j_U[\lambda]$  carries a tail uniform  $\kappa$ -complete ultrafilter. Equivalently, there is a  $\kappa$ -complete ultrafilter  $Z$  on  $j_U(\lambda)$  such that  $\delta_Z = \sup j_U[\lambda]$ . Let  $W = j_U^{-1}[Z]$ . Then  $\text{CRT}(j_W) \geq \text{CRT}(j_Z^{M_U} \circ j_U)$  (by Lemma 3.2.17), so  $W$  is  $\kappa$ -complete. Moreover since  $\delta_Z = \sup j_U[\lambda]$ ,  $\delta_W = \lambda$ . Thus  $W$  is a  $\kappa$ -complete uniform ultrafilter on  $\lambda$ . Since  $Z$  concentrates on  $\sup j_U[\lambda] \leq a_U$ ,  $W <_{\mathbb{k}} U$  by the definition of the Ketonen order (Lemma 3.3.4). This contradicts the  $<_{\mathbb{k}}$ -minimality of  $U$ .  $\square$

We can now prove a local version of Ketonen's theorem, which fits into the list of reformulations of  $\lambda$ -strong compactness from Theorem 7.2.10:

**Theorem 7.2.15** (Ketonen). *Suppose  $\kappa \leq \lambda$  are regular uncountable cardinals. Then the following are equivalent:*

- (1)  $\kappa$  is  $\lambda$ -strongly compact.
- (2) Every regular cardinal in the interval  $[\kappa, \lambda]$  carries a uniform  $\kappa$ -complete ultrafilter.
- (3)  $\lambda$  carries a  $\kappa$ -complete ultrafilter  $U$  such that  $j_U$  is  $(\lambda, \delta)$ -tight for some  $\delta < j_U(\kappa)$ .

*Proof.* (1) implies (2): Note that the Fréchet filter on a regular cardinal  $\delta$  is  $\delta$ -complete. Thus (2) follows from (1) as an immediate consequence of the filter extension property of strongly compact cardinals (Theorem 7.2.10 (3)).

(2) implies (3): Assume (2). By Theorem 7.2.14, there is a  $\kappa$ -complete ultrafilter  $U$  on  $\lambda$  such that  $\sup j_U[\lambda]$  carries no tail uniform  $\kappa$ -complete ultrafilter in  $M_U$ . Therefore by Proposition 7.2.13,  $j_U$  is  $(\lambda, \delta)$ -tight for some  $\delta < j_U(\kappa)$ .

(3) implies (1): See Theorem 7.2.10 (5).  $\square$

## Solovay's Theorem

In this section we give a proof of a local version of Solovay's theorem that we use throughout this dissertation.

**Theorem 7.2.16** (Solovay). *Suppose  $\kappa \leq \lambda$  are uncountable cardinals,  $\lambda$  is regular, and  $\kappa$  is  $\lambda$ -strongly compact. Then  $\lambda^{<\kappa} = \lambda$ .*

We need the following lemma, which is in a sense an analog of Proposition 4.2.31, though much easier:

**Lemma 7.2.17.** *Suppose  $U$  is a countably complete ultrafilter. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $U$ . Then for any  $\eta \geq \lambda_U^+$ ,  $j$  is not  $(\eta, \delta)$ -tight for any  $M$ -cardinal  $\delta < j(\eta)$ .*

*Proof.* We may assume by induction that  $\eta$  is a successor cardinal. In particular,  $\eta$  is regular, so by Lemma 3.5.32,  $j(\eta) = \sup j[\eta]$ . Suppose towards a contradiction that  $\delta < j(\eta)$  is an  $M$ -cardinal such that  $j$  is  $(\eta, \delta)$ -tight. By Theorem 7.2.12,  $\text{cf}^M(j(\eta)) = \text{cf}^M(\sup j[\eta]) \leq \delta < j(\eta)$ . This contradicts that  $\eta$  is regular in  $M$  by elementarity.  $\square$

**Lemma 7.2.18.** *Suppose  $\kappa \leq \gamma$  are cardinals. Suppose  $\gamma$  is singular and*

$$\sup_{\eta < \gamma} \eta^{<\kappa} \leq \gamma \tag{7.1}$$

*Suppose  $\gamma^+$  carries a uniform  $\kappa$ -complete ultrafilter  $U$ . Then  $\gamma^{<\kappa} \leq \gamma^+$ .*

*Proof.* Let  $\lambda = \gamma^+$ . We will prove the equivalent statement that  $\lambda^{<\kappa} = \lambda$ .

Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $U$ . Let  $\delta = \text{cf}^M(\sup j[\lambda])$ . Note that  $\delta < j(\lambda)$ , so  $\delta \leq j(\gamma)$ . In fact, since  $j(\gamma)$  is singular in  $M$ ,  $\delta < j(\gamma)$ . Therefore by (7.1) and the elementarity of  $j$ :

$$(\delta^{<\kappa})^M \leq (\delta^{<j(\kappa)})^M \leq j(\gamma) \tag{7.2}$$

By Theorem 7.2.12,  $j$  is  $(\lambda, \delta)$ -tight, so we can fix  $B \in M$  with  $j[\lambda] \subseteq B$ . Now  $j[P_\kappa(\lambda)] \subseteq B^{<\kappa}$ , and since  $M$  is closed under  $\kappa$ -sequences,  $B^{<\kappa} \in M$ . Lemma 7.2.6 now implies that  $j$  is  $(\lambda^{<\kappa}, (\delta^{<\kappa})^M)$ -tight.

Assume towards a contradiction that  $\lambda^{<\kappa} \geq \lambda^+$ . Then  $j$  is  $(\lambda^+, (\delta^{<\kappa})^M)$ -tight. Since  $\lambda_U = \lambda$ , it follows from Lemma 7.2.17 that  $(\delta^{<\kappa})^M \geq j(\lambda^+)$ , contradicting (7.2).  $\square$

We now prove Solovay's theorem:

*Proof of Theorem 7.2.16.* Suppose  $\kappa$  is  $\lambda$ -strongly compact. Assume by induction that for all regular  $\iota < \lambda$ ,  $\iota^{<\kappa} = \iota$ . Since  $\lambda$  is regular, every element of  $P_\kappa(\lambda)$  is bounded below  $\lambda$ , so  $P_\kappa(\lambda) = \bigcup_{\eta < \lambda} P_\kappa(\eta)$ . Thus computing cardinalities:

$$\lambda^{<\kappa} = \sup_{\eta < \lambda} \eta^{<\kappa}$$

If  $\lambda$  is a limit cardinal, it follows immediately from our induction hypothesis that  $\lambda^{<\kappa} = \lambda$ . Therefore assume  $\lambda$  is a successor cardinal. If the cardinal predecessor of  $\lambda$  is a regular cardinal  $\iota$ , then applying our induction hypothesis we obtain:

$$\lambda^{<\kappa} = \sup_{\eta < \lambda} \eta^{<\kappa} = \lambda \cdot \iota^{<\kappa} = \lambda$$

Therefore assume the cardinal predecessor of  $\lambda$  is a singular cardinal  $\gamma$ . Then  $\sup_{\eta < \gamma} \eta^{<\kappa} \leq \gamma$ . In this case, by Lemma 7.2.18,  $\lambda^{<\kappa} = \lambda$ .  $\square$

## 7.3 Fréchet cardinals and the least ultrafilter $\mathcal{K}_\lambda$

### Fréchet cardinals

In this section, we begin our systematic study of strong compactness assuming UA. We will ultimately prove that UA implies that strong compactness and supercompactness coincide to the extent that this is possible. (A theorem of Menas shows that assuming sufficiently large cardinals, not all strongly compact cardinals are supercompact; see Section 8.1.) An

oddity of the proof is that it requires a preliminary analysis of the first strongly compact cardinal. Indeed, to obtain the strongest results, one must enact a hyperlocal analysis of essentially the weakest ultrafilter-theoretic forms of strong compactness.

With this in mind, we introduce the following central concept:

**Definition 7.3.1.** An uncountable cardinal  $\lambda$  is *Fréchet* if  $\lambda$  carries a countably complete uniform ultrafilter.

Fréchet cardinals almost certainly do not appear in the work of Fréchet. Their name derives from the fact that  $\lambda$  is Fréchet if and only if the Fréchet filter on  $\lambda$  extends to a countably complete ultrafilter.

The following proposition is almost tautological:

**Proposition 7.3.2.** *A cardinal  $\lambda$  is Fréchet if and only if  $\lambda = \lambda_U$  for some countably complete ultrafilter  $U$ .* □

For regular cardinals  $\lambda$ , we have the following obvious characterizations of Fréchetness:

**Proposition 7.3.3.** *Suppose  $\lambda$  is a regular uncountable cardinal. The following are equivalent:*

- (1)  $\lambda$  is Fréchet.
- (2) There is a countably complete tail uniform ultrafilter on  $\lambda$ .
- (3) Some ordinal of cofinality  $\lambda$  carries a tail uniform ultrafilter.
- (4) Every ordinal of cofinality  $\lambda$  carries a tail uniform ultrafilter.
- (5) There is an elementary embedding  $j : V \rightarrow M$  that is discontinuous at  $\lambda$ .

*Proof.* (1) implies (2): Since  $\lambda$  is a cardinal, any uniform ultrafilter on  $\lambda$  is tail uniform. Thus since there is a countably complete uniform ultrafilter on  $\lambda$ , there is a countably complete tail uniform ultrafilter on  $\lambda$ .

(2) *implies* (3): Trivial.

(3) *implies* (4): Recall that two ordinals  $\alpha$  and  $\beta$  have the same cofinality if and only if there is a weakly order preserving cofinal function  $f : (\alpha, \leq) \rightarrow (\beta, \leq)$ . In particular,  $f_*(T_\alpha) = T_\beta$  where  $T_\alpha$  is the tail filter on  $\alpha$ . Thus if  $\alpha$  carries a countably complete tail uniform ultrafilter  $U$ , then so does  $\beta$ , namely  $f_*(U)$ .

(4) *implies* (5): Suppose  $U$  is a countably complete tail uniform ultrafilter on  $\lambda$ . Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $U$ . Note that for any  $\alpha < \lambda$ ,  $j(\alpha) < a_U$  since  $\alpha < \delta_U$ . Thus  $\sup j[\lambda] \leq a_U < j(\lambda)$ . In other words,  $j$  is discontinuous at  $\lambda$ .  $\square$

Singular Fréchet cardinals are more subtle, especially when one does not assume the Generalized Continuum Hypothesis. The following fact gives a sense of how singular Fréchet cardinals should arise:

**Proposition 7.3.4.** *Suppose  $\lambda$  is a singular limit of Fréchet cardinals. Let  $\iota$  be the cofinality of  $\lambda$ . Then  $\lambda$  is Fréchet if and only if  $\iota$  is Fréchet.*

*Proof.* If  $\lambda$  is Fréchet, then  $\iota$  is Fréchet by Proposition 7.3.3 (4), and this does not require that  $\lambda$  is a limit of Fréchet cardinals.

We now turn to the converse. Let  $\langle \lambda_\alpha : \alpha < \iota \rangle$  be an increasing cofinal sequence of Fréchet cardinals less than  $\lambda$ . Let  $U_\alpha$  be a countably complete ultrafilter on  $\lambda$  with  $\lambda_{U_\alpha} = \lambda_\alpha$ . Let  $D$  be a countably complete uniform ultrafilter on  $\iota$ . Let

$$U = D\text{-}\lim_{\alpha < \iota} U_\alpha$$

Clearly  $U$  is a countably complete ultrafilter on  $\lambda$ . We claim that  $U$  is uniform, or in other words that every set  $X \in U$  has cardinality  $\lambda$ . Suppose  $X \subseteq \lambda$  is such a set. By the definition of ultrafilter limits,  $\{\alpha < \iota : X \in U_\alpha\} \in D$ . Since  $D$  is a uniform ultrafilter, the set  $\{\alpha < \iota : X \in U_\alpha\}$  is unbounded in  $\iota$ . Therefore  $X \in U_\alpha$  for unboundedly many  $\alpha < \iota$ , and in particular  $|X| \geq \lambda_{U_\alpha} = \lambda_\alpha$  for unboundedly many  $\alpha < \iota$ . Thus  $|X| \geq \sup_{\alpha < \iota} \lambda_\alpha = \lambda$ , as desired. Since  $\lambda$  carries a countably complete uniform ultrafilter, follows that  $\lambda$  is a Fréchet cardinal.  $\square$



Proposition 7.3.4 tells us that when  $\lambda$  is a singular limit of Fréchet cardinals, whether  $\lambda$  is Fréchet depends only on whether the regular cardinal  $\text{cf}(\lambda)$  is Fréchet. One might therefore hope to reduce problems about Fréchet cardinals in general to the regular case, where we have a bit more information. It is not provable in ZFC, however, that a singular Fréchet cardinal must be a limit of Fréchet cardinals. The Fréchet cardinals where this fails are called *isolated cardinals*, and arise as a major issue in our analysis of strong compactness under UA. Isolated cardinals are studied in Section 7.4 and especially Section 7.5.

## Ketonen ultrafilters

The following definition is inspired by the proof of Theorem 7.2.15, which turned on the existence of a  $\kappa$ -complete ultrafilter  $U$  on  $\lambda$  such that  $\text{sup } j_U[\lambda]$  carries no  $\kappa$ -complete tail uniform ultrafilter in  $M_U$ .

Recall from Lemma 4.4.17 that a uniform ultrafilter  $U$  on a regular cardinal  $\lambda$  is weakly normal if and only if letting  $j : V \rightarrow M$  be the ultrapower of the universe by  $U$ ,  $a_U = \text{sup } j[\lambda]$ . Equivalently,  $U$  is weakly normal if it is closed under decreasing diagonal intersections.

**Definition 7.3.5.** If  $\lambda$  is a regular cardinal, an ultrafilter  $U$  on  $\lambda$  is a *Ketonen ultrafilter* if the following hold:

- $U$  is countably complete and weakly normal.
- $U$  concentrates on ordinals that carry no countably complete tail uniform ultrafilter.

By Lemma 4.4.17 and Proposition 7.3.3, we have the following characterization of Ketonen ultrafilters on regular cardinals:

**Lemma 7.3.6.** *Suppose  $\lambda$  is a regular cardinal and  $U$  is a countably complete ultrafilter on  $\lambda$ . Then  $U$  is Ketonen if and only if  $a_U = \text{sup } j_U[\lambda]$  and either of the following equivalent statements holds:*

- $\text{sup } j_U[\lambda]$  carries no countably complete tail uniform ultrafilter in  $M_U$ .

- $\text{cf}^{M_U}(\sup j_U[\lambda])$  is not Fréchet in  $M_U$ . □

In this way the key ordinal  $\text{cf}^{M_U}(\sup j_U[\lambda])$  from Theorem 7.2.12 arises immediately in the study of Ketonen ultrafilters on regular cardinals.

The following theorem asserts that Ketonen ultrafilters are analogous to  $\lambda$ -minimal ultrafilters of Section 4.4, except that Ketonen ultrafilters are minimal in the Ketonen order rather than merely being minimal in the Rudin-Keisler order.

**Lemma 7.3.7.** *Suppose  $\lambda$  is a regular cardinal. Then  $U$  is a Ketonen ultrafilter on  $\lambda$  if and only if  $U$  is a  $<_{\mathbb{k}}$ -minimal element of the set of countably complete uniform ultrafilters on  $\lambda$ .*

*Proof.* Suppose first that  $U$  is a Ketonen ultrafilter. Let

$$\alpha = a_U = \sup j_U[\lambda]$$

Suppose  $W <_{\mathbb{k}} U$ . We will show that  $\lambda_W < \lambda$ . By the definition of the Ketonen order (Lemma 3.3.4), there is some  $Z \in \mathcal{B}^{M_U}(j_U(\lambda), \sup j_U[\lambda])$  such that  $j_U^{-1}[Z] = W$ . Since  $\sup j_U[\lambda]$  does not carry a countably complete tail uniform ultrafilter in  $M_U$ , there is some  $\beta < \sup j_U[\lambda]$  such that  $Z$  concentrates on  $\beta$ . Fix  $\alpha < \lambda$  such that  $j_U(\alpha) \geq \beta$ . Then  $j_U(\alpha) \in Z$ , so  $\alpha \in W$ . Thus  $\lambda_W < \lambda$  as desired.

Conversely, assume  $U$  is a  $<_{\mathbb{k}}$ -minimal element of the set of uniform ultrafilters on  $\lambda$ . In particular,  $U$  is an  $<_{\text{rk}}$ -minimal element of the set of uniform ultrafilters on  $\lambda$ , which by Lemma 4.4.20 is equivalent to being weakly normal.

Finally, fix  $Z \in \mathcal{B}^{M_U}(j_U(\lambda))$ , and we will show that  $\delta_Z < \sup j_U[\lambda]$ . Let  $W = j_U^{-1}[Z]$ . Then  $W <_{\mathbb{k}} U$  by the definition of the Ketonen order. It from the minimality of  $U$  that  $\delta_W < \lambda$ , so for some  $\alpha < \lambda$ ,  $\alpha \in W$ . Now  $j_U(\alpha) \in Z$ , so  $\delta_Z \leq j_U(\alpha) < \sup j_U[\lambda]$ , as desired.

It follows that  $\sup j_U[\lambda]$  does not carry a countably complete tail uniform ultrafilter in  $M_U$ , so  $U$  is Ketonen by Lemma 7.3.6. □

Reflecting on Lemma 7.3.7, we obtain a definition of Ketonen ultrafilters on arbitrary cardinals:

**Definition 7.3.8.** Suppose  $\lambda$  is a Fréchet cardinal. An ultrafilter  $U$  on  $\lambda$  is *Ketonen* if  $U$  is a  $<_{\mathbb{k}}$ -minimal element of the set of countably complete uniform ultrafilters on  $\lambda$ .

The wellfoundedness of the Ketonen order (Theorem 3.3.8) immediately yields the existence of Ketonen ultrafilters:

**Theorem 7.3.9.** *Every Fréchet cardinal carries a Ketonen ultrafilter.* □

When  $\lambda$  is singular, it is important that the definition of a Ketonen ultrafilter demands minimality only among uniform ultrafilters and not among the broader class of tail uniform ultrafilters, since an ultrafilter on  $\lambda$  that is minimal in this stronger sense is essentially the same thing as a Ketonen ultrafilter on  $\text{cf}(\lambda)$ :

**Lemma 7.3.10.** *Suppose  $\gamma$  is an ordinal and  $U$  is a  $<_{\mathbb{k}}$ -minimal among countably complete ultrafilters  $W$  with  $\delta_W = \gamma$ . Let  $\lambda = \text{cf}(\gamma)$  and let  $f : \lambda \rightarrow \gamma$  be a continuous cofinal function. Then  $U = f_*(D)$  for some Ketonen ultrafilter  $D$  on  $\lambda$ .*

*Proof.* Since  $U$  is  $<_{\mathbb{k}}$ -minimal among countably complete ultrafilters  $W$  with  $\delta_W = \gamma$ , in particular  $U$  is  $<_{\text{rk}}$ -minimal, so every function  $g : \gamma \rightarrow \gamma$  that is regressive on a set in  $U$  is bounded on a set in  $U$ . It follows that  $U$  contains every closed cofinal  $C \subseteq \gamma$ : letting  $A = \gamma \setminus C$  and  $g(\alpha) = \sup(C \cap \alpha)$ ,  $g$  is regressive on  $A$  and unbounded on any cofinal subset of  $A$ .

Let  $C = f[\lambda]$ . Then  $C \in U$ . Let  $g : C \rightarrow \lambda$  be the inverse of  $f$ . Let  $D = g_*(U)$ . Clearly  $U = f_*(D)$ . We must show that  $D$  is Ketonen. Suppose  $W <_{\mathbb{k}} D$ . We claim  $f_*(W) <_{\mathbb{k}} U$ . Given this, it follows that  $\delta_{f_*(W)} < \gamma$  and hence  $\delta_W < \lambda$ . It follows that  $D$  is a  $<_{\mathbb{k}}$ -minimal element of the set of countably complete uniform ultrafilters on  $\lambda$ , so  $D$  is Ketonen.

We finally verify  $f_*(W) <_{\mathbb{k}} U$ . (The proof will show that if  $f : \lambda \rightarrow \gamma$  is an order preserving function, then the pushforward map  $f_*$  is Ketonen order preserving.) Fix  $I \in D$  and  $\langle W_\alpha : \alpha \in I \rangle$  such that  $W = D\text{-}\lim_{\alpha \in I} W_\alpha$  and  $\delta_\alpha \leq \alpha$  for all  $\alpha \in I$ . Let  $J = f[I]$ , so

$J \in U$  and moreover:

$$f_*(W) = U\text{-}\lim_{\beta \in J} f_*(W_{g(\beta)})$$

Moreover  $\delta_{f_*(W_{g(\beta)})} \leq \sup f[\delta_{W_{g(\beta)}}] \leq \sup f[g(\beta)] \leq \beta$ . Thus the sequence  $\langle f_*(W_{g(\beta)}) : \beta \in J \rangle$  witnesses  $f_*(W) <_{\mathbb{k}} U$ , as desired.  $\square$

## Introducing $\mathcal{K}_\lambda$

Under the Ultrapower Axiom, the Ketonen order is linear, so there is a canonical Ketonen ultrafilter on each Fréchet cardinal  $\lambda$ :

**Definition 7.3.11 (UA).** For any Fréchet cardinal  $\lambda$ , the *least ultrafilter on  $\lambda$* , denoted by  $\mathcal{K}_\lambda$ , is the unique Ketonen ultrafilter on  $\lambda$ .

The analysis of supercompactness under UA proceeds by first completely analyzing the ultrafilters  $\mathcal{K}_\lambda$  and then propagating the structure of  $\mathcal{K}_\lambda$  to all ultrafilters.

Let us begin with some simple examples. Let  $\kappa_0$  be the least measurable cardinal. Then without assuming UA, it is easy to prove that an ultrafilter on  $\kappa_0$  is Ketonen if and only if it is normal. Assuming UA,  $\mathcal{K}_{\kappa_0}$  is the unique normal ultrafilter on  $\kappa_0$ .

Moving up to the second measurable cardinal  $\kappa_1$ , it is *not* provable in ZFC that the Ketonen ultrafilters on  $\kappa_1$  are normal, or even that there is a normal Ketonen ultrafilter on  $\kappa_1$ . This is because it is consistent that  $\kappa_0$  is  $\kappa_1$ -strongly compact. Under this assumption, if  $U$  is a normal ultrafilter on  $\kappa_1$ ,  $\kappa_0$  is  $j_U(\kappa_1)$ -strongly compact in  $M_U$ , and hence  $U$  concentrates on ordinals that carry  $\kappa_0$ -complete uniform ultrafilters. In fact, under this hypothesis, if  $W$  is a Ketonen ultrafilter on  $\kappa_1$ , then  $j_W$  is  $(\kappa_1, \delta)$ -tight for some  $\delta < j_W(\kappa)$ , and hence witnesses the  $\kappa_1$ -strong compactness of  $\kappa_0$ .

Of course, under UA,  $\kappa_0$  is not  $\kappa_1$ -strongly compact, since by Theorem 5.3.21, every countably complete ultrafilter in  $V_{\kappa_1}$  is isomorphic to  $\mathcal{K}_{\kappa_0}^n$  for some  $n < \omega$ . In fact, once again  $\mathcal{K}_{\kappa_1}$  is the unique normal ultrafilter on  $\kappa_1$ . To see this, one can apply Theorem 5.3.8 and the following lemma:

**Lemma 7.3.12** (UA). *For any regular cardinal  $\lambda$ ,  $\mathcal{K}_\lambda$  is an irreducible ultrafilter.*

*Proof.* Suppose  $D <_{\text{RF}} \mathcal{K}_\lambda$ . Then since  $D <_{\text{RK}} \mathcal{K}_\lambda$  and  $\mathcal{K}_\lambda$  is weakly normal,  $\lambda_D < \lambda$ . Therefore by Lemma 3.5.32,

$$j_D(\lambda) = \sup j_D[\lambda]$$

Assume towards a contradiction that  $D$  is nonprincipal. Then by Proposition 5.4.5,  $t_D(\mathcal{K}_\lambda) <_{\mathfrak{k}} j_D(\mathcal{K}_\lambda)$ , so  $\delta_{t_D(\mathcal{K}_\lambda)} < j_D(\lambda)$  by Lemma 7.3.7 applied in  $M_D$ . But  $\mathcal{K}_\lambda = j_D^{-1}[t_D(\mathcal{K}_\lambda)]$ , so

$$\delta_{\mathcal{K}_\lambda} = \min\{\delta : j_D(\delta) > \delta_{t_D(\mathcal{K}_\lambda)}\} < \lambda$$

This contradicts that  $\mathcal{K}_\lambda$  is a uniform ultrafilter on  $\lambda$ . □

We do not know whether this lemma is provable in ZFC, although it does follow from Theorem 5.3.17.

If  $\lambda$  is singular, then  $\mathcal{K}_\lambda$  is not necessarily irreducible. (In fact, we will show under UA that for strong limit singular cardinals  $\lambda$ ,  $\mathcal{K}_\lambda$  is *never* irreducible.) For example, suppose  $\lambda_0$  is the least singular cardinal that carries a uniform countably complete ultrafilter. Of course, assuming just ZFC, one cannot prove much about  $\lambda_0$ : it is consistent that  $\lambda_0 = \kappa_0^{+\kappa_0}$ , or that  $\lambda_0$  is not a limit of regular cardinals that carry uniform countably complete ultrafilters.

Assuming UA, it is not hard to give a complete analysis of  $\lambda_0$  and  $\mathcal{K}_{\lambda_0}$ . Let  $\langle \kappa_\alpha : \alpha < \kappa_0 \rangle$  enumerate the first  $\kappa_0$  measurable cardinals in increasing order. Then  $\lambda_0 = \sup_{\alpha < \kappa_0} \kappa_\alpha$ , and

$$\mathcal{K}_{\lambda_0} = \mathcal{K}_{\kappa_0}\text{-}\lim_{\alpha < \kappa_0} \mathcal{K}_{\kappa_\alpha} \upharpoonright \lambda_0$$

The sets  $A_\alpha = \kappa_\alpha \setminus \sup_{\beta < \alpha} \kappa_\beta$  witness that the sequence  $\langle \mathcal{K}_{\kappa_\alpha} \upharpoonright \lambda_0 : \alpha < \kappa_0 \rangle$  is discrete, so  $\mathcal{K}_{\kappa_0} <_{\text{RF}} \mathcal{K}_{\lambda_0}$ . In other words,  $\mathcal{K}_{\lambda_0}$  is produced by the iterated ultrapower  $\langle \mathcal{K}_\kappa, \mathcal{K}_{\lambda_0}^{M_{\mathcal{K}_\kappa}} \rangle$ .

Of course all this is closely related to Proposition 7.3.4. For singular cardinals  $\lambda$ ,  $\mathcal{K}_\lambda$  is of greatest interest if  $\lambda$  is not a limit of Fréchet cardinals, since in this case  $\mathcal{K}_\lambda$  cannot be represented in terms of ultrafilters on smaller cardinals.

## The universal property of $\mathcal{K}_\lambda$

The main result of this section is a universal property of the least ultrafilter  $\mathcal{K}_\lambda$  on a regular Fréchet cardinal:

**Theorem 7.3.13** (UA). *Suppose  $\lambda$  is a regular Fréchet cardinal. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Suppose  $i : V \rightarrow N$  is an ultrapower embedding. Then the following are equivalent:*

- (1) *There is an internal ultrapower embedding  $k : M \rightarrow N$  such that  $k \circ j = i$ .*
- (2)  *$\sup i[\lambda]$  carries no tail uniform ultrafilter in  $N$ .*
- (3)  *$\text{cf}^N(\sup i[\lambda])$  is not Fréchet in  $N$ .*

While the proof is quite simple, the result has profound consequences for the structure of the ultrafilters  $\mathcal{K}_\lambda$ . In fact, this universal property is ultimately responsible for all of our results on supercompactness under UA.

Before proving Theorem 7.3.13 (which is not that difficult), let us show how it can be used to give a complete analysis of the internal ultrapower embeddings of  $M_{\mathcal{K}_\lambda}$  when  $\lambda$  is regular.

**Theorem 7.3.14** (UA). *Suppose  $\lambda$  is a regular Fréchet cardinal. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Suppose  $k : M \rightarrow N$  is an ultrapower embedding. Then the following are equivalent:*

- (1)  *$k$  is an internal ultrapower embedding.*
- (2)  *$k$  is continuous at  $\sup j[\lambda]$ .*
- (3)  *$k$  is continuous at  $\text{cf}^M(\sup j[\lambda])$ .*

*Proof.* (1) implies (2): Since  $\sup j[\lambda]$  carries no tail uniform countably complete ultrafilter in  $M$ , every elementary embedding of  $M$  that is close to  $M$  is continuous at  $\sup j[\lambda]$ . In particular, every internal ultrapower embedding of  $M$  is continuous at  $\sup j[\lambda]$ .

(2) *implies* (1): Let  $i = k \circ j$ . Then  $\sup i[\lambda] = \sup k[\sup j[\lambda]] = k(\sup j[\lambda])$  since  $k$  is continuous at  $\sup j[\lambda]$ . It follows that  $\sup i[\lambda]$  carries no tail uniform countably complete ultrafilter in  $N$ . Therefore by Theorem 7.3.13, there is an internal ultrapower embedding  $k' : M \rightarrow N$  such that  $k' \circ j = i$ .

We claim  $k' = k$ . First of all,  $k' \circ j = k \circ j$ . In other words,  $k' \upharpoonright j[V] = k \upharpoonright j[V]$ . Moreover since  $k'$  is  $M$ -internal  $k'(\sup j[\lambda]) = \sup i[\lambda] = k(\sup j[\lambda])$ . But  $M = H^M(j[V] \cup \{a_{\mathcal{K}_\lambda}\}) = H^M(j[V] \cup \{\sup j[\lambda]\})$  since  $\mathcal{K}_\lambda$  is weakly normal. Since we have shown  $k' \upharpoonright j[V] \cup \{\sup j[\lambda]\} = k \upharpoonright j[V] \cup \{\sup j[\lambda]\}$ , it follows that  $k' = k$ .

Since  $k'$  is an internal ultrapower embedding, so is  $k$ , as desired.

The equivalence of (2) and (3) is trivial (and does not require UA).  $\square$

The notion of *indecomposable ultrafilters* is an important part of infinite combinatorics. We will need the following relativized version of this concept:

**Definition 7.3.15.** Suppose  $M$  is a transitive model of ZFC and  $U$  is an  $M$ -ultrafilter on  $X$ . Suppose  $\delta$  is an  $M$ -cardinal. Then  $U$  is  $\delta$ -*indecomposable* if for any partition  $\langle X_\alpha : \alpha < \delta \rangle \in M$  of  $X$ , there is some  $S \subseteq \delta$  in  $M$  with  $|S|^M < \delta$  and  $\bigcup_{\alpha \in S} X_\alpha \in U$ .

As a corollary of Theorem 7.3.14, every  $\lambda$ -indecomposable ultrafilter is internal to  $\mathcal{K}_\lambda$ :

**Corollary 7.3.16** (UA). *Suppose  $\lambda$  is a regular Fréchet cardinal. Suppose  $D$  is a countably complete  $\lambda$ -indecomposable ultrafilter, then  $D \sqsubset \mathcal{K}_\lambda$ . In particular, if  $D$  is a countably complete ultrafilter such that  $\lambda_D < \lambda$ , then  $D \sqsubset \mathcal{K}_\lambda$ .*

*Proof.* Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . To show  $D \sqsubset \mathcal{K}_\lambda$ , we need to show that  $j_D \upharpoonright M$  is an internal ultrapower embedding of  $M_U$ . By Lemma 5.5.9,  $j_D \upharpoonright M$  is an ultrapower embedding. Since  $D$  is  $\lambda$ -indecomposable,  $j_D$  is continuous at all ordinals of cofinality  $\lambda$ , and in particular,  $j_D$  is continuous at  $\sup j[\lambda]$ . Thus  $j_D \upharpoonright M$  is an ultrapower embedding of  $M$  that is continuous at  $\sup j[\lambda]$ , and it follows from Theorem 7.3.14 that  $j_D \upharpoonright M$  is an internal ultrapower embedding of  $M$ , as desired.  $\square$

This fact is highly reminiscent of Corollary 4.3.28, the theorem that analyzes which ultrafilters lie Mitchell below a Dodd solid ultrafilter. In fact, we will show that  $\mathcal{K}_\lambda$  gives rise to a supercompact ultrapower precisely by leveraging the fact that so many ultrapower embeddings are internal to it. (See Section 7.3, Section 7.3, and especially Proposition 7.3.32.)

If  $M$  is a transitive model of ZFC and  $\delta$  is an  $M$ -regular cardinal, then an  $M$ -ultrafilter  $U$  is  $\delta$ -indecomposable if and only if  $j_U^M$  is continuous at  $\delta$ , and we therefore obtain the following combinatorial characterization of the countably complete  $M$ -ultrafilters that belong to  $M$  when  $M$  is the ultrapower of the universe by a Ketonen ultrafilter on a regular cardinal:

**Theorem 7.3.17 (UA).** *Suppose  $\lambda$  is a regular Fréchet cardinal. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Let  $\delta = \text{cf}^M(\sup j[\lambda])$ . Suppose  $U$  is a countably complete  $M$ -ultrafilter. Then the following are equivalent:*

(1)  $U$  is  $\delta$ -indecomposable.

(2)  $U \in M$ .

*In particular, if  $U$  is a countably complete  $M$ -ultrafilter on a cardinal  $\gamma < \delta$ , then  $U \in M$ .  $\square$*

In summary, the universal property of  $\mathcal{K}_\lambda$  is a powerful tool for analyzing the model  $M_{\mathcal{K}_\lambda}$ . Let us therefore prove it:

*Proof of Theorem 7.3.13. (1) implies (2):* First,  $k(\sup j[\lambda])$  carries no tail uniform countably complete ultrafilter in  $N$  by elementarity, since  $\sup j[\lambda]$  carries no tail uniform countably complete ultrafilter in  $M$ . Note also that  $k : M \rightarrow N$  is continuous at  $\sup j[\lambda]$  since  $\sup j[\lambda]$  carries no tail uniform countably complete ultrafilter in  $M$ . Therefore  $k(\sup j[\lambda]) = \sup k \circ j[\lambda] = \sup i[\lambda]$ . Hence  $\sup i[\lambda]$  carries no tail uniform countably complete ultrafilter in  $N$ .

*(2) implies (1):* Let  $(e, h) : (M, N) \rightarrow P$  be an internal ultrapower comparison of  $(j, i)$ . Then

$$e(\sup j[\lambda]) = \sup e \circ j[\lambda] = \sup h \circ i[\lambda] = h(\sup i[\lambda])$$



The theorem is now an immediate consequence of Lemma 3.5.27:  $M = H^M(j[V] \cup \{\sup j[\lambda]\})$  and  $(e, h)$  witnesses  $(j, \sup j[\lambda]) =_S (i, \sup i[\lambda])$ , so there is an internal ultrapower embedding  $k : M \rightarrow N$  such that  $k \circ j = i$ .

The equivalence of (2) and (3) is trivial (and does not require UA). □

## Independent families and the Hamkins properties

A key intuition in the theory of forcing is that forcing does not create new large cardinals. The Levy-Solovay Theorem [30] establishes this for small forcing, but various counterintuitive forcing constructions of the next few decades demonstrate that in general, the intuition is just not correct. The earliest example, due to Kunen, shows that it is consistent that there is a forcing that makes a measurable cardinal out of a cardinal that is not even weakly compact. Woodin’s  $\Sigma_2$ -Resurrection Theorem ([8], Theorem 2.5.10) yields even more striking examples: for example, if there is a proper class of Woodin cardinals and there is a huge cardinal, then arbitrarily large cardinals can be forced to be huge cardinals.

Hamkins isolated two closure properties of inner models: the *approximation* and *covering* properties, or collectively the *Hamkins properties*. If an inner model  $M$  has the Hamkins properties, then many of the large cardinal properties of the ambient universe of sets are downwards absolute to  $M$ . For many forcing extensions  $V[G]$ , the universe  $V$  satisfies the Hamkins properties inside  $V[G]$ , and therefore the large cardinals of  $V[G]$  “already exist” in  $V$ .

Somewhat unexpectedly, the Hamkins properties have turned out to be relevant outside of the domain of forcing, in the provenance of inner model theory. Woodin has shown that inherits a supercompact cardinal  $\kappa$  from the ambient universe in a natural way necessarily satisfies the Hamkins properties at  $\kappa$ , and therefore inherits all large cardinals from the ambient universe. Such models are called *weak extender models* for the supercompactness of  $\kappa$ . A canonical inner model with a supercompact cardinal is expected to be a weak extender model, and therefore Woodin conjectures that if there is a canonical inner model with a

supercompact cardinal, in fact this is the *ultimate inner model*, a canonical inner model that satisfies all true large cardinal axioms.

In our work on UA, the Hamkins properties rear their heads once again. Here they arise in relation with (generalizations of) the Mitchell order, which can be seen as yet another instantiation of the downwards absoluteness of large cardinal properties to inner models. Recall that we are trying to show that the ultrapower of the universe by  $\mathcal{K}_\lambda$  has closure properties. All we know so far is that this ultrapower absorbs many countably complete ultrafilters (Theorem 7.3.17). To transform this into a model theoretic closure property of the ultrapower, for example closure under  $\lambda$ -sequences, we prove a converse to Hamkins and Woodin's absoluteness theorems for models with the Hamkins properties. This converse says that any inner model that inherits enough ultrafilters from the ambient universe must satisfy the Hamkins properties. In our context, this will lead to a proof that the ultrapower  $\mathcal{K}_\lambda$  is (roughly) closed under  $\lambda$ -sequences.

The ultrapowers we consider do not satisfy the (relevant) Hamkins properties in full, but rather satisfy local versions of these properties, introduced here for the first time:

**Definition 7.3.18.** Suppose  $M$  is an inner model,  $\kappa$  is a cardinal, and  $\lambda$  is an ordinal.

- $M$  has the  $\kappa$ -covering property at  $\lambda$  if every  $\sigma \in P_\kappa(\lambda)$  there is some  $\tau \in P_\kappa(\lambda) \cap M$  with  $\sigma \subseteq \tau$ .
- $M$  has the  $\kappa$ -approximation property at  $\lambda$  if any  $A \subseteq \lambda$  with  $A \cap \sigma \in M$  for all  $\sigma \in P_\kappa(\lambda) \cap M$  is an element of  $M$ .

We say  $M$  has the  $\kappa$ -covering property if  $M$  has the  $\kappa$ -covering property at all  $M$ -cardinals, and  $M$  has the  $\kappa$ -approximation property if  $M$  has the  $\kappa$ -approximation property at all  $M$  cardinals.

In this section, we identify necessary and sufficient conditions for the  $\kappa$ -covering and approximation properties that involve the absorption of filters. We are working in slightly

more generality than we will need, but we think the results are quite interesting and hopefully lead to a clearer exposition than would arise by working in a more specific case.

The condition equivalent to the covering property essentially comes from Woodin's proof of the covering property for weak extender models:

**Proposition 7.3.19.** *Suppose  $M$  is an inner model. Then  $M$  has the  $\kappa$ -covering property at  $\lambda$  if and only if there is a  $\kappa$ -complete fine filter on  $P_\kappa(\lambda)$  that concentrates on  $M$ .*

*Proof.* First assume there is a  $\kappa$ -complete fine filter  $\mathcal{F}$  on  $P_\kappa(\lambda)$  that concentrates on  $M$ . Fix  $\sigma \in P_\kappa(\lambda)$ , and we will find  $\tau \in P_\kappa(\lambda) \cap M$  such that  $\sigma \subseteq \tau$ . For each  $\alpha < \lambda$ , let  $A_\alpha = \{\tau \in P_\kappa(\lambda) : \alpha \in \tau\}$ , so that  $A_\alpha \in \mathcal{F}$  by the definition of a fine filter. Then suppose  $\sigma \in P_\kappa(\lambda)$ . The set

$$\{\tau \in P_\kappa(\lambda) : \sigma \subseteq \tau\} = \bigcap_{\alpha \in \sigma} \{A_\alpha : \alpha \in \sigma\} \in \mathcal{F}$$

since  $\mathcal{F}$  is  $\kappa$ -complete. Since  $\mathcal{F}$  concentrates on  $M$ ,  $\{\tau \in P_\kappa(\lambda) : \sigma \subseteq \tau\} \cap M \in \mathcal{F}$ , and in particular this set is nonempty. Any  $\tau$  that belongs to this set satisfies  $\tau \in P_\kappa(\lambda) \cap M$  and  $\sigma \subseteq \tau$ , as desired.

Conversely, assume  $M$  has the  $\kappa$ -covering property at  $\lambda$ . Let  $\mathcal{B} = \{A_\alpha \cap M : \alpha < \lambda\}$ . Then  $\mathcal{B}$  is a  $\kappa$ -complete filter base: for any  $S \subseteq \mathcal{B}$  with  $|S| < \kappa$ , we have  $S = \{A_\alpha \cap M : \alpha \in \sigma\}$  for some  $\sigma \in P_\kappa(\lambda)$ , and so fixing  $\tau \in P_\kappa(\lambda) \cap M$  such that  $\sigma \subseteq \tau$ , we have  $\tau \in \bigcap_{\alpha \in \sigma} A_\alpha$ . Therefore  $\mathcal{B}$  extends to a  $\kappa$ -complete filter  $\mathcal{G}$ . Let

$$\mathcal{F} = \mathcal{G} \upharpoonright P_\kappa(\lambda) = \{A \subseteq P_\kappa(\lambda) : A \cap M \in \mathcal{G}\}$$

be the canonical extension of  $\mathcal{G}$  to an filter on  $P_\kappa(\lambda)$ . Then  $\mathcal{F}$  is  $\kappa$ -complete and concentrates on  $M$ . Moreover,  $A_\alpha \in \mathcal{F}$  for all  $\alpha < \lambda$ , so  $\mathcal{F}$  is fine. Thus we have produced a  $\kappa$ -complete fine filter on  $P_\kappa(\lambda)$  that concentrates on  $M$ , as desired.  $\square$

One ultrafilter theoretic characterization of the approximation property is given by the following theorem:

**Theorem 7.3.20.** *Suppose  $\kappa$  is strongly compact and  $M$  is an inner model with the  $\kappa$ -covering property. Then  $M$  has the  $\kappa$ -approximation property if and only if every  $\kappa$ -complete  $M$ -ultrafilter belongs to  $M$ .*

We will actually prove a local version of this theorem that requires no large cardinal assumptions. The locality of this theorem is important in our analysis of the ultrafilters  $\mathcal{U}_\lambda$ . For the statement, we need use the following definition:

**Definition 7.3.21.** Suppose  $X$  is a set and  $\Sigma$  is an algebra of subsets of  $X$ . A set  $U \subseteq \Sigma$  is said to be an *ultrafilter over  $\Sigma$*  if  $U$  is closed under intersections and for any  $A \in \Sigma$ ,  $A \in U$  if and only if  $X \setminus A \notin U$ . An ultrafilter  $U$  over  $\Sigma$  is said to be  $\kappa$ -complete if for any  $\sigma \in P_\kappa(U)$ ,  $\bigcap \sigma \neq \emptyset$ .

What we call an ultrafilter over  $\Sigma$  is commonly referred to as an *ultrafilter on the Boolean algebra  $\Sigma$* , but we are being a bit pedantic: we do not want to confuse this with an ultrafilter with underlying set  $\Sigma$ , which in our terminology is a family of subsets of  $\Sigma$  rather than a subset of  $\Sigma$ . Notice that for us a  $\kappa$ -complete ultrafilter over  $\Sigma$  is the same thing as an ultrafilter over  $\Sigma$  that is a  $\kappa$ -complete filter base. (It is not the same thing as being  $\kappa$ -complete ultrafilter on the Boolean algebra  $\Sigma$ .)

**Theorem 7.3.22.** *Suppose  $M$  is an inner model,  $\kappa$  is a cardinal,  $\lambda$  is an  $M$ -cardinal, and  $M$  has the  $\kappa$ -covering property at  $\lambda$ . Then the following are equivalent:*

- (1)  *$M$  has the  $\kappa$ -approximation property at  $\lambda$ .*
- (2) *Suppose  $\Sigma \in M$  is an algebra of sets of  $M$ -cardinality  $\lambda$ . Then every  $\kappa$ -complete ultrafilter over  $\Sigma$  belongs to  $M$ .*

To simplify notation, we use the following lemma (analogous in flavor to Lemma 7.2.6) characterizing the approximation property at  $\lambda$ :

**Lemma 7.3.23.** *Suppose  $M$  is an inner model,  $\kappa$  is a cardinal, and  $\lambda$  is an  $M$ -cardinal. Then the following are equivalent:*

- (1)  $M$  has the  $\kappa$ -approximation property at  $\lambda$
- (2) For all  $\Sigma \in M$  such that  $|\Sigma|^M \leq \lambda$ , for all  $B \subseteq \Sigma$  such that  $B \cap \sigma \in M$  for all  $\sigma \in P_\kappa(\Sigma) \cap M$ ,  $B \in M$ .
- (3) For some  $\Sigma \in M$  such that  $|\Sigma|^M = \lambda$ , for all  $B \subseteq \Sigma$  such that  $B \cap \sigma \in M$  for all  $\sigma \in P_\kappa(\Sigma) \cap M$ ,  $B \in M$ . □

The following notation will be convenient (although of course it is a bit ambiguous):

**Definition 7.3.24.** Suppose  $X$  is a set and  $\sigma$  is a family of subsets of  $X$ . Then the *dual* of  $\sigma$  in  $X$  is the family  $\sigma^* = \{X \setminus A : A \in \sigma\}$ .

We point out that the dualizing operation depends implicitly on the underlying set  $X$ .

**Definition 7.3.25.** Suppose  $\kappa$  is a cardinal and  $X$  is a set. A family  $\Gamma$  of subsets of  $X$  is  $\kappa$ -independent if for any disjoint sets  $\tau_0, \tau_1 \in P_\kappa(\Gamma)$ ,  $\bigcap \tau_0 \cap \bigcap \tau_1^* \neq \emptyset$ .

Equivalently,  $\Gamma$  is  $\kappa$ -independent if for any disjoint sets  $X, Y \subseteq \Gamma$ , the collection  $X \cup Y^*$  is a  $\kappa$ -complete filter base. Note that a  $\kappa$ -complete family of subsets of  $X$  is never an algebra of sets, since if  $A \in \Gamma$ , then  $X \setminus A \notin \Gamma$ .

**Theorem 7.3.26** (Hausdorff). *Suppose  $\kappa$  and  $\lambda$  are cardinals. Then there is a  $\kappa$ -independent family of subsets of  $X = \{(\sigma, s) : \sigma \in P_\kappa(\lambda) \text{ and } s \in P_\kappa(P(\sigma))\}$  of cardinality  $2^\lambda$ .*

*Proof.* Define  $f : P(\lambda) \rightarrow P(X)$  by

$$f(A) = \{(\sigma, s) \in X : \sigma \cap S \in s\}$$

Let  $\Gamma = \text{ran}(f)$ . Suppose  $\tau_0, \tau_1 \in P_\kappa(P(\lambda))$  are disjoint. We claim that the set

$$S = \bigcap f[\tau_0] \cap \bigcap f[\tau_1]^*$$

is nonempty. This simultaneously shows that  $f$  is injective and  $\Gamma$  is  $\kappa$ -independent. Therefore  $\Gamma$  is a  $\kappa$ -independent family of cardinality  $2^\lambda$ .

Let  $\sigma \in P_\kappa(\lambda)$  be large enough that  $\sigma \cap A_0 \neq \sigma \cap A_1$  for any  $A_0 \in \tau_0$  and  $A_1 \in \tau_1$ . Let

$$s = \{\sigma \cap A : A \in \tau_0\}$$

We claim that  $(\sigma, s) \in S$ .

First we show that  $(\sigma, s) \in \bigcap f[\tau_0]$ . Suppose  $A \in \tau_0$ . We will show that  $(\sigma, s) \in f(A)$ . By the definition of  $s$ , since  $A \in \tau_0$ ,  $\sigma \cap A \in s$ . Therefore by the definition of  $f$ ,  $(\sigma, s) \in f(A)$ , as desired. This shows  $(\sigma, s) \in \bigcap f[\tau_0]$ .

Next we show that  $(\sigma, s) \in \bigcap f[\tau_1]^*$ . Suppose  $B \in \tau_1$ , and we will show that  $(\sigma, s) \in X \setminus B$ . By the choice of  $\sigma$ ,  $\sigma \cap B \neq \sigma \cap A$  for any  $A \in \tau_0$ . Therefore by the definition of  $s$ ,  $\sigma \cap B \notin s$ . Finally, by the definition of  $f$ , it follows that  $(\sigma, s) \notin f(B)$ , or in other words,  $(\sigma, s) \in X \setminus B$ . Hence  $(\sigma, s) \in \bigcap f[\tau_1]^*$ .

Since  $(\sigma, s) \in \bigcap f[\tau_0]$  and  $(\sigma, s) \in \bigcap f[\tau_1]^*$ , it follows that  $(\sigma, s) \in S$ . Thus  $S$  is nonempty, which completes the proof.  $\square$

Computing cardinalities, Hausdorff's theorem implies the existence of  $\kappa$ -independent sets that are as large as possible:

**Corollary 7.3.27** (Hausdorff). *Suppose  $\kappa$  and  $\lambda$  are cardinals such that  $\lambda^{<\kappa} = \lambda$ . Then there is a  $\kappa$ -independent family of subsets of  $\lambda$  of cardinality  $2^\lambda$ .*

*Proof.* Let  $X = \{(\sigma, s) : \sigma \in P_\kappa(\lambda) \text{ and } s \in P_\kappa(P(\sigma))\}$ . In other words,

$$X = \coprod_{\sigma \in P_\kappa(\lambda)} P_\kappa(P(\sigma))$$

Thus

$$|X| = |P_\kappa(\lambda)| \cdot \sup_{\sigma \in P_\kappa(\lambda)} |P_\kappa(P(\sigma))| = \lambda^{<\kappa} \cdot (2^{<\kappa})^{<\kappa} = \lambda^{<\kappa} = \lambda$$

By Theorem 7.3.26, there is a  $\kappa$ -independent family of subsets of  $X$  of cardinality  $2^\lambda$ , and therefore there is a  $\kappa$ -independent family of subsets of  $\lambda$  of cardinality  $2^\lambda$ .  $\square$

We now establish our characterization of the approximation property.

*Proof of Theorem 7.3.22. (1) implies (2):* Assume (1), and we will prove (2). Suppose  $\Sigma \in M$  is an algebra of subsets of  $X$  of  $M$ -cardinality  $\lambda$  and  $U$  is a  $\kappa$ -complete ultrafilter over  $\Sigma$ . Fix  $\sigma \in P_\kappa(\Sigma) \cap M$  and we will show that  $\sigma \cap U \in M$ . Since  $U$  is  $\kappa$ -complete,

$$S = \bigcap \{A : A \in \sigma \cap U\} \cap \bigcap \{X \setminus A : A \in \sigma \setminus U\}$$

is nonempty. Therefore fix  $a \in X$  with  $a \in S$ . By the choice of  $a$ ,  $\sigma \cap U = \{A \in \sigma : a \in A\}$ . Thus  $\sigma \cap U \in M$ .

By the  $\kappa$ -approximation property at  $\lambda$  (using Lemma 7.3.23), it follows that  $U \in M$ .

*(2) implies (1):* Fix  $\Gamma \in M$  such that  $M$  satisfies that  $\Gamma$  is a  $\kappa$ -independent family of subsets of some set  $X$  and  $|\Gamma|^M = \lambda$ . Suppose  $C \subseteq \Gamma$  is such that  $C \cap \sigma \in M$  for all  $\sigma \in P_\kappa(\Gamma) \cap M$ . We will show that  $C \in M$ . This verifies the condition of Lemma 7.3.23 (3), and so implies that  $M$  satisfies the  $\kappa$ -approximation property at  $\lambda$ .

Let

$$\mathcal{B} = C \cup (\Gamma \setminus C)^*$$

We claim that  $\mathcal{B}$  is a  $\kappa$ -complete filter base on  $X$ . Suppose  $\sigma \in P_\kappa(\mathcal{B})$ . We must show that  $\bigcap \sigma \neq \emptyset$ . Using the  $\kappa$ -covering property at  $\lambda$ , fix  $\tau \in P_\kappa(\Gamma) \cap M$  such that  $\sigma \subseteq \tau \cup \tau^*$ .

By our assumption on  $C$ ,  $\tau \cap C \in M$ . Let  $\tau_0 = \tau \cap C$  and let  $\tau_1 = \tau \setminus C = \tau \setminus \tau_0 \in M$ . Since  $\sigma \subseteq \mathcal{B} = C \cup (\Gamma \setminus C)^*$ , we have  $\sigma \subseteq \tau_0 \cup \tau_1^*$ . Since  $\Gamma$  is  $\kappa$ -independent in  $M$ ,

$$\bigcap \tau_0 \cap \bigcap \tau_1^* \neq \emptyset$$

But  $\bigcap \tau_0 \cap \bigcap \tau_1^* = \bigcap (\tau_0 \cup \tau_1) \subseteq \bigcap \sigma$ , and hence  $\bigcap \sigma \neq \emptyset$ , as desired. This shows  $\mathcal{B}$  is a  $\kappa$ -complete filter base.

Let  $\Sigma$  be the algebra on  $X$  generated by  $\Gamma$  and let  $U$  be the ultrafilter over  $\Sigma$  generated by  $\mathcal{B}$ . Then  $U$  is  $\kappa$ -complete because  $\mathcal{B}$  is  $\kappa$ -complete. Therefore  $U \in M$  by our assumption on  $M$ . But  $C = \Gamma \cap \mathcal{B} = \Gamma \cap U$ , so  $C \in M$ , as desired. Thus  $M$  has the  $\kappa$ -approximation property at  $\lambda$ . □

The proof of Theorem 7.3.22 has the following corollary, which will be important going forward:

**Proposition 7.3.28.** *Suppose  $M$  is an inner model,  $\kappa$  is a cardinal,  $\lambda$  is an  $M$ -cardinal, and  $M$  has the  $\kappa$ -covering property at  $\lambda$ . Then the following are equivalent:*

- (1)  *$M$  has the  $\kappa$ -approximation property at  $\lambda$ .*
- (2) *There is a  $\kappa$ -independent family  $\Gamma$  of  $M$  with  $M$ -cardinality  $\lambda$  such that every  $\kappa$ -complete ultrafilter over the algebra generated by  $\Gamma$  belongs to  $M$ . □*

## The strength and supercompactness of $\mathcal{K}_\lambda$

**Definition 7.3.29.** For any Fréchet cardinal  $\lambda$ ,  $\kappa_\lambda$  denotes the completeness of  $\mathcal{K}_\lambda$ .

In other words,  $\kappa_\lambda = \text{CRT}(j_{\mathcal{K}_\lambda})$ . In Section 7.4, we will prove the following theorem:

**Theorem 7.3.30 (UA).** *Suppose  $\lambda$  is a Fréchet cardinal that is either a successor cardinal or a strongly inaccessible cardinal. Then  $\kappa_\lambda$  is  $\lambda$ -strongly compact.*

This is one of the harder theorems of this chapter, so we will just work under this hypothesis for a while. The following theorem begins to show why it is a useful assumption:

**Theorem 7.3.31 (UA).** *Suppose  $\lambda$  is a regular Fréchet cardinal and  $\kappa_\lambda$  is  $\lambda$ -strongly compact. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Then  $P(\gamma) \subseteq M$  for all  $\gamma < \lambda$ .*

Because we will occasionally need to use this argument in a more general context, let us instead prove the following:

**Proposition 7.3.32.** *Suppose  $\kappa \leq \gamma$  are cardinals,  $\kappa$  is  $\gamma$ -strongly compact, and  $M$  is an inner model that is closed under  $<\kappa$ -sequences. Assume every  $\kappa$ -complete ultrafilter on  $\gamma$  is amenable to  $M$ . Then  $P(\gamma) \subseteq M$ . Moreover if  $\text{cf}(\gamma) \geq \kappa$  then  $P(\eta) \subseteq M$  for all  $\eta \leq 2^\gamma$  such that  $\kappa$  is  $\eta$ -strongly compact.*

*Proof.* We may assume by induction that  $P(\alpha) \subseteq M$  for all ordinals  $\alpha < \gamma$ . Let  $\nu = \text{cf}(\gamma)$ .



Assume first that  $\nu < \kappa$ . Let  $\langle \gamma_\alpha : \alpha < \nu \rangle \in M$  be an increasing sequence cofinal in  $\gamma$ . Suppose  $A \subseteq \gamma$ . Let  $A_\alpha = A \cap \gamma_\alpha$ , so  $A_\alpha \in M$  for all  $\alpha < \nu$  by our inductive assumption. Then  $\langle A_\alpha : \alpha < \nu \rangle \in M$  since  $M$  is closed under  $<\kappa$ -sequences. Therefore  $A = \bigcup_{\alpha < \nu} A_\alpha \in M$ . It follows that  $P(\gamma) \subseteq M$ , which finishes the proof in this case.

Therefore we may assume that  $\nu \geq \kappa$ . We claim that  $\kappa$  is  $\gamma$ -strongly compact in  $M$ . Fix an ordinal  $\alpha \in [\kappa, \gamma]$  such that  $\text{cf}^M(\alpha) \geq \kappa$ . Then  $\text{cf}(\alpha) \geq \kappa$  since  $M$  is closed under  $\kappa$ -sequences. Since  $\kappa$  is  $\gamma$ -strongly compact, there is a  $\kappa$ -complete tail uniform ultrafilter  $U$  on  $\alpha$ . But  $U \cap M \in M$ , so in  $M$  there is a tail uniform  $\kappa$ -complete ultrafilter on  $\alpha$ . In particular, every  $M$ -regular cardinal  $\iota \in [\kappa, \lambda]$  carries a  $\kappa$ -complete ultrafilter in  $M$ , so by Theorem 7.2.15,  $\kappa$  is  $\gamma$ -strongly compact in  $M$ .

Therefore by Corollary 6.3.2,  $(\gamma^{<\kappa})^M = \gamma$ , so by Corollary 7.3.27,  $M$  satisfies that there is a  $\kappa$ -independent family of subsets of  $\gamma$  of cardinality  $(2^\gamma)^M$ .

Let  $\Gamma \in M$  be such that  $M$  satisfies that  $\Gamma$  is a  $\kappa$ -independent family of subsets of  $\gamma$  of cardinality  $\gamma$ . Let  $\Sigma$  be the algebra of subsets of  $\gamma$  generated by  $\Gamma$ . If  $U_0$  is a  $\kappa$ -complete ultrafilter over  $\Sigma$ , then  $U_0$  extends to a  $\kappa$ -complete ultrafilter  $U$  on  $\gamma$  by Theorem 7.2.10, since  $\kappa$  is  $\gamma$ -strongly compact and  $U_0$  is a  $\kappa$ -complete filter base of cardinality  $\gamma$ . It follows from Proposition 7.3.28 that  $M$  has the  $\kappa$ -approximation property at  $\gamma$ . Since  $M$  is closed under  $<\kappa$ -sequences, it follows from this that  $P(\gamma) \subseteq M$ .

We can now find larger independent families: since  $P(\gamma) \subseteq M$ ,  $(2^\gamma)^M \geq 2^\gamma$ , and in particular,  $M$  satisfies that there is a  $\kappa$ -independent family of subsets of  $\gamma$  of cardinality  $(2^\gamma)^V$ .

Assume finally that  $\delta \leq 2^\gamma$  is a cardinal and  $\kappa$  is  $\delta$ -strongly compact. Then let  $\Gamma \in M$  be a  $\kappa$ -independent family of subsets of  $\gamma$  in  $M$  with cardinality  $\delta$ . As in the previous paragraph, any  $\kappa$ -complete ultrafilter over the algebra generated by  $\Gamma$  belongs to  $M$ , so  $M$  has the  $\kappa$ -approximation property at  $\delta$  by Proposition 7.3.28. Since  $M$  is closed under  $<\kappa$ -sequences, it follows from this that  $P(\delta) \subseteq M$ . □

*Proof of Theorem 7.3.31.* By Theorem 7.3.17, every countably complete  $M$ -ultrafilter  $U$  on

$\gamma < \lambda$  belongs to  $M$ . Therefore if  $\gamma < \lambda$ , our strong compactness assumption on  $\kappa_\lambda$  implies the hypotheses of Proposition 7.3.32 hold at  $\gamma$ , and so  $P(\gamma) \subseteq M$ .  $\square$

Having proved that  $\mathcal{K}_\lambda$  has some strength, let us now turn to the supercompactness properties of  $\mathcal{K}_\lambda$ .

**Theorem 7.3.33.** *Suppose  $\lambda$  is a regular Fréchet cardinal and  $\kappa_\lambda$  is  $\lambda$ -strongly compact. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Then*

- $j$  is  $\lambda$ -tight.
- $j$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ .

*In other words,  $M^\gamma \subseteq M$  for all  $\gamma < \lambda$  and  $M$  has the  $\leq \lambda$ -covering property.*

*Proof.* Suppose towards a contradiction that  $j$  is not  $\lambda$ -tight. By Theorem 7.2.12, it follows that  $\delta = \text{cf}^M(\sup j[\lambda]) > \lambda$ . By Theorem 7.3.17, any countably complete  $M$ -ultrafilter  $U$  on  $\lambda$  belongs to  $M$ . But then by Proposition 7.3.32,  $P(\lambda) \subseteq M$ . But then  $\mathcal{K}_\lambda$  itself is a countably complete  $M$ -ultrafilter on  $\lambda$ , so  $\mathcal{K}_\lambda \in M$ . This contradicts the irreflexivity of the Mitchell order (Lemma 4.2.40).

Now that we know  $j$  is  $\lambda$ -tight, let us show that  $j$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ . We may assume by induction that  $j$  is  $< \gamma$ -supercompact. Then if  $\gamma$  is singular, it is easy to see that  $j$  is  $\gamma$ -supercompact. Therefore assume  $\gamma$  is regular. Let  $\gamma' = \text{cf}^M(\sup j[\gamma])$ . Then  $\gamma' \leq \lambda$  since  $j$  is  $\lambda$ -tight and hence  $j$  is  $(\gamma, \lambda)$ -tight. Since  $\gamma < \lambda$ , in fact  $\gamma' < \lambda$ . Thus  $P(\gamma') \subseteq M$  by Theorem 7.3.31. By Theorem 7.2.12,  $j$  is  $(\gamma, \gamma')$ -tight, so fix  $A \in M$  with  $|A|^M = \gamma'$  and  $j[\gamma] \subseteq A$ . Note that since  $|A|^M = \gamma'$ ,  $P(A) \subseteq M$ . Therefore  $j[\gamma] \subseteq M$ . Therefore  $j$  is  $\gamma$ -supercompact, as desired.

That  $M^\gamma \subseteq M$  for all  $\gamma < \lambda$  is an immediate consequence of Corollary 4.2.21. That  $M$  has the  $\leq \lambda$ -covering property is an immediate consequence of Lemma 7.2.7.  $\square$

Finally, if  $\lambda$  is not a strongly inaccessible cardinal, we can show that  $j_{\mathcal{K}_\lambda}$  is precisely as supercompact as it should be:

**Theorem 7.3.34 (UA).** *Suppose  $\lambda$  is a regular Fréchet cardinal and  $\kappa_\lambda$  is  $\lambda$ -strongly compact. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . If  $\lambda$  is not strongly inaccessible then  $j$  is  $\lambda$ -supercompact.*

*Proof.* Let  $\kappa = \kappa_\lambda$  for ease of notation. We split into two cases:

**Case 1.** For some  $\gamma < \lambda$  with  $\text{cf}(\gamma) \geq \kappa$ ,  $2^\gamma \geq \lambda$ .

*Proof in Case 1.* Since  $\gamma < \lambda$ , by Theorem 7.3.17 every countably complete  $M$ -ultrafilter on  $\gamma$  belongs to  $M$ . Since  $\text{cf}(\gamma) \geq \kappa$ ,  $\lambda \leq 2^\gamma$ , and  $\kappa$  is  $\lambda$ -strongly compact, we can therefore apply the second part of Proposition 7.3.32 to conclude that  $P(\lambda) \subseteq M$ .

Given that  $j$  is  $\lambda$ -tight by Theorem 7.3.33, it now follows easily that  $j$  is  $\lambda$ -supercompact: fix  $A \in M$  with  $|A|^M = \lambda$  and  $j[\lambda] \subseteq A$ ; then  $P(A) \subseteq M$  so  $j[\lambda] \in M$ , as desired.  $\square$

**Case 2.** For all  $\gamma < \lambda$  with  $\text{cf}(\gamma) \geq \kappa$ ,  $2^\gamma < \lambda$ .

*Proof in Case 2.* Since  $\lambda$  is not inaccessible, there is some  $\eta < \lambda$  such that  $2^\eta \geq \lambda$ . Let  $\gamma = \eta^{<\kappa}$ . Then  $\text{cf}(\gamma) \geq \kappa$  and  $2^\gamma \geq 2^\eta \geq \lambda$ . Therefore by our case hypothesis,  $\lambda \leq \gamma$ . By Theorem 7.3.33,  $j$  is  $\eta$ -supercompact. By Lemma 4.2.25,  $j$  is  $\eta^{<\kappa}$ -supercompact. Therefore  $j$  is  $\lambda$ -supercompact as desired.  $\square$

Thus in either case  $j$  is  $\lambda$ -supercompact, which completes the proof.  $\square$

## 7.4 Fréchet cardinals

### The Fréchet successor

Given the results of Section 7.3, to analyze  $\mathcal{K}_\lambda$  when  $\lambda$  is a regular Fréchet cardinal, it would be enough to show that its completeness  $\kappa_\lambda$  is  $\lambda$ -strongly compact. The following easy generalization of Ketonen's Theorem (Theorem 7.2.15) reduces this to the analysis of Fréchet cardinals in the interval  $[\kappa_\lambda, \lambda]$ :

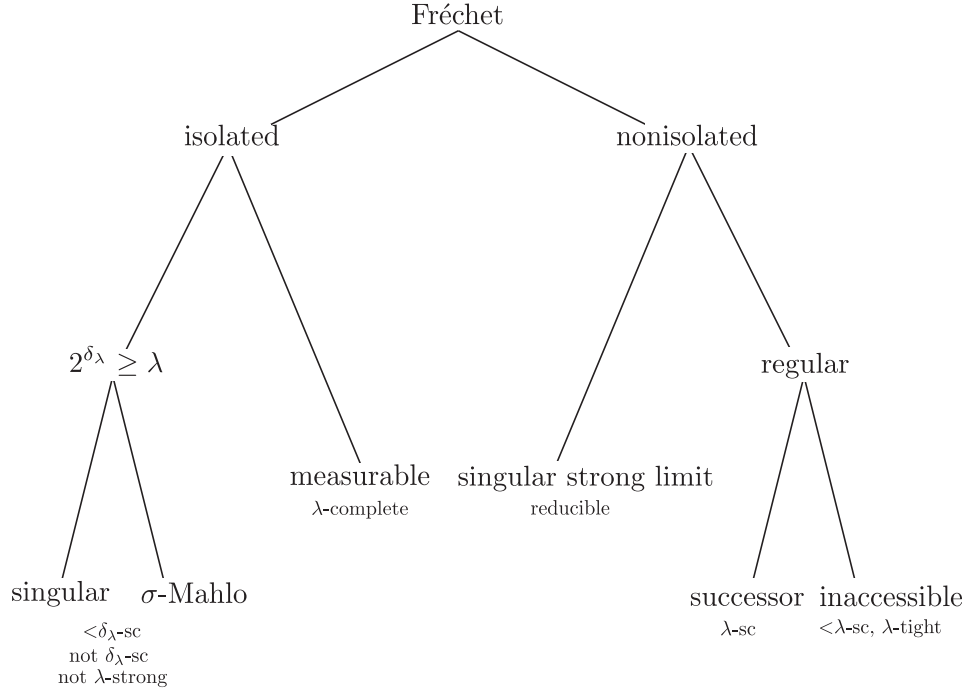


Figure 7.1: Types of Fréchet cardinals.

**Proposition 7.4.1.** *Suppose  $\lambda$  is a regular Fréchet cardinal. Suppose  $j : V \rightarrow M$  is the ultrapower of the universe by a Ketonen ultrafilter  $U$  on  $\lambda$ . Suppose  $\kappa \leq \lambda$  is a cardinal and every regular cardinal in the interval  $[\kappa, \lambda]$  is Fréchet. Then  $j$  is  $(\lambda, \delta)$ -tight for some  $\delta < j(\kappa)$ . In particular, if  $\kappa = \text{CRT}(j)$  then  $\kappa$  is  $\lambda$ -strongly compact.*

*Proof.* Since  $U$  is Ketonen, the  $M$ -cardinal  $\delta = \text{cf}^M(\sup j[\lambda])$  is not Fréchet in  $M$ . Therefore by elementarity  $\delta \notin j([\kappa, \lambda])$ . Since  $\delta < j(\lambda)$ , we must have  $\delta < j(\kappa)$ . Theorem 7.2.12 implies that  $j$  is  $(\lambda, \delta)$ -tight, proving the proposition.  $\square$

Suppose  $\lambda$  is a regular Fréchet cardinal. To obtain that every regular cardinal in the interval  $[\kappa_\lambda, \lambda)$  is Fréchet, it actually suffices to show that every *successor* cardinal in the interval  $(\kappa_\lambda, \lambda]$  is Fréchet. (See Corollary 7.4.5.) Our approach to this problem is as follows. Fix an ordinal  $\gamma \in [\kappa_\lambda, \lambda)$ . We consider the *Fréchet successor* of  $\gamma$ :

**Definition 7.4.2.** Suppose  $\gamma$  is an ordinal. Then the *Fréchet successor* of  $\gamma$ , denoted  $\gamma^\sigma$ , is the least Fréchet cardinal strictly greater than  $\gamma$ .

We will attempt to use the fact that  $\gamma$  lies in the interval  $[\kappa_\lambda, \lambda)$  to show that  $\gamma^\sigma = \gamma^+$ . Since  $\gamma^\sigma$  is Fréchet by definition, this would show  $\gamma^+$  is Fréchet. In this way, we would establish that every successor cardinal in the interval  $(\kappa_\lambda, \lambda]$  is Fréchet, as desired.

The following classic result of Prikry [33] shows in particular that there is nontrivial structure to the Fréchet cardinals even if we do not assume UA:

**Theorem 7.4.3** (Prikry). *Suppose  $\lambda$  is a cardinal and  $U$  is a  $\lambda^+$ -decomposable ultrafilter. Then  $U$  is  $\text{cf}(\lambda)$ -decomposable.  $\square$*

A key part of our analysis of Fréchet cardinals is the following generalization of Theorem 7.4.3:

**Proposition 7.4.4.** *Suppose  $\eta$  is a cardinal such that  $\eta^+$  is Fréchet. Either  $\eta$  is Fréchet or  $\eta$  is a singular cardinal and all sufficiently large regular cardinals below  $\eta$  are Fréchet.*

*Proof.* Suppose  $\gamma^\sigma = \eta^+$ . We will show that either  $\eta$  is Fréchet or  $\eta$  is a limit of Fréchet cardinals. Fix a countably complete uniform ultrafilter  $U$  on  $\eta^+$ , and let  $j : V \rightarrow M$  be the ultrapower of the universe by  $U$ . Let

$$U_* = \{A \in j(P(\eta^+)) : j^{-1}[A] \in U\}$$

Thus  $U_*$  is an  $M$ -ultrafilter. Note that  $\lambda_{U_*} < j(\eta^+)$  since for example  $\sup j[\eta^+] \in U_*$ . Thus  $\lambda_{U_*} \leq j(\eta)$ .

The proof now splits into two cases:

**Case 1.**  $\lambda_{U_*} \geq \sup j[\eta]$ .

*Proof in Case 1.* Let  $\lambda = \lambda_{U_*}$ . Then  $\sup j[\eta] \leq \lambda \leq j(\eta)$ . Let  $W_*$  be an  $M$ -ultrafilter on  $j(\eta)$  that concentrates on  $\lambda$  and is isomorphic to  $U_*$ . In other words, there is a set  $X \in U_*$  and a bijection  $f : \lambda \rightarrow X$  with  $f \in M$  such that  $W_* = \{f^{-1}[A] : A \in U_*\}$ . All we need about  $W_*$  is that  $\lambda_{W_*} = \lambda \geq \sup j[\eta]$ . Let

$$W = j^{-1}[W_*]$$

Then  $W$  is a countably complete ultrafilter on  $\eta$ .

We claim that  $W$  is uniform. Suppose  $A \in W$ . Then  $j(A) \in W_*$  so  $|j(A)|^M = \lambda$ . In particular, since  $\lambda \geq \sup j[\eta]$ , for any cardinal  $\kappa < \eta$ ,  $|j(A)|^M > j(\kappa)$ , and therefore  $|A| > \kappa$ . It follows that  $|A| \geq \eta$ . Thus  $W$  is uniform.  $\square$

**Case 2.**  $\lambda_{U_*} < \sup j[\eta]$ .

*Proof in Case 2.* Fix  $\kappa < \eta$  and  $B \in U_*$  such that letting  $\delta = |B|^M$ , we have  $\delta < j(\kappa)$ . Let  $A = j^{-1}[B]$ . Then  $A \in U$  so  $|A| = \eta^+$  since  $U$  is a uniform ultrafilter on  $\eta^+$ . Since  $j[A] \subseteq B$ , it follows that  $j$  is  $(\eta^+, \delta)$ -tight.

We claim that  $j$  is discontinuous at every regular cardinal  $\iota$  in the interval  $[\kappa, \eta^+]$ . To see this, note that  $j(\iota) > \delta$  is a regular cardinal of  $M$ . On the other hand,  $j[\iota]$  is contained in a set  $C \in M$  such that  $|C|^M \leq \delta$  since  $j$  is  $(\iota, \delta)$ -tight. Therefore  $C$  is not cofinal in  $j(\iota)$ , and hence neither is  $j[\iota]$ . It follows that  $j$  is discontinuous at  $\iota$ .

Since  $j$  is discontinuous at every regular cardinal in the interval  $[\kappa, \eta^+]$ , which contains  $\eta$ , it follows that either  $\eta$  is a regular Fréchet cardinal or  $\eta$  is a singular cardinal and all sufficiently large regular cardinals below  $\eta$  are Fréchet.  $\square$

Thus in either case, the conclusion of the proposition holds.  $\square$

An interesting feature of Proposition 7.4.4 is that it *does not* seem to show that every  $\eta^+$ -decomposable ultrafilter  $U$  is either  $\eta$ -decomposable or  $\iota$ -decomposable for all sufficiently large  $\iota < \eta$ . Instead the proof shows that this is true of  $U^2$ . (Under UA, we can in fact prove that every  $\eta^+$ -decomposable countably complete ultrafilter  $U$  is either  $\eta$ -decomposable or  $\iota$ -decomposable for all sufficiently large  $\iota < \eta$ .)

Proposition 7.4.4 has two important consequences. The first is our claim above that one need only show that all successor cardinals in  $[\kappa_\lambda, \lambda]$  are Fréchet to conclude that all regular cardinals in  $[\kappa_\lambda, \lambda]$  are. (This is really just a consequence of Theorem 7.4.3.)

**Corollary 7.4.5.** *Suppose  $\kappa \leq \lambda$  are cardinals and every successor cardinal in the interval  $(\kappa, \lambda]$  is Fréchet. Then every regular cardinal in the interval  $[\kappa, \lambda]$  is Fréchet.*

*Proof.* Suppose  $\iota$  is a regular cardinal in the interval  $[\kappa, \lambda)$ . Then  $\iota^+ \in (\kappa, \lambda]$ , so  $\iota^+$  is a Fréchet cardinal. Therefore  $\iota$  is a Fréchet cardinal by Proposition 7.4.4.  $\square$

The consequence of Proposition 7.4.4 that is ultimately most important here is a constraint on the Fréchet successor operation:

**Corollary 7.4.6.** *Suppose  $\gamma$  is an ordinal and  $\gamma^\sigma$  is a successor cardinal. Then  $\gamma^\sigma = \gamma^+$ .*

*Proof.* Suppose towards a contradiction that  $\gamma^\sigma = \eta^+$  for some cardinal  $\eta > \gamma$ . Since  $\eta^+$  is Fréchet, by Proposition 7.4.4,  $\eta$  is either Fréchet or a limit of Fréchet cardinals. Either way, there is a Fréchet cardinal in the interval  $(\gamma, \eta^+)$ . But the definition of  $\gamma^\sigma$  implies that there are no Fréchet cardinals in  $(\gamma, \gamma^\sigma)$ . This is a contradiction.

Thus  $\gamma^\sigma = \eta^+$  for some cardinal  $\eta \leq \gamma$ . In other words,  $\gamma^\sigma = \gamma^+$ .  $\square$

The problematic cases in the analysis of the Fréchet successor function therefore occur when  $\gamma^\sigma$  is a limit cardinal:

**Definition 7.4.7.** A cardinal  $\lambda$  is *isolated* if the following hold:

- $\lambda$  is Fréchet.
- $\lambda$  is a limit cardinal.
- $\lambda$  is not a limit of Fréchet cardinals.

By Proposition 7.4.4,  $\lambda$  is isolated if and only if  $\lambda = \gamma^\sigma$  for some ordinal  $\gamma$  such that  $\gamma^+ < \lambda$ . Our analysis of Fréchet cardinals would be essentially complete if we could prove the following conjecture:

**Conjecture 7.4.8 (UA).** *A cardinal  $\lambda$  is isolated if and only if  $\lambda$  is a measurable cardinal,  $\lambda$  is not a limit of measurable cardinals, and no cardinal  $\kappa < \lambda$  is  $\lambda$ -supercompact.*

Proposition 7.5.4 below shows that Conjecture 7.4.8 is a consequence of UA + GCH, so to some extent this problem is solved in the most important case. But assuming UA alone,

we do not know how to rule out, for example, the existence of singular isolated cardinals. Enacting an analysis of isolated cardinals under UA that is as complete as possible allows us to prove our main results without cardinal arithmetic assumptions.

## The strong compactness of $\kappa_\lambda$

In this section we will prove the following theorem:

**Theorem 7.4.9** (UA). *Suppose  $\lambda$  is a nonisolated regular Fréchet cardinal. Then  $\kappa_\lambda$  is  $\lambda$ -strongly compact.*

This yields the following corollary, which gives a complete analysis of Fréchet successor cardinals:

**Corollary 7.4.10** (UA). *Suppose  $\lambda$  is a Fréchet successor cardinal. Then  $\kappa_\lambda$  is  $\lambda$ -supercompact and in fact the ultrapower embedding associated to  $\mathcal{K}_\lambda$  is  $\lambda$ -supercompact.*

*Proof.* This is an immediate consequence of Theorem 7.4.9 and Theorem 7.3.34. □

In general, we only obtain

**Proposition 7.4.11** (UA). *Suppose  $\lambda$  is a nonisolated regular Fréchet cardinal. Then  $\kappa_\lambda$  is  $<\lambda$ -supercompact and  $\lambda$ -strongly compact. In fact, the ultrapower embedding associated to  $\mathcal{K}_\lambda$  is  $<\lambda$ -supercompact and  $\lambda$ -tight.*

*Proof.* This is an immediate consequence of Theorem 7.4.9 and Theorem 7.3.33. □

As we have sketched above, the proof of Theorem 7.4.9 will follow from an analysis of Fréchet cardinals in the interval  $[\kappa_\lambda, \lambda]$ :

**Lemma 7.4.12.** *Suppose  $\kappa \leq \lambda$  are cardinals and there are no isolated cardinals in the interval  $(\kappa, \lambda]$ . Suppose that for all  $\gamma \in [\kappa, \lambda)$ , there is a Fréchet cardinal in the interval  $(\gamma, \lambda]$ . Then every regular cardinal in the interval  $[\kappa, \lambda)$  is Fréchet.*



*Proof.* Since  $\lambda$  is Fréchet, we need only show that every regular cardinal in the interval  $[\kappa, \lambda)$  is Fréchet. By Corollary 7.4.5, for this it is enough to show that every successor cardinal in the interval  $(\kappa, \lambda]$  is Fréchet. In other words, it suffices to show that for any ordinal  $\gamma \in [\kappa, \lambda)$ ,  $\gamma^+$  is Fréchet. Therefore fix  $\gamma \in [\kappa, \lambda)$ . By assumption,  $\gamma^\sigma \in (\gamma, \lambda]$ , so in particular  $\gamma^\sigma$  is not isolated. Therefore  $\gamma^\sigma$  is not a limit cardinal. It follows that  $\gamma^\sigma$  is a successor cardinal, so by Proposition 7.4.4,  $\gamma^\sigma = \gamma^+$ , as desired.  $\square$

Our goal now it to prove the following lemma:

**Lemma 7.4.13 (UA).** *Suppose  $\lambda$  is a Fréchet cardinal that is either regular or isolated. Then there are no isolated cardinals in the interval  $[\kappa_\lambda, \lambda)$ .*

Given this, we could complete the proof of Theorem 7.4.9 as follows:

*Proof of Theorem 7.4.9 assuming Lemma 7.4.13.* By Lemma 7.4.13, there are no isolated cardinals in the interval  $[\kappa_\lambda, \lambda)$ . Since  $\lambda$  is not isolated, there are no isolated cardinals in the interval  $[\kappa_\lambda, \lambda]$ . Therefore applying Lemma 7.4.12, every regular cardinal in the interval  $[\kappa_\lambda, \lambda]$  is Fréchet. By Proposition 7.4.1, it follows that  $\kappa_\lambda$  is  $\lambda$ -strongly compact.  $\square$

We now proceed to the proof of Lemma 7.4.13. We will first need to improve our understanding of isolated cardinals. The first step is to provide some criteria that guarantee a cardinal's nonisolation:

**Lemma 7.4.14.** *Suppose  $\eta$  is a limit cardinal. Suppose  $U$  is a countably complete uniform ultrafilter on  $\eta$ . Suppose  $W$  is a countably complete ultrafilter such that  $j_W$  is discontinuous at  $\eta$  and  $U \sqsubset W$ . Then  $\eta$  is a limit of Fréchet cardinals.*

*Proof.* Let  $i : V \rightarrow N$  be the ultrapower of the universe by  $W$ . Let

$$U_* = s_W(U) = \{B \in i(P(\lambda)) : i^{-1}[B] \in U\}$$

By Lemma 5.5.9,  $U_* \in N$ .

**Case 1.**  $\lambda_{U_*} \geq \sup i[\eta]$

*Proof in Case 1.* Working in  $N$ ,  $\lambda_{U_*}$  is a Fréchet cardinal  $\lambda$  with  $\sup i[\eta] \leq \lambda < i(\eta)$ . It follows that for any  $\kappa < \eta$ ,  $N$  satisfies that there is a Fréchet cardinal strictly between  $i(\kappa)$  and  $i(\eta)$ , and so by elementarity there is a Fréchet cardinal strictly between  $\kappa$  and  $\eta$ . It follows that  $\eta$  is a limit of Fréchet cardinals.  $\square$

**Case 2.**  $\lambda_{U_*} < \sup i[\eta]$

*Proof in Case 2.* Fix  $\kappa < \eta$  and  $B \in U_*$  such that letting  $\delta = |B|^N$ ,  $\delta < i(\kappa)$ . Let  $A = i^{-1}[B]$ . Then  $A \in U$ , so  $|A| = \eta$  by the uniformity of  $U$ . Since  $|A| = \eta$  and  $i[A] \subseteq B$ ,  $i$  is  $(\eta, \delta)$ -tight by Theorem 7.2.12. It follows that  $i$  is discontinuous at every regular cardinal in the interval  $[\kappa, \eta]$ . (See the proof of Proposition 7.4.4.) In particular,  $\eta$  is a limit of Fréchet cardinals.  $\square$

In either case,  $\eta$  is a limit of Fréchet cardinals, as desired.  $\square$

The second nonisolation lemma brings in a bit more of the theory of the internal relation:

**Lemma 7.4.15** (UA). *Suppose  $\eta$  is a Fréchet limit cardinal. Suppose there is a countably complete ultrafilter  $W$  such that  $\mathcal{K}_\eta \sqsubset W$  but  $W \not\sqsubset \mathcal{K}_\eta$ . Then  $\eta$  is a limit of Fréchet cardinals.*

*Proof.* By Lemma 7.4.14, if  $j_W$  is discontinuous at  $\eta$ , then  $\eta$  is a limit of Fréchet cardinals. Therefore assume without loss of generality that  $j_W$  is continuous at  $\eta$ .

By the basic theory of the internal relation (Lemma 5.5.13), since  $\mathcal{K}_\eta \sqsubset W$ , the translation  $t_W(\mathcal{K}_\eta)$  is equal to the pushforward  $s_W(\mathcal{K}_\eta)$ .

Since  $W \not\sqsubset \mathcal{K}_\eta$ , the theory of the internal relation (Lemma 5.5.13) implies that in  $M_W$ ,  $t_W(\mathcal{K}_\eta) <_{\mathbb{k}} j_W(\mathcal{K}_\eta)$ . Since  $M_W$  satisfies that  $j_W(\mathcal{K}_\eta)$  is the  $<_{\mathbb{k}}$ -least uniform ultrafilter on  $j_W(\eta)$ , it follows that

$$\lambda_{t_W(\mathcal{K}_\eta)} < j_W(\eta)$$

But  $t_W(\mathcal{K}_\eta) = s_W(\mathcal{K}_\eta)$  and  $j_W(\eta) = \sup j_W[\eta]$  by our assumption that  $j_W$  is continuous at  $\eta$ . Thus

$$\lambda_{s_W(\mathcal{K}_\eta)} < \sup j_W[\eta]$$

Fix  $\kappa < \eta$  and  $B \in s_W(\mathcal{K}_\eta)$  such that  $\delta = |B|^{M_W} < j_W(\kappa)$ . Let  $A = j_W^{-1}[B]$ . Then  $A \in \mathcal{K}_\eta$ , so  $|A| = \eta$ . Moreover  $j_W[A] \subseteq B \in M_W$ , so  $j_W$  is  $(\eta, \delta)$ -tight. In particular,  $j_W$  is discontinuous at every regular cardinal in the interval  $[\kappa, \eta]$ . (See the proof of Proposition 7.4.4.) Therefore  $\eta$  is a limit of Fréchet cardinals.  $\square$

Finally, we need a version of Theorem 7.3.14 that applies at singular cardinals.

We use a lemma that follows immediately from the ultrafilter sum construction:

**Lemma 7.4.16.** *Suppose  $U$  is a countably complete ultrafilter on a cardinal  $\lambda$  and  $U'$  is a countably complete  $M_U$ -ultrafilter with  $\lambda_{U'} \leq j_U(\lambda)$ . Then there is a countably complete ultrafilter  $W$  on  $\lambda$  such that  $j_W = j_{U'}^{M_U} \circ j_U$ .*  $\square$

**Proposition 7.4.17 (UA).** *Suppose  $\lambda$  is an isolated cardinal. Then  $\mathcal{K}_\lambda$  is  $\lambda$ -internal.*

*Proof.* Suppose  $D$  is a countably complete ultrafilter on a cardinal  $\gamma < \lambda$ . We will show  $D \sqsubset \mathcal{K}_\lambda$ . Since  $\lambda$  is isolated, by increasing  $\gamma$ , we may assume  $\lambda = \gamma^\sigma$ .

Assume towards a contradiction that in  $M_D$ ,

$$t_D(\mathcal{K}_\lambda) <_{\mathbb{k}} j_D(\mathcal{K}_\lambda)$$

Then  $\lambda_{t_D(\mathcal{K}_\lambda)} < j_D(\lambda)$ , and so since  $\lambda_{t_D(\mathcal{K}_\lambda)}$  is a Fréchet cardinal of  $M_D$ ,  $\lambda_{t_D(\mathcal{K}_\lambda)} \leq j_D(\gamma)$ . Therefore, there is an ultrafilter  $W$  on  $\gamma$  such that

$$j_W = j_{t_D(\mathcal{K}_\lambda)}^{M_D} \circ j_D = j_{t_{\mathcal{K}_\lambda}(D)}^{M_{\mathcal{K}_\lambda}} \circ j_{\mathcal{K}_\lambda}$$

It follows from the basic theory of the Rudin-Keisler order (Lemma 3.4.4) that  $\mathcal{K}_\lambda \leq_{\text{RK}} W$ , which contradicts that  $\lambda_{\mathcal{K}_\lambda} = \lambda > \gamma \geq \lambda_W$ .

Thus our assumption was false, and in fact,  $j_D(\mathcal{K}_\lambda) \leq_{\mathbb{k}} t_D(\mathcal{K}_\lambda)$  in  $M_D$ . By the theory of the internal relation (Lemma 5.5.13), this implies that  $D \sqsubset \mathcal{K}_\lambda$ .  $\square$

In Section 7.5, we prove a much stronger version of this theorem that constitutes a complete generalization of Theorem 7.3.13 to isolated cardinals.

**Lemma 7.4.18** (UA). *Suppose  $\eta < \lambda$  is are Fréchet cardinals that are regular or isolated. Then either  $\eta < \kappa_\lambda$  or  $\mathcal{K}_\lambda \not\sqsubset \mathcal{K}_\eta$ .*

*Proof.* By Theorem 7.3.14 or Proposition 7.4.17,  $\mathcal{K}_\eta$  and  $\mathcal{K}_\lambda$  are  $\lambda$ -internal.

Assume  $\mathcal{K}_\lambda \sqsubset \mathcal{K}_\eta$ . Note that we also have  $\mathcal{K}_\eta \sqsubset \mathcal{K}_\lambda$  since  $\mathcal{K}_\lambda$  is  $\lambda$ -uniform. By Proposition 5.5.26,  $\eta < \kappa_\lambda$ . □

We can finally prove Lemma 7.4.13.

*Proof of Lemma 7.4.13.* Suppose towards a contradiction that  $\eta \in [\kappa_\lambda, \lambda)$  is isolated. Then by Lemma 7.4.18,  $\mathcal{K}_\lambda \not\sqsubset \mathcal{K}_\eta$ . Therefore by Lemma 7.4.15,  $\eta$  is a limit of Fréchet cardinals, contrary to the assumption that  $\eta$  is isolated. □

Since we will use it repeatedly, it is worth noting that  $\kappa_\lambda$  can be characterized in terms of isolated cardinals:

**Lemma 7.4.19** (UA). *Suppose  $\lambda$  is a nonisolated regular Fréchet cardinal. Then  $\kappa_\lambda$  is the supremum of the isolated cardinals less than  $\lambda$ .*

*Proof.* Let  $\kappa$  be the supremum of the isolated cardinals less than  $\lambda$ . By Lemma 7.4.13, there are no isolated cardinals in the interval  $[\kappa_\lambda, \lambda)$ , so  $\kappa \leq \kappa_\lambda$ .

Since there are no isolated cardinals in the interval  $(\kappa, \lambda]$ , Lemma 7.4.12 implies that every regular cardinal in the interval  $[\kappa, \lambda]$  is Fréchet. By Proposition 7.4.1, it follows that  $\kappa_\lambda \leq \kappa$ . Thus  $\kappa_\lambda = \kappa$ , as desired. □

## The first supercompact cardinal

In this subsection, we show how the theory of the internal relation can be used to characterize the least supercompact cardinal (and its local instantiations).

**Theorem 7.4.20** (UA). *Suppose  $\lambda$  is a successor cardinal and  $\kappa$  is the least  $(\omega_1, \lambda)$ -strongly compact cardinal. Then  $\kappa$  is  $\lambda$ -supercompact. In fact,  $\kappa = \kappa_\lambda$ .*

*Proof.* Since  $\kappa$  is  $(\omega_1, \lambda)$ -strongly compact, every regular cardinal in the interval  $[\kappa, \lambda]$  is Fréchet. By Proposition 7.4.1,  $\kappa_\lambda \leq \kappa$ . By Corollary 7.4.10,  $\kappa_\lambda$  is  $\lambda$ -supercompact. In particular,  $\kappa_\lambda$  is  $(\omega_1, \lambda)$ -strongly compact. Therefore  $\kappa \leq \kappa_\lambda$ , and hence  $\kappa = \kappa_\lambda$ . Thus  $\kappa$  is  $\lambda$ -supercompact, as desired.  $\square$

**Corollary 7.4.21 (UA).** *Suppose  $\lambda$  is a successor cardinal and  $\kappa$  is the least  $\lambda$ -strongly compact cardinal. Then  $\kappa$  is  $\lambda$ -supercompact. In fact,  $\kappa = \kappa_\lambda$ .*  $\square$

**Corollary 7.4.22 (UA).** *The least  $(\omega_1, \text{Ord})$ -strongly compact cardinal  $\kappa$  is supercompact.*

*Proof.* No cardinal  $\delta < \kappa$  is  $(\omega_1, \kappa)$ -strongly compact. In particular, for any successor cardinal  $\lambda > \kappa$ ,  $\kappa$  is the least  $(\omega_1, \lambda)$ -strongly compact cardinal. Therefore  $\kappa$  is  $\lambda$ -supercompact by Theorem 7.4.20.  $\square$

**Theorem 7.4.23 (UA).** *The least strongly compact cardinal is supercompact.*  $\square$

Lemma 3.5.43 identifies the following ordinals as key thresholds in the structure theory of countably complete ultrafilters:

**Definition 7.4.24.** The *ultrapower threshold* is the least cardinal  $\kappa$  such that for all  $\alpha$ , there is an ultrapower embedding  $j : V \rightarrow M$  such that  $j(\kappa) > \alpha$ .

Suppose  $\gamma$  is an ordinal. The  $\gamma$ -*threshold* is the least ordinal  $\kappa \leq \gamma$  such that for all  $\alpha < \gamma$  is an ultrapower embedding  $j : V \rightarrow M$  such that  $j(\kappa) > \alpha$ .

The ultrapower threshold cannot be proved to exist without large cardinal assumptions, but for any ordinal  $\gamma$ , the  $\gamma$ -threshold exists and is less than or equal to  $\gamma$ .

**Lemma 7.4.25.** *Suppose  $\kappa$  is a cardinal. If  $\kappa$  is the  $\gamma$ -threshold for some ordinal  $\gamma$  then  $\kappa$  is the  $\kappa$ -threshold.*

*Proof.* We may assume without loss of generality that  $\kappa < \gamma$ . Let  $\nu \leq \kappa$  be the  $\kappa$ -threshold.

We claim that for any  $\alpha < \gamma$ , there is an ultrapower embedding  $h : V \rightarrow N$  such that  $h(\nu) > \alpha$ . Fix  $\alpha < \gamma$ . Let  $j : V \rightarrow M$  be such that  $j(\kappa) > \alpha$ . In  $M$ ,  $j(\nu)$  is the  $j(\kappa)$ -threshold, so since  $\alpha < j(\kappa)$ , there is an internal ultrapower embedding  $i : M \rightarrow N$  such that  $i(j(\nu)) > \alpha$ . Let  $h = i \circ j$ . Then  $h : V \rightarrow N$  is an ultrapower embedding such that  $h(\nu) > \alpha$ , as desired.

By the minimality of the  $\gamma$ -threshold,  $\kappa \leq \nu$ . Hence  $\kappa = \nu$  as desired.  $\square$

**Theorem 7.4.26 (UA).** *Suppose  $\lambda$  is a strong limit cardinal and  $\kappa < \lambda$  is the  $\lambda$ -threshold. Then  $\kappa$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ .*

The proof uses the following lemma, an often-useful approximation to Conjecture 7.4.8:

**Lemma 7.4.27 (UA).** *Suppose  $\lambda_0$  is an isolated cardinal and  $\lambda_1 = (\lambda_0)^\sigma$ . Then  $\lambda_1$  is measurable.*

*Proof.* Note that  $\kappa_{\lambda_1} > \lambda_0$ : otherwise  $\lambda_0 \in [\kappa_{\lambda_1}, \lambda_1)$  contrary to the fact that there are no isolated cardinals in the interval  $[\kappa_{\lambda_1}, \lambda_1)$  by Lemma 7.4.13. Since  $\kappa_{\lambda_1}$  is measurable,  $\kappa_{\lambda_1}$  is Fréchet. Hence  $\lambda_1 = (\lambda_0)^\sigma \leq \kappa_{\lambda_1}$ . Obviously  $\kappa_{\lambda_1} \leq \lambda_1$ , so  $\kappa_{\lambda_1} = \lambda_1$ . Therefore  $\lambda_1$  is measurable.  $\square$

*Proof of Theorem 7.4.26.* By induction, we may assume that the theorem holds for all strong limit cardinals  $\bar{\lambda} < \lambda$ .

Suppose  $\alpha < \lambda$ . We claim that there is a countably complete ultrafilter  $D$  with  $\lambda_D < \lambda$  such that  $j_D(\kappa) > \alpha$ . To see this, fix an ultrapower embedding  $j : V \rightarrow M$  such that  $j(\kappa) > \alpha$ . Then by Lemma 5.5.27, one can find a countably complete ultrafilter  $D$  such that  $\lambda_D \leq 2^{|\alpha|} < \lambda$  and an elementary embedding  $k : M_D \rightarrow M$  such that  $k \circ j_D = j$  and  $\text{CRT}(k) > \alpha$ . Since  $k(j_D(\kappa)) = j(\kappa) > \alpha = k(\alpha)$ , by the elementarity of  $k$ ,  $j_D(\kappa) > \alpha$ .

Next, we show that  $\lambda$  is a limit of Fréchet cardinals. Suppose  $\delta$  is a cardinal with  $\kappa \leq \delta < \lambda$ . We will find a Fréchet cardinal in the interval  $(\delta, \lambda)$ . By the previous paragraph, there is a countably complete ultrafilter  $D$  such that  $j_D(\kappa) \geq (2^\delta)^+$  and  $\lambda_D < \lambda$ . On the

other hand  $\delta < \lambda_D$  since  $2^\delta < |j_D(\kappa)| \leq \kappa^{\lambda_D} = 2^{\lambda_D}$ . Thus  $\lambda_D$  is a Fréchet cardinal in the interval  $(\delta, \lambda)$ , as desired.

We claim that every regular cardinal in the interval  $[\kappa, \lambda)$  is Fréchet. By Lemma 7.4.12, it suffices to show that there are no isolated cardinals in the interval  $[\kappa, \lambda)$ . Suppose  $\lambda_0 \in [\kappa, \lambda)$  is isolated. Let  $\lambda_1 = (\lambda_0)^\sigma$ . Lemma 7.4.27 implies that  $\lambda_1$  is measurable. Since  $\lambda$  is a limit of Fréchet cardinals,  $\lambda_1 < \lambda$ . Note that for all  $\alpha < \lambda_1$ , there is an ultrapower embedding  $j : V \rightarrow M$  such that  $j(\kappa) > \alpha$ , so the  $\lambda_1$ -threshold  $\kappa'$  is less than  $\lambda_1$ . By our induction hypothesis,  $\kappa'$  is  $\gamma$ -supercompact for all  $\gamma < \lambda_1$ . This contradicts that  $\lambda_1 = (\lambda_0)^\sigma$  is not a limit of Fréchet cardinals.

We finally claim that  $\kappa$  is  $\delta$ -supercompact for any successor cardinal  $\delta \in (\kappa, \lambda)$ , which proves the theorem. Suppose  $\delta \in (\kappa, \lambda)$  is a successor cardinal. Then  $\kappa_\delta$  is  $\delta$ -supercompact by Corollary 7.4.10. Since  $\kappa_\delta$  is the limit of the isolated cardinals below  $\delta$  (Lemma 7.4.19),  $\kappa_\delta \leq \kappa$ . On the other hand, by Lemma 7.4.25,  $\kappa$  is the  $\kappa$ -threshold, so in particular, no  $\nu < \kappa$  is  $\kappa$ -supercompact. Hence  $\kappa_\delta \not\leq \kappa$ . It follows that  $\kappa = \kappa_\delta$ , as desired.  $\square$

## The number of countably complete ultrafilters

We close this section with a result that just barely uses the analysis of  $\mathcal{K}_\lambda$  given by Theorem 7.3.14 and Proposition 7.4.17. Recall that  $\mathcal{B}(X)$  denotes the set of countably complete ultrafilters on  $X$ . The main result is a bound on the cardinality of  $\mathcal{B}(X)$ :

**Theorem 7.4.28** (UA). *For any set  $X$ ,  $|\mathcal{B}(X)| \leq (2^{|X|})^+$ .*

The theorem is proved by a generalizing Solovay's Proposition 6.3.6. To do this, we need to generalize the notion of the Mitchell rank of an ultrafilter:

**Definition 7.4.29.** Suppose  $\delta$  is an ordinal and  $W$  is a countably complete ultrafilter on  $\delta$ .

- $\mathcal{B}_W(\delta)$  denotes the set of countably complete ultrafilters  $U$  on  $\delta$  such that  $U <_{\mathbb{k}} W$ .
- $\sigma(W)$  denotes the rank of  $(\mathcal{B}_W(\delta), <_{\mathbb{k}})$ .

- $\sigma(\delta)$  denotes the rank of  $(\mathcal{B}(\delta), <_{\mathbf{k}})$ .
- $\sigma(<\delta) = \sup_{\alpha < \delta} \sigma(\alpha) + 1$ .

Since the Ultrapower Axiom implies that the Ketonen order is linear, the rank of an ultrafilter completely determines its position in the Ketonen order:

**Lemma 7.4.30** (UA). *Suppose  $U$  and  $W$  are countably complete ultrafilters on ordinals. Then  $U \leq_{\mathbf{k}} W$  if and only if  $\sigma(U) \leq \sigma(W)$ .* □

The following lemma relates  $\sigma^V$  to  $\sigma^{M_U}$ :

**Lemma 7.4.31** (UA). *Suppose  $U$  is a countably complete ultrafilter and  $W$  is a countably complete ultrafilter on an ordinal  $\delta$ . Then  $\sigma(W) \leq \sigma^{M_U}(t_U(W))$ .*

*Proof.* It suffices to show that there is a Ketonen order preserving embedding from  $\mathcal{B}_W(\delta)$  to  $\mathcal{B}_{t_U(W)}^{M_U}(j_U(\delta))$ . By Theorem 5.4.42, the translation function  $t_U$  restricts to such a function. □

We briefly mention that a version of Lemma 7.4.31 is provable in ZFC. Suppose  $Z$  is a countably complete ultrafilter and  $W$  is an ultrafilter on an ordinal  $\delta$ . If  $\langle W_i : i \in I \rangle$  is sequence of countably complete ultrafilters on  $\delta$  such that  $W = Z\text{-}\lim_{i \in I} W_i$ , then

$$\sigma(W) \leq [\langle \sigma(W_i) : i \in I \rangle]_Z$$

We omit the proof, which is an application of Lemma 3.3.10.

**Corollary 7.4.32** (UA). *Suppose  $U$  is a countably complete ultrafilter and  $W$  is a countably complete ultrafilter on an ordinal. If  $j_U(\sigma(W)) = \sigma(W)$  then  $U \sqsubset W$ .*

*Proof.* Assume  $j_U(\sigma(W)) = \sigma(W)$ . Then

$$\sigma^{M_U}(j_U(W)) = j_U(\sigma(W)) = \sigma(W) \leq \sigma^{M_U}(t_U(W))$$

For the final inequality, we use Lemma 7.4.31. By Lemma 7.4.30, it follows that  $j_U(W) \leq_{\mathbf{k}} t_U(W)$  in  $M_U$ . By the theory of the internal relation (Lemma 5.5.13), this implies  $U \sqsubset W$ . □



**Lemma 7.4.33** (UA). *Suppose  $\gamma$  is an ordinal. Then for any ordinal  $\xi \in [\sigma(<\gamma), \sigma(\gamma))$ , there is a countably complete tail uniform ultrafilter  $U$  on  $\gamma$  with  $j_U(\xi) > \xi$ .*

*Proof.* Let  $U$  be unique element of  $\mathcal{B}(\gamma)$  with  $\sigma(U) = \xi$ . Since  $\xi \geq \eta$ ,  $U$  does not concentrate on  $\alpha$  for any  $\alpha < \eta$ . Therefore  $U$  is a nonprincipal tail uniform ultrafilter on  $\gamma$ . Since  $U$  is nonprincipal,  $U \not\sqsubset U$ . Therefore  $j_U(\sigma(U)) > \sigma(U)$  by Corollary 7.4.32. In other words,  $j_U(\xi) > \xi$ .  $\square$

The following fact is ultimately equivalent to Theorem 7.5.44 below:

**Lemma 7.4.34** (UA). *Suppose  $\xi$  and  $\delta$  are ordinals and  $U$  is the  $<_{\mathbb{k}}$ -minimum countably complete ultrafilter on  $\delta$  such that  $j_U(\xi) > \xi$ . Then for any countably complete ultrafilter  $D$  such that  $j_D(\xi) = \xi$ ,  $D \sqsubset U$ .*

*Proof.* Since  $j_D$  is elementary and  $j_D(\xi) = \xi$ ,  $j_D(U)$  is the  $<_{\mathbb{k}}^{M_D}$ -minimum countably complete ultrafilter  $Z$  of  $M_D$  on  $j_D(\delta)$  such that  $j_Z^{M_D}(\xi) > \xi$ . On the other hand,  $t_D(U)$  is a countably complete ultrafilter of  $M_D$  on  $j_D(\delta)$  such that

$$j_{t_D(U)}^{M_D}(\xi) = j_{t_D(U)}^{M_D}(j_D(\xi)) = j_{t_U(D)}^{M_U}(j_U(\xi)) \geq j_U(\xi) > \xi$$

Hence by the linearity of the Ketonen order,  $j_D(U) \leq_{\mathbb{k}} t_D(U)$  in  $M_D$ . Now the basic theory of the internal relation (Lemma 5.5.13) implies that  $D \sqsubset U$ .  $\square$

The central combinatorial argument of Theorem 7.4.28 appears in the following proposition:

**Proposition 7.4.35** (UA). *Suppose  $\lambda$  is a Fréchet cardinal. Then for any ordinal  $\gamma < \lambda$ ,  $|\mathcal{B}(\gamma)| \leq 2^\lambda$ .*

*Proof.* Assume towards a contradiction that  $\lambda$  is the least Fréchet cardinal at which the theorem fails. In particular,  $\lambda$  is not a limit of Fréchet cardinals, so by Theorem 7.3.14 or Proposition 7.4.17,  $\mathcal{H}_\lambda$  is  $\lambda$ -internal. Let  $\gamma < \lambda$  be the least ordinal such that  $|\mathcal{B}(\gamma)| > 2^\lambda$ . Then in particular,  $\gamma$  is the least ordinal such that  $\sigma(\gamma) \geq (2^\lambda)^+$ , so  $\sigma(<\gamma) < (2^\lambda)^+$ .

Let  $\xi$  be an ordinal with the following properties:

- $\sigma(<\gamma) \leq \xi < (2^\lambda)^+$ .
- For all  $\alpha < \gamma$ , for all  $D \in \mathcal{B}(\alpha)$ ,  $j_D(\xi) = \xi$ .
- $j_{\mathcal{K}_\lambda}(\xi) = \xi$ .

To see that such an ordinal  $\xi$  exists, let  $S = \bigcup_{\alpha < \gamma} \mathcal{B}(\alpha) \cup \{\mathcal{K}_\lambda\}$ . Note that  $|S| \leq 2^\lambda$  by the minimality of  $\gamma$ . For each  $D \in S$ , the collection of fixed points of  $j_D$  is  $\omega$ -closed unbounded in  $(2^\lambda)^+$ . Therefore the intersection of the fixed points of  $j_D$  for all  $D \in S$  is  $\omega$ -closed unbounded in  $(2^\lambda)^+$ .

Since  $\xi \in [\sigma(<\gamma), \sigma(\gamma))$ , Lemma 7.4.33 implies that there is a countably complete tail uniform ultrafilter  $U$  on  $\gamma$  with  $j_U(\xi) > \xi$ . Let  $U$  be the  $<_{\mathbb{k}}$ -least countably complete ultrafilter on  $\gamma$  such that  $j_U(\xi) > \xi$ . By Lemma 7.4.34,  $U$  is  $\lambda$ -internal, and moreover  $\mathcal{K}_\lambda \sqsubset U$ .

Since  $\lambda_U < \lambda$ ,  $U \sqsubset \mathcal{K}_\lambda$ . Thus  $U \sqsubset \mathcal{K}_\lambda$  and  $\mathcal{K}_\lambda \sqsubset U$ , so by Theorem 5.5.22,  $U$  and  $\mathcal{K}_\lambda$  commute. Since  $U$  is  $\lambda_U$ -internal and  $\mathcal{K}_\lambda$  is  $\lambda$ -internal, we can apply the converse to Kunen's commuting ultrapowers lemma (Proposition 5.5.26) to obtain  $U \in V_{\kappa_\lambda}$ . (Obviously  $\mathcal{K}_\lambda$  is not in  $V_\kappa$  where  $\kappa$  is the completeness of  $U$ .) In particular  $\gamma < \kappa_\lambda$ . But then  $|\mathcal{B}(\gamma)|^+ < \kappa_\lambda$  since  $\kappa_\lambda$  is inaccessible. This contradicts that  $\kappa_\lambda \leq \lambda < (2^\lambda)^+ \leq \sigma(\gamma) < |\mathcal{B}(\gamma)|^+$ .  $\square$

The proof above is a bit mysterious, and the situation can be clarified by doing a bit more work than the bare minimum required to prove the theorem. In fact one can prove the following. Suppose  $\lambda$  is a Fréchet cardinal that is either regular or isolated. Let  $\xi$  be the first fixed point of  $j_{\mathcal{K}_\lambda}$  above  $\kappa_\lambda$ . Then for any  $D <_{\mathbb{k}} \mathcal{K}_\lambda$ ,  $j_D(\xi) = \xi$ . The  $<_{\mathbb{k}}$ -minimum countably complete ultrafilter  $U$  on  $\lambda$  such that  $j_U(\xi) > \xi$ , if it exists, is isomorphic the  $\triangleleft$ -least normal fine ultrafilter  $\mathcal{U}$  on  $P_{\kappa_\lambda}(\lambda)$  such that  $\mathcal{K}_\lambda \triangleleft \mathcal{U}$ . This is related to Proposition 8.4.19.

Incidentally, Proposition 7.4.35 yields an alternate proof of instances of GCH from UA plus large cardinals. For example, assume  $|\mathcal{B}(\kappa)| = 2^{2^\kappa}$ ,  $|\mathcal{B}(\kappa^+)| > 2^{(\kappa^+)}$ , and  $\kappa^{++}$  is Fréchet.

Then

$$2^{2^\kappa} = |\mathcal{B}(\kappa)| \leq 2^{(\kappa^+)} < |\mathcal{B}(\kappa^+)| \leq 2^{(\kappa^{++})}$$

Thus  $2^{2^\kappa} < 2^{(\kappa^{++})}$ , and in particular  $2^\kappa < \kappa^{++}$ . In other words,  $2^\kappa = \kappa^+$ . (This result is not as strong as Theorem 6.3.15.)

Proposition 7.4.35 admits the following self-improvement:

**Theorem 7.4.36** (UA). *For any Fréchet cardinal  $\lambda$ , for any  $W \in \mathcal{B}(\lambda)$ ,  $|\mathcal{B}_W(\lambda)| \leq 2^\lambda$ . Hence  $\sigma(\lambda) \leq (2^\lambda)^+$ .*

*Proof.* For  $\alpha \leq \lambda$ , let  $\mathcal{B}(\lambda, \alpha) = \{Z \in \mathcal{B}(\lambda) : \delta_Z \leq \alpha\}$ . By the definition of the Ketonen order, every element of  $\mathcal{B}_W(\lambda)$  is of the form  $W\text{-}\lim_{\alpha \in I} U_\alpha$  for some  $I \in W$  and  $\langle U_\alpha : \alpha \in I \rangle \in \prod_{\alpha \in I} \mathcal{B}(\lambda, \alpha)$ . Thus  $|\mathcal{B}_W(\lambda)| \leq |\prod_{I \subseteq \lambda} \prod_{\alpha \in I} \mathcal{B}(\lambda, \alpha)|$ . It therefore suffices show that the cardinality of  $\prod_{I \subseteq \lambda} \prod_{\alpha \in I} \mathcal{B}(\lambda, \alpha)$  is at most  $2^\lambda$ . Since

$$|\prod_{I \subseteq \lambda} \prod_{\alpha \in I} \mathcal{B}(\lambda, \alpha)| = 2^\lambda \cdot \sup_{I \subseteq \lambda} \prod_{\alpha \in I} |\mathcal{B}(\lambda, \alpha)|$$

it suffices to show that  $|\mathcal{B}(\lambda, \alpha)| \leq 2^\lambda$  for all  $\alpha < \lambda$ . But there is a one-to-one correspondence between  $\mathcal{B}(\lambda, \alpha)$  and  $\mathcal{B}(\alpha)$ , and by Proposition 7.4.35,  $|\mathcal{B}(\alpha)| \leq \sigma(\alpha) < (2^\lambda)^+$ . Thus  $|\mathcal{B}(\lambda, \alpha)| \leq 2^\lambda$ , which completes the proof.  $\square$

We finally prove  $|\mathcal{B}(X)| \leq (2^{|X|})^+$ :

*Proof of Theorem 7.4.28.* For  $A \subseteq X$ , let

$$\mathcal{B}(X, A) = \{U \in \mathcal{B}(X) : A \in U\}$$

For any  $A \subseteq X$  of cardinality  $\lambda$ ,  $|\mathcal{B}(X, A)| = |\mathcal{B}(A)| = |\mathcal{B}(\lambda)|$ . Since every ultrafilter  $U$  concentrates on a set whose cardinality is a Fréchet cardinal, we have

$$\mathcal{B}(X) = \bigcup \{\mathcal{B}(X, A) : A \subseteq X \text{ and } |A| \text{ is Fréchet}\}$$

Hence

$$|\mathcal{B}(X)| \leq 2^{|X|} \cdot \sup\{|\mathcal{B}(\lambda)| : \lambda \leq |X| \text{ is Fréchet}\} \tag{7.3}$$

By Theorem 7.4.36, for any Fréchet cardinal  $\lambda$  such that  $\lambda \leq |X|$ ,  $|\mathcal{B}(\lambda)| \leq (2^\lambda)^+ \leq (2^{|X|})^+$ . Hence by (7.3),  $|\mathcal{B}(X)| \leq 2^{|X|} \cdot (2^{|X|})^+ = (2^{|X|})^+$ , as desired.  $\square$

## 7.5 Isolation

In this section, we study isolated cardinals more deeply. The main objects of study are nonmeasurable isolated cardinals, yet we have the feeling that a clever argument might prove that these objects simply do not exist. (See Conjecture 7.4.8.) So far, however, we have been unable to rule them out. In this section, we show that there are significant constraints on their structure, and this allows us to prove the linearity of the Mitchell order on normal fine ultrafilters from UA without using any cardinal arithmetic assumptions (Theorem 7.5.39).

### Isolated measurable cardinals

Recall that isolated cardinals are Fréchet limit cardinals that are not limits of Fréchet cardinals. We begin by giving a full analysis of cardinals that are limits of Fréchet cardinals. This will lead us to a proof of Conjecture 7.4.8 for strong limit isolated cardinals.

**Lemma 7.5.1** (UA). *Suppose  $\lambda$  is a limit of Fréchet cardinals. Let  $\kappa$  be the supremum of the isolated cardinals less than  $\lambda$ , and assume  $\kappa < \lambda$ . Then  $\kappa$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ . In fact,  $\kappa = \kappa_\iota$  for all regular cardinals  $\iota \in [\kappa, \lambda)$ .*

*Proof.* Since there are no isolated cardinals in the interval  $[\kappa, \lambda)$ , Lemma 7.4.12 implies that every regular cardinal in the interval  $[\kappa, \lambda)$  is Fréchet. Assume  $\iota \in [\kappa, \lambda)$  is a regular cardinal. Then  $\iota$  is a nonisolated Fréchet cardinal. Since  $\kappa$  is the supremum of the isolated cardinals below  $\iota$ ,  $\kappa = \kappa_\iota$  by Lemma 7.4.19. Now by Proposition 7.4.11,  $\kappa$  is  $\gamma$ -supercompact for all  $\gamma < \iota$ . Since  $\iota$  was an arbitrary regular cardinal in  $[\kappa, \lambda)$  and  $\lambda$  is a limit cardinal,  $\kappa$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ .  $\square$

**Corollary 7.5.2** (UA). *Suppose  $\lambda$  is a cardinal. Then the following are equivalent:*



In particular, it follows that every limit of Fréchet cardinals is a strong limit cardinal: if  $\lambda$  is a limit of measurable cardinals, this is immediate; on the other hand, if some  $\kappa < \lambda$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ , then Theorem 6.3.12 implies that for all  $\gamma \in [\kappa, \lambda)$ ,  $2^\gamma = \gamma^+$ .

**Lemma 7.5.3** (UA). *Suppose  $\lambda$  is a strong limit cardinal such that no cardinal  $\kappa < \lambda$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ . Then for all ultrapower embeddings  $j : V \rightarrow M$ ,  $j[\lambda] \subseteq \lambda$ . In fact, no ordinal  $\kappa < \lambda$  can be mapped arbitrarily high below  $\lambda$  by ultrapower embeddings.*

*Proof.* This is immediate from Theorem 7.4.26. □

The following proposition shows that all the mystery of isolated cardinals falls away if one assumes the Generalized Continuum Hypothesis.

**Proposition 7.5.4** (UA). *Suppose  $\lambda$  is cardinal. Then the following are equivalent:*

- (1)  $\lambda$  is a strong limit isolated cardinal.
- (2)  $\lambda$  is a measurable cardinal,  $\lambda$  is not a limit of measurable cardinals, and no cardinal  $\kappa < \lambda$  is  $\lambda$ -supercompact.

*Proof.* (1) implies (2): Since  $\lambda$  is not a limit of Fréchet cardinals, clearly  $\lambda$  is not a limit of measurable cardinals and no  $\kappa < \lambda$  is  $\lambda$ -supercompact. It remains to show that  $\lambda$  is measurable. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Note that  $j[\lambda] \subseteq \lambda$  by Lemma 7.5.3. By Proposition 7.4.17,  $D \sqsubset \mathcal{K}_\lambda$  for all  $D$  with  $\lambda_D < \lambda$ . Therefore by Lemma 5.5.28,  $\mathcal{K}_\lambda$  is  $\lambda$ -complete. Since there is a  $\lambda$ -complete uniform ultrafilter on  $\lambda$ ,  $\lambda$  is measurable.

(2) implies (1): Since  $\lambda$  is measurable,  $\lambda$  is a strong limit cardinal. It remains to show that  $\lambda$  is isolated. Note that no cardinal  $\kappa < \lambda$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ : since  $\lambda$  is measurable, this would imply  $\kappa$  is  $\lambda$ -supercompact, contrary to assumption. Corollary 7.5.2 now implies that  $\lambda$  is not a limit of Fréchet cardinals. Therefore  $\lambda$  is isolated. □

The main application of isolated measurable cardinals is factoring ultrapower embeddings:

**Theorem 7.5.5** (UA). *Suppose  $\kappa$  is a strong limit cardinal that is not a limit of Fréchet cardinals. Suppose  $U$  is a countably complete ultrafilter. Then there is a countably complete ultrafilter  $D$  such that  $\lambda_D < \kappa$  admitting an internal ultrapower embedding  $h : M_D \rightarrow M_U$  such that  $h \circ j_D = j_U$  and  $\text{CRT}(h) \geq \kappa$  if  $h$  is nontrivial.*

*Proof.* Fix  $\gamma < \kappa$  such that  $\kappa = \gamma^\sigma$ . By Lemma 5.5.27, one can find a countably complete ultrafilter  $D$  such that  $\lambda_D < \kappa$  and there is an elementary embedding  $e : M_D \rightarrow M_U$  such that  $\text{CRT}(e) > \beth_{10}(\gamma)$  and  $e \circ j_D = j_U$ . Let  $\lambda = \lambda_D$ . We may assume without loss of generality that  $\lambda$  is the underlying set of  $D$ . Since  $\lambda < \kappa$  is a Fréchet cardinal,  $\lambda \leq \gamma$ . Let  $\lambda' = j_D(\lambda)$ . Then  $\lambda' < (2^\lambda)^+$ , so  $2^{2^{\lambda'}} < \beth_{10}(\gamma)$ . Since  $e : M_D \rightarrow M_U$  has critical point above  $2^{2^{\lambda'}}$ ,

$$P(P(\lambda')) \cap M_D = P(P(\lambda')) \cap M_U$$

Thus the following hold where  $\mathcal{B}(X)$  denotes the set of countably complete ultrafilters on  $X$ :

- $\lambda' = j_D(\lambda) = e(\lambda') = j_U(\lambda)$ .
- $\mathcal{B}^{M_D}(\lambda') = \mathcal{B}^{M_U}(\lambda')$ .
- $\leq_{\mathbb{k}}^{M_D} \upharpoonright \mathcal{B}^{M_D}(\lambda') = \leq_{\mathbb{k}}^{M_U} \upharpoonright \mathcal{B}^{M_U}(\lambda')$
- $j_D \upharpoonright P(\lambda) = j_U \upharpoonright P(\lambda)$ .

By Theorem 5.4.40,  $t_D(D)$  is the  $\leq_{\mathbb{k}}^{M_D}$ -least element  $D' \in \mathcal{B}^{M_D}(\lambda')$  such that  $j_D^{-1}[D'] = D$ . By Theorem 5.4.40,  $t_U(D)$  is the  $\leq_{\mathbb{k}}^{M_U}$ -least element  $D' \in \mathcal{B}^{M_U}(\lambda')$  such that  $j_U^{-1}[D'] = D$ . By the agreement set out in the bullet points above, it therefore follows that  $t_D(D) = t_U(D)$ . On the other hand, by Lemma 5.4.39,  $t_D(D)$  is principal in  $M_D$ . Thus  $t_U(D)$  is principal in  $M_U$ . Therefore by Lemma 5.4.39,  $D \leq_{\text{RF}} U$ .

Let  $h : M_D \rightarrow M_U$  be the internal ultrapower embedding with  $h \circ j_D = j_U$ . By Theorem 3.5.10,  $h(\alpha) \leq e(\alpha)$  for all  $\alpha \in \text{Ord}$ , so  $\text{CRT}(h) \geq \text{CRT}(e) > \beth_{10}(\gamma) > j_D(\gamma)$ . Since  $h$  is an internal ultrapower embedding of  $M_D$ , if  $h$  is nontrivial then  $\text{CRT}(h)$  is a measurable

cardinal of  $M_D$  above  $j_D(\gamma)$ . Since there are no measurable cardinals in the interval  $(\gamma, \kappa)$ , there are no measurable cardinals of  $M_D$  in the interval  $(j_D(\gamma), j_D(\kappa))$ . Therefore if  $h$  is nontrivial, then  $\text{CRT}(h) \geq \kappa$ .  $\square$

## Ultrafilters on an isolated cardinal

In this subsection, which is perhaps the most technical of this dissertation, we enact a very detailed analysis of the countably complete ultrafilters on an isolated cardinal  $\lambda$ . One of the goals is to prove the following theorem:

**Lemma 7.5.6.** *Suppose  $\lambda$  is an isolated cardinal and  $W$  is a countably complete ultrafilter. Then  $\mathcal{K}_\lambda \leq_{\text{RF}} W$  if and only if  $W$  is  $\lambda$ -decomposable and  $W \sqsubset \mathcal{K}_\lambda$  if and only if  $W$  is  $\lambda$ -indecomposable.*

This should be seen as a generalization of the universal property for  $\mathcal{K}_\lambda$  when  $\lambda$  is regular to isolated cardinals  $\lambda$ .

We begin with the following fact:

**Theorem 7.5.7 (UA).** *Suppose  $\lambda$  is an isolated cardinal. Then  $\mathcal{K}_\lambda$  is the unique countably complete weakly normal ultrafilter on  $\lambda$ .*

It turns out to be easier to prove something that is a priori slightly stronger. Recall the notion of the Dodd parameter  $p(j)$  of an elementary embedding  $j$ , defined in Definition 4.3.17 in the general context of elementary embeddings, and once again in Definition 5.4.24 in the more relevant special case of ultrapower embeddings.

**Proposition 7.5.8 (UA).** *Suppose  $\lambda$  is an isolated cardinal. Then  $\mathcal{K}_\lambda$  is the unique countably complete incompressible ultrafilter  $U$  on  $\lambda$  such that  $|p(j_U)| = 1$ .*

*Proof.* Suppose towards a contradiction that the proposition fails. Let  $U$  be the  $<_{\mathbb{k}}$ -least countably complete incompressible ultrafilter on  $\lambda$  such that  $p(j_U) = 1$  and  $U \neq \mathcal{K}_\lambda$ . Since  $\mathcal{K}_\lambda$  is the  $<_{\mathbb{k}}$ -least uniform ultrafilter on  $\lambda$ ,  $\mathcal{K}_\lambda <_{\mathbb{k}} U$ .



Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$  and let  $\nu = a_{\mathcal{K}_\lambda}$ . Let  $i : V \rightarrow N$  be the ultrapower of the universe by  $U$  and let  $\xi = a_U$ . By the incompressibility of  $U$ ,  $p(j_U) = \{\xi\}$ .

Let  $(k, h) : (M, N) \rightarrow P$  be the pushout of  $(j, i)$ . Since  $\mathcal{K}_\lambda <_{\mathbf{k}} U$ ,

$$k(\nu) < h(\xi) \tag{7.4}$$

We claim that  $h(\xi)$  is a generator of  $k : M \rightarrow P$ , or in other words that

$$h(\xi) \notin H^P(k[M] \cup h(\xi))$$

Since  $\xi$  is a generator of  $i$ ,  $h(\xi)$  is a generator of  $h \circ i$  by Lemma 5.4.25. Since  $k \circ j = h \circ i$ ,  $h(\xi)$  is a generator of  $k \circ j$ . Since  $M = H^M(j[V] \cup \{\nu\})$ ,

$$H^P(k[M] \cup h(\xi)) = H^P(k \circ j[V] \cup \{k(\nu)\} \cup h(\xi)) = H^P(k \circ j[V] \cup h(\xi))$$

The final equality follows from (7.4). Therefore since  $h(\xi) \notin H^P(k \circ j[V] \cup h(\xi))$ ,  $h(\xi) \notin H^P(k[M] \cup h(\xi))$ , as desired.

Let  $Z = t_{\mathcal{K}_\lambda}(U)$ , so  $Z$  is the  $M$ -ultrafilter on  $j(\lambda)$  derived from  $k$  using  $h(\xi)$ . Then  $Z$  is a countably complete ultrafilter on  $j(\lambda)$  and  $a_Z = h(\xi)$  is a generator of  $j_Z^M = k$ .

We claim that  $Z$  is an incompressible ultrafilter on  $j(\lambda)$  in  $M$ . Since  $a_Z = h(\xi)$  is a generator of  $j_Z^M$ , it suffices to show that  $Z$  is tail uniform, or in other words,  $\delta_Z = j(\lambda)$ . Since  $a_Z$  is a generator of  $j_Z^M$ ,  $\delta_Z = \lambda_Z$  is a Fréchet cardinal in  $M$ . By (7.4),  $\delta_Z > a_{\mathcal{K}_\lambda}$ . Since  $U$  is on  $\lambda$ ,  $\xi < i(\lambda)$ , so  $h(\xi) < h(i(\lambda)) = k(j(\lambda))$ , which implies  $\delta_Z \leq j(\lambda)$ . Thus  $\delta_Z \in (a_{\mathcal{K}_\lambda}, j(\lambda)]$ . Since  $\lambda$  is isolated, no Fréchet cardinal of  $M$  lies in the interval  $[\sup j[\lambda], j(\lambda))$ . Therefore  $\delta_Z = j(\lambda)$ , as desired.

It follows that in  $M$ ,  $Z$  is a countably complete incompressible ultrafilter on  $j(\lambda)$ . Moreover  $p(j_Z^M) = \{h(\xi)\}$  by Lemma 5.4.26, so  $p(j_Z^M)$  has cardinality 1.

We claim that  $Z \neq j(\mathcal{K}_\lambda)$ . The reason is that  $j^{-1}[Z] = U$  (since  $Z = t_{\mathcal{K}_\lambda}(U)$ ) while  $j^{-1}[j(\mathcal{K}_\lambda)] = \mathcal{K}_\lambda$ .

Thus we have shown that in  $M$ ,  $Z$  is a countably complete incompressible ultrafilter on  $j(\lambda)$  such that  $|p(j_Z^M)| = 1$  and  $Z \neq j(\mathcal{K}_\lambda)$ . By the definition of  $U$  and the elementarity of  $j$ , it follows that  $j(U) \leq_{\mathbb{k}} Z$  in  $M$ . Lemma 5.5.13 now implies that  $\mathcal{K}_\lambda \sqsubset U$ . But  $j_U$  is discontinuous at  $\lambda$  since  $\lambda_U = \lambda$ . Thus by Lemma 7.4.14,  $\lambda$  is not isolated. This is a contradiction.  $\square$

*Proof of Theorem 7.5.7.* If  $U$  is a countably complete weakly normal ultrafilter on  $\lambda$ , then  $U$  is incompressible and  $p(j_U) = \{a_U\}$  by Proposition 4.4.23. Therefore we can apply Proposition 7.5.8.  $\square$

We now investigate the iterated ultrapowers of  $\mathcal{K}_\lambda$ .

**Definition 7.5.9.** If  $\lambda$  is an isolated cardinal, then the *iterated ultrapower* of  $\mathcal{K}_\lambda$  is the iterated ultrapower

$$\mathcal{I}_\lambda = \langle M_n^\lambda, j_{mn}^\lambda, U_m^\lambda : m \leq n < \omega \rangle$$

formed by setting  $U_m^\lambda = j_{0m}^\lambda(\mathcal{K}_\lambda)$  for all  $m < \omega$ . For  $n < \omega$ , let  $p^{n\lambda} = p(j_{0n}^\lambda)$ , and let  $\mathcal{K}_\lambda^n$  be the ultrafilter on  $[\lambda]^\ell$  derived from  $j_{0n}^\lambda$  using  $p^{n\lambda}$  where  $\ell = |p^{n\lambda}|$ .

Thus  $j_{0,n}^\lambda : V \rightarrow M_n^\lambda$  is the ultrapower of the universe by  $\mathcal{K}_\lambda^n$ . We now analyze the parameters  $p^n$ :

**Lemma 7.5.10.** *Suppose  $\lambda$  is an isolated cardinal. Let  $\langle M_n, j_{m,n}, U_m : m \leq n < \omega \rangle$  be the iterated ultrapower of  $\mathcal{K}_\lambda$ . For  $n < \omega$ , let  $p^n = p^{n\lambda}$ . Then for all  $n < \omega$ ,  $|p^n| = n$  and*

$$\begin{aligned} p^{n+1} \upharpoonright n &= j_{01}(p(j_{0n})) \\ p_n^{n+1} &= j_{01}(j_{0n})(a_{\mathcal{K}_\lambda}) \end{aligned} \tag{7.5}$$

*Proof.* Note that the conclusion of the lemma holds when  $n = 0$ . Assume  $m \geq 1$  and that the conclusion of the lemma holds when  $n = m - 1$ . We will prove that the conclusion of the lemma holds when  $n = m$ .

Note that

$$j_{0m+1} = j_{1m+1} \circ j_{01} = j_{01}(j_{0m}) \circ j_{01} = j_{01} \circ j_{0m}$$

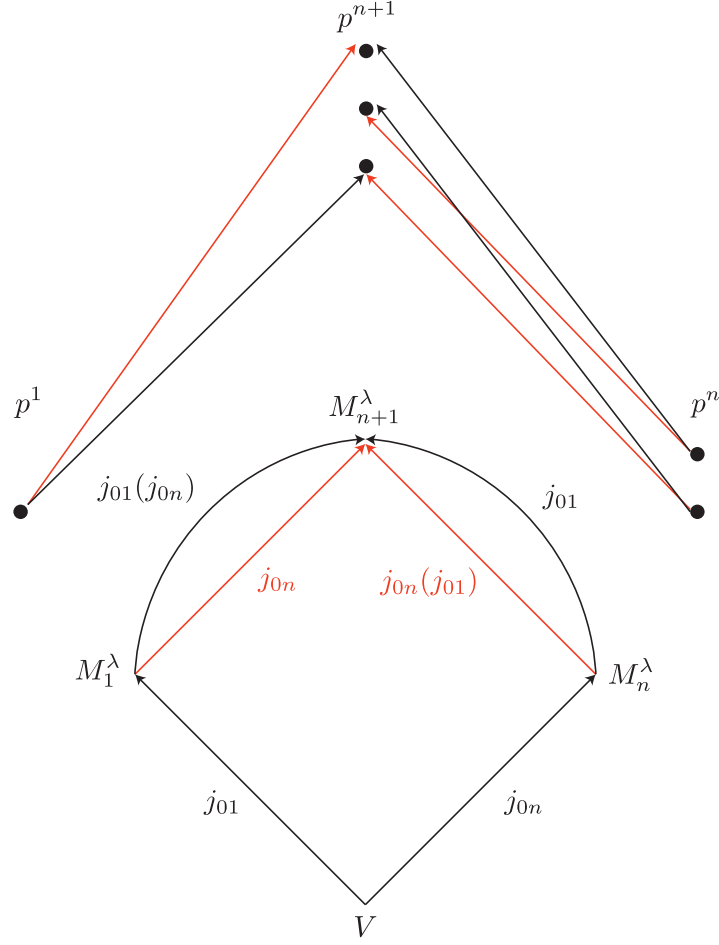


Figure 7.3: The iterated ultrapower of  $\mathcal{K}_\lambda$ .

Since  $p^m$  is the Dodd parameter of  $j_{0m}$ ,  $p^{m+1}$  is the Dodd parameter of  $j_{0m+1}$ , and  $j_{0m+1} = j_{01} \circ j_{0m}$ ,  $j_{01}(p^m) < p^{m+1}$  in the parameter order, and hence  $j_{01}(p^m) \leq p^{m+1} \upharpoonright m$ .

Let  $W$  be the  $M_m$ -ultrafilter derived from  $j_{01}$  using  $j_{01}(j_{0m})(a_{\mathcal{K}_\lambda})$ . Then by the basic theory of the internal relation (Lemma 5.5.9),

$$W = s_{\mathcal{K}_\lambda^m}(\mathcal{K}_\lambda)$$

$j_W^{M_m} = j_{01} \upharpoonright M_n$ ,  $a_W = j_{01}(j_{0m})(a_{\mathcal{K}_\lambda})$ , and

$$M_{n+1} = H^{M_{n+1}}(j_{0n+1}[V] \cup j_{01}(p^m) \cup \{j_{01}(j_{0m})(a_{\mathcal{K}_\lambda})\})$$

It follows from the minimality of the Dodd parameter that

$$p^{m+1} \leq j_{01}(p^m) \cup \{j_{01}(j_{0m})(a_{\mathcal{K}_\lambda})\} \tag{7.6}$$

By our induction hypothesis,

$$\min p^m = j_{01}(j_{0m-1})(a_{\mathcal{K}_\lambda}) = j_{1m}(a_{\mathcal{K}_\lambda}) \geq \sup j_{1m}[\sup j_{01}[\lambda]] = \sup j_{0m}[\lambda]$$

Therefore  $j_{0m}(\xi) < \min p^m$  for all  $\xi < \lambda$ , so by Loś's Theorem,

$$j_{01}(j_{0m})(a_{\mathcal{K}_\lambda}) < \min j_{01}(p^m)$$

Combining this with (7.6), we can conclude that  $p^{m+1} \upharpoonright m \leq j_{01}(p^m)$ .

Putting these two inequalities together, we have shown  $p^{m+1} \upharpoonright m = j_{01}(p^m)$ .

By Lemma 5.4.26, to show (7.5), it suffices to show that  $j_{01}(j_{0m})(a_{\mathcal{K}_\lambda})$  is the largest  $j_{01}(p^m)$ -generator of  $j_{0m+1}$ . By (7.6), it in fact suffices to show that  $j_{01}(j_{0m})(a_{\mathcal{K}_\lambda})$  is a  $j_{01}(p^m)$ -generator of  $j_{0m+1}$ .

We claim that an ordinal  $\xi$  is a  $j_{01}(p^m)$ -generator of  $j_{0m+1}$  if and only if  $\xi$  is a generator of  $j_{01} \upharpoonright M_m$ . This follows immediately from the fact that

$$H^{M_{m+1}}(j_{0m+1}[V] \cup j_{01}(p^m) \cup \xi) = H^{M_{m+1}}(j_{01}[j_{0m}[V] \cup p^m] \cup \xi) = H^{M_{m+1}}(j_{1m}[M_m] \cup \xi)$$

for any ordinal  $\xi$ .

Thus to finish, we must show that  $j_{01}(j_{0m})(a_{\mathcal{K}_\lambda})$  is a generator of  $j_{01} \upharpoonright M_m$ .

Let  $\lambda' = \sup j_{0m}[\lambda]$ . We first show that  $\lambda_W = \lambda'$ . By the definition of  $s_{\mathcal{K}_\lambda^n}(\mathcal{K}_\lambda)$ ,  $\lambda' \in W$ : note that  $j_{0m}^{-1}[\lambda'] = \lambda \in \mathcal{K}_\lambda$ . It follows that  $\lambda_W \leq \lambda'$ . Thus we are left to show that  $\lambda' \leq \lambda_W$ . Assume to the contrary that there is a set  $B \in W$  such that for some  $\kappa < \lambda$ , letting  $\delta = |B|^{M_m}$ ,  $\delta < j_{0m}(\kappa)$ . Then  $j_{0m}$  is  $(\lambda, \delta)$ -tight, and it follows that  $j_{0m}$  is discontinuous at all regular cardinals in the interval  $[\kappa, \lambda]$ . (See the proof of Proposition 7.4.4.) Therefore  $\lambda$  is a limit of Fréchet cardinals, which contradicts that  $\lambda$  is isolated.

Since  $\lambda_W = \lambda'$ ,  $j_{01} \upharpoonright M_m$  must have a generator in the interval  $[\sup j_{01}[\lambda'], j_{01}(\lambda')]$ . Let  $\xi$  be the least such generator. Then

$$\xi \leq j_{01}(j_{0m})(a_{\mathcal{K}_\lambda})$$

by (7.6). Let  $U$  be the ultrafilter derived from  $j_{0m+1}$  using  $\xi$  and let  $k : M_U \rightarrow M_{m+1}$  be the factor embedding with  $k \circ j_U = j_{0m+1}$  and  $k(a_U) = \xi$ . Clearly  $\xi$  is a generator of  $j_{0m+1}$ , so

$a_U$  is a generator of  $j_U$ . Therefore  $\lambda_U = \delta_U = \lambda$ , since  $\sup j_{0_{m+1}}[\lambda] = \sup j_{0_1}[\lambda'] \leq \xi$ . Thus  $U$  is a uniform countably complete ultrafilter on  $\lambda$  and  $p(j_U) = \{a_U\}$ , so by Proposition 7.5.8,  $U = \mathcal{K}_\lambda$ .

Thus  $k$  and  $j_{0_1}(j_{0_m})$  are elementary embeddings from  $M_1$  to  $M_{m+1}$ . Since  $j_{0_1}(j_{0_m})$  is an internal ultrapower embedding of  $M_1$ , it follows from Theorem 3.5.10 that

$$j_{0_1}(j_{0_m})(a_{\mathcal{K}_\lambda}) \leq k(a_{\mathcal{K}_\lambda}) = k(a_U) = \xi$$

Putting these inequalities together,  $\xi = j_{0_1}(j_{0_m})(a_{\mathcal{K}_\lambda})$ , and therefore  $j_{0_1}(j_{0_m})(a_{\mathcal{K}_\lambda})$  is a generator of  $j_{0_1} \upharpoonright M_m$ , as desired.  $\square$

A key parameter in the theory of Fréchet cardinals is the strict cardinal supremum of a cardinal's Fréchet predecessors:

**Definition 7.5.11.** For any cardinal  $\lambda$ ,  $\delta_\lambda = \sup\{\eta^+ : \eta < \lambda \text{ and } \eta \text{ is Fréchet}\}$ .

If  $\lambda$  is a Fréchet cardinal, then  $\lambda$  is isolated if and only if  $\delta_\lambda < \lambda$ .

We have the following immediate corollary:

**Lemma 7.5.12 (UA).** *Suppose  $\lambda$  is an isolated cardinal. Let  $\langle M_n, j_{m,n}, U_m : m \leq n < \omega \rangle$  be the iterated ultrapower of  $\mathcal{K}_\lambda$ . Suppose  $i : V \rightarrow N$  is an ultrapower embedding of the form  $i = d \circ j_{0_n}$  where  $d : M_n \rightarrow N$  is the ultrapower of  $M_n$  by a countably complete ultrafilter  $D$  of  $M_n$  with  $\lambda_D < j_{0_n}(\delta_\lambda)$ . Then  $p(i) \setminus i(\delta_\lambda) = d(p^{n,\lambda})$ .*

The following theorem amounts to a complete analysis of the ultrafilters on an isolated cardinal:

**Theorem 7.5.13 (UA).** *Suppose  $\lambda$  is an isolated cardinal. Let  $\langle M_n, j_{m,n}, U_m : m \leq n < \omega \rangle$  be the iterated ultrapower of  $\mathcal{K}_\lambda$ . Suppose  $i : V \rightarrow N$  is the ultrapower by a countably complete ultrafilter on  $\lambda$ . Then for some  $n < \omega$ ,  $i = d \circ j_{0_n}$  where  $d : M_n \rightarrow N$  is the ultrapower of  $M_n$  by a countably complete ultrafilter  $D$  of  $M_n$  with  $\lambda_D < j_{0_n}(\delta_\lambda)$ .*

*Proof.* Suppose  $U$  is a countably complete ultrafilter on  $\lambda$ . Assume by induction that the proposition holds when  $i = j_W$  for an ultrafilter  $W <_{\mathbb{k}} U$ .

Let  $i : V \rightarrow N$  be the ultrapower of the universe by  $U$ , and we will show that the theorem is true for  $i$ .

If  $\lambda_U < \lambda$ , then the theorem is vacuously true. Therefore we may assume  $\lambda_U = \lambda$ .

Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Let  $\nu = a_{\mathcal{K}_\lambda}$ .

Let  $(k, h) : (M, N) \rightarrow P$  be the pushout of  $(j, i)$ . Since  $(k, h)$  is the pushout of  $(j, i)$ ,  $k$  is the ultrapower embedding of  $M$  associated to  $t_{\mathcal{K}_\lambda}(U)$ . Since  $\lambda_U = \lambda$ ,  $j_U$  is discontinuous at  $\lambda$ . Hence by Lemma 7.4.14,  $\mathcal{K}_\lambda \not\sqsubseteq U$ . Therefore by Lemma 5.5.13,  $t_{\mathcal{K}_\lambda}(U) <_{\mathbb{k}} j(U)$  in  $M$ . We can now apply our induction hypothesis, shifted by  $j$  to  $M$ , to the ultrafilter  $t_{\mathcal{K}_\lambda}(U)$  of  $M$ . We conclude that for some  $\ell < \omega$ ,  $k = d \circ j(j_{0\ell})$  where  $d : j(M_\ell) \rightarrow P$  is the ultrapower of  $j(M_\ell)$  by a countably complete ultrafilter  $D$  of  $j(M_\ell)$  such that  $\lambda_D < j(j_{0\ell})(j(\delta_\lambda))$ . Let

$$n = \ell + 1$$

Then  $j(j_{0\ell}) = j_{1n}$ ,  $j(M_\ell) = M_n$ , and  $j(j_{0\ell})(j(\delta_\lambda)) = j_{1n}(j(\delta_\lambda)) = j_{0n}(\delta_\lambda)$ . Thus

$$k = d \circ j_{1n}$$

where  $d : M_n \rightarrow P$  is the ultrapower of  $M_n$  by a countably complete ultrafilter  $D$  of  $M_n$  such that  $\lambda_D < j_{0n}(\delta_\lambda)$ . Note that

$$k \circ j = d \circ j_{0n}$$

has the form we want to show that  $i$  has.

Let  $p^\ell = p^{\ell, \lambda}$  and let  $p^n = p^{n, \lambda}$ , so that

$$p^n = j(p^\ell) \cup \{j_{1n}(\nu)\} \tag{7.7}$$

by Lemma 7.5.10 and the fact that  $j_{1n} = j_{01}(j_{0\ell})$ .

Let  $q' = p(k) \setminus k(j(\delta_\lambda))$ . By Lemma 7.5.12 applied in  $M$ ,  $q' = d(j(p^\ell))$ . Since  $k \circ j = d \circ j_{0n}$ ,

$$\begin{aligned} p(k \circ j) \setminus k(j(\delta_\lambda)) &= p(d \circ j_{0n}) \setminus d(j_{0n}(\delta_\lambda)) \\ &= d(p^n) \end{aligned} \tag{7.8}$$

$$= d(j(p^\ell) \cup \{j_{1n}(\nu)\}) \tag{7.9}$$

$$= q' \cup \{k(\nu)\} \tag{7.10}$$

Here (7.8) follows from Lemma 7.5.12; (7.9) follows from (7.7); (7.10) follows from the fact that  $d(j(p^\ell)) = q'$  and  $d \circ j_{1n} = k$ .

Let  $\xi$  be the least generator of  $i$  such that  $\sup i[\lambda] \leq \xi < i(\lambda)$ .

**Claim 1.**  $k(\nu) = h(\xi)$ .

*Proof of Claim 1.* By Proposition 4.4.23, the ultrafilter derived from  $i$  using  $\xi$  is a countably complete weakly normal ultrafilter on  $\lambda$ , and hence is equal to  $\mathcal{K}_\lambda$ . Let  $e : M \rightarrow N$  be the factor embedding with  $e(\nu) = \xi$  and  $e \circ j = i$ . The comparison  $(e, \text{id})$  witnesses  $(j, \nu) \leq_{\mathbb{k}} (i, \xi)$ . Since  $(j, \nu) \leq_{\mathbb{k}} (i, \xi)$ , we must have  $k(\nu) \leq h(\xi)$ .

Assume towards a contradiction that  $k(\nu) \neq h(\xi)$ , so  $k(\nu) < h(\xi)$ .

Let  $q = p(i) \setminus \sup i[\lambda]$ . We claim that  $h(q) = p(k) \upharpoonright |q|$ . The proof is by induction. Assume  $m < |q|$  and  $h(q) \upharpoonright m = p(k) \upharpoonright m$ . By Lemma 5.4.26,  $q_m$  is the largest  $q \upharpoonright m$ -generator of  $i$ . Hence  $h(q_m)$  is the largest  $h(q \upharpoonright m)$ -generator of  $h \circ i$ . Replacing like terms,  $h(q_m)$  is the largest  $p(k) \upharpoonright m$ -generator of  $k \circ j$ . Since  $q_m$  is a generator of  $i$  above  $\sup i[\lambda]$ ,  $q_m \geq \xi$ . Hence  $h(q_m) \geq h(\xi) > k(\nu)$  by our assumption that  $h(\xi) > k(\nu)$ . Therefore  $h(q_m)$  is not only a  $p(k) \upharpoonright m$ -generator of  $k \circ j$  but also a  $p(k) \upharpoonright m \cup \{k(\nu)\}$ -generator of  $k \circ j$ . In other words,  $h(q_m)$  is a  $p(k) \upharpoonright m$ -generator of  $k$ , and it must therefore be the largest  $p(k) \upharpoonright m$ -generator of  $k$ . By Lemma 5.4.26,  $h(q_m) = p(k)_m$ .

Since  $q$  has no elements below  $\sup i[\lambda]$ , in particular,  $q$  has no elements below  $i(\delta_\lambda)$ . Therefore  $h(q)$  has no elements below  $h(i(\delta_\lambda)) = k(j(\delta_\lambda))$ . Since  $h(q) \subseteq p(k)$  by the previous paragraph, it follows that  $h(q) \subseteq p(k) \setminus k(j(\delta_\lambda)) = q'$ .

We now claim that  $k(\nu)$  is a generator of  $h$ . To show this, it suffices to show that  $k(\nu)$  is a  $h(p(i))$ -generator of  $h \circ i$ . Let  $r = p(i) \cap \sup i[\lambda]$ . Thus  $p(i) = q \cup r$ . Note that  $h(r) \subseteq \sup h \circ i[\lambda] = \sup k \circ j[\lambda] \leq k(\nu)$ , since  $\sup j[\lambda] \leq \nu$ . Hence  $h(r) \subseteq k(\nu)$ . Thus to show that  $k(\nu)$  is a  $k(p(i))$ -generator of  $h \circ i$ , it suffices to show that  $k(\nu)$  is a  $h(q)$ -generator of  $h \circ i$ . Since  $h(q) \subseteq q'$ , it suffices to show that  $k(\nu)$  is a  $q'$ -generator of  $k \circ j$ . This is an immediate consequence of (7.10): by Lemma 5.4.26,  $k(\nu)$  is the largest  $q'$ -generator of  $k \circ j$ .

Thus  $k(\nu)$  is a generator of  $h$ . Let  $W$  be the tail uniform  $N$ -ultrafilter derived from  $h$  using  $k(\nu)$ . Then  $W$  is an incompressible ultrafilter. We have  $\sup i[\lambda] \leq \delta_W$  since  $\sup h[\sup i[\lambda]] = \sup k \circ j[\lambda] \leq k(\nu)$ . Moreover  $\delta_W \leq \xi$  since  $k(\nu) < h(\xi)$ . Since  $W$  is incompressible,  $\lambda_W = \delta_W$ . But  $\lambda_W$  is a Fréchet cardinal of  $N$  and  $\sup i[\lambda] \leq \lambda_W \leq \xi < i(\lambda)$ . This contradicts that  $\lambda$  is isolated.

It follows that our assumption that  $k(\nu) \neq h(\xi)$  was false. This proves Claim 1.  $\square$

Since  $k(\nu) = h(\xi)$ , it follows from Lemma 5.3.10 that  $\mathcal{K}_\lambda \leq_{\text{RF}} U$ . Let  $k' : M \rightarrow N$  be an internal ultrapower embedding. Then  $(k', \text{id})$  is a pushout of  $(j, i)$ . By the uniqueness of pushouts,  $k = k'$  and  $h = \text{id}$ . Hence  $k : M \rightarrow N$  is the unique internal ultrapower embedding such that  $k \circ j = i$ . Thus  $i = k \circ j = d \circ j_{0n}$ . Since  $d : M_n \rightarrow N$  is the ultrapower of  $M_n$  by a countably complete ultrafilter  $D$  of  $M_n$  with  $\lambda_D < j_{0n}(\lambda)$ , this proves the proposition.  $\square$

**Corollary 7.5.14** (UA). *Suppose  $\lambda$  is an isolated cardinal. Let  $\langle M_n, j_{m,n}, U_m : m \leq n < \omega \rangle$  be the iterated ultrapower of  $\mathcal{K}_\lambda$ . Then for any ultrapower embedding  $k : V \rightarrow P$ , there is some  $n < \omega$  such that*

$$k = h \circ d \circ j_{0n}$$

where  $M_n \xrightarrow{d} N \xrightarrow{h} P$  are ultrapower embeddings with the following properties:

- $d : M_n \rightarrow N$  is the ultrapower of  $M_n$  by a countably complete ultrafilter  $D$  of  $M_n$  with  $\lambda_D < j_{0n}(\delta_\lambda)$ .
- $h : N \rightarrow P$  is an internal ultrapower embedding of  $N$  with  $\text{CRT}(h) > d(j_{0n}(\lambda))$  if  $h$  is nontrivial.



*Proof.* We claim there is a strong limit cardinal  $\kappa > \lambda$  such that there are no Fréchet cardinals in the interval  $(\lambda, \kappa)$ . If there are no Fréchet cardinals above  $\lambda$ , let  $\kappa = \beth_\omega(\lambda)$ . Otherwise, let  $\kappa = \lambda^\sigma$ . By Lemma 7.4.27,  $\kappa$  is measurable, and in particular,  $\kappa$  is a strong limit cardinal.

By Theorem 7.5.5, there is a countably complete ultrafilter  $U$  with  $\lambda_U < \kappa$  such that there is an internal ultrapower embedding  $h : M_U \rightarrow P$  with  $h \circ j_U = j$  and  $\text{CRT}(h) \geq \kappa$ . Since  $\lambda_U < \kappa$  is Fréchet and there are no Fréchet cardinals in the interval  $[\lambda, \kappa]$ ,  $\lambda_U \leq \lambda$ . Therefore we may assume that  $U$  is a countably complete ultrafilter on  $\lambda$ . In particular  $\text{CRT}(h) \geq \kappa > j_U(\lambda)$ .

Let  $i = j_U$ . By Theorem 7.5.13, for some  $n < \omega$ ,  $i = d \circ j_{0n}$  where  $d : M_n \rightarrow N$  is the ultrapower of  $M_n$  by a countably complete ultrafilter  $D$  of  $M_n$  with  $\lambda_D < j_{0n}(\delta_\lambda)$ . Putting everything together,

$$j = h \circ d \circ j_{0n}$$

and this proves the corollary. □

It is not a priori obvious that  $p^n$  contains all the generators  $\xi$  of  $\mathcal{K}_\lambda^n$  with  $\xi \geq \sup j_{0n}[\lambda]$ . In fact this is true:

**Proposition 7.5.15 (UA).** *Suppose  $\lambda$  is an isolated cardinal and  $n < \omega$ . Then  $\mathcal{K}_\lambda^n$  is the unique countably complete ultrafilter  $W$  on  $[\lambda]^n$  such that  $a_W$  is the set of generators  $\xi$  of  $j_W$  with  $\xi \geq \sup j_W[\lambda]$ .*

*Proof.* Assume by induction that the corollary is true when  $n = m$ , and we will prove it when  $n = m + 1$ .

Therefore assume  $W$  is a countably complete ultrafilter on  $[\lambda]^{m+1}$  such that  $a_W$  is the set of generators  $\xi$  of  $j_W$  with  $\xi \geq \sup j_W[\lambda]$ . Let  $q$  be the first  $m$  generators of  $j_W$  above  $\sup j_W[\lambda]$ . Let  $U$  be the ultrafilter derived from  $j_W$  using  $q$ . Then by our induction hypothesis,  $U = \mathcal{K}_\lambda^m$ . Let  $d : M_m \rightarrow M_W$  be the factor embedding with  $d \circ j_{0m} = j_W$  and  $d(p^m) = q$ . By Theorem 7.5.13, there is an internal ultrapower embedding  $d' : M_m \rightarrow M_W$ . Note that  $d'(p^m)$  is a set of generators of  $d' \circ j_{0m} = d \circ j_{0m}$ , so  $d'(p^m) \geq d(p^m)$ . On the other hand,

$d'(p^m) \leq d(p^m)$  by Theorem 3.5.10. Thus  $d'(p^m) = d(p^m)$ . Since  $d' \circ j_{0m} = d \circ j_{0m}$ , we have  $d' = d$ . Thus  $d$  is an internal ultrapower embedding.

Let  $\xi$  be the largest generator of  $j_W$ . Thus  $d(q) \subseteq \xi$ , so  $\xi$  is a  $d(q)$ -generator of  $j_W$  and hence  $\xi$  is a generator of  $d$ . Let  $Z$  be the tail uniform  $M_m$ -ultrafilter derived from  $d$  using  $\xi$ . Then  $Z$  is an incompressible ultrafilter of  $M_m$  and  $\delta_Z \in [\sup j_{0m}[\lambda], j_{0m}(\lambda)]$ . Since  $\delta_Z = \lambda_Z$  is a Fréchet cardinal of  $M_m$ , it follows that  $\delta_Z = j_{0m}(\lambda)$ . Therefore by Theorem 7.5.7,  $Z = j_{0m}(\mathcal{K}_\lambda)$ .

Since  $M_W = H^{M_W}(j_W[V] \cup q \cup \{\xi\}) = H^{M_W}(d[M_m] \cup \{\xi\})$ , we have  $d = j_Z^{M_m} = j_{mm+1}$ . Thus  $d \circ j_{0m} = j_{0m+1}$ . Thus  $j_W = j_{0m+1}$ .

Since  $p^{m+1}$  consists solely of generators of  $j_{0m+1}$  above  $\sup j_{0m+1}[\lambda]$ ,  $p^{m+1} \subseteq a_W$ . Since  $|a_W| = |p^{m+1}|$ , it follows that  $p^{m+1} = a_W$ . Therefore  $W = \mathcal{K}_\lambda^{m+1}$ , as desired.  $\square$

**Proposition 7.5.16** (UA). *Suppose  $\lambda$  is an isolated cardinal. Then  $\mathcal{K}_\lambda^n$  is the unique countably complete ultrafilter  $W$  on  $[\lambda]^n$  such that  $a_W$  is a set of generators of  $j_W$  disjoint from  $\sup j_W[\lambda]$ .*

*Proof.* Suppose  $W$  is such an ultrafilter. Let  $p$  be the set of all generators of  $\xi$  of  $j_W$  with  $\xi \geq \sup j_W[\lambda]$ . Let  $m = |p|$ . By Proposition 7.5.15, the ultrafilter derived from  $j_W$  using  $p$  is  $\mathcal{K}_\lambda^m$ . It follows that  $j_W = j_{0m}$  and  $p = p^m$ . Therefore  $p \leq a_W$  by the minimality of the Dodd parameter. On the other hand,  $a_W \subseteq p$  since  $p$  consists of all the generators of  $j_W$  above  $\sup j_W[\lambda]$ . Therefore  $a_W = p$ . Hence  $m = n$  and  $W = \mathcal{K}_\lambda^n$ , as desired.  $\square$

We also have an analog of Theorem 7.3.17 at isolated cardinals:

**Theorem 7.5.17** (UA). *Suppose  $\lambda$  is an isolated cardinal. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$  and let  $\nu = a_{\mathcal{K}_\lambda}$ . Suppose  $Z$  is a countably complete  $M$ -ultrafilter that is  $\delta$ -indecomposable for all  $M$ -cardinals  $\delta \in [\sup j[\lambda], \nu]$ . Then  $Z \in M$ .*

*Proof.* Let  $e : M \rightarrow P$  be the ultrapower of  $M$  by  $Z$ . Then  $e(\nu)$  is a generator of  $e \circ j$  by Lemma 5.4.25. Since  $Z$  is  $\delta$ -indecomposable for all  $\delta \in [\sup j[\lambda], \nu]$ ,  $e$  has no generators in

the interval  $[\sup e \circ j[\lambda], e(\nu)]$ . In other words,  $e \circ j$  has no  $e(\nu)$ -generators in the interval  $[\sup e \circ j[\lambda], e(\nu)]$ .

Let  $k = e \circ j$ . Applying Corollary 7.5.14, there is some  $n < \omega$  such that

$$k = h \circ d \circ j_{0n}$$

where  $M_n \xrightarrow{d} N \xrightarrow{h} P$  are ultrapower embeddings with the following properties:

- $d : M_n \rightarrow N$  is the ultrapower of  $M_n$  by a countably complete ultrafilter  $D$  of  $M_n$  with  $\lambda_D < j_{0n}(\delta_\lambda)$ .
- $h : N \rightarrow P$  is an internal ultrapower embedding of  $N$  with  $\text{CRT}(h) > d(j_{0n}(\lambda))$  if  $h$  is nontrivial.

Let  $e' = h \circ d \circ j_{1n}$ , so that  $e' : M \rightarrow P$  is an internal ultrapower embedding with  $e' \circ j = k = e \circ j$ . We claim  $e'(\nu) = e(\nu)$ . By Theorem 3.5.10,  $e'(\nu) \leq e(\nu)$ .

Suppose towards a contradiction  $e'(\nu) < e(\nu)$ . Then  $e'(\nu)$  is not an  $e(\nu)$ -generator of  $e \circ j = e' \circ j$ . Note that  $h(d(j_{1n}(\nu))) = e'(\nu)$  and  $h(e(\nu)) = e(\nu)$ , so  $d(j_{1n}(\nu))$  is not an  $e(\nu)$ -generator of  $d \circ j_{0n}$ . But consider the ultrafilter  $U$  on  $[\lambda]^2$  derived from  $d \circ j_{0n}$  using  $\{d(j_{1n}(\nu)), e(\nu)\}$ . Since  $d(j_{1n}(\nu))$  and  $e(\nu)$  are generators  $\xi$  of  $d \circ j_{0n}$  with  $\xi \geq \sup d \circ j_{0n}[\lambda]$ ,  $a_U$  consists of generators  $\xi$  of  $j_U$  with  $\xi \geq \sup j_U[\lambda]$ . Thus  $U = \mathcal{K}_\lambda^2$ . But then by Lemma 7.5.10,  $\min(a_U)$  is a  $j_U$ -generator. This contradicts that  $d(j_{1n}(\nu))$  is not an  $e(\nu)$ -generator.  $\square$

The main application of Theorem 7.5.17 is the following fact:

**Lemma 7.5.18** (UA). *Assume  $\lambda$  is isolated and let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Either  $j[\lambda] \subseteq \lambda$  or  $\mathcal{K}_\lambda \cap M \in M$ .*

*Proof.* Assume  $\sup j[\lambda] > \lambda$ . Then  $\mathcal{K}_\lambda \cap M$  is not  $\gamma$ -decomposable for any  $M$ -cardinal  $\gamma \in [\sup j[\lambda], j(\lambda))$ . Therefore  $\mathcal{K}_\lambda \cap M \in M$ .  $\square$

Theorem 7.5.17 gives a coarse bound on the strength of  $\mathcal{K}_\lambda$  when  $\lambda$  is isolated.

**Proposition 7.5.19 (UA).** *Suppose  $\lambda$  is isolated and let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Then  $P(\lambda) \subseteq M$  if and only if  $\mathcal{K}_\lambda$  is  $\lambda$ -complete.*

*Proof.* Assume  $P(\lambda) \subseteq M$ . Since  $\mathcal{K}_\lambda \notin M$ ,  $\mathcal{K}_\lambda \cap M \notin M$ , so by Lemma 7.5.18,  $\sup j[\lambda] \subseteq \lambda$ . By the Kunen Inconsistency Theorem (Theorem 4.2.37), this implies  $\text{CRT}(j) \geq \lambda$ . In other words,  $\mathcal{K}_\lambda$  is  $\lambda$ -complete.  $\square$

Assume  $\lambda$  is a nonmeasurable isolated cardinal. One would expect to get a much better bound on the strength of  $\mathcal{K}_\lambda$  than  $\lambda$ . When  $\delta_\lambda$  is a successor cardinal, one can in fact prove that  $P(\delta_\lambda) \not\subseteq M_{\mathcal{K}_\lambda}$ , which determines the strength of  $j_{\mathcal{K}_\lambda}$  exactly (since  $j_{\mathcal{K}_\lambda}$  is  $<\delta_\lambda$ -supercompact by Proposition 7.5.20 below). When  $\delta_\lambda$  is inaccessible, however, we do not know whether  $P(\delta_\lambda) \subseteq M_{\mathcal{K}_\lambda}$  is possible.

## Isolated cardinals and the GCH

By Proposition 7.5.4, the existence of isolated cardinals that are not measurable is paired with failures of the Generalized Continuum Hypothesis. In this section, we study precisely how GCH fails below a nonmeasurable isolated cardinal. Here the cardinal  $\delta_\lambda$  (see Definition 7.5.11) takes the center stage.

**Proposition 7.5.20 (UA).** *Suppose  $\lambda$  is an isolated cardinal that is not measurable. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Let  $\kappa = \kappa_\lambda$  and  $\delta = \delta_\lambda$ . Then the following hold:*

- (1) *Every regular cardinal  $\iota \in [\kappa, \delta)$  is Fréchet.*
- (2)  *$j$  is  $<\delta$ -supercompact.*
- (3) *If  $\delta$  is a limit cardinal then  $\delta$  is strongly inaccessible.*
- (4) *Otherwise  $\delta$  is the successor of a cardinal  $\gamma$  of cofinality at least  $\kappa_\lambda$ . In fact, no cardinal in the interval  $(\text{cf}(\gamma), \gamma)$  is  $\gamma$ -strongly compact.*

*Proof.* We first prove (1). Let  $\eta \in [\iota, \delta)$  be a Fréchet cardinal. Then for any  $\gamma \in [\kappa, \eta)$ , then there is a Fréchet cardinal in  $(\gamma, \eta]$ . By Lemma 7.4.13, there are no isolated cardinals in  $[\kappa, \lambda)$ . Lemma 7.4.12 implies that every regular cardinal in  $[\kappa, \eta)$  is Fréchet. In particular,  $\iota$  is Fréchet.

We now prove (2). Fix a regular cardinal  $\iota \in [\kappa, \delta)$ , and we will show that  $j$  is  $\iota$ -supercompact. (This suffices since the Recall that there are no isolated cardinals in  $[\kappa, \lambda)$  (Lemma 7.4.13). Thus  $\kappa_\iota \leq \kappa$  as a consequence of Lemma 7.4.19. Moreover, by Theorem 7.4.9,  $\kappa_\iota$  is  $\iota$ -strongly compact. We can therefore apply our technique for converting amenability of ultrafilters into strength (Proposition 7.3.32) to conclude that  $P(\iota) \subseteq M$ :  $\kappa_\iota$  is  $\iota$ -strongly compact,  $M$  is closed under  $\kappa_\iota$ -sequences, and every countably complete ultrafilter on  $\iota$  is amenable to  $M$  (Proposition 7.4.17), so  $P(\iota) \subseteq M$ .

By Theorem 7.3.33,  $j_{\mathcal{K}_\iota}$  is  $\iota$ -tight. Moreover  $j_{\mathcal{K}_\iota}(\kappa) \geq j_{\mathcal{K}_\iota}(\kappa_\iota) > \iota$ . By Proposition 7.4.17,  $\mathcal{K}_\iota \sqsubset \mathcal{K}_\lambda$ . We now use the following fact:

**Lemma.** *Suppose  $\kappa \leq \iota$  are cardinals,  $U$  and  $W$  are countably complete ultrafilters,  $U$  is  $\iota$ -tight,  $j_U(\kappa) > \iota$ ,  $W$  is  $\kappa$ -complete, and  $U \sqsubset W$ . Then  $j_W$  is  $\iota$ -tight.*

*Proof.* Since  $j_U(W)$  is  $\iota$ -complete in  $M_U$ ,  $\text{Ord}^\iota \cap M_U \subseteq M_{j_U(W)}^{M_U} = j_U(M_W) \subseteq M_W$ . (The final containment uses  $U \sqsubset W$ .) Therefore since  $M_U$  has the  $\leq \iota$ -covering property, so does  $M_W$ . Thus  $j_W$  is  $\iota$ -tight.  $\square$

We can apply the fact to  $U = \mathcal{K}_\iota$  and  $W = \mathcal{K}_\lambda$ . It follows that  $j$  is  $\iota$ -tight. Since  $j$  is  $\iota$ -tight and  $P(\iota) \subseteq M$ ,  $j$  is  $\iota$ -supercompact.

We now prove (3). Suppose towards a contradiction that  $\delta$  is singular. Then by (2),  $j$  is  $\delta$ -supercompact. If  $\text{cf}(\delta) \geq \kappa_\lambda$ , it follows that  $\delta$  is Fréchet, contrary to the definition of  $\delta_\lambda$ . Therefore  $\text{cf}(\delta) < \kappa_\lambda$ . But then by Lemma 4.2.25,  $j$  is  $\delta^+$ -supercompact. Then  $\delta^+$  is Fréchet. The definition of  $\delta$  implies that no cardinal in  $[\delta, \lambda)$  is Fréchet, so it must be that  $\delta^+ = \lambda$ . This contradicts that  $\lambda$  is isolated (and in particular is a limit cardinal).

For (4), assume towards a contradiction that some cardinal  $\nu$  in the interval  $(\text{cf}(\gamma), \gamma)$  is  $\gamma$ -strongly compact. Then  $\nu$  is  $\gamma^+$ -strongly compact by Lemma 4.2.25. But  $\gamma^+ = \delta$  is not Fréchet, and this is contradiction.  $\square$

Suppose  $\lambda$  is an isolated cardinal, and let  $\delta = \delta_\lambda$ . Must  $2^{<\delta} = \delta$ ? By Proposition 7.5.20 (3), this is true if  $\delta$  is a limit cardinal, but we are unable to answer the question when  $\delta$  is a successor. The following bound is sufficient for most applications:

**Theorem 7.5.21.** *Suppose  $\lambda$  is isolated and  $\delta = \delta_\lambda$ . Then  $2^{<\delta} < \lambda$ .*

*Proof.* Assume by induction that the theorem holds for all isolated cardinals below  $\lambda$ . Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Then  $j$  is  $<\delta$ -supercompact (Proposition 7.5.20). Thus  $2^{<\delta} \leq (2^{<\delta})^M$ , so it suffices to show that  $(2^{<\delta})^M < \lambda$ .

**Claim 1.**  $(\delta^\sigma)^M \leq \lambda$ .

*Proof of Claim 1.* There are two cases.

First assume  $\sup j[\lambda] = \lambda$ . Since  $j$  is  $<\delta$ -supercompact, Kunen's Inconsistency Theorem (Lemma 4.2.38) implies that there is a measurable cardinal  $\iota < \delta$  such that  $j(\iota) > \delta$ . Now  $j(\iota) < \lambda$  is a measurable cardinal of  $M$ , so  $(\delta^\sigma)^M \leq j(\iota) < \lambda$ , as desired.

Assume instead that  $\lambda < \sup j[\lambda]$ . Then  $\mathcal{K}_\lambda \cap M \in M$  by Theorem 7.5.17. Thus  $\lambda$  is Fréchet in  $M$ , so  $(\delta^\sigma)^M \leq \lambda$ .  $\square$

If  $\delta^{+M}$  is Fréchet in  $M$ , then  $(2^{<\delta})^M = \delta$  by Theorem 6.3.15. Assume therefore that  $\delta^{+M}$  is not Fréchet in  $M$ . Let  $\eta = (\delta^\sigma)^M$ . Then  $\eta$  is isolated in  $M$  by Proposition 7.4.4. Moreover  $\eta \leq \lambda < j(\lambda)$ , so our induction hypothesis shifted to  $M$  applies at  $\eta$ . Notice that  $\delta \leq (\delta_\eta)^M$ : indeed, by Proposition 7.5.20,  $M$  is correct about cardinals below  $\delta$ , and by Proposition 7.4.17, all sufficiently large cardinals below  $\delta$  are Fréchet in  $M$ . Thus

$$(2^{<\delta})^M \leq (2^{<\delta_\eta^M})^M < \eta \leq \lambda$$

In particular  $(2^{<\delta})^M < \lambda$ , as desired.  $\square$

The following closely related fact can be seen as an ultrafilter-theoretic version of SCH:

**Proposition 7.5.22 (UA).** *Suppose  $\lambda$  is a regular isolated cardinal. Suppose  $D$  is a countably complete ultrafilter such that  $\lambda_D < \lambda$ . Then  $j_D(\lambda) = \lambda$ .*

*Proof.* Suppose towards a contradiction that the theorem fails, and let  $\lambda$  be the least counterexample. Let  $j : V \rightarrow M$  be the ultrapower by  $\mathcal{K}_\lambda$ . Let  $\delta = \delta_\lambda$  be the strict supremum of the Fréchet cardinals below  $\lambda$ . By Proposition 7.5.20,  $M^{<\delta} \subseteq M$ , and by Proposition 7.4.17,  $M$  satisfies that there is a countably complete ultrafilter  $D$  is  $\lambda_D < \delta$  such that  $j_D(\lambda) \neq \lambda$ .

Suppose first that  $\lambda < \sup j[\lambda]$ . Then  $\mathcal{K}_\lambda \cap M \in M$  by Theorem 7.5.17. Therefore  $\lambda$  is a regular Fréchet cardinal in  $M$ . Clearly  $\lambda$  is a limit cardinal in  $M$ . Since  $\lambda < j(\lambda)$ ,  $\lambda$  is not a counterexample to the proposition in  $M$ . Therefore  $\lambda$  is not isolated in  $M$ , so  $\lambda$  is strongly inaccessible in  $M$  by Corollary 7.5.2. But this contradicts that there is a countably complete ultrafilter  $D$  is  $\lambda_D < \delta$  such that  $j_D(\lambda) \neq \lambda$ .

Suppose instead that  $\lambda = \sup j[\lambda]$ . Let  $\kappa = \kappa_\lambda$ . We claim that for any countably complete ultrafilter  $U \in V_\kappa$ ,  $j_U(\lambda) = \lambda$ . Fix such an ultrafilter  $U$ . Since  $2^{<\delta} < \lambda$ ,  $j_U(\delta) < \lambda$ . By elementarity there are no Fréchet cardinals of  $M_U$  in the interval  $[j_U(\delta), j_U(\lambda))$ . But  $\mathcal{K}_\lambda \sqsubset U$  (by Kunen's commuting ultrapowers lemma, Theorem 5.5.20), so  $\mathcal{K}_\lambda \cap M_U \in M_U$ , and hence  $\lambda$  is Fréchet in  $M_U$ . Thus  $\lambda$  is a Fréchet cardinal of  $M_U$  in the interval  $[j_U(\delta), j_U(\lambda)]$ , so we must have  $j_U(\lambda) = \lambda$ , as claimed.

Let  $\eta$  be the least ordinal such that for some ultrafilter  $D$  with  $\lambda_D < \lambda$ ,  $j_D(\eta) > \lambda$ . (Note that  $\eta$  exists since  $\lambda$  is regular.) We claim  $j(\eta) = \eta$ . To see this, note that if  $D$  is an ultrafilter with  $\delta$ , then  $j_D[\eta] \subseteq \eta$ . (Otherwise we would contradict the minimality of  $\eta$  as in Lemma 7.4.25.) If  $j(\eta) > \eta$ , however, then since  $j(\eta) < \lambda$ , there is an ultrafilter  $D$  on  $\delta$  such that  $j_D(\eta) > j(\eta)$ . Since  $Z \sqsubset \mathcal{K}_\lambda$ , this contradicts that  $M$  thinks  $j(\eta)$  is closed under ultrapower embeddings associated to ultrafilters on  $j(\delta)$ .

Suppose  $\xi$  is a fixed point of  $j$ . Let  $\gamma$  be the least cardinal that carries a countably complete ultrafilter  $U$  such that  $j_U(\eta) > \xi$ . Then  $\gamma < \delta$  by assumption. We claim  $j(\gamma) = \gamma$ . The reason is that  $M$  is closed under  $\gamma$  sequences and contains every ultrafilter on  $\gamma$ , so  $M$

satisfies that there is an ultrafilter  $U$  on  $\gamma$  such that  $j_U(\eta) > \xi$ . Since  $j(\eta) = \eta$  and  $j(\xi) = \xi$ , it follows that  $j(\gamma)$  is the least  $M$  cardinal carrying such an ultrafilter  $U$ , and hence  $j(\gamma) = \gamma$ . Since  $\gamma$  is a fixed point of  $j$  below its supercompactness,  $\gamma < \kappa$  by the Kunen inconsistency theorem.

It follows that  $\eta$  is mapped arbitrarily high below  $\lambda$  by ultrafilters in  $V_\kappa$ . Since  $\lambda$  is regular, there must be a single ultrafilter  $U \in V_\kappa$  such that  $j_U(\eta) \geq \lambda$ . This contradicts that for all  $U \in V_\kappa$ ,  $j_U(\lambda) = \lambda$ .  $\square$

Our next theorem shows that the problematic isolated cardinals  $\lambda$  suffer a massive failure of GCH precisely at  $\delta_\lambda$ :

**Theorem 7.5.23.** *Suppose  $\lambda$  is a nonmeasurable isolated cardinal and  $\delta = \delta_\lambda$ . Then  $2^\delta \geq \lambda$ .*

It is not clear whether it is possible that  $2^\delta = \lambda$ . This of course implies that  $\lambda$  is regular and hence weakly Mahlo by Theorem 7.5.36 below.

This theorem requires an analysis of indecomposable ultrafilters due to Silver. His analysis can be seen as an improvement of Lemma 5.5.27 in a the key special case of indecomposable ultrafilters.

**Theorem 7.5.24** (Silver). *indexIndecomposable ultrafilter!Silver's Theorem Suppose  $\delta$  is a regular cardinal and  $U$  is a countably complete ultrafilter that is  $\lambda$ -indecomposable for all  $\lambda \in [\delta, 2^\delta]$ . Then there is an ultrafilter  $D$  with  $\lambda_D < \delta$  such that there is an elementary embedding  $k : M_D \rightarrow M_U$  with  $j_U = k \circ j_D$  and  $\text{CRT}(k) > j_D((2^\delta)^+)$  if  $k$  is nontrivial.*

The proof does not really use that  $U$  is countably complete, and this was important in Silver's original work. Since we only need the theorem when  $U$  is countably complete, we make this assumption. (This is for notational convenience: the notion of the critical point of  $k$  does not really make sense if  $M_D$  is illfounded.)

We begin by describing a correspondence between partitions of ultrafilters and points in the ultrapower embedding that is implicit in Silver's proof.



**Definition 7.5.25.** Suppose  $P$  is a partition of a set  $X$  and  $A$  is a subset of  $X$ . Then the *restriction of  $P$  to  $A$*  is the partition  $P \upharpoonright A$  defined by

$$P \upharpoonright A = \{A \cap S : S \in P \text{ and } A \cap S \neq \emptyset\}$$

**Definition 7.5.26.** Suppose  $U$  is an ultrafilter on a set  $X$  and  $\lambda$  is a cardinal.

- $\mathbb{Q}_U$  denotes the preorder on the collection of partitions of  $X$  defined by setting  $P \leq Q$  if there exists some  $A \in U$  such that  $Q \upharpoonright A$  refines  $P \upharpoonright A$ .
- $\mathbb{Q}_U^\lambda \subseteq \mathbb{Q}_U$  consists of those  $P$  such that  $|P \upharpoonright A| < \lambda$  for some  $A \in U$ .
- $\mathbb{P}_U$  denotes the preorder on  $M_U$  defined by setting  $x \leq y$  if  $x$  is definable in  $M$  from  $y$  and parameters in  $j_U[V]$ .
- $\mathbb{P}_U^\lambda \subseteq \mathbb{P}_U$  is the restriction of  $\mathbb{P}_U$  to  $H^{M_U}(j_U[V] \cup \text{sup } j_U[\lambda])$ .

The following lemma, which is ultimately just an instance of the correspondence between partitions of  $X$  and surjective functions on  $X$ , shows that the preorders  $\mathbb{Q}_U$  and  $\mathbb{P}_U$  are equivalent preorders:

**Lemma 7.5.27.** *Suppose  $U$  is an ultrafilter on a set  $X$ . For  $P \in \mathbb{Q}_U$ , let  $\Phi(P)$  be the unique  $S \in j_U(P)$  such that  $a_U \in S$ . Then the following hold:*

- (1)  $\Phi$  is order-preserving: for any  $P, Q \in \mathbb{Q}_U$ ,  $P \leq Q$  if and only if  $\Phi(P) \leq \Phi(Q)$ .
- (2)  $\Phi$  is surjective on equivalence classes: for any  $x \in \mathbb{P}_U$ , there is some  $P \in \mathbb{Q}_U$  such that  $x$  and  $\Phi(P)$  are equivalent in  $\mathbb{P}_U$ .
- (3) For any cardinal  $\lambda$ ,  $\Phi[\mathbb{Q}_U^\lambda] \subseteq \mathbb{P}_U^\lambda$ .
- (4) Suppose  $P \in \mathbb{Q}_U$ . Let  $D = \{A \subseteq P : \bigcup A \in U\}$ . Then there is a unique elementary embedding  $k : M_D \rightarrow M_U$  such that  $k \circ j_D = j_U$  and  $k(a_D) = \Phi(D)$ .

*Proof. Proof of (1):* Suppose  $P, Q \in \mathbb{Q}_U$  and  $P \leq Q$ . Fix  $A \in U$  such that  $Q \upharpoonright A$  refines  $P \upharpoonright A$ . Then  $\Phi(P)$  is definable in  $M_U$  from the parameters  $\Phi(Q), j_U(P), j_U(A)$  as the unique  $S \in j_U(P)$  such that  $\Phi(Q) \cap j_U(A) \subseteq S \cap j_U(A)$ . In other words,  $\Phi(P) \leq \Phi(Q)$ .

Conversely suppose  $\Phi(P) \leq \Phi(Q)$ , so that  $\Phi(P) = j_U(f)(\Phi(Q))$  for some  $f : Q \rightarrow P$ . Let  $A \subseteq X$  consist of those  $x \in X$  such that  $x \in f(S)$  where  $S$  is the unique element of  $Q$  with  $x \in S$ . Then  $A \in U$  since  $a_U \in j_U(f)(S)$  where  $S = \Phi(Q)$  is the unique  $S \in j_U(Q)$  such that  $a_U \in S$ . Moreover for any  $S \in Q$ ,  $S \cap A \subseteq f(S) \cap A$ , so  $Q \upharpoonright A$  refines  $P \upharpoonright A$ . In other words,  $P \leq Q$ .

*Proof of (2):* Fix  $x \in \mathbb{P}_U$ . Fix  $f : X \rightarrow V$  such that  $x = j_U(f)(a_U)$ . Let

$$P = \{f^{-1}[\{y\}] : y \in \text{ran}(f)\}$$

Then  $\Phi(P)$  is interdefinable with  $x$  over  $M_U$  using parameters in  $j_U[V]$ :  $\Phi(P)$  is the unique  $S \in j_U(P)$  such that  $x \in j_U(f)[S]$ ; and since  $j_U(f)[\Phi(P)] = \{x\}$ ,  $x = \bigcup j_U(f)[\Phi(P)]$ .

*Proof of (3):* Suppose  $P \in \mathbb{Q}_U^\lambda$ . Fix  $\delta < \lambda$  and a surjection  $f : \delta \rightarrow P$ . Then  $\Phi(P) = j_U(f)(\xi)$  for some  $\xi < j_U(\delta) \leq \sup j_U[\lambda]$ . Hence  $\Phi(P) \in H^{M_U}(j_U[V] \cup \sup j_U[\lambda])$ , as desired.

*Proof of (4):* Define  $g : X \rightarrow P$  by setting  $g(a)$  equal to the unique  $S \in P$  such that  $a \in S$ . Then  $g_*(U) = D$  and  $j_U(g)(a_U) = \Phi(P)$ . Therefore by the basic theory of the Rudin-Keisler order (Corollary 5.2.8), there is a unique elementary embedding  $k : M_D \rightarrow M_U$  with  $k \circ j_D = j_U$  and  $k(a_D) = \Phi(P)$ .  $\square$

For Silver's theorem, it is useful to reformulate indecomposability in terms of  $\mathbb{Q}_U$ :

**Lemma 7.5.28.** *Suppose  $U$  is an ultrafilter on  $X$  and  $\lambda$  is a cardinal. Then  $U$  is  $\lambda$ -indecomposable if every partition of  $X$  into  $\lambda$  pieces is equivalent in  $\mathbb{Q}_U$  to a partition of  $X$  into fewer than  $\lambda$  pieces.*  $\square$

We now prove Silver's theorem.

*Proof of Theorem 7.5.24.* Let  $(\mathbb{Q}, \leq) = \mathbb{Q}_U^{(2^\delta)^+}$  be the preorder of  $U$ -refinement on the set of partitions of  $X$  of size at most  $2^\delta$ . Let  $\preceq$  be the preorder of refinement on  $\mathbb{Q}$ , so  $P \preceq Q$  implies  $Q$  refines  $P$ . Thus  $(\mathbb{Q}, \leq)$  extends  $(\mathbb{Q}, \preceq)$ .

Note that  $\preceq$  is  $\leq\delta$ -directed. Indeed, suppose  $\mathcal{S} \subseteq \mathbb{Q}$  has cardinality  $\delta$ . Then

$$P = \left\{ \bigcap \mathcal{C} : \mathcal{C} \text{ meets each element of } \mathcal{S} \text{ and } \bigcap \mathcal{C} \neq \emptyset \right\}$$

refines every partition in  $\mathcal{S}$ , and  $|P| \leq |\prod \mathcal{S}| \leq 2^\delta$ . The partition  $P$  is called the *least common refinement* of  $\mathcal{S}$ .

We claim that  $(\mathbb{Q}, \leq)$  has a maximum element (up to equivalence). Since  $(\mathbb{Q}, \preceq)$  is directed,  $(\mathbb{Q}, \leq)$  is directed, and thus it suffices to show that  $(\mathbb{Q}, \leq)$  has a maximal element. Assume the contrary, towards a contradiction. Then since  $(\mathbb{Q}, \preceq)$  is  $\leq\delta$ -directed, we can produce a sequence  $\langle P_\alpha : \alpha \leq \delta \rangle$  of elements of  $\mathbb{Q}$  such that for all  $\alpha < \beta \leq \delta$ ,  $P_\alpha \preceq P_\beta$  and  $P_\beta \not\leq P_\alpha$ .

Since  $U$  is  $\lambda$ -indecomposable for all  $\lambda \in [\delta, 2^\delta]$ , there is some  $A \in U$  such that  $|P_\delta \upharpoonright A| < \delta$ . For each  $\alpha \leq \delta$ , let  $Q_\alpha = P_\alpha \upharpoonright A$ . We use the following general fact:

**Claim.** *Suppose  $\delta$  is a regular cardinal,  $A$  is a set of size less than  $\delta$ , and  $\langle Q_\alpha : \alpha < \delta \rangle$  is a sequence of partitions of  $A$  such that for all  $\alpha < \beta < \delta$ ,  $Q_\beta$  refines  $Q_\alpha$ . Then for all sufficiently large  $\alpha < \beta < \delta$ ,  $Q_\alpha = Q_\beta$ .*

*Proof.* Let  $Q$  be the least common refinement of  $\{Q_\alpha : \alpha < \delta\}$ . Suppose  $S \in Q$ . We claim that  $S \in Q_\alpha$  for some  $\alpha < \delta$ . Consider the sequence  $\langle S_\alpha : \alpha < \delta \rangle$  where  $S_\alpha \in Q_\alpha$  is the unique element of  $Q_\alpha$  containing  $S$ . Thus  $S = \bigcap_{\alpha < \delta} S_\alpha$ . Note that  $\langle S_\alpha : \alpha < \delta \rangle$  is a decreasing sequence of sets, each of cardinality less than  $\delta$ . Thus for all sufficiently large  $\alpha < \delta$ ,  $S_\alpha = S$ , and in particular,  $S \in Q_\alpha$ .

For each  $S \in Q$ , fix  $\alpha_S < \delta$  such that  $S \in Q_{\alpha_S}$ . Let  $\gamma = \sup_{S \in Q} \alpha_S$ . Then  $\gamma < \delta$  since  $|Q| < \delta$  and  $\delta$  is regular. By definition,  $Q \subseteq Q_\gamma$ , so  $Q_\gamma = Q$ . If  $\alpha \in [\gamma, \delta)$ , then  $Q$  refines  $Q_\alpha$  which refines  $Q_\gamma = Q$ , and hence  $Q = Q_\alpha$ . This proves the claim.  $\square$

Thus for all sufficiently large  $\alpha < \beta < \delta$ ,  $Q_\alpha = Q_\beta$ , or in other words,  $P_\alpha \upharpoonright A = P_\beta \upharpoonright A$ . It follows that  $P_\beta \leq P_\alpha$ , and this contradicts our choice of  $P_\beta$ . Thus our assumption that  $(\mathbb{Q}, \leq)$  has no maximum element was false.

Let  $P$  be a maximum element of  $(\mathbb{Q}, \leq)$ . By the indecomposability of  $U$ , we may assume  $|P| < \delta$  by replacing  $P$  with an equivalent element of  $(\mathbb{Q}, \leq)$ . We now apply Lemma 7.5.27. Let  $D$  be the ultrafilter corresponding to  $P$  as in Lemma 7.5.27 (4):

$$D = \{\mathcal{A} \subseteq P : \bigcup \mathcal{A} \in U\}$$

Let  $k : M_D \rightarrow M_U$  be unique elementary embedding with  $k \circ j_D = j_U$  and  $k(a_D) = \Phi(P)$ . We have  $\lambda_D < \delta$  since  $|P| < \delta$ .

Let  $\eta = (2^\delta)^+$ . We will show  $\text{CRT}(k) > j_U(\eta)$  if  $k$  is nontrivial, or in other words, that  $j_U(\eta) \subseteq k[M_D]$ . Since  $P$  is a maximum element of  $\mathbb{Q}_U^\eta$ , Lemma 7.5.27 (1), (2), and (3) imply that  $\Phi(P)$  is a maximum element of  $\mathbb{P}_U^\eta$ . In other words, if  $x \in H^{M_U}(j_U[V] \cup \text{sup } j_U[\eta])$ , then  $x$  is definable in  $M_U$  from  $\Phi(P)$  and parameters in  $j_U[V]$ , or in other words  $x \in k[M_D]$ . In particular,  $\text{sup } j_U[\eta] \subseteq k[M_D]$ .

We finish by showing that  $\text{sup } j_U[\eta] = j_U(\eta)$ . Suppose not. Then since  $\eta$  is regular,  $U$  is  $\eta$ -decomposable. Since  $\eta = (2^\delta)^+$ , Theorem 7.4.3 implies that  $U$  is  $\lambda$ -decomposable where  $\lambda = \text{cf}(2^\delta)$ . But by König's Theorem,  $\lambda \in [\delta, 2^\delta]$ , and this is a contradiction.  $\square$

We can finally prove Theorem 7.5.23:

*Proof of Theorem 7.5.23.* Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Assume  $2^\delta < \lambda$ . We will show that  $\text{CRT}(j) \geq \lambda$ , so  $\mathcal{K}_\lambda$  is a  $\lambda$ -complete uniform ultrafilter on  $\lambda$ , and hence  $\lambda$  is measurable.

Since  $\lambda$  is isolated and  $2^\delta < \lambda$ ,  $\mathcal{K}_\lambda$  is  $\gamma$ -indecomposable for all cardinals in the interval  $[\delta, 2^\delta]$ . By Proposition 7.5.20,  $\delta$  is regular. Therefore we can apply Theorem 7.5.24. Fix  $D$  with  $\lambda_D < \delta$  such that there is an elementary embedding  $k : M_D \rightarrow M_{\mathcal{K}_\lambda}$  with  $k \circ j_D = j$  and  $\text{CRT}(k) > j_D(\delta)$  if  $k$  is nontrivial.

By Proposition 7.4.17,  $D \sqsubset \mathcal{K}_\lambda$ . Therefore  $j_D \upharpoonright \delta \in M$ . But  $j_D \upharpoonright \delta = j \upharpoonright \delta$  since  $\text{CRT}(k) > j_D(\delta)$ . It follows that  $j$  is  $\delta$ -supercompact. Since  $\delta$  is regular and  $\mathcal{K}_\lambda$  is  $\delta$ -indecomposable,  $j(\delta) = \text{sup } j[\delta]$ . Since  $j$  is  $\delta$ -supercompact and  $j(\delta) = \text{sup } j[\delta]$ , the Kunen Inconsistency (Theorem 4.4.32) implies that  $\text{CRT}(j) \geq \delta$ . There are no measurable cardinals

in the interval  $[\delta, \lambda)$  since in fact there are no Fréchet cardinals in  $[\delta, \lambda)$ . The fact that  $\text{CRT}(j) \geq \delta$  therefore implies  $\text{CRT}(j) \geq \lambda$ , as desired.  $\square$

Theorem 7.5.24 can be combined with Theorem 7.4.28 to prove a strengthening of Theorem 7.5.5:

**Theorem 7.5.29 (UA).** *Suppose  $\delta$  is a regular cardinal and  $U$  is a countably complete ultrafilter that is  $\lambda$ -indecomposable for all  $\lambda \in [\delta, 2^\delta]$ . Then there is an ultrafilter  $D$  with  $\lambda_D < \delta$  such that there is an internal ultrapower embedding  $h : M_D \rightarrow M_U$  with  $h \circ j_D = j_U$  and  $\text{CRT}(h) > j_D(\delta)$  if  $h$  is nontrivial.*

*Proof.* Using Silver's theorem, fix a uniform countably complete ultrafilter  $D$  on a cardinal  $\eta < \delta$  such that there is an elementary embedding  $k : M_D \rightarrow M_U$  with  $k \circ j_D = j_U$  and  $\text{CRT}(k) > j_D((2^\delta)^+)$  if  $k$  is nontrivial.

Recall that  $\mathcal{B}(X)$  denotes the set of countably complete ultrafilters on  $X$ . Theorem 7.4.28 implies that  $|\mathcal{B}(\eta)| \leq (2^\eta)^+$ . Thus  $j_D(\mathcal{B}(\eta))$  has cardinality less than or equal to  $j_D((2^\eta)^+)$  in  $M_U$ . Since  $\text{CRT}(k) > j_D((2^\eta)^+)$ ,  $k$  restricts to an isomorphism from  $j_D(\mathcal{B}(\eta), <_{\mathbf{k}})$  to  $j_U(\mathcal{B}(\eta), <_{\mathbf{k}})$ . Moreover, for any  $Z \in j_D(\mathcal{B}(\eta))$ ,

$$j_D^{-1}[Z] = j_U^{-1}[k(Z)]$$

We now use the fact that  $k$  is an isomorphism conjugating  $j_D^{-1}$  to  $j_U^{-1}$  to conclude that  $k(t_D(D)) = t_U(D)$ . By Theorem 5.4.40,  $t_D(D)$  is the least element of  $j_D(\mathcal{B}(\eta), <_{\mathbf{k}})$  with  $j_D^{-1}[Z] = D$ . Therefore since  $k$  is an order-isomorphism that conjugates  $j_D^{-1}$  to  $j_U^{-1}$ ,  $k(t_D(D))$  is the least element  $Z$  of  $j_U(\mathcal{B}(\eta), <_{\mathbf{k}})$  with  $j_U^{-1}[Z] = D$ . But by Theorem 5.4.40, the least such  $Z$  is equal to  $t_U(D)$ . Thus  $k(t_D(D)) = t_U(D)$ .

Recall the characterization of the Rudin-Frolík order in terms of translation functions (Lemma 5.4.39): if  $W$  and  $Z$  are countably complete ultrafilters, then  $W \leq_{\text{RF}} Z$  if and only if  $t_Z(W)$  is principal in  $M_Z$ . Applying this characterization in one direction to  $D \leq_{\text{RF}} D$ ,  $t_D(D)$  is principal in  $M_D$ . Therefore  $t_U(D) = k(t_D(D))$  is principal in  $M_U$ , so applying the characterization in the other direction, it follows that  $D \leq_{\text{RF}} U$ .

Let  $h : M_D \rightarrow M_U$  be the unique internal ultrapower embedding such that  $h \circ j_D = j_U$ . By Lemma 5.4.39,  $t_D(D)$  is the principal ultrafilter concentrated at  $a_D$  and  $t_U(D)$  is the principal ultrafilter concentrated at  $h(a_D)$ . Since  $k(t_D(D)) = t_U(D)$ , it follows that  $k(a_D) = h(a_D)$ . Since  $k \circ j_D = j_U$ , in fact  $k \upharpoonright j_D[V] \cup \{a_D\} = h \upharpoonright j_D[V] \cup \{a_D\}$ . Thus  $k = h$ , since  $M_D = H^{M_D}(j_D[V] \cup \{a_D\})$ . It follows that  $h : M_D \rightarrow M_U$  is an internal ultrapower embedding with  $h \circ j_D = j_U$  and  $\text{CRT}(h) > j_D(\delta)$  if  $h$  is nontrivial.  $\square$

Our work on isolated cardinals leads to some relatively simple criteria for the completeness of an ultrafilter in terms of a local version of irreducibility that will become important when we analyze larger supercompact cardinals:

**Definition 7.5.30.** Suppose  $\lambda$  is a cardinal and  $U$  is a countably complete ultrafilter.

- $U$  is  $\lambda$ -irreducible if for all  $D \leq_{\text{RF}} U$  with  $\lambda_D < \lambda$ ,  $D$  is principal.
- $U$  is  $\leq \lambda$ -irreducible if  $U$  is  $\lambda^+$ -irreducible.

Note that  $U$  is  $\leq \lambda$ -irreducible if and only if  $U$  is  $\lambda^\sigma$ -irreducible.

At isolated cardinals, we have the following fact which is often useful:

**Theorem 7.5.31 (UA).** *Suppose  $\lambda$  is a cardinal and  $U$  is a countably complete ultrafilter.*

- (1) *If  $\lambda$  is a strong limit cardinal that is not a limit of Fréchet cardinals, then  $U$  is  $\lambda$ -irreducible if and only if  $U$  is  $\lambda$ -complete.*
- (2) *If  $\lambda$  is a strong limit cardinal and no cardinal  $\kappa < \lambda$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ , then  $U$  is  $\lambda$ -irreducible if and only if  $U$  is  $\lambda$ -complete.*
- (3) *If  $\lambda$  is isolated, then  $U$  is  $\lambda^+$ -irreducible if and only if  $U$  is  $\lambda^+$ -complete.*

*Proof.* (1) is immediate from Theorem 7.5.5.

(2) follows from (1). By Corollary 7.5.2, either  $\lambda$  is not a limit of Fréchet cardinals or  $\lambda$  is a limit of isolated cardinals. The former case is precisely (1). In the latter case, we can

apply (1) at each isolated cardinal below  $\lambda$ . Thus we conclude that  $U$  is  $\bar{\lambda}$ -complete for all isolated cardinals  $\bar{\lambda} < \lambda$ . It follows that  $U$  is  $\lambda$ -complete as desired.

(3) also follows from (1). Since  $U$  is  $\lambda^+$ -irreducible,  $U$  is  $\lambda^\sigma$ -irreducible, and by Lemma 7.4.27,  $\lambda^\sigma$  is measurable. Thus  $U$  is  $\lambda^\sigma$ -complete by (1) and in particular,  $U$  is  $\lambda^+$ -complete.  $\square$

Working in a bit more generality but with a stronger irreducibility assumption, we have the following completeness result:

**Theorem 7.5.32 (UA).** *Suppose  $\delta$  is a regular cardinal such that no cardinal  $\kappa \leq \delta$  is  $\delta$ -supercompact. Then a countably complete ultrafilter  $U$  is  $\delta^+$ -complete if and only if it is  $\leq 2^\delta$ -irreducible.*

*Proof.* The forward direction is trivial, so let us prove the converse.

Suppose that  $U$  is  $\leq 2^\delta$ -irreducible. We claim that  $U$  is  $\lambda$ -irreducible where  $\lambda > \delta$  is a strong limit cardinal that is not a limit of Fréchet cardinals. An immediate consequence of the factorization theorem for isolated measurable cardinal (Theorem 7.5.5) is that any  $\lambda$ -irreducible ultrafilter is  $\lambda$ -complete, and this proves the theorem.

If  $\delta^\sigma$  does not exist, then the  $\leq \delta$ -irreducibility of  $U$  implies that  $U$  itself is principal, so  $U$  is  $\lambda$ -irreducible and  $\lambda$ -complete for any cardinal  $\lambda$ . Thus assume  $\delta^\sigma$  exists.

There are two cases. Suppose first that  $\delta$  is a Fréchet cardinal. Let  $\lambda = \delta^\sigma$ . Since  $U$  is  $\leq \delta$ -irreducible,  $U$  is  $\lambda$ -irreducible. We claim that  $\lambda$  is an isolated measurable cardinal. First note that  $\lambda > \delta^+$  since otherwise  $\kappa_{\delta^+}$  is  $\delta$ -supercompact by Corollary 7.4.10. Thus by Proposition 7.4.4,  $\lambda$  is isolated. Assume towards a contradiction that  $\lambda$  is not measurable. Then by Proposition 7.5.20,  $\kappa_\lambda$  is  $< \delta_\lambda$ -supercompact. But  $\delta < \delta_\lambda$  since  $\delta < \lambda$  is Fréchet, and hence  $\kappa_\lambda$  is  $\delta$ -supercompact, a contradiction. Hence  $\lambda$  is measurable.

Suppose instead that  $\delta$  is not a Fréchet cardinal. If  $\delta^\sigma$  is measurable, let  $\lambda = \delta^\sigma$ . Suppose  $\delta^\sigma$  is not measurable. By Theorem 7.5.23,  $\delta^\sigma \leq 2^\delta$ , so in particular  $U$  is  $\leq \delta^\sigma$ -irreducible. Let  $\lambda = \delta^{\sigma\sigma}$ . (If  $\delta^{\sigma\sigma}$  does not exist, then again since  $U$  is  $\leq \delta^\sigma$ -irreducible,  $U$  is principal.)

By Lemma 7.4.27,  $\lambda$  is measurable; here, one must check that  $\delta^\sigma$  is isolated. Since  $U$  is  $\leq 2^\delta$ -irreducible,  $U$  is  $\leq \delta^\sigma$ -irreducible, so  $U$  is  $< \lambda$ -irreducible.  $\square$

One might expect a strengthening of this theorem to be true: if  $U$  is just  $\leq \delta$ -irreducible and no  $\kappa \leq \delta$  is  $\delta$ -supercompact, then  $U$  should be  $\delta^+$ -complete. The main issue is that if  $\lambda = \delta^\sigma$  is an isolated nonmeasurable cardinal, then  $U = \mathcal{K}_\lambda$  is a counterexample. If instead  $\lambda$  is measurable, then  $\leq \delta$ -irreducibility indeed suffices. What about  $\delta$ -irreducibility? If  $\delta$  is the least cardinal such that  $\mathcal{K}_\delta$  exists and does not have a  $\delta$ -supercompact ultrapower, then  $U = \mathcal{K}_\delta$  is a counterexample.

A similar theorem is true for singular cardinals:

**Theorem 7.5.33 (UA).** *Suppose  $U$  is a countably complete ultrafilter and  $\gamma$  is a singular cardinal such that no  $\kappa \leq \gamma$  is  $\gamma^+$ -supercompact. Then  $U$  is  $\gamma^+$ -complete if and only if  $U$  is  $\leq 2^\delta$ -irreducible for all  $\delta < \gamma$ .*

*Proof.* Let  $\delta = \sup\{\gamma^+ : \gamma < \lambda \text{ is a Fréchet cardinal}\}$ .

Suppose first that  $\delta$  is regular. Since  $\delta$  is not Fréchet, no cardinal  $\kappa \leq \delta$  is  $\delta$ -supercompact. Since  $U$  is  $\leq 2^\delta$ -irreducible, we are in a position to apply Theorem 7.5.32. We can conclude that  $U$  is  $\delta^+$ -complete. Since there are no measurable cardinals in the interval  $(\delta, \gamma)$ , it follows that  $U$  is  $\gamma^+$ -complete.

Suppose instead that  $\delta$  is singular. If  $\delta^\sigma$  does not exist, then it is easy to see that  $U$  is principal, and thus we are done. Therefore assume  $\delta^\sigma$  exists, and let  $\lambda = \delta^\sigma$ . Note that  $\lambda > \delta^+$ : if  $\delta < \gamma$  this follows from the fact that  $\delta^+$  is not Fréchet, while if  $\delta = \gamma$ , this follows from the fact that no cardinal is  $\gamma^+$ -supercompact. Thus  $\lambda$  is isolated. Note that  $\delta_\lambda = \delta$  is singular. Therefore by Proposition 7.5.20,  $\lambda$  is measurable. Since  $U$  is  $\leq \delta$ -irreducible,  $U$  is  $< \lambda$ -irreducible, and therefore as an immediate consequence of the factorization theorem for isolated measurable cardinal (Theorem 7.5.5),  $U$  is  $\lambda$ -complete.  $\square$

Let us close this subsection with a remark about the size of regular isolated cardinals.



**Definition 7.5.34.** A regular cardinal  $\kappa$  is  $\sigma$ -Mahlo if there is a countably complete weakly normal ultrafilter on  $\kappa$  that concentrates on regular cardinals.

**Proposition 7.5.35.** *If  $\kappa$  is  $\sigma$ -Mahlo then  $\kappa$  is weakly Mahlo.* □

In fact,  $\sigma$ -Mahlo cardinals are “greatly weakly Mahlo.” A theorem of Gitik shows that it is consistent that there is a  $\sigma$ -Mahlo cardinal that does not have the tree property.

**Theorem 7.5.36 (UA).** *Suppose  $\kappa$  is a regular isolated cardinal. Then  $\kappa$  is  $\sigma$ -Mahlo. In fact,  $\mathcal{K}_\kappa$  concentrates on regular cardinals.*

*Proof.* Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\kappa$ . Let  $\kappa_* = \sup j[\kappa]$ . Let  $\delta = \text{cf}^M(\kappa_*)$ . By Theorem 7.3.33,  $j$  is  $(\kappa, \delta)$ -tight, so  $j$  is discontinuous at any regular cardinal  $\iota \leq \kappa$  such that  $\delta < j(\iota)$ . Since  $\kappa$  is isolated,  $j$  is continuous at all sufficiently large cardinals less than  $\kappa$ . Putting these observations together, it follows that there are no regular cardinals  $\iota < \kappa$  such that  $j(\iota) > \delta$ . In other words,  $\sup j[\kappa] \leq \delta$ . Thus  $\kappa_* = \delta$ , so  $\kappa_*$  is regular. Since  $\mathcal{K}_\kappa$  is weakly normal,  $\kappa_* = a_{\mathcal{K}_\kappa}$ , so by Los’s Theorem,  $\mathcal{K}_\kappa$  concentrates on regular cardinals. □

This fact has a converse: assuming UA, any  $\sigma$ -Mahlo cardinal that is not measurable is isolated. It is not clear that singular Fréchet cardinals must be very large. For example, we do not know how to rule out that the least Fréchet cardinal  $\lambda$  that is neither measurable nor a limit of measurables is in fact equal to  $\kappa^{+\kappa}$  for some measurable  $\kappa < \lambda$ .

## The linearity of the Mitchell order without GCH

Theorem 4.4.2 states that assuming UA + GCH, the Mitchell order is linear on normal fine ultrafilters on  $P_{\text{bd}}(\lambda)$ , the collection of bounded subsets of  $\lambda$ . Here we prove essentially the same theorem using UA alone. Instead of using  $P_{\text{bd}}(\lambda)$  as our underlying set, we use the following variant:

**Definition 7.5.37.** For any cardinal  $\lambda$ , let  $P_*(\lambda) = \{\sigma \in P_{\text{bd}}(\lambda) : |\sigma|^+ < \lambda\}$ .

The following obvious characterization of  $P_*(\lambda)$  is often more useful than the definition above:

$$P_*(\lambda) = \begin{cases} P_{\text{bd}}(\lambda) & \text{if } \lambda \text{ is a limit cardinal} \\ P_\gamma(\lambda) & \text{if } \lambda \text{ is a successor cardinal and } \gamma \text{ is its cardinal predecessor} \end{cases}$$

**Definition 7.5.38.** For any cardinal  $\lambda$ , let  $\mathcal{U}_\lambda$  denote the set of normal fine ultrafilters on  $P_*(\lambda)$ . Let  $\mathcal{U} = \bigcup_{\lambda \in \text{Card}} \mathcal{U}_\lambda$ .

The main theorem of this subsection is the following:

**Theorem 7.5.39 (UA).** *The class  $\mathcal{U}$  is linearly ordered by the Mitchell order.*

Due to the following fact, Theorem 7.5.39 can be seen as a precise formulation of the (literally false) statement that the Mitchell order is linear on normal fine ultrafilters:

**Proposition 7.5.40.** *Every normal fine ultrafilter is isomorphic to a unique element of  $\mathcal{U}$ .*

*Proof.* Recall that for any cardinal  $\lambda$ ,  $\mathcal{N}_\lambda$  denotes the set of normal fine ultrafilters on  $P_{\text{bd}}(\lambda)$  and  $\mathcal{N} = \bigcup_{\lambda \in \text{Card}} \mathcal{N}_\lambda$ . Also recall Proposition 4.4.12, which states that every normal fine ultrafilter is isomorphic to a unique element of  $\mathcal{N}$ . Therefore to prove the proposition, it suffices to show that there is a bijection  $\phi : \mathcal{N} \rightarrow \mathcal{U}$  such that  $\phi(\mathcal{U}) \cong \mathcal{U}$  for all  $\mathcal{U} \in \mathcal{N}$ .

In fact, if  $\mathcal{U} \in \mathcal{N}_\lambda$ , we will just set  $\phi(\mathcal{U}) = \mathcal{U} \upharpoonright P_*(\lambda)$ . It is clear that  $\phi$  is as desired as long as  $P_*(\lambda) \in \mathcal{U}$ . We now establish that this holds. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{U}$ . Then  $a_{\mathcal{U}} = j[\lambda]$  by Lemma 4.4.9. Of course  $|j[\lambda]|^M = \lambda$ , but note also that  $\lambda^{+M} < j(\lambda)$ : by Lemma 4.2.38, there is an inaccessible cardinal  $\kappa \leq \lambda$  such that  $\lambda < j(\kappa)$ , so  $\lambda^{+M} < j(\kappa) \leq j(\lambda)$ . Thus  $|j[\lambda]|^{+M} < j(\lambda)$ . By Łoś's Theorem, it follows that  $\{\sigma \in P_{\text{bd}}(\lambda) : |\sigma|^+ < \lambda\} \in \mathcal{U}$ . That is,  $P_*(\lambda) \in \mathcal{U}$ .  $\square$

The reason we use  $P_*(\lambda)$  as an underlying set rather than sticking with  $P_{\text{bd}}(\lambda)$  is that without assuming GCH, we cannot prove  $|P_{\text{bd}}(\lambda)| = \lambda$ . Therefore  $P_{\text{bd}}(\lambda)$  may be too large to use as an underlying set. On the other hand, we can prove  $|P_*(\lambda)| = \lambda$  in the relevant cases:

**Proposition 7.5.41** (UA). *Suppose  $\lambda$  is a cardinal such that  $\mathcal{U}_\lambda$  is nonempty. Then  $|P_*(\lambda)| = \lambda$ .*

*Proof.* Since  $\mathcal{U}_\lambda$  is nonempty, there is a normal fine ultrafilter on  $P_*(\lambda)$ , and hence there is a cardinal  $\kappa \leq \lambda$  that is  $\lambda$ -supercompact.

There are now two cases.

Suppose first that  $\lambda$  is a limit cardinal. Then  $P_*(\lambda) = P_{\text{bd}}(\lambda)$ . Moreover by Theorem 6.3.12,  $2^{<\lambda} = \lambda$ . Thus  $|P_*(\lambda)| = |P_{\text{bd}}(\lambda)| = 2^{<\lambda} = \lambda$ .

Suppose instead that  $\lambda$  is a successor cardinal. Let  $\gamma$  be the cardinal predecessor of  $\lambda$ . Then  $P_*(\lambda) = P_\gamma(\lambda)$ , so  $|P_*(\lambda)| = \lambda^{<\gamma}$ . Since  $\lambda$  is regular,  $\lambda^{<\gamma} = \lambda \cdot \gamma^{<\gamma}$ . To finish, it therefore suffices to show  $\gamma^{<\gamma} \leq \lambda$ . By Theorem 6.3.15,  $2^{<\gamma} = \gamma$ . If  $\gamma$  is singular, then  $\gamma$  is a singular strong limit cardinal, so by Solovay's Theorem on SCH above a strongly compact cardinal (Corollary 6.3.2),  $\gamma^{<\gamma} \leq \gamma^\gamma = \gamma^+ = \lambda$ . Otherwise,  $\gamma^{<\gamma} = 2^{<\gamma} = \gamma$ .  $\square$

Recall that an ultrafilter  $U$  on a set  $X$  is *hereditarily uniform* if  $|\text{tc}(X)| = \lambda_U$ . We observed that the generalized Mitchell order is well-behaved on hereditarily uniform ultrafilters: for example it is isomorphism invariant (Lemma 4.2.14) and transitive (Proposition 4.2.44). Under UA, it follows that the Mitchell order is well-behaved on  $\mathcal{U}$ :

**Lemma 7.5.42** (UA). *Every ultrafilter in  $\mathcal{U}$  is hereditarily uniform.*

*Proof.* Suppose  $\mathcal{U} \in \mathcal{U}$ . Fix a cardinal  $\lambda$  with  $\mathcal{U} \in \mathcal{U}_\lambda$ . Since  $P_*(\lambda)$  is the underlying set of  $\mathcal{U}$ , to show that  $\mathcal{U}$  is hereditarily transitive, we must show that  $|\text{tc}(P_*(\lambda))| \leq \lambda_{\mathcal{U}}$ . Of course,  $\text{tc}(P_*(\lambda)) = P_*(\lambda) \cup \lambda$ , which has cardinality  $\lambda$  by Proposition 7.5.41. Since  $j_{\mathcal{U}}$  is  $\lambda$ -supercompact, Proposition 4.2.31 implies that  $\lambda \leq \lambda_{\mathcal{U}}$ . Thus  $|\text{tc}(P_*(\lambda))| \leq \lambda_{\mathcal{U}}$ , as desired.  $\square$

Recall that an ultrafilter  $U$  on a cardinal  $\lambda$  is *isonormal* if  $U$  is weakly normal and  $j_U$  is  $\lambda$ -supercompact. Recall Theorem 4.4.37, which states that every normal fine ultrafilter is isomorphic to an isonormal ultrafilter. Combined with the isomorphism invariance of the

Mitchell order on hereditarily uniform ultrafilters, the following theorem therefore easily implies Theorem 7.5.39:

**Theorem 7.5.43** (UA). *Suppose  $U$  is an isonormal ultrafilter. Then for any  $D <_{\mathbf{k}} U$ ,  $D \triangleleft U$ . In particular, the Mitchell order is linear on isonormal ultrafilters.*

Note that a strong version of this theorem (Corollary 4.3.28) follows from GCH. Let us explain in full detail how to prove the linearity of the Mitchell order on  $\mathscr{U}$  (Theorem 7.5.39) from Theorem 7.5.43:

*Proof of Theorem 7.5.39.* Suppose  $\mathcal{U}_0$  and  $\mathcal{U}_1$  are elements of  $\mathscr{U}$ . We must show that either  $\mathcal{U}_0 \triangleleft \mathcal{U}_1$ ,  $\mathcal{U}_0 = \mathcal{U}_1$ , or  $\mathcal{U}_0 \triangleright \mathcal{U}_1$ . Since every normal fine ultrafilter is isomorphic to an isonormal ultrafilter (Theorem 4.4.37), there are isonormal ultrafilters  $U_0$  and  $U_1$  such that  $U_0 \cong \mathcal{U}_0$  and  $U_1 \cong \mathcal{U}_1$ . By Theorem 7.5.43, either  $U_0 = U_1$ ,  $U_0 \triangleleft U_1$ , or  $U_0 \triangleright U_1$ . If  $U_0 = U_1$ , then  $\mathcal{U}_0 \cong \mathcal{U}_1$ . Therefore by the uniqueness clause of Proposition 7.5.40,  $\mathcal{U}_0 = \mathcal{U}_1$ . If  $U_0 \triangleleft U_1$ , then since the Mitchell order is isomorphism invariant on hereditarily uniform ultrafilters (Lemma 4.2.14),  $\mathcal{U}_0 \triangleleft \mathcal{U}_1$ . (All the ultrafilters we are considering are hereditarily uniform; the nontrivial part of this is Lemma 7.5.42.) Similarly, if  $U_0 \triangleright U_1$ , then  $\mathcal{U}_0 \triangleright \mathcal{U}_1$ .  $\square$

We therefore proceed to the proof of Theorem 7.5.43. This requires a general fact from the theory of the internal relation which is of independent interest. Here is the idea. Since no nonprincipal ultrafilter  $U$  satisfies  $U \triangleleft U$ , under UA there is a least  $W$  in the Ketonen order such that  $W \not\triangleleft U$ . What is the relationship between  $U$  and  $W$ ? Perhaps  $W \leq_{\text{RF}} U$ , but this is an open question. It turns out that one can make some headway if one considers instead the  $<_{\mathbf{k}}$ -least  $W$  such that  $W \not\triangleleft U$ . (Proposition 8.3.36 shows that this actually defines the same ultrafilter.)

**Theorem 7.5.44** (UA). *Suppose  $U$  is a nonprincipal countably complete ultrafilter and  $W$  is the  $<_{\mathbf{k}}$ -least countably complete uniform ultrafilter on an ordinal such that  $W \not\triangleleft U$ . Then for any  $D \sqsubset U$ ,  $D \sqsubset W$ .*

To prove Theorem 7.5.44, we use the following closure property of the internal relation:

**Lemma 7.5.45.** *Suppose  $D \sqsubset U$  is an ultrafilter on a set  $X$  and  $\langle W_i : i \in X \rangle$  is a sequence of ultrafilters on a set  $Y$  such that  $W_i \sqsubset U$  for all  $i \in X$ . Then  $D\text{-}\sum_{i \in X} W_i \sqsubset U$  and  $D\text{-}\lim_{i \in X} W_i \sqsubset U$ .*

*Proof.* Since  $D\text{-}\lim_{i \in X} W_i \leq_{\text{RK}} D\text{-}\sum_{i \in X} W_i$ , if we show that  $D\text{-}\sum_{i \in X} W_i \sqsubset U$ , we obtain  $D\text{-}\lim_{i \in X} W_i \sqsubset U$  as a consequence of Corollary 5.5.11.

Let  $j : V \rightarrow N$  be the ultrapower of the universe by  $D$ . Let  $W = [\langle W_i : i \in X \rangle]_D$  and let  $k : N \rightarrow P$  be the ultrapower of  $M$  by  $W$ . Thus  $k \circ j$  is the ultrapower embedding associated to  $D\text{-}\sum_{i \in X} W_i$ , so to prove the lemma, we must show that  $k \circ j \upharpoonright M_U$  is an internal ultrapower embedding of  $M_U$ .

Since  $D \sqsubset U$ ,  $j$  is an internal ultrapower embedding of  $M_U$ . Therefore to show  $k \circ j \upharpoonright M_U$  is an internal ultrapower embedding of  $M_U$ , it suffices to show that  $k \upharpoonright j(M_U)$  is an internal ultrapower embedding of  $j(M_U)$ . Note that by the elementarity of  $j : V \rightarrow N$ ,  $j(M_U) = (M_{j(U)})^N$ . Since  $k = (j_W)^N$ , to show that  $k \upharpoonright j(M_U)$  is an internal ultrapower embedding of  $j(M_U)$ , it suffices to show that  $W \sqsubset j(U)$  in  $N$ . But  $W_i \sqsubset U$  for all  $i \in X$ , so  $W \sqsubset j(U)$  in  $N$  by Łoś's Theorem.  $\square$

*Proof of Theorem 7.5.44.* Suppose  $D \sqsubset U$ . By Lemma 5.5.13,  $t_D(U) = j_D(U)$ . We claim  $j_D(W) \leq_{\mathbb{k}}^{M_D} t_D(W)$ . Suppose towards a contradiction that this fails, so  $t_D(W) <_{\mathbb{k}}^{M_D} j_D(W)$ . Let  $X$  be the underlying set of  $D$ , and fix  $\langle W_i : i \in X \rangle$  such that  $t_D(W) = [\langle W_i : i \in X \rangle]_D$ . Then since  $W_i <_{\mathbb{k}} W$  for all  $i \in X$ , in fact  $W_i \sqsubset U$  for all  $i \in X$ . Thus  $D\text{-}\lim_{i \in X} W_i \sqsubset U$  by Lemma 7.5.45. But  $W = D\text{-}\lim_{i \in X} W_i$ , and this contradicts the definition of  $W$ .  $\square$

*Proof of Theorem 7.5.43.* Let  $\lambda = \lambda_U$ . If  $2^{<\lambda} = \lambda$ , then  $U$  is Dodd sound (Theorem 4.4.25), so for all  $D <_{\mathbb{k}} U$ , we have  $D \triangleleft U$  (Corollary 4.3.28), and thus we are done. We therefore assume that  $2^{<\lambda} > \lambda$ . (It is not clear whether this assumption is consistent. We will not try to reach a contradiction, however, but rather to prove that the theorem is true even if  $2^{<\lambda} > \lambda$ .)

Since  $j_U$  witnesses that some cardinal  $\kappa \leq \lambda$  is  $\lambda$ -supercompact, the local version of the theorem that GCH holds above a supercompact under UA (Theorem 6.3.12) implies that  $2^{<\lambda} = \lambda$  if  $\lambda$  is a limit cardinal. Therefore by our assumption that  $2^{<\lambda} > \lambda$ ,  $\lambda$  is a successor cardinal.

Let  $\gamma$  be the cardinal predecessor of  $\lambda$ . To simplify notation, we will from now on refer to  $\lambda$  only as  $\gamma^+$ . We therefore reformulate our assumption that  $2^{<\lambda} > \lambda$ :

$$2^\gamma > \gamma^+ \tag{7.11}$$

Since  $\gamma^+$  is Fréchet, our local result on GCH (Theorem 6.3.15) yields that  $2^{<\gamma} = \gamma$ . If  $\gamma$  is singular, then since  $2^{<\gamma} = \gamma$ ,  $\gamma$  is a singular strong limit cardinal, so the fact that  $2^\gamma > \gamma^+$  contradicts the local version of Solovay's Theorem that SCH holds above a strongly compact cardinal (Corollary 6.3.2). Therefore  $\gamma$  is regular.

**Claim 1.**  $M_U$  satisfies that  $2^{2^\gamma} = (2^\gamma)^+$

*Proof.* Let  $\mathcal{D}$  be the normal fine ultrafilter on  $P_{\text{bd}}(\gamma)$  derived from  $j_U$  using  $j_U[\gamma]$ . Since  $M_U$  is closed under  $\gamma^+$ -sequences, every ultrafilter on  $\gamma$  belongs to  $M_U$  (Proposition 6.3.9). Therefore since  $P_{\text{bd}}(\gamma)$  has hereditary cardinality  $2^{<\gamma} = \gamma$ , we have  $\mathcal{D} \in M_U$ . Therefore by a generalization of Solovay's argument that a  $2^\kappa$ -supercompact cardinal carries  $2^{2^\kappa}$  normal ultrafilters (Lemma 6.3.4),  $M_U$  satisfies that every subset of  $P(\gamma)$  belongs to  $M_{\mathcal{W}}$  for some normal fine ultrafilter  $\mathcal{W}$  on  $P_{\text{bd}}(\gamma)$ . By Proposition 6.3.6 applied in  $M_U$ ,  $M_U$  satisfies that  $2^{2^\gamma} = (2^\gamma)^+$ . (Alternately one can use Theorem 7.4.28.)  $\square$

Now let  $\eta = ((\gamma^+)^\sigma)^{M_U}$  be the least Fréchet cardinal above  $\gamma^+$  as computed in  $M_U$ . The following claim is a consequence of our analysis of isolated cardinals:

**Claim 2.**  $\eta$  is a measurable cardinal of  $M_U$ .

*Proof.* Since  $P(\gamma) \subseteq M_U$ ,  $(2^\gamma)^{M_U} \geq (2^\gamma)^V > \gamma^+$ , and therefore  $M_U$  satisfies that  $2^\gamma > \gamma^+$ .

We now work in  $M_U$  to avoid a profusion of superscripts. We cannot have  $\eta = \gamma^{++}$ : otherwise  $\gamma^{++}$  is Fréchet and hence  $2^\gamma = \gamma^+$  by Theorem 6.3.15, contradicting that  $2^\gamma > \gamma^+$ . Therefore  $\eta > \gamma^+$  and so by Proposition 7.4.4,  $\eta$  is isolated.

Let  $\delta = \delta_\eta$ . Then since  $\eta = (\gamma^+)^\sigma$ ,  $\delta \leq \gamma^{++} \leq 2^\gamma$ . The final inequality uses the fact that  $2^\gamma > \gamma^+$ . Thus  $2^\delta \leq 2^{2^\gamma} = (2^\gamma)^+$  by Claim 1. But  $2^\gamma < \eta$  by our results on the continuum function below an isolated cardinal (Theorem 7.5.21). Therefore  $(2^\gamma)^+ < \eta$  since  $\eta$  is isolated (and therefore is a limit cardinal). It follows that  $2^\delta < \eta$ . Therefore  $\eta$  is measurable by Theorem 7.5.23.  $\square$

Let  $W$  be the  $<_{\mathbb{k}}$ -least countably complete ultrafilter on an ordinal such that  $W \not\sqsubseteq U$ . Then  $W \leq_{\mathbb{k}} U$ . To prove the theorem, we must show  $U = W$ .

Since every countably complete ultrafilter on  $\gamma$  belongs to  $M_U$  and hence is internal to  $U$ , we have  $\lambda_W = \gamma^+$ . Let

$$(k, h) : (M_W, M_U) \rightarrow N$$

be the pushout of  $(j_W, j_U)$ .

**Claim 3.** *If  $h$  is nontrivial, then  $\text{CRT}(h) \geq \eta$ .*

*Proof.* Let  $W' = t_U(W)$ , so  $h : M_U \rightarrow N$  is the ultrapower of  $M_U$  by  $W'$ .

Suppose  $D$  is a countably complete ultrafilter of  $M_U$  with  $\lambda_D < \eta$ . We claim that  $D \sqsubset W'$  in  $M_U$ . Since  $\lambda_D$  is a Fréchet cardinal of  $M_U$  below  $\eta = ((\gamma^+)^\sigma)^{M_U}$ ,  $\lambda_D \leq \gamma^+$ . We may therefore assume that the underlying set of  $D$  is  $\gamma^+$ . Since  $j_U$  is  $\gamma^+$ -supercompact,  $P(\gamma^+) \subseteq M_U$ . Thus  $D$  is an ultrafilter on  $\gamma^+$  (in  $V$ ). Since  $M_U$  is closed under  $\gamma^+$ -sequences,  $j_D \upharpoonright M_U = j_D^{M_U}$ , so in fact  $D \sqsubset U$ . By Theorem 7.5.44,  $D \sqsubset W$ . Thus  $j_D \upharpoonright N$  is amenable to both  $M_U$  and  $M_W$ . By our characterization of the internal ultrapower embeddings of a pushout (Theorem 5.4.19),  $j_D \upharpoonright N$  is an internal ultrapower embedding of  $N$ . Equivalently  $j_D^{M_U} \upharpoonright N$  is an internal ultrapower embedding of  $N$ , or in other words,  $D \sqsubset W'$  in  $M_U$ .

By Lemma 7.5.3,  $h[\eta] \subseteq \eta$ . Working in  $M_U$ ,  $\eta$  is a strong limit cardinal,  $h[\eta] \subseteq \eta$ , and for all  $D$  with  $\lambda_D < \eta$ ,  $D \sqsubset W'$ . Applying in  $M_U$  our criterion for the completeness of an

ultrafilter in terms of the internal relation (Lemma 5.5.28), it follows that  $W'$  is  $\eta$ -complete. Since  $h$  is the ultrapower of  $M_U$  by  $W'$ , if  $h$  is nontrivial then  $\text{CRT}(h) \geq \eta$ .  $\square$

Since  $U$  is a weakly normal ultrafilter on  $\gamma^+$ ,  $a_U = \sup j_U[\gamma^+]$  (Lemma 4.4.17). Since  $h$  is the identity or  $\text{CRT}(h) \geq \eta > \gamma^+$ ,  $h$  is continuous at ordinals of  $M_U$ -cofinality  $\gamma^+$ . Since  $M_U$  is closed under  $\gamma^+$ -sequences,  $\sup j_U[\gamma^+]$  is on ordinal of  $M_U$ -cofinality  $\gamma^+$ . Therefore

$$h(a_U) = h(\sup j_U[\gamma^+]) = \sup h \circ j_U[\gamma^+] \leq \sup k \circ j_W[\gamma^+] \leq k(a_W)$$

The final inequality follows from the fact that  $\lambda_W = \gamma^+$  and hence  $\sup j_W[\gamma^+] \leq a_W$ . Therefore  $(k, h)$  witnesses that  $U \leq_{\mathbf{k}} W$ . Since  $U \leq_{\mathbf{k}} W$  and  $W \leq_{\mathbf{k}} U$ ,  $U = W$ , as desired.  $\square$



# Chapter 8

## Higher Supercompactness

### 8.1 Introduction

#### Obstructions to the supercompactness analysis

The main result of Chapter 7 is that under UA, the first strongly compact is supercompact. What about the second? What about all of the other strongly compact cardinals? This chapter answers all these questions and more. In this introductory section, we explain in broad strokes the obstructions to generalizing the theory of Chapter 7 and the technique that ultimately overcomes them.

#### Menas's Theorem

The first obstruction to generalizing the results of Chapter 7 is that not every strongly compact cardinal is supercompact. This is a consequence of the following theorem of Menas:

**Theorem 8.1.1** (Menas). *The least strongly compact limit of strongly compact cardinals is not supercompact.*

In order to explain the proof, we introduce an auxiliary notion:

**Definition 8.1.2.** Suppose  $\kappa$  and  $\lambda$  are cardinals. A cardinal  $\kappa$  is *almost  $\lambda$ -strongly compact* if for any  $\alpha < \kappa$ , there is an elementary embedding  $j : V \rightarrow M$  such that  $\text{CRT}(j) > \alpha$  and  $M$  has the  $(\leq \lambda, < j(\kappa))$ -covering property;  $\kappa$  is *almost strongly compact* if  $\kappa$  is almost  $\lambda$ -strongly compact for all cardinals  $\lambda$ .

As in Theorem 7.2.10, there is a characterization of almost strong compactness in terms of fine ultrafilters:

**Lemma 8.1.3.** *A cardinal  $\kappa$  is almost  $\lambda$ -strongly compact if and only if for every  $\alpha < \kappa$ , there is an  $\alpha^+$ -complete fine ultrafilter on  $P_\kappa(\lambda)$ .  $\square$*

Unlike strongly compact cardinals, it is easy to see that almost strongly compact cardinals form a closed class:

**Lemma 8.1.4.** *Any limit of almost  $\lambda$ -strongly compact cardinals is almost strongly compact. In particular, every limit of strongly compact cardinals is almost strongly compact.  $\square$*

The following proposition shows that almost strongly compact cardinals really almost are strongly compact:

**Proposition 8.1.5.** *A cardinal  $\kappa$  is  $\lambda$ -strongly compact if and only if  $\kappa$  is measurable and almost  $\lambda$ -strongly compact.*

*Proof.* Since  $\kappa$  is measurable, there is a  $\kappa$ -complete uniform ultrafilter  $U$  on  $\kappa$ . Since  $\kappa$  is almost strongly compact, for each  $\alpha < \kappa$ , there is an  $\alpha^+$ -complete fine ultrafilter  $\mathcal{W}_\alpha$  on  $P_\kappa(\lambda)$ . Let

$$\mathcal{W} = U\text{-}\lim_{\gamma < \kappa} \mathcal{W}_\gamma$$

It is immediate that  $\mathcal{W}$  is a fine ultrafilter on  $P_\kappa(\lambda)$ .

We claim that  $\mathcal{W}$  is  $\kappa$ -complete. Suppose  $\nu < \kappa$  and  $\{A_i : i < \nu\} \subseteq \mathcal{W}$ . For each  $i < \nu$ , let  $S_i = \{\alpha < \kappa : A_i \in \mathcal{W}_\alpha\}$ . Since  $A_i \in \mathcal{W}$ ,  $S_i \in U$  by the definition of an ultrafilter limit. Since  $U$  is  $\kappa$ -complete,  $\bigcap_{i < \nu} S_i$  belongs to  $U$ . Since  $U$  is uniform,  $\bigcap_{i < \nu} S_i \setminus \nu \in U$ .

Suppose  $\alpha \in \bigcap_{i < \nu} S_i \setminus \nu$ . Then  $\{A_i : i < \nu\} \in \mathcal{W}_\alpha$ . Therefore since  $\mathcal{W}_\alpha$  is  $\alpha^+$ -complete,  $\bigcap_{i < \nu} A_i \in \mathcal{W}_\alpha$ . Thus

$$\bigcap_{i < \nu} S_i \setminus \nu \subseteq \left\{ \alpha < \kappa : \bigcap_{i < \nu} A_i \in \mathcal{W}_\alpha \right\}$$

It follows that  $\{\alpha < \kappa : \bigcap_{i < \nu} A_i \in \mathcal{W}_\alpha\} \in U$ . In other words,  $\bigcap_{i < \nu} A_i \in \mathcal{W}$ . □

**Corollary 8.1.6** (Menas). *Every measurable limit of strongly compact cardinals is strongly compact.* □

The least strongly compact limit of strongly compact cardinals is therefore in a sense accessible from below:

**Lemma 8.1.7** (Menas). *Let  $\kappa$  be the least strongly compact limit of strongly compact cardinals. Then the set of measurable cardinals below  $\kappa$  is nonstationary in  $\kappa$ . Therefore  $\kappa$  has Mitchell rank 1. In particular,  $\kappa$  is not  $\mu$ -measurable, let alone  $2^\kappa$ -strong, let alone  $2^\kappa$ -supercompact.*

*Proof.* Let  $C$  be the set of limits of strongly compact cardinals less than  $\kappa$ . Since  $\kappa$  is a regular limit of strongly compact cardinals,  $C$  is unbounded in  $\kappa$ . Moreover,  $C$  is closed by definition. We claim that  $C$  contains no measurable cardinals. Suppose  $\delta \in C$  is measurable. Then by Corollary 8.1.6,  $\delta$  is strongly compact. This contradicts that  $\kappa$  is the least strongly compact limit of strongly compact cardinals. It follows that the class of measurable cardinals is nonstationary in  $\kappa$ . □

A strongly compact cardinal  $\kappa$  always carries  $2^{2^\kappa}$ -many  $\kappa$ -complete ultrafilters. But Menas's theorem shows that the Mitchell order may be trivial on  $\kappa$ . Under UA, this has the following surprising consequence:

**Theorem 8.1.8** (UA). *The least strongly compact limit of strongly compact cardinals carries a unique normal ultrafilter.*

*Proof.* Let  $\kappa$  be the least strongly compact limit of strongly compact cardinals. By Menas's Theorem (Lemma 8.1.7), the rank of the Mitchell order on normal ultrafilters on  $\kappa$  is 1.

By Theorem 2.3.11, the Mitchell order linearly orders these ultrafilters. Therefore  $\kappa$  carries exactly one normal ultrafilter.  $\square$

## Complete UA

The second obstruction to generalizing the results of Chapter 7 is much more subtle: UA alone does not seem to suffice to enact a direct generalization of the structure of the least supercompact cardinal to the higher ones. In order to shed light on the underlying issue, we introduce a principle called the Complete Ultrapower Axiom (CUA), which does suffice.

**Definition 8.1.9.** Suppose  $\kappa$  is an uncountable cardinal. Then  $\text{UA}(\kappa)$  denotes the following statement. Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  are ultrapower embeddings with  $\text{CRT}(j_0) \geq \kappa$  and  $\text{CRT}(j_1) \geq \kappa$ . Then there is a comparison  $(i_0, i_1) : (M_0, M_1) \rightarrow N$  of  $(j_0, j_1)$  such that  $\text{CRT}(i_0) \geq \kappa$  and  $\text{CRT}(i_1) \geq \kappa$ .

Thus the usual Ultrapower Axiom is equivalent to  $\text{UA}(\omega_1)$ . Notice that  $\text{UA}(\kappa)$  is equivalent to the assertion that the Rudin-Frolík order is directed on  $\kappa$ -complete ultrafilters.

**Complete Ultrapower Axiom.**  $\text{UA}(\kappa)$  holds for all uncountable cardinals  $\kappa$ .

Assuming CUA, one can generalize all the proofs in the previous section to obtain results about the higher supercompact cardinals. In fact, one does not even need to dig into the details to see that this is possible:

**Proposition 8.1.10 (CUA).** *Suppose  $\kappa$  is strongly compact. Either  $\kappa$  is supercompact or  $\kappa$  is a limit of supercompact cardinals.*

*Sketch.* Suppose first that  $\kappa$  is not a limit of strongly compact cardinals. We will show that  $\kappa$  is supercompact. Let  $\delta < \kappa$  be the supremum of the strongly compact cardinals below  $\kappa$ . Let  $G \subseteq \text{Col}(\omega, \delta)$  be  $V$ -generic. Then in  $V[G]$ ,  $\kappa$  is the least strongly compact cardinal. Moreover, since  $\text{UA}(\delta^+)$  holds in  $V$ , UA holds in  $V[G]$ . Therefore by the analysis of the

least strongly compact cardinal under UA (Theorem 7.4.23),  $\kappa$  is supercompact in  $V[G]$ . It follows that  $\kappa$  is supercompact in  $V$ , as desired.

Suppose instead  $\kappa$  is a limit of strongly compact cardinals. Then by the result of the previous paragraph, every successor strongly compact cardinal below  $\kappa$  is supercompact, so  $\kappa$  is a limit of supercompact cardinals.  $\square$

The issue now is that there is no inner model theoretic reason whatsoever to believe that CUA is consistent with very large cardinals, but it cannot be that easy to refute:

**Proposition 8.1.11** (UA). *Suppose  $j_0 : V \rightarrow M_0$  and  $j_1 : V \rightarrow M_1$  witness that CUA is false and  $\lambda = \min\{\text{CRT}(j_0), \text{CRT}(j_1)\}$ . Then some cardinal  $\kappa < \lambda$  is  $\lambda$ -supercompact.*

*Sketch.* Since  $\lambda$  is measurable, it suffices to show that some  $\kappa < \lambda$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ . Assume towards a contradiction that no cardinal below  $\lambda$  has this property. Then by Corollary 7.5.2, for any ultrapower embedding  $i : V \rightarrow N$ ,  $i[\lambda] \subseteq \lambda$ . Let

$$(i_0, i_1) : (M_0, M_1) \rightarrow N$$

be the pushout of  $(j_0, j_1)$ . Let  $W$  be a countably complete ultrafilter such that  $M_W = N$  and  $j_W = i_0 \circ j_0 = i_1 \circ j_1$ . By the analysis of ultrafilters internal to a pushout (Theorem 5.4.19),  $W$  is  $\lambda$ -internal. Thus  $j_W[\lambda] \subseteq \lambda$  and  $W$  is  $\lambda$ -internal, so the internal relation theoretic criterion for completeness (Lemma 5.5.28) implies that  $W$  is  $\lambda$ -complete. Thus  $\text{CRT}(i_0) \geq \text{CRT}(i_0 \circ j_0) = \text{CRT}(j_W) = \kappa$ , and similarly  $\text{CRT}(i_1) \geq \kappa$ . This contradicts that  $j_0$  and  $j_1$  witness the failure of CUA.  $\square$

One can do a bit better using the following fact, whose proof we omit:

**Proposition 8.1.12** (UA). *Suppose CUA fails. Then there are irreducible ultrafilters  $U_0$  and  $U_1$  such that  $j_{U_0}$  and  $j_{U_1}$  witness the failure of CUA.*  $\square$

Since UA implies the linearity of the Mitchell order on normal ultrafilters (Theorem 2.3.11), CUA cannot fail for a pair of normal ultrafilters, and hence the analysis of normality and irreducible ultrafilters (Theorem 5.3.11) implies that  $\min\{\text{CRT}(j_{U_0}), \text{CRT}(j_{U_1})\}$  is a  $\mu$ -measurable

cardinal. One can push this quite a bit further, but not far enough to answer the following question:

**Question 8.1.13.** Is CUA consistent with the existence of cardinals  $\kappa < \lambda$  that are both  $\lambda^+$ -supercompact?

The most interesting possibility is that large cardinals refute CUA. In any case, unless one can prove CUA from UA (or Weak Comparison), it is far from well-justified. The analysis of the second strongly compact cardinal therefore requires a different approach.

## Irreducible ultrafilters and supercompactness

Given the techniques of the previous chapter, the obvious approach is to study the  $\kappa$ -complete generalizations of Fréchet cardinals and the ultrafilters  $\mathcal{K}_\lambda$ .

**Definition 8.1.14.** Suppose  $\kappa \leq \lambda$  are uncountable cardinals. Then  $\lambda$  is  $\kappa$ -Fréchet if there is a  $\kappa$ -complete uniform ultrafilter on  $\lambda$ .

**Definition 8.1.15 (UA).** Suppose  $\lambda$  is a  $\kappa$ -Fréchet cardinal. Then  $\mathcal{K}_\lambda^\kappa$  denotes the minimum  $\kappa$ -complete uniform ultrafilter on  $\lambda$  in the Ketonen order.

Most of the key properties of  $\mathcal{K}_\lambda$  do not directly generalize to  $\mathcal{K}_\lambda^\kappa$ : the proofs seem to require UA( $\kappa$ ). Essentially the only nontrivial UA result that lifts is Lemma 7.3.12, the fact that  $\mathcal{K}_\lambda$  is irreducible for regular  $\lambda$ .

**Lemma 8.1.16 (UA).** *Suppose  $\kappa \leq \lambda$  and  $\lambda$  is  $\kappa$ -Fréchet. Then  $\mathcal{K}_\lambda^\kappa$  is weakly normal.*

*Proof.* Recall Lemma 4.4.20, which asserts that a uniform ultrafilter  $U$  on a cardinal  $\lambda$  is weakly normal if and only if for all  $W <_{\text{rk}} U$ ,  $\lambda_W < \lambda$ . We will show that this holds for  $U = \mathcal{K}_\lambda^\kappa$ . Suppose  $W <_{\text{rk}} \mathcal{K}_\lambda^\kappa$ . Since  $W \leq_{\text{RK}} \mathcal{K}_\lambda^\kappa$ ,  $W$  is  $\kappa$ -complete, and since  $W <_{\text{rk}} \mathcal{K}_\lambda^\kappa$ ,  $W <_{\mathbb{k}} \mathcal{K}_\lambda^\kappa$ . By the minimality of  $\mathcal{K}_\lambda^\kappa$ ,  $\lambda_W < \lambda$ .  $\square$

**Proposition 8.1.17 (UA).** *Suppose  $\nu < \lambda$  and  $\lambda$  is a  $\nu^+$ -Fréchet regular cardinal.<sup>1</sup> Then  $\mathcal{K}_\lambda^{\nu^+}$  is irreducible.*

*Proof.* Let  $\mathcal{K} = \mathcal{K}_\lambda^{\nu^+}$ . Suppose  $D <_{\text{RF}} \mathcal{K}$ . We must show that  $D$  is principal. Since  $\mathcal{K}$  is  $\nu^+$ -complete and  $D \leq_{\text{RK}} \mathcal{K}$ ,  $D$  is  $\nu^+$ -complete, and in particular  $j_D(\nu) = \nu$ . Since  $\mathcal{K}$  is weakly normal and  $D <_{\text{RK}} \mathcal{K}$ ,  $\lambda_D < \lambda$  by Proposition 4.4.22. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}$  and let  $h : M_D \rightarrow M$  be the unique internal ultrapower embedding such that  $h \circ j_D = j$ . Then  $h$  is the ultrapower of  $M_D$  by  $t_D(\mathcal{K})$ , and  $\text{CRT}(h) \geq \text{CRT}(j) > \nu = j_D(\nu)$ . Thus  $t_D(\mathcal{K})$  is  $j_D(\nu^+)$ -complete in  $M_D$ .

Assume towards a contradiction that  $D$  is nonprincipal. By Proposition 5.4.5,  $t_D(\mathcal{K}) <_{\mathbb{k}} j_D(\mathcal{K})$  in  $M_D$ . Since  $t_D(\mathcal{K})$  is  $j_D(\nu^+)$ -complete, the  $<_{\mathbb{k}}^{M_D}$ -minimality of  $j_D(\mathcal{K})$  among  $j_D(\nu^+)$ -complete uniform ultrafilters on  $j_D(\lambda)$  implies that  $\lambda_{t_D(\mathcal{K})} < j_D(\lambda)$ . Since  $j_D(\lambda)$  is  $M_D$ -regular, it follows that  $\delta_{t_D(\mathcal{K})} < j_D(\lambda)$ . Since  $\lambda_D < \lambda$  and  $\lambda$  is regular,  $j_D(\lambda) = \sup j_D[\lambda]$  by Lemma 3.5.32. Therefore there is some ordinal  $\alpha < \lambda$  such that  $\delta_{t_D(\mathcal{K})} < j_D(\alpha)$ . But  $\alpha \in j_D^{-1}[t_D(\mathcal{K})] = \mathcal{K}$ , contradicting that  $\mathcal{K}$  is uniform. Thus our assumption was false, and in fact  $D$  is principal. This shows that  $\mathcal{K}$  is irreducible, as desired.  $\square$

Beyond Proposition 8.1.17, the ultrafilters  $\mathcal{K}_\lambda^\kappa$  turn out to be a bit of a red herring. The analysis of higher supercompact cardinals does not proceed by generalizing the theory of  $\mathcal{K}_\lambda$  to the ultrafilters  $\mathcal{K}_\lambda^\kappa$  but instead by propagating the  $\lambda$ -supercompactness of  $\mathcal{K}_\lambda$  itself to arbitrary irreducible ultrafilters. Recall that an ultrafilter  $U$  is  $\lambda$ -irreducible if every ultrafilter  $D \leq_{\text{RF}} U$  such that  $\lambda_D < \lambda$  is principal. The main theorems of this chapter, to which we refer collectively as the *Irreducibility Theorem*, show that supercompactness and irreducibility are equivalent:

**Theorem 8.2.21 (UA).** *Suppose  $\lambda$  is a successor cardinal or a strong limit singular cardinal and  $U$  is a countably complete uniform ultrafilter on  $\lambda$ . Then the following are equivalent:*

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<sup>1</sup>It is necessary here to restrict to consideration of  $\mathcal{K}_\lambda^{\nu^+}$ , rather than considering  $\mathcal{K}_\lambda^\kappa$  in general. In fact,  $\mathcal{K}_\lambda^\kappa$  is irreducible if and only if there is some  $\nu < \kappa$  such that  $\mathcal{K}_\lambda^\kappa = \mathcal{K}_\lambda^{\nu^+}$ . This is closely related to Menas's Theorem (Theorem 8.1.1).

(1)  $j_U$  is  $\lambda$ -irreducible.

(2)  $j_U$  is  $\lambda$ -supercompact.

It does not seem to be possible to generalize this to the case that  $\lambda$  is inaccessible, and instead we obtain the following theorem:

**Theorem 8.2.22** (UA). *Suppose  $\lambda$  is an inaccessible cardinal and  $U$  is a countably complete ultrafilter on  $\lambda$ . Then the following are equivalent:*

(1)  $j_U$  is  $\lambda$ -irreducible.

(2)  $j_U$  is  $<\lambda$ -supercompact and  $\lambda$ -tight.

We will use these two theorems to give a complete characterization of strongly compact cardinals assuming UA:

**Theorem 8.3.9** (UA). *Suppose  $\kappa$  is a strongly compact cardinal. Either  $\kappa$  is a supercompact cardinal or  $\kappa$  is a measurable limit of supercompact cardinals*

## Outline of Chapter 8

We now outline the rest of this chapter.

**Section 8.2.** We prove the main structural result of the section, called the *Irreducibility Theorem*, from which all the other theorems flow. The Irreducibility Theorem refers to a cluster of results (especially Theorem 8.2.18 and Corollary 8.2.20) that show an equivalence between irreducibility and supercompactness.

**Section 8.3.** We use the Irreducibility Theorem to resolve the Identity Crisis for strongly compact cardinals under UA. We also use it in Section 8.3 to completely characterize the internal relation in terms of the Mitchell order.

**Section 8.4.** We discuss the relationship between UA and very large cardinals. We begin by (partially) analyzing the relationship between hugeness and non-regular ultrafilters



under UA (Theorem 8.4.5). We then turn to the topic of cardinal preserving embeddings. We show that UA rules out such embeddings (Lemma 8.4.10), and more generally that local cardinal preservation hypotheses are equivalent to rank-into-rank large cardinal large cardinal axioms under UA (Theorem 8.4.12). Finally in Section 8.4, we discuss the structure of supercompactness at inaccessible cardinals, and in particular the prospect that the local equivalence of strong compactness and supercompactness breaks down there.

## 8.2 The Irreducibility Theorem

In this section, we prove the central Irreducibility Theorem (Theorem 8.2.21 and Theorem 8.2.22). We begin in Section 8.2 by proving the forward implication from supercompactness to irreducibility. This raises a central open question (Question 8.2.4) that will be discussed at greater length in Section 8.4. The next two sections are devoted to proving the preliminary lemmas necessary for the proof of the Irreducibility Theorem. In Section 8.2, we prove two key lemmas regarding the comparison of  $\mathcal{K}_\lambda$  with an arbitrary ultrafilter. In the very short Section 8.2, we prove two theorems on the combinatorics of normal ultrafilters that show up in the proof of the Irreducibility Theorem. Finally, Section 8.2 contains the proof of the Irreducibility Theorem as well as some slightly more general theorems.

### Pseudocompactness and irreducibility

In this short subsection, we prove the easy direction of the irreducibility theorem:  $\lambda$ -supercompactness implies  $\lambda$ -irreducibility. In fact, we will prove something slightly stronger. The following property is a priori somewhat weaker than  $\lambda$ -supercompactness, but already implies  $\lambda$ -irreducibility.

**Definition 8.2.1.** Suppose  $\lambda$  is a cardinal. An elementary embedding  $j : V \rightarrow M$  is said to be  $\lambda$ -*pseudocompact* if  $j$  is  $\gamma$ -tight for every cardinal  $\gamma \leq \lambda$ .

**Lemma 8.2.2.** *An ultrapower embedding  $j : V \rightarrow M$  is  $\lambda$ -pseudocompact if and only if  $M$  has the  $\leq\gamma$ -covering property for all  $\gamma \leq \lambda$ .*

*Proof.* This is an immediate consequence of the self-strengthening of tightness that holds for ultrapower embeddings (Lemma 7.2.7).  $\square$

**Proposition 8.2.3.** *Suppose  $\lambda$  is a cardinal and  $U$  is a countably complete ultrafilter. If  $j_U$  is  $\lambda$ -pseudocompact, then  $U$  is  $\lambda$ -irreducible.*

*Proof.* Suppose  $D \leq_{\text{RF}} U$  and  $\lambda_D < \lambda$ . We must show that  $D$  is principal. We first show that  $j_D$  is  $\lambda$ -pseudocompact. Since  $j_U$  is  $\lambda$ -pseudocompact,  $M_U$  has the  $\leq\gamma$ -covering property for all  $\gamma \leq \lambda$ . Since  $D \leq_{\text{RF}} U$ ,  $M_U \subseteq M_D$ . It follows that  $M_D$  has the  $\leq\gamma$ -covering property for all  $\gamma \leq \lambda$ : suppose  $\gamma \leq \lambda$  and  $A$  is a set of ordinals of cardinality  $\gamma$ ; then  $A$  is contained in a set  $B \in M_U$  such that  $|B|^{M_U} \leq \gamma$ , and since  $M_U \subseteq M_D$ , we have  $B \in M_D$  and  $|B|^{M_D} \leq \gamma$ , as desired. Thus  $j_D$  is  $\lambda$ -pseudocompact.

In particular, since  $\lambda_D < \lambda$ ,  $D$  is  $\lambda_D^+$ -tight. Assume towards a contradiction that  $D$  is nonprincipal. By Lemma 4.2.32,  $j_D(\lambda_D) > \lambda_D^+$ . Thus  $D$  is  $(\lambda_D^+, \delta)$ -tight where  $\delta = \lambda_D^+ < j_D(\lambda_D)$ . This contradicts Lemma 7.2.17, which states that if  $\eta$  is a cardinal and  $Z$  is a nonprincipal countably complete ultrafilter such that  $\lambda_Z < \eta$ , then  $Z$  is not  $(\eta, \delta)$ -tight for any  $\delta < j_Z(\eta)$ . Thus  $D$  is principal, as desired.  $\square$

The only known instances of  $\lambda$ -pseudocompact elementary embeddings that are not  $\lambda$ -supercompact come from large cardinal axioms at the level of rank-into-rank cardinals. Specifically, assume the axiom  $I_2$ . Thus there is a cardinal  $\lambda$  and an elementary embedding  $j : V \rightarrow M$  such that  $\text{CRT}(j) < \lambda$ ,  $j(\lambda) = \lambda$ , and  $V_\lambda \subseteq M$ . The embedding  $j$  is not  $\lambda$ -supercompact by the Kunen Inconsistency Theorem, but  $j$  is trivially  $\lambda$ -pseudocompact since  $j[\lambda] \subseteq \lambda$ . In fact,  $j$  is  $\lambda^{+\alpha}$ -pseudocompact for all  $\alpha < \text{CRT}(j)$ . On the other hand, there are no known examples of *ultrapower* embeddings that are  $\lambda$ -pseudocompact but not  $\lambda$ -supercompact. In fact, it is not known whether it is consistent that such an example exists:

**Question 8.2.4** (ZFC). Suppose  $\lambda$  is a cardinal and  $j : V \rightarrow M$  is a  $\lambda$ -pseudocompact ultrapower embedding. Must  $j$  be  $\lambda$ -supercompact?

The natural inclination is to conjecture that the answer is no: typically large cardinal properties formulated in terms of covering do not imply supercompactness in ZFC. But the problem turns out to be much more subtle than one might expect.

We highlight below the most basic instance of this problem (in simple English):

**Question 8.2.5.** Suppose  $j : V \rightarrow M$  is an elementary embedding with critical point  $\kappa$  such that  $\text{cf}^M(\sup j[\kappa^+]) = \kappa^+$ . Must  $j[\kappa^+]$  belong to  $M$ ?

We devote the final section of this dissertation (Section 8.4) to the relationship between Question 8.2.4 and the Inner Model Problem.

On this topic, let us mention an interesting way in which tightness can act as a stand-in for strength:

**Lemma 8.2.6.** *Suppose  $j : V \rightarrow M$  is an elementary embedding,  $\lambda$  is a cardinal,  $\delta$  is an  $M$ -cardinal, and  $j$  is  $(\lambda, \delta)$ -tight. Then  $2^\lambda \leq (2^\delta)^M$ .*

*Proof.* Fix  $B \in M$  such that  $|B|^M = \delta$  and  $j[\lambda] \subseteq B$ . Then the map  $f : P(\lambda) \rightarrow P(B) \cap M$  defined by  $f(S) = j(S) \cap B$  is an injection: if  $S \neq T$ , then fix  $\alpha \in S \triangle T$ , and note that since  $j[\lambda] \subseteq B$ ,  $j(\alpha) \in (j(S) \triangle j(T)) \cap B = f(S) \triangle f(T)$ . Since  $|P(B) \cap M|^M = (2^\delta)^M$  it follows that  $2^\lambda \leq |(2^\delta)^M| \leq (2^\delta)^M$ .  $\square$

As a sample application (and a brief diversion), suppose  $\kappa$  is a cardinal such that for all cardinals  $\lambda \geq \kappa$ , there is a  $\lambda$ -tight embedding  $j : V \rightarrow M$  such that  $j(\kappa) > \lambda$ . Then the Generalized Continuum Hypothesis cannot fail first above  $\kappa$ . To see this, assume that for all cardinals  $\gamma < \kappa$ ,  $2^\gamma = \gamma^+$ . Fix  $\lambda \geq \kappa$ . Let  $j : V \rightarrow M$  be a  $\lambda$ -tight embedding with  $j(\kappa) > \lambda$ . Then in  $M$ ,  $2^\lambda = \lambda^+$ . Therefore  $2^\lambda \leq (2^\lambda)^M \leq (\lambda^+)^M \leq \lambda^+$ , so  $2^\lambda = \lambda^+$ .

## Translations of $\mathcal{K}_\lambda$

Suppose  $U$  is a  $\lambda$ -irreducible uniform ultrafilter on a successor cardinal  $\lambda$ . The Irreducibility Theorem asserts that  $j_U$  is  $\lambda$ -supercompact. The proof proceeds by analyzing the pushout comparison of  $(j_{\mathcal{K}_\lambda}, j_U)$  where  $\lambda$  is a Fréchet successor cardinal. In this section, we will prove a number of lemmas regarding this pushout that amount to pieces of this analysis.

The universal property of  $\mathcal{K}_\lambda$  (Theorem 7.3.13) identifies the pushout of  $(j_{\mathcal{K}_\lambda}, j_U)$  when  $\text{cf}^{M_U}(\sup j_U[\lambda])$  is not Fréchet in  $M_U$ : in fact,  $\mathcal{K}_\lambda \leq_{\text{RF}} U$ , so the pushout is given by the unique internal ultrapower embedding  $h : M_{\mathcal{K}_\lambda} \rightarrow M_U$ . It turns out that the universal property is powerful enough to yield an analysis of this comparison even when  $\text{cf}^{M_U}(\sup j_U[\lambda])$  is a Fréchet cardinal of  $M_U$ . The following lemma tells us which ultrafilter is hit on the  $M_U$ -side of the comparison:

**Lemma 8.2.7 (UA).** *Suppose  $\lambda$  is a regular Fréchet cardinal and  $U$  is a countably complete ultrafilter. Let  $\delta = \text{cf}^{M_U}(\sup j_U[\lambda])$ .*

- *Suppose  $\delta$  is not Fréchet in  $M_U$ . Then  $t_U(\mathcal{K}_\lambda)$  is principal in  $M_U$ .*
- *Suppose  $\delta$  is Fréchet in  $M_U$ . Then  $t_U(\mathcal{K}_\lambda) \cong (\mathcal{K}_\delta)^M$ .*

*Proof.* The first bullet point is immediate from the universal property of  $\mathcal{K}_\lambda$  (Theorem 7.3.13): we have  $\mathcal{K}_\lambda \leq_{\text{RF}} U$ , so by Lemma 5.4.39,  $t_U(\mathcal{K}_\lambda)$  is principal in  $M_U$ . Therefore assume instead that  $\delta$  is Fréchet in  $M_U$ .

Let  $Z = t_U(\mathcal{K}_\lambda)$ . We claim that in  $M_U$ ,  $Z$  is a  $<_{\mathbb{k}}$ -minimal element of the set of countably complete ultrafilters  $W$  on  $j_U(\lambda)$  with  $\delta_W \geq \sup j_U[\lambda]$ . Clearly  $\delta_Z \geq \sup j_U[\lambda]$ , since otherwise  $\delta_{j_U^{-1}[Z]} < \lambda$  contradicting that  $j_U^{-1}[Z] = \mathcal{K}_\lambda$ . Suppose  $W \in j_U(\mathcal{B}(\lambda))$  and  $W <_{\mathbb{k}} Z$  in  $M_U$ , and we will show  $\delta_W < \sup j_U[\lambda]$ . Let  $\bar{W} = j_U^{-1}[W]$ . Then  $t_U(\bar{W}) \leq_{\mathbb{k}} W <_{\mathbb{k}} Z = t_U(\mathcal{K}_\lambda)$ . By Theorem 5.4.42, it follows that  $\bar{W} <_{\mathbb{k}} \mathcal{K}_\lambda$ . Since  $\mathcal{K}_\lambda$  is the  $<_{\mathbb{k}}$ -least uniform ultrafilter on the regular cardinal  $\lambda$ ,  $\delta_{\bar{W}} < \lambda$ . But  $j_U(\delta_{\bar{W}}) \in W$ , so  $\delta_W \leq j_U(\delta_{\bar{W}}) < \sup j_U[\lambda]$ .

Applying the analysis of  $<_{\mathbb{k}}$ -minimal tail uniform ultrafilters (Lemma 7.3.10) in  $M_U$ , it follows that in  $M_U$ , there is a Ketonen ultrafilter  $D$  on  $\text{cf}^{M_U}(\sup j_U[\lambda]) = \delta$  that is isomorphic to  $Z$ . Applying UA in  $M_U$ ,  $D = \mathcal{K}_\delta$ , the unique Ketonen ultrafilter on  $\delta$ .  $\square$

The analysis of the  $M_{\mathcal{K}_\lambda}$ -side of the comparison is much more subtle, and uses the following fact:

**Lemma 8.2.8** (UA). *Suppose  $\lambda$  is a nonisolated regular Fréchet cardinal. Let  $M = M_{\mathcal{K}_\lambda}$ . Suppose  $i : M \rightarrow N$  is an internal ultrapower embedding. Then there is a countably complete ultrafilter  $D$  of  $M$  with  $\lambda_D < \lambda$  such that there is an internal ultrapower embedding  $h : (M_D)^M \rightarrow N$  with  $h \circ j_D = i$  and  $\text{CRT}(h) > j_D(\lambda)$ .*

The proof uses an analysis of  $\lambda^\sigma$  in  $M_{\mathcal{K}_\lambda}$  which is similar to Claim 2 of Theorem 7.5.43:

**Lemma 8.2.9** (UA). *Suppose  $\lambda$  is a nonisolated regular Fréchet cardinal. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . Then  $(\lambda^\sigma)^M$  is a measurable cardinal of  $M$ .*

*Proof.* By Theorem 7.3.33 and Theorem 7.4.9,  $j$  is  $\lambda$ -tight and therefore  $\text{cf}^M(\sup j[\lambda]) = \lambda$ . Therefore by the definition of  $\mathcal{K}_\lambda$  (or more precisely, Lemma 7.3.6),  $\lambda$  is not Fréchet in  $M$ .

Let  $\eta = (\lambda^\sigma)^M$ . Assume towards a contradiction that  $\eta$  is not measurable. Let  $i : M \rightarrow N$  be the ultrapower of  $M$  by  $(\mathcal{K}_\eta)^M$  and let  $a = a_{(\mathcal{K}_\eta)^M}$ .

We claim that every countably complete  $N$ -ultrafilter  $D$  on  $\lambda$  belongs to  $M$ . For any such  $D$ ,  $j_D^N \circ i$  is continuous at  $\lambda$ :  $i$  is continuous at  $\lambda$  because  $i$  is internal to  $M$  and  $\lambda$  is not Fréchet in  $M$ , while  $j_D^N$  is continuous at  $i(\lambda)$  since  $i(\lambda)$  is an  $N$ -regular cardinal with  $i(\lambda) > \lambda \geq \lambda_D$ , and combining these observations:

$$j_D^N(i(\lambda)) = \sup j_D^N[i(\lambda)] = \sup j_D^N[\sup i[\lambda]] = \sup j_D^N \circ i[\lambda]$$

Thus by the characterization of internal ultrapower embeddings of  $M_{\mathcal{K}_\lambda}$  (Theorem 7.3.14),  $j_D^N \circ i$  an internal ultrapower embedding of  $M$ . Since  $j_D^N$  can be defined at a typical element of  $N$  by setting

$$j_D^N([f]_{(\mathcal{K}_\eta)^M}) = j_D^N \circ i(f)(j_D^N(a))$$

it follows that  $j_D^N$  is definable over  $M$ . Thus  $D \in M$ . Applying inside  $M$  the characterization of countably complete ultrafilters amenable to an isolated ultrapower (Theorem 7.5.17), we have that  $D \in N$ .

Proposition 7.3.32 states that if  $\kappa$  is  $\lambda$ -strongly compact and  $Q$  is a  $<\kappa$ -closed inner model such that every  $\kappa$ -complete ultrafilter  $U$  on  $\lambda$  is amenable to  $Q$ , then  $P(\lambda) \subseteq Q$ . By Theorem 7.4.9,  $\kappa_\lambda$  is  $\lambda$ -strongly compact. Moreover  $N$  is a  $<\kappa_\lambda$ -closed (indeed  $<\lambda$ -closed by Proposition 7.5.20) inner model such that every countably complete  $N$ -ultrafilter on  $\lambda$  belongs to  $\lambda$ . It therefore follows that  $P(\lambda) \subseteq N$ . But then  $\mathcal{K}_\lambda$  itself is an  $N$ -ultrafilter, so  $\mathcal{K}_\lambda \in N$ . Since  $N \subseteq M$ , this implies  $\mathcal{K}_\lambda \in M = M_{\mathcal{K}_\lambda}$ , so  $\mathcal{K}_\lambda \triangleleft \mathcal{K}_\lambda$ , contradicting the irreflexivity of the Mitchell order (Lemma 4.2.40).  $\square$

*Proof of Lemma 8.2.8.* By Lemma 8.2.9,  $\eta = (\lambda^\sigma)^M$  is a measurable cardinal that is not a limit of Fréchet cardinals. The theorem follows by applying in  $M$  the fact that ultrapower embeddings can be factored across strong limit cardinals that are not limits of Fréchet cardinals (Theorem 7.5.5).  $\square$

Lemma 8.2.8 has the following curious and sometimes useful corollary:

**Lemma 8.2.10** (UA). *Suppose  $\lambda$  is a strongly inaccessible cardinal such that one of the following holds:*

- $\lambda$  is Fréchet.
- $\lambda^\sigma$  is measurable.

*Then every ultrapower embedding is  $\lambda$ -tight.*

*Proof.* Suppose  $U$  is a countably complete ultrafilter. We will show that  $j_U$  is  $\lambda$ -tight.

Assume first that  $\lambda$  is not Fréchet. Then by assumption  $\eta = \lambda^\sigma$  is measurable. By Theorem 7.5.5, there is a countably complete ultrafilter  $D$  with  $\lambda_D < \eta$  such that there is an elementary embedding  $k : M_D \rightarrow M_U$  with  $k \circ j_D = j_U$  and  $\text{CRT}(k) \geq \eta$ . Since  $\lambda_D < \eta$ , in fact

$\lambda_D < \lambda$ , so  $j_D(\lambda) = \lambda$  since  $\lambda$  is inaccessible. But since  $\text{CRT}(k) > j_D(\lambda)$ ,  $j_U(\lambda) = j_D(\lambda) = \lambda$ . Therefore  $j_D$  is vacuously  $\lambda$ -tight.

Assume instead that  $\lambda$  is Fréchet. Let  $(h, i) : (M_U, M_{\mathcal{K}_\lambda}) \rightarrow N$  be the pushout of  $(j_U, j_{\mathcal{K}_\lambda})$ . Applying Lemma 8.2.8,  $i$  factors in such a way that we can conclude that  $i(\lambda) = \lambda$  by the argument of the previous paragraph. Since  $\mathcal{K}_\lambda$  is  $\lambda$ -tight by Proposition 7.4.11 and  $i$  is vacuously  $\lambda$ -tight,  $i \circ j_{\mathcal{K}_\lambda}$  is  $\lambda$ -tight. In other words,  $N$  has the  $\leq \lambda$ -covering property. Since  $N \subseteq M_U$  and  $N$  has the  $\leq \lambda$ -covering property,  $M_U$  has the  $\leq \lambda$ -covering property. Therefore  $j_U$  is  $\lambda$ -tight, as desired.  $\square$

## Elementary embeddings and normal filters

In this short subsection, we prove some combinatorial constraints on comparisons involving normal filters. Suppose  $U$  and  $W$  are countably complete ultrafilters on a cardinal  $\kappa$ . A question that often arises in the context of UA is what sort of  $M_W$ -ultrafilters  $Z$  on  $j_W(\kappa)$  pull back to  $U$  in the sense that  $U = j_W^{-1}[Z]$ . Such  $M_W$ -ultrafilters arise from any comparison of  $(j_U, j_W)$ . Focusing on a more specific question, assume  $U$  is normal, and suppose  $Z$  is a tail uniform  $M_W$ -ultrafilter on  $j_W(\kappa)$  with  $j_W^{-1}[Z] = U$ . Must  $Z = j_W(U)$ ? The following lemma, which has almost certainly been discovered before, tells us that the answer is yes:

**Lemma 8.2.11.** *Suppose  $\mathcal{F}$  is a normal fine filter on a set  $Y$ , and  $W$  is an ultrafilter on  $X = \bigcup Y$ . Then  $j_W(\mathcal{F})$  is the unique  $M$ -filter on  $j_W(Y)$  that extends  $j_W[F]$  and concentrates on  $\{\sigma \in j_W(Y) : a_W \in \sigma\}$ . In particular,  $j_W(\mathcal{F})$  is the unique fine  $M$ -filter on  $j_W(Y)$  extending  $j_W[F]$ .*

*Proof.* Suppose  $A \in j_W(\mathcal{F})$ . We will find  $B \in \mathcal{F}$  such that

$$j_W(B) \cap \{\sigma \in j_W(Y) : a_W \in \sigma\} \subseteq A$$

Fix a function  $G : X \rightarrow \mathcal{F}$  such that  $A = j_W(G)(a_W)$ . Let

$$B = \Delta_{x \in X} G(x)$$

Suppose  $\tau \in j_W(B) \cap \{\sigma \in j_W(Y) : a_W \in \sigma\}$ . We will show that  $\tau \in A$ . Since  $\tau$  belongs to  $j_W(B) = \Delta_{x \in j_W(X)} j_W(G)(x)$ , the definition of the diagonal intersection operation implies that  $\tau \in j_W(G)(x)$  for all  $x \in \tau$ . But  $a_W \in \tau$ , and hence  $\tau \in j_W(G)(a_W) = A$ .  $\square$

In general, one must adjoin the set  $\{\sigma \in j(Y) : a_W \in \sigma\}$  in order to generate all of  $\mathcal{F}$ . Suppose  $\mathcal{F}$  is a normal fine ultrafilter on  $Y$  and  $W$  is an ultrafilter on  $X = \bigcup Y$ . Then  $j_W[\mathcal{F}]$  generates  $j_W(\mathcal{F})$  if and only if there is some  $\tau \in Y$  such that  $W$  concentrates on  $\tau$  and  $\mathcal{F}$  concentrates on  $\{\sigma \in Y : \tau \subseteq \sigma\}$ .

To better explain how this lemma is related to UA, we offer a sample corollary:

**Corollary 8.2.12 (UA).** *Suppose  $F$  is a normal filter on a cardinal  $\kappa$ . Let  $U$  be the  $<_{\mathbb{k}}$ -least countably complete ultrafilter on  $\kappa$  that extends  $F$ . Then for all  $D <_{\mathbb{k}} U$ ,  $D \sqsubset U$ .*

*Proof.* Suppose  $D <_{\mathbb{k}} U$ . We claim  $j_D(U) \leq_{\mathbb{k}} t_D(U)$  in  $M_D$ , which implies  $D \sqsubset U$  by the theory of the internal relation (Lemma 5.5.13). Since  $j_D(U)$  is the  $<_{\mathbb{k}}^{M_D}$ -least countably complete ultrafilter of  $M_D$  that extends  $j_D(F)$ , it suffices to show that  $j_D(F) \subseteq t_D(U)$ . Of course  $j_D[F] \subseteq t_D(U)$  since  $j_D^{-1}[t_D(U)] = U$ . Moreover since  $D <_{\mathbb{k}} U$ , we must have that  $t_D(U)$  concentrates on ordinals greater than  $a_D$  (since otherwise  $t_D(U)$  witnesses  $U \leq_{\mathbb{k}} D$ ). In other words,  $\{\alpha < j_D(\kappa) : a_D \in \alpha\} \in t_D(U)$ . Therefore by Lemma 8.2.11,  $j_D(F) \subseteq t_D(U)$ , as desired.  $\square$

Here is an intriguing consequence of Corollary 8.2.12. Suppose  $\kappa$  is a regular cardinal and  $F$  is the  $\omega$ -club filter on  $\kappa$ . Suppose  $F$  extends to a countably complete ultrafilter. Mitchell [34] showed that this hypothesis is equiconsistent with a measurable cardinal of Mitchell order  $\omega$ , but assuming UA, it implies that there is a  $\mu$ -measurable cardinal and quite a bit more. The reason is that Corollary 8.2.12 shows that the  $<_{\mathbb{k}}$ -least extension of  $F$  is irreducible; clearly it is not normal, so we can apply the dichotomy between normal ultrafilters and  $\mu$ -measurability (Theorem 5.3.8). But how strong is this hypothesis? (Inner model theory suggests that it is beyond a superstrong cardinal.)



As a corollary of Lemma 8.2.11, we have a similar unique extension theorem for isonormal ultrafilters on regular cardinals. We begin with a corollary of Solovay's Lemma (Theorem 4.4.27) that explains the statement of Lemma 8.2.14:

**Lemma 8.2.13.** *Suppose  $\lambda$  is a regular cardinal and  $W$  is a countably complete weakly normal ultrafilter on  $\lambda$ . Suppose  $\langle S_\xi : \xi < \lambda \rangle$  is a partition of  $S_\omega^\lambda$  into stationary sets. Then for any  $\xi < \lambda$ ,  $W$  concentrates on the set of  $\alpha < \lambda$  such that  $S_\xi$  is stationary in  $\alpha$ .*

*Proof.* Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $W$ . Then since  $W$  is weakly normal,  $a_W = \sup j[\lambda]$ . Let  $\langle T_\xi : \xi < j(\lambda) \rangle = j(\langle S_\alpha : \alpha < \lambda \rangle)$ . By Solovay's Lemma (Lemma 4.4.29),

$$j[\lambda] = \{\xi < j(\lambda) : T_\xi \text{ is stationary in } \sup j[\lambda]\}$$

In particular, if  $\xi < \lambda$ , then  $M$  satisfies that  $T_{j(\xi)}$  is stationary in  $a_W$ , and so by Loś's Theorem,  $W$  concentrates on the set of  $\alpha < \lambda$  such that  $S_\xi$  is stationary in  $\alpha$ .  $\square$

**Lemma 8.2.14.** *Suppose  $\lambda$  is a regular cardinal,  $W$  is an isonormal ultrafilter on  $\lambda$ , and  $D$  is a countably complete ultrafilter on  $\lambda$ . Let  $\langle S_\xi : \xi < \lambda \rangle$  be a partition of  $S_\omega^\lambda$  into stationary sets, and let  $\langle T_\xi : \xi < j_D(\lambda) \rangle = j_D(\langle S_\xi : \xi < \lambda \rangle)$ . Let*

$$A = \{\alpha < j_D(\lambda) : M_D \models T_{a_D} \text{ is stationary in } \alpha\}$$

*Then  $j_D(W)$  is the unique  $M_D$ -filter on  $j_D(\lambda)$  that extends  $j_D[W]$  and concentrates on  $A$ .*

*Proof.* Let  $\mathcal{U}$  be the normal fine ultrafilter on  $P(\lambda)$  isomorphic to  $W$ . Let  $g : P(\lambda) \rightarrow \lambda + 1$  be the sup function

$$g(\sigma) = \sup \sigma$$

By Solovay's Lemma (Corollary 4.4.28),  $g_*(\mathcal{U}) = W$ . Let  $f : \lambda \rightarrow P(\lambda)$  be the function defined by

$$f(\alpha) = \{\xi < \lambda : S_\xi \text{ is stationary in } \alpha\}$$

By the proof of Solovay's Lemma, for any  $A \subseteq \lambda$ ,  $f[A]$  and  $g^{-1}[A]$  are equal modulo  $\mathcal{U}$ . Thus since  $W = g_*(\mathcal{U})$ ,

$$W = \{A \subseteq \lambda : f[A] \in \mathcal{U}\}$$

By Lemma 8.2.11,  $j_D(\mathcal{U})$  is the unique  $M_D$ -filter on  $j_D(Y)$  that extends  $j_D[\mathcal{U}]$  and concentrates on  $\{\sigma \in j_D(P(\lambda)) : a_D \in \sigma\}$ . Since  $j_D(W) = \{A \subseteq \lambda : j_D(f)[A] \in j_D(\mathcal{U})\}$ , it follows that  $j_D(W)$  is the unique  $M_D$ -filter on  $j_D(\lambda)$  that extends  $\{A : j_D(f)[A] \in j_D[\mathcal{U}]\}$  and concentrates on

$$\{\alpha < \lambda : a_D \in j_D(f)(\alpha)\} = \{\alpha < \lambda : M_D \models T_{a_D} \text{ is stationary in } \alpha\}$$

In other words,  $j_D(W)$  is the unique  $M_D$ -filter on  $j_D(\lambda)$  that extends  $j_D[W]$  and concentrates on  $A$ , as desired.  $\square$

Let us include one more useful combinatorial fact, this time about pullbacks of weakly normal ultrafilters. To state the lemma in the generality we will need, we introduce a relativized version of the notion of a weakly normal ultrafilter.

**Definition 8.2.15.** Suppose  $M$  is a transitive model of ZFC,  $\lambda$  is an  $M$ -regular cardinal, and  $F$  is an  $M$ -filter on  $\lambda$ . Then  $F$  is *weakly normal* if for all sequences  $\langle A_\alpha : \alpha < \lambda \rangle \in M$  of subsets of  $\lambda$  such that  $A_\alpha \in F$  for all  $\alpha < \lambda$  and  $A_\alpha \supseteq A_\beta$  for all  $\alpha \leq \beta < \lambda$ , the diagonal intersection  $\Delta_{\alpha < \lambda} A_\alpha$  belongs to  $F$ .

We will really only need this notion for  $M$ -ultrafilters, in which case it has the following familiar formulation:

**Lemma 8.2.16.** *If  $M$  is a transitive model of ZFC,  $\lambda$  is an  $M$ -regular cardinal, and  $U$  is an  $M$ -ultrafilter on  $\lambda$ , then  $U$  is weakly normal if and only if  $a_U = \sup j_U^M[\lambda]$ .*  $\square$

**Lemma 8.2.17.** *Suppose  $\lambda$  is a regular cardinal and  $j : V \rightarrow M$  is an elementary embedding that is continuous at  $\lambda$ . Suppose  $F$  is a weakly normal  $M$ -filter on  $j(\lambda)$ . Then  $j^{-1}[F]$  is a weakly normal filter on  $\lambda$ .*

*Proof.* Suppose  $\langle A_\alpha : \alpha < \lambda \rangle$  is a decreasing sequence of subsets of  $\lambda$  such that  $A_\alpha \in j^{-1}[F]$  for all  $\alpha < \lambda$ . We must show that  $\Delta_{\alpha < \lambda} A_\alpha \in j^{-1}[F]$ . Let  $\langle B_\beta : \beta < j(\lambda) \rangle = j(\langle A_\alpha : \alpha < \lambda \rangle)$ . Since  $j(\Delta_{\alpha < \lambda} A_\alpha) = \Delta_{\beta < j(\lambda)} B_\beta$ , it suffices to show that  $\Delta_{\beta < j(\lambda)} B_\beta \in F$ .

By the elementarity of  $j$ ,  $\langle B_\beta : \beta < j(\lambda) \rangle$  is a decreasing sequence of subsets of  $j(\lambda)$ . We claim that for all  $\beta < j(\lambda)$ ,  $B_\beta \in F$ . To see this, fix  $\beta < j(\lambda)$ . Since  $j$  is continuous at  $\lambda$ , there is some  $\alpha < \lambda$  such that  $\beta \leq j(\alpha)$ . Now  $B_{j(\alpha)} = j(A_\alpha) \in F$  since  $A_\alpha \in j^{-1}[F]$ . But  $B_{j(\alpha)} \subseteq B_\beta$  since  $\beta \leq j(\alpha)$  and  $\langle B_\beta : \beta < j(\lambda) \rangle$  is a decreasing sequence. Therefore  $B_\beta \in F$ , as claimed. Since  $F$  is weakly normal, it follows that  $\Delta_{\beta < j(\lambda)} B_\beta \in F$ .  $\square$

## Proof of the Irreducibility Theorem

We will obtain the Irreducibility Theorem as an immediate consequence of the following slightly more general fact:

**Theorem 8.2.18** (UA). *Suppose  $U$  is a countably complete ultrafilter and  $\lambda$  is a Fréchet successor cardinal. Then there is a countably complete ultrafilter  $D$  with  $\lambda_D < \lambda$  and an internal ultrapower embedding  $e : M_D \rightarrow M_U$  that is  $j_D(\lambda)$ -supercompact in  $M_D$ .*

*Proof.* Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$  and let  $i : V \rightarrow N$  be the ultrapower of the universe of by  $U$ . Let

$$(i_*, j_*) : (M, N) \rightarrow P$$

be the pushout of  $(j, i)$ . Note that  $i_*$  denotes the embedding on the  $M$ -side of the comparison and  $j_*$  denotes the embedding on the  $N$ -side of the comparison. The proof amounts to an analysis of  $(i_*, j_*)$ .

We first characterize  $j_*$ . By definition (Lemma 5.4.34),  $j_*$  is the ultrapower of  $N$  by  $t_U(\mathcal{K}_\lambda)$ . Let

$$\delta = \text{cf}^N(\sup i[\lambda])$$

By the analysis of translations of  $\mathcal{K}_\lambda$  (Lemma 8.2.7), one of the following holds in  $N$ :

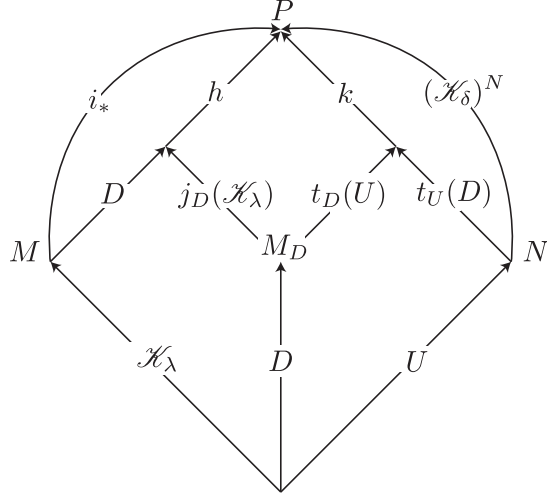


Figure 8.1: Diagram of the Irreducibility Theorem.

- $\delta$  is not Fréchet and  $t_U(\mathcal{K}_\lambda)$  is principal.
- $\delta$  is Fréchet and  $t_U(\mathcal{K}_\lambda)$  is isomorphic to  $(\mathcal{K}_\delta)^N$ .

The hard part of the proof is the analysis of  $i_*$ , the embedding on the  $M$ -side of the comparison of  $(j, i)$ . Let  $\eta$  be the least measurable cardinal of  $M$  above  $\lambda$ . Applying Lemma 8.2.8, let  $D$  be a countably complete ultrafilter of  $M$  with  $\lambda_D < \lambda$  such that there is an internal ultrapower embedding  $h : (M_D)^M \rightarrow P$  with  $\text{CRT}(h) \geq \eta$  and  $i_* = h \circ j_D^M$ . We may assume without loss of generality that the underlying set of  $D$  is the cardinal  $\lambda_D$ . Recall Corollary 7.4.10, which states that  $M^\lambda \subseteq M$ . In particular,  $P(\gamma) \subseteq M$ , so  $D$  truly is an ultrafilter.

The following are the two key claims:

**Claim 1.**  $\delta = j_D(\lambda)$  and  $\text{Ord}^{j_D(\lambda)} \cap N = \text{Ord}^{j_D(\lambda)} \cap P = \text{Ord}^{j_D(\lambda)} \cap M_D$ .

**Claim 2.**  $D \leq_{\text{RF}} U$ .

Assuming these claims, the conclusion of the theorem is immediate: by Claim 2, let  $e : M_D \rightarrow N$  be the unique internal ultrapower embedding such that  $e \circ j_D = i$ ; then  $e$  is  $j_D(\lambda)$ -supercompact in  $M_D$  since  $\text{Ord}^{j_D(\lambda)} \cap N = \text{Ord}^{j_D(\lambda)} \cap M_D$  by Claim 1.

We therefore focus on proving these two claims.

*Proof of Claim 1.* We begin by showing  $\text{Ord}^{j_D(\lambda)} \cap M_D = \text{Ord}^{j_D(\lambda)} \cap P$ . Since  $j : V \rightarrow M$  is a  $\lambda$ -supercompact ultrapower embedding,  $\text{Ord}^\lambda = \text{Ord}^\lambda \cap M$ . By the elementarity of  $j_D$ ,

$$\text{Ord}^{j_D(\lambda)} \cap M_D = \text{Ord}^{j_D(\lambda)} \cap j_D(M) = \text{Ord}^{j_D(\lambda)} \cap (M_D)^M$$

The final equality follows from the fact that  $M$  is closed under  $\lambda$ -sequences and hence correctly computes the ultrapower of  $M$  by  $D$ . But  $h : (M_D)^M \rightarrow P$  is an internal ultrapower embedding such that  $\text{CRT}(h) \geq \eta > j_D(\lambda)$ . Hence  $\text{Ord}^{j_D(\lambda)} \cap (M_D)^M = \text{Ord}^{j_D(\lambda)} \cap P$ . Putting all this together, we have shown

$$\text{Ord}^{j_D(\lambda)} \cap M_D = \text{Ord}^{j_D(\lambda)} \cap P$$

One consequence of the agreement between  $M_D$  and  $P$ , which we set down now for future use, is that  $j_D(\lambda)$  is a successor cardinal of  $P$ :  $\lambda$  is a successor cardinal, so by elementarity,  $j_D(\lambda)$  is a successor cardinal of  $M_D$ , and therefore since  $\text{Ord}^{j_D(\lambda)} \cap M_D = \text{Ord}^{j_D(\lambda)} \cap P$ ,  $j_D(\lambda)$  is a successor cardinal of  $P$ .

Next, we show that  $\delta = j_D(\lambda)$ . To do this, we calculate the  $P$ -cofinality of the ordinal  $\sup j_* \circ i[\lambda]$  in two different ways.

On the one hand, we claim

$$\text{cf}^P(\sup j_* \circ i[\lambda]) = j_D(\lambda) \tag{8.1}$$

We have that  $j_* \circ i = h \circ j_D \circ j = h \circ j_D(j) \circ j_D$ . Since  $\lambda_D < \lambda$ , and  $\lambda$  is regular,  $j_D(\lambda) = \sup j_D[\lambda]$  (Lemma 3.5.32). Now we calculate:

$$\begin{aligned} \text{cf}^P(\sup h \circ j_D(j) \circ j_D[\lambda]) &= \text{cf}^P(\sup h \circ j_D(j)[\sup j_D[\lambda]]) \\ &= \text{cf}^P(\sup h \circ j_D(j)[j_D(\lambda)]) \\ &= \text{cf}^{M_D}(\sup h \circ j_D(j)[j_D(\lambda)]) \\ &= j_D(\lambda) \end{aligned}$$

The second-to-last equality uses the fact that  $\text{Ord}^{j_D(\lambda)} \cap P = \text{Ord}^{j_D(\lambda)} \cap M_D$ . The final equality uses the fact that  $h \circ j_D(j)$  is increasing and definable over  $M_D$  and  $j_D(\lambda)$  is regular in  $M_D$ . Putting everything together yields that  $\text{cf}^P(\sup j_* \circ i[\lambda]) = j_D(\lambda)$ , as claimed.

On the other hand, we claim

$$\text{cf}^P(\sup j_* \circ i[\lambda]) = \text{cf}^P(\sup j_*[\delta]) \quad (8.2)$$

Since  $\delta = \text{cf}^N(\sup i[\lambda])$ , there is an increasing cofinal function  $f : \delta \rightarrow \sup i[\lambda]$  with  $f \in N$ . Now  $\sup j_* \circ i[\lambda] = \sup j_*[\sup f[\delta]] = \sup j_*(f)[\sup j_*[\delta]]$ . Thus  $j_*(f) \in P$  restricts to an increasing cofinal function from  $\sup j_*[\delta]$  to  $\sup j_* \circ i[\lambda]$ . It follows that  $\text{cf}^P(\sup j_* \circ i[\lambda]) = \text{cf}^P(\sup j_*[\delta])$ , as desired.

Combining (8.1) and (8.2), we have shown  $\text{cf}^P(\sup j_*[\delta]) = j_D(\lambda)$ . To show  $\delta = j_D(\lambda)$ , we must show  $\text{cf}^P(\sup j_*[\delta]) = \delta$ . In other words (applying the easy direction of Theorem 7.3.33), we must show  $j_*$  is  $\delta$ -tight.

Recall that  $j_*$  is the ultrapower of  $N$  by  $t_U(\mathcal{K}_\lambda)$ . If  $t_U(\mathcal{K}_\lambda)$  is principal, then trivially  $j_*$  is  $\delta$ -tight. Therefore assume  $t_U(\mathcal{K}_\lambda)$  is nonprincipal. By the second paragraph of this proof,  $N$  satisfies that  $\delta$  is Fréchet and  $t_U(\mathcal{K}_\lambda)$  is isomorphic to  $(\mathcal{K}_\delta)^N$ .

It suffices to show that  $\delta$  is not isolated in  $N$ . Then applying in  $N$  the analysis of  $\mathcal{K}_\delta$  at nonisolated cardinals  $\delta$  (Proposition 7.4.11),  $j_*$  is  $\delta$ -tight.

Thus assume towards a contradiction that  $\delta$  is isolated in  $N$ . In particular,  $\delta$  is a regular limit cardinal in  $N$ . Moreover, by Theorem 7.5.36,  $(\mathcal{K}_\delta)^N$  concentrates on  $N$ -regular cardinals, so by Łoś's Theorem,  $a_{(\mathcal{K}_\delta)^N} = \sup j_*[\delta]$  is regular in  $P$ . Thus by (8.2),  $\text{cf}^P(\sup j_* \circ i[\lambda]) = \sup j_*[\delta]$ , and so by (8.1),  $\sup j_*[\delta] = j_D(\lambda)$ . Since  $\delta$  is a limit cardinal of  $N$ ,  $\sup j_*[\delta]$  is a limit cardinal of  $P$ . This contradicts the fact (set down earlier) that  $j_D(\lambda)$  is a successor cardinal of  $P$ . Thus our assumption that  $\delta$  is isolated in  $N$  was false. It follows that  $\delta$  is not isolated and hence  $j_*$  is  $\delta$ -tight, and hence  $\text{cf}^P(\sup j_*[\delta]) = \delta$ , and hence by (8.1) and (8.2),  $j_D(\lambda) = \delta$ .

We finally show that  $\text{Ord}^\delta \cap N = \text{Ord}^\delta \cap P$ . If  $t_U(\mathcal{K}_\lambda)$  is principal then  $P = N$ , so this is obvious. If not, then  $j_* : N \rightarrow P$  is the ultrapower embedding associated to  $(\mathcal{K}_\delta)^N$ . Note

that  $\delta = j_D(\lambda)$  is a successor cardinal of  $P$ , and so since  $P \subseteq N$ ,  $\delta$  is a successor cardinal of  $N$ . Thus by the analysis of Ketonen ultrafilters on successor cardinals (Corollary 7.4.10) applied in  $N$ ,  $j_*$  is  $\delta$ -supercompact. In particular,  $\text{Ord}^\delta \cap N = \text{Ord}^\delta \cap P$ .  $\square$

We now turn to the proof that  $D \leq_{\text{RF}} U$ .

*Proof of Claim 2.* To show  $D \leq_{\text{RF}} U$ , it suffices (by the definition of translation functions, or Lemma 5.4.39) to show that  $t_U(D)$  is principal in  $N$ .

Let us first show that

$$t_U(D) \leq_{\text{RF}} t_U(\mathcal{K}_\lambda)$$

in  $N$ . Note that

$$(h \circ j_D(j), j_*) : (M_D, N) \rightarrow P$$

is an internal ultrapower comparison of  $(j_D, i)$ . Since

$$(j_{t_D(U)}^{M_D}, j_{t_U(D)}^{M_U}) : (M_D, N) \rightarrow (M_{t_U(D)})^N$$

is the pushout of  $(j_D, i)$  (by Lemma 5.4.34), it follows that there is an internal ultrapower embedding  $k : (M_{t_U(D)})^N \rightarrow P$  such that  $k \circ j_{t_D(U)}^{M_D} = h \circ j_D(j)$  and  $k \circ j_{t_U(D)}^{M_U} = j_* = j_{t_U(\mathcal{K}_\lambda)}^N$ . The latter equation is equivalent to the statement that  $t_U(D) \leq_{\text{RF}} t_U(\mathcal{K}_\lambda)$  in  $N$ .

Since  $t_U(\mathcal{K}_\lambda)$  is either principal or isomorphic to the ultrafilter  $(\mathcal{K}_{j_D(\lambda)})^N$ , which is irreducible by Lemma 7.3.12, one of the following must hold:

- (1)  $j_D(\lambda)$  is Fréchet in  $N$  and  $N \models t_U(D) \cong (\mathcal{K}_{j_D(\lambda)})^N$ .
- (2)  $t_U(D)$  is principal in  $N$ .

Our goal is to show that (2) holds, so to finish the proof of the claim, it suffices to show that (1) fails. Towards this, we will prove the following subclaim:

**Subclaim 1.** *Assume  $j_D(\lambda)$  is Fréchet in  $N$ . Then  $(\mathcal{K}_{j_D(\lambda)})^N = j_D(\mathcal{K}_\lambda)$ .*

*Proof of Subclaim 1.* We plan to prove the claim by applying our unique extension lemma for isonormal ultrafilters. By Corollary 7.4.10,  $\mathcal{K}_\lambda$  is an isonormal ultrafilter on  $\lambda$ . By Claim 1,  $(\mathcal{K}_{j_D(\lambda)})^N$  is an  $M_D$ -filter on  $j_D(\lambda)$ . Let  $\langle S_\xi : \xi < \lambda \rangle$  be a partition of  $S_\omega^\lambda$  into stationary sets. Let  $\langle T_\xi : \xi < j_D(\lambda) \rangle = j_D(\langle S_\xi : \xi < \lambda \rangle)$ . By Lemma 8.2.14, to show that  $j_D(\mathcal{K}_\lambda) = (\mathcal{K}_{j_D(\lambda)})^N$ , it suffices to show that the following hold:

- (i)  $\{\alpha < j_D(\lambda) : M_D \models T_{a_D} \text{ is stationary in } \alpha\} \in (\mathcal{K}_{j_D(\lambda)})^N$ .
- (ii)  $j_D[\mathcal{K}_\lambda] \subseteq (\mathcal{K}_{j_D(\lambda)})^N$ .

(i) will be proved by applying Lemma 8.2.13. Note that  $\langle T_\xi : \xi < j_D(\lambda) \rangle$  belongs to  $N$  and  $N$  satisfies that  $\langle T_\xi : \xi < j_D(\lambda) \rangle$  is a stationary partition of  $S_\omega^{j_D(\lambda)}$ : this follows from the fact that  $P(j_D(\lambda)) \cap N = P(j_D(\lambda)) \cap M_D$  by Claim 1 and  $\langle T_\xi : \xi < j_D(\lambda) \rangle$  is a stationary partition of  $S_\omega^{j_D(\lambda)}$  in  $M_D$ . Since  $(\mathcal{K}_{j_D(\lambda)})^N$  is a countably complete weakly normal ultrafilter of  $N$ , Lemma 8.2.13 implies that  $(\mathcal{K}_{j_D(\lambda)})^N$  concentrates on  $\{\alpha < j_D(\lambda) : M_D \models T_\xi \text{ is stationary in } \alpha\}$  for any  $\xi < j_D(\lambda)$ , and in particular  $\{\alpha < j_D(\lambda) : M_D \models T_{a_D} \text{ is stationary in } \alpha\} \in (\mathcal{K}_{j_D(\lambda)})^N$ , as desired.

Towards (ii), let  $W = j_D^{-1}[(\mathcal{K}_{j_D(\lambda)})^N]$ . It suffices to show that  $W = \mathcal{K}_\lambda$ . It is clear that  $W$  is a countably complete uniform ultrafilter on  $\lambda$ . Recall that  $\mathcal{K}_\lambda$  is the unique Ketonen ultrafilter on  $\lambda$ . Let  $A$  be the set of ordinals below  $\lambda$  that carry no countably complete tail uniform ultrafilter. By the definition of a Ketonen ultrafilter on a regular cardinal (Definition 7.3.5), to show  $W = \mathcal{K}_\lambda$ , it suffices to show that the following hold:

- $A \in W$ .
- $W$  is weakly normal.

Let us show that  $A \in W$ . In other words, we must show that  $j_D(A) \in (\mathcal{K}_\delta)^N$ . Note that  $j_D(A)$  is the set of ordinals less than  $j_D(\lambda) = \delta$  that carry no countably complete tail uniform ultrafilter in  $M_D$ . By the definition of a Ketonen ultrafilter on a regular cardinal (Definition 7.3.5) applied in  $N$ ,  $(\mathcal{K}_\delta)^N$  concentrates on the set of ordinals less than  $\delta$  that



carry no countably complete tail uniform ultrafilter in  $N$ . Thus to show that  $j_D(A) \in (\mathcal{K}_\delta)^N$ , it suffices to show that if an ordinal less than  $\delta$  carries no countably complete tail uniform ultrafilter in  $N$ , then it carries no countably complete tail uniform ultrafilter in  $M_D$ . In fact we will show that for any ordinal  $\alpha < \delta$ ,

$$\mathcal{B}^{M_D}(\alpha) = \mathcal{B}^N(\alpha)$$

where  $\mathcal{B}(X)$  denotes the set of countably complete ultrafilters on  $X$ .

This is an application Proposition 6.3.9, which asserts that if  $\gamma$  is a cardinal and  $Q$  is an ultrapower of the universe that is closed under  $\gamma$ -sequences, then for any ordinal  $\alpha < \gamma$ ,  $\mathcal{B}(\alpha) = \mathcal{B}^Q(\alpha)$ . Fix an ordinal  $\alpha < \delta$ . Applying Proposition 6.3.9 in  $M_D$  to the ultrapower  $P$  of  $M_D$ , which satisfies  $\text{Ord}^\delta \cap P = \text{Ord}^\delta \cap M_D$  by Claim 1,

$$\mathcal{B}^{M_D}(\alpha) = \mathcal{B}^P(\alpha)$$

Similarly, applying Proposition 6.3.9 and Claim 1 in  $N$  to  $P$ ,

$$\mathcal{B}^N(\alpha) = \mathcal{B}^P(\alpha)$$

Hence  $\mathcal{B}^{M_D}(\alpha) = \mathcal{B}^N(\alpha)$ , as desired. This shows  $A \in W$ .

We now show that  $W$  is weakly normal. We do this by applying Lemma 8.2.17. Note that  $(\mathcal{K}_{j_D(\lambda)})^N$  is a weakly normal  $M_D$ -ultrafilter since it is a weakly normal ultrafilter of  $N$  and  $P(j_D(\lambda)) \cap M_D = P(j_D(\lambda)) \cap N$ . Therefore since  $j_D : V \rightarrow M_D$  is continuous at  $\lambda$ , Lemma 8.2.17 implies that  $j_D^{-1}[(\mathcal{K}_{j_D(\lambda)})]$  is weakly normal. In other words,  $W$  is weakly normal.

Thus we have shown that  $W$  is a Ketonen ultrafilter on  $\lambda$ , so  $W = \mathcal{K}_\lambda$ . This implies (ii).

As we explained above, (i), (ii), and Lemma 8.2.14 together imply  $(\mathcal{K}_{j_D(\lambda)})^N = j_D(\mathcal{K}_\lambda)$ , which proves the subclaim.  $\square$

Using Subclaim 1, we show that (1) above does not hold. If  $j_D(\lambda)$  is not Fréchet in  $N$ , then obviously (1) does not hold, so assume instead that  $j_D(\lambda)$  is Fréchet in  $N$ . Let  $\mathcal{K} = (\mathcal{K}_{j_D(\lambda)})^N = j_D(\mathcal{K}_\lambda)$ . Thus  $\mathcal{K} \in M_D \cap N$ .

Recall that  $M_{t_U(D)}^N$  is the target model of the pushout of  $(j_D, i)$ . Thus by the analysis of ultrafilters amenable to a pushout (Theorem 5.4.19),  $\mathcal{K} \cap M_{t_U(D)}^N \in M_{t_U(D)}^N$ . On the other hand, we will show that  $\mathcal{K} \cap P \notin P$ . By the strictness of the Mitchell order on nonprincipal ultrafilters (Lemma 4.2.40),

$$\mathcal{K} \notin M_{\mathcal{K}}^{M_D} = j_D(M_{\mathcal{K}\lambda}) = j_D(M)$$

Recall that  $h : j_D(M) \rightarrow P$  is an internal ultrapower embedding, so in particular  $P \subseteq j_D(M)$ , and hence  $\mathcal{K} \notin P$  since  $\mathcal{K} \notin j_D(M)$ . Since  $P(j_D(\lambda)) \cap N = P(j_D(\lambda)) \cap P$  by Claim 1, it follows that  $\mathcal{K} = \mathcal{K} \cap P$ , and so  $\mathcal{K} \cap P \notin P$ .

We have  $\mathcal{K} \cap M_{t_U(D)}^N \in M_{t_U(D)}^N$  and  $\mathcal{K} \cap P \notin P$ , so  $M_{t_U(D)}^N \neq P$ . Since  $P = M_{\mathcal{K}}^N$ , it follows that  $t_U(D)$  and  $\mathcal{K}$  are not isomorphic in  $N$ : they have different ultrapowers. In other words, (1) above fails.

Thus (2) holds, which proves  $D \leq_{\text{RF}} U$ , establishing the claim.  $\square$

Having proved Claim 1 and Claim 2, the theorem follows, as we explained after the statement of Claim 2.  $\square$

An immediate corollary of Theorem 8.2.18 is the following fact, which will imply the Irreducibility Theorem:

**Corollary 8.2.19** (UA). *Suppose  $\lambda$  is a Fréchet successor cardinal and  $U$  is a  $\lambda$ -irreducible ultrafilter. Then  $j_U$  is  $\lambda$ -supercompact.*

*Proof.* We begin with the case that  $\lambda$  is a successor cardinal. By Theorem 8.2.18, there is an ultrafilter  $D$  with  $\lambda_D < \lambda$  such that there is an internal ultrapower embedding  $e : M_D \rightarrow M_U$  with  $e \circ j_D = j_U$  that is  $j_D(\lambda)$ -supercompact in  $M_D$ . Since  $U$  is  $\lambda$ -irreducible,  $D$  is principal, and hence  $j_U = e \circ j_D = e$  is  $\lambda$ -supercompact as desired.  $\square$

**Corollary 8.2.20** (UA). *Suppose  $\lambda$  is a strong limit cardinal and  $U$  is a  $\lambda$ -irreducible ultrafilter. Then  $j_U$  is  $<\lambda$ -supercompact. If  $\lambda$  is singular, then  $j_U$  is  $\lambda$ -supercompact. If  $\lambda$  is regular and Fréchet, then  $j_U$  is  $\lambda$ -tight.*

*Proof.* We start by showing that  $j_U$  is  $<\lambda$ -supercompact. Fix a successor cardinal  $\delta < \lambda$ . If  $\delta$  is Fréchet, then  $j_U$  is  $\delta$ -supercompact by Corollary 8.2.19. If  $\delta$  is not Fréchet, then we can apply Theorem 7.5.32: no cardinal  $\kappa \leq \delta$  is  $\delta$ -supercompact and  $U$  is  $\leq 2^\delta$ -irreducible, so  $U$  is  $\delta^+$ -complete and  $j_U$  is vacuously  $\delta$ -supercompact. Thus  $j_U$  is  $<\lambda$ -supercompact.

Since  $j_U$  is a  $<\lambda$ -supercompact ultrapower embedding,  $(M_U)^{<\lambda} \subseteq M_U$ . If  $\lambda$  is singular, this immediately implies  $(M_U)^\lambda \subseteq M_U$ . Therefore  $j_U$  is  $\lambda$ -supercompact.

If  $\lambda$  is regular and Fréchet, we can apply Lemma 8.2.10 to conclude that  $j_U$  is  $\lambda$ -tight.  $\square$

As a corollary, we can finally prove the Irreducibility Theorem.

**Theorem 8.2.21 (UA).** *Suppose  $\lambda$  is a successor cardinal or a strong limit singular cardinal and  $U$  is a countably complete uniform ultrafilter on  $\lambda$ . Then the following are equivalent:*

(1)  $j_U$  is  $\lambda$ -irreducible.

(2)  $j_U$  is  $\lambda$ -supercompact.

*Proof.* (1) implies (2): Follows from Theorem 8.2.18 and Corollary 8.2.20.

(2) implies (1): Follows from Proposition 8.2.3.  $\square$

**Theorem 8.2.22 (UA).** *Suppose  $\lambda$  is an inaccessible cardinal and  $U$  is a countably complete ultrafilter on  $\lambda$ . Then the following are equivalent:*

(1)  $j_U$  is  $\lambda$ -irreducible.

(2)  $j_U$  is  $<\lambda$ -supercompact and  $\lambda$ -tight.

*Proof.* (1) implies (2): Follows from Corollary 8.2.20 and Lemma 8.2.10.

(2) implies (1): Follows from Proposition 8.2.3.  $\square$

It is sometimes easier to use a version of the Irreducibility Theorem in the form of Theorem 8.2.18. This follows from Corollary 8.2.20 using the structure of the Rudin-Frolík order (Theorem 5.3.17).

**Lemma 8.2.23** (UA). *Suppose  $U$  is a countably complete ultrafilter and  $\lambda$  is a cardinal. Then there is a countably complete ultrafilter  $D \leq_{\text{RF}} U$  with  $\lambda_D < \lambda$  such that  $t_D(U)$  is  $\lambda_*$ -irreducible in  $M_D$  where  $\lambda_* = \sup j_D[\lambda]$ .*

*Proof.* By the local ascending chain condition for the Rudin-Frolík order (Theorem 5.3.17), there is an  $\leq_{\text{RF}}$ -maximal  $D \leq_{\text{RF}} U$  such that  $\lambda_D < \lambda$ . Let  $i : M_D \rightarrow M_U$  be the unique internal ultrapower embedding such that  $i \circ j_D = j_U$ . Then  $i$  is the ultrapower of  $M_D$  by  $t_D(U)$ .

Suppose towards a contradiction that  $t_D(U)$  is not  $\lambda_*$ -irreducible in  $M_D$ . Fix a cardinal  $\gamma < \lambda$  and a countably complete ultrafilter  $Z$  of  $M_D$  on  $j_D(\gamma)$  such that  $Z \leq_{\text{RF}} t_D(U)$ . Then the iteration  $\langle D, W \rangle$  is given by an ultrafilter  $D'$  on  $\lambda_D \cdot \gamma$ . Now  $\lambda_{D'} \leq \lambda_D \cdot \gamma < \lambda$  but  $D <_{\text{RF}} D' \leq_{\text{RF}} U$ . This contradicts the maximality of  $D$ .  $\square$

Combining this with the Irreducibility Theorem immediately yields the following fact:

**Corollary 8.2.24** (UA). *Suppose  $U$  is a countably complete ultrafilter.*

- *If  $\lambda$  is a Fréchet successor cardinal, then there is an ultrafilter  $D \leq_{\text{RF}} U$  with  $\lambda_D < \lambda$  such that the unique internal ultrapower embedding  $h : M_D \rightarrow M_U$  with  $h \circ j_D = j_U$  is  $j_D(\lambda)$ -supercompact in  $M_D$ .*
- *If  $\lambda$  is a Fréchet inaccessible cardinal, then there is an ultrafilter  $D \leq_{\text{RF}} U$  with  $\lambda_D < \lambda$  such that the unique internal ultrapower embedding  $h : M_D \rightarrow M_U$  with  $h \circ j_D = j_U$  is  $< \lambda$ -supercompact and  $\lambda$ -tight in  $M_D$ .*
- *If  $\lambda$  is a strong limit singular cardinal, then there is an ultrafilter  $D \leq_{\text{RF}} U$  with  $\lambda_D < \lambda$  such that the unique internal ultrapower embedding  $h : M_D \rightarrow M_U$  with  $h \circ j_D = j_U$  is  $\lambda_*$ -supercompact in  $M_D$  where  $\lambda_* = \sup j_D[\lambda]$ .*  $\square$

### 8.3 Resolving the identity crisis

In this section, we characterize all strongly compact cardinals assuming UA. This begins with an analysis of the  $\kappa$ -complete analog of  $\mathcal{K}_\lambda$ , denoted  $\mathcal{K}_\lambda^\kappa$ .

#### The equivalence of strong compactness and supercompactness

Recall that if  $\lambda$  is  $\kappa$ -Fréchet, then  $\mathcal{K}_\lambda^\kappa$  is the  $<_{\mathbb{k}}$ -least  $\kappa$ -complete uniform ultrafilter on  $\lambda$ . Applying the Irreducibility Theorem, Proposition 8.1.17 yields a generalization of our analysis of  $\mathcal{K}_\lambda$  for successor  $\lambda$  (Corollary 7.4.10) to these more complete ultrafilters:

**Corollary 8.3.1** (UA). *Suppose  $\kappa < \lambda$  and  $\lambda$  is a  $\kappa^+$ -Fréchet successor cardinal. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda^{\kappa^+}$ . Then  $M^\lambda \subseteq M$ .*

*Proof.* By Proposition 8.1.17,  $\mathcal{K} = \mathcal{K}_\lambda^{\kappa^+}$  is irreducible. Since  $\lambda_{\mathcal{K}} = \lambda$ ,  $\mathcal{K}$  is  $\lambda$ -irreducible. Therefore by the Irreducibility Theorem (Corollary 8.2.19),  $M^\lambda \subseteq M$ .  $\square$

**Corollary 8.3.2** (UA). *Suppose  $\kappa < \lambda$  and  $\lambda$  is a  $\kappa^+$ -Fréchet successor cardinal. Then there is a  $\lambda$ -supercompact cardinal  $\delta$  such that  $\kappa < \delta < \lambda$ .*  $\square$

As in the case of the first supercompact cardinal, if  $\lambda$  is strongly inaccessible, it is not clear whether  $\mathcal{K}_\lambda^{\kappa^+}$  witnesses full  $\lambda$ -supercompactness:

**Corollary 8.3.3** (UA). *Suppose  $\kappa < \lambda$  and  $\lambda$  is a  $\kappa^+$ -Fréchet inaccessible cardinal. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda^{\kappa^+}$ . Then  $M^{<\lambda} \subseteq M$  and  $M$  has the  $\leq \lambda$ -covering property.*

*Proof.* By Proposition 8.1.17,  $\mathcal{K} = \mathcal{K}_\lambda^{\kappa^+}$  is irreducible. Since  $\lambda_{\mathcal{K}} = \lambda$ ,  $\mathcal{K}$  is  $\lambda$ -irreducible. Therefore by the Irreducibility Theorem (Corollary 8.2.20),  $M^{<\lambda} \subseteq M$  and  $M$  has the  $\leq \lambda$ -covering property.  $\square$

Let us now analyze  $\mathcal{K}_\lambda^\kappa$  for general  $\kappa$ :

**Theorem 8.3.4** (UA). *Suppose  $\kappa \leq \lambda$  and  $\lambda$  is a  $\kappa$ -Fréchet regular cardinal. Let  $\mathcal{K} = \mathcal{K}_\lambda^\kappa$ .*

(1) Suppose  $\kappa$  is not a measurable limit of  $\lambda$ -strongly compact cardinals. Then  $\mathcal{K}$  is irreducible.

(2) Suppose  $\kappa$  is a measurable limit of  $\lambda$ -strongly compact cardinals. Let  $D$  be the  $\triangleleft$ -least normal ultrafilter on  $\kappa$ . Then  $D \leq_{\text{RF}} \mathcal{K}$ .

*Proof.* *Proof of (1):* Assume first that  $\kappa$  is not measurable. Since  $\mathcal{K}$  is  $\kappa$ -complete, it is  $\kappa^+$ -complete. Hence  $\mathcal{K} = \mathcal{K}_\lambda^{\kappa^+}$  is irreducible by Proposition 8.1.17.

Assume instead that  $\kappa$  is not a limit of  $\lambda$ -strongly compact cardinals. Let  $\nu < \kappa$  be the supremum of the  $\lambda$ -strongly compact cardinals below  $\kappa$ . Note that  $\lambda$  is  $\nu^+$ -Fréchet since  $\lambda$  is  $\kappa$ -Fréchet and  $\nu \leq \kappa$ . Moreover,  $\mathcal{K}$  is  $\nu^+$ -complete since  $\nu^+ \leq \kappa$ . Since  $\mathcal{K}_\lambda^{\nu^+}$  is the  $<_{\mathbb{k}}$ -least  $\nu^+$ -complete uniform ultrafilter on  $\lambda$ ,  $\mathcal{K}_\lambda^{\nu^+} \leq_{\mathbb{k}} \mathcal{K}_\lambda^\kappa$ . On the other hand,  $\mathcal{K}_\lambda^{\nu^+}$  is  $\kappa$ -complete: by Corollary 8.3.1 and Corollary 8.3.3, the completeness of  $\mathcal{K}_\lambda^{\nu^+}$  is a  $\lambda$ -strongly compact cardinal in the interval  $(\nu, \lambda)$ , and by choice of  $\nu$ , the completeness is at least  $\kappa$ . Since  $\mathcal{K}_\lambda^{\nu^+}$  is a  $\kappa$ -complete uniform ultrafilter on  $\lambda$  and  $\mathcal{K} = \mathcal{K}_\lambda^\kappa$  is the  $<_{\mathbb{k}}$ -least such ultrafilter,  $\mathcal{K} \leq_{\mathbb{k}} \mathcal{K}_\lambda^{\nu^+}$ . By the antisymmetry of the Ketonen order,  $\mathcal{K} = \mathcal{K}_\lambda^{\nu^+}$ , and in particular  $\mathcal{K}$  is irreducible by Proposition 8.1.17.

*Proof of (2):* Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}$ .

We first claim that  $\kappa$  is not measurable in  $M$ . Since  $\mathcal{K}$  is  $\kappa$ -complete,  $\text{CRT}(j) \geq \kappa$ . Therefore if  $\delta < \kappa$  is  $\lambda$ -strongly compact, then  $\delta$  is  $j(\lambda)$ -strongly compact in  $M$ . Suppose towards a contradiction that  $\kappa$  is measurable in  $M$ . Then  $\kappa$  is a measurable limit of  $j(\lambda)$ -strongly compact cardinals in  $M$ , so  $\kappa$  is  $j(\lambda)$ -strongly compact in  $M$  by Menas's Theorem (Corollary 8.1.6). But by the minimality of  $\mathcal{K}_\lambda^\kappa$  (see Theorem 7.2.14),  $\text{cf}^M(\sup j[\lambda])$  is not  $\kappa$ -Fréchet in  $M$ , contradicting that  $\kappa$  is  $\text{cf}^M(\sup j[\lambda])$ -strongly compact in  $M$ . Thus our assumption was false and so  $\kappa$  is not measurable in  $M$ .

Since  $\kappa$  is measurable in  $V$  but not in  $M$ , it follows that  $\text{CRT}(j) \leq \kappa$ , so  $\text{CRT}(j) = \kappa$ . Let  $D$  be the ultrafilter on  $\kappa$  derived from  $j$  using  $\kappa$ . Since  $D$  is a normal ultrafilter and  $\kappa$  is not measurable in  $M_D$ ,  $D$  is the  $\triangleleft$ -least ultrafilter on  $\kappa$  (by the linearity of the Mitchell order, Theorem 2.3.11). Recall that our analysis of derived normal ultrafilters (Theorem 5.3.11)

implies that either  $D \triangleleft \mathcal{K}$  or  $D \leq_{\text{RF}} \mathcal{K}$ . Since  $\kappa$  is not measurable in  $M = M_{\mathcal{K}}$ , it cannot be that  $D \triangleleft \mathcal{K}$ , and therefore we can conclude that  $D \leq_{\text{RF}} \mathcal{K}$ .  $\square$

It is not hard to show that in the situation of Theorem 8.3.4 (2), in fact  $\mathcal{K}_{\kappa}^{\lambda}$  is one of the ultrafilters defined in the proof of Menas's Theorem (Corollary 8.1.6):

$$\mathcal{K}_{\kappa}^{\lambda} = D\text{-}\lim_{\alpha < \kappa} \mathcal{K}_{\lambda}^{\alpha+}$$

Moreover, there is a set  $I \in D$  such that the sequence  $\langle \mathcal{K}_{\lambda}^{\alpha+} : \alpha \in I \rangle$  is discrete, which explains why  $D \leq_{\text{RF}} \mathcal{K}_{\kappa}^{\lambda}$ .

We now characterize the critical point of  $\mathcal{K}_{\lambda}^{\nu}$ .

**Definition 8.3.5.** Suppose  $\nu \leq \lambda$  are uncountable cardinals and  $\lambda$  is  $\nu$ -Fréchet. Then  $\kappa_{\lambda}^{\nu}$  denotes the completeness of  $\mathcal{K}_{\lambda}^{\nu}$ .

To analyze  $\kappa_{\lambda}^{\nu}$ , we use the following generalization of Proposition 7.4.1:

**Lemma 8.3.6.** *Suppose  $\nu \leq \lambda$  are cardinals and  $\lambda$  is regular. Suppose  $\kappa \leq \lambda$  is the least  $(\nu, \lambda)$ -strongly compact cardinal. Suppose  $j : V \rightarrow M$  is an elementary embedding such that  $\text{cf}^M(\sup j[\lambda])$  is not  $j(\nu)$ -Fréchet in  $M$ . Then  $j$  is  $(\lambda, \delta)$ -tight for some  $M$ -cardinal  $\delta < j(\kappa)$ .*

*Proof.* Since  $\kappa$  is  $(\nu, \lambda)$ -strongly compact, every cardinal in the interval  $[\kappa, \lambda]$  is  $\nu$ -Fréchet. Thus in  $M$ , every cardinal in the interval  $j([\kappa, \lambda])$  is  $j(\nu)$ -Fréchet. Let  $\delta = \text{cf}^M(\sup j[\lambda])$ . By Theorem 7.2.12,  $j$  is  $(\lambda, \delta)$ -tight. Moreover  $\delta \leq \sup j[\lambda] \leq j(\lambda)$  and  $\delta \notin j([\kappa, \lambda])$  since  $\delta$  is not  $j(\nu)$ -Fréchet. Thus  $\delta < j(\kappa)$ . This proves the lemma.  $\square$

The following proposition shows that under UA, all the ultrafilter-theoretic generalizations of strong compactness collapse to a single concept:

**Proposition 8.3.7 (UA).** *Suppose  $\nu \leq \kappa \leq \lambda$  are cardinals,  $\lambda$  is a regular cardinal, and  $\kappa$  is the least  $(\nu, \lambda)$ -strongly compact cardinal. Then  $\kappa = \kappa_{\lambda}^{\nu}$  and  $\kappa$  is  $\lambda$ -strongly compact.*

*Proof.* Since there is a  $(\nu, \lambda)$ -strongly compact cardinal  $\kappa \leq \lambda$ , there is some cardinal below  $\lambda$  that is  $\lambda$ -supercompact. Thus if  $\lambda$  is a limit cardinal then  $\lambda$  is strongly inaccessible by

our results on GCH (Theorem 6.3.12). In particular, we are in a position to apply the Irreducibility Theorem.

By Theorem 8.3.4, either  $\kappa'_\lambda$  is a measurable limit of  $\lambda$ -strongly compact cardinals or  $\mathcal{K}'_\lambda$  is irreducible. In the former case  $\kappa'_\lambda$  is  $\lambda$ -strongly compact by Theorem 8.1.1. In the latter case,  $\mathcal{K}'_\lambda$  witnesses that  $\kappa'_\lambda$  is  $<\lambda$ -supercompact and  $\lambda$ -strongly compact by the Irreducibility Theorem (Corollary 8.2.19 and Corollary 8.2.20).

In particular,  $\kappa'_\lambda$  is  $(\nu, \lambda)$ -strongly compact, so  $\kappa \leq \kappa'_\lambda$ .

On the other hand, by Lemma 8.3.6,  $\kappa \leq \kappa'_\lambda$ .

Thus  $\kappa = \kappa'_\lambda$ , and in particular  $\kappa$  is  $\lambda$ -strongly compact. □

**Corollary 8.3.8 (UA).** *Suppose  $\kappa \leq \lambda$  are cardinals,  $\lambda$  is a successor cardinal, and  $\kappa$  is  $\lambda$ -strongly compact. Then either  $\kappa$  is  $\lambda$ -supercompact or  $\kappa$  is a measurable limit of  $\lambda$ -supercompact cardinals.*

*Proof.* Assume by induction that the theorem is true for  $\bar{\kappa} < \kappa$ . By Proposition 8.3.7,  $\kappa = \kappa'_\lambda$ . By Theorem 8.3.4, either  $\kappa$  is a measurable limit of  $\lambda$ -strongly compact cardinals or  $\mathcal{K}'_\lambda$  is irreducible. If  $\kappa$  is a limit of  $\lambda$ -strongly compact cardinals, then by our induction hypothesis,  $\kappa$  is a measurable limit of  $\lambda$ -supercompact cardinals. If instead  $\mathcal{K}'_\lambda$  is irreducible, then by Theorem 8.2.18,  $\mathcal{K}'_\lambda$  witnesses that  $\kappa$  is  $\lambda$ -supercompact. □

This implies our converse to Menas's Theorem, stating that under UA, a strongly compact cardinal is either a supercompact cardinal or a measurable limit of supercompact cardinals:

**Theorem 8.3.9 (UA).** *Suppose  $\kappa$  is a strongly compact cardinal. Either  $\kappa$  is a supercompact cardinal or  $\kappa$  is a measurable limit of supercompact cardinals*

*Proof.* Suppose  $\kappa$  is strongly compact. By the Pigeonhole Principle, there is a cardinal  $\gamma \geq \kappa$  such that a cardinal  $\bar{\kappa} \leq \kappa$  is supercompact if and only if  $\bar{\kappa}$  is  $\gamma$ -supercompact. Since  $\kappa$  is  $\gamma^+$ -strongly compact, Corollary 8.3.8 implies that either  $\kappa$  is  $\gamma^+$ -supercompact or  $\kappa$  is a limit of  $\gamma^+$ -supercompact cardinals. By our choice of  $\gamma$ , it follows that either  $\kappa$  is supercompact or  $\kappa$  is a limit of supercompact cardinals, as desired. □



The use of the Pigeonhole Principle is unnecessary here, since the cardinal  $\gamma$  turns out to equal  $\kappa$ ; a more careful argument appears in the proof of Corollary 8.3.15.

Before generalizing our results on ultrapower thresholds (Theorem 7.4.26), it is worth noting that our large cardinal assumptions now put us in a local GCH context. For example, we have the following lemma:

**Lemma 8.3.10 (UA).** *Suppose  $\lambda$  is a regular Fréchet cardinal. Suppose  $\lambda$  is also  $\kappa_\lambda^+$ -Fréchet.<sup>2</sup> Then for all cardinals  $\gamma \in [\kappa_\lambda, \lambda]$ ,  $2^\gamma = \gamma^+$ .*

*Proof.* Let  $\kappa = \kappa_\lambda$ . Let  $\mathcal{K}_0 = \mathcal{K}_\lambda$  and  $\mathcal{K}_1 = \mathcal{K}_\lambda^{\kappa^+}$ . Then  $\mathcal{K}_1$  is  $\lambda$ -decomposable yet since  $\mathcal{K}_1$  is  $\kappa^+$ -complete,  $\mathcal{K}_0 \not\leq_{\text{RF}} \mathcal{K}_1$ . Therefore Theorem 7.5.13 implies that  $\lambda$  is not isolated. It follows that  $\kappa$  is  $<\lambda$ -supercompact. In particular, applying our results on GCH (namely Theorem 6.3.12), either  $\lambda$  is a successor cardinal or  $\lambda$  is a strongly inaccessible cardinal. Thus we are in a position to apply Corollary 8.3.1 and Corollary 8.3.3.

A weak consequence of the conjunction of these two theorems is that there is an elementary embedding  $j : V \rightarrow M$  such that  $\text{CRT}(j) > \kappa$ ,  $j(\lambda) > \lambda^{++M}$ , and  $j$  is  $\lambda$ -pseudocompact (or in other words,  $j$  is  $\gamma$ -tight for all  $\gamma \leq \lambda$ ). Since  $j(\kappa) = \kappa$  and  $j(\lambda) > \lambda^{++M}$ ,  $\kappa$  is  $\lambda^{++M}$ -supercompact in  $M$ . Thus by our results on GCH (Theorem 6.3.12) applied in  $M$ ,  $M$  satisfies that for all  $\gamma \in [\kappa, \lambda]$ ,  $2^\gamma = \gamma^+$ . But for all  $\gamma \leq \lambda$ , the  $\gamma$ -tightness of  $j$  implies that  $2^\gamma \leq (2^\gamma)^M$  (by Lemma 8.2.6), and hence

$$2^\gamma \leq (2^\gamma)^M \leq \gamma^{+M} \leq \gamma^+$$

as desired. □

**Definition 8.3.11.** Suppose  $\nu \leq \lambda$  are uncountable cardinals. The  $(\nu, \lambda)$ -*threshold* is the least ordinal  $\kappa$  such that for all  $\alpha < \lambda$ , there is an ultrapower embedding  $j : V \rightarrow M$  such that  $\text{CRT}(j) \geq \nu$  and  $j(\kappa) > \alpha$ .

The following theorem is proved in ZFC and has nothing to do with UA.

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<sup>2</sup>By the proof of the lemma, this hypothesis can be reformulated as the statement that there are distinct  $\lambda$ -strongly compact cardinals.

**Theorem 8.3.12.** *Suppose  $\kappa \leq \lambda$  are cardinals,  $\lambda$  is regular, and  $\kappa$  is the  $(\nu, \leq \lambda^+)$ -threshold. Assume  $2^\gamma = \gamma^+$  for all cardinals  $\gamma \in [\kappa, \lambda]$ . Then  $\kappa$  is  $(\nu, \lambda)$ -strongly compact.*

*Proof.* Let  $U$  be a  $\nu$ -complete ultrafilter such that  $j_U(\kappa) \geq \lambda^+$ . Suppose  $\gamma$  is a regular cardinal in the interval  $[\kappa, \lambda]$ . Suppose towards a contradiction that  $U$  is  $\gamma$ -indecomposable and  $\gamma^+$ -indecomposable. Since  $2^\gamma = \gamma^+$ , we can apply Silver's Theorem (Theorem 7.5.24). This yields an ultrafilter  $D$  with  $\lambda_D < \gamma$  such that there is an elementary embedding  $k : M_D \rightarrow M_U$  with  $k \circ j_D = j_U$  and  $\text{CRT}(k) > j_D(\gamma^+)$ . Since  $j_D(\kappa) \leq j_D(\gamma^+)$ ,

$$j_D(\kappa) = k(j_D(\kappa)) = j_U(\kappa) \geq \lambda^+$$

But  $j_D(\kappa) < (\kappa^{\lambda_D})^+ \leq (\gamma^{<\gamma})^+ = \gamma^+ \leq \lambda^+$ , which is a contradiction.

Therefore  $U$  is either  $\gamma$ -decomposable or  $\gamma^+$ -decomposable. But if  $U$  is  $\gamma^+$ -decomposable, then since  $\gamma$  is regular, in fact,  $U$  is  $\gamma$ -decomposable (by Prikry's Theorem [33], or the proof of Proposition 7.4.4). In particular, every regular cardinal in the interval  $[\kappa, \lambda]$  carries a  $\nu$ -complete uniform ultrafilter, which implies that  $\kappa$  is  $(\nu, \lambda)$ -strongly compact.  $\square$

Let us point out that this answers a question of Hamkins [17] assuming GCH. Hamkins defines a cardinal  $\kappa$  to be *strongly tall* if  $\kappa$  is the  $(\kappa, \text{Ord})$ -threshold, and asks about the relationship between strongly tall and strongly compact cardinals:

**Theorem 8.3.13** (GCH). *If  $\kappa$  is strongly tall, then  $\kappa$  is strongly compact.*  $\square$

**Theorem 8.3.14** (UA). *Suppose  $\lambda$  is a regular Fréchet cardinal. Suppose  $\kappa \leq \lambda$  is the  $(\nu, \leq \lambda^+)$ -threshold for some  $\nu > \kappa_\lambda$ . Then  $\kappa$  is  $\lambda$ -strongly compact.*

*Proof.* The following is the main claim:

**Claim.**  $\lambda$  is  $\nu$ -Fréchet.

*Sketch.* We first claim that there is some  $\nu$ -Fréchet cardinal in the interval  $[\lambda, 2^\lambda]$ . Assume towards a contradiction that this fails. Fix  $U$  such that  $j_U(\kappa) \geq \lambda^+$ . By Silver's Theorem (Theorem 7.5.24), there is an ultrafilter  $D$  with  $\lambda_D < \lambda$  such that there is an elementary

embedding  $k : M_D \rightarrow M_U$  with  $\text{CRT}(k) > j_D((2^\lambda)^+)$ . In particular,  $j_D(\kappa) \geq \lambda^+$ . In particular, it follows that  $\lambda$  is not isolated by Proposition 7.5.22. Let  $\gamma = \lambda_D$ . We claim that  $2^\gamma = \gamma^+$ . If  $\gamma$  is singular, this follows from Theorem 6.3.12: note that  $\gamma \in [\kappa_\lambda, \lambda]$  so some cardinal is  $\gamma$ -supercompact by Theorem 7.4.9, and hence  $2^\gamma = \gamma^+$  by Theorem 6.3.12. If  $\gamma$  is regular, then this follows from Lemma 8.3.10 since by Lemma 7.4.19,  $\kappa_\gamma \leq \kappa_\lambda \leq \nu$ . Thus  $2^\gamma = \gamma^+$  in either case. From this (and Theorem 6.3.12) it follows that  $\lambda^\gamma = \lambda$ . This contradicts that  $j_D(\lambda) \geq \lambda^+$ . Thus our assumption was false, so there is a  $\nu$ -Fréchet cardinal in the interval  $[\lambda, 2^\lambda]$ .

Now let  $\lambda'$  be the least  $\nu$ -Fréchet cardinal greater than or equal to  $\lambda$ . Suppose towards a contradiction that  $\lambda' > \lambda$ .

We claim  $\lambda'$  is an isolated cardinal. Clearly  $\lambda'$  is Fréchet. By the proof of Proposition 7.4.4,  $\lambda'$  is a limit cardinal. Finally,  $\lambda'$  is not a limit of Fréchet cardinals: otherwise by Corollary 7.5.2,  $\lambda'$  is a strong limit cardinal, contradicting that  $\lambda < \lambda' \leq 2^\lambda$ . Thus  $\lambda'$  is isolated, as claimed.

Theorem 7.5.13 implies  $\mathcal{K}_{\lambda'} \leq_{\text{RF}} \mathcal{K}_{\lambda'}^\nu$ , which implies that  $\mathcal{K}_{\lambda'}$  is  $\nu$ -complete, or in other words  $\kappa_{\lambda'} \geq \nu$ . Since  $\lambda \geq \kappa_{\lambda'}$ , Lemma 7.4.18 implies  $\mathcal{K}_{\lambda'} \not\sqsubseteq \mathcal{K}_\lambda$ . By the characterization of internal ultrapower embeddings of  $M_{\mathcal{K}_\lambda}$  (Theorem 7.3.14),  $\mathcal{K}_{\lambda'}$  must be discontinuous at  $\lambda$ . But this implies  $\lambda$  is  $\kappa_{\lambda'}$ -Fréchet, and hence  $\lambda$  is  $\nu$ -Fréchet. This contradicts our assumption that  $\lambda' > \lambda$  is the least  $\nu$ -Fréchet cardinal greater than or equal to  $\lambda$ .  $\square$

Since  $\lambda$  is  $\nu$ -Fréchet and  $\nu > \kappa_\lambda$ , we are in the situation of Lemma 8.3.10. Therefore for all cardinals  $\gamma \in [\kappa_\lambda, \lambda]$ ,  $2^\gamma = \lambda^+$ . This yields the cardinal arithmetic hypothesis of Theorem 8.3.12, so we can conclude that  $\kappa$  is the least  $(\nu, \lambda)$ -strongly compact cardinal. By Proposition 8.3.7, it follows that  $\kappa$  is  $\lambda$ -strongly compact.  $\square$

Of course, if one works below a strong limit cardinal, one obtains the complete generalization of Theorem 7.4.26:

**Corollary 8.3.15** (UA). *If  $\lambda$  is a strong limit cardinal and  $\kappa < \lambda$  is the  $(\nu, \lambda)$ -threshold, then  $\kappa$  is  $\gamma$ -strongly compact for all  $\gamma < \lambda$ . Therefore one of the following holds:*

- $\kappa$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ .
- $\kappa$  is a measurable limit of cardinals that are  $\gamma$ -supercompact for all  $\gamma < \lambda$ .

*Proof.* Let  $\kappa_0$  be the  $\lambda$ -threshold. By Theorem 7.4.26,  $\kappa_0$  is  $<\lambda$ -supercompact. If  $\nu \leq \kappa_0$ , then  $\kappa_0$  is the  $(\nu, \lambda)$ -threshold, so  $\kappa = \kappa_0$ , which proves the corollary.

Therefore assume  $\nu > \kappa$ . Suppose  $\delta \in [\kappa, \lambda]$  is a regular cardinal. By the proof of Theorem 7.4.26,  $\kappa_0 = \kappa_\delta$ . Moreover  $\kappa$  is the  $(\nu, \delta^+)$ -threshold by Lemma 7.4.25. Therefore we can apply Theorem 8.3.14 to obtain that  $\kappa$  is  $\delta$ -strongly compact.

The final two bullet points are immediate from Corollary 8.3.8. Suppose  $\kappa$  is not  $\delta$ -supercompact for some  $\delta < \lambda$ . By Corollary 8.3.8,  $\kappa$  is a measurable limit of  $\gamma$ -supercompact cardinals for all  $\gamma \in [\delta, \lambda)$ . Now suppose  $\kappa_0 < \kappa$  is  $\kappa$ -supercompact. We claim  $\kappa_0$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ . Fix  $\gamma < \lambda$ . There is some  $\kappa_1 \in (\kappa_0, \kappa]$  that is  $\gamma$ -supercompact. But  $\kappa_0$  is  $\kappa_1$ -supercompact, so in fact,  $\kappa_0$  is  $\gamma$ -supercompact, as desired.  $\square$

## Level-by-level equivalence at singular cardinals

A well-known theorem of Apter-Shelah [35] shows the consistency of *level-by-level equivalence* of strong compactness and supercompactness: it is consistent with very large cardinals that for all regular  $\lambda$ , a cardinal  $\kappa$  is  $\lambda$ -strongly compact if and only if it is  $\lambda$ -supercompact or a measurable limit of  $\lambda$ -supercompact cardinals. (By Corollary 8.1.6, this is best possible.) We showed this is a consequence of UA assuming  $\lambda$  is a successor cardinal; when  $\lambda$  is inaccessible, we ran into the usual problems.

When  $\lambda$  is singular, level-by-level equivalence is in general false. This is a consequence of the following observation:

**Lemma 8.3.16.** *Suppose  $\kappa \leq \lambda$  are cardinals.*

- If  $\text{cf}(\lambda) < \kappa$ , then  $\kappa$  is  $\lambda$ -strongly compact if and only if  $\kappa$  is  $\lambda^+$ -strongly compact.
- If  $\kappa \leq \text{cf}(\lambda) < \lambda$ , then  $\kappa$  is  $\lambda$ -strongly compact if and only if  $\kappa$  is  $<\lambda$ -strongly compact.

The first bullet point shows that if level-by-level equivalence holds at successor cardinals, it also holds at singular cardinals of small cofinality. But by the second bullet point, it need not hold at singular cardinals of larger cofinality:

**Proposition 8.3.17.** *Suppose  $\kappa$  is the least cardinal  $\delta$  that  $\beth_\delta(\delta)$ -strongly compact. Then  $\kappa$  is not  $\beth_\kappa(\kappa)$ -supercompact.*

*Proof.* In fact, if  $\delta$  is  $\beth_\delta(\delta)$ -supercompact, then  $\delta$  is a limit of cardinals  $\bar{\delta} < \delta$  that are  $\beth_{\bar{\delta}}(\bar{\delta})$ -strongly compact. To see this, let  $j : V \rightarrow M$  be an elementary embedding such that  $\text{CRT}(j) = \delta$ ,  $j(\delta) > \beth_\delta(\delta)$ , and  $M^{\beth_\delta(\delta)} \subseteq M$ . Then  $\delta$  is  $<\beth_\delta(\delta)$ -supercompact in  $M$ . It follows from Lemma 8.3.16 that  $\delta$  is  $\beth_\delta(\delta)$ -strongly compact in  $M$ . Therefore by the usual reflection argument,  $\delta$  is a limit of cardinals  $\bar{\delta} < \delta$  that are  $\beth_{\bar{\delta}}(\bar{\delta})$ -strongly compact.  $\square$

Upon further thought, however, Proposition 8.3.17 *does not* rule out that a version of level-by-level equivalence that holds at singular cardinals, but rather shows that the conventional localization of strong compactness degenerates at singular cardinals of large cofinality. We therefore introduce an alternate localization of strong compactness:

**Definition 8.3.18.** A cardinal  $\kappa$  is  $\lambda$ -club compact if there is a  $\kappa$ -complete ultrafilter on  $P_\kappa(\lambda)$  that extends the closed unbounded filter.

If  $\kappa$  is  $\lambda$ -supercompact, then  $\kappa$  is  $\lambda$ -club compact: a normal fine ultrafilter always extends the closed unbounded filter. On the other hand, if every  $\kappa$ -complete filter on  $P_\kappa(\lambda)$  extends to a  $\kappa$ -complete ultrafilter, then in particular, the closed unbounded filter on  $P_\kappa(\lambda)$  extends to a  $\kappa$ -complete ultrafilter, so  $\kappa$  is  $\lambda$ -club compact.

**Question 8.3.19 (ZFC).** Suppose  $\lambda$  is a regular cardinal and  $\kappa$  is  $\lambda$ -strongly compact. Must  $\kappa$  be  $\lambda$ -club compact?

To state stronger results, we introduce the Bagaria-Magidor versions of club compactness as well:

**Definition 8.3.20.** A cardinal  $\kappa$  is  $(\nu, \lambda)$ -club compact if there is a  $\nu$ -complete ultrafilter on  $P_\kappa(\lambda)$  that extends the closed unbounded filter.  $\kappa$  is *almost  $\lambda$ -club compact* if  $\kappa$  is  $(\nu, \lambda)$ -club compact for all  $\nu < \kappa$ .

As is typical in the Bagaria-Magidor notation, if  $\kappa$  is  $(\nu, \lambda)$ -club compact, then every cardinal greater than  $\kappa$  is  $(\nu, \lambda)$ -club compact.

Menas's Theorem (Corollary 8.1.6) carries over to club compactness:

**Lemma 8.3.21.** *Suppose  $\lambda$  is a cardinal. Any limit of  $\lambda$ -club compact cardinals is almost  $\lambda$ -club compact. An almost  $\lambda$ -club compact cardinal is  $\lambda$ -club compact if and only if it is measurable. Thus every measurable limit of  $\lambda$ -club compact cardinals is  $\lambda$ -club compact.  $\square$*

The main theorem of this section is that under UA, level-by-level equivalence holds for club compactness at singular cardinals.

**Theorem 8.3.22 (UA).** *Suppose  $\kappa \leq \lambda$  are cardinals and  $\lambda$  is singular. Then the following are equivalent:*

- (1)  $\kappa$  is  $\lambda$ -club compact.
- (2)  $\kappa$  is the least  $(\nu, \lambda)$ -club compact cardinal for some  $\nu \leq \kappa$ .
- (3)  $\kappa$  is  $\lambda$ -supercompact or a measurable limit of  $\lambda$ -supercompact cardinals.

For the proof, we use the following much more general lemma:

**Definition 8.3.23.** The *Katětov order* is defined on filters  $F$  and  $G$  by setting  $F \leq_{\text{Kat}} G$  if there is a function  $f$  on a set in  $G$  such that  $F \subseteq f_*(G)$ .

Thus  $F \leq_{\text{Kat}} G$  if and only if there is an extension  $F'$  of  $F$  below  $G$  in the Rudin-Keisler order.

**Lemma 8.3.24** (UA). *Suppose  $\nu < \lambda$  are cardinals. Suppose  $\mathcal{F}$  is a normal fine filter on a set  $Y$  such that  $\lambda \subseteq Y \subseteq P(\lambda)$ . Suppose  $A$  is a set of ordinals and  $U$  is the  $<_{\mathbb{k}}$ -least  $\nu^+$ -complete ultrafilter on  $A$  such that  $\mathcal{F} \leq_{\text{Kat}} U$ . Then  $U$  is  $\lambda$ -irreducible.*

*Proof.* Suppose  $D \leq_{\text{RF}} U$  and  $\lambda_D < \lambda$ . We must show that  $D$  is principal. To do this, we will show that  $j_D(U) \leq_{\mathbb{k}} t_D(U)$  in  $M_D$ . By Proposition 5.4.5, it then follows that  $D$  is principal. As usual, to show  $j_D(U) \leq_{\mathbb{k}} t_D(U)$  in  $M_D$ , we verify that  $t_D(U)$  satisfies the properties for which  $U$  was minimized hold for  $t_D(U)$  with the parameters shifted by  $j_D$ . In other words, we show that  $M_D$  satisfies the following:

- $t_D(U)$  is a  $j_D(\nu^+)$ -complete ultrafilter on  $j_D(A)$ .
- $j_D(\mathcal{F}) \leq_{\text{Kat}} t_D(U)$ .

The first bullet point is rather easy. By definition,  $t_D(U)$  is an ultrafilter on  $j_D(A)$ . Moreover,  $t_D(U)$  is  $j_D(\nu^+)$ -complete in  $M_D$  since

$$\text{CRT}(k) \geq \text{CRT}(j) > \nu = j(\nu) \geq j_D(\nu)$$

The second bullet point is a bit more subtle. Since  $\mathcal{F} \leq_{\text{Kat}} U$ , there is some  $B \in M_U$  such that  $\mathcal{F}$  is contained in the ultrafilter derived from  $j_U$  using  $B$ . In other words, for all  $S \in \mathcal{F}$ ,  $B \in j_U(S)$ . Note that for any  $f : \lambda \rightarrow \lambda$ ,  $B$  is closed under  $j_U(f)$ : by normality,  $\{\sigma \in Y : \sigma \text{ is closed under } f\} \in \mathcal{F}$ , and hence  $B \in j_U(\{\sigma \in Y : \sigma \text{ is closed under } f\})$ , or in other words,  $B$  is closed under  $j_U(f)$ . We will use this fact in an application of Lemma 6.3.11.

Let  $k : M_D \rightarrow M$  be the unique internal ultrapower embedding with  $k \circ j_D = j_U$ . Thus  $k$  is the ultrapower of  $M_D$  by  $t_D(U)$ . Let

$$\mathcal{W} = \{S \in j_D(P(Y)) : B \in k(S)\}$$

Thus  $\mathcal{W}$  is the  $M_D$ -ultrafilter on  $j_D(Y)$  derived from  $k$  using  $B$ . In particular,  $\mathcal{W} \leq_{\text{RK}} t_D(U)$  by the characterization of the Rudin-Keisler order in terms of derived embeddings

(Lemma 3.4.4). We claim that  $j_D(\mathcal{F}) \subseteq \mathcal{W}$ . Clearly  $j_D[\mathcal{F}] \subseteq \mathcal{W}$ . The key point is that by Lemma 6.3.11,  $k(a_D) \in B$ . In other words,

$$\{\sigma \in j_D(Y) : a_D \in \sigma\} \in \mathcal{W}$$

Therefore by our unique extension lemma for normal filters (Lemma 8.2.11),  $j_D(\mathcal{F}) \subseteq \mathcal{W}$ , as desired.

Now  $j_D(\mathcal{F}) \subseteq \mathcal{W} \leq_{\text{RK}} t_D(U)$ , or in other words  $j_D(\mathcal{F}) \leq_{\text{Kat}} t_D(U)$ . □

*Proof of Theorem 8.3.22. (1) implies (2):* Trivial.

*(2) implies (3):* Clearly  $\lambda$  is a limit of Fréchet cardinals, so by Corollary 7.5.2,  $\lambda$  is a strong limit cardinal.

We first handle the case in which there is some  $\nu < \kappa$  such that  $\kappa$  is the least  $(\nu, \lambda)$ -club compact cardinal. Note that  $\nu$  is either not measurable or not almost  $\lambda$ -club compact, since otherwise  $\nu$  would be the least  $(\nu, \lambda)$ -club compact cardinal. If  $\nu$  is not almost  $\lambda$ -club compact, then there is some  $\bar{\nu} < \nu$  such that  $\kappa$  is the least  $(\bar{\nu}^+, \lambda)$ -club compact cardinal. If  $\nu$  is not measurable, then  $\kappa$  is the least  $(\nu^+, \lambda)$ -club compact cardinal. In either case, we can fix  $\eta < \kappa$  such that  $\kappa$  is the least  $(\eta^+, \lambda)$ -club compact cardinal.

Let  $\mathcal{F}$  be the closed unbounded filter on  $P_\kappa(\lambda)$ . Let  $U$  be the least  $\eta^+$ -complete ultrafilter on an ordinal such that  $\mathcal{F} \leq_{\text{Kat}} U$ . Then  $U$  is  $\lambda$ -irreducible. Since  $\lambda$  is a singular strong limit cardinal, by Corollary 8.2.20,  $(M_U)^\lambda \subseteq M_U$ . Thus  $\text{CRT}(j_U)$  is  $\lambda$ -supercompact. Note that  $\text{CRT}(j_U) \leq \kappa$  since  $\mathcal{F} \leq_{\text{Kat}} U$  and  $\mathcal{F}$  is not  $\kappa^+$ -complete. On the other hand  $\text{CRT}(j_U) > \eta$ , so  $\text{CRT}(j_U)$  is an  $(\eta^+, \lambda)$ -club compact cardinal, and hence  $\text{CRT}(j_U) \leq \kappa$ . Thus  $\kappa = \text{CRT}(j_U)$  is  $\lambda$ -supercompact.

We now handle the case in which  $\kappa$  is  $(\kappa, \lambda)$ -club compact but there is no  $\nu < \kappa$  such that  $\kappa$  is the least  $(\nu, \lambda)$ -club compact cardinal. Since  $\kappa$  is  $(\nu, \lambda)$ -club compact for all  $\nu < \kappa$ , it follows that for each  $\nu < \lambda$ , the least  $(\nu, \lambda)$ -club compact cardinal lies strictly below  $\kappa$ . Thus by the previous case,  $\kappa$  is a limit of  $\lambda$ -supercompact cardinals. Moreover,  $\kappa$  is measurable



since  $\kappa$  is  $(\kappa, \lambda)$ -club compact. Thus  $\kappa$  is a measurable limit of  $\lambda$ -club compact cardinals, as desired.

(3) *implies (1)*: This follows from Lemma 8.3.21. □

## The Mitchell order, the internal relation, and coherence

Assume UA and suppose  $U$  is a normal ultrafilter on  $\kappa$ . Can  $P(\kappa^+) \subseteq M_U$ ? The question remains open in general, but the following theorem shows that if  $\kappa^+$  is Fréchet, this cannot occur:

**Theorem 8.3.25** (UA). *Suppose  $\lambda$  is a Fréchet cardinal. Suppose  $U$  is a countably complete ultrafilter such that  $P(\lambda) \subseteq M_U$ . Then  $(M_U)^\lambda \subseteq M_U$ .*

*Proof.* Assume by induction that the theorem holds for cardinals below  $\lambda$ . If  $\lambda$  is a limit of Fréchet cardinals, we then have  $(M_U)^{<\lambda} \subseteq M_U$ . In particular, if  $\lambda$  is a singular limit of Fréchet cardinals, then  $(M_U)^\lambda \subseteq M_U$ . Thus we may assume that  $\lambda$  is either regular or isolated. This puts the analysis of  $\mathcal{K}_\lambda$  (especially Theorem 7.3.14 and Proposition 7.4.17) at our disposal.

We first show that  $U$  is  $\lambda$ -irreducible. Suppose towards a contradiction that there is a uniform ultrafilter  $D \leq_{\text{RF}} U$  on an infinite cardinal  $\gamma < \lambda$ . Since  $M_U \subseteq M_D$ , so in particular  $P(\lambda) \subseteq M_D$ . A general bound on the strength of ultrapowers (Lemma 4.2.41) implies that

$$\lambda < j_D(\gamma)$$

Assume first that  $\lambda$  is isolated. By Proposition 7.4.17,  $D \sqsubset \mathcal{K}_\lambda$ , and by Proposition 7.5.20,  $P(\gamma) \subseteq M_{\mathcal{K}_\lambda}$ . Thus

$$P(\lambda) \subseteq j_D(P(\lambda_D)) \subseteq M_{\mathcal{K}_\lambda}$$

Therefore by our bound on the strength of  $j_{\mathcal{K}_\lambda}$  for nonmeasurable isolated cardinals  $\lambda$  (Proposition 7.5.19),  $\lambda$  is measurable. Since  $\lambda$  is a strong limit,  $D \in H(\lambda) \subseteq M_D$ , and this is a contradiction.

Assume instead that  $\lambda$  is a nonisolated regular cardinal. We use an argument similar to the one from the local proof of GCH (Theorem 6.3.12). Let  $M = M_{\mathcal{K}_\lambda}$  and let  $N = (M_D)^M$ . Consider the embedding  $j_{\mathcal{K}_\lambda}^N \circ j_D^M$ . (Note:  $j_{\mathcal{K}_\lambda}^N$  denotes the ultrapower formed by using functions in  $N$  modulo the  $N$ -ultrafilter  $\mathcal{K}_\lambda$ , *not* the ultrafilter  $(\mathcal{K}_\lambda)^N$ , which we have not proved to exist.) This is an ultrapower embedding from  $M$ , and we claim that it is internal to  $M$ . By our analysis of internal ultrapower embeddings of  $M$  (Theorem 7.3.14), it suffices to show that  $j_{\mathcal{K}_\lambda}^N \circ j_D^M$  is continuous at  $\text{cf}^M(\sup j[\lambda]) = \lambda$ . (To compute the cofinality of  $\sup j[\lambda]$  in  $M$ , we use Proposition 7.4.11.) Clearly  $j_D^M(\lambda) = \sup j_D^M[\lambda]$  since  $\lambda$  is regular and  $D$  lies on  $\gamma < \lambda$ . Moreover  $j_D^M(\lambda)$  is regular in  $N$  and is larger than  $\lambda$  since  $j_D^M(\gamma) = j_D(\gamma) > \lambda$ . Thus  $j_{\mathcal{K}_\lambda}^N(j_D^M(\lambda)) = \sup j_{\mathcal{K}_\lambda}^N[j_D^M(\lambda)]$ . Putting it all together,

$$j_{\mathcal{K}_\lambda}^N \circ j_D^M(\lambda) = \sup j_{\mathcal{K}_\lambda}^N[j_D^M(\lambda)] = \sup j_{\mathcal{K}_\lambda}^N[\sup j_D^M[\lambda]] = \sup j_{\mathcal{K}_\lambda}^N \circ j_D^M[\lambda]$$

Thus  $j_{\mathcal{K}_\lambda}^N \circ j_D^M$  is an internal ultrapower embedding of  $M$ .

In fact,  $j_{\mathcal{K}_\lambda}^N$  itself is definable over  $M$ : for any  $f \in M^\gamma$ ,

$$j_{\mathcal{K}_\lambda}^N([f]_D^M) = j_{\mathcal{K}_\lambda}^N \circ j_D^M(f)(a_{\mathcal{K}_\lambda}^N)$$

Thus  $j_{\mathcal{K}_\lambda}^N$  is definable over  $M$ . Since  $P(\lambda) \subseteq N$ , we have  $\mathcal{K}_\lambda = \{A \subseteq \lambda : a_{\mathcal{K}_\lambda}^N \in j_{\mathcal{K}_\lambda}^N(A)\}$ .

Thus  $\mathcal{K}_\lambda$  is definable over  $M$ , and it follows that  $\mathcal{K}_\lambda \in M$ , or in other words,  $\mathcal{K}_\lambda \triangleleft \mathcal{K}_\lambda$ .

This is a contradiction.

Thus our assumption was false, and in fact  $U$  is  $\lambda$ -irreducible.

To finish the proof, we break once again into cases.

Suppose first that  $\lambda$  is a nonmeasurable isolated cardinal. We will show that  $U$  is  $\lambda^+$ -complete. We claim that  $\mathcal{K}_\lambda \not\leq_{\text{RF}} U$ : otherwise,  $P(\lambda) \subseteq M_U \subseteq M_{\mathcal{K}_\lambda}$ , and hence  $\mathcal{K}_\lambda$  is  $\lambda$ -complete by Proposition 7.5.19, contradicting that  $\lambda$  is not measurable. Since  $\mathcal{K}_\lambda \not\leq_{\text{RF}} U$ , our factorization theorem for isolated cardinals (Theorem 7.5.13) implies that  $U$  is  $\lambda^+$ -irreducible. Therefore by Theorem 7.5.31,  $U$  is  $\lambda^+$ -complete, as claimed.

If  $\lambda$  is not a nonmeasurable isolated cardinal, then  $\lambda$  is either a Fréchet successor cardinal or a Fréchet inaccessible cardinal. Since  $U$  is  $\lambda$ -irreducible, the Irreducibility Theorem

(Corollary 8.2.19 and Corollary 8.2.20) implies that  $j_U[\lambda]$  is contained in a set  $A \in M_U$  such that  $|A|^{M_U} = \lambda$ . Since  $P(\lambda) \subseteq M_U$  and  $|A|^{M_U} = \lambda$ , in fact  $P(A) \subseteq M_U$ . In particular, the subset  $j_U[\lambda] \subseteq A$  belongs to  $M_U$ , so  $j_U$  is  $\lambda$ -supercompact, and hence  $(M_U)^\lambda \subseteq M_U$ .  $\square$

A consequence of the coincidence of strength and supercompactness at Fréchet cardinals is that under UA, the generalized Mitchell order is very well-behaved.

**Theorem 8.3.26** (UA). *Suppose  $U$  and  $W$  are countably complete ultrafilters such that  $U \triangleleft W$ . Then  $(j_U)^{M_W} = j_U \upharpoonright M_W$ . In fact,  $(M_W)^{\lambda_U} \subseteq M_W$ .*

*Proof.* Let  $\lambda = \lambda_U$ . Fix  $A \in U$  with  $|A| = \lambda$ . Since  $U \in M_W$ ,  $P(A) \subseteq M_W$ , and hence  $P(\lambda) \subseteq M_W$ . Since  $\lambda = \lambda_U$ ,  $\lambda$  is Fréchet. Hence  $(M_W)^\lambda \subseteq M_W$  by Theorem 8.3.25. By Proposition 4.2.29, this implies  $(j_U)^{M_W} = j_U \upharpoonright M_W$ .  $\square$

As a consequence, UA implies that the internal relation and the seed order extend the Mitchell order:

**Corollary 8.3.27** (UA). *Suppose  $U$  and  $W$  are countably complete ultrafilters such that  $U \triangleleft W$ . Then  $U \sqsubset W$ . Assume moreover that  $\lambda_U$  is the underlying set of  $U$  and  $W$  concentrates on ordinals. Then  $U <_S W$ .*

*Proof.* By Theorem 8.3.26,  $U \sqsubset W$ . Moreover,  $j_W$  is  $\lambda_U$ -supercompact, so by Proposition 4.2.31,  $\lambda_U \leq \lambda_W$ . Thus if  $\lambda_U$  is the underlying set of  $U$  and  $W$  concentrates on ordinals, then

$$\delta_U = \lambda_U \leq \lambda_W \leq \delta_W$$

Therefore by Theorem 5.5.15, we have  $U <_S W$ .  $\square$

Using the Irreducibility Theorem, we prove some converses of Corollary 8.3.27 that demystify the internal relation. This requires an argument we have seen before but which we now make explicit:

**Lemma 8.3.28.** *Suppose  $W$  is a countably complete ultrafilter such that  $j_W$  is  $<\lambda$ -strong and  $\lambda$ -tight.<sup>3</sup> Suppose that there is a countably complete ultrafilter  $U$  on  $\lambda$  such that  $U \sqsubset W$  and  $\sup j_U[\lambda] < j_U(\lambda)$ . Then  $j_W$  is  $\lambda$ -supercompact.*

*Proof.* We first show that  $P(\lambda) \subseteq M_W$ . Since  $W$  is  $<\lambda$ -strong,  $P(\alpha) \subseteq M_W$  for all  $\alpha < \lambda$ . Therefore by the elementarity of  $j_U$ ,  $M_U$  satisfies that  $P(\sup j_U[\lambda]) \subseteq j_U(M_W)$ . In other words,  $P^{M_U}(\sup j_U[\lambda]) \subseteq j_W(M_U)$ . Since  $U \sqsubset W$ ,  $j_U(M_W) \subseteq M_W$ , and therefore  $P^{M_U}(\sup j_U[\lambda]) \subseteq M_W$ . Now fix  $A \subseteq \lambda$ . We have  $j_U(A) \cap \sup j_U[\lambda] \in P^{M_U}(\sup j_U[\lambda]) \subseteq M_W$ . Moreover  $j_U \upharpoonright \lambda \in M_W$  since  $U \sqsubset W$ . Hence

$$A = j_U^{-1}[j_U(A) \cap \sup j_U[\lambda]] \in M_W$$

This shows that  $P(\lambda) \subseteq M_W$ , as claimed.

Now suppose  $B$  is a subset of  $M_W$  of cardinality at most  $\lambda$ . Since  $j_W$  is  $\lambda$ -tight, there is a set  $C \in M_W$  of  $M_W$ -cardinality at most  $\lambda$  such that  $B \subseteq C$ . Since  $P(\lambda) \subseteq M_W$  and  $|C|^{M_W} \leq \lambda$ ,  $P(C) \subseteq M_W$ . Thus  $B \in M_W$ . It follows that  $j_W$  is  $\lambda$ -supercompact.  $\square$

**Theorem 8.3.29 (UA).** *Suppose  $W$  is a countably complete ultrafilter and  $U$  is a countably complete uniform ultrafilter on a set  $X \subseteq M_W$ . Then the following are equivalent:*

- (1)  $U \triangleleft W$ .
- (2)  $U \sqsubset W$  and  $W$  is  $|X|$ -irreducible.

*Proof.* Let  $\lambda = \lambda_U = |X|$ .

(1) *implies* (2): Suppose  $U \triangleleft W$ . Then  $j_W$  is  $\lambda$ -supercompact by Theorem 8.3.25, so  $W$  is  $\lambda$ -irreducible by Proposition 8.2.3. Moreover by Corollary 8.3.27,  $U \sqsubset W$ . This shows that (2) holds.

(2) *implies* (1): Suppose  $U \sqsubset W$  and  $W$  is  $\lambda$ -irreducible.

Suppose first that  $\lambda$  is an isolated cardinal. We claim that  $W$  is  $\lambda^+$ -complete. Note that  $j_W$  must be continuous at  $\lambda$  by Lemma 7.4.14. It follows that  $W$  is  $\lambda^+$ -irreducible. Hence  $W$

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<sup>3</sup>Equivalently,  $j_W$  is  $<\lambda$ -supercompact and  $\lambda$ -tight.

is  $\lambda^\sigma$ -irreducible. But  $\lambda^\sigma$  is measurable (by Lemma 7.4.27), so by Theorem 7.5.32 it follows that  $W$  is  $\lambda^+$ -complete. As an immediate consequence,  $U \triangleleft W$ .

Suppose instead that  $\lambda$  is not isolated. We can then apply the Irreducibility Theorem (Corollary 8.2.19 and Corollary 8.2.20) to conclude that  $W$  is  $<\lambda$ -supercompact and  $\lambda$ -tight. Since  $U \sqsubset W$ , Lemma 8.3.28 yields that  $j_W$  is  $\lambda$ -supercompact. In particular,  $P(\lambda) \subseteq M_W$ , so  $U \triangleleft W$ , as desired.  $\square$

We can reformulate Theorem 8.3.29 slightly to characterize the internal relation in terms of the Mitchell order:

**Theorem 8.3.30 (UA).** *Suppose  $U$  and  $W$  are hereditarily uniform irreducible ultrafilters. Then the following are equivalent:*

- (1)  $U \sqsubset W$ .
- (2) Either  $U \triangleleft W$  or  $W \in V_\kappa$  where  $\kappa = \text{CRT}(j_U)$ .

For this, we need the following theorem, which shows that the notions of  $\lambda$ -irreducible,  $\lambda$ -Mitchell, and  $\lambda$ -internal ultrafilters (Definition 7.5.30, Definition 6.3.8, Definition 5.5.25 respectively) coincide under UA:

**Theorem 8.3.31 (UA).** *Suppose  $U$  is an ultrafilter and  $\lambda$  is a cardinal. Then the following are equivalent:*

- (1)  $U$  is  $\lambda$ -irreducible.
- (2)  $U$  is  $\lambda$ -Mitchell.
- (3)  $U$  is  $\lambda$ -internal.

*Proof.* (1) implies (2): Assume  $U$  is  $\lambda$ -irreducible. We may assume by induction that for all  $U' \leq_{\mathbb{k}} U$  and  $\lambda' \leq \lambda$  with  $U' <_{\mathbb{k}} U$  or  $\lambda' < \lambda$ , if  $U'$  is  $\lambda'$ -irreducible then  $U'$  is  $\lambda'$ -Mitchell. Thus  $U$  is  $\lambda'$ -Mitchell for all  $\lambda' < \lambda$ . In particular,  $U$  is automatically  $\lambda$ -Mitchell unless  $\lambda$  is

a successor cardinal and the cardinal predecessor  $\gamma$  of  $\lambda$  is Fréchet. Therefore we can assume  $\lambda = \gamma^+$  where  $\gamma$  is a Fréchet cardinal.

We may also assume that  $\gamma^\sigma$  exists, since otherwise the  $\lambda$ -irreducibility of  $U$  implies  $U$  is principal, so (2) holds automatically. Let  $\eta = \gamma^\sigma$ .

Assume first that  $\eta = \gamma^+$ . Then  $\gamma^+$  is Fréchet, so by the Irreducibility Theorem (Corollary 8.2.19),  $U$  is  $\gamma^+$ -supercompact. Therefore every countably complete ultrafilter on  $\gamma$  belongs to  $M_U$  by Proposition 6.3.9. In other words,  $U$  is  $\gamma^+$ -Mitchell.

This leaves us with the case that  $\eta > \gamma^+$ . In other words, by Proposition 7.4.4,  $\eta$  is isolated.

Assume first that  $\mathcal{K}_\eta \not\leq_{\text{RF}} U$ . Then by Theorem 7.5.13,  $U$  is  $\eta$ -indecomposable, and so in particular  $U$  is  $\eta^+$ -irreducible. By Theorem 7.5.31 (3),  $U$  is  $\eta^+$ -complete, which easily implies that  $U$  is  $\gamma^+$ -Mitchell.

Assume finally that  $\mathcal{K}_\eta \leq_{\text{RF}} U$ . Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\eta$ . Let  $h : M \rightarrow M_U$  be the unique internal ultrapower embedding with  $h \circ j = j_U$ .

Recall that  $t_{\mathcal{K}_\eta}(U)$  is the canonical ultrafilter  $Z$  of  $M$  such that  $j_Z^M = h$ . We claim that  $t_{\mathcal{K}_\eta}(U)$  is  $\gamma^+$ -irreducible in  $M$ . Suppose  $M$  satisfies that  $D$  is an ultrafilter on  $\gamma$  with  $D \leq_{\text{RF}} t_{\mathcal{K}_\eta}(U)$ . Let  $i : (M_D)^M \rightarrow M_U$  be the unique internal ultrapower embedding such that

$$i \circ j_D^M = h$$

We will show  $D$  is principal by showing that  $D \leq_{\text{RF}} U$ . By Proposition 7.5.20,  $M$  is closed under  $\gamma$ -sequences. In particular,  $P(\gamma) \subseteq M$ , so  $D$  really is an ultrafilter on  $\gamma$ , and hence the question of whether  $D \leq_{\text{RF}} U$  makes sense. Moreover  $j_D \upharpoonright M = j_D^M$ , and so  $j_D^M \circ j = j_D(j) \circ j_D$ . Now

$$i \circ j_D(j) \circ j_D = i \circ j_D^M \circ j = h \circ j = j_U$$

Thus  $i \circ j_D(j) : M_D \rightarrow M_U$  is an internal ultrapower embedding witnessing  $D \leq_{\text{RF}} U$ . It follows that  $D$  is principal since  $U$  is  $\gamma^+$ -irreducible.

Thus  $t_{\mathcal{X}_\eta}(U)$  is  $\gamma^+$ -irreducible in  $M$ . Moreover by Proposition 5.4.5,  $t_{\mathcal{X}_\eta}(U) <_{\mathbf{k}} j(U)$  in  $M$ . Our induction hypothesis yields that for all  $U' <_{\mathbf{k}} U$  and all  $\lambda' \leq \gamma^+$ , if  $U'$  is  $\lambda'$ -irreducible then  $U'$  is  $\lambda'$ -Mitchell. Shifting this hypothesis by the elementary embedding  $j : V \rightarrow M$ , we have that for all  $U' <_{\mathbf{k}} j(U)$  and all  $\lambda' \leq j(\gamma^+)$ , if  $U'$  is  $\lambda'$ -irreducible in  $M$  then  $U'$  is  $\lambda'$ -Mitchell in  $M$ . Applying this with  $U' = t_{\mathcal{X}_\lambda}(U)$  and  $\lambda' = \gamma^+$ , it follows that  $t_{\mathcal{X}_\lambda}(U)$  is  $\gamma^+$ -Mitchell in  $M$ . Thus every countably complete ultrafilter of  $M$  on  $\gamma$  belongs to  $(M_{t_{\mathcal{X}_\eta}(U)})^M = M_U$ . But by Proposition 7.4.17 and Proposition 7.5.20, every countably complete ultrafilter on  $\gamma$  belongs to  $M$ . Hence every countably complete ultrafilter on  $\gamma$  belongs to  $M_U$ . In other words,  $U$  is  $\gamma^+$ -Mitchell as desired.

(2) *implies* (3): Immediate from Corollary 8.3.27.

(3) *implies* (4): Assume  $U$  is  $\lambda$ -internal. Suppose  $D \leq_{\text{RF}} U$  and  $\lambda_D < \lambda$ . We will show  $D$  is principal. Since  $\lambda_D < \lambda$ ,  $D \sqsubset U$ . Thus  $D \leq_{\text{RF}} U \sqsupset D$ , so  $D \sqsubset D$  by Proposition 5.5.12. Since the internal relation is irreflexive on nonprincipal ultrafilters,  $D$  is principal.  $\square$

*Proof of Theorem 8.3.30.* (1) *implies* (2): Suppose  $U \sqsubset W$ .

Assume first that  $\lambda_U \leq \lambda_W$ . Then since  $W$  is irreducible,  $W$  is  $\lambda_U$ -irreducible. By Theorem 8.3.29,  $U \triangleleft W$ .

Assume instead that  $\lambda_W < \lambda_U$ . Then by Theorem 8.3.31,  $W \sqsubset U$ . Since  $U \sqsubset W$  and  $W \sqsubset U$ , Theorem 5.5.22 implies that  $U$  and  $W$  are commuting ultrafilters in the sense of Kunen's commuting ultrapowers lemma (Theorem 5.5.20). Moreover, again by Theorem 8.3.31,  $U$  is  $\lambda_U$ -internal and  $W$  is  $\lambda_W$ -internal. We can therefore apply our converse to Kunen's commuting ultrapowers lemma, from which it follows that  $W \in V_\kappa$  where  $\kappa = \text{CRT}(j_U)$ .

(2) *implies* (1): If  $U \triangleleft W$ , then  $U \sqsubset W$  by Corollary 8.3.27. If  $W \in V_\kappa$  where  $\kappa = \text{CRT}(j_U)$ , then  $U \sqsubset W$  by Kunen's commuting ultrapowers lemma (Theorem 5.5.20).  $\square$

We now reformulate UA in terms of a form of coherence:

**Definition 8.3.32.** Suppose  $C$  is a class of countably complete ultrafilters.

- Suppose  $\mathcal{I} = \langle M_n, j_{nm}, U_n : n < m \leq \ell \rangle$  is a finite iterated ultrapower.
  - A countably complete ultrafilter  $U$  is *given by*  $\mathcal{I}$  if  $j_U = j_{0\ell}$ .
  - $\mathcal{I}$  is a *C-iteration* if  $U_n \in j_{0n}(C)$  for all  $n < \ell$ .
- $C$  is *cofinal* if the class of ultrafilters given by  $C$ -iterations is Rudin-Frolík cofinal.
- $C$  is *coherent* if for any distinct ultrafilters  $U$  and  $W$  of  $C$ , either  $U \in j_W(C)$  and  $(M_W)^{\lambda_U} \subseteq M_W$ , or  $W \in j_U(C)$  and  $(M_U)^{\lambda_W} \subseteq M_U$ .

**Theorem 8.3.33.** *The following are equivalent:*

- (1) *There is a coherent cofinal class of countably complete ultrafilters.*
- (2) *The Ultrapower Axiom holds.*

For one direction of the theorem, we show that under UA, there is a canonical coherent cofinal class of ultrafilters:

**Definition 8.3.34.** An ultrafilter  $D$  is a *Mitchell point* if for all uniform countably complete ultrafilters  $U$ , if  $U <_{\mathbb{k}} D$ , then  $U \triangleleft D$ .

Dodd sound ultrafilters are Mitchell points by Corollary 4.3.28. Under UA, isonormal ultrafilters are Mitchell points by Theorem 7.5.43. The following fact is trivial:

**Lemma 8.3.35** (UA). *The Mitchell points form a coherent class of ultrafilters.*

*Proof.* Let  $C$  be the class of Mitchell points. Since the Ketonen order is linear,  $C$  is linearly ordered by  $<_{\mathbb{k}}$ , and hence by the definition of a Mitchell point,  $C$  is linearly ordered by the Mitchell order. The property of being a Mitchell point is absolute, so if  $U \triangleleft W$  are Mitchell points, then  $U \in j_W(C)$ . Moreover Theorem 8.3.26,  $(M_W)^{\lambda_U} \subseteq M_W$ . Thus  $C$  is coherent. □

We next show that under UA, the Mitchell points form a cofinal class. The first step is to give an alternate characterization in terms of the internal relation:



**Proposition 8.3.36 (UA).** *Suppose  $U$  is a nonprincipal countably complete tail uniform ultrafilter on an ordinal  $\delta$ . The following are equivalent:*

- (1) *For all countably complete uniform ultrafilters  $U$ , if  $U <_{\mathbb{k}} D$ , then  $U \sqsubset D$ .*
- (2)  *$D$  is a Mitchell point*
- (3) *For all Mitchell points  $D'$ , if  $D' <_{\mathbb{k}} D$ , then  $D' \triangleleft D$ .*

*Proof.* (1) implies (2): Note that (1) implies in particular that  $U$  is  $\delta$ -internal. Thus  $U$  is a uniform ultrafilter on  $\delta$ . There are two cases. Suppose first that  $D = \mathcal{K}_\delta$ . Then Theorem 8.3.31,  $D$  is  $\delta$ -Mitchell, which is what (2) asserts. Assume instead that  $D \neq \mathcal{K}_\delta$ , so  $\mathcal{K}_\delta <_{\mathbb{k}} D$  since  $\mathcal{K}_\delta$  is the least uniform ultrafilter on  $\delta$ . By (1),  $\mathcal{K}_\delta \sqsubset U$ , and in particular by Lemma 7.4.14,  $\delta$  is not isolated. By Theorem 8.3.31,  $U$  is  $\delta$ -irreducible, and therefore by the Irreducibility Theorem,  $U$  is  $<\delta$ -supercompact and  $\delta$ -tight. Since  $\mathcal{K}_\delta \sqsubset D$ , Lemma 8.3.28 yields that  $j_D$  is  $\delta$ -supercompact. In particular,  $P(\delta) \subseteq M_D$ , and so for any countably complete ultrafilter  $U$  on  $\delta$  with  $U \sqsubset D$ ,  $U \triangleleft D$ . Given (1), this implies (2).

(2) implies (3): Immediate.

(3) implies (1): Let  $D'$  be the  $<_{\mathbb{k}}$ -least tail uniform ultrafilter that is not internal to  $D$ . To show that (1) holds, we must show  $D' = D$ . Clearly  $D' \leq_{\mathbb{k}} D$  (since a nonprincipal ultrafilter is never internal to itself). By Corollary 8.3.27, the internal relation extends the Mitchell order, so  $D' \not\triangleleft D$ . Theorem 7.5.44 asserts that  $D'$  has the following property: for any  $U \sqsubset D$ , in fact  $U \sqsubset D'$ . In particular, for any  $U <_{\mathbb{k}} D'$ , by the minimality of  $D'$ , we have  $U \sqsubset D$ , and so we can conclude that  $U \sqsubset D'$ . Since we have shown that (1) implies (3), we can conclude that  $D'$  is a Mitchell point. Since  $D'$  is a Mitchell point and  $D' \not\triangleleft D$ , (3) implies that  $D' \not<_{\mathbb{k}} D$ . Since  $D' \leq_{\mathbb{k}} D$ , it follows that  $D = D'$ , as desired.  $\square$

**Definition 8.3.37.** For any countably complete ultrafilter  $W$ , the *Mitchell point of  $W$* , denoted  $D(W)$ , is the  $<_{\mathbb{k}}$ -least tail uniform ultrafilter  $D$  such that  $D \not\triangleleft W$ .

The proof of Proposition 8.3.36 yields the following fact:

**Theorem 8.3.38** (UA). *Suppose  $W$  is a nonprincipal countably complete ultrafilter and  $D = D(W)$ . Then the following hold:*

- $D$  is a Mitchell point.
- $\{U : U \triangleleft W\} = \{U : U \triangleleft D\}$ .
- If  $U$  is a countably complete ultrafilter such that  $U \sqsubset W$ , then  $U \sqsubset D$ .
- $D \not\sqsubset W$ . □

**Theorem 8.3.39** (UA). *The Mitchell points form a cofinal class of ultrafilters.*

*Sketch.* Suppose  $U$  is a countably complete ultrafilter. We will show that there is an ultrafilter  $U'$  given by a Mitchell point iteration such that  $U \leq_{\text{RF}} U'$ . By induction, we may assume that this statement is true for all  $\bar{U} <_{\mathbf{k}} U$ . Let  $D = D(U)$ . Since  $D \not\sqsubset U$ ,  $t_D(U) <_{\mathbf{k}} j_D(U)$  in  $M_D$ . Therefore by our induction hypothesis,  $M_D$  satisfies that there is an ultrafilter  $W'$  given by a Mitchell point iteration of such that  $t_D(U) \leq_{\text{RF}} W'$ . Let  $U'$  be such that  $j_{U'} = j_{W'}^{M_D} \circ j_D$ . It is easy to see that  $U'$  is given by a Mitchell point iteration and  $U \leq_{\text{RF}} U'$ . □

We now turn to the other direction of Theorem 8.3.33. It would be enough to prove the following fact:

**Proposition 8.3.40.** *Suppose  $C$  is a coherent class of countably complete ultrafilters. Then the restriction of the Rudin-Frolík order to the class of ultrafilters given by  $C$ -iterations is directed.*

*Proof.* The idea of the proof is that the ultrafilters in  $C$  can be compared by the comparisons given by the internal relation Lemma 5.5.6, and then this can be propagated to compare arbitrary  $C$ -iterations by recursion. This is quite easy to see (given the right definition of a coherent class), but we nevertheless include a very detailed proof.<sup>4</sup>

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<sup>4</sup>We caution, however, that as usual it may be easier for the reader work out the details than to read them.

We use the following convention: if  $\mathcal{I}$  is an iterated ultrapower of length  $\ell$ , then  $j^{\mathcal{I}} = j_{0\ell}^{\mathcal{I}}$ .

We begin with a one-step claim:

**Claim 1.** *Suppose  $D \in C$ . For any  $C$ -iteration  $\mathcal{I}$ , there is a  $C$ -iteration  $\mathcal{J}$  such that  $U_0^{\mathcal{J}} = D$  and a  $C$ -iteration  $\mathcal{I}'$  extending  $\mathcal{I}$  such that  $j^{\mathcal{I}'} = j^{\mathcal{J}}$ .*

*Proof of Claim 1.* The proof is by induction on the length of  $\mathcal{I}$ .

If  $U_0^{\mathcal{I}} = D$ , then we can take  $\mathcal{I} = \mathcal{J}$ .

Therefore assume  $U_0^{\mathcal{I}} \neq D$ . Since  $C$  is coherent, either  $D \triangleleft U_0^{\mathcal{I}}$  or  $U_0^{\mathcal{I}} \triangleleft D$ . Define

$$D_* = \begin{cases} j_{01}^{\mathcal{I}}(D) & \text{if } U_0^{\mathcal{I}} \triangleleft D \\ D & \text{if } D \triangleleft U_0^{\mathcal{I}} \end{cases}$$

and

$$U_1^* = \begin{cases} U_0^{\mathcal{I}} & \text{if } U_0^{\mathcal{I}} \triangleleft D \\ j_D(U_0^{\mathcal{I}}) & \text{if } D \triangleleft U_0^{\mathcal{I}} \end{cases}$$

The key point is that by the definition of a coherent class of ultrafilters,  $D_* \in j_{01}^{\mathcal{I}}(C)$ ,  $U_1^* \in j_D(C)$ , and

$$j_{U_1^*}^{M_D} \circ j_D = j_{D_*}^{M_{\mathcal{I}}} \circ j_{01}^{\mathcal{I}}$$

Let  $\mathcal{I}_* = \mathcal{I} \upharpoonright [1, \infty)$ , which is a  $j_{01}^{\mathcal{I}}(C)$ -iteration of  $M_{\mathcal{I}}^{\mathcal{I}}$ . By our induction hypothesis applied in  $M_{\mathcal{I}}^{\mathcal{I}}$  to the and the ultrafilter  $D_* \in j_{01}^{\mathcal{I}}(C)$ , there is a  $j_{01}^{\mathcal{I}}(C)$ -iteration  $\mathcal{J}_*$  with  $U_0^{\mathcal{J}_*} = D_*$  and a  $j_{01}^{\mathcal{I}}(C)$ -iteration  $\mathcal{I}'_*$  extending  $\mathcal{I}_*$  such that  $j^{\mathcal{I}'_*} = j^{\mathcal{J}_*}$ .

Let  $\mathcal{I}'$  be the iterated ultrapower of  $V$  given by  $U_0^{\mathcal{I}}$  followed by  $\mathcal{I}'_*$ . Clearly  $\mathcal{I}'$  is a  $C$ -iteration extending  $\mathcal{I}$ . Let  $\ell = \text{lth}(\mathcal{J}_*)$ , and define a  $C$ -iteration  $\mathcal{J}$  of length  $\ell + 1$  in terms of the ultrafilters  $U_n^{\mathcal{J}}$ :

$$U_0^{\mathcal{J}} = D$$

$$U_1^{\mathcal{J}} = U_1^*$$

$$U_n^{\mathcal{J}} = U_{n-1}^{\mathcal{J}_*}$$

Then

$$j^{\mathcal{J}} = j_{1\ell-1}^{\mathcal{J}_*} \circ j_{U_1^*}^{M_D} \circ j_D = j_{1\ell-1}^{\mathcal{J}_*} \circ j_{D_*}^{M_1^{\mathcal{I}}} \circ j_{01}^{\mathcal{I}} = j^{\mathcal{J}_*} \circ j_{01}^{\mathcal{I}} = j^{\mathcal{I}'} \circ j_{01}^{\mathcal{I}} = j^{\mathcal{I}'}$$

This verifies the induction step, and proves the claim.  $\square$

We now turn to the multi-step claim:

**Claim 2.** *For any  $C$ -iteration  $\mathcal{H}$ , for any  $C$ -iteration  $\mathcal{I}$ , there are  $C$ -iterations  $\mathcal{H}^*$  and  $\mathcal{I}^*$  extending  $\mathcal{H}$  and  $\mathcal{I}$  respectively such that  $j^{\mathcal{H}^*} = j^{\mathcal{I}^*}$ .*

*Proof of Claim 2.* The proof is by induction on the length  $\ell$  of  $\mathcal{H}$ : thus our induction hypothesis is that for any  $C$ -iteration  $\bar{\mathcal{H}}$  of length less  $\ell$ , for any  $C$ -iteration  $\mathcal{I}$ , there are  $C$ -iterations  $\bar{\mathcal{H}}^*$  and  $\mathcal{I}'$  extending  $\bar{\mathcal{H}}$  and  $\mathcal{I}$  respectively such that  $j^{\bar{\mathcal{H}}^*} = j^{\mathcal{I}'}$ .

Let  $D = U_0^{\mathcal{H}}$ . By our first claim, there is a  $C$ -iteration  $\mathcal{J}$  such that  $U_0^{\mathcal{J}} = D$  and a  $C$ -iteration  $\mathcal{I}'$  extending  $\mathcal{I}$  such that  $j^{\mathcal{I}'} = j^{\mathcal{J}}$ . Now we work in  $M_D$ . Let  $\bar{\mathcal{H}} = \mathcal{H} \upharpoonright [1, \infty)$ . Thus  $\bar{\mathcal{H}}$  is a  $j_D(C)$ -iteration of  $M_D$  of length less than  $\ell$ . Let  $\bar{\mathcal{J}} = \mathcal{J} \upharpoonright [1, \infty)$ , so that  $\bar{\mathcal{J}}$  is also a  $j_D(C)$ -iteration of  $M_D$ .

By our induction hypothesis applied in  $M_D$ , there are  $j_D(C)$ -iterations  $\bar{\mathcal{H}}^*$  and  $\bar{\mathcal{J}}^*$  of  $M_D$  extending  $\bar{\mathcal{H}}$  and  $\bar{\mathcal{J}}$  respectively such that  $j^{\bar{\mathcal{H}}^*} = j^{\bar{\mathcal{J}}^*}$ . Define

$$\begin{aligned}\mathcal{H}^* &= D \frown \bar{\mathcal{H}}^* \\ \mathcal{I}^* &= \mathcal{I}' \frown \mathcal{K}\end{aligned}$$

where  $\mathcal{K}$  is the iteration such that  $\bar{\mathcal{J}}^* = \bar{\mathcal{J}} \frown \mathcal{K}$ .

Obviously  $\mathcal{H}^*$  and  $\mathcal{I}^*$  are  $C$ -iterations extending  $\mathcal{H}$  and  $\mathcal{I}$  respectively. Moreover

$$j^{\mathcal{H}^*} = j^{\bar{\mathcal{H}}^*} \circ j_D = j^{\bar{\mathcal{J}}^*} \circ j_D = j^{\mathcal{K}} \circ j^{\bar{\mathcal{J}}} \circ j_D = j^{\mathcal{K}} \circ j^{\mathcal{J}} = j^{\mathcal{K}} \circ j^{\mathcal{I}'} = j^{\mathcal{I}'}$$

This proves the claim  $\square$

It follows easily from Claim 2 that the restriction of the Rudin-Frolík order to the class of ultrafilters given by  $C$ -iterations is directed.  $\square$

We finally prove our characterization of UA in terms of coherent cofinal sequences.

*Proof of Theorem 8.3.33. (1) implies (2):* This is immediate from Lemma 8.3.35 and Theorem 8.3.39.

*(2) implies (1):* Let  $C$  be a coherent cofinal class of ultrafilters. Since  $C$  is coherent, Proposition 8.3.40 implies that the restriction of the Rudin-Frolík order to the class of ultrafilters  $C'$  given by  $C$ -iterations is directed. Since  $C$  is cofinal,  $C'$  is cofinal in the Rudin-Frolík order. Since the Rudin-Frolík order has a cofinal directed subset, the Rudin-Frolík order is itself directed. This implies that the Ultrapower Axiom holds (by Corollary 5.2.9).  $\square$

## 8.4 Very large cardinals

### Huge cardinals

The notion of  $(\kappa, \lambda)$ -regularity is a two cardinal generalization of  $\kappa^+$ -incompleteness that has already shown up implicitly in this dissertation:

**Definition 8.4.1.** Suppose  $\kappa \leq \lambda$  are cardinals. An ultrafilter  $U$  is  $(\kappa, \lambda)$ -regular if there is a set  $F \subseteq U$  of cardinality  $\lambda$  such that  $\bigcap \sigma \notin U$  for any  $\sigma \subseteq F$  of cardinality at least  $\kappa$ .

The combinatorial definition of  $(\kappa, \lambda)$ -regularity defined above obscures its true significance:

**Lemma 8.4.2.** *Suppose  $\kappa \leq \lambda$  are cardinals and  $U$  is an ultrafilter. Then the following are equivalent:*

- (1)  $U$  is  $(\kappa, \lambda)$ -regular.
- (2) For some fine ultrafilter  $\mathcal{U}$  on  $P_\kappa(\lambda)$ ,  $\mathcal{U} \leq_{\text{RK}} U$ .
- (3)  $j_U$  is  $(\lambda, \delta)$ -tight for some  $M_U$ -cardinal  $\delta < j_U(\kappa)$ .

*Proof. (1) implies (2):* Fix a set  $F \subseteq U$  of cardinality  $\lambda$  such that  $\bigcap \sigma \notin U$  for any  $\sigma \subseteq F$  of cardinality at least  $\kappa$ . Let  $X$  be the underlying set of  $U$ . Define  $f : X \rightarrow P_\kappa(F)$  by setting

$f(x) = \{A \in F : x \in A\}$ . Let  $\mathcal{U} = f_*(U)$ . We claim  $\mathcal{U}$  is a fine ultrafilter on  $P_\kappa(F)$ . Suppose  $A \in F$ . We must show  $\{\sigma \in P_\kappa(F) : A \in \sigma\} \in \mathcal{U}$ . But by the definition of  $f$ ,  $A \in f(x)$  if and only if  $x \in A$ . Thus

$$f^{-1}[\{\sigma \in P_\kappa(F) : A \in \sigma\}] = A \in U$$

and so  $\{\sigma \in P_\kappa(F) : A \in \sigma\} \in \mathcal{U}$ .

(2) implies (3): Fix a fine ultrafilter  $\mathcal{U}$  on  $P_\kappa(\lambda)$  such that  $\mathcal{U} \leq_{\text{RK}} U$ . Let  $A = a_{\mathcal{U}}$ . Then  $j_{\mathcal{U}}[\lambda] \subseteq A$  by Lemma 4.4.9, and  $|A|^{M_{\mathcal{U}}} < j_{\mathcal{U}}(\kappa)$  by Łoś's Theorem. Let  $k : M_{\mathcal{U}} \rightarrow M_U$  be an elementary embedding such that  $k \circ j_{\mathcal{U}} = j_U$ . Then  $j_U[\lambda] = k[j_{\mathcal{U}}[\lambda]] \subseteq k(A)$  and  $|k(A)|^{M_U} < k(j_{\mathcal{U}}(\kappa)) = j_U(\kappa)$ . Let  $\delta = |k(A)|^{M_U}$ . Then  $k(A)$  witnesses that  $j_U$  is  $(\lambda, \delta)$ -tight, as desired.

(3) implies (1): Fix  $A \in M_U$  such that  $|A|^{M_U} < j_U(\kappa)$  and  $j_U[\lambda] \subseteq A$ . Let  $f$  be a function such that  $A = [f]_U$ . By Łoś's Theorem, there is a set  $X \in U$  such that  $f[X] \subseteq P_\kappa(\lambda)$ . Let  $S_\alpha = \{x \in X : \alpha \in f(x)\}$ . Let  $F = \{S_\alpha : \alpha < \lambda\}$ . We claim that  $\bigcap_{\alpha \in \sigma} S_\alpha = \emptyset$  for any  $\sigma \subseteq \lambda$  of cardinality at least  $\kappa$ . Suppose towards a contradiction that  $x \in \bigcap_{\alpha \in \sigma} S_\alpha$ . Then  $\sigma \subseteq f(x)$ , so  $|f(x)| \geq \kappa$ , contradicting that  $f(x) \in P_\kappa(\lambda)$ . Thus  $F$  witnesses that  $U$  is  $(\kappa, \lambda)$ -regular.  $\square$

Another way of stating (2) above is to say that the minimum fine filter on  $P_\kappa(\lambda)$  lies below  $U$  in the Katětov order.

**Definition 8.4.3.** If  $\kappa \leq \lambda$  are cardinals, then  $P^\kappa(\lambda)$  denotes the collection of subsets of  $\lambda$  of cardinality exactly  $\kappa$ .

$$\text{Thus } P^\kappa(\lambda) = P_{\kappa^+}(\lambda) \setminus P_\kappa(\lambda).$$

**Definition 8.4.4.** A cardinal  $\kappa$  is *huge* if there is an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  such that  $M^{j(\kappa)} \subseteq M$ .

A question raised in [11] is the relationship between nonregular ultrafilters and huge cardinals. Assuming UA, we can almost show an equivalence:

**Theorem 8.4.5 (UA).** *Suppose  $\kappa < \lambda$  are cardinals and  $\lambda$  is regular. The following are equivalent:*

- (1) *There is a countably complete fine ultrafilter on  $P^\kappa(\lambda)$  that cannot be pushed forward to a fine ultrafilter on  $P_\kappa(\lambda)$ .*
- (2) *There is a countably complete ultrafilter that is  $(\kappa^+, \lambda)$ -regular but not  $(\kappa, \lambda)$ -regular.*
- (3) *There is an elementary embedding  $j : V \rightarrow M$  such that  $j(\kappa) = \lambda$ ,  $M^{<\lambda} \subseteq M$ , and  $M$  has the  $\leq \lambda$ -covering property.*

*If  $\lambda$  is a successor cardinal, then we can add to the list:*

- (4) *There is an elementary embedding  $j : V \rightarrow M$  such that  $j(\kappa) = \lambda$  and  $M^\lambda \subseteq M$ .*
- (5) *There is a normal fine ultrafilter on  $P^\kappa(\lambda)$ .*

*Proof.* The equivalence of (1) and (2) is immediate from Lemma 8.4.2. We now turn to the equivalence of (2) and (3). Before we begin, we point out that the property of being  $(\kappa^+, \lambda)$ -regular but not  $(\kappa, \lambda)$ -regular can be reformulated in terms of ultrapowers:

$$U \text{ is } (\kappa^+, \lambda)\text{-regular but not } (\kappa, \lambda)\text{-regular if and only if } \text{cf}^{M_U}(\sup j_U[\lambda]) = j_U(\kappa).$$

This is an immediate consequence of Lemma 8.4.2 (3) and Ketonen's analysis of tight embeddings in terms of cofinality (Theorem 7.2.12).

(2) *implies (3):* Let  $U$  be the  $<_{\mathbf{k}}$ -least countably complete ultrafilter concentrating on ordinals that is  $(\kappa^+, \lambda)$ -regular but not  $(\kappa, \lambda)$ -regular.

We claim that  $U$  is  $\lambda$ -irreducible. (In fact,  $U$  is an irreducible weakly normal ultrafilter on  $\lambda$ , but this is not relevant to the proof.) Suppose  $D \leq_{\text{RF}} U$  and  $\lambda_D < \lambda$ . We must show that  $D$  is principal. We claim  $t_D(U)$  is  $(j_D(\kappa^+), j_D(\lambda))$ -regular but not  $(j_D(\kappa), j_D(\lambda))$ -regular. Let  $i : M_D \rightarrow M_U$  be the unique internal ultrapower embedding with  $i \circ j_D = j_U$ . Thus  $i : M_D \rightarrow M_U$  is the ultrapower of  $M_D$  by  $t_D(U)$ . Therefore to show that  $t_D(U)$  is

$(j_D(\kappa), j_D(\lambda))$ -regular it suffices (by our remark at the beginning of the proof) to show that  $\text{cf}^{M_U}(\sup i[j_D(\lambda)]) = i(j_D(\kappa))$ . Since  $\lambda_D < \lambda$ , by Lemma 3.5.32,

$$\sup i[j_D(\lambda)] = \sup i \circ j_D[\lambda] = \sup j_U[\lambda]$$

Furthermore, since  $U$  is  $(j_D(\kappa^+), j_D(\lambda))$ -regular but not  $(j_D(\kappa), j_D(\lambda))$ -regular, applying our remark at the beginning of the proof again,

$$\text{cf}^{M_U}(\sup j_U[\lambda]) = j_U(\kappa) = i(j_D(\kappa))$$

Thus  $\text{cf}^{M_U}(\sup i[j_D(\lambda)]) = i(j_D(\kappa))$ , as desired.

By elementarity  $j_D(U)$  is the  $<_{\mathbf{k}}$ -least ultrafilter that is  $(j_D(\kappa^+), j_D(\lambda))$ -regular but not  $(j_D(\kappa), j_D(\lambda))$ -regular. Hence  $j_D(U) \leq_{\mathbf{k}} t_D(U)$ . Recall Proposition 5.4.5, which states that if  $D$  is nonprincipal and  $D \leq_{\text{RF}} U$ , then  $t_D(U) <_{\mathbf{k}} j_D(U)$ . It follows that  $D$  is principal.

Since  $U$  is  $\lambda$ -irreducible, and now we would like to apply the Irreducibility Theorem. For this, we need that  $\lambda$  is either a successor cardinal or an inaccessible cardinal. Assume  $\lambda$  is a limit cardinal, and we will show that  $\lambda$  is a strong limit cardinal. Since  $\kappa < \lambda$ , we have  $\kappa^+ < \lambda$ . Since  $U$  is  $(\kappa^+, \lambda)$ -regular,  $U$  is  $\delta$ -decomposable for all regular cardinals  $\delta \in [\kappa^+, \lambda]$ . Therefore  $\lambda$  is a limit of Fréchet cardinals, and hence by Corollary 7.5.2,  $\lambda$  is a strong limit cardinal, as desired.

To summarize,  $j_U : V \rightarrow M_U$  is an elementary embedding such that  $j_U(\kappa) = \lambda$ ,  $M_U^{<\lambda} \subseteq M_U$  and  $M_U$  has the  $\leq \lambda$ -covering property. This shows that (3) holds.

(3) implies (2): Let  $U$  be the ultrafilter on  $\lambda$  derived from  $j$  using  $\sup j[\lambda]$ , and let  $k : M_U \rightarrow M$  be the factor embedding with  $k \circ j_U = j$  and  $k(a_U) = \sup j[\lambda]$ . Then  $a_U = \sup j_U[\lambda]$ , and  $k(\text{cf}^{M_U}(a_U)) = \text{cf}^M(\sup j[\lambda]) = \lambda = j(\kappa) = k(j_U(\kappa))$ . By the elementarity of  $k$ ,

$$\text{cf}^{M_U}(\sup j_U[\lambda]) = \text{cf}^{M_U}(a_U) = j_U(\kappa)$$

Thus by our remark at the beginning of the proof,  $U$  is  $(\kappa^+, \lambda)$ -regular but not  $(\kappa, \lambda)$ -regular. This shows that (1) holds.



Assuming  $\lambda$  is a successor cardinal, the argument that (2) implies (3) shows that in fact (2) implies (4), since the Irreducibility Theorem leads to full  $\lambda$ -supercompactness in the case that  $\lambda$  is a successor cardinal.

Finally, (4) and (5) are equivalent (in general) by an easy argument using derived ultrafilters and ultrapowers (Lemma 4.4.10).  $\square$

We cannot show that  $M^\lambda \subseteq M$  in the key case that  $\lambda$  is inaccessible, which blocks proving the equivalence between huge cardinals and nonregular countably complete ultrafilters.

## Cardinal preserving elementary embeddings

In this section, we turn to even larger large cardinal axioms.

**Definition 8.4.6.** An elementary embedding  $j : V \rightarrow M$  is *weakly cardinal preserving* if whenever  $\kappa$  is a cardinal,  $j(\kappa)$  is also a cardinal.

The following question, due to Caicedo, essentially asks whether the Kunen Inconsistency Theorem can be strengthened to rule out cardinal preserving elementary embeddings:

**Question 8.4.7.** Is it consistent that there is a nontrivial weakly cardinal preserving elementary embedding?

Under UA, we will show that there are no nontrivial weakly cardinal preserving embeddings.

**Lemma 8.4.8 (UA).** *Suppose  $U$  is a countably complete uniform ultrafilter on  $\kappa^+$  such that  $j_U[\kappa] \subseteq \kappa$ . Either  $\kappa$  is  $\kappa^+$ -supercompact or  $\kappa$  is a limit of  $\kappa^+$ -supercompact cardinals.*

*Proof.* By Corollary 8.2.24, there is some  $D \leq_{\text{RF}} U$  with  $\lambda_D < \kappa^+$  such that there is an internal ultrapower embedding  $i : M_D \rightarrow M_U$  with  $i \circ j_D = j_U$  that is  $j_D(\kappa^+)$ -supercompact in  $M_D$ . Note that  $\sup j_D[\kappa] \subseteq \kappa$  and  $\sup i[\kappa] \subseteq \kappa$ , since both  $i$  and  $j_D$  are bounded on the ordinals by  $j_U$ .

We claim that  $\text{CRT}(i) \in [\kappa, j_D(\kappa)]$ . To see this, note that  $\sup i[\kappa] \subseteq \kappa$  and  $i$  is  $\kappa$ -supercompact, so by the Kunen Inconsistency Theorem (Theorem 4.2.37),  $\text{CRT}(i) \geq \kappa$ . On the other hand,  $i$  is given by an ultrafilter on  $j_D(\kappa^+)$ , so  $\text{CRT}(i) \leq j_D(\kappa)$ .

Now  $i$  witnesses that  $\text{CRT}(i)$  is  $j_D(\kappa^+)$ -supercompact in  $M_D$ . If  $\text{CRT}(i) = j_D(\kappa)$ , then  $\kappa$  is  $\kappa^+$ -supercompact by elementarity. Otherwise  $\sup j_D[\kappa] = \kappa \leq \text{CRT}(i) < j_D(\kappa)$ , so  $\kappa$  is a limit of  $\kappa^+$ -supercompact cardinals by a standard reflection argument.  $\square$

The following observation is due to Caicedo:

**Lemma 8.4.9.** *Suppose  $j : V \rightarrow M$  and  $\gamma$  is a cardinal. If  $j(\gamma^+) \neq \gamma^+$  and  $j$  is continuous at  $\gamma^+$ , then  $j(\gamma^+)$  is not a cardinal.*

*Proof.* Note that  $j(\gamma^+)$  is a singular ordinal since  $j[\gamma^+]$  is cofinal in  $j(\gamma^+)$ . Moreover  $j(\gamma) < j(\gamma^+) = j(\gamma)^{+M} \leq j(\gamma)^+$ . There are no singular cardinals between  $j(\gamma)$  and  $j(\gamma)^+$ , so  $j(\gamma^+)$  is not a cardinal.  $\square$

**Lemma 8.4.10 (UA).** *Suppose  $j : V \rightarrow M$  is a nontrivial elementary embedding with critical point  $\kappa$ . Let  $\gamma$  be a cardinal above  $\kappa$  with  $j(\gamma) = \gamma$ . Then  $j$  is continuous at  $\gamma^{+\kappa+1}$  and therefore  $j(\gamma^{+\kappa+1})$  is not a cardinal.*

*Proof.* We begin the proof by making some general observations about the action of  $j$  on cardinals in the vicinity of  $\gamma$ . First, for all  $\alpha < \kappa$ ,  $j(\gamma^{+\alpha}) = (\gamma^{+\alpha})^M \leq \gamma^{+\alpha}$ . It follows that  $j(\gamma^{+\alpha}) = \gamma^{+\alpha}$ . Hence  $\sup j[\gamma^{+\kappa}] = \gamma^{+\kappa}$ .

Next, we claim that  $(\gamma^{+\kappa+1})^M = \gamma^{+\kappa+1}$ . This is proved by following the argument of Lemma 4.2.32: fix  $\alpha < \gamma^{+\kappa+1}$ , and we will show that  $\alpha < (\gamma^{+\kappa+1})^M$ . Let  $(\gamma^{+\kappa}, \prec)$  be a wellorder of order type  $\alpha$ . Then  $(\gamma^{+\kappa}, j(\prec))$  is a wellorder of  $\gamma^{+\kappa}$  that belongs to  $M$ . Since  $j[\gamma^{+\kappa}] \subseteq \gamma^{+\kappa}$ ,  $j$  embeds  $(\gamma^{+\kappa}, \prec)$  into  $(\gamma^{+\kappa}, j(\prec))$ , so

$$\alpha \leq \text{ot}(\gamma^{+\kappa}, \prec) \leq \text{ot}(\gamma^{+\kappa}, j(\prec)) < (\gamma^{+\kappa+1})^M$$

as desired.

It follows that

$$j(\gamma^{+\kappa+1}) > j(\gamma^{+\kappa}) = (\gamma^{+j(\kappa)})^M > (\gamma^{+\kappa+1})^M = \gamma^{+\kappa+1}$$

Thus to prove  $j(\gamma^{+\kappa+1})$  is not a cardinal, by Lemma 8.4.9 it suffices to show  $j$  is continuous at  $\gamma^{+\kappa+1}$ .

Suppose towards a contradiction that  $j$  is discontinuous at  $\gamma^{+\kappa+1}$ . Let  $U$  be the ultrafilter on  $\gamma^{+\kappa+1}$  derived from  $j$  using  $\sup j[\gamma^{+\kappa+1}]$ . Then  $U$  is a countably complete uniform ultrafilter on  $\gamma^{+\kappa+1}$ . Moreover,

$$\sup j_U[\gamma^{+\kappa}] \leq \sup j[\gamma^{+\kappa}] = \gamma^{+\kappa}$$

Therefore by Lemma 8.4.8,  $\gamma^{+\kappa}$  is either  $\gamma^{+\kappa+1}$ -supercompact or else a limit of  $\gamma^{+\kappa+1}$ -supercompact cardinals. This is impossible since there are no inaccessible cardinals in the interval  $(\gamma, \gamma^{+\kappa}]$ . Thus our assumption was false, and in fact  $j$  is continuous at  $\gamma^{+\kappa+1}$ .

Now  $j$  is continuous at  $\gamma^{+\kappa+1}$  and  $j(\gamma^{+\kappa+1}) > \gamma^{+\kappa+1}$ . Therefore by Lemma 8.4.9,  $j(\gamma^{+\kappa+1})$  is not a cardinal. □

**Corollary 8.4.11** (UA). *Any weakly cardinal preserving elementary embedding of the universe is the identity.*

We now investigate the relationship between cardinal preservation and rank-into-rank axioms.

**Theorem 8.4.12** (UA). *Assume  $\lambda$  is an ordinal,  $M \subseteq V_\lambda$  is a transitive set, and  $j : V_\lambda \rightarrow M$  is an elementary embedding with critical point  $\kappa$  that has no fixed points above  $\kappa$ . Suppose that  $\text{Card}^M \cap \lambda = \text{Card} \cap \lambda$ . Then  $M = V_\lambda$ .*

If the assumption that  $\text{Card}^M \cap \lambda = \text{Card} \cap \lambda$  is weakened to the assumption that  $j$  is weakly cardinal preserving below  $\lambda$  (or in other words that  $j[\text{Card} \cap \lambda] \subseteq \text{Card} \cap \lambda$ ), then the resulting statement is false. Let us provide a counterexample. Suppose  $j : V \rightarrow M$  is an elementary embedding with critical point  $\kappa$ . Let  $\lambda$  be the first cardinal fixed point of  $j$  above

$\kappa$ . Assume  $V_\lambda \subseteq M$ , so  $j$  witnesses the axiom  $I_2$ . Suppose  $U$  is a  $\kappa$ -complete ultrafilter on  $\kappa$ . Then by Corollary 5.5.35,  $j_U^M \circ j : V \rightarrow (M_U)^M$  has the property that  $j_U^M \circ j \upharpoonright \text{Ord} = j \upharpoonright \text{Ord}$ , so in particular  $j_U^M \circ j[\text{Card} \cap \lambda] = j[\text{Card} \cap \lambda] \subseteq \text{Card} \cap \lambda$ . But of course  $(M_U)^M$  does not contain  $V_\lambda$ .

One of the key lemmas is the following curiosity, a close cousin of Lemma 8.2.10:

**Lemma 8.4.13** (UA). *Suppose  $U$  is a countably complete ultrafilter and  $\delta$  is a successor cardinal. Then  $\text{cf}^{M_U}(\sup j_U[\delta])$  is a successor cardinal of  $M_U$ .*

*Proof.* If  $\sup j_U[\delta] = j_U(\delta)$ , then  $\sup j_U[\delta]$  is itself a successor cardinal of  $M_U$ , so of course its  $M_U$ -cofinality (which is again  $\sup j_U[\delta]$ ) is a successor cardinal of  $M_U$ . We may therefore assume that  $\sup j_U[\delta] < j_U(\delta)$ .

Hence  $\delta$  is Fréchet, and so we are in a position to apply Theorem 8.2.18. By Theorem 8.2.18, there is an ultrafilter  $D$  with  $\lambda_D < \delta$  such that there is an internal ultrapower embedding  $h : M_D \rightarrow M_U$  such that  $h$  is  $j_D(\delta)$ -supercompact in  $M_D$ . Since  $\lambda_D < \delta$ ,  $j_D(\delta) = \sup j_D[\delta]$  by Lemma 3.5.32. Thus

$$\text{cf}^{M_U}(\sup j_U[\delta]) = \text{cf}^{M_U}(\sup h[j_D(\delta)]) = \text{cf}^{M_D}(j_D(\delta)) = j_D(\delta)$$

Since  $j_D(\delta)$  is a successor cardinal of  $M_D$ , and  $\text{Ord}^{j_D(\delta)} \cap M_D \subseteq M_U$ ,  $j_D(\delta)$  is a successor cardinal of  $M_U$ . □

We now turn to the proof of Theorem 8.4.12.

*Proof of Theorem 8.4.12.* For  $n < \omega$ , let  $\kappa_n = \kappa_n^j$  be the  $n$ th element of the critical sequence of  $j$  (Definition 4.2.35), and note that  $\lambda = \sup_{n < \omega} \kappa_n$  since  $j$  has no fixed points above  $\kappa$ .

Let us make some preliminary remarks about the interaction between ultrapowers and the structure  $V_\lambda$ . Suppose that  $U$  is a countably complete ultrafilter on a set  $X \in V_\lambda$ . Then any function  $f : X \rightarrow V_\lambda$  is bounded on a set in  $U$ . In particular,

$$j_U(V_\lambda) = \{[f]_U : f \in V_\lambda \text{ and } \text{dom}(f) = X\}$$

In other words,  $V_\lambda$  correctly computes the ultrapower by  $U$ . We will go to great lengths, however, not to work inside  $V_\lambda$ , which we have not yet proved to be a model of ZFC.

Suppose  $X \in V_\lambda$ ,  $a \in j(X)$ , and  $U$  is the ultrafilter on  $X$  derived from  $j$  using  $a$ . Then  $U$  is countably complete, so the remark of the previous paragraph applies. Thus we can define a factor embedding  $k : j_U(V_\lambda) \rightarrow M$  by setting  $k([f]_U) = j(f)(a)$  whenever  $f \in V_\lambda$  is a function on  $X$ . The usual argument shows that  $k$  is well-defined and elementary. Moreover,  $k \circ (j_U \upharpoonright V_\lambda) = j$  and  $k(a_U) = a$ .

Suppose  $\delta < \lambda$  is a successor cardinal. Let  $U$  be the uniform ultrafilter derived from  $j$  using  $\sup j[\delta]$ , and let  $k : j_U(V_\lambda) \rightarrow M$  be the factor embedding. We claim:

- $\text{cf}^M(\sup j[\delta]) = \delta$ .
- $j_U$  is  $\delta$ -tight.
- $k(\delta) = \delta$ .

By Lemma 8.4.13,  $\sup j_U[\delta]$  is a successor cardinal of  $M_U$ . Thus  $\sup j_U[\delta]$  is a successor cardinal of  $j_U(V_\lambda)$ , so  $k(\sup j_U[\delta]) = \text{cf}^M(\sup j[\delta])$  is a successor cardinal of  $M$ . Since  $M$  is correct about cardinals below  $\lambda$ ,  $\text{cf}^M(\sup j[\delta])$  is a successor cardinal (in  $V$ ). In particular,  $\text{cf}^M(\sup j[\delta])$  is regular. Thus  $\text{cf}^M(\sup j[\delta]) = \text{cf}(\text{cf}^M(\sup j[\delta])) = \text{cf}(\sup j[\delta]) = \delta$ , as desired.

It follows that  $j_U$  is  $\delta$ -tight:

$$\text{cf}^{M_U}(\sup j_U[\delta]) = \text{cf}^{j_U(V_\lambda)}(\sup j_U[\delta]) \leq k(\text{cf}^{j_U(V_\lambda)}(\sup j_U[\delta])) = \text{cf}^M(\sup j[\delta]) = \delta$$

so  $\text{cf}^{M_U}(\sup j_U[\delta]) = \delta$ , and hence  $j_U$  is  $\delta$ -tight by Theorem 7.2.12.

Repeating the same argument, it now follows that  $k(\delta) = \delta$ :

$$k(\delta) = k(\text{cf}^{M_U}(\sup j_U[\delta])) = k(\text{cf}^{j_U(V_\lambda)}(\sup j_U[\delta])) = \text{cf}^M(\sup j[\delta]) = \delta$$

We recall an argument due to Caicedo-Woodin ([36]) that shows that  $\kappa_n$  is strongly inaccessible for all  $n < \omega$ . Suppose by induction that  $\kappa_n$  is strongly inaccessible, and we will show that  $\kappa_{n+1}$  is strongly inaccessible. Since  $\kappa_n$  is strongly inaccessible and  $j(\kappa_n) = \kappa_{n+1}$ ,  $M$

satisfies that  $\kappa_{n+1}$  is strongly inaccessible. Since  $M$  is cardinal correct below  $\lambda$  and satisfies that  $\kappa_{n+1}$  is a limit cardinal,  $\kappa_{n+1}$  is a limit cardinal (in  $V$ ). It therefore suffices to show that for all successor cardinals  $\delta < \kappa_{n+1}$ ,  $2^\delta < \kappa_{n+1}$ . Let  $U$  be the ultrafilter on  $V_\lambda$  derived from  $j$  using  $\sup j[\delta]$ , and let  $k : j_U(V_\lambda) \rightarrow M$  be the factor embedding. Since  $j_U$  is  $\delta$ -tight,  $2^\delta \leq (2^\delta)^{M_U}$  by Lemma 8.2.6. Since  $\delta < \kappa_{n+1}$  and  $\kappa_{n+1}$  is strongly inaccessible in  $M$ ,  $M$  satisfies that  $2^\delta$  exists, and  $(2^\delta)^M < \kappa_{n+1}$ . Since  $k(\delta) = \delta$ , by elementarity  $j_U(V_\lambda)$  satisfies that  $2^\delta$  exists, and hence  $(2^\delta)^{j_U(V_\lambda)} = (2^\delta)^{M_U}$ . Thus

$$2^\delta \leq (2^\delta)^{M_U} = (2^\delta)^{j_U(V_\lambda)} \leq k((2^\delta)^{j_U(V_\lambda)}) = (2^\delta)^M < \kappa_{n+1}$$

Thus  $\lambda = \sup_{n < \omega} \kappa_n$  is a limit of strongly inaccessible cardinals.

Suppose  $\eta \in [\kappa, \lambda]$  is a strongly inaccessible cardinal. We will show  $V_\eta \subseteq M$ .

Let  $U$  be the ultrafilter on  $\eta$  derived from  $j$  using  $\sup j[\eta]$ . Let  $k : j_U(V_\lambda) \rightarrow M$  be the factor embedding.

By Corollary 8.2.24, there is an ultrafilter  $D$  with  $\lambda_D < \eta$  such that there is an internal ultrapower embedding  $h : M_D \rightarrow M_U$  with  $h \circ j_D = j_U$  that is  $< j_D(\eta)$ -supercompact in  $M_D$ . Since  $\lambda_D < \eta$  and  $\eta$  is strongly inaccessible,  $j_D(\eta) = \eta$ .

In particular, we have that  $V_\eta \cap M_D = V_\eta \cap M_U$ . We can therefore define an elementary embedding  $i : V_\eta \rightarrow V_\eta \cap M$ : for  $x \in V_\eta$ , set  $i(x) = k(j_D(x))$ . Note that  $i$  is a weakly cardinal preserving embedding of  $V_\eta$ :

$$\text{Card}^{V_\eta \cap M} = \text{Card}^{M_D} \cap \eta = \text{Card} \cap \eta$$

The second-order structure  $(V_\eta, V_{\eta+1})$  is a model of NBG + UA, so we can apply Lemma 8.4.10 in  $(V_\eta, V_{\eta+1})$  to conclude that  $i$  is the identity. In particular,  $V_\eta \cap M = V_\eta$ , and therefore  $V_\eta \subseteq M$ , as desired.

Since  $\eta < \lambda$  was an arbitrary inaccessible cardinal and  $\lambda$  is a limit of inaccessible cardinals,  $V_\lambda \subseteq M$ . Hence  $M = V_\lambda$ , as desired.  $\square$

The following question remains open:

**Question 8.4.14.** Suppose there is a weakly cardinal preserving elementary embedding from  $V_\lambda$  into a transitive set  $M \subseteq V_\lambda$ . Must there be an elementary embedding  $j : V_\lambda \rightarrow V_\lambda$ ?

This cannot be entirely trivial: an application of Corollary 5.5.35 shows that a weakly cardinal preserving embedding itself need not have target model  $V_\lambda$ . Suppose  $\kappa < \lambda$  are cardinals,  $j : V \rightarrow M$  is an elementary embedding with critical point  $\kappa$ ,  $j(\lambda) = \lambda$ , and  $V_\lambda \subseteq M$ . Let  $U$  a  $\kappa$ -complete ultrafilter on  $\kappa$ . Then  $j_U \circ j \upharpoonright \text{Ord} = j$  by Corollary 5.5.35, and it follows that  $j_U \circ j \upharpoonright V_\lambda$  is weakly cardinal preserving, even though its target model is  $M_U \cap V_\lambda$  and not  $V_\lambda$ .

## Supercompactness at inaccessible cardinals

The following are probably the most interesting questions left open by our work:

**Question 8.4.15 (UA).** Suppose  $\lambda$  is an inaccessible cardinal and  $\kappa$  is the least  $\lambda$ -strongly compact cardinal. Must  $\kappa$  be  $\lambda$ -supercompact? More generally, if  $\kappa$  is  $\lambda$ -strongly compact, must  $\kappa$  be  $\lambda$ -supercompact or a measurable limit of  $\lambda$ -supercompact cardinals?

This final chapter consists of some inconclusive observations regarding this problem.

The whole question, it turns out, reduces to the analysis of  $\mathcal{K}_\lambda$ :

**Lemma 8.4.16 (UA).** *Assume  $\lambda$  is an inaccessible Fréchet cardinal. Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ , and let  $\kappa$  be the least measurable cardinal of  $M$  above  $\lambda$ . Then for any  $\lambda$ -irreducible ultrafilter  $U$ ,  $\text{Ord}^\kappa \cap M \subseteq M_U$ .*

*Proof.* Let  $(k, h) : (M, M_U) \rightarrow P$  be the pushout of  $(j, j_U)$ , and let  $W$  be such that  $P = M_W$ . By the analysis of ultrafilters internal to a pushout, for any  $D$  with  $\lambda_D < \lambda$ , since  $D \sqsubset U$  and  $D \sqsubset \mathcal{K}_\lambda$ , in fact,  $D \sqsubset W$ . In particular,  $W$  is  $\lambda$ -irreducible, so  $V_\lambda \subseteq M_W = P$  by Corollary 8.2.20. By our factorization lemma for embeddings of  $M$  (Lemma 8.2.8), it follows that  $\text{CRT}(k) \geq \kappa$ . (Otherwise  $k$  would factor through an ultrapower by an ultrafilter in  $V_\lambda$ , contrary to the fact that  $V_\lambda \subseteq P$ .) Therefore  $\text{Ord}^\kappa \cap M \subseteq P \subseteq M_U$ , as desired.  $\square$

**Corollary 8.4.17** (UA). *Suppose  $\lambda$  is a Fréchet inaccessible cardinal. Let  $M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ , and assume  $M$  is closed under  $\lambda$ -sequences. Then for any  $\lambda$ -irreducible ultrafilter  $U$ ,  $M_U$  is closed under  $\lambda$ -sequences.*

*Proof.* By Lemma 8.4.16,  $\text{Ord}^\lambda = \text{Ord}^\lambda \cap M \subseteq M_U$ , so  $M_U$  is closed under  $\lambda$ -sequences.  $\square$

We now show that the  $<_{\mathbb{k}}$ -second irreducible ultrafilter on an inaccessible cardinal  $\lambda$  always witnesses  $\lambda$ -supercompactness. This is a bit surprising given that we cannot prove the supercompactness of  $\mathcal{K}_\lambda$ .

We use the following lemma, extracted from Ketonen's proof that the Ketonen order is wellfounded on weakly normal ultrafilters.

**Lemma 8.4.18.** *Suppose  $\lambda$  is a regular cardinal. Suppose  $W$  is a countably complete ultrafilter on  $\lambda$  that extends the closed unbounded filter. Suppose  $U <_{\mathbb{k}} W$ . Then  $\delta_{t_U(W)} = j_U(\lambda)$ . In fact,  $t_U(W)$  extends the closed unbounded filter on  $j_U(\delta)$ .*

*Proof.* Let  $F$  be the closed unbounded filter on  $\lambda$ . Clearly  $j_U[F] \subseteq t_U(W)$ . Moreover  $\{\alpha < j_U(\delta) : a_U \in \alpha\} \in t_U(W)$  since

$$j_{t_U(W)}^{M_U}(a_U) < j_{t_W(U)}^{M_W}(a_W) = a_{t_U(W)}$$

Thus by Lemma 8.2.11,  $j_U(F) \subseteq t_U(W)$ , as claimed.  $\square$

We choose not to cite the Irreducibility Theorem in the proof of the following proposition since it predates the Irreducibility Theorem and is really much easier:

**Proposition 8.4.19** (UA). *Suppose  $\lambda$  is a regular cardinal. The following are equivalent:*

- (1)  $\lambda$  carries distinct uniform irreducible ultrafilters.
- (2) There is a countably complete uniform ultrafilter  $U$  such that  $\mathcal{K}_\lambda \not\leq_{\text{RF}} U$  and  $U \not\leq \mathcal{K}_\lambda$ .
- (3)  $\lambda$  carries a countably complete weakly normal ultrafilter that concentrates on ordinals that carry countably complete tail uniform ultrafilters.



(4)  $\lambda$  carries distinct countably complete weakly normal ultrafilters.

(5)  $\lambda$  carries distinct countably complete ultrafilters extending the closed unbounded filter.

(6) There is a normal fine  $\kappa_\lambda$ -complete ultrafilter  $\mathcal{U}$  on  $P_{\kappa_\lambda}(\lambda)$  such that  $\mathcal{K}_\lambda \triangleleft \mathcal{U}$ .

*Proof.* (1) implies (2): Suppose  $U \neq \mathcal{K}_\lambda$  is an irreducible ultrafilter on  $\lambda$ . By irreducibility,  $\mathcal{K}_\lambda \not\leq_{\text{RF}} U$ . Since  $\text{sup } j_{\mathcal{K}_\lambda}[\lambda]$  carries no countably complete tail uniform ultrafilter in  $M_{\mathcal{K}_\lambda}$ ,  $j_U \upharpoonright M_{\mathcal{K}_\lambda}$  is not internal to  $M_{\mathcal{K}_\lambda}$ , since it is discontinuous at  $\text{sup } j_{\mathcal{K}_\lambda}[\lambda]$ . In other words  $U \not\sqsubseteq \mathcal{K}_\lambda$ .

(2) implies (3): Suppose  $U$  is a countably complete ultrafilter such that  $\mathcal{K}_\lambda \not\leq_{\text{RF}} U$  and  $U \not\sqsubseteq \mathcal{K}_\lambda$ . Since  $U \not\sqsubseteq \mathcal{K}_\lambda$ , by the characterization of internal ultrapower embeddings of  $M_{\mathcal{K}_\lambda}$  (Theorem 7.3.14),  $j_U$  must be discontinuous at  $\lambda$ . Since  $\mathcal{K}_\lambda \not\leq_{\text{RF}} U$ , by the universal property of  $\mathcal{K}_\lambda$ ,  $\text{sup } j_U[\lambda]$  carries a countably complete tail uniform ultrafilter in  $M_U$ . Let  $W$  be the ultrafilter on  $\lambda$  derived from  $j_U$  using  $\text{sup } j_U[\lambda]$ . Then  $W$  is weakly normal (by Corollary 4.4.18) and  $W$  concentrates on ordinals carrying countably complete tail uniform ultrafilters by the definition of a derived ultrafilter.

(3) implies (4): If  $\lambda$  carries a countably complete uniform ultrafilter, then  $\lambda$  carries a countably complete weakly normal ultrafilter that *does not* concentrate on ordinals carrying countably complete tail uniform ultrafilters (by Theorem 7.2.14); in the context of UA, this is  $\mathcal{K}_\lambda$ . Thus if (3) holds,  $\lambda$  carries distinct countably complete weakly normal ultrafilters.

(4) implies (5): Immediate given the fact that weakly normal ultrafilters extend the closed unbounded filter.

(5) implies (6): Assume (5) holds. Let  $U$  be the  $<_{\mathbb{k}}$ -least countably complete ultrafilter that extends the closed unbounded filter on  $\lambda$  and is not equal to  $\mathcal{K}_\lambda$ . We claim that for all  $D <_{\mathbb{k}} U$ ,  $D \sqsubset U$ . We will verify the criterion for showing  $D \sqsubset U$  given by Lemma 5.5.13 by showing that  $j_D(U) \leq_{\mathbb{k}} t_D(U)$  in  $M_D$ .

Let  $U' = t_D(U)$ . By Lemma 8.4.18,  $U'$  extends the closed unbounded filter on  $j_D(\lambda)$ .

Moreover we claim that  $j_D(\mathcal{K}_\lambda) \neq U'$ . To see this, note that

$$j_D^{-1}[j_D(\mathcal{K}_\lambda)] = \mathcal{K}_\lambda \neq U = j_D^{-1}[U']$$

Thus  $j_D(\mathcal{K}_\lambda) \neq U'$ , as claimed.

By elementarity, in  $M_D$ ,  $j_D(U)$  is the  $\leq_{\mathbf{k}}$ -least countably complete ultrafilter that extends the closed unbounded filter on  $j_D(\lambda)$  and is not equal to  $j_D(\mathcal{K}_\lambda)$ . It follows that  $j_D(U) \leq_{\mathbf{k}} U'$  in  $M_D$ . Lemma 5.5.13 now implies that  $D \sqsubset U$ , as claimed.

Let  $\kappa = \kappa_\lambda$ . Since  $\lambda$  is not isolated, by Lemma 7.4.19,  $\kappa$  is a limit of isolated cardinals. By Lemma 7.5.3, for all isolated cardinals  $\gamma < \kappa$ ,  $j_U[\gamma] \subseteq \gamma$ , and hence  $j_U[\kappa] \subseteq \kappa$ . Lemma 5.5.28 states that if  $\kappa$  is a strong limit cardinal such that  $j_U[\kappa] \subseteq \kappa$  and  $D \sqsubset U$  for all countably complete ultrafilters  $D$  with  $\lambda_D < \kappa$ , then  $U$  is  $\kappa$ -complete. Thus  $U$  is  $\kappa$ -complete. In particular,  $\text{Ord}^\kappa \subseteq M_U$ . Since  $\mathcal{K}_\lambda \sqsubset U$ ,  $j_{\mathcal{K}_\lambda}(\text{Ord}^\kappa) = \text{Ord}^{j_{\mathcal{K}_\lambda}(\kappa)} \cap M_{\mathcal{K}_\lambda} \subseteq M_U$ . As  $j_{\mathcal{K}_\lambda}(\kappa) > \lambda$  by Proposition 7.4.1, it follows that  $\text{Ord}^\lambda \cap M_{\mathcal{K}_\lambda} \subseteq M_U$ .

Now suppose  $A \in \text{Ord}^\lambda$ . Then  $j_{\mathcal{K}_\lambda}[A]$  is contained in a set  $B \in [\text{Ord}]^\lambda \cap M_{\mathcal{K}_\lambda}$ . Hence  $B \in M_U$ . We may assume  $B \subseteq j_{\mathcal{K}_\lambda}(A)$ , so that  $j_{\mathcal{K}_\lambda}^{-1}[B] = A$ . Since  $\mathcal{K}_\lambda \sqsubset U$ ,  $j_{\mathcal{K}_\lambda} \upharpoonright \alpha \in M_U$  for all ordinals  $\alpha$ . Hence  $A = j_{\mathcal{K}_\lambda}^{-1}[B] \in M_U$ . Thus  $\text{Ord}^\lambda \subseteq M_U$ .

If  $Z$  is a countably complete ultrafilter extending the closed unbounded filter on  $\lambda$  such that  $Z \triangleleft \mathcal{U}_0$ , then  $Z \sqsubset U$  so  $Z <_{\mathbf{k}} U$  by Lemma 5.5.14 and consequently by the minimality of  $U$ ,  $Z = \mathcal{K}_\lambda$ . In particular, no cardinal less than or equal to  $\lambda$  can be  $2^\lambda$ -supercompact in  $M_U$ . It follows that  $j_U(\kappa) > \lambda$ : otherwise  $j_U(\kappa) < \lambda$  is  $j_U(\lambda)$ -supercompact and since  $2^\lambda < j_U(\lambda)$ , we contradict the previous sentence.

Thus  $U$  is  $\kappa$  complete and  $j_U(\kappa) > \lambda$ . Let  $\mathcal{U}$  be the normal fine  $\kappa$ -complete ultrafilter on  $P_\kappa(\lambda)$  derived from  $j_U$  using  $j_U[\lambda]$ . It is easy to see that  $\mathcal{K}_\lambda \triangleleft \mathcal{U}$  (and in fact  $U \cong \mathcal{U}$ ). This completes the proof.  $\square$

We now turn to the question of pseudocompact cardinals first raised in Section 8.2. Recall that an elementary embedding is  $\lambda$ -pseudocompact if it is  $\gamma$ -tight for all  $\gamma \leq \lambda$ . Our main question asked whether  $\lambda$ -pseudocompactness and  $\lambda$ -supercompactness coincide below

rank-into-rank cardinals. If  $\lambda$  is the least cardinal where this fails, then it has the following property:

**Definition 8.4.20.** A cardinal  $\lambda$  is said to be *pathological* if there is an elementary embedding  $j : V \rightarrow M$  that is  $<\lambda$ -supercompact and  $\lambda$ -tight but not  $\lambda$ -supercompact. The embedding  $j$  is said to *witness the pathology of  $\lambda$* .

Equivalently,  $j : V \rightarrow M$  witnesses the pathology of  $\lambda$  if  $H(\lambda) \subseteq M$  and  $j[\lambda]$  can be covered by a set of size  $\lambda$  in  $M$ , yet  $j[\lambda] \notin M$ . The axiom  $I_2(\lambda)$  asserts that there is an elementary embedding  $j : V \rightarrow M$  with critical point less than  $\lambda$  such that  $j(\lambda) = \lambda$  and  $V_\lambda \subseteq M$ . By the Kunen Inconsistency Theorem (Theorem 4.2.37),  $j[\lambda] \notin M$ . Thus if  $I_2(\lambda)$  holds, then  $\lambda$  is pathological.

**Question 8.4.21.** Suppose  $\lambda$  is pathological. Must  $\text{cf}(\lambda) = \omega$ ? Must  $I_2(\lambda)$ ?

Our guess is that the answer is no.

We begin by establishing a dichotomy: pathological cardinals are either regular or of countable cofinality. For the proof we use the following fact, a generalization of the Kunen inconsistency theorem that is based on an observation due to Foreman [37].

**Theorem 8.4.22** (Foreman). *Suppose  $\lambda$  is a cardinal. Suppose  $Q$  is a transitive structure that is closed under countable sequences. Suppose  $k : Q \rightarrow H(\lambda)$  is a nontrivial elementary embedding. Suppose  $k$  has a fixed point above its critical point. Let  $\gamma$  be the supremum of the critical sequence of  $k$ . Then  $\lambda = \gamma^+$ .*

*Proof.* Suppose  $k$  has a fixed point above its critical point, and let  $\gamma$  be the least. Let  $\langle \kappa_n : n < \omega \rangle$  be the critical sequence of  $j$ . By a standard argument (which uses the closure of  $Q$  under countable sequences)  $\gamma = \sup_{n < \omega} \kappa_n$ . In particular,  $\gamma$  has countable cofinality.

Assume towards a contradiction that  $\gamma^+ < \lambda$ .

We will show  $\gamma^{+\omega} = \gamma^+$ . Shelah's Representation Theorem [27] yields a sequence  $\langle \delta_n : n < \omega \rangle$  of regular cardinals less than  $\gamma$  and a sequence  $\langle f_\alpha : \alpha < \gamma^+ \rangle$  that is cofinal in the

preorder  $(\prod_{n < \omega} \delta_n, \leq_{\text{bd}})$  where  $\leq_{\text{bd}}$  is the relation of eventual domination. Since  $\gamma^+ < \lambda$ ,  $\langle f_\alpha : \alpha < \gamma^+ \rangle \in H(\lambda)$ . By elementarity,  $Q$  satisfies that there is a sequence  $\langle g_\alpha : \alpha < \gamma^{+Q} \rangle$  cofinal in  $(\prod_{n < \omega} \delta_n, \leq_{\text{bd}})$ . Since  $Q$  is closed under countable sequences,  $\langle g_\alpha : \alpha < \gamma^{+Q} \rangle$  really is cofinal. (This is where we make substantial use of the hypothesis that  $Q$  is closed under countable sequences.) But the cofinality of  $(\prod_{n < \omega} \delta_n, \leq_{\text{bd}})$  is  $\gamma^+$ , and so it follows that  $\gamma^{+Q} = \gamma^+$ .

We now follow Woodin's proof of Kunen's inconsistency theorem. The structure  $Q$  satisfies that there is a partition of  $S_\omega^{\gamma^+}$  into stationary sets  $\langle S_\alpha : \alpha < \gamma^+ \rangle$ . Let  $\langle T_\alpha : \alpha < \gamma^+ \rangle = k(\langle S_\alpha : \alpha < \gamma^+ \rangle)$  (since  $H(\lambda)$  does and  $k$  is elementary). Note that  $k[\gamma^+]$  is an  $\omega$ -club in  $\gamma^+$ . The fact that  $k[\gamma^+]$  is  $\omega$ -closed follows from the fact that  $Q$  is closed under countable sequences, which implies that  $k$  is continuous at ordinals of countable cofinality.

Let  $\kappa$  be the critical point of  $k$ . Then  $k[\gamma^+] \cap T_\kappa \neq \emptyset$  since  $T_\kappa$  is stationary in  $\gamma^+$ . Therefore fix  $\xi < \gamma^+$  such that  $k(\xi) \in T_\kappa$ . Since  $k(\xi) \in T_\kappa$ ,  $\xi \in S_\omega^{\gamma^+}$ . Hence for some  $\alpha < \gamma^+$ ,  $\xi \in S_\alpha$ . Therefore  $k(\xi) \in k(S_\alpha) = T_{k(\alpha)}$ . In particular,  $T_{k(\alpha)} \cap T_\kappa \neq \emptyset$ . Since  $\langle T_\alpha : \alpha < \gamma^+ \rangle$  is a partition, it follows that  $k(\alpha) = \kappa$ . This contradicts that  $\kappa$  is the critical point of  $k$ .

Thus our assumption that  $\gamma^+ < \lambda$  was false. Hence  $\lambda = \gamma^+$ , so  $\gamma$  is the largest cardinal fixed by  $k$ . □

**Lemma 8.4.23.** *Suppose  $\lambda$  is a pathological cardinal of uncountable cofinality and  $j : V \rightarrow M$  witnesses the pathology of  $\lambda$ . Let  $A \in M$  be a cover of  $j[\lambda]$  of  $M$ -cardinality  $\lambda$ , and let  $\mathcal{U}$  be the fine ultrafilter on  $P(\lambda)$  derived from  $j$  using  $A$ . Let  $k : M_{\mathcal{U}} \rightarrow M$  be the factor embedding. Then  $\text{CRT}(k) > \lambda$  and therefore  $j_{\mathcal{U}}$  witnesses the pathology of  $\lambda$ .*

*Proof.* Let  $k : M_{\mathcal{U}} \rightarrow M$  be the factor embedding. We must show that  $\text{CRT}(k) > \lambda$ . Let  $\bar{A} = a_{\mathcal{U}}$ , so  $k(\bar{A}) = A$ . Clearly  $j_{\mathcal{U}}[\lambda] \subseteq \bar{A}$ , so  $|\bar{A}|^{M_{\mathcal{U}}} \geq |\bar{A}| \geq \lambda$ . On the other hand,  $|\bar{A}|^{M_{\mathcal{U}}} \leq k(|\bar{A}|^{M_{\mathcal{U}}}) = |A|^M = \lambda$ . Thus  $|\bar{A}|^{M_{\mathcal{U}}} = \lambda$ , so

$$k(\lambda) = k(|\bar{A}|^{M_{\mathcal{U}}}) = |A|^M = \lambda$$

Assume towards a contradiction that  $\text{CRT}(k) < \lambda$ . Since  $j$  is  $<\lambda$ -supercompact,  $j$  is  $<\lambda$ -strong, and therefore  $H(\lambda) \cap M = H(\lambda)$ . Thus  $k$  restricts to a nontrivial elementary embedding  $k : H(\lambda) \cap M_{\mathcal{U}} \rightarrow H(\lambda)$ . Since  $M_{\mathcal{U}}$  is closed under countable sequences, we can apply Foreman's inconsistency theorem. Since  $\lambda$  has uncountable cofinality and  $k(\lambda) = \lambda$ ,  $k$  has a fixed point in the interval  $(\text{CRT}(k), \lambda)$ . Therefore by Foreman's theorem (Theorem 8.4.22),  $\lambda = \gamma^+$  where  $\gamma$  is the supremum of the critical sequence of  $k$ . But  $j$  is  $\gamma$ -supercompact and  $j$  is continuous at  $\gamma$ , so by Lemma 4.2.25,  $j$  is  $\gamma^+$ -supercompact. Since  $j$  witnesses the pathology of  $\lambda$ ,  $j$  is not  $\lambda$ -supercompact. This contradicts that  $\lambda = \gamma^+$ .

Thus our assumption was false, and in fact  $\text{CRT}(k) \geq \lambda$ . Since  $k(\lambda) = \lambda$ , it follows that  $\text{CRT}(k) > \lambda$ . We finally show that this implies  $j_{\mathcal{U}}$  witnesses the pathology of  $\lambda$ .

The set  $\bar{A}$  witnesses that  $j_{\mathcal{U}}$  is  $\lambda$ -tight.

Assume towards a contradiction that  $j_{\mathcal{U}}$  is  $\lambda$ -supercompact. Since  $\text{CRT}(k) > \lambda$ ,  $k(j_{\mathcal{U}}[\lambda]) = k \circ j_{\mathcal{U}}[\lambda] = j[\lambda]$ , so  $j$  is  $\lambda$ -supercompact, which is a contradiction.

We finally show that  $j_{\mathcal{U}}$  is  $<\lambda$ -supercompact. Since  $j_{\mathcal{U}}$  is an ultrapower embedding, it suffices to show that  $j_{\mathcal{U}}$  is  $\delta$ -supercompact for all regular cardinals  $\delta < \lambda$ . To do this, it is enough to show that  $j[\delta] \in k[M_{\mathcal{U}}]$ , since then  $k^{-1}(j[\delta]) = j_{\mathcal{U}}[\delta]$  belongs to  $M_{\mathcal{U}}$ . By Solovay's Lemma (Lemma 4.4.29),  $j[\delta]$  is definable in  $M$  from  $\sup j[\delta]$  and parameters in  $j[V]$ . Since  $j[V] \subseteq k[M_{\mathcal{U}}]$  and  $k[M_{\mathcal{U}}]$  is closed under definability in  $M$ , to show  $j[\delta] \in k[M_{\mathcal{U}}]$ , it suffices to show that  $\sup j[\delta] \in k[M_{\mathcal{U}}]$ . To finish, we show that  $k(\sup j_{\mathcal{U}}[\delta]) = \sup j[\delta]$ , or in other words that  $k$  is continuous at  $\sup j_{\mathcal{U}}[\delta]$ . Since  $\text{CRT}(k) > \lambda$ , it is enough to show that  $\text{cf}^{M_{\mathcal{U}}}(\sup j_{\mathcal{U}}[\delta]) \leq \lambda$ . Since  $j_{\mathcal{U}}$  is  $\lambda$ -tight,  $j_{\mathcal{U}}$  is  $(\delta, \lambda)$ -tight, so by the easy direction of Theorem 7.2.12,  $\text{cf}^{M_{\mathcal{U}}}(\sup j_{\mathcal{U}}[\delta]) \leq \lambda$ , as desired.  $\square$

As a corollary, we eliminate many pathologies which a priori might have seemed plausible:

**Corollary 8.4.24.** *Suppose  $\lambda$  is a pathological cardinal. Either  $\lambda$  is regular or  $\lambda$  has countable cofinality.*

*Proof.* Assume  $\lambda$  has uncountable cofinality, and we will show that  $\lambda$  is regular. By Lemma 8.4.23,

the pathology of  $\lambda$  is witnessed by an ultrapower embedding  $i : V \rightarrow N$ . Since  $i$  is a  $<\lambda$ -supercompact ultrapower embedding,  $N$  is closed under  $<\lambda$ -sequences. If  $\lambda$  is singular, it follows that  $N$  is closed under  $\lambda$ -sequences, contradicting that  $i$  is not  $\lambda$ -supercompact. Therefore  $\lambda$  is regular.  $\square$

**Corollary 8.4.25.** *Suppose  $\lambda$  is a regular pathological cardinal. Suppose  $j : V \rightarrow M$  witnesses the pathology of  $\lambda$ . Let  $U$  be the ultrafilter on  $\lambda$  derived from  $j$  using  $\sup j[\lambda]$ , and let  $k : M_U \rightarrow M$  be the factor embedding. Then  $\text{CRT}(k) > \lambda$  and  $j_U$  witnesses the pathology of  $\lambda$ .*

*Proof.* Since  $\lambda$  is regular and  $j$  is  $\lambda$ -tight,  $\text{cf}^M(\sup j[\lambda]) = \lambda$ . Note that  $a_U = \sup j_U[\lambda]$ , so  $k(\sup j_U[\lambda]) = \sup j[\lambda]$ . We have  $\text{cf}^{M_U}(\sup j_U[\lambda]) \geq \text{cf}(\sup j_U[\lambda]) = \lambda$  on the one hand, and  $\text{cf}^{M_U}(\sup j_U[\lambda]) \leq k(\text{cf}^{M_U}(\sup j_U[\lambda])) = \text{cf}^M(\sup j[\lambda]) = \lambda$  on the other. Thus  $\text{cf}^{M_U}(\sup j_U[\lambda]) = \lambda$ . It follows that  $k(\lambda) = k(\text{cf}^{M_U}(\sup j_U[\lambda])) = \text{cf}^M(\sup j[\lambda]) = \lambda$ .

Given that  $k(\lambda) = \lambda$ , one can finish the proof as in Lemma 8.4.23. Instead of redoing this proof, however, we note that the corollary follows from an *application* of Lemma 8.4.23. Using Theorem 7.2.12, fix a cover  $\bar{A} \subseteq \sup j_U[\lambda]$  of  $j_U[\lambda]$  of  $M_U$ -cardinality  $\lambda$ . Let  $A = k(\bar{A})$ . Thus  $|A|^M = k(\lambda) = \lambda$ . Moreover, it is easy to see that

$$H^M(j[V] \cup \{\sup j[\lambda]\}) = H^M(j[V] \cup \{A\})$$

The left-to-right inclusion follows from the fact that  $\sup j[\lambda] = \sup A$  is definable from  $A$  in  $M$ , while the right-to-left inclusion follows from the fact that  $A = k(\bar{A})$  and  $k[M_U] = H^M(j[V] \cup \{\sup j[\lambda]\})$ . Therefore  $j_U = j_{\mathcal{U}}$  and the factor embeddings from  $M_{\mathcal{U}}$  into  $M$  is equal to  $k$ . Therefore by Lemma 8.4.23,  $\text{CRT}(k) > \lambda$  and  $j_U$  witnesses the pathology of  $\lambda$ .  $\square$

Pathological cardinals of countable cofinality, on the other hand, have a property that is a lot like  $I_2(\lambda)$ :

**Proposition 8.4.26.** *Suppose  $\lambda$  is a pathological cardinal of countable cofinality. Then there is a countably complete fine ultrafilter  $\mathcal{U}$  on  $P(\lambda)$  such that there is a nontrivial elementary*

embedding  $k : M_{\mathcal{U}} \rightarrow M$  such that  $k \circ j_{\mathcal{U}} = j$  and  $\lambda$  is the supremum of the critical sequence of  $k$ .

*Proof.* Immediate from the proof of Lemma 8.4.23. □

If the ultrafilter  $\mathcal{U}$  of the previous lemma is principal, then  $I_2(\lambda)$  holds. Under UA, there is a way to make this conclusion:

**Theorem 8.4.27** (UA). *Suppose  $\lambda$  is a pathological cardinal of countable cofinality. Then  $I_2(\lambda)$ .*

*Proof.* Let  $j : V \rightarrow M$  witness the pathology of  $\lambda$ . Then  $j$  witnesses that some cardinal  $\kappa < \lambda$  is  $\gamma$ -supercompact for all  $\gamma < \lambda$ . In particular, by our results on GCH (Theorem 6.3.12),  $\lambda$  is a strong limit cardinal.

Applying Proposition 8.4.26, fix a countably complete fine ultrafilter  $\mathcal{U}$  on  $P(\lambda)$  and a nontrivial elementary embedding  $k : M_{\mathcal{U}} \rightarrow M$  such that  $k \circ j_{\mathcal{U}} = j$  and  $\lambda$  is the supremum of the critical sequence of  $k$ .

By Corollary 8.2.24, fix a countably complete ultrafilter  $D$  with  $\lambda_D < \lambda$  and an elementary embedding  $k : M_D \rightarrow M_{\mathcal{U}}$  such that  $k$  is  $j_D(\lambda)$ -supercompact in  $M_D$ . Since  $\lambda$  is a strong limit cardinal of countable cofinality,  $j_D(\lambda) = \lambda$ . In particular,  $V_{\lambda} \cap M_D = V_{\lambda} \cap M_{\mathcal{U}}$ . Since  $(j_D \upharpoonright V_{\lambda}) : V_{\lambda} \rightarrow V_{\lambda} \cap M_D$  and  $(k \upharpoonright V_{\lambda} \cap M_D) : V_{\lambda} \cap M_D \rightarrow V_{\lambda}$  are elementary embeddings,

$$i = (k \upharpoonright V_{\lambda} \cap M_D) \circ (j_D \upharpoonright V_{\lambda})$$

is an elementary embedding from  $V_{\lambda}$  to  $V_{\lambda}$ . Moreover, suppose  $A \subseteq V_{\lambda}$  is a wellfounded relation. Then  $i(A) = \bigcup_{\alpha < \lambda} i(A \cap V_{\alpha})$  is also wellfounded since  $i(A) = k(j_D(A))$ , and  $k$  and  $j_D$  preserve wellfoundedness. Thus  $i$  extends to an elementary embedding  $i^* : V \rightarrow N$  where  $N$  is wellfounded, and it follows that  $I_2(\lambda)$  holds. □

Under UA, regular pathological cardinals are inaccessible:

**Proposition 8.4.28** (UA). *Suppose  $\lambda$  is a regular pathological cardinal. Then  $\lambda$  is strongly inaccessible and  $\mathcal{K}_{\lambda}$  witnesses the pathology of  $\lambda$ .*

*Proof.* By Lemma 8.4.23, there is a countably complete ultrafilter  $U$  such that  $j_U$  witnesses the pathology of  $\lambda$ . In particular,  $j_U$  is  $<\lambda$ -supercompact and  $\lambda$ -tight. It follows that  $j_U$  is  $\lambda$ -pseudocompact, since this just means  $j_U$  is  $\gamma$ -tight for all cardinals  $\gamma \leq \lambda$ . In particular,  $U$  is  $\lambda$ -irreducible by Proposition 8.2.3.

Note that  $j_U$  witnesses that  $\lambda$  is Fréchet. Suppose towards a contradiction that  $\lambda$  is a successor cardinal. Then by the Irreducibility Theorem (Corollary 8.2.19),  $j_U$  is  $\lambda$ -supercompact, contradicting that  $U$  witnesses the pathology of  $\lambda$ .

Thus  $\lambda$  is a limit cardinal. But  $j_U$  is  $<\lambda$ -supercompact, so by our results on GCH (Theorem 6.3.12),  $\lambda$  is a strong limit cardinal. Therefore  $\lambda$  is strongly inaccessible.

Finally we show that  $\mathcal{K}_\lambda$  witnesses the pathology of  $\lambda$ . Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $\mathcal{K}_\lambda$ . It suffices to show that  $j$  is not  $\lambda$ -supercompact, since by Theorem 7.3.33,  $j$  is  $<\lambda$ -supercompact and  $\lambda$ -tight. Suppose towards a contradiction that  $j$  is  $\lambda$ -supercompact. Then by Corollary 8.4.17, *every* ultrapower by a  $\lambda$ -irreducible ultrafilter is  $\lambda$ -supercompact, contradicting that  $j_U$  is not  $\lambda$ -supercompact. Thus  $\mathcal{K}_\lambda$  witnesses the pathology of  $\lambda$ . □

To summarize, under UA, if a cardinal is pathological, it is pathological for good reason:

**Theorem 8.4.29** (UA). *If  $\lambda$  is a pathological cardinal, then one of the following holds:*

- $\lambda$  is a strong limit singular cardinal of countable cofinality and  $I_2(\lambda)$  holds.
- $\lambda$  is a strongly inaccessible cardinal and  $\mathcal{K}_\lambda$  witnesses the pathology of  $\lambda$ . □

We now turn to the question of whether regular pathological cardinals can exist at all (without assuming UA). This is equivalent to the existence of pseudocompact embeddings that are not supercompact:

**Proposition 8.4.30.** *Suppose  $j : V \rightarrow M$  is an elementary embedding such that  $M$  has the  $\leq \gamma$ -covering property for all  $\gamma \leq \lambda$ . Then one of the following holds:*

- $M^\lambda \subseteq M$ .



- $j$  witnesses the pathology of a regular cardinal  $\delta \leq \lambda$ .

*Proof.* We claim that if  $\gamma \leq \lambda$  and  $j$  is  $\gamma$ -supercompact, then  $M^\gamma \subseteq M$ . (In fact, it suffices that  $P(\gamma) \subseteq M$  which follows from  $\gamma$ -supercompactness by Lemma 4.2.20.) To see this, suppose  $A \subseteq M$  and  $|A| \leq \gamma$ . Using the  $\leq \gamma$ -covering property, fix  $B \in M$  with  $A \subseteq B$  and  $|B|^M \leq \gamma$ . Then since  $P(\gamma) \subseteq M$ ,  $P(B) \subseteq M$ , and hence  $A \in M$ , as desired.

Therefore let  $\delta$  be the least cardinal such that  $j$  is not  $\delta$ -supercompact. Note that  $\delta$  is the least cardinal such that  $M^\delta \not\subseteq M$ , and therefore  $\delta$  is regular. If  $\delta \leq \lambda$ , then  $j$  witnesses that  $\delta$  is pathological. Otherwise  $\delta > \lambda$ , and hence  $j$  is  $\lambda$ -supercompact, so  $M^\lambda \subseteq M$  by the previous paragraph.  $\square$

Recall Woodin's Ultimate  $L$  Conjecture, which in a weak form states that if  $\delta$  is extendible then there is an inner model with the  $\delta$ -approximation and  $\delta$ -covering properties that satisfies the axiom  $V = \text{Ultimate } L$ . The motivation is that the canonical inner model with a supercompact cardinal should give rise to such an inner model. The same intuition motivates the *UA Conjecture*, which we now define.

**Definition 8.4.31.** We say that the *UA Hypothesis* holds at a cardinal  $\delta$  if there is an inner model of UA with the  $\delta$ -cover and  $\delta$ -approximation properties.

**Conjecture 8.4.32** (UA Conjecture). *ZFC proves that if  $\delta$  is an extendible cardinal, then the UA Hypothesis holds at  $\delta$ .*

It is a plausible conjecture that the axiom  $V = \text{Ultimate } L$  implies UA. If this is the case, then the Ultimate  $L$  Conjecture implies the UA Conjecture. On the other hand, the UA Conjecture implies the HOD Conjecture.

Our next theorem, due to Woodin in the case that  $\lambda$  is strongly inaccessible, shows that the pathologies we are studying are in a sense absolute.

**Definition 8.4.33.** Suppose  $\kappa < \lambda$  are cardinals. Then  $\lambda$  is  $\kappa$ -*pathological* if there is an elementary embedding  $j : V \rightarrow M$  with critical point  $\kappa$  that witnesses the pathology of  $\lambda$ .

**Theorem 8.4.34.** *Suppose  $\delta < \kappa < \lambda$  are cardinals and  $\lambda$  is regular. Suppose  $N$  is an inner model with the  $\delta$ -cover and  $\delta$ -approximation properties. Suppose  $\lambda$  is  $\kappa$ -pathological. Then  $\lambda$  is  $\kappa$ -pathological in  $N$ .*

The proof uses several facts from the remarkable theory of models with the approximation and covering properties. First, we will need Hamkins's uniqueness theorem for models with the approximation and covering properties:

**Theorem 8.4.35** (Hamkins). *Suppose  $\delta$  is a cardinal and  $N_0$  and  $N_1$  are inner models of ZFC with the  $\delta$ -approximation and  $\delta$ -covering properties at an ordinal  $\alpha$ . If  $N_0 \cap H(\delta^+) = N_1 \cap H(\delta^+)$  then  $N_0 \cap P(\alpha) = N_1 \cap P(\alpha)$ .  $\square$*

Second we need the Hamkins-Reitz theorem on the propagation of the covering property:

**Theorem 8.4.36** (Hamkins-Reitz). *Suppose  $\delta$  is a cardinal and  $N$  is an inner model of ZFC with the  $\delta$ -approximation and  $\delta$ -covering properties. Then  $N$  has the  $\lambda$ -covering property for every cardinal  $\lambda \geq \delta$ .  $\square$*

With these tools in hand, we can prove Theorem 8.4.34.

*Proof of Theorem 8.4.34.* Applying Corollary 8.4.25, let  $U$  be a  $\kappa$ -complete weakly normal ultrafilter on  $\lambda$  such that  $j_U$  witnesses the pathology of  $\lambda$ . Let  $W = U \cap N$ . By Theorem 7.3.22,  $W$  belongs to  $N$  (and in fact every  $\delta$ -complete  $N$ -ultrafilter belongs to  $N$ ).

Let  $j : V \rightarrow M$  be the ultrapower of the universe by  $U$ . Let  $i : N \rightarrow P$  be the ultrapower of  $N$  by  $W$ . Let  $k : P \rightarrow j(N)$  be the factor embedding, defined by

$$k(i(f)(\sup i[\lambda])) = j(f)(\sup j[\lambda])$$

Thus  $k \circ i = j \upharpoonright N$  and  $k(\sup i[\lambda]) = \sup j[\lambda]$ .

We now show that

$$\text{cf}^{j(N)}(\sup j[\lambda]) = \lambda$$

Since  $j$  is  $\lambda$ -tight,  $\text{cf}^M(\sup j[\lambda]) = \lambda$ . Since  $j(N)$  has the  $\delta$ -approximation and  $\delta$ -covering properties in  $M$ , in fact  $j(N)$  has the  $\lambda$ -covering property in  $M$  (by Theorem 8.4.36). Therefore  $j(N)$  correctly computes the cofinality of  $\sup j[\lambda]$  in  $M$ , and it follows that  $\text{cf}^{j(N)}(\sup j[\lambda]) = \lambda$ .

We now claim that  $k(\lambda) = \lambda$  and  $\text{cf}^N(\sup i[\lambda]) = \lambda$ . The argument is by now familiar. Since  $k(\sup i[\lambda]) = \sup j[\lambda]$ ,

$$k(\text{cf}^N(\sup i[\lambda])) = \text{cf}^{j(N)}(\sup j[\lambda]) = \lambda$$

Since  $\lambda \leq \text{cf}^N(\sup i[\lambda])$ ,  $\lambda \leq k(\lambda) \leq k(\text{cf}^N(\sup i[\lambda])) = \lambda$ . Thus  $k(\lambda) = \lambda$ . Similarly  $\lambda \leq \text{cf}^N(\sup i[\lambda]) \leq k(\text{cf}^N(\sup i[\lambda])) \leq \lambda$ , so  $\text{cf}^N(\sup i[\lambda]) = \lambda$ .

We claim  $j(N) \cap P(\alpha) = N \cap P(\alpha)$  for all  $\alpha < \lambda$ . The argument is due to Hamkins [17]. Fix  $\alpha < \lambda$ . Since  $j$  is  $<\lambda$ -supercompact,  $M \cap P(\alpha) = P(\alpha)$ . By elementarity,  $j(N)$  has the  $\delta$ -approximation and  $\delta$ -covering properties in  $M$ , and in particular  $j(N)$  has the  $\delta$ -approximation and  $\delta$ -covering properties at  $\alpha$ . Similarly,  $N$  has the  $\delta$ -approximation and  $\delta$ -covering properties at  $\alpha$ . But  $N \cap H(\delta^+) = j(N) \cap H(\delta^+)$  since  $\text{CRT}(j) > \delta$ . Thus  $j(N) \cap P(\alpha) = N \cap P(\alpha)$  by the uniqueness theorem (Theorem 8.4.35).

We claim  $j[\alpha] \in j(N)$  for all  $\alpha < \lambda$ . This follows from the  $\delta$ -approximation property for  $j(N)$  in  $M$  and the  $\delta$ -covering property for  $N$  in  $V$ . Let  $\alpha_* = \sup j[\alpha]$ . Suppose  $\sigma \in P_\delta(\alpha_*) \cap j(N)$ . Fix  $\tau \in P_\delta(\alpha)$  such that  $j^{-1}[\sigma] \subseteq \tau$ . Then  $\sigma \cap j[\alpha] = \sigma \cap j[\tau] = \sigma \cap j(\tau) \in j(N)$ .

Since  $P(\alpha) \cap N = P(\alpha) \cap j(N)$  for all  $\alpha < \lambda$ , we have that  $H(\lambda) \cap N = H(\lambda) \cap j(N)$ . Let  $H = H(\lambda) \cap N$  and let  $Q = H(\lambda) \cap P$ . We claim that  $k \upharpoonright Q \in N$ . The proof is a generalization of the proof of Woodin's Universality Theorem for models with the approximation property. Since  $Q$  is transitive,  $k \upharpoonright Q$  is the inverse of the transitive collapse of  $k[Q]$ , and therefore it suffices to show that  $k[Q] \in N$ . Since  $N$  satisfies the  $\delta$ -approximation property, it suffices to show that  $k[Q] \cap \sigma \in N$  for any  $\sigma \in P_\delta(H) \cap N$ . Fix such a  $\sigma$ . Since  $N$  has the  $\delta$ -cover property, there is some  $\tau \in P_\delta(Q) \cap N$  with  $k^{-1}[\sigma] \subseteq \tau$ . Since  $P^\delta \cap N \subseteq P$ ,  $\tau \in P$ , and

hence  $\tau \in Q$ . Since  $\text{CRT}(k) \geq \delta$ ,  $k(\tau) = k[\tau]$ . Thus

$$k[Q] \cap \sigma = k[\tau] \cap \sigma = k(\tau) \cap \sigma \in N$$

Thus  $k[Q] \cap \sigma \in N$ . By the  $\delta$ -approximation property,  $k[Q] \in N$ , and hence  $k \upharpoonright Q \in N$ .

We now apply Foreman's inconsistency result in  $N$ . Assume towards a contradiction that  $\text{CRT}(k) < \lambda$ . Note that  $k$  restricts to an elementary embedding from  $Q$  to  $H(\lambda) \cap N$  that belongs to  $N$ . Moreover  $Q$  is closed under  $\omega$ -sequences in  $N$ . Then it follows from Theorem 8.4.22 applied in  $N$  that  $\lambda = \gamma^+$  where  $\gamma$  is the supremum of the critical sequence of  $k$ . It follows that  $\lambda$  is the successor of a singular cardinal  $\gamma$  of countable cofinality in  $N$ .

Recall that  $j[\gamma] \in j(N)$  since  $j[\alpha] \in j(N)$  for all  $\alpha < \lambda$ . Note that

$$j[P_\kappa^N(\gamma)] = \{j[\sigma] : \sigma \in P_\kappa^N(\gamma)\} = P_\kappa^{j(N)}(j[\gamma])$$

In particular,  $j[P_\kappa^N(\gamma)] \in j(N)$  since it is definable over  $j(N)$  from  $j[\gamma] \in j(N)$ . Recall that  $\lambda = \gamma^{+N}$ . Therefore by König's theorem, there is a surjection  $f : P_\kappa^N(\gamma) \rightarrow \lambda$  in  $N$ . Then  $j(f)[j[P_\kappa^N(\gamma)]] = j[\lambda]$ . Thus  $j[\lambda] \in j(N)$ . In particular,  $j[\lambda] \in M$ . This contradicts the fact that  $j$  is not  $\lambda$ -supercompact.

Thus our assumption that  $\text{CRT}(k) < \lambda$  was false. Moreover since  $k(\lambda) = \lambda$ , it follows that  $\text{CRT}(k) > \lambda$ .

We established the following:

- $k : P \rightarrow j(N)$  is an elementary embedding with critical point above  $\lambda$
- $j[\alpha] \in j(N)$  for all  $\alpha < \lambda$ .
- $j[\lambda] \notin j(N)$ .

Since  $\text{CRT}(k) > \lambda$ ,  $\text{CRT}(i) = \kappa$ . The proof that  $i[\alpha] \in P$  for all  $\alpha < \lambda$ , and  $i[\lambda] \notin P$  now proceeds exactly as in Lemma 8.4.23. Thus  $i$  witnesses that  $\lambda$  is  $\kappa$ -pathological in  $N$ .  $\square$

The UA Conjecture rules out certain kinds of pathological cardinals that are not obviously ruled out in ZFC alone:

**Theorem 8.4.37.** *Suppose  $\delta$  is a cardinal and the UA Hypothesis holds at  $\delta$ . If  $\gamma$  is a singular cardinal, then  $\gamma^+$  is not  $\kappa$ -pathological for any  $\kappa > \delta$ .*

*Proof.* Let  $N$  be an inner model of UA with the  $\delta$ -covering and  $\delta$ -approximation properties. Assume towards a contradiction that  $\gamma^+$  is  $\kappa$ -pathological for some  $\kappa > \delta$ . By Theorem 8.4.34,  $\gamma^+$  is pathological in  $N$ . But by Theorem 8.4.36,  $\gamma^+$  is a successor cardinal in  $N$ : this follows from the fact that every ordinal in the interval  $[\gamma, \gamma^+]$  has cofinality less than  $\gamma$  in  $N$  by the  $\gamma$ -covering property. Since  $N$  satisfies UA, by Proposition 8.4.28, no successor cardinal is pathological in  $N$ . This is a contradiction.  $\square$

**Question 8.4.38 (ZFC).** Can the successor of a singular cardinal be pathological?

Finally let us tie this all back up with the question of whether UA implies that  $\lambda$ -irreducible ultrafilters are  $\lambda$ -supercompact.

**Theorem 8.4.39.** *Suppose  $\delta$  is a cardinal and assume there is an inner model  $N$  with the  $\delta$ -approximation and  $\delta$ -covering properties that has no regular pathological cardinals above  $\delta$ . Suppose  $\lambda > \delta$  is a cardinal. Suppose  $j : V \rightarrow M$  is an elementary embedding with critical point above  $\delta$  such that  $M$  has the  $\leq \gamma$ -covering property for all  $\gamma \leq \lambda$ . Then  $M^\lambda \subseteq M$ .*

*Proof.* Suppose not. Then by Proposition 8.4.30,  $j$  witnesses that some regular cardinal  $\gamma \leq \lambda$  is  $\text{CRT}(j)$ -pathological. By Theorem 8.4.34,  $\gamma$  is  $\text{CRT}(j)$ -pathological in  $N$ , contrary to our assumption that  $N$  has no regular pathological cardinals above  $\delta$ .  $\square$

Thus granting the UA Conjecture, either pseudocompactness principles are (eventually) equivalent to supercompactness or else UA is consistent with regular pathological cardinals. it seems more reasonable to make the following conjecture:

**Conjecture 8.4.40.** *UA is consistent with the existence of a regular pathological cardinal.*

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