Real Orientations of Lubin--Tate Spectra and the Slice Spectral Sequence of a C4-Equivariant Height-4 Theory

Real Orientations of Lubin–Tate Spectra and the Slice Spectral Sequence of a C_4-Equivariant Height-4 Theory

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Abstract

In this thesis, we show that Lubin–Tate spectra at the prime 2 are Real oriented and Real Landweber exact. The proof is by application of the Goerss–Hopkins–Miller theorem to algebras with involution. For each height n, we compute the entire homotopy fixed point spectral sequence for E_n with its C_2-action given by the formal inverse. We study, as the height varies, the Hurewicz images of the stable homotopy groups of spheres in the homotopy of these C_2-fixed points.

We completely compute the slice spectral sequence of the C_4-spectrum $BP^{(\!(C_4)\!)}\langle 2\rangle$. After periodization and $K(4)$-localization, this spectrum is equivalent to a height-4 Lubin–Tate theory E_4 with C_4-action induced from the Goerss–Hopkins–Miller theorem. In particular, our computation shows that $E_4^{hC_{12}}$ is 384-periodic.

Contents

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1 Real orientations of Lubin–Tate spectra

The first part of this thesis (Sections 1–7) is based on joint work with Jeremy Hahn.

1.1 Motivation and main results

Topological K-theory is a remarkably useful cohomology theory that has produced important homotopy-theoretic invariants in topology. Many deep facts in topology have surprisingly simple proofs using topological K-theory. For instance, Adams's original solution [1] to the Hopf invariant one problem used ordinary cohomology and secondary cohomology operations, but, together with Atiyah [3], he later discovered a much simpler solution using complex K-theory and its Adams operations. He also studied the real K-theory of real projective spaces [2] and used it to resolve the vector fields on spheres problem.

In 1966, Atiyah [7] formalized the connection between complex K-theory, KU, and real K-theory, KO. The complex conjugation action on complex vector bundles induces a natural C_2-action on KU. Under this action, the C_2-fixed points and the homotopy fixed points of KU are both KO:

$$KU^{C_2} \simeq KU^{hC_2} \simeq KO.$$

Furthermore, there is a homotopy fixed point spectral sequence computing the homotopy groups of KO, starting from the action of C_2 on the homotopy groups of KU:

$$E_2^{s,t} = H^s(C_2; \pi_t KU) \Longrightarrow \pi_{t-s} KO.$$

The spectrum KU, equipped with this C_2-action and considered as a C_2-spectrum, is called Atiyah's Real K-theory $K_{\mathbb{R}}$.

The spectrum KU is a complex oriented cohomology theory, which means that there is a map $MU \longrightarrow KU$, where MU is the complex cobordism spectrum. Early work on MU due to Milnor [55], Novikov [56, 57, 58], and Quillen [60] established the complex cobordism spectrum as a critical tool in modern stable homotopy theory, with deep connections to algebraic geometry and number theory through the theory of formal groups [42, 64, 50, 59]. The complex orientation of KU induces a map of rings

$$\pi_* MU \longrightarrow \pi_* KU$$

on the level of homotopy groups. Quillen's [60] calculation of $\pi_* MU$ shows that the map above produces a one dimensional formal group law over $\pi_* KU$, which turns out to be the multiplicative formal group law $\mathbb{G}_m(x, y) = x + y - xy$.

Analogously as in the case of KU, the complex conjugation action on complex manifolds induces a natural C_2-action on MU. This action produces the Real cobordism spectrum $MU_{\mathbb{R}}$ of Landweber [46], Fujii [25], and Araki [6]. The underlying spectrum of $MU_{\mathbb{R}}$ is MU, with the C_2-action given by complex conjugation.

Complex conjugation acts on KU and MU by coherently commutative ($\mathbb{E}_\infty$) maps, making $K_{\mathbb{R}}$ and $MU_{\mathbb{R}}$ commutative C_2-spectra. The complex orientation of KU is compatible with the complex conjugation action, and it can be refined to a *Real orientation*

$$MU_{\mathbb{R}} \to K_{\mathbb{R}}.$$

Complex K-theory belongs to a more general class of spectra — the Lubin–Tate spectra — central to the study of chromatic homotopy theory and the stable homotopy groups of spheres. These spectra are reverse-engineered from algebra as follows. Given a formal group $\mathbb{G}$ of finite height n over a perfect field k of characteristic p, Lubin and Tate [49] showed that $\mathbb{G}$ admits a universal deformation defined over a complete local ring R with residue field k. The ring R is non-canonically isomorphic to $W(k)[[u_1, u_2, \ldots, u_{n-1}]]$, over which the formal group law is characterized by a map

$$MU_* \longrightarrow W(k)[[u_1, u_2, \ldots, u_{n-1}]][u^{\pm}].$$

This map can be shown to be Landweber exact. Applying the Landweber exact functor theorem yields a complex oriented homology theory represented by a homotopy commutative ring spectrum $E_{(k,\mathbb{G})}$. When $k = \mathbb{F}_{p^n}$ and $\mathbb{G}$ is the Honda formal group law, the resulting Lubin–Tate spectrum is commonly called E_n, the height n Morava E-theory. Since the height 1 Morava E-theory is $KU_p^\wedge$, the Lubin–Tate spectra can be thought of as the higher height analogues of K-theory.

To endow the Lubin–Tate theories $E_{(k,\mathbb{G})}$ with coherent multiplicative structures, Goerss, Hopkins, and Miller computed the moduli space of $\mathbb{A}_\infty$- and $\mathbb{E}_\infty$-structures on $E_{(k,\mathbb{G})}$. The group $\mathbb{G}_n$ of automorphisms of the formal group $\mathbb{G}$ naturally acts on $\pi_* E_{(k,\mathbb{G})}$, and the Goerss–Hopkins–Miller computation demonstrates that there is in fact an action of $\mathbb{G}_n$ on $E_{(k,\mathbb{G})}$ by $\mathbb{E}_\infty$-ring automorphisms.

For any subgroup $G \subseteq \mathbb{G}_n$, one can use the Goerss–Hopkins–Miller action to construct a homotopy fixed point spectrum $E_{(k,\mathbb{G})}^{hG} := F(EG_+, E_{(k,\mathbb{G})})^G$. There are homotopy fixed point spectral sequences of the form

$$E_2^{s,t} = H^s(G; \pi_t(E_{(k,\mathbb{G})})) \Longrightarrow \pi_{t-s}(E_{(k,\mathbb{G})}^{hG}).$$

The spectra E_n^{hG} turn out to be the essential building blocks of the p-local stable homotopy category. In particular, the homotopy groups $\pi_* E_n^{hG}$ assemble to the stable homotopy groups of spheres. To be more precise, the chromatic convergence theorem of Hopkins and Ravenel [64] exhibits the p-local sphere spectrum $\mathbb{S}_{(p)}$ as the inverse limit of the chromatic tower

$$\cdots \longrightarrow L_{E_n}\mathbb{S} \longrightarrow L_{E_{n-1}}\mathbb{S} \longrightarrow \cdots \longrightarrow L_{E_0}\mathbb{S},$$

where each $L_{E_n}\mathbb{S}$ is assembled via the chromatic fracture square

$$\begin{array}{ccc} L_{E_n}\mathbb{S} & \longrightarrow & L_{K(n)}\mathbb{S} \\ \downarrow & & \downarrow \\ L_{E_{n-1}}\mathbb{S} & \longrightarrow & L_{E_{n-1}}L_{K(n)}\mathbb{S}, \end{array}$$

where $K(n)$ is the nth Morava K-theory.

Devinatz and Hopkins [23] proved that $L_{K(n)}\mathbb{S} \simeq E_n^{h\mathbb{G}_n}$, and, furthermore, that the Adams–Novikov spectral sequence computing $L_{K(n)}\mathbb{S}$ can be identified with the associated homotopy fixed point spectral sequence for $E_n^{h\mathbb{G}_n}$. The fixed point spectrum $E_n^{h\mathbb{G}_n}$ admits resolutions by $\{E_n^{hG} \,|\, G \subset \mathbb{G}_n\}$, where G ranges over finite subgroups of $\mathbb{G}_n$.

For the rest of the paper, we designate $p = 2$. When $k = \mathbb{F}_2$, and $\mathbb{G}$ is the multiplicative formal group $G_m(x, y) = x + y - xy$, we find that $E_{(\mathbb{F}_2, G_m)}^{hC_2} = KO_2^{\wedge}$, the 2-adic completion of real K-theory. For this reason, the spectra E_n^{hG} are commonly called the *higher real K-theories*.

At height 2, these homotopy fixed points are known as TMF and TMF with level structures. Computations of the homotopy groups of these spectra are done by Hopkins–Mahowald [43], Bauer [11], Mahowald–Rezk [52], Behrens–Ormsby [17], Hill–Hopkins–Ravenel [40], and Hill–Meier [41].

For higher heights $n > 2$, the homotopy fixed points E_n^{hG} are notoriously difficult to compute. Prior to the present work, essentially no progress had been made. One of the chief reasons that these homotopy fixed points are so difficult to compute is because the group actions are constructed purely from obstruction theory. This stands in contrast to the cases of Atiyah's Real K-theory $K_{\mathbb{R}}$ and Real cobordism $MU_{\mathbb{R}}$, whose actions come from geometry. The main theorem of this work establishes the first known connection between the obstruction-theoretic actions on Lubin–Tate theories and the geometry of complex conjugation:

Theorem 1.1. *Let k be a perfect field of characteristic 2, $\mathbb{G}$ a height n formal group over k, and $E_{(k,\mathbb{G})}$ the corresponding Lubin–Tate theory. Suppose G is a finite subgroup of the Morava stabilizer group that contains the central subgroup C_2. Then there is a G-equivariant map*

$$N_{C_2}^G MU_{\mathbb{R}} \longrightarrow E_{(k,\mathbb{G})},$$

where $N_{C_2}^G(-)$ is the Hill–Hopkins–Ravenel norm functor.

In particular, when $G = C_2$, Theorem 1.1 implies that for all height $n \geq 1$, the classical complex orientation $MU \to E_n$ can be refined to a Real orientation

$$MU_{\mathbb{R}} \longrightarrow E_n.$$

The presence of geometry, aside from its intrinsic interest, has tremendous computational consequences. Hu and Kriz [44] were able to completely compute the homotopy fixed point spectral sequence for $MU_{\mathbb{R}}$. Combining our main theorem with the Hu–Kriz computation, we obtain the first calculations for $E_n^{hC_2}$, valid for arbitrarily large heights n.

Theorem 1.2. *The E_2-page of the $RO(C_2)$-graded homotopy fixed point spectral sequence* *of E_n is*

$$E_2^{s,t}(E_n^{hC_2}) = W(\mathbb{F}_{2^n})[[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-1}]][\bar{u}^{\pm}] \otimes \mathbb{Z}[u_{2\sigma}^{\pm}, a_\sigma]/(2a_\sigma).$$

The classes $\bar{u}_1$, $\ldots$, $\bar{u}_{n-1}$, $\bar{u}^{\pm}$, and a_σ are permanent cycles. All the differentials in the spectral sequence are determined by the differentials

$$\begin{aligned} d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) &= \bar{u}_k \bar{u}^{2^k-1} a_\sigma^{2^{k+1}-1}, \ \ 1 \leq k \leq n-1, \\ d_{2^{n+1}-1}(u_{2\sigma}^{2^{n-1}}) &= \bar{u}^{2^n-1} a_\sigma^{2^{n+1}-1}, \ \ k = n, \end{aligned}$$

and multiplicative structures.

The existence of equivariant orientations renders computations that rely on the slice spectral sequence tractable. This observation was first made in the solution of the Kervaire invariant problem by Hill, Hopkins, and Ravenel in 2009.

In their landmark paper [39], Hill, Hopkins, and Ravenel established that the Kervaire invariant elements θ_j do not exist for $j \geq 7$ (see also [54, 37, 38] for surveys on the result). A key construction in their proof is the spectrum Ω, which detects the Kervaire invariant elements in the sense that if $\theta_j \in \pi_{2^{j+1}-2}\mathbb{S}$ is an element of Kervaire invariant 1, then the Hurewicz image of θ_j under the map $\pi_*\mathbb{S} \to \pi_*\Omega$ is nonzero.

The detecting spectrum Ω is constructed using equivariant homotopy theory as the C_8-fixed point of a genuine C_8-spectrum $\Omega_{\mathbb{O}}$, which in turn is an equivariant localization of $MU^{((C_8))} := N_{C_2}^{C_8} MU_{\mathbb{R}}$. In particular, there is a C_8-equivariant orientation

$$MU^{((C_8))} \to \Omega_{\mathbb{O}}.$$

For $G = C_{2^n}$, the G-spectrum $MU^{((G))}$ and its equivariant localizations are amenable to computations. To analyze the C_8-equivariant homotopy groups of $\Omega_{\mathbb{O}}$, Hill, Hopkins, and

Ravenel generalized the C_2-equivariant filtration of Hu–Kriz [44] and Dugger [24] to a G-equivariant Postnikov filtration for all finite groups G. They called this the *slice filtration*. Given any G-equivariant spectrum X, the slice filtration produces the slice tower $\{P^*X\}$, whose associated slice spectral sequence strongly converges to the $RO(G)$-graded homotopy groups $\pi_\bigstar^G X$.

Using the slice spectral sequence, Hill, Hopkins, and Ravenel proved the Gap Theorem and the Periodicity Theorem, which state, respectively, that $\pi_i^{C_8}\Omega_{\mathbb{O}} = 0$ for $-4 < i < 0$, and that there is an isomorphism $\pi_*^{C_8}\Omega_{\mathbb{O}} \cong \pi_{*+256}^{C_8}\Omega_{\mathbb{O}}$. The two theorems together imply that

$$\pi_{2^{j+1}-2}\Omega = \pi_{2^{j+1}-2}^{C_8}\Omega_{\mathbb{O}} = 0$$

for all $j \geq 7$, from which the nonexistence of the corresponding Kervaire invariant elements follows.

Analogues of the Kervaire invariant elements exist at odd primes. In 1978, Ravenel [61] computed the C_p-homotopy fixed points of the Lubin–Tate spectrum E_{p-1} and proved that the p-primary Kervaire invariant elements do not exist for all $p \geq 5$.

In light of Ravenel's work, Hill, Hopkins, and Ravenel had hoped that the homotopy fixed points of a certain Lubin–Tate theory would entail the nonexistence of the bona fide Kervaire invariant elements. Indeed, they mentioned in [38] that the Detection Theorem held for $E_4^{hC_8}$, which made it a promising candidate to resolve the Kervaire invariant problem. However, because of the computational difficulties surrounding the homotopy fixed point spectral sequence, they could not prove the Gap Theorem and the Periodicity Theorem for $E_4^{hC_8}$.

Instead, in [39], they opted to consider $\Omega_{\mathbb{O}}^{C_8}$, which serves to mimic $E_4^{hC_8}$, but benefits from the geometric rigidity it inherits from $MU^{((C_8))}$: once the theory of slice filtrations is set up, the Gap Theorem and the Periodicity Theorem are immediate.

Despite the computational access granted via $MU^{((G))}$, its localizations are unsuitable for chromatic homotopy theory because the E_2-pages of their slice spectral sequences are too large and contain many unnecessary classes. Thus, one cannot hope to resolve the $K(n)$-local sphere by fixed points of the localizations of $MU^{((G))}$.

To address this situation, one could hope to quotient $MU^{((G))}$ by generators in its equivariant stable homotopy group in order to cut down the size of its slice spectral sequence and its coefficient group. This can been done at heights ≤ 2. At higher heights, however, the quotienting process fails to preserve the higher coherent structure ($\mathbb{E}_\infty$-ness) of the spectrum.

For example, even at $G = C_2$, the spectra $BP_{\mathbb{R}}\langle n\rangle$ and the Real Johnson–Wilson theories $E\mathbb{R}(n)$ are not known to be rings when $n \geq 3$ (see [45] and [41, Remark 4.19]). They also have no clear connection to the Lubin–Tate spectra E_n. Therefore, despite its computability, it is difficult to use $E\mathbb{R}(n)$ to obtain information about the higher real K-theories E_n^{hG} and the $K(n)$-local sphere.

Theorem 1.1 and Theorem 1.2 combine the computational power of the slice spectral sequence with the import of the Lubin–Tate spectra. Preponderant in chromatic homotopy theory, the Lubin–Tate spectra have smaller coefficient rings than the localizations of $MU^{((G))}$, so they are ideal candidates for resolving the $K(n)$-local sphere.

It is a consequence of Theorem 1.2 that E_n is an even C_2-spectrum, and, in particular, has pure and isotropic slice cells. Theorem 1.1 and Theorem 1.2 can be used to compute the slice filtration of E_n for all n, considered as a G-spectrum, where G is a cyclic group of order a power of 2. It will follow that E_n has pure and isotropic G-slice cells.

Once this is established, the proofs in [39] are applicable to E_n^{hG}. Hence E_n^{hG} satisfies a

Gap Theorem and a Periodicity Theorem, and, moreover, there is a factorization

$$\begin{array}{ccc} MU^{((G))} & \longrightarrow & E_n \\ \downarrow & \nearrow & \\ D^{-1}MU^{((G))} & & \end{array} \quad .$$

In particular, there is a C_8-equivariant map from the detection spectrum $\Omega_{\mathbb{O}} \longrightarrow E_4$.

1.2 Summary of contents

We now turn to a more detailed summary of the contents. To prove Theorem 1.1, we first consider a specific Lubin–Tate spectra. Let $\widehat{E(n)}$ denote the 2-periodic completed Johnson–Wilson theory, with

$$\pi_*(\widehat{E(n)}) = \mathbb{Z}_2[[v_1, v_2, ..., v_{n-1}]][u^{\pm}], \ |u| = 2.$$

This spectrum is a version of Morava E-theory. In particular, it is a complex-oriented $\mathbb{E}_\infty$-ring spectrum. Work of Goerss, Hopkins, and Miller [28, 65] identifies the space of $\mathbb{E}_\infty$-ring automorphisms of $\widehat{E(n)}$, and in particular ensures the existence of a central Galois C_2-action by $\mathbb{E}_\infty$-ring maps. At the level of homotopy groups, C_2 acts as the formal inverse of the canonical formal group law.

There is also a natural C_2-action on MU, by complex-conjugation. To this end, we first prove the following:

Theorem 1.3. *The spectrum $\widehat{E(n)}$, with its central Galois C_2-action, is **Real oriented**. That is to say, it receives a C_2-equivariant map*

$$MU_{\mathbb{R}} \longrightarrow \widehat{E(n)}$$

from the Real cobordism spectrum $MU_{\mathbb{R}}$.

Leveraging the Hill–Hopkins–Ravenel norm functor [39], Theorem 1.1 is a formal consequence of Theorem 1.3.

To prove Theorem 1.3 it will be helpful to sketch a construction of $\widehat{E(n)}$ as a ring spectrum, not yet worrying about any C_2-actions. Recall that there is a periodic version of complex cobordism, denoted MUP, that is an $\mathbb{E}_\infty$-ring spectrum. We denote the symmetric monoidal category of MUP-module spectra by MUP-**Mod**. The subgroupoid spanned by the unit and its automorphisms is the space $BGL_1(MUP)$, which is naturally an infinite loop space. Associated to any map of spaces $f : X \to BGL_1(MUP)$ is a Thom MUP-module Thom(f) [4]. The category of spaces over $BGL_1(MUP)$ is symmetric monoidal, and an associative algebra object in this category gives rise to an $\mathbb{A}_\infty$-algebra structure on its Thom spectrum [5].

Consider now the following diagram of categories:

$$\begin{array}{ccccc} \mathbb{A}_\infty\left(\mathbf{Spaces}_{/BGL_1(MUP)}\right) & \xrightarrow{Thom} & \mathbb{A}_\infty(\text{MUP-}\mathbf{Mod}) & \xrightarrow{Forget} & \mathbb{A}_\infty(\mathbf{Spectra}) \\ \Omega\uparrow & & \downarrow L_{K(n)} & & \downarrow L_{K(n)} \\ \mathbf{Spaces}_{/B^2GL_1(MUP)} & & \mathbb{A}_\infty(\text{MUP-}\mathbf{Mod}) & \xrightarrow{Forget} & \mathbb{A}_\infty(\mathbf{Spectra}) \\ & & & & \uparrow Forget \\ & & & & \mathbb{E}_\infty(\mathbf{Spectra}). \end{array} \qquad (\star)$$

In Section 2, we will construct a certain map of spaces $X \to B^2GL_1(MUP)$. Applying Ω and then the Thom spectrum construction, we obtain an $\mathbb{A}_\infty$-MUP-algebra $E(n)$ that is a 2-periodic version of Johnson–Wilson theory. The $K(n)$-localization of $E(n)$ is $\widehat{E(n)}$, equipped with the structure of an $\mathbb{A}_\infty$-MUP-algebra.

It is a consequence of work of Goerss, Hopkins, and Miller [28, 65] that we may lift the $\mathbb{A}_\infty$-ring spectrum underlying $\widehat{E(n)}$ to an $\mathbb{E}_\infty$-ring spectrum. Indeed, letting $\mathcal{C}^{\simeq}$ denote the maximal subgroupoid of a category $\mathcal{C}$, they prove that the path-component of $\mathbb{A}_\infty(\mathbf{Spectra})^{\simeq}$ containing $\widehat{E(n)}$ is *equivalent* to a path component in $\mathbb{E}_\infty(\mathbf{Spectra})^{\simeq}$, with the equivalence given by the forgetful functor.

Our strategy for the proof of Theorem 1.3 is to produce a Real orientation $MU_{\mathbb{R}} \to \overline{E(n)}$ into *some* ring spectrum $\overline{E(n)}$ with C_2-action. The $\overline{E(n)}$ we produce is obviously equivalent to $\widehat{E(n)}$ as a spectrum, and the C_2-action is obviously the Galois one up to homotopy. However, it is *not* at all obvious that the full, coherent C_2-action on $\overline{E(n)}$ is the Galois action. To prove it, we must make full use of the Goerss–Hopkins–Miller theorem.

We produce $\overline{E(n)}$ via a C_2-equivariant lift of the above construction of $\widehat{E(n)}$:

Construction 1.4. *In section 3, each of the categories in the diagram* ($\star$) *will be equipped with a C_2-action, yielding an equivariant diagram:*

$$\begin{array}{ccccc}
 & & & & \overset{op}{\circlearrowleft} \\
\circlearrowleft & & \circlearrowleft & & \\
\mathbb{A}_\infty\left(\boldsymbol{Spaces}_{/BGL_1(MUP)}\right) & \xrightarrow{Thom} & \mathbb{A}_\infty(MUP\text{-}\boldsymbol{Mod}) & \xrightarrow{Forget} & \mathbb{A}_\infty(\boldsymbol{Spectra}) \\
\Omega^\sigma \uparrow & & \downarrow L_{K(n)} & & \downarrow L_{K(n)} \\
\boldsymbol{Spaces}_{/B^\rho GL_1(MUP)} & & \mathbb{A}_\infty(MUP\text{-}\boldsymbol{Mod}) & \xrightarrow{Forget} & \mathbb{A}_\infty(\boldsymbol{Spectra}) \circlearrowleft op \\
\circlearrowleft & & \circlearrowleft & & \uparrow Forget \\
 & & & & \mathbb{E}_\infty(\boldsymbol{Spectra}). \\
 & & & & \underset{trivial}{\circlearrowleft}
\end{array} \qquad (\star\star)$$

*The action on $\mathbb{E}_\infty$(**Spectra**) will be the trivial C_2-action. The action on $\mathbb{A}_\infty$(**Spectra**) will be the non-trivial op action that takes an algebra to its opposite.*

Remark 1.5. By a homotopy fixed point in a category $\mathcal{C}$ with C_2-action we mean an object in the category $\mathcal{C}^{hC_2}$. For example, a homotopy fixed point in $\mathbb{E}_\infty(\mathbf{Spectra})$ with its trivial action is just an $\mathbb{E}_\infty$-ring spectrum with C_2-action by $\mathbb{E}_\infty$-ring maps. A homotopy fixed point for the op action on $\mathbb{A}_\infty(\mathbf{Spectra})$ is an $\mathbb{A}_\infty$-algebra A equipped with an *involution*, meaning a coherent algebra map $\sigma : A \to A^{\mathrm{op}}$. We believe the use of algebras with involutions to be the most interesting feature of our construction.

In Sections 4 and 5, we will refine our map $X \to B^2GL_1(MUP)$ to an equivariant map $X \to B^\rho GL_1(MU_{\mathbb{R}}P)$. Applying Ω^σ produces a homotopy fixed point of $\mathbb{A}_\infty\left(\mathbf{Spaces}_{/BGL_1(MUP)}\right)$, which in turn equips $E(n)$ with an $\mathbb{A}_\infty$-involution. After $K(n)$-localizing, we obtain a C_2-action on $\widehat{E(n)}$ by $\mathbb{A}_\infty$-involutions. The Goerss–Hopkins–Miller Theorem [28, 65] proves that any such action on $\widehat{E(n)}$ may be lifted to one by $\mathbb{E}_\infty$-ring maps. Since Goerss, Hopkins, and Miller furthermore calculate the entire space of $\mathbb{E}_\infty$-ring automorphisms of $\widehat{E(n)}$, we may determine any $\mathbb{E}_\infty$-C_2-action on $\widehat{E(n)}$ by its effect on homotopy groups.

In Section 6, we look towards computational applications of the above results. For simplicity, we use a specific Morava E-theory E_n that is defined via a lift of the height n Honda formal group law over $\mathbb{F}_{2^n}$. Its homotopy groups are

$$\pi_* E_n = W(\mathbb{F}_{2^n})[[u_1, u_2, \ldots, u_{n-1}]][u^{\pm}].$$

Using Theorem 1.1 and leveraging Hu and Kriz's computation of the homotopy fixed point spectral sequence for $MU_{\mathbb{R}}$ [44], we prove Theorem 1.2. As a corollary, we learn that as a C_2-spectrum, E_n is *strongly even* and *Real Landweber exact* in the sense of Hill–Meier [41].

Theorem 1.6. *E_n is strongly even and Real Landweber exact. More precisely, $\underline{\pi}_{k\rho-1}E_n = 0$ and $\underline{\pi}_{k\rho}E_n$ is a constant Mackey functor for all $k \in \mathbb{Z}$. The Real orientation $MU_{\mathbb{R}} \to E_n$*

induces a map

$$MU_{\mathbb{R}\bigstar}(X) \otimes_{MU_{2*}} (E_n)_{2*} \to E_{n\bigstar}(X)$$

which is an isomorphism for every C_2-spectrum X.

The second author's detection theorem for $MU_{\mathbb{R}}^{hC_2}$, joint with Li, Wang, and Xu [47], allows us to conclude a detection theorem for $E_n^{hC_2}$. Roughly speaking, as the height grows, an increasing amount of the Kervaire and $\bar{\kappa}$-families in the stable homotopy groups of spheres are detected by $\pi_* E_n^{hC_2}$. More precisely, we prove in Section 7 the following:

Theorem 1.7 (Detection theorem for $E_n^{hC_2}$)**.**

1. *For $1 \leq i, j \leq n$, if the element $h_i \in Ext_{\mathcal{A}_*}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ or $h_j^2 \in Ext_{\mathcal{A}_*}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hC_2}$ is nonzero.*

2. *For $1 \leq k \leq n-1$, if the element $g_k \in Ext_{\mathcal{A}_*}^{4,2^{k+2}+2^{k+3}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hC_2}$ is nonzero.*

Remark 1.8. We freely use the language of ∞-categories throughout this work, and will refer to an ∞-category simply as a category. If $\mathcal{C}$ is a symmetric monoidal category, we use $\mathbb{A}_\infty(\mathcal{C})$ to denote the category of associative algebra objects in $\mathcal{C}$, and similarly use $\mathbb{E}_\infty(\mathcal{C})$ to denote commutative algebra objects. We will use **Spaces** to denote the symmetric monoidal category of *pointed* spaces under cartesian product.

2 Thom spectra and Johnson–Wilson theory

In this section we will describe a non-equivariant construction of $\widehat{E(n)}$, a Landweber exact Morava E-theory with

$$\pi_*(\widehat{E(n)}) \cong \mathbb{Z}_2[[v_1, v_2, \cdots, v_{n-1}]][u^{\pm}].$$

Our construction is a riff on Theorem 1.4 of [10].

We begin with a brief review of the classical theory of Thom spectra. Useful references, in the language of ∞-categories we espouse here, include [4] and [5].

If R is an $\mathbb{E}_\infty$-ring spectrum, then the category of R-modules acquires a symmetric monoidal structure. The full subcategory consisting of the unit and its automorphisms is denoted $BGL_1(R)$. The symmetric monoidal structure equips $BGL_1(R)$ with an infinite loop space structure, and we write $BGL_1(R) \simeq \Omega^\infty \Sigma gl_1(R)$. The space $GL_1(R) \simeq \Omega^\infty gl_1(R)$ sits in a pullback square

$$\begin{array}{ccc} GL_1(R) & \longrightarrow & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R), \end{array}$$

where $\pi_0(R)^\times$ is the subset of units of $\pi_0(R)$ under multiplication. From this latter description of $GL_1(R)$, it is clear that

$$\pi_*(BGL_1(R)) \cong \pi_{*-1}(GL_1(R)) \cong \pi_{*-1}(R), \text{ for } * > 1.$$

Given a map of spaces $X \to BGL_1(R)$, we can form the Thom R-module by taking the colimit of the composite functor $X \to BGL_1(R) \subset R$-**Mod**. If X is a loop space and $X \to BGL_1(R)$ is a loop map, then the main theorem of [5] shows that the associated Thom spectrum is an $\mathbb{A}_\infty$-R-algebra. Similarly, if X is an infinite loop space and $X \to BGL_1(R)$ an infinite loop map, then [5] shows that the associated Thom spectrum is an $\mathbb{E}_\infty$-R-algebra.

Given two maps $f_1 : X_1 \to BGL_1(R)$ and $f_2 : X_2 \to BGL_1(R)$, we may use the infinite loop space structure on $BGL_1(R)$ to produce a product map

$$(f_1, f_2) : X_1 \times X_2 \to BGL_1(R) \times BGL_1(R) \to BGL_1(R).$$

The Thom R-module $\mathrm{Thom}(f_1, f_2)$ is the R-module smash product $\mathrm{Thom}(f_1) \wedge_R \mathrm{Thom}(f_2)$.

We may speak not only of $BGL_1(R)$, but also of the infinite loop space $\mathrm{Pic}(R)$. As a symmetric monoidal category, $\mathrm{Pic}(R)$ is the full subcategory of R-$\mathbf{Mod}^{\simeq}$ spanned by the invertible R-modules. It is a union of path components each of which is equivalent to $BGL_1(R)$. Again, [5] explains that the colimit of an infinite loop map $X \to \mathrm{Pic(R)} \subset R$-$\mathbf{Mod}$ is an $\mathbb{E}_\infty$-R-algebra. Our only use of this more general construction is to recall the following classical example:

Example 2.1. *The complex J-homomorphism is an infinite loop map $BU \times \mathbb{Z} \to Pic(\mathbb{S})$, obtained via the algebraic K-theory construction on $\coprod BU(n) \to Pic(\mathbb{S})$. The resulting Thom $\mathbb{E}_\infty$-ring spectrum is the periodic complex cobordism spectrum, denoted MUP. The 2-connective cover of spectra $bu \to ku$ is an infinite loop map $BU \to BU \times \mathbb{Z}$, which induces a map of Thom $\mathbb{E}_\infty$-ring spectra $MU \to MUP$.*

The map $J : BU \times \mathbb{Z} \to Pic(\mathbb{S})$ decomposes as a product of the infinite loop map $BU \to BGL_1(\mathbb{S})$ and the loop map $\mathbb{Z} \to Pic(\mathbb{S})$. This yields an equivalence of Thom $\mathbb{A}_\infty$-ring spectra

$$MUP \simeq MU \wedge \left(\bigvee_{n \in \mathbb{Z}} S^{2n} \right) \simeq \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MU,$$

which allows us to calculate $\pi_*(MUP) \cong \pi_*(MU)[u^{\pm}] \cong \mathbb{Z}[x_1, x_2, \cdots][u^{\pm}]$, where $|u| = 2$

and $|x_i| = 2i$. *The complex-conjugation action on* $BU \times \mathbb{Z}$ *by infinite loop maps yields a* C_2*-action on* MUP *by* $\mathbb{E}_\infty$*-ring homomorphisms; we will make no use of this action in the current section, but much use of it in Sections 3 and 4.*

We now specialize the discussion and embark on our construction of $E(n)$. Suppose that we choose a non-zero $\alpha \in \pi_2(MUP) \cong \pi_3(BGL_1(MUP))$. Then, e.g. by [10, Theorem 5.6] or [5, Theorem 4.10], there is an equivalence of MUP-module spectra

$$\mathrm{Thom}(\alpha) \simeq \mathrm{Cofiber}(\Sigma^2 MUP \xrightarrow{\alpha} MUP) \simeq MUP/\alpha.$$

If we choose a sequence of elements $(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \pi_2(MUP)$, we may produce a map

$$S^3 \times S^3 \times \cdots S^3 \to BGL_1(MUP)$$

and an associated Thom MUP-module

$$\begin{aligned} \mathrm{Thom}(\alpha_1, \alpha_2, \cdots, \alpha_n) &\simeq (MUP/\alpha_1) \wedge_{MUP} (MUP/\alpha_2) \wedge_{MUP} \cdots \wedge_{MUP} (MUP/\alpha_n) \\ &\simeq MUP/(\alpha_1, \alpha_2, \cdots, \alpha_n). \end{aligned}$$

If the sequence $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is *regular* in $\pi_*(MUP)$, then the usual cofiber sequences imply that

$$\pi_*(MUP/(\alpha_1, \alpha_2, \cdots, \alpha_n)) \cong \pi_*(MUP)/(\alpha_1, \alpha_2, \cdots, \alpha_n).$$

Finally, we may even mod out an infinite regular sequence $(\alpha_1, \alpha_2, \cdots)$ by using the natural maps

$$S^3 \to S^3 \times S^3 \to S^3 \times S^3 \times S^3 \to \cdots$$

to produce a filtered colimit of MUP-modules

$$MUP/\alpha_1 \to MUP/(\alpha_1, \alpha_2) \to MUP/(\alpha_1, \alpha_2, \alpha_3) \to \cdots \to MUP/(\alpha_1, \alpha_2, \cdots).$$

Proposition 2.2. *Each map $\alpha_i : S^3 \to BGL_1(MUP)$ can be given the structure of a loop map. In other words, the above construction of the MUP-module $MUP/(\alpha_1, \alpha_2, \cdots)$ can be refined to a construction of an $\mathbb{A}_\infty$-MUP-algebra.*

Proof. It will suffice to construct a map $\widetilde{\alpha_i} : BS^3 \to B^2GL_1(MUP)$ such that $\Omega\widetilde{\alpha_i} \simeq \alpha_i$. This is equivalent to asking that the precomposition of the map $\widetilde{\alpha_i} : BS^3 \to B^2GL_1(MUP)$ with the inclusion $S^4 \to BS^3$ be adjoint to the map $\alpha_i : S^3 \to BGL_1(MUP)$. In fact, any map $S^4 \to B^2GL_1(MUP)$ automatically admits at least one factorization through BS^3. The reason is that BS^3 admits an even cell decomposition: there is a filtered colimit

$$S^4 = Y_1 \to Y_2 \to Y_3 \to \cdots \to BS^3$$

and pushouts

$$\begin{array}{ccc} S^{4n-1} & \longrightarrow & Y_{n-1} \\ \downarrow & & \downarrow \\ D^{4n} & \longrightarrow & Y_n. \end{array}$$

This cell decomposition is easily seen from the model $BS^3 \simeq \mathbb{HP}^\infty$, the infinite dimensional quaternionic projective space, where it is the canonical cell-decomposition corresponding to the inclusions of the $\mathbb{HP}^\ell$. The obstructions to factoring a map $Y_{n-1} \to B^2GL_1(MUP)$ through Y_n therefore live in $\pi_{4n-1}(MUP)$. This group is 0, as explained in Example 2.1. □

To summarize, if we choose any regular sequence $(\alpha_1, \alpha_2, \cdots) \in \pi_*(MUP) \cong \mathbb{Z}[x_1, x_2, \cdots][u^\pm]$, each element of which lies in degree 2, then we may construct the quotient MUP-module

$MUP/(\alpha_1, \alpha_2, \cdots)$ as an $\mathbb{A}_\infty$-MUP-algebra. The following standard lemma allows us to use Proposition 2.2 to build Morava E-theories as $\mathbb{A}_\infty$ algebras:

Lemma 2.3. *Let $\mathbb{G}$ denote a formal group of height n over the field $\mathbb{F}_2$, and E the associated Morava E-theory. Then there is a map $MUP \to E$, classifying a universal deformation of $\mathbb{G}$, which may be described as first taking the quotient of MUP by a regular sequence $(\alpha_1, \alpha_2, \cdots)$ of degree 2 classes and then performing $K(n)$-localization.*

Remark 2.4. If the reader prefers, they will lose no intuition by thinking of the regular sequence

$$(\alpha_1, \alpha_2, \cdots) = (x_{2^n-1}u^{2-2^n} - u, x_2u^{-1}, x_4u^{-3}, x_5u^{-4}, x_6u^{-5}, x_8u^{-7}, \cdots),$$

where the classes x_iu^{-i+1} that are included are those such that either

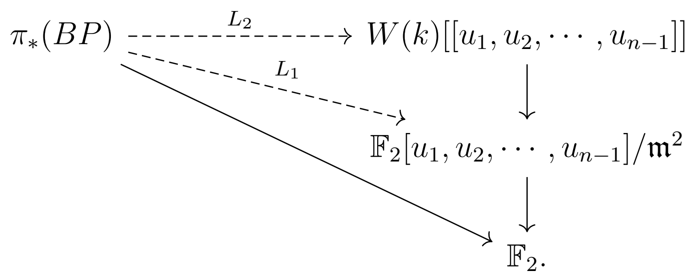
- i is not one less than a power of 2, or
- i is larger that $2^n - 1$.

However, since there are non-isomorphic formal groups over $\mathbb{F}_2$, not every Morava E-theory is obtained by quotienting out this particular sequence.

Definition 2.5. We denote by $E(n)$ the quotient of MUP by the regular sequence $(\alpha_1, \alpha_2, \cdots)$ of Lemma 2.3, and say that $E(n)$ is a 2-periodic form of Johnson–Wilson theory. Proposition 2.2 provides a (not necessarily unique) construction of $E(n)$ as an $\mathbb{A}_\infty$ MUP-algebra. We are deliberately vague about which formal group $\mathbb{G}$ defines $E(n)$, so that we may handle all cases at once.

Proof of Lemma 2.3. The formal group $\mathbb{G}$ is classified by some map of (ungraded) rings

$\pi_*(BP) \to \mathbb{F}_2$. View this map as the solid arrow in the diagram of ring homomorphisms



According to [65, §5.10], as long as the lift L_1 is chosen correctly, any further lift L_2 will classify a universal deformation. Furthermore, we may assume that v_i maps to u_i for $i \leq n-1$ under L_1, while v_n maps to 1. Each v_j for $j > n$ then maps to some $L_1(v_j)$ that is an $\mathbb{F}_2$-linear combination of $L_1(v_1), L_1(v_2), \cdots L_1(v_n)$. Write $\phi(v_j)$ to denote the element of $\pi_*(BP)$ that is given by the same linear combination of $v_1, v_2, \cdots v_n$. Then the map L_2 can be chosen to be the quotient by the regular sequence $(v_n - 1, v_{n+1} - \phi(v_{n+1}), v_{n+2} - \phi(v_{n+2}), \cdots)$.

Using the invertible element u to move elements by even degrees, we may identify $\pi_2(MUP)$ with $\pi_*(MU)$. Inside of $\pi_*(MU)$ we identify $\pi_*(BP)$ by viewing v_i as x_{2^i-1}. This allows us to talk about $\phi(v_j)$ as a class in $\pi_2(MUP)$.

To obtain the lemma, one mods out by the regular sequence $(\alpha_1, \alpha_2, \cdots) \in \pi_2(MUP)$, where one mods out, in order of $i \in \mathbb{N}$:

- All $x_i u^{-i+1}$ with i not one less than a power of 2.
- The class $x_i u^{-i+1} - u$ where $i = 2^n - 1$.
- The classes $x_i u^{-i+1} - \phi(v_j)$ where $i = 2^j - 1$ for $j > n$.

$\square$

3 Categories with involutions

In this section, we will construct a diagram of categories with C_2-action and equivariant functors between them:

$$
\begin{array}{ccccc}
 & & & & op \\
\mathbb{A}_\infty\left(\mathbf{Spaces}_{/BGL_1(MUP)}\right) & \xrightarrow{Thom} & \mathbb{A}_\infty(MUP\text{-}\mathbf{Mod}) & \xrightarrow{Forget} & \mathbb{A}_\infty(\mathbf{Spectra}) \\
\Omega^\sigma \uparrow & & \downarrow L_{K(n)} & & \downarrow L_{K(n)} \\
\mathbf{Spaces}_{/B^\rho GL_1(MUP)} & & \mathbb{A}_\infty(MUP\text{-}\mathbf{Mod}) & \xrightarrow{Forget} & \mathbb{A}_\infty(\mathbf{Spectra}) \quad \text{op} \\
 & & & & \uparrow Forget \\
 & & & & \mathbb{E}_\infty(\mathbf{Spectra}). \\
 & & & & trivial
\end{array}
\qquad (\star\star)
$$

Remark 3.1. An equivariant functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between categories with C_2-action is an arrow in the functor category $\mathrm{Hom}(BC_2, \mathbf{Cat}_\infty)$. Such an arrow contains a substantial amount of data, and is in particular *not* determined by its underlying functor $F_{\text{underlying}}$ of non-equivariant categories. For example, using

$$\sigma_i : \mathcal{C}_i \to \mathcal{C}_i$$

to denote the C_2-action on $\mathcal{C}_i$, part of the data of F is a choice of natural isomorphism

$$\sigma_2 \circ F_{\text{underlying}} \simeq F_{\text{underlying}} \circ \sigma_1.$$

Nonetheless, for notational convenience we will be somewhat fast and loose regarding the distinction between F and $F_{\text{underlying}}$.

Let $\mathbf{MonCat}_{Lax}$ denote the category of monoidal categories and lax monoidal functors. In the language of [51, §4.1], this is the category of coCartesian fibrations of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{A}\mathrm{ss}^{\otimes}$, with morphisms maps of ∞-operads $\mathcal{C}_1 \to \mathcal{C}_2$ over $\mathcal{A}\mathrm{ss}^{\otimes}$ that are not required to preserve coCartesian arrows. Remark 4.1.1.8 in [51] constructs a C_2-action *rev* on $\mathbf{MonCat}_{Lax}$. If $(\mathcal{C}, \otimes)$ is a monoidal ∞-category, then $(\mathcal{C}_{rev}, \otimes_{rev})$ has the same underlying category as $\mathcal{C}$ but the *opposite* $\otimes$-structure, with $X \otimes_{rev} Y$ in $\mathcal{C}_{rev}$ calculated as $Y \otimes X$ in $\mathcal{C}$. We call a homotopy fixed point for this rev action a monoidal category $(\mathcal{C}, \otimes)$ *with involution.* Such a category is equipped with a coherent equivalence $\mathcal{C} \xrightarrow{\simeq} \mathcal{C}_{rev}$.

Remark 4.1.1.8 also constructs an equivalence between $\mathbb{A}_\infty$-algebra objects A in $\mathcal{C}$ and $\mathbb{A}_\infty$-algebra objects A^{rev} in $\mathcal{C}_{rev}$. If $\mathcal{C}$ is equipped with an involution, then there is an induced C_2-action on $\mathbb{A}_\infty(\mathcal{C})$. In other words, there is an equivariant functor

$$\overset{rev}{\circlearrowleft}\mathbf{MonCat}_{Lax} \xrightarrow{\mathbb{A}_\infty(-)} \overset{trivial}{\circlearrowleft}\mathbf{Cat}_\infty,$$

and so a homotopy fixed point in $\mathbf{MonCat}_{Lax}$ is sent to one in $\mathbf{Cat}_\infty$.

Finally, we also consider the category $\mathbf{SymMonCat}_{Lax}$ of symmetric monoidal categories and lax functors. The last paragraph of Remark 4.1.1.8 of [51] ensures that the sequence of forgetful functors

$$\mathbf{SymMonCat}_{Lax} \xrightarrow{\text{Forget}} \mathbf{MonCat}_{Lax} \xrightarrow{\text{Forget}} \mathbf{Cat}_\infty$$

is equivariant, with the trivial C_2-action on $\mathbf{SymMonCat}_{Lax}$, the rev action on $\mathbf{MonCat}_{Lax}$, and the trivial action on $\mathbf{Cat}_\infty$.

Example 3.2. *Consider the category* ***Set*** *of sets, equipped with the cartesian symmetric monoidal structure. The trivial C_2-action on* ***Set*** *by symmetric monoidal identity* *functors allows us to view **Set** as a homotopy fixed point for the trivial C_2-action on $\boldsymbol{SymMonCat}_{Lax}$. This equips the underlying monoidal category of **Set** with a canonical involution, which in turn equips the category of monoids with a C_2-action. This is the classical op action that takes a monoid M to its opposite monoid M^{op}, which has the same underlying set but the opposite multiplication.*

Example 3.3. *More generally, if $\mathcal{C}$ is any symmetric monoidal category, then the trivial action on $\mathcal{C}$ by symmetric monoidal identity functors induces an involution, and therefore an op action on $\mathbb{A}_\infty(\mathcal{C})$. There is an equivariant sequence of categories*

$$\mathbb{E}_\infty(\mathcal{C}) \xrightarrow{Forget} \mathbb{A}_\infty(\mathcal{C}) \xrightarrow{Forget} \mathcal{C},$$

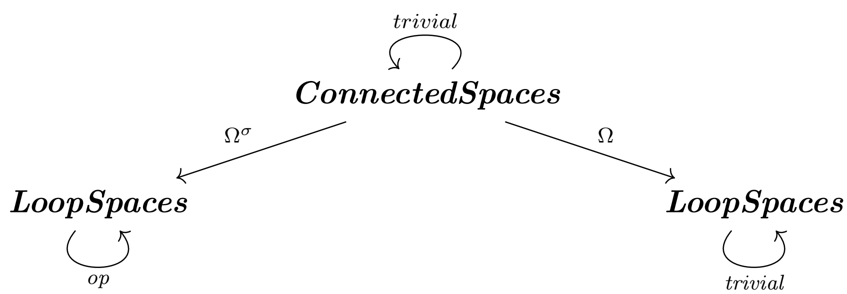
where $\mathbb{E}_\infty(\mathcal{C})$ and $\mathcal{C}$ are given the trivial C_2-actions and $\mathbb{A}_\infty(\mathcal{C})$ is given the op action.

Taking $\mathcal{C}$ in Example 3.3 to be the category **Spaces** of pointed spaces, we obtain the op action on $\mathbb{A}_\infty$(**Spaces**). We call a homotopy fixed point for this action an $\mathbb{A}_\infty$-space *with involution*; such spaces, considered as groupoids, are special cases of categories with involution. Any spectrum E with C_2-action has an underlying $\mathbb{A}_\infty$-space with involution $\Omega^\infty E$.

Example 3.4. *Suppose that X is an $\mathbb{A}_\infty$-space with involution. Then the monoidal category $\boldsymbol{Spaces}_{/X}$ is equipped with an involution. Concretely, this involution takes an algebra map $A \to X$ to the natural algebra map $A^{op} \to X^{op} \xrightarrow{inv} X$.*

Remark 3.5. If a monoid M happens to be a group, then there is a canonical equivalence $M \simeq M^{\mathrm{op}}$ defined by the inverse homomorphism $m \mapsto m^{-1}$. Our next few observations exploit an analogue of this equivalence for *grouplike* $\mathbb{A}_\infty$-spaces. We denote by **LoopSpaces** the full subcategory of grouplike objects in $\mathbb{A}_\infty$(**Spaces**). Notice that the property of being grouplike is preserved under the op action on $\mathbb{A}_\infty(\textbf{Spaces})$, so there is an op action on **LoopSpaces**.

Construction 3.6. *There is a diagram of equivalences of equivariant categories*



The equivariant functors Ω *and* Ω^σ *share the same underlying, non-equivariant functor.*

Proof. It is classical that Ω and the bar construction provide inverse equivalences of the non-equivariant categories **ConnectedSpaces** and **LoopSpaces**. This category has a universal property: it is the initial pointed category with all connected colimits. As such, any C_2-action on it admits an essentially unique equivalence with the trivial C_2-action. $\square$

Corollary 3.7. *Suppose* X *is a grouplike* $\mathbb{A}_\infty$*-space with involution. Then there exists some connected space with* C_2*-action* $B^\sigma X$ *such that* $\Omega^\sigma B^\sigma X \simeq X$*. There is a natural* C_2*-equivariant functor*

$$\textbf{\textit{Spaces}}_{/B^\sigma X} \xrightarrow{\Omega^\sigma} \mathbb{A}_\infty\left(\textbf{\textit{Spaces}}_{/X}\right),$$

where the latter object is the category with C_2*-action underlying the monoidal category with involution from Example 3.4.*

Consider the sequence of right adjoints

$$\textbf{Spaces} \xrightarrow{(-)_0} \textbf{ConnectedSpaces} \xrightarrow{\Omega^\sigma} \mathbb{A}_\infty(\textbf{Spaces}) \xrightarrow{Forget} \textbf{Spaces}$$

By Example 3.3, this is an equivariant functor from **Spaces** with trivial action to **Spaces** with trivial action. As such it sends any space with C_2-action X to some other space with C_2-action, which by abuse of notation we denote $\Omega^\sigma X$.

Proposition 3.8. *Suppose X is a space with C_2-action. Then the space with C_2-action $\Omega^\sigma X$ is the equivariant function space $Hom(S^\sigma, X)$. In other words, the action on a loop $S^1 \to X$ is given by both precomposing with the antipode on S^1 and postcomposing with the action on X.*

Proof. By the Yoneda lemma, $\Omega^\sigma X$ must be the equivariant function space $\mathrm{Hom}(S^1, X)$ for *some* C_2-action on S^1. To determine that this action is by the antipode, and not the trivial action, we look at the sequence of equivariant functors

$$\mathbf{Groups} \xrightarrow{Bar} \mathbf{ConnectedSpaces} \xrightarrow{\Omega^\sigma} \mathbf{LoopSpaces} \xrightarrow{\pi_0} \mathbf{Groups},$$

which connects groups with trivial action to groups with op action. The only *natural* isomorphism between a group and its opposite is given by $g \mapsto g^{-1}$, which is non-trivial on underlying sets. □

Specializing the discussion, recall that MUP is the Thom spectrum of the J homomorphism $BU \times \mathbb{Z} \xrightarrow{J} BGL_1(\mathbb{S})$. Since the J homomorphism is an infinite loop map, MUP acquires the structure of an $\mathbb{E}_\infty$-ring spectrum. The complex-conjugation action by infinite loop maps on $BU \times \mathbb{Z}$ gives MUP a C_2-action by $\mathbb{E}_\infty$-ring maps. This in turn induces C_2-actions by symmetric monoidal functors on the categories MUP**-Mod** and

$\mathbf{Spaces}_{/BGL_1(MUP)}$. There is a diagram of lax symmetric monoidal functors

$$\begin{array}{ccccc} \mathbf{Spaces}_{/BGL_1(MUP)} & \xrightarrow{Thom} & \text{MUP-}\mathbf{Mod} & \xrightarrow{Forget} & \mathbf{Spectra} \\ & & \downarrow{\scriptstyle L_{K(n)}} & & \downarrow{\scriptstyle L_{K(n)}} \\ & & \text{MUP-}\mathbf{Mod} & \xrightarrow{Forget} & \mathbf{Spectra}. \end{array}$$

Since the C_2-action on MUP is unital, the entire diagram becomes C_2-equivariant once we equip **Spectra** with the trivial C_2-action. We may thus view the diagram as one of morphisms in the category of homotopy fixed points of $\mathbf{SymMonCat}_{Lax}$ with trivial action. This in turn induces a diagram in the homotopy fixed point category of $\mathbf{MonCat}_{Lax}$ with rev action, which yields a diagram of C_2-equivariant categories

$$\begin{array}{ccccc} \mathbb{A}_\infty\left(\mathbf{Spaces}_{/BGL_1(MUP)}\right) & \xrightarrow{Thom} & \mathbb{A}_\infty(\text{MUP-}\mathbf{Mod}) & \xrightarrow{Forget} & \mathbb{A}_\infty(\mathbf{Spectra}) \\ & & \downarrow{\scriptstyle L_{K(n)}} & & \downarrow{\scriptstyle L_{K(n)}} \\ & & \mathbb{A}_\infty(\text{MUP-}\mathbf{Mod}) & \xrightarrow{Forget} & \mathbb{A}_\infty(\mathbf{Spectra}). \end{array}$$

This is nearly all of our diagram ($\star\star$). To complete the diagram, we use Corollary 3.7 for $X \simeq BGL_1(MUP)$ and Example 3.3 for $\mathcal{C}$ the symmetric monoidal category of **Spectra**.

Remark 3.9. In the sequel, we will denote the space with C_2-action $B^\sigma BGL_1(MU_\mathbb{R}P)$ by $B^\rho GL_1(MU_\mathbb{R}P)$.

4 An equivariant map to $B^\rho GL_1(MU_\mathbb{R}P)$

The previous Section 3 constructs a C_2-equivariant functor

$$\mathbf{Spaces}_{/B^\rho GL_1(MUP)} \to \mathbb{A}_\infty(\mathbf{Spectra}),$$

which remains C_2-equivariant after composing with $K(n)$-localization. Here, $\mathbf{Spaces}_{/B^\rho GL_1(MUP)}$ is granted its C_2-action via the one on the space $B^\rho GL_1(MUP) = B^\rho GL_1(MU_{\mathbb{R}}P)$. The category $\mathbb{A}_\infty(\mathbf{Spectra})$ is equipped with the op action of Example 3.3.

A homotopy fixed point for $\mathbf{Spaces}_{/B^\rho GL_1(MUP)}$ is just a map of spaces with C_2-action $X \to B^\rho GL_1(MUP)$, and such a map therefore gives rise to a homotopy fixed point of $\mathbb{A}_\infty(\mathbf{Spectra})$. In other words, an equivariant map of spaces with C_2-action $f : X \to B^\rho GL_1(MUP)$ gives rise to a Thom $\mathbb{A}_\infty$-algebra with involution $(\Omega^\sigma X)^{\Omega^\sigma f}$.

One can apply the construction to $* \to B^\rho GL_1(MUP)$ to obtain MUP itself as an $\mathbb{A}_\infty$-algebra with involution. Using the equivariant map $* \to X$, we obtain a map of $\mathbb{A}_\infty$-rings with involution $MUP \to (\Omega^\sigma X)^{\Omega^\sigma f}$. The canonical Real orientation

$$\Sigma^{-2}\mathbb{CP}^\infty \to MU_{\mathbb{R}} \to MU_{\mathbb{R}}P$$

then equips $(\Omega^\sigma X)^{\Omega^\sigma f}$ with a Real orientation. If $(\Omega^\sigma X)^{\Omega^\sigma f}$ happens to also be a C_2-equivariant homotopy commutative ring, then [44, Theorem 2.25] implies that it receives an equivariant homotopy commutative ring map from $MU_{\mathbb{R}}$.

In this section we will be concerned with the construction of a particular map of spaces with C_2-action into $B^\rho GL_1(MU_{\mathbb{R}}P)$; the underlying map of spaces will be the morphism

$$BS^3 \times BS^3 \times \cdots \to B^2 GL_1(MUP)$$

constructed in Section 2. Our aim is to construct both 2-periodic Johnson–Wilson theory and Morava E-theory as $\mathbb{A}_\infty$-rings with involution.

Remark 4.1. Recall that, among spaces with C_2-action, we may identify certain representation spheres $S^{a+b\sigma}$ as the one-point compactifications of real C_2-representations. We use σ to denote the sign representation, 1 to denote the trivial representation, and the shorthand ρ to denote the regular representation $1+\sigma$. If X is a space or spectrum with C_2-action, then we use $\pi_{a+b\sigma}(X)$ to denote π_0 of the space of equivariant maps $S^{a+b\sigma} \to X$. Of interest to us, Proposition 3.8 implies that $\pi_{a+b\sigma}(B^\sigma X) \cong \pi_{a+(b-1)\sigma}(X)$.

In [44], the equivariant homotopy groups of $MU_{\mathbb{R}}P$ are computed. For each n, $\pi_{n\rho-1}(MU_{\mathbb{R}}P) \cong 0$. Additionally, there is a ring isomorphism

$$\pi_{*\rho}(MU_{\mathbb{R}}P) \cong \mathbb{Z}[\bar{x}_1, \bar{x}_2, \cdots][\bar{u}^{\pm}],$$

where $\bar{x}_i$ is in degree $i\rho$ and $\bar{u}$ is in degree ρ. The forgetful map from the equivariant to ordinary homotopy groups $\pi_{*\rho}(MU_{\mathbb{R}}P) \to \pi_{2*}(MUP)$ takes $\bar{x}_i$ to x_i and $\bar{u}$ to u.

Since $GL_1(MU_{\mathbb{R}}P)$ is defined via a pullback square of spaces with C_2-action

$$\begin{array}{ccc} GL_1(MU_{\mathbb{R}}P) & \longrightarrow & \Omega^\infty MUP \\ \downarrow & & \downarrow \\ \pi_0(MU_{\mathbb{R}}P)^\times & \longrightarrow & \pi_0(MU_{\mathbb{R}}P), \end{array}$$

we learn that $\pi_{a+b\sigma}(B^\rho GL_1(MU_{\mathbb{R}}P)) \cong \pi_{(a-1)+(b-1)\sigma}(MU_{\mathbb{R}}P)$ whenever $a, b > 1$.

Our next task is to understand the C_2-equivariant space $B^\sigma S^{\rho+1}$. The analogue of the even cell structure that played a prominent role in Section 2 is the following:

Proposition 4.2. *The equivariant space $B^\sigma S^{\rho+1}$ arises as a filtered colimit*

$$Y_1 = S^{2\rho} \to Y_2 \to Y_3 \to \cdots,$$

where there are homotopy pushout square of spaces with C_2-action

$$\begin{array}{ccc} S^{2n\rho-1} & \longrightarrow & Y_{n-1} \\ \downarrow & & \downarrow \\ * & \longrightarrow & Y_n. \end{array}$$

Proof. This cell decomposition is due to Mike Hopkins. Recall that, non-equivariantly, the cellular filtration on BS^3 agrees with the standard filtration

$$\mathbb{HP}^1 \to \mathbb{HP}^2 \to \cdots \to \mathbb{HP}^\infty \simeq BS^3,$$

where $\mathbb{HP}^\infty$ is the infinite-dimensional quaternionic projective space. For us, the relevant C_2-action on this space is by conjugation by i. In other words, we act on a point

$$[z_0 : z_1 : z_2 : \cdots]$$

by sending it to

$$[iz_0i^{-1} : iz_1i^{-1} : \cdots].$$

From the expression $i(a+bi+cj+dk)i^{-1} = a+bi-cj-dk$ we learn both that the action is well-defined and that the C_2-cells attached are multiples of 2ρ. The non-equivariant map $S^4 \to BS^3 \simeq \mathbb{HP}^\infty$ adjoint to the identity is lifted to an equivariant map $S^{2\rho} \to \mathbb{HP}^\infty$, given by the inclusion $\mathbb{HP}^1 \to \mathbb{HP}^\infty$ under the described C_2-action. This shows in particular that $\mathbb{HP}^\infty \simeq B^\sigma S^{\rho+1}$. □

As a corollary, exactly as in Section 2, we learn that any map $S^{2\rho} \to B^\rho GL_1(MU_{\mathbb{R}}P)$ factors through $B^\sigma S^{\rho+1}$. Using the symmetric monoidal structure on $\mathbf{Spaces}_{/B^\rho GL_1(MUP)}$, which commutes with the C_2-action, we may construct from any sequence $(\alpha_1, \alpha_2, \cdots) \in$

$\pi_\rho MU_\mathbb{R}P$ a map

$$S^{\rho+1} \times S^{\rho+1} \times \cdots \to BGL_1(MU_\mathbb{R}P).$$

This then factors through at least one equivariant map

$$B^\sigma S^{\rho+1} \times B^\sigma S^{\rho+1} \times \cdots \to B^\rho GL_1(MU_\mathbb{R}P).$$

We choose for $(\alpha_0, \alpha_1, \cdots)$ the same sequence as in Lemma 2.3, with the x_i replaced by $\bar{x}_i$. The reader may prefer to consider the special case in which the sequence is

$$(\bar{x}_{2^n-1}\bar{u}^{2-2^n} - \bar{u}, \bar{x}_2\bar{u}^{-1}, \bar{x}_4\bar{u}^{-3}, \bar{x}_5\bar{u}^{-4}, \bar{x}_6\bar{u}^{-5}, \bar{x}_8\bar{u}^{-7}, \cdots),$$

where the classes $\bar{x}_i\bar{u}^{-i+1}$ that are included in the sequence are all those such that either

- i is not one less than a power of 2.
- i is greater than $2^n - 1$.

In any case, applying Ω^σ and then the Thom construction we obtain a homotopy fixed point of the category $\mathbb{A}_\infty(MUP - \mathbf{Mod})$. The underlying $\mathbb{A}_\infty$-ring is $E(n)$, the 2-periodic version of Johnson–Wilson theory constructed in Section 2. Our constructions produce a coherent $\mathbb{A}_\infty$-ring map $E(n) \xrightarrow{\simeq} E(n)^{\mathrm{op}}$ lifting the complex-conjugation C_2-action $E(n) \longrightarrow E(n)$. We denote this ring with involution by $\mathbb{E}(n)$.

Remark 4.3. From this work, it seems that the natural action on $E(n)$ is by $\mathbb{A}_\infty$-involutions rather than $\mathbb{A}_\infty$-algebra maps. However, we can sketch an approach to producing an action by $\mathbb{A}_\infty$-algebra maps in the spirit of this paper.

Using Theorem 1.4 of [10], $E(n)$ can be built as a Thom spectrum of a map $SU \to BGL_1(MUP)$. Obstruction theory easily lifts this to a map $BSU \to B^2GL_1(MUP)$, which produces the same involution we see above. We may go further though, and note that B^3SU also has an even cell structure. This means that it is easy to produce maps $B^3SU \to B^4GL_1(MUP)$, but as noted in [22, §6] it is not so easy to know which maps $BSU \to B^2GL_1(MUP)$ these lie over. If one could produce $E(n)$ as a Thom $\mathbb{E}_3$-MUP-algebra in this way, non-equivariantly, it seems likely that one could produce an $\mathbb{E}_{2\sigma+1}$-structure on the equivariant $E(n)$. In particular, this would mean the C_2-action on $E(n)$ is by $\mathbb{A}_\infty$-ring homomorphisms.

This may be of interest in light of [45], in which Kitchloo, Lorman, and Wilson provide a homotopy commutative and associative ring structure *up to phantom maps* on Real Johnson-Wilson theory. We thank Kitchloo for pointing out to us that the difficulty with phantom maps disappears after $K(n)$-localization.

5 Proof of Theorem 1.1

In the previous section, we constructed an $\mathbb{A}_\infty$-ring spectrum $E(n)$ with a C_2-action by $\mathbb{A}_\infty$-involutions. After $K(n)$-localizing, we obtain a C_2-action by involutions on Morava E-theory $\widehat{E(n)}$.

Now, consider the equivariant sequence of forgetful functors

$$\mathbb{E}_\infty(\mathbf{Spectra}) \to \mathbb{A}_\infty(\mathbf{Spectra}) \to \mathbf{Spectra},$$

where both $\mathbb{E}_\infty(\mathbf{Spectra})$ and $\mathbf{Spectra}$ are given the trivial C_2-action, but $\mathbb{A}_\infty(\mathbf{Spectra})$ is given the op action. We may restrict this sequence to an equivariant sequence of subcategories

$$\mathcal{C}_3 \to \mathcal{C}_2 \to \mathcal{C}_1,$$

where

- $\mathcal{C}_1$ is the category of all spectra equivalent to $\widehat{E(n)}$ and equivalences between them.
- $\mathcal{C}_2$ is the category of $\mathbb{A}_\infty$-ring spectra with underlying spectrum $\widehat{E(n)}$, and equivalences between them.
- $\mathcal{C}_3$ is the category of $\mathbb{E}_\infty$-ring spectra with underlying spectrum $\widehat{E(n)}$, and equivalences between them.

Note that a map of categories with C_2-action is equivalence if and only if the underlying non-equivariant functor is an equivalence of non-equivariant categories. The Goerss–Hopkins–Miller theorem [28, 65] says that the map $\mathcal{C}_3 \to \mathcal{C}_2$ is an equivalence of categories. It follows that any homotopy fixed point of $\mathcal{C}_2$ may uniquely be lifted to one of $\mathcal{C}_3$. Thus, the C_2-action on $\widehat{E(n)}$ by $\mathbb{A}_\infty$-involutions has a unique lift to a C_2-action by $\mathbb{E}_\infty$-ring automorphisms. According to Goerss–Hopkins–Miller [65], the categories $\mathcal{C}_3$ and $\mathcal{C}_2$ are equivalent to $B\mathbb{G}$, where $\mathbb{G}$ is the Morava stabilizer group. A C_2-action on $\widehat{E(n)}$ by $\mathbb{E}_\infty$-ring maps is therefore the data of a map $BC_2 \to B\mathbb{G}$, which is the data of a group homomorphism $C_2 \to \mathbb{G}$. It follows by direct calculation that any C_2-action by $\mathbb{E}_\infty$-ring maps is determined by its effect on homotopy groups. The Real orientation $MU_{\mathbb{R}} \to \mathbb{E}(n) \to \widehat{E(n)}$ determines that the C_2-action we have constructed is the central one that acts by the formal inverse, proving Theorem 1.3.

Remark 5.1. Our discussion of algebras with involution, and our use of the Goerss–Hopkins–Miller Theorem, may both be entirely avoided if one only wants to know that the C_2-action on $\widehat{E(n)}$ is the Galois one *in the homotopy category of spectra.* It is, however, not a priori clear that there is a unique lift of this homotopy C_2-action on $\widehat{E(n)}$ to a fully coherent C_2-action.

To prove Theorem 1.1, recall that the assignment $(k, \mathbb{G}) \mapsto E_{(k,\mathbb{G})}$ is a functor to $\mathbb{E}_\infty(\mathbf{Spectra})$. In particular, for any field extension $\mathbb{F}_2 \subset k$ there are induced C_2-equivariant homotopy ring maps $MU_\mathbb{R} \to \widehat{E(n)} \to E_{(k,\mathbb{G})}$ involving some version of $\widehat{E(n)}$. If G is a finite subgroup of the $\mathbb{E}_\infty$-ring automorphisms of $E_{(k,\mathbb{G})}$ containing the central C_2, there then arises a sequence of homotopy ring maps

$$N_{C_2}^G MU_\mathbb{R} \longrightarrow N_{C_2}^G E_{(k,\mathbb{G})} \longrightarrow E_{(k,\mathbb{G})}.$$

The existence of the last homomorphism follows from the fact that the norm is an adjunction between $\mathbb{E}_\infty$-rings with C_2-action and $\mathbb{E}_\infty$-rings with G-action (see [41, §2.2]).

6 Real Landweber exactness and proof of Theorem 1.2

In the remainder of the paper, for simplicity, we use a specific Morava E-theory E_n that is defined via a lift of the height n Honda formal group law over $\mathbb{F}_{2^n}$. Its homotopy groups are

$$\pi_* E_n = W(\mathbb{F}_{2^n})[[u_1, u_2, \dots, u_{n-1}]][u^\pm].$$

and the 2-typical formal group law over $\pi_* E_n$ is determined by the map $\pi_* BP \to \pi_* E_n$ sending

$$v_i \mapsto \begin{cases} u_i u^{2^i-1} & 1 \leq i \leq n-1 \\ u^{2^n-1} & i = n \\ 0 & i > n. \end{cases}$$

Our results are all easily generalized to other variants of Morava E-theory.

In this section, we will show that E_n, as a C_2-spectrum, is Real Landweber exact in the sense of [41]. We do so by completely computing the $RO(C_2)$-graded homotopy fixed point spectral sequence of E_n.

6.1 $RO(C_2)$-graded homotopy fixed point spectral sequence of E_n

So far, we have constructed a C_2-equivariant map from

$$MU_{\mathbb{R}} \to E_n.$$

Here, the C_2-action on $MU_{\mathbb{R}}$ is by complex conjugation, and the C_2-action on E_n is by the Goerss–Hopkins–Miller E_∞-action. The existence of this equivariant map will help us in computing the C_2-homotopy fixed point spectral sequence of E_n. In particular, the map $MU_{\mathbb{R}} \to E_n$ induces the map of spectral sequences

$$C_2\text{-}\operatorname{HFPSS}(MU_{\mathbb{R}}) \to C_2\text{-}\operatorname{HFPSS}(E_n)$$

of C_2-equivariant homotopy fixed point spectral sequences. Since both the complex conjugation action on $MU_{\mathbb{R}}$ and the Galois C_2-action on E_n are by E_∞-ring maps, both spectral sequences are multiplicative (the map between them is not necessarily a multiplicative map, but this is perfectly fine). At this point, we will replace $MU_{\mathbb{R}}$ by $BP_{\mathbb{R}}$ because everything is 2-local, and argument below is exactly the same regardless of whether we are using $MU_{\mathbb{R}}$ or $BP_{\mathbb{R}}$. Moreover, since $MU_{\mathbb{R}}$ splits as a wedge of suspensions of $BP_{\mathbb{R}}$'s, the homotopy fixed point spectral sequence of $BP_{\mathbb{R}}$ has the advantage of having less classes than $MU_{\mathbb{R}}$ but yet still retaining the important 2-local information that we need.

By [41, Corollary 4.7], the E_2-pages of the $RO(C_2)$-graded homotopy fixed point spectral sequences of $BP_{\mathbb{R}}$ and E_n are

$$E_2^{s,t}(BP_{\mathbb{R}}^{hC_2}) = \mathbb{Z}[\bar{v}_1, \bar{v}_2, \ldots] \otimes \mathbb{Z}[u_{2\sigma}^{\pm}, a_\sigma]/(2a_\sigma)$$
$$E_2^{s,t}(E_n^{hC_2}) = W(\mathbb{F}_{2^n})[[\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-1}]][\bar{u}^{\pm}] \otimes \mathbb{Z}[u_{2\sigma}^{\pm}, a_\sigma]/(2a_\sigma).$$

On the E_2-page, the class $\bar{v}_i$ is in stem $|\bar{v}_i| = i\rho$ for $i \geq 1$; the class $\bar{u}_i$ is in stem $|\bar{u}_i| = 0$ for $1 \leq i \leq n-1$; and the class $\bar{u}$ is in stem $|\bar{u}| = \rho$. The classes $u_{2\sigma}$ and a_σ are in stems $2 - 2\sigma$ and $-\sigma$, respectively. They can be defined more generally as follows:

Definition 6.1 (a_V and u_V)**.** Let V be a representation of G of dimension d.

1. $a_V \in \pi_{-V}^G S^0$ is the map corresponding to the inclusion $S^0 \hookrightarrow S^V$ induced by $\{0\} \subset V$.
2. If V is oriented, $u_V \in \pi_{d-V}^G H\underline{\mathbb{Z}}$ is the class of the generator of $H_d^G(S^V; H\underline{\mathbb{Z}})$.

Proof of Theorem 1.2. In [44], Hu and Kriz completely computed the C_2-homotopy fixed point spectral sequence of $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$. In particular, the classes $\bar{v}_i$ for all $i \geq 1$ and the class a_σ are permanent cycles. All the differentials are determined by the differentials

$$d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) = \bar{v}_k a_\sigma^{2^{k+1}-1},\ k \geq 1$$

and multiplicative structures. There are no nontrivial extension problems on the E_∞-page.

On the E_2-page, the map

$$C_2\text{- HFPSS}(BP_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

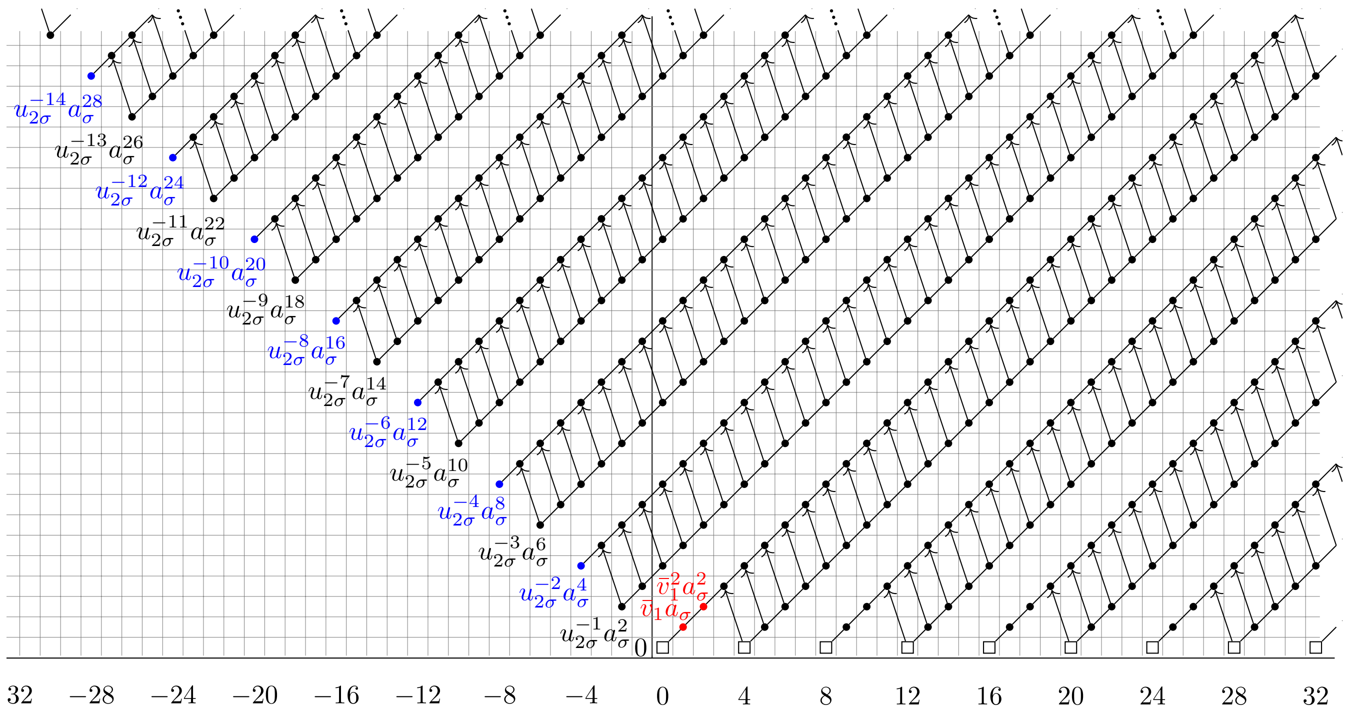


Figure 1: Important d_3-differentials and surviving torsion classes on the E_3-page.

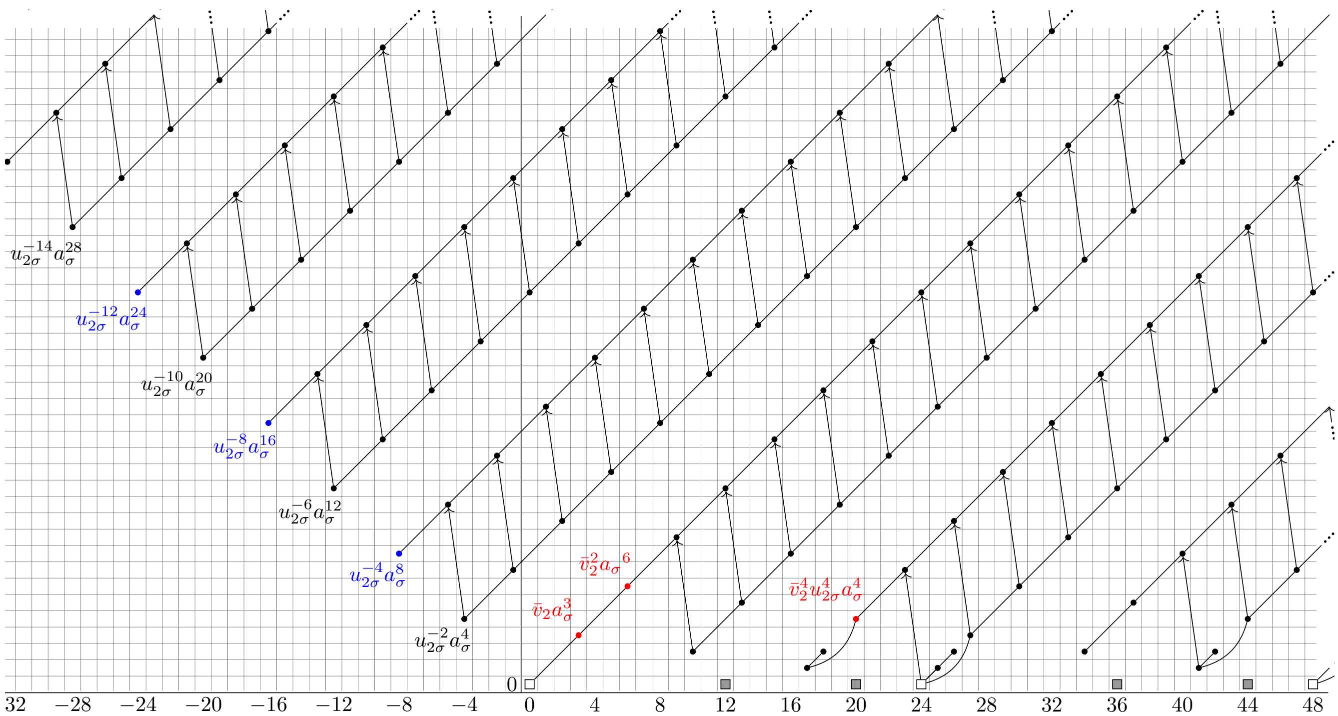


Figure 2: Important d_7-differentials and surviving torsion classes on the E_7-page.

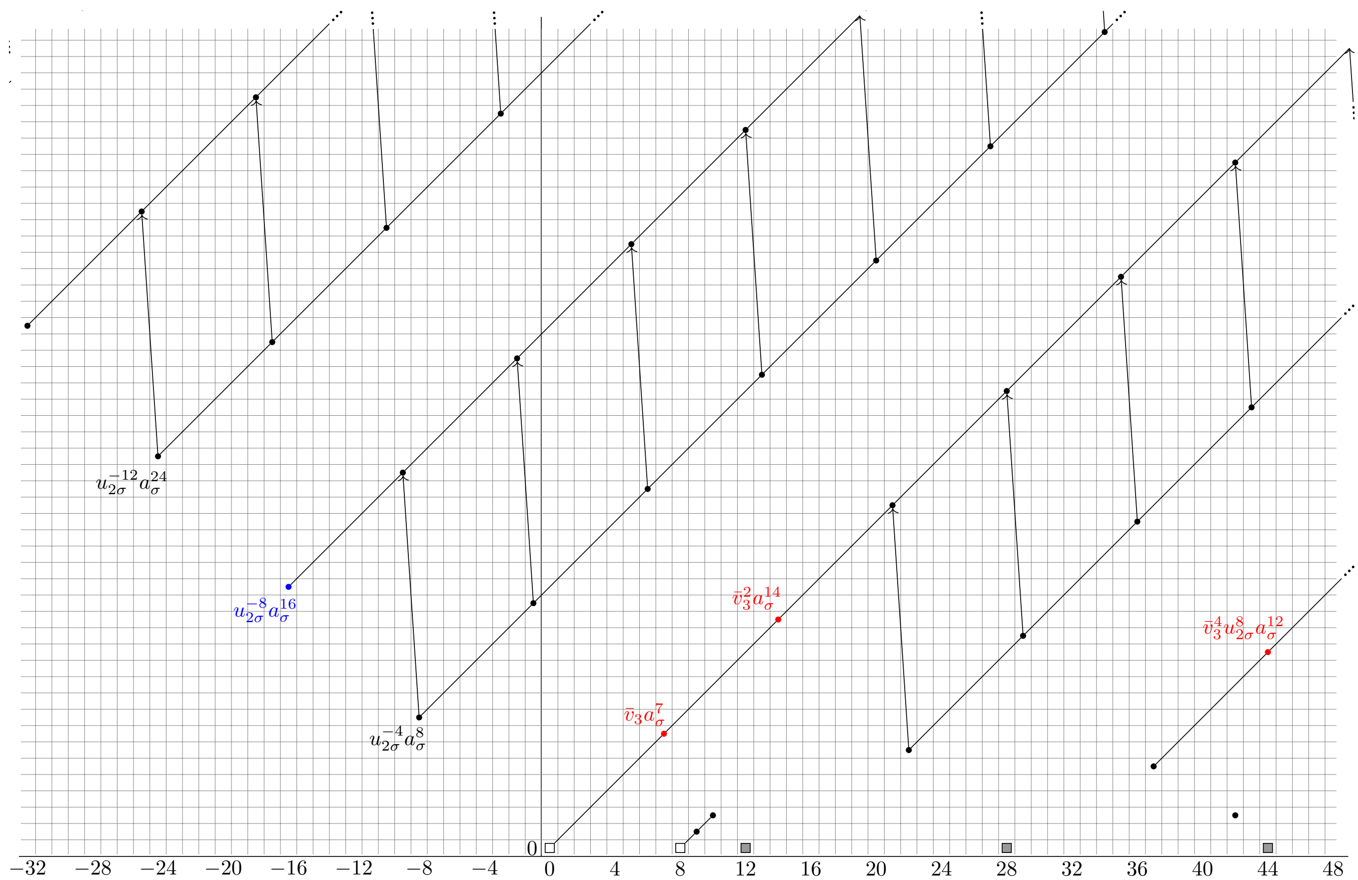


Figure 3: Important d_{15}-differentials and surviving torsion classes on the E_{15}-page.

of spectral sequences sends the classes $u_{2\sigma} \mapsto u_{2\sigma}$, $a_\sigma \mapsto a_\sigma$, and

$$\bar{v}_i \mapsto \begin{cases} \bar{u}_i \bar{u}^{2^i-1} & 1 \leq i \leq n-1 \\ \bar{u}^{2^n-1} & i = n \\ 0 & i > n. \end{cases}$$

We will first prove that the classes $\bar{u}_1, \ldots, \bar{u}_{n-1}$, $\bar{u}^{\pm}$, and a_σ are permanent cycles in C_2- HFPSS(E_n). Since the classes $\bar{v}_i$, $i \geq 1$, and a_σ are permanent cycles in C_2- HFPSS$(BP_{\mathbb{R}})$, their images are also permanent cycles in C_2- HFPSS(E_n). This shows that the classes $\bar{u}_i \bar{u}^{2^i-1}$, $1 \leq i \leq n-1$, $\bar{u}^{2^n-1}$, and a_σ are permanent cycles in C_2- HFPSS(E_n).

Now, consider the non-equivariant map

$$u : S^2 \to i_e^* E_n.$$

Applying the Hill–Hopkins–Ravenel norm functor $N_e^{C_2}(-)$ ([39]) produces the equivariant map

$$N_e^{C_2}(u) = \bar{u}^2 : S^{2\rho} \to N_e^{C_2} i_e^* E_n \to E_n,$$

where the last map is the co-unit map of the norm–restriction adjunction

$$N_e^{C_2} : \text{Commutative } C_2\text{-spectra} \rightleftarrows \text{Commutative spectra} : i_e^*.$$

Since the element $N_e^{C_2}(u) = \bar{u}^2$ is an actual element in $\pi_{\bigstar}^{C_2} E_n$, it is a permanent cycle. This, combined with the fact that $\bar{u}^{2^n-1}$ is a permanent cycle, shows that $\bar{u} = \bar{u}^{2^n-1} \cdot (\bar{u}^{-2})^{2^{n-1}}$ is a permanent cycle. It follows from the previous paragraph that the classes $\bar{u}_1, \ldots, \bar{u}_{n-1}$, and $\bar{u}^{\pm}$ are all permanent cycles in C_2- HFPSS(E_n).

It remains to produce the differentials in C_2- HFPSS(E_n). We will show by induction on k, $1 \leq k \leq n$, that all the differentials in C_2- HFPSS(E_n) are determined by the differentials

$$\begin{aligned} d_{2^{k+1}-1}(u_{2\sigma}^{2^{k-1}}) &= \bar{u}_k \bar{u}^{2^k-1} a_\sigma^{2^{k+1}-1}, \ \ 1 \leq k \leq n-1, \\ d_{2^{n+1}-1}(u_{2\sigma}^{2^{n-1}}) &= \bar{u}^{2^n-1} a_\sigma^{2^{n+1}-1}, \ \ k = n \end{aligned}$$

and multiplicative structures.

For the base case, when $k = 1$, there is a d_3-differential

$$d_3(u_{2\sigma}) = \bar{v}_1 a_\sigma^3$$

in C_2- HFPSS$(BP_{\mathbb{R}})$. Under the map

$$C_2\text{- HFPSS}(BP_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

of spectral sequences, the the source is mapped to $u_{2\sigma}$ and the target is mapped to $\bar{u}_1 \bar{u} a_\sigma^3$. It follows that there is a d_3-differential

$$d_3(u_{2\sigma}) = \bar{u}_1 \bar{u} a_\sigma^3$$

in C_2- HFPSS(E_n). Multiplying this differential by the permanent cycles produced before determines the rest of the d_3-differentials. These are all the d_3-differentials because there are no more room for other d_3-differentials after these differentials.

Suppose now that the induction hypothesis holds for all $1 \leq k \leq r-1 < n$. For degree reasons, after the d_{2^r-1}-differentials, the next possible differential is of length $d_{2^{r+1}-1}$. In

C_2- HFPSS($BP_{\mathbb{R}}$), there is a $d_{2^{r+1}-1}$-differential

$$d_{2^{r+1}-1}(u_{2\sigma}^{2^{r-1}}) = \bar{v}_r a_\sigma^{2^{r+1}-1}.$$

The map

$$C_2\text{- HFPSS}(BP_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

of spectral sequences sends the source to $u_{2\sigma}^{2^{r-1}}$ and the target to

$$\bar{v}_r a_\sigma^{2^{r+1}-1} \mapsto \begin{cases} \bar{u}_r \bar{u}^{2^r-1} a_\sigma^{2^{r+1}-1} & r < n \\ \bar{u}^{2^n-1} a_\sigma^{2^{n+1}-1} & r = n. \end{cases}$$

In particular, both images are not zero. Moreover, the image of the target must be killed by a differential of length at most $2^{r+1}-1$. By degree reasons, the image of the target cannot be killed by a shorter differential. It follows that there is a $d_{2^{r+1}-1}$-differential

$$d_{2^{r+1}-1}(u_{2\sigma}^{2^{r-1}}) = \begin{cases} \bar{u}_r \bar{u}^{2^r-1} a_\sigma^{2^{r+1}-1} & r < n \\ \bar{u}^{2^n-1} a_\sigma^{2^{n+1}-1} & r = n. \end{cases}$$

The rest of the $d_{2^{r+1}-1}$-differentials are produced by multiplying this differential with permanent cycles. After these differentials, there are no room for other $d_{2^{r+1}-1}$-differentials by degree reasons. This concludes the proof of the theorem. □

Remark 6.2. As an example, Figures 4–7 show the differentials in the integer-graded part of C_2- HFPSS(E_3). The spectral sequence converges after the E_{15}-page and we learn that $\pi_* E_3^{hC_2}$ is 32-periodic.

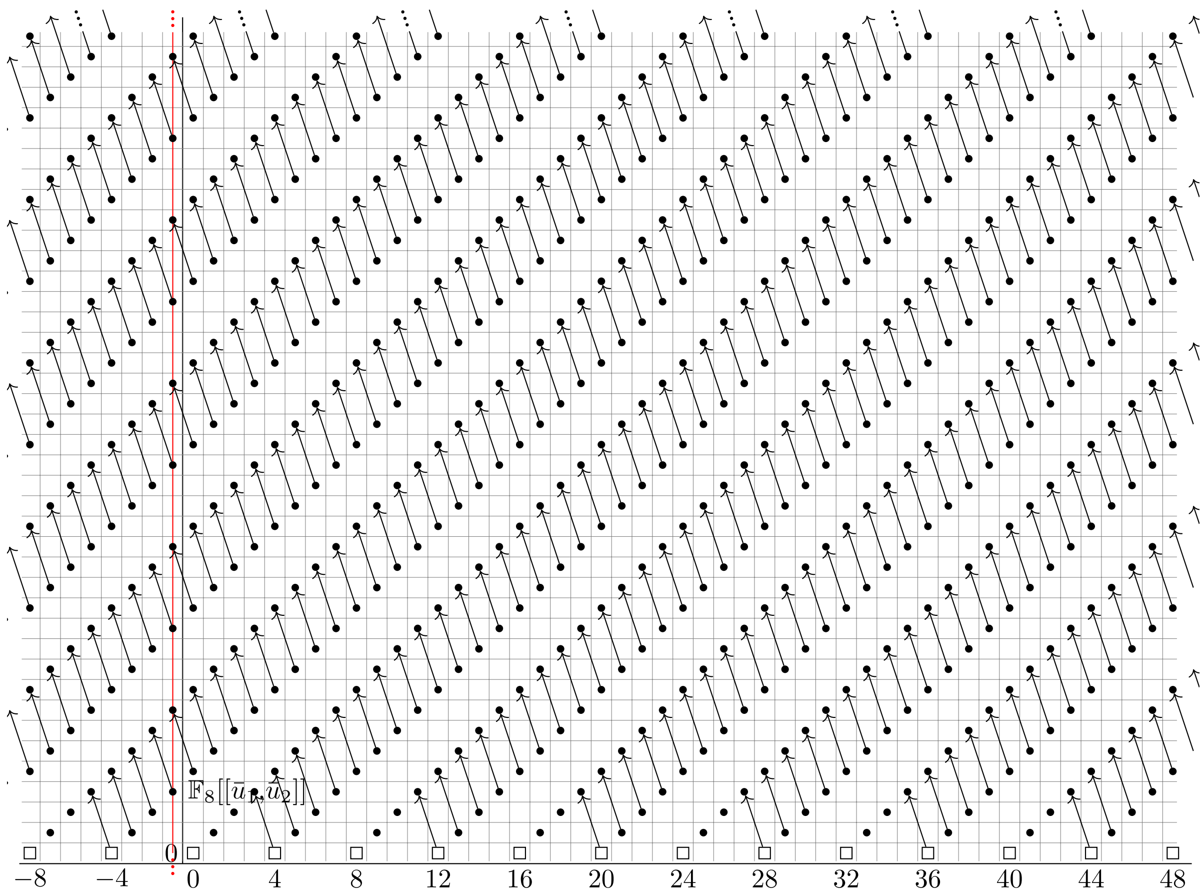


Figure 4: d_3-differentials in the integer graded part of C_2- HFPSS(E_3).

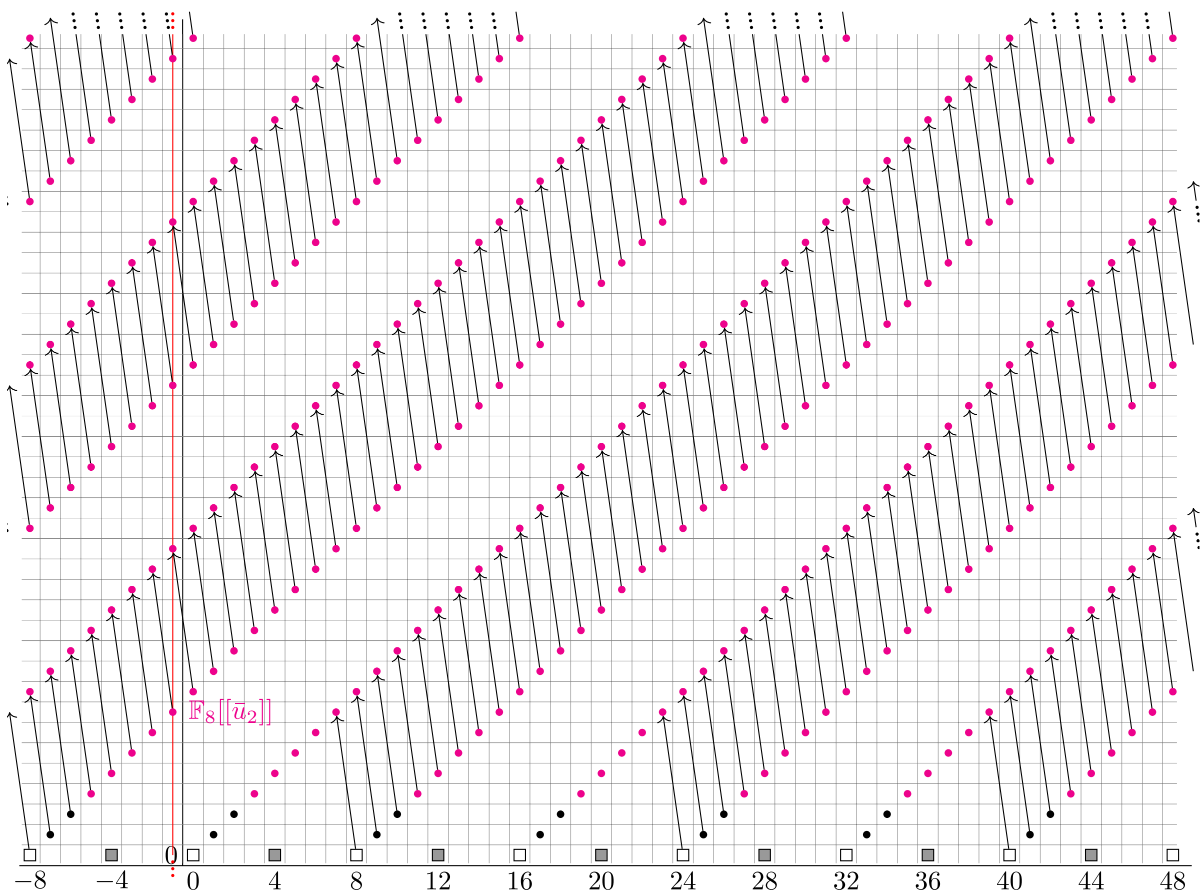


Figure 5: d_7-differentials in the integer graded part of C_2- HFPSS(E_3).

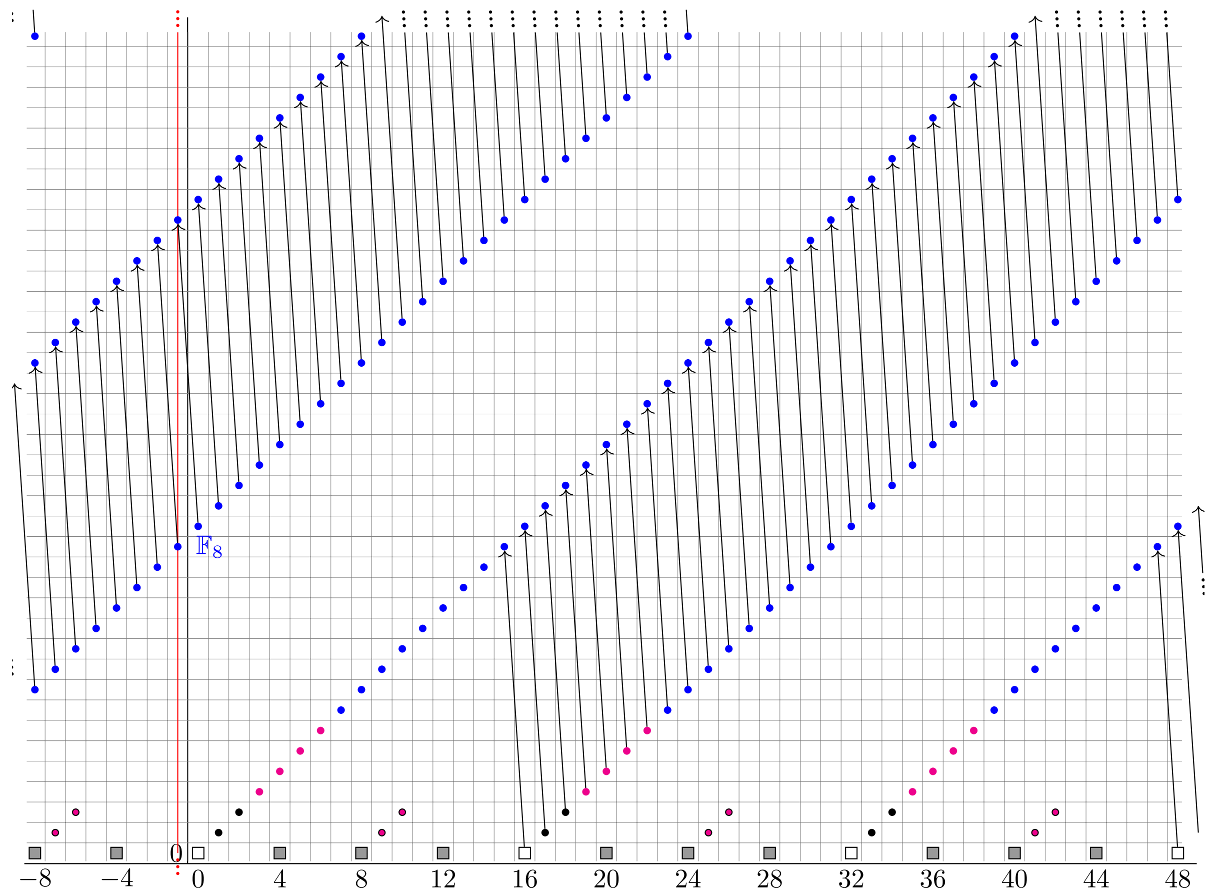


Figure 6: d_{15}-differentials in the integer graded part of C_2- HFPSS(E_3).

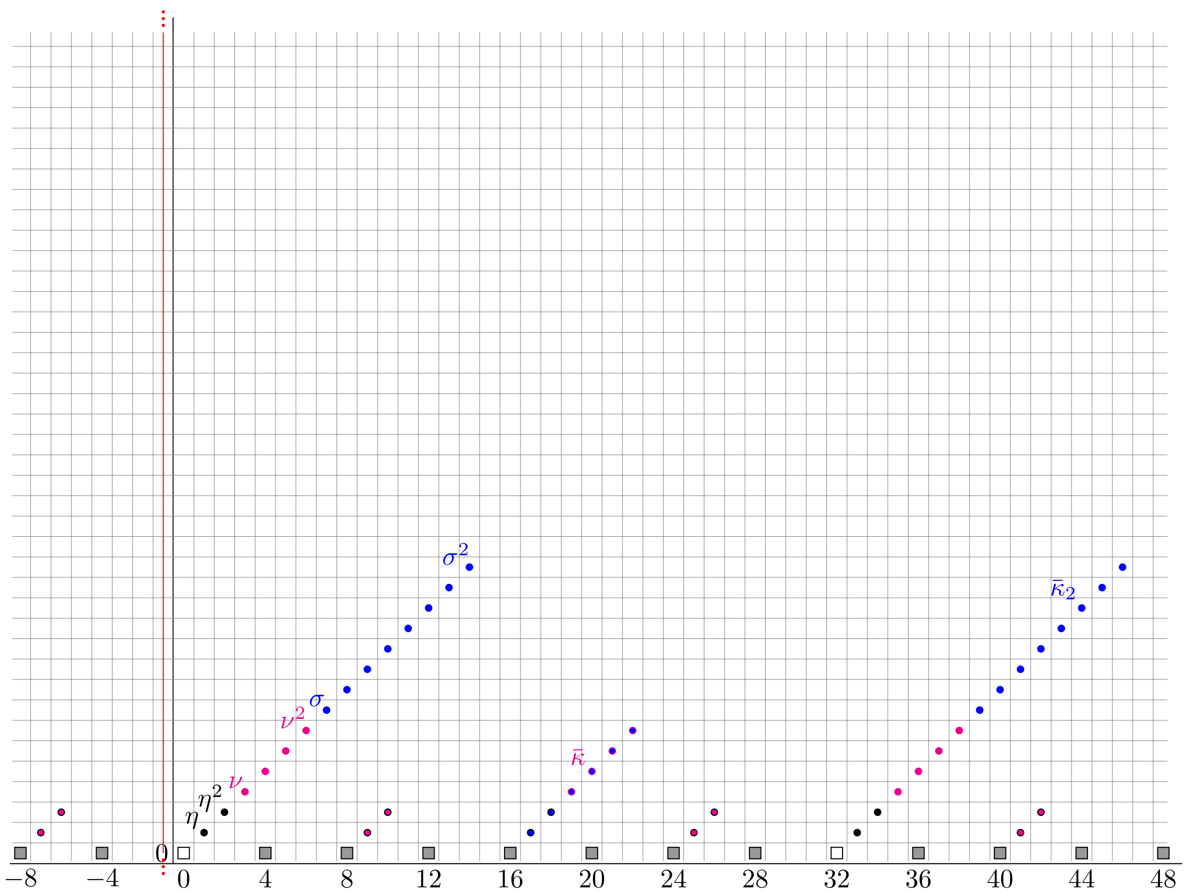


Figure 7: E_∞-page of the integer graded part of C_2- HFPSS(E_3).

6.2 Real Landweber exactness

We will now use the C_2-homotopy fixed point spectral sequence of E_n to show that E_n is Real Landweber exact. First, we will recall some definitions and theorems from [41].

Definition 6.3. [[6]]Let E be a C_2-equivariant homotopy commutative ring spectrum. A *Real orientation* of E is a class $\bar{x} \in \widetilde{E}^{\rho}_{C_2}(\mathbb{CP}^\infty)$ whose restriction to

$$\widetilde{E}^{\rho}_{C_2}(\mathbb{CP}^1) = \widetilde{E}^{\rho}_{C_2}(S^\rho) \cong E^0_{C_2}(pt)$$

is the unit. Here, we are viewing $\mathbb{CP}^n$ as a C_2-space via complex conjugation.

By [44, Theorem 2.25], Real orientations of E are in one-to-one correspondence with homotopy commutative maps $MU_{\mathbb{R}} \to E$ of C_2-ring spectra.

Definition 6.4. ([41, Definition 3.1]). A C_2-spectrum $E\mathbb{R}$ is *even* if $\underline{\pi}_{k\rho-1}E\mathbb{R} = 0$ for all $k \in \mathbb{Z}$. It is called *strongly even* if additionally $\underline{\pi}_{k\rho}E\mathbb{R}$ is a constant Mackey functor for all $k \in \mathbb{Z}$, i.e., if the restriction

$$\pi^{C_2}_{k\rho}E\mathbb{R} \to \pi^{e}_{k\rho}E\mathbb{R} \cong \pi^{e}_{2k}E\mathbb{R}$$

is an isomorphism.

Even spectra satisfy very nice properties. In particular, Hill–Meier further proved ([41, Lemma 3.3]) that if a C_2-spectrum $E\mathbb{R}$ is even, then $E\mathbb{R}$ is Real orientable. They proved this by showing that all the obstructions to having a Real orientation lie in the groups $\pi_{2k-1}E\mathbb{R}$ and $\pi^{C_2}_{k\rho-1}E\mathbb{R}$, which are all 0 by definition.

Definition 6.5. ([41, Definition 3.5]). Let $E\mathbb{R}$ be a strongly even C_2-spectrum with underlying spectrum E. Then $E\mathbb{R}$ is called *Real Landweber exact* if for every Real orientation

$MU_{\mathbb{R}} \to E\mathbb{R}$ the induced map

$$MU_{\mathbb{R}\bigstar}(X) \otimes_{MU_{2*}} E_{2*} \to E\mathbb{R}_{\bigstar}(X)$$

is an isomorphism for every C_2-spectrum X.

Here, we are treating $MU_{\mathbb{R}\bigstar}$ as a graded MU_{2*}-module because the restriction map $(MU_{\mathbb{R}})_{k\rho} \to MU_{2k}$ is an isomorphism, and it defines a graded ring morphism $MU_{2*} \to MU_{\mathbb{R}\bigstar}$ by sending elements of degree $2k$ to elements of degree $k\rho$.

Theorem 6.6 ([41], Real Landweber exact functor theorem)**.** *Let $E\mathbb{R}$ be a strongly even C_2-spectrum whose underlying spectrum E is Landweber exact. Then $E\mathbb{R}$ is Real Landweber exact.*

For E_n, its underlying spectrum is clearly Landweber exact. In light of Theorem 6.6, we prove the following:

Theorem 6.7. *E_n is a Real Landweber exact spectrum.*

Proof. By Theorem 6.6, it suffices to show that E_n is strongly even. By Thereom 1.2, the classes $\bar{u}_1, \ldots, \bar{u}_{n-1}$, and $\bar{u}^{\pm}$ are permanent cycles in C_2- HFPSS(E_n). The restriction of these classes to $\pi^e_{2*}E_n$ are $u_1, \ldots, u_{n-1}$, and $u^{\pm}$, respectively. Furthermore, there are no other classes in $\pi^{C_2}_{*\rho}E_n$. This shows that the restriction map

$$\pi^{C_2}_{*\rho}E_n \to \pi^e_{2*}E_n$$

is an isomorphism, hence $\underline{\pi}_{k\rho}E_n$ is a constant Mackey functor for all $k \in \mathbb{Z}$.

Classically, we already know that $\pi^e_{2k-1}E_n = 0$. The following lemma shows that $\underline{\pi}_{k\rho-1}E_n = 0$ for all $k \in \mathbb{Z}$. □

Lemma 6.8. *The groups* $\pi^{C_2}_{k\rho-1}E_n = 0$ *for all* $k \in \mathbb{Z}$.

Proof. In C_2- HFPSS(E_n), the classes $\bar{u}^{\pm}$ are permanent cycles. Since $|\bar{u}| = \rho$, multiplying by $\bar{u}^k$ produces an isomorphism

$$\pi^{C_2}_{\bigstar}E_n \xrightarrow{\cong} \pi^{C_2}_{\bigstar+k\rho}E_n.$$

It follows that in order to show $\pi^{C_2}_{k\rho-1}E_n = 0$ for $k \in \mathbb{Z}$, it suffices to prove $\pi^{C_2}_{-1}E_n = 0$.

Recall that the E_2-page of C_2- HFPSS(E_n) is

$$E_2^{s,t}(E_n^{hC_2}) = W(\mathbb{F}_{2^n})[[\bar{u}_1, \ldots, \bar{u}_{n-1}]][\bar{u}^{\pm}] \otimes \mathbb{Z}[u^{\pm}_{2\sigma}, a_\sigma]/(2a_\sigma).$$

As in Figure 7, every class on the 0-line is of the form

$$W(\mathbb{F}_{2^n})[[\bar{u}_1, \ldots, \bar{u}_{n-1}]]\bar{u}^a u^b_{2\sigma},$$

where $a, b \in \mathbb{Z}$, and every class of filtration greater than 0 is of the form

$$\mathbb{F}_{2^n}[[\bar{u}_1, \ldots, \bar{u}_{n-1}]]\bar{u}^a u^b_{2\sigma} a^c_\sigma,$$

where $a, b \in \mathbb{Z}$, and $c > 0$. For degree reasons, the classes on the (-1)-stem are all of the form

$$\mathbb{F}_{2^n}[[\bar{u}_1, \ldots, \bar{u}_{n-1}]]\bar{u}^{2\ell-1} u^{-\ell}_{2\sigma} a^{4\ell-1}_\sigma,$$

where $\ell \geq 1$. The relevant differentials that have source or target in the (-1)-stem are all generated by

$$d_{2^{r+1}-1}(u_{2\sigma}^{-2^{r-1}}) = d_{2^{r+1}-1}(u_{2\sigma}^{-2^r} \cdot u_{2\sigma}^{2^{r-1}}) = u_{2\sigma}^{-2^r} \cdot d_{2^{r+1}-1}(u_{2\sigma}^{2^{r-1}}) = \begin{cases} \bar{u}_r \bar{u}^{2^r-1} u_{2\sigma}^{-2^r} a_\sigma^{2^{r+1}-1} & 0 < r < n \\ \bar{u}^{2^n-1} u_{2\sigma}^{-2^n} a_\sigma^{2^{n+1}-1} & r = n. \end{cases}$$

We will analyze these differentials one-by-one:

(1) The relevant d_3-differentials are all generated by the differential

$$d_3(u_{2\sigma}^{-1}) = \bar{u}_1 \bar{u} u_{2\sigma}^{-2} a_\sigma^3.$$

The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 1 \pmod 2$, are the sources of these differentials, and hence they die after the E_3-page. The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 0 \pmod 2$, are the targets. These differentials quotient out the principal ideal $(\bar{u}_1)$ at these targets. The remaining classes at these targets are of the form

$$\mathbb{F}_{2^n}[[\bar{u}_2, \ldots, \bar{u}_{n-1}]] \bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1},$$

with $\ell \equiv 0 \pmod 2$.

(2) The relevant d_7-differentials are all generated by the differential

$$d_7(u_{2\sigma}^{-2}) = \bar{u}_2 \bar{u}^3 u_{2\sigma}^{-4} a_\sigma^7.$$

The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 2 \pmod 4$, are the sources of these differentials, and hence they die after the E_7-page. The classes at $\bar{u}^{2\ell-1} u_{2\sigma}^{-\ell} a_\sigma^{4\ell-1}$, with $\ell \equiv 0 \pmod 4$, are the targets. These differentials quotient out the principal ideal $(\bar{u}_2)$ at these targets. The remaining classes at these targets are of the form

$$\mathbb{F}_{2^n}[[\bar{u}_3, \ldots, \bar{u}_{n-1}]]\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1},$$

with $\ell \equiv 0 \pmod 4$.

(3) In general, for $0 < r < n$, the relevant $d_{2^{r+1}-1}$-differentials are all generated by the differential

$$d_{2^{r+1}-1}(u_{2\sigma}^{-2^{r-1}}) = \bar{u}_r\bar{u}^{2^r-1}u_{2\sigma}^{-2^r}a_\sigma^{2^{r+1}-1}.$$

The classes at $\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1}$, with $\ell \equiv 2^{r-1} \pmod{2^r}$, are the sources of these differentials, and hence they die after the $E_{2^{r+1}-1}$-page. The classes at $\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1}$, with $\ell \equiv 0 \pmod{2^r}$, are the targets. These differentials quotient out the principal ideal $(\bar{u}_r)$ at these targets. The remaining classes at these targets are of the form

$$\mathbb{F}_{2^n}[[\bar{u}_{r+1}, \ldots, \bar{u}_{n-1}]]\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1},$$

with $\ell \equiv 0 \pmod{2^r}$.

(4) The relevant $d_{2^{n+1}-1}$-differentials are all generated by the differential

$$d_{2^{n+1}-1}(u_{2\sigma}^{-2^{n-1}}) = \bar{u}^{2^n-1}u_{2\sigma}^{-2^n}a_\sigma^{2^{n+1}-1}.$$

The classes at $\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1}$, with $\ell \equiv 2^{n-1} \pmod{2^n}$, are the sources of these differentials, and hence they die after the $E_{2^{n+1}-1}$-page. The classes at $\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1}$, with $\ell \equiv 0$

(mod 2^n), are the targets. They also die after these differentials because the only classes at these targets now are $\bar{u}^{2\ell-1}u_{2\sigma}^{-\ell}a_\sigma^{4\ell-1}$.

It follows that every class at the (-1)-stem vanish after the $E_{2^{n+1}-1}$-page. This implies $\pi_{-1}^{C_2}E_n = 0$, as desired.

□

7 Hurewicz images

In this section, we will prove that $\pi_* E_n^{hC_2}$ detects the Hopf elements, the Kervaire classes, and the $\bar{\kappa}$-family. The case when $n = 1$ and $n = 2$ are previously known. When $n = 1$, $E_1 = KU_2^\wedge$ and $E_1^{hC_2} = KO_2^\wedge$. It is well-known that $\pi_* KO_2^\wedge$ detects $\eta \in \pi_1\mathbb{S}$ and $\eta^2 \in \pi_2\mathbb{S}$ ([7]). When $n = 2$, the Mahowald–Rezk transfer argument ([52]) shows that $\pi_* E_2^{hC_2}$ detects η, η^2, $\nu \in \pi_3\mathbb{S}$, $\nu^2 \in \pi_6\mathbb{S}$, and $\bar{\kappa} \in \pi_{20}\mathbb{S}$.

The Hopf elements are represented by the elements

$$h_i \in \mathrm{Ext}_{\mathcal{A}_*}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$$

on the E_2-page of the classical Adams spectral sequence at the prime 2. By Adam's solution of the Hopf invariant one problem [1], only h_0, h_1, h_2, and h_3 survive to the E_∞-page. By Browder's work [20], the Kervaire classes $\theta_j \in \pi_{2^{j+1}-2}\mathbb{S}$, if they exist, are represented by the elements

$$h_j^2 \in \mathrm{Ext}_{\mathcal{A}_*}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$$

on the E_2-page. For $j \leq 5$, h_j^2 survive. The case $\theta_4 \in \pi_{30}\mathbb{S}$ is due to Barratt–Mahowald–

Tangora [53, 9], and the case $\theta_5 \in \pi_{62}\mathbb{S}$ is due to Barratt–Jones–Mahowald [8]. The fate of h_6^2 is unknown. Hill–Hopkins–Ravenel's result [39] shows that the h_j^2, for $j \geq 7$, do not survive to the E_∞-page.

To introduce the $\bar{\kappa}$-family, we appeal to Lin's complete classification of $\mathrm{Ext}_{\mathcal{A}_*}^{\leq 4,t}(\mathbb{F}_2, \mathbb{F}_2)$ in [48]. In his classification, Lin showed that there is a family $\{g_k \,|\, k \geq 1\}$ of indecomposable elements with

$$g_k \in \mathrm{Ext}_{\mathcal{A}_*}^{4,2^{k+2}+2^{k+3}}(\mathbb{F}_2, \mathbb{F}_2).$$

The first element of this family, g_1, is in bidegree $(4, 24)$. It survives the Adams spectral sequence to become $\bar{\kappa} \in \pi_{20}\mathbb{S}$. It is for this reason that we name this family the $\bar{\kappa}$-family. The element g_2 also survives to become the element $\bar{\kappa}_2 \in \pi_{44}\mathbb{S}$. For $k \geq 3$, the fate of g_k is unknown.

In [47], the second author, together with Li, Wang, and Xu, proved detection theorems for the Hurewicz images of $MU_{\mathbb{R}}^{C_2} \approx MU_{\mathbb{R}}^{hC_2}$ and $BP_{\mathbb{R}}^{C_2} \approx BP_{\mathbb{R}}^{hC_2}$ (the equivalences between the C_2-fixed points and the C_2-homotopy fixed points for $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$ are due to Hu and Kriz [44, Theorem 4.1]).

Theorem 7.1. *(Li–Shi–Wang–Xu, Detection Theorems for $MU_{\mathbb{R}}$ and $BP_{\mathbb{R}}$). The Hopf elements, the Kervaire classes, and the $\bar{\kappa}$-family are detected by the Hurewicz maps $\pi_*\mathbb{S} \to \pi_* MU_{\mathbb{R}}^{C_2} \cong \pi_* MU_{\mathbb{R}}^{hC_2}$ and $\pi_*\mathbb{S} \to \pi_* BP_{\mathbb{R}}^{C_2} \cong \pi_* BP_{\mathbb{R}}^{hC_2}$.*

Given the discussion above, Theorem 7.1 shows that the elements η, ν, σ, and θ_j, for $1 \leq j \leq 5$, are detected by $\pi_*^{C_2} MU_{\mathbb{R}}^{hC_2}$ and $\pi_*^{C_2} BP_{\mathbb{R}}^{hC_2}$. The last unknown Kervaire class θ_6 and the classes g_k for $k \geq 3$ will also be detected, should they survive the Adams spectral sequence.

The proof of Theorem 7.1 requires the C_2-equivariant Adams spectral sequence developed by Greenlees [29, 30, 31] and Hu–Kriz [44]. Since $MU_{\mathbb{R}}$ splits as a wedge of suspensions of $BP_{\mathbb{R}}$ 2-locally, we only need to focus on $BP_{\mathbb{R}}$. There is a map of Adams spectral sequences

$$\begin{array}{ccc}
\text{classical Adams spectral sequence of } \mathbb{S} & \Longrightarrow & (\pi_*\mathbb{S})^\wedge_2 \\
\downarrow & & \downarrow \\
C_2\text{-equivariant Adams spectral sequence of } \mathbb{S} & \Longrightarrow & (\pi^{C_2}_\bigstar F(EC_{2+}, \mathbb{S}))^\wedge_2 \\
\downarrow & & \downarrow \\
C_2\text{-equivariant Adams spectral sequence of } BP_{\mathbb{R}} & \Longrightarrow & (\pi^{C_2}_\bigstar F(EC_{2+}, BP_{\mathbb{R}}))^\wedge_2.
\end{array}$$

It turns out that for degree reasons, the C_2-equivariant Adams spectral sequence for $BP_{\mathbb{R}}$ degenerates at the E_2-page. From this, Theorem 7.1 follows easily from the following algebraic statement:

Theorem 7.2 (Li–Shi–Wang–Xu, Algebraic Detection Theorem)**.** *The images of the elements* $\{h_i \,|\, i \geq 1\}$*,* $\{h_j^2 \,|\, j \geq 1\}$*, and* $\{g_k \,|\, k \geq 1\}$ *on the* E_2*-page of the classical Adams spectral sequence of* $\mathbb{S}$ *are nonzero on the* E_2*-page of the* C_2*-equivariant Adams spectral sequence of* $BP_{\mathbb{R}}$*.*

The proof of Theorem 7.2 requires an analysis of the algebraic maps

$$\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}_{\mathcal{A}^{cc}_\bigstar}(H^c_\bigstar, H^c_\bigstar) \to \mathrm{Ext}_{\Lambda^{cc}_\bigstar}(H^c_\bigstar, H^c_\bigstar).$$

These are the maps on the E_2-pages of the Adams spectral sequences above. Here, $\mathcal{A}_* := (H\mathbb{F}_2 \wedge H\mathbb{F}_2)_*$ is the classical dual Steenrod algebra; $H^c_\bigstar := F(EC_{2+}, H\mathbb{F}_2)_\bigstar$ is the Borel C_2-equivariant Eilenberg–MacLane spectrum; $\mathcal{A}^{cc}_\bigstar := F(EC_{2+}, H\mathbb{F}_2 \wedge H\mathbb{F}_2)_\bigstar$ is the Borel C_2-equivariant dual Steenrod algebra; and $\Lambda^{cc}_\bigstar$ is a quotient of $\mathcal{A}^{cc}_\bigstar$. Hu and Kriz [44] studied $\mathcal{A}^{cc}_\bigstar$ and completely computed the Hopf algebroid structure of $(H^c_\bigstar, \mathcal{A}^{cc}_\bigstar)$. Using their formulas, it is possible to compute the map

$$(H\mathbb{F}_2, \mathcal{A}_*) \to (H^c_\bigstar, \mathcal{A}^{cc}_\bigstar) \to (H^c_\bigstar, \Lambda^{cc}_\bigstar)$$

of Hopf-algebroids. Filtering these Hopf algebroids compatibly produces maps of May spectral sequences:

$$\begin{array}{ccc}
\text{May spectral sequence of } \mathbb{S} & \Longrightarrow & \mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2) \\
\downarrow & & \downarrow \\
C_2\text{-equivariant May spectral sequence of } \mathbb{S} & \Longrightarrow & \mathrm{Ext}_{\mathcal{A}^{cc}_\bigstar}(H^c_\bigstar, H^c_\bigstar) \\
\downarrow & & \downarrow \\
C_2\text{-equivariant May spectral sequence of } BP_\mathbb{R} & \Longrightarrow & \mathrm{Ext}_{\Lambda^{cc}_\bigstar}(H^c_\bigstar, H^c_\bigstar).
\end{array}$$

There is a connection between the C_2-equivariant May spectral sequence of $BP_\mathbb{R}$ and the homotopy fixed point spectral sequence of $BP_\mathbb{R}$:

Theorem 7.3 (Li–Shi–Wang–Xu)**.** *The C_2-equivariant May spectral sequence of $BP_\mathbb{R}$ is isomorphic to the associated-graded homotopy fixed point spectral sequence of $BP_\mathbb{R}$ as $RO(C_2)$-graded spectral sequences.*

By the "associated-graded homotopy fixed point spectral sequence", we mean that whenever we see a $\mathbb{Z}$-class on the E_2-page, we replace it by a tower of $\mathbb{Z}/2$-classes. Since the equivariant Adams spectral sequence of $BP_\mathbb{R}$ degenerates, the E_∞-page of the C_2-equivariant May spectral sequence of $BP_\mathbb{R}$ is an associated-graded of $\pi^{C_2}_\bigstar F(EC_{2+}, BP_\mathbb{R})$. The isomorphism in Theorem 7.3 allows us to identify the classes in C_2- HFPSS(E_n) that detects the Hopf elements, the Kervaire classes, and the $\bar{\kappa}$-family. This is crucial for tackling detection theorems of $E_n^{hC_2}$.

Using Hu–Kriz's formulas, one can compute the maps on the E_2-pages of the May spectral sequences above, as well as all the differentials in the C_2-equivariant May spectral sequence of $BP_{\mathbb{R}}$.

Theorem 7.4 (Li–Shi–Wang–Xu). *On the E_2-page of the map*

$$MaySS(\mathbb{S}) \to C_2\text{-}MaySS(\mathbb{S}) \to C_2\text{-}MaySS(BP_{\mathbb{R}}) \cong C_2\text{-}HFPSS(BP_{\mathbb{R}}),$$

The classes

$$\begin{aligned} h_i &\mapsto \bar{v}_i a_\sigma^{2^i-1}, \\ h_j^2 &\mapsto \bar{v}_j^2 a_\sigma^{2(2^j-1)}, \\ h_{2k}^4 &\mapsto \bar{v}_{k+1}^4 u_{2\sigma}^{2^{k+1}} a_\sigma^{4(2^k-1)}. \end{aligned}$$

These classes all survive to the E_∞-page in C_2-HFPSS$(BP_{\mathbb{R}})$.

Since E_n is Real oriented and everything is 2-local, a Real orientation gives us a C_2-equivariant homotopy commutative map

$$BP_{\mathbb{R}} \to E_n,$$

which induces a multiplicative map

$$C_2\text{-}\mathrm{HFPSS}(BP_{\mathbb{R}}) \to C_2\text{-}\mathrm{HFPSS}(E_n)$$

of spectral sequences. On the E_2-page, this map sends the classes $u_{2\sigma} \mapsto u_{2\sigma}$, $a_\sigma \mapsto a_\sigma$, and

$$\bar{v}_i \mapsto \begin{cases} \bar{u}_i \bar{u}^{2^i - 1} & 1 \leq i \leq n-1 \\ \bar{u}^{2^n - 1} & i = n \\ 0 & i > n. \end{cases} \tag{7.1}$$

Theorem 7.5 (Detection Theorem for $E_n^{hC_2}$)**.**

1. *For $1 \leq i, j \leq n$, if the element $h_i \in Ext_{\mathcal{A}_*}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ or $h_j^2 \in Ext_{\mathcal{A}_*}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_* \mathbb{S} \to \pi_* E_n^{hC_2}$ is nonzero.*

2. *For $1 \leq k \leq n-1$, if the element $g_k \in Ext_{\mathcal{A}_*}^{4,2^{k+2}+2^{k+3}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_* \mathbb{S} \to \pi_* E_n^{hC_2}$ is nonzero.*

Proof. By Theorem 7.4 and (7.1), the composite map

$$\text{MaySS}(\mathbb{S}) \to C_2\text{-MaySS}(BP_{\mathbb{R}}) \cong C_2\text{- HFPSS}(BP_{\mathbb{R}}) \to C_2\text{- HFPSS}(E_n)$$

on the E_2-pages sends the classes

$$h_i \mapsto \bar{v}_i a_\sigma^{2^i-1} \quad \mapsto \quad \begin{cases} \bar{u}_i \bar{u}^{2^i-1} a_\sigma^{2^i-1} & 1 \leq i \leq n-1 \\ \bar{u}^{2^n-1} a_\sigma^{2^n-1} & i = n \\ 0 & i > n, \end{cases}$$

$$h_j^2 \mapsto \bar{v}_j^2 a_\sigma^{2(2^j-1)} \quad \mapsto \quad \begin{cases} \bar{u}_j^2 \bar{u}^{2(2^j-1)} a_\sigma^{2(2^j-1)} & 1 \leq j \leq n-1 \\ \bar{u}^{2(2^n-1)} a_\sigma^{2(2^n-1)} & j = n \\ 0 & j > n, \end{cases}$$

$$h_{2k}^4 \mapsto \bar{v}_{k+1}^4 u_{2\sigma}^{2^{k+1}} a_\sigma^{4(2^k-1)} \quad \mapsto \quad \begin{cases} \bar{u}_{k+1}^4 \bar{u}^{4(2^{k+1}-1)} u_{2\sigma}^{2^{k+1}} a_\sigma^{4(2^k-1)} & 1 \leq k \leq n-2 \\ \bar{u}^{4(2^n-1)} u_{2\sigma}^{2^n} a_\sigma^{4(2^{n-1}-1)} & k = n-1 \\ 0 & k > n-1. \end{cases}$$

We know all the differentials in C_2- HFPSS(E_n) from Section 6. From these differentials, it is clear that all the nonzero images on the E_2-page survive to the E_∞-page to represent elements in $\pi_* E_n^{hC_2}$. The statement of the theorem follows. □

Corollary 7.6 (Detection Theorem for E_n^{hG})**.** *Let G be a finite subgroup of the Morava stabilizer group $\mathbb{G}_n$ containing the centralizer subgroup C_2.*

1. *For $1 \leq i, j \leq n$, if the element $h_i \in Ext_{\mathcal{A}_*}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ or $h_j^2 \in Ext_{\mathcal{A}_*}^{2,2^{j+1}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hG}$ is nonzero.*

2. *For $1 \leq k \leq n-1$, if the element $g_k \in Ext_{\mathcal{A}_*}^{4,2^{k+2}+2^{k+3}}(\mathbb{F}_2, \mathbb{F}_2)$ survives to the E_∞-page of the Adams spectral sequence, then its image under the Hurewicz map $\pi_*\mathbb{S} \to \pi_* E_n^{hG}$ is nonzero.*

Proof. Consider the following factorization of the unit map $\mathbb{S} \to E_n^{hC_2}$:

$$\begin{array}{ccc} E_n^{hG} = F(EG_+, E_n)^G & \longrightarrow & F(EG_+, E_n)^{C_2} = E_n^{hC_2} \\ \uparrow & \nearrow & \\ \mathbb{S} & & \end{array}$$

The claims now follow easily from Theorem 7.5. □

8 The slice spectral sequence of a C_4-equivariant height-4 theory

The second part of this thesis (Sections 8–20) is based on joint work with Michael A. Hill, Guozhen Wang, and Zhouli Xu. The main result of the second part of this thesis is the following:

Theorem 8.1. *There exists a height-4 Lubin–Tate theory E_4 such that the C_4-equivariant orientation $\mathrm{BP}^{((C_4))} \longrightarrow E_4$ factors through $\mathrm{BP}^{((C_4))}\langle 2\rangle$:*

$$\begin{array}{ccc} \mathrm{BP}^{((C_4))} & \longrightarrow & E_4 \\ \downarrow & \nearrow & \\ \mathrm{BP}^{((C_4))}\langle 2\rangle & & \end{array}$$

Furthermore, after inverting a certain element $D_2 \in \pi_{24\rho_4}^{C_4}\mathrm{BP}^{((C_4))}\langle 2\rangle$ and applying $K(4)$-localization, there is an equivalence

$$(L_{K(4)}D_2^{-1}\mathrm{BP}^{((C_4))}\langle 2\rangle)^{C_4} \simeq E_4^{h\,\mathrm{Gal}(\mathbb{F}_{2^4}/\mathbb{F}_2)\ltimes C_{12}}.$$

We completely compute the slice spectral sequence of $\mathrm{BP}^{((C_4))}\langle 2\rangle$ (see Figure 8). The spectral *sequence degenerates after the E_{61}-page and has a horizontal vanishing line of filtration 61. Furthermore, the C_4-fixed points of the C_4-spectrum $D_2^{-1}\mathrm{BP}^{((C_4))}\langle 2\rangle$ is 384-periodic.*

Roughly speaking, the C_4-spectrum $\mathrm{BP}^{((C_4))}\langle 2\rangle$ encodes the universal example of a height-4 formal group law with a C_4-action extending the formal inversion action.

At height 2, Hill, Hopkins, and Ravenel [40] studied the slice spectral sequence of the spectrum $D_1^{-1}\mathrm{BP}^{((C_4))}\langle 1\rangle$. They showed that $D_1^{-1}\mathrm{BP}^{((C_4))}\langle 1\rangle$ is 32-periodic and is closely related to a height-2 Lubin–Tate theory, which has also been studied by Behrens–Ormsby [17] as $\mathrm{TMF}_0(5)$.

The spectrum $D_2^{-1}\mathrm{BP}^{((C_4))}\langle 2\rangle$, which is 384-periodic, is a height-4 generalization of $D_1^{-1}\mathrm{BP}^{((C_4))}\langle 1\rangle$. This can be viewed as a different perspective than that of Behrens and Lawson [16] to generalizing TMF with level structures to higher heights.

8.1 Motivation and main results

In 2009, Hill, Hopkins, and Ravenel [39] proved that the Kervaire invariant elements θ_j do not exist for $j \geq 7$. A key construction in their proof is the spectrum Ω, which detects all the Kervaire invariant elements in the sense that if $\theta_j \in \pi_{2^{j+1}-2}\mathbb{S}$ is an element of Kervaire invariant 1, then the Hurewicz image of θ_j under the map $\pi_*\mathbb{S} \to \pi_*\Omega$ is nonzero (see also [54, 37, 38] for surveys on the result).

The detecting spectrum Ω is constructed using equivariant homotopy theory as the fixed points of a C_8-spectrum $\Omega_{\mathbb{O}}$, which in turn is a chromatic-type localization of $\mathrm{MU}^{((C_8))} := N_{C_2}^{C_8}\mathrm{MU}_{\mathbb{R}}$. Here, $N_{C_2}^{C_8}(-)$ is the Hill–Hopkins–Ravenel norm functor and $\mathrm{MU}_{\mathbb{R}}$ is the Real cobordism spectrum of Landweber [46], Fujii [25], and Araki [6]. The underlying spectrum of $\mathrm{MU}_{\mathbb{R}}$ is MU, with the C_2-action coming from the complex conjugation action on complex manifolds.

To analyze the G-equivariant homotopy groups of $\mathrm{MU}^{((G))}$, Hill, Hopkins, and Ravenel generalized the C_2-equivariant filtration of Hu–Kriz [44] and Dugger [24] to a G-equivariant Postnikov filtration for all finite groups G. They called this the *slice filtration*. Given any G-equivariant spectrum X, the slice filtration produces the slice tower $\{P^*X\}$, whose associated slice spectral sequence strongly converges to the $RO(G)$-graded homotopy groups $\pi_\bigstar^G X$.

For $G = C_{2^n}$, the G-spectrum $\mathrm{MU}^{((G))}$ are amenable to computations. Hill, Hopkins, and Ravenel proved that the slice spectral sequences for $\mathrm{MU}^{((G))}$ and its equivariant localizations have especially simple E_2-terms. Furthermore, they proved the Gap Theorem and the Periodicity Theorem, which state, respectively, that $\pi_i^{C_8}\Omega_{\mathbb{O}} = 0$ for $-4 < i < 0$, and that there is an isomorphism $\pi_*^{C_8}\Omega_{\mathbb{O}} \cong \pi_{*+256}^{C_8}\Omega_{\mathbb{O}}$. The two theorems together imply that

$$\pi_{2^{j+1}-2}\Omega = \pi_{2^{j+1}-2}^{C_8}\Omega_{\mathbb{O}} = 0$$

for all $j \geq 7$, from which the nonexistence of the corresponding Kervaire invariant elements follows.

The solution of the Kervaire Invariant One problem gives us a motivating slogan:

Slogan. *The homotopy groups of the fixed points of* $\mathrm{MU}^{((G))}$ *as* $|G|$ *grows are increasingly good approximations to the stable homotopy groups of spheres.*

To explain the slogan some, we unpack some of the algebraic geometry around $\mathrm{MU}^{((G))}$ when $G = C_{2^n}$. The spectrum underlying $\mathrm{MU}^{((G))}$ is the smash product of 2^{n-1}-copies of MU, and so the underlying homotopy ring co-represents the functor which associates to a (graded) commutative ring a formal group law and a sequence of $(2^{n-1}-1)$ isomorphisms:

$$F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} \dots \xrightarrow{f_{2^{n-1}-1}} F_{2^{n-1}}.$$

The underlying homotopy ring has an action of C_{2^n}, and by canonically enlarging our moduli problem, we can record this as well. We extend our sequence of isomorphisms by one final isomorphism from the final formal group law back to the first, composing the inverses to the isomorphism already given with the formal inversion. This gives us our moduli problem: maps from the underlying homotopy of $\mathrm{MU}^{(\!(C_{2^n})\!)}$ to a graded commutative C_{2^n}-equivariant ring R are given by a formal group law F together with isomorphisms

$$f_{i+1}\colon {\gamma^{i}}^{*}F \to {\gamma^{(i+1)}}^{*}F, \;\; 0 \leq i \leq 2^{n-1}-1$$

such that the composite of all of the f_i is the formal inversion on F.

If F is a formal group law over a ring R that has an action of C_{2^k} extending the action of C_2 given by formal inversion, then F canonically defines a sequence of formal groups as above. Simply take all of the maps f_i to be the identity unless we pass a multiple of 2^{n-k}, in which case, take the corresponding element of C_{2^k}. In this way, we see that the stack $\mathrm{Spec}(\pi_*^e \mathrm{MU}^{(\!(C_{2^n})\!)})//C_{2^n}$ provides a cover of the moduli stack of formal groups in a way that reflects the automorphisms groups which extend the formal inversion action and which are isomorphic to subgroups of C_{2^n}.

As an immediate, important example, we consider the universal deformation Γ_m of a fixed height-m formal group law F_m over an algebraically closed field k of characteristic p. Lubin and Tate [49] showed that the space of deformations is Ind-representable by a pro-ring abstractly isomorphic to

$$\mathbb{W}(k)[\![u_1, \ldots, u_{m-1}]\!][u^{\pm 1}] =: E_{m*},$$

over which Γ_m is defined. Here, $\mathbb{W}(k)$ is the p-typical Witt vectors of k, $|u_i| = 0$, and

$|u| = 2$.

By naturality, the ring E_{m*} is acted on by the Morava stabilizer group $\mathbb{S}_m$, the automorphism group of F_m. Hewett [34] showed that if $m = 2^{n-1}(2r+1)$, then there is a subgroup of the Morava stabilizer group isomorphic to C_{2^n}. In particular, associated to Γ_m and the action of a generator of C_{2^n}, we have a C_{2^n}-equivariant map

$$\pi_*^e \mathrm{MU}^{((C_{2^n}))} \longrightarrow E_{m*}.$$

Topologically, this entire story can be lifted. The formal group law Γ_m is Landweber exact, and hence there is a complex orientable spectrum E_m which carries the universal deformation Γ_m. The Goerss–Hopkins–Miller Theorem [65, 28] says that E_m is a commutative ring spectrum and that automorphism group of E_m as a commutative ring spectrum is homotopy equivalent to the Morava stabilizer group. In particular, we may view E_m as a commutative ring object in naive G-spectra. The functor

$$X \longmapsto F(EG_+, X)$$

takes naive equivalences to genuine equivariant equivalences, and hence allows us to view E_m as a genuine G-equivariant spectrum. The commutative ring spectrum structure on E_m gives an action of a trivial E_∞-operad on $F(EG_+, E_m)$. Work of Blumberg–Hill [18] shows that this is sufficient to ensure that $F(EG_+, E_m)$ is actually a genuine equivariant commutative ring spectrum, and hence it has norm maps.

The spectra E_m^{hG} turn out to be the essential building blocks of the p-local stable homotopy category. In particular, the homotopy groups $\pi_* E_m^{hG}$ assemble to the stable homotopy groups of spheres. To be more precise, the chromatic convergence theorem [63] exhibits the p-local sphere spectrum $\mathbb{S}_{(p)}$ as the inverse limit of the chromatic tower

$$\cdots \longrightarrow L_{E_m}\mathbb{S} \longrightarrow L_{E_{m-1}}\mathbb{S} \longrightarrow \cdots \longrightarrow L_{E_0}\mathbb{S},$$

where each $L_{E_m}\mathbb{S}$ is assembled via the chromatic fracture square

$$\begin{array}{ccc} L_{E_m}\mathbb{S} & \longrightarrow & L_{K(m)}\mathbb{S} \\ \downarrow & & \downarrow \\ L_{E_{m-1}}\mathbb{S} & \longrightarrow & L_{E_{m-1}}L_{K(m)}\mathbb{S}. \end{array}$$

Here, $K(m)$ is the mth Morava K-theory.

Devinatz and Hopkins [23] proved that $L_{K(m)}\mathbb{S} \simeq E_m^{h\mathbb{G}_m}$, and, furthermore, that the Adams–Novikov spectral sequence computing $L_{K(m)}\mathbb{S}$ can be identified with the associated homotopy fixed point spectral sequence for $E_m^{h\mathbb{G}_m}$. The fixed point spectrum $E_m^{h\mathbb{G}_m}$ admits resolutions by $\{E_m^{hG} \,|\, G \subset \mathbb{G}_m\}$, where G ranges over finite subgroups of $\mathbb{G}_m$.

At height 1, $E_1^{hC_2} = KO_2^{\wedge}$, the 2-adic completion of real K-theory. For this reason, the spectra E_m^{hG} are commonly called the *higher real K-theories*. The Morava stabilizer group $\mathbb{G}_1$ is isomorphic to $\mathbb{Z}_2^{\times}$. Adams, Baird, and Ravenel [62] showed that there is a fiber sequence

$$L_{K(1)}S^0 = E_1^{h\mathbb{G}_1} \longrightarrow E_1^{hC_2} \xrightarrow{\psi^3-1} E_1^{hC_2},$$

where ψ^3 is a topological generator of $\mathbb{Z}_2^{\times}/\{\pm 1\} \cong \mathbb{Z}_2$.

At height 2, these homotopy fixed points are known as TMF and TMF with level structures. Computations of the homotopy groups of these spectra are done by Hopkins–Mahowald [43], Bauer [11], Mahowald–Rezk [52], Behrens–Ormsby [17], Hill–Hopkins–Ravenel [40], and Hill–Meier [41]. For works on the resolution of the $K(2)$-local sphere, see papers of Goerss–Henn–Mahowald [26], Goerss–Henn–Mahowald–Rezk [27], Behrens

[15], Henn–Karamanov–Mahowald [33], Behrens–Ormsby [17], Beaudry [12, 13], Bobkova–Goerss [19], and Beaudry–Goerss–Henn [14].

For higher heights $m > 2$, the homotopy fixed points E_m^{hG} are notoriously difficult to compute. One of the chief reasons that these homotopy fixed points are so difficult to compute is because the group actions are constructed purely from obstruction theory. This stands in contrast to the norms of $\mathrm{MU}_{\mathbb{R}}$, whose actions are induced from geometry.

Recent work of Hahn–Shi [32] establishes the first known connection between the obstruction-theoretic actions on Lubin–Tate theories and the geometry of complex conjugation. More specifically, there is a Real orientation for any of the E_m: there are C_2-equivariant maps

$$\mathrm{MU}_{\mathbb{R}} \longrightarrow i_{C_2}^* E_m.$$

Using the norm-forget adjunction, such a map can be promoted to a G-equivariant map

$$\mathrm{MU}^{((G))} \longrightarrow N_{C_2}^G i_{C_2}^* E_m \longrightarrow E_m.$$

By construction, since the original map $\mathrm{MU}_{\mathbb{R}} \to E_m$ classified Γ_m as a Real formal group law, this G-equivariant map exactly recovers the algebraic map.

As a consequence of the Real orientation theorem, the fixed point spectra $(\mathrm{MU}^{((C_{2^n}))})^{C_{2^n}}$ and $E_{2^{n-1}m}^{hC_{2^n}}$ can be assembled into the following diagram:

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & (\mathrm{MU}^{((C_{2^n}))})^{C_{2^n}} & \longrightarrow & E_{2^{n-1}m}^{hC_{2^n}} \\
 & \nearrow & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
\mathbb{S} & \longrightarrow & (\mathrm{MU}^{((C_8))})^{C_8} & \longrightarrow & E_{4m}^{hC_8} \\
 & \searrow & \downarrow & & \\
 & & (\mathrm{MU}^{((C_4))})^{C_4} & \longrightarrow & E_{2m}^{hC_4} \\
 & \searrow & \downarrow & & \\
 & & (\mathrm{MU}_{\mathbb{R}})^{C_2} & \longrightarrow & E_m^{hC_2}.
\end{array}
\tag{8.1}
$$

The existence of equivariant orientations renders computations that rely on the slice spectral sequence tractable. Using differentials in the slice spectral sequence of $\mathrm{MU}_{\mathbb{R}}$ and the Real orientation $\mathrm{MU}_{\mathbb{R}} \to E_m$, Hahn–Shi computed $E_m^{hC_2}$, valid for arbitrarily large heights m.

An example of a Real orientable theory that was previously known is Atiyah's Real K-theory. In 1966, Atiyah [7] formalized the connection between complex K-theory (KU) and real K-theory (KO). Analogous as in the case of $MU_{\mathbb{R}}$, the complex conjugation action on complex vector bundles induces a natural C_2-action on KU, and this produces a C_2-spectrum $K_{\mathbb{R}}$ called Atiyah's Real K-theory. The theory $K_{\mathbb{R}}$ interpolates between complex and real K-theory in the sense that the underlying spectrum of $K_{\mathbb{R}}$ is KU, and its C_2-fixed points is KO. The $RO(C_2)$-graded homotopy groups $\pi_{\bigstar}^{C_2} K_{\mathbb{R}}$ has two periodicities: a ρ_2-periodicity that corresponds to the complex Bott-periodicity, and a 8-periodicity that corresponds to the real Bott-periodicity.

In [40], Hill, Hopkins, and Ravenel computed the slice spectral sequence of a C_4-

equivariant height-2 theory that is analogous to $K_{\mathbb{R}}$. To introduce this theory, note that the height of the formal group law Γ_m is at most m and the ring E_{m*} is 2-local. We can therefore pass to 2-typical formal group laws (and hence BP), and our map

$$\mathrm{BP} \longrightarrow E_m$$

classifying the formal group law descends to a map

$$E(m) \longrightarrow E_m,$$

where $E(m)$ is the height-m Johnson–Wilson theory. Equivariantly, we have a similar construction, which we review in more detail in Section 9. The C_{2^n}-equivariant map

$$\mathrm{BP}^{(\!(C_{2^n})\!)} \longrightarrow E_m$$

will factor through a localization of a quotient of $\mathrm{BP}^{(\!(C_{2^n})\!)}$. To study the Hurewicz image, it therefore suffices to study these localizations of quotients, and for these, it suffices to study the quotients. Hill–Hopkins–Ravenel restricted their attention to computing the homotopy Mackey functors of

$$\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle.$$

There exists a height-2 Lubin–Tate theory E_2 such that the C_4-equivariant orientation $\mathrm{BP}^{(\!(C_4)\!)} \to E_2$ factors through $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$:

$$\begin{array}{ccc} \mathrm{BP}^{(\!(C_4)\!)} & \longrightarrow & E_2 \\ \downarrow & \nearrow & \\ \mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle & & \end{array}$$

Furthermore, after inverting the element

$$D_1 := N(\bar{v}_2)N(\bar{r}_1) \in \pi_{4\rho_4}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$$

and applying $K(2)$-localization, there is an equivalence

$$(L_{K(2)}D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)^{C_4} \simeq E_2^{h\,\mathrm{Gal}(\mathbb{F}_{2^2}/\mathbb{F}_2)\ltimes C_4}.$$

The slice spectral sequence of $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ degenerates after the E_{13}-page and has a horizontal vanishing line of filtration 13.

The C_4-spectrum $D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ has three periodicities:

1. $S^{\rho_4} \wedge D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle \simeq D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$;
2. $S^{4-4\sigma} \wedge D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle \simeq D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$;
3. $S^{8+8\sigma-8\lambda} \wedge D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle \simeq D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$.

These three periodicities combine to imply that $D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ and $E_2^{hC_4}$ are 32-periodic theories.

To this end, the goal of this paper is to give a complete computation of the slice spectral sequence of the C_4-fixed points of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$.

Theorem 8.2 (Theorem 9.12)**.** *There exists a height-4 Lubin–Tate theory E_4 such that the C_4-equivariant orientation $\mathrm{BP}^{(\!(C_4)\!)} \longrightarrow E_4$ factors through $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$:*

$$\begin{array}{ccc} \mathrm{BP}^{(\!(C_4)\!)} & \longrightarrow & E_4 \\ \downarrow & \nearrow & \\ \mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle & & \end{array}$$

Furthermore, after inverting the element

$$D_2 := N(\bar{v}_4)N(\bar{r}_3)N(\bar{r}_3^2 + \bar{r}_3(\gamma\bar{r}_3) + (\gamma\bar{r}_3)^2) \in \pi_{24\rho_4}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$$

and applying $K(4)$-localization, there is an equivalence

$$(L_{K(4)}D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)^{C_4} \simeq E_4^{h\,\mathrm{Gal}(\mathbb{F}_{2^4}/\mathbb{F}_2)\ltimes C_{12}}.$$

Theorem 8.3. *We completely compute the slice spectral sequence of* $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ *(see Figure 8). The slice spectral sequence degenerates after the E_{61}-page and has a horizontal vanishing line of filtration 61.*

Theorem 8.4. *After inverting the element $D_2 \in \pi_{24\rho_4}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$, the C_4-spectrum $D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ has three periodicities:*

1. $S^{3\rho_4} \wedge D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \simeq D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$*;*

2. $S^{24-24\sigma} \wedge D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \simeq D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$*;*

3. $S^{32+32\sigma-32\lambda} \wedge D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \simeq D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$.

Together, these three periodicities imply that $(D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)^{C_4}$ and $E_4^{hC_{12}}$ are 384-periodic theories.

When $G = C_2$, Li–Shi–Wang–Xu [47] analyzed the bottom layer of tower (8.1) and showed that the Hopf-, Kervaire-, and $\bar{\kappa}$-families in the stable homotopy groups of spheres are detected by the map

$$\mathbb{S} \longrightarrow (\mathrm{MU}_{\mathbb{R}})^{C_2}.$$

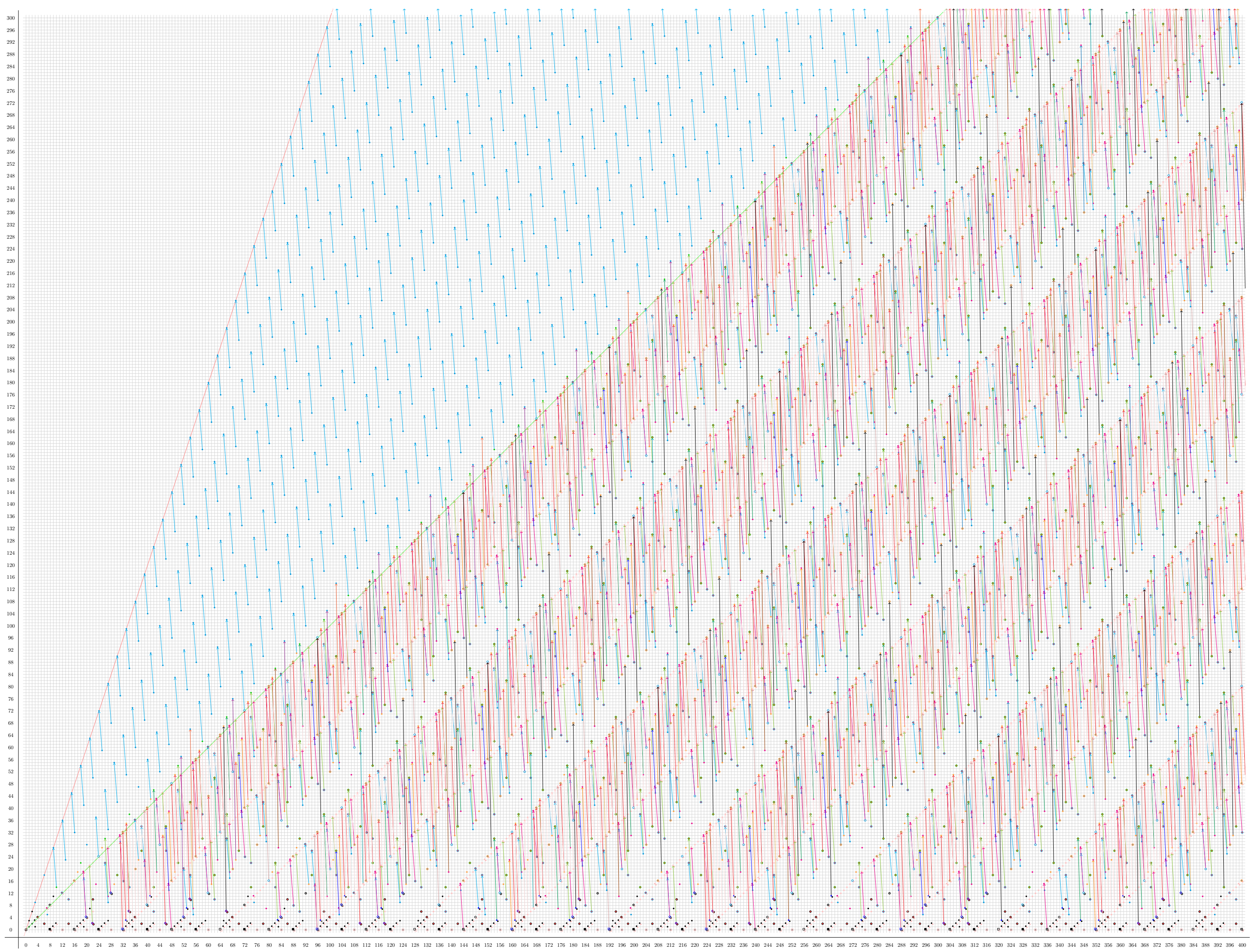


Figure 8: The slice spectral sequence of $BP^{((C_4))}\langle 2\rangle$.

As we increase the height m, an increasing subset of the elements in these families is detected by the map

$$\mathbb{S} \longrightarrow E_m^{hC_2}.$$

Since $(\mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle)^{C_4}$ is closely related to $\mathrm{TMF}_0(5)$, one can study its Hurewicz images via the Hurewicz images of TMF (see [17, 40]). In particular, there are elements detected by the C_4-fixed points $(\mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle)^{C_4}$ that are not detected by $(\mathrm{MU}_{\mathbb{R}})^{C_2}$.

In general, it is difficult to determine all the Hurewicz images of $(\mathrm{BP}^{(\!(G)\!)})^G$. Computations of Hill [35] have shown that the class $\eta^3 \in \pi_3\mathbb{S}$ is not detected by $(\mathrm{BP}^{(\!(C_{2^n})\!)})^{C_{2^n}}$ for any $n \geq 1$. However, this element is detected by $(\mathrm{BP}^{(\!(Q_8)\!)})^{Q_8}$, where Q_8 is the quaternion group. It is a current project to understand the Hurewicz images of the G-fixed points of $\mathrm{BP}^{(\!(G)\!)}$ and its various quotients when $G = C_{2^n}$ and Q_8. In particular, since we have completely computed the slice spectral sequence of $(\mathrm{BP}^{(\!(C_4)\!)}\langle 2 \rangle)^{C_4}$, the following question is of immediate interest:

Question 8.5. *What are the Hurewicz images of* $(\mathrm{BP}^{(\!(C_4)\!)}\langle 2 \rangle)^{C_4}$*?*

8.2 Summary of the contents

We now turn to a summary of the contents for the second part of the thesis. Section 9 provides the necessary background on $\mathrm{MU}^{(\!(G)\!)}$. In particular, we define the Hill–Hopkins–Ravenel theories $\mathrm{BP}^{(\!(G)\!)}\langle m \rangle$ (Defintion 9.1) and describe the E_2-pages of their slice spectral sequences. Theorem 9.12 shows that after periodization and completion, the C_4-fixed points of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2 \rangle$ is equivalent to the C_{12}-fixed points of a height-4 Lubin–Tate theory E_4.

In Section 10, we review Hill–Hopkins–Ravenel's computation of $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle)$. Our proofs for some of the differentials are slightly different than those appearing in [40]. The computation is presented in a way that will resemble our subsequent computation for

SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$).

Section 11 describes the slice filtration of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$. We organize the slice cells of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ into collections called $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-*truncations* and $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-*truncations*. This is done to facilitate later computations. In Section 12, we compute the C_2-slice spectral sequence of $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$.

From Section 13 foward, we focus our attention on computing the C_4-slice spectral sequence of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$. Section 13 proves that all the differentials in C_4-SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$) of length ≤ 12, as well as some of the d_{13}-differentials, can be induced from C_4-SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$) via the quotient map $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \to \mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$. In Section 14 we prove all the d_{13} and d_{15} differentials by using the restriction map, the transfer map, and multiplicative structures.

In Section 15, we prove differentials on the classes $u_{2\lambda}a_\sigma$, $u_{4\lambda}a_\sigma$, $u_{8\lambda}a_\sigma$, and $u_{16\lambda}a_\sigma$ by norming up C_2-equivariant differentials in C_2-SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$). Using these differentials, we prove the Vanishing Theorem (Theorem 18.1), which states that a large portion of the classes that are above filtration 96 on the E_2-page must die on or before the E_{61}-page. The Vanishing Theorem is of great importance for us because it establishes a bound on the differentials that can possibly occur on a class.

Sections 17, 18, and 19 prove all of the the remaining differentials in the slice spectral sequence. The slice spectral sequence degenerates after the E_{61}-page and has a horizontal vanishing line of filtration 61 at the E_∞-page. Section 20 gives a summary of all the differentials.

9 Preliminaries

9.1 The slice spectral sequence of $\mathrm{MU}^{((G))}$

Let $\mathrm{MU}_{\mathbb{R}}$ be the Real cobordism spectrum, and G be the cyclic group of order 2^n. The spectrum $\mathrm{MU}^{((G))}$ is defined as

$$\mathrm{MU}^{((G))} := N_{C_2}^{G}\mathrm{MU}_{\mathbb{R}},$$

where $N_H^G(-)$ is the Hill–Hopkins–Ravenel norm functor [39]. The underlying spectrum of $\mathrm{MU}^{((G))}$ is the smash product of 2^{n-1}-copies of MU.

Hill, Hopkins, and Ravenel [39, Section 5] constructed generators

$$\bar{r}_i \in \pi_{i\rho_2}^{C_2}\mathrm{MU}^{((G))}$$

such that

$$\pi_{*\rho_2}^{C_2}\mathrm{MU}^{((G))} \cong \mathbb{Z}_{(2)}[\bar{r}_1, \gamma\bar{r}_1, \ldots, \gamma^{2^{n-1}-1}\bar{r}_1, \bar{r}_2, \ldots].$$

Here γ is a generator of C_{2^n}, and the Weyl action is given by

$$\gamma \cdot \gamma^j \bar{r}_i = \begin{cases} \gamma^{j+1}\bar{r}_i & 0 \leq j \leq 2^{n-1}-2 \\ (-1)^i \bar{r}_i & j = 2^{n-1}-1. \end{cases}$$

Adjoint to the maps

$$\bar{r}_i \colon S^{i\rho_2} \to i_{C_2}^{*}\mathrm{MU}^{((G))}$$

are associative algebra maps from free associative algebras

$$\mathbb{S}^0[\bar{r}_i] = \bigvee_{j\geq 0} \left(S^{i\rho_2}\right)^{\wedge j} \to i^*_{C_2}\mathrm{MU}^{((G))},$$

and hence G-equivariant associative algebra maps

$$\mathbb{S}^0[G\cdot\bar{r}_i] = N_{C_2}^G \mathbb{S}^0[\bar{r}_i] \to \mathrm{MU}^{((G))}.$$

Smashing these all together gives an associative algebra map

$$A := \mathbb{S}^0[G\cdot\bar{r}_1,\dots] = \bigwedge_{i=1}^{\infty} S^0[G\cdot\bar{r}_i] \to \mathrm{MU}^{((G))}.$$

For $\mathrm{MU}^{((G))}$ and the quotients below, the slice filtration is the filtration associated to the powers of the augmentation ideal of A, by the Slice Theorem of [39].

The classical Quillen idempotent map $\mathrm{MU} \longrightarrow \mathrm{BP}$ can be lifted to a C_2-equivariant map

$$\mathrm{MU}_{\mathbb{R}} \longrightarrow \mathrm{BP}_{\mathbb{R}},$$

where $\mathrm{BP}_{\mathbb{R}}$ is the Real Brown–Peterson spectrum. Taking the norm $N_{C_2}^G(-)$ of this map produces a G-equivariant map

$$\mathrm{MU}^{((G))} \to \mathrm{BP}^{((G))} =: N_{C_2}^G \mathrm{BP}_{\mathbb{R}}.$$

Using the techniques developed in [39], it follows that $\mathrm{BP}^{((G))}$ has refinement

$$\mathbb{S}^0[G\cdot\bar{r}_1, G\cdot\bar{r}_3, G\cdot\bar{r}_7,\dots] \longrightarrow \mathrm{BP}^{((G))}.$$

We can also produce truncated versions of these norms of $\mathrm{BP}_{\mathbb{R}}$, wherein we form quotients by all of the $\bar{r}_{2^m-1}$ for all m sufficiently large. For each $m \geq 0$, let

$$A_m = \bigwedge_{j=m}^{\infty} \mathbb{S}^0[G \cdot \bar{r}_{2^j-1}].$$

Definition 9.1 (Hill–Hopkins–Ravenel theories)**.** *For each $m \geq 0$, let*

$$\mathrm{BP}^{(\!(G)\!)}\langle m \rangle = \mathrm{BP}^{(\!(G)\!)} \wedge_{A_m} S^0.$$

The Reduction Theorem of [39] says that for all G, $\mathrm{BP}^{(\!(G)\!)}\langle 0 \rangle = H\underline{\mathbb{Z}}$, and [40] studied the spectrum $\mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle$ (a computation we review below).

Remark 9.2. *Although the underlying homotopy groups of $\mathrm{BP}^{(\!(G)\!)}\langle m \rangle$ is a polynomial ring:*

$$\pi_*^e \mathrm{BP}^{(\!(G)\!)}\langle m \rangle \cong \mathbb{Z}_{(2)}[r_1, \gamma r_1, \ldots, \gamma^{2^{n-1}-1} r_1, \ldots, \gamma^{2^{n-1}-1} r_{2^m-1}],$$

we do not know that $\mathrm{BP}^{(\!(G)\!)}\langle m \rangle$ has even an associative multiplication. It is, however, canonically an $\mathrm{MU}^{(\!(G)\!)}$-module, and hence the slice spectral sequence will be a spectral sequence of modules over the slice spectral sequence for $\mathrm{MU}^{(\!(G)\!)}$.

The same arguments as for $\mathrm{BP}^{(\!(G)\!)}$ allow us to determine the slice associated graded for $\mathrm{BP}^{(\!(G)\!)}\langle m \rangle$ for any m.

Theorem 9.3. *The slice associated graded for $\mathrm{BP}^{(\!(G)\!)}\langle m \rangle$ is the graded spectrum*

$$\mathbb{S}^0[G \cdot \bar{r}_1, \ldots, G \cdot \bar{r}_{2^m-1}] \wedge H\underline{\mathbb{Z}},$$

where the degree of a summand corresponding to a polynomial in the $\bar{r}_i$ and their conjugates *is just the underlying degree.*

Corollary 9.4. *The slice spectral sequence for the $RO(G)$-graded homotopy of* $\mathrm{BP}^{(\!(G)\!)}\langle m\rangle$ *has E_2-term the $RO(G)$-graded homology of $\mathbb{S}^0[G\cdot\bar{r}_1,\ldots,G\cdot\bar{r}_{2^m-1}]$ with coefficients in $\underline{\mathbb{Z}}$, the constant Mackey functor $\mathbb{Z}$.*

Since the slice filtration is an equivariant filtration, the slice spectral sequence is a spectral sequence of $RO(G)$-graded Mackey functors. Moreover, the slice spectral sequence for $\mathrm{MU}^{(\!(G)\!)}$ is a multiplicative spectral sequence, and the slice spectral sequence for $\mathrm{BP}^{(\!(G)\!)}\langle m\rangle$ is a spectral sequence of modules over it in Mackey functors.

9.2 The slice spectral sequence for $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$

From now on, we restrict attention to the case $G = C_4$ and $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$. We will use the slice spectral sequence to compute the integer graded homotopy Mackey functors of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$. To describe this, we describe in more detail the E_2-term of the slice spectral sequence.

Notation 9.5. *Let σ denote the 1-dimensional sign representation of C_4, and let λ denote the 2-dimensional irreducible representation of C_4 given by rotation by $\pi/2$. Let σ_2 denote the 1-dimensional sign representation of C_2. Finally, let 1 denote the trivial representation of dimension 1.*

The homology groups of a representation sphere with coefficients in $\underline{\mathbb{Z}}$ are generated by certain products of Euler classes and orientation classes for irreducible representations.

Definition 9.6. *For any representation V for which $V^G = \{0\}$, let $a_V\colon S^0 \to S^V$ denote the Euler class of the representation V. Let a_V also denote the corresponding Hurewicz image in $\pi_{-V}H\underline{\mathbb{Z}}$.*

Definition 9.7. *If V is an orientable representation of G, then let*

$$u_V \in H_{\dim V}(S^V; \underline{\mathbb{Z}}) \cong \mathbb{Z}$$

be the generator which restricts to the element 1 under the suspension isomorphism

$$H_{\dim V}(S^{\dim V}; \mathbb{Z}) \cong \tilde{H}_0(S^0; \mathbb{Z}).$$

For the group C_4, these elements satisfy a number of relations:

1. $2a_\sigma = 2a_{\sigma_2} = 4a_\lambda = 0$;
2. $res^{C_4}_{C_2}(a_\sigma) = 0$, $res^{C_4}_{C_2}(a_\lambda) = a_{2\sigma_2}$, $res^{C_4}_{C_2}(u_{2\sigma}) = 1$, $res^{C_4}_{C_2}(u_\lambda) = u_{2\sigma_2}$;
3. $u_\lambda a_{2\sigma} = 2a_\lambda u_{2\sigma}$ (gold relation);

These allow us to identify all of the elements in the homology groups of representation spheres.

9.3 Tambara structure

A multiplicative spectral sequence of Mackey functors can equivalently be thought of a a kind of Mackey functor object in spectral sequences. In particular, we can view this as being 3 spectral sequences:

1. a multiplicative spectral sequence computing the C_4-fixed points,
2. a multiplicative spectral sequence computing the C_2-fixed points, and
3. a (collapsing) multiplicative spectral sequence computing the underlying homotopy.

The restriction and transfer maps in the Mackey functors can then be viewed as maps of spectral sequences connecting these, with the restriction maps being maps of DGAs, and the transfer maps being maps of DGMs over these DGAs.

For commutative ring spectra like $\mathrm{MU}^{((G))}$, we have additional structure on the $RO(G)$-graded homotopy groups given by the norms. If R is a G-equivariant commutative ring spectrum, then we have a multiplicative map

$$N_H^G \colon \pi_V^H(R) \to \pi^G_{Ind_H^G V}(R)$$

which takes a map

$$S^V \to i_H^* R$$

to the composite

$$S^{Ind_H^G V} \cong N_H^G(S^V) \to N_H^G i_H^* R \to R,$$

where the final map is the counit of the norm-forget adjunction. The norm maps are not additive, but they do satisfy certain explicitly describable formulae which encode the norms of sums and of transfers. At the level of π_0, this data is traditionally called a "Tambara functor", studied by Brun for equivariant commutative ring spectra, and more generally, this $RO(G)$-graded version was used by Hill, Hopkins, and Ravenel in their analysis of the slice spectral sequence [21, 39, 40].

In the slice spectral sequence, the norms play a more subtle role. The norm from H to G scales slice filtration by $|G/H|$, just as multiplication scales degree. In particular, it will not simply commute with the differentials. We have a formula, however, for the differentials on key multiples of norms.

Theorem 9.8. *Let $d_r(x) = y$ be a d_r-differential in the C_2-slice spectral sequence. If both*

$a_\sigma N_{C_2}^{C_4}x$ *and* $N_{C_2}^{C_4}y$ *survive to the* E_{2r-1}*-page, then* $d_{2r-1}(a_\sigma N_{C_2}^{C_4}x) = N_{C_2}^{C_4}y$ *in* C_4-SliceSS(X) *(see [40, Corollary 4.8]).*

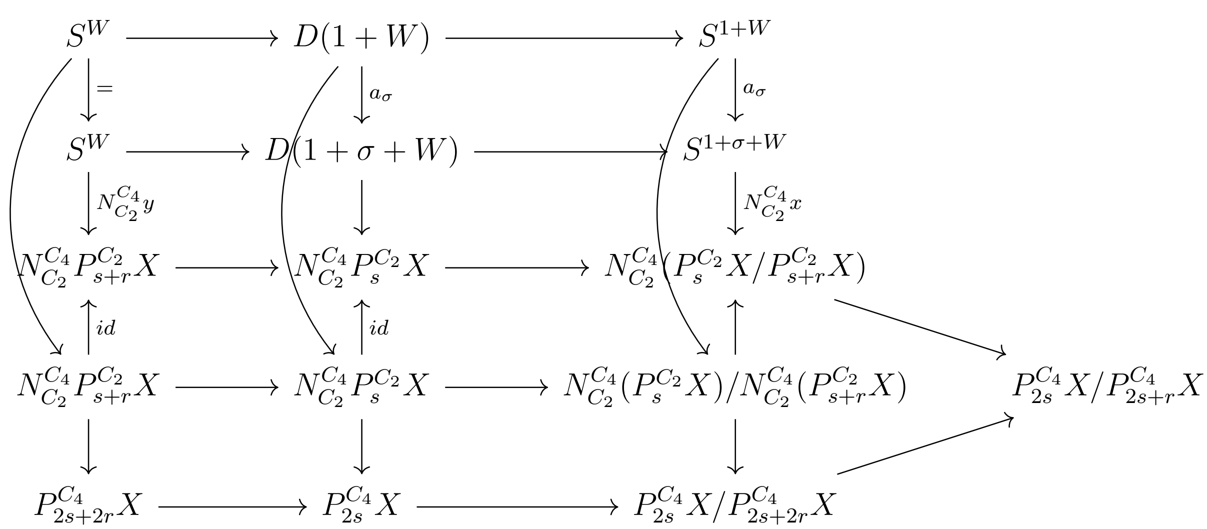
Proof. The d_r-differential can be represented by the diagram

$$\begin{array}{ccccc} S^V & \longrightarrow & D(1+V) & \longrightarrow & S^{1+V} \\ \downarrow{\scriptstyle y} & & \downarrow & & \downarrow{\scriptstyle x} \\ P^{C_2}_{s+r}X & \longrightarrow & P^{C_2}_{s}X & \longrightarrow & P^{C_2}_{s}X/P^{C_2}_{s+r}X. \end{array}$$

Applying the norm functor $N_{C_2}^{C_4}(-)$ yields the new diagram

$$\begin{array}{ccccc} S^W & \longrightarrow & D(1+\sigma+W) & \longrightarrow & S^{1+\sigma+W} \\ \downarrow{\scriptstyle N_{C_2}^{C_4}y} & & \downarrow & & \downarrow{\scriptstyle N_{C_2}^{C_4}x} \\ N_{C_2}^{C_4}P^{C_2}_{s+r}X & \longrightarrow & N_{C_2}^{C_4}P^{H}_{s}X & \longrightarrow & N_{C_2}^{C_4}(P^{C_2}_{s}X/P^{H}_{s+r}X). \end{array}$$

Both rows of the this diagram are no longer cofiber sequences. We can enlarge this diagram so that both the top and the bottom rows are cofiber sequences:



The first, fourth, and fifth rows are cofiber sequences. The third vertical map from the fourth row to the third row is induced by the first two vertical maps. The third long vertical map from the first row to the fourth row is induced from the first two long vertical maps.

The composite map from the first row to the fifth row predicts a d_{2r-1}-differential in the C_4-slice spectral sequence. The predicted target is $N_{C_2}^{C_4} y$. Therefore, this class must die on or before the E_{2r-1}-page. If both this class and $a_\sigma N_{C_2}^{C_4} x$ survive to the E_{2r-1}-page, then

$$d_{2r-1}(a_\sigma N_{C_2}^{C_4} x) = N_{C_2}^{C_4} y.$$

□

Remark 9.9. *The slice spectral sequence is actually a spectral sequence of graded Tambara functors in the sense that the differentials are actually genuine equivariant differentials in the sense of [36]. We will not need this in what follows, however.*

9.4 Formal group law formulas

Consider the C_2-equivariant map

$$\mathrm{BP}_{\mathbb{R}} \longrightarrow i_{C_2}^* N_{C_2}^{C_4} \mathrm{BP}_{\mathbb{R}} = i_{C_2}^* \mathrm{BP}^{((C_4))}$$

coming from the norm-restriction adjunction. Post-composing with the quotient map $\mathrm{BP}^{((C_4))} \to \mathrm{BP}^{((C_4))}\langle 2 \rangle$ produces the C_2-equivariant map

$$\mathrm{BP}_{\mathbb{R}} \to i_{C_2}^* \mathrm{BP}^{((C_4))}\langle 2 \rangle,$$

which, after taking $\pi_{*\rho_2}^{C_2}(-)$, is a map

$$\mathbb{Z}[\bar{v}_1, \bar{v}_2, \ldots] \longrightarrow \mathbb{Z}[\bar{r}_1, \gamma\bar{r}_1, \bar{r}_3, \gamma\bar{r}_3]$$

of polynomial algebras.

Let $S := \pi^{C_2}_{*\rho_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle = \mathbb{Z}[\bar{r}_1, \gamma\bar{r}_1, \bar{r}_3, \gamma\bar{r}_3]$. By an abuse of notation, let $\bar{v}_i \in S$ denote the image of $\bar{v}_i \in \pi^{C_2}_{(2^i-1)\rho_2}\mathrm{BP}_{\mathbb{R}}$ under the map above. Our next goal is to relate the $\bar{v}_i$-generators to the $\bar{r}_i$-generators.

Let $\bar{F}$ be the C_2-equivariant formal group law corresponding to the map $\pi^{C_2}_{*\rho_2}\mathrm{BP}_{\mathbb{R}} \to \pi^{C_2}_{*\rho_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$. By definition, its 2-series is

$$[2]_{\bar{F}}(\bar{x}) = 2\bar{x} +_{\bar{F}} \bar{v}_1\bar{x}^2 +_{\bar{F}} \bar{v}_2\bar{x}^4 +_{\bar{F}} \bar{v}_3\bar{x}^8 +_{\bar{F}} \bar{v}_4\bar{x}^{16} + \cdots .$$

Let $\bar{m}_i \in 2^{-1}S$ be the coefficients of the logarithm of $\bar{F}$:

$$\log_{\bar{F}}(\bar{x}) = \bar{x} + \bar{m}_1\bar{x}^2 + \bar{m}_2\bar{x}^4 + \bar{m}_3\bar{x}^8 + \bar{m}_4\bar{x}^{16} + \cdots .$$

Taking the logarithm of both sides of the 2-series produces the equation

$$2\log_{\bar{F}}(\bar{x}) = \log_{\bar{F}}(2\bar{x}) + \log_{\bar{F}}(\bar{v}_1\bar{x}^2) + \log_{\bar{F}}(\bar{v}_2\bar{x}^4) + \log_{\bar{F}}(\bar{v}_3\bar{x}^8) + \cdots .$$

Expanding both sides of the equation using the power series expansion of the logarithm and comparing coefficients, we obtain the equations

$$\begin{aligned} 2\bar{m}_1 &= 4\bar{m}_1 + \bar{v}_1 \\ 2\bar{m}_2 &= 16\bar{m}_2 + \bar{m}_1\bar{v}_1^2 + \bar{v}_2 \\ 2\bar{m}_3 &= 2^8\bar{m}_3 + \bar{m}_2\bar{v}_1^4 + \bar{m}_1\bar{v}_2^2 + \bar{v}_3 \\ 2\bar{m}_4 &= 2^{16}\bar{m}_4 + \bar{m}_3\bar{v}_1^8 + \bar{m}_2\bar{v}_2^4 + \bar{m}_1\bar{v}_3^2 + \bar{v}_4 \\ &\vdots \end{aligned} \tag{9.1}$$

Rearranging, we obtain the relation

$$\bar{v}_i = 2\bar{m}_i \pmod{M_i} \tag{9.2}$$

for all $i \geq 1$. Here, M_i is the S-submodule of $2^{-1}S$ (regarded as a S-module) that is generated by the elements 2, $\bar{m}_1$, $\bar{m}_2$, …, $\bar{m}_{i-1}$. In other words, an element in M_i is of the form

$$s_0 \cdot 2 + s_1 \cdot \bar{m}_1 + \cdots + s_{i-1}\bar{m}_{i-1}$$

where $s_j \in S$ for all $0 \leq j \leq i-1$.

Lemma 9.10. *Let* $I_i \in S$ *denote the ideal* $(2, \bar{v}_1, \cdots \bar{v}_{i-1})$. *Then*

$$M_i \cap S = I_i.$$

Proof. We will prove the claim by using induction on i. The base case when $i \geq 1$ is straight forward: an element in M_1 is of the form $s_0 \cdot 2$, where $s_0 \in S$. Therefore $M_1 \cap S = (2) = I_1$.

Now, suppose that $M_{i-1} \cap S = I_{i-1}$. Furthermore, suppose that the element

$$m = s_0 \cdot 2 + s_1 \cdot \bar{m}_1 + \cdots + s_{i-1}\bar{m}_{i-1} \in M_i$$

is also in S. From the equations in (9.1), it is straightforward to see that $\bar{m}_k$ has denominator exactly 2^k for all $k \geq 1$. In the expression for m, only the last term $s_{i-1}\bar{m}_{i-1}$ has denominator 2^{i-1}. All the other terms have denominators at most 2^{i-2}. Since $m \in S$, s_{i-1} must be divisible by 2. In other words, $s_{i-1} = 2s'_{i-1}$ for some $s'_{i-1} \in S$. Using equation (9.2),

m can be rewritten as

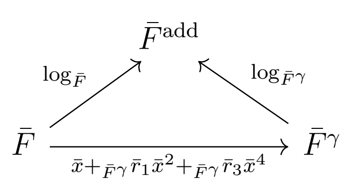
$$\begin{aligned} m &= s_0 \cdot 2 + s_1 \cdot \bar{m}_1 + \cdots + s_{i-2}\bar{m}_{i-2} + 2s'_{i-1}\bar{m}_{i-1} \\ &= s_0 \cdot 2 + s_1 \cdot \bar{m}_1 + \cdots + s_{i-2}\bar{m}_{i-2} + s'_{i-1}(2\bar{m}_{i-1}) \\ &\in s_0 \cdot 2 + s_1 \cdot \bar{m}_1 + \cdots + s_{i-2}\bar{m}_{i-2} + s'_{i-1}(v_{i-1} + M_{i-1}) \\ &\in M_{i-1} + s'_{i-1}(v_{i-1} + M_{i-1}) \\ &= M_{i-1} + s'_{i-1}v_{i-1}. \end{aligned}$$

Therefore, $m = x + s'_{i-1}v_{i-1}$ for some $x \in M_{i-1}$. Since $m \in S$ and $s'_{i-1}v_{i-1} \in S$, $x \in S$ as well. The induction hypothesis now implies that $x \in I_{i-1}$. It follows from this that $m \in I_i$, as desired. □

Theorem 9.11. *We have the following relations:*

$$\begin{aligned} \bar{v}_1 &= \bar{r}_1 + \gamma\bar{r}_1 \pmod{2}, \\ \bar{v}_2 &= \bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3 \pmod{2, \bar{v}_1}, \\ \bar{v}_3 &= \bar{r}_1(\bar{r}_3^2 + \bar{r}_3(\gamma\bar{r}_3) + (\gamma\bar{r}_3)^2) \pmod{2, \bar{v}_1, \bar{v}_2}, \\ \bar{v}_4 &= \bar{r}_3^4(\gamma\bar{r}_3) \pmod{2, \bar{v}_1, \bar{v}_2, \bar{v}_3}. \end{aligned}$$

Proof. To obtain the formulas in the statement of the theorem, we need to establish relations between the generators $\{\bar{r}_1, \gamma\bar{r}_1, \bar{r}_3, \gamma\bar{r}_3\}$ and the $\bar{m}_i$-generators. The $\bar{r}_i$ generators, by definition, are the coefficients of the strict isomorphism from $\bar{F}$ to $\bar{F}^\gamma$ (see [39, Section 5]):



Here, $\log_{\bar{F}^\gamma}$ is the logarithm for the formal group law $\bar{F}^\gamma$, and its power-series expansion is

$$\log_{\bar{F}^\gamma}(x) = \bar{x} + (\gamma\bar{m}_1)\bar{x}^2 + (\gamma\bar{m}_2)\bar{x}^4 + (\gamma\bar{m}_3)\bar{x}^8 + (\gamma\bar{m}_4)\bar{x}^{16} + \cdots .$$

The commutativity of the diagram implies that

$$\begin{aligned}\log_{\bar{F}}(\bar{x}) &= \log_{\bar{F}^\gamma}(\bar{x} +_{\bar{F}^\gamma} \bar{r}_1\bar{x}^2 +_{\bar{F}^\gamma} \bar{r}_3\bar{x}^4) \\ &= \log_{\bar{F}^\gamma}(\bar{x}) + \log_{\bar{F}^\gamma}(\bar{r}_1\bar{x}^2) + \log_{\bar{F}^\gamma}(\bar{r}_3\bar{x}^4)\end{aligned}$$

Expanding both sides according to the logarithm formulas, we get

$$\begin{aligned}\bar{x} + \bar{m}_1\bar{x}^2 + \bar{m}_2\bar{x}^4 + \bar{m}_3\bar{x}^8 + \bar{m}_4\bar{x}^{16} + \cdots &= \bar{x} + (\gamma\bar{m}_1 + \bar{r}_1)\bar{x}^2 + (\gamma\bar{m}_2 + (\gamma\bar{m}_1)\bar{r}_1^2 + \bar{r}_3)\bar{x}^4 \\ &+ (\gamma\bar{m}_3 + (\gamma\bar{m}_2)\bar{r}_1^4 + (\gamma\bar{m}_1)\bar{r}_3^2)\bar{x}^8 \\ &+ (\gamma\bar{m}_4 + (\gamma\bar{m}_3)\bar{r}_1^8 + (\gamma\bar{m}_2)\bar{r}_3^4)\bar{x}^{16} + \cdots\end{aligned}$$

Comparing coefficients, we obtain the relations

$$\begin{aligned}\bar{m}_1 - \gamma\bar{m}_1 &= \bar{r}_1 \\ \bar{m}_2 - \gamma\bar{m}_2 &= (\gamma\bar{m}_1)\bar{r}_1^2 + \bar{r}_3 \\ \bar{m}_3 - \gamma\bar{m}_3 &= (\gamma\bar{m}_2)\bar{r}_1^4 + (\gamma\bar{m}_1)\bar{r}_3^2 \\ \bar{m}_4 - \gamma\bar{m}_4 &= (\gamma\bar{m}_3)\bar{r}_1^8 + (\gamma\bar{m}_2)\bar{r}_3^4\end{aligned}$$

We can also apply γ to the relations above to obtain more relations

$$\begin{aligned}
\gamma\bar{m}_1 + \bar{m}_1 &= \gamma\bar{r}_1 \\
\gamma\bar{m}_2 + \bar{m}_2 &= -\bar{m}_1(\gamma\bar{r}_1)^2 + \gamma\bar{r}_3 \\
\gamma\bar{m}_3 + \bar{m}_3 &= -\bar{m}_2(\gamma\bar{r}_1)^4 - \bar{m}_1(\gamma\bar{r}_3)^2 \\
\gamma\bar{m}_4 + \bar{m}_4 &= -\bar{m}_3(\gamma\bar{r}_1)^8 - \bar{m}_2(\gamma\bar{r}_3)^4.
\end{aligned}$$

These relations together produce the following formulas:

$$
\begin{aligned}
\bar{v}_1 &= 2\bar{m}_1 \pmod{M_1} \\
&= \bar{r}_1 + \gamma\bar{r}_1 \pmod{M_1}; \\
\bar{v}_2 &= 2\bar{m}_2 \pmod{M_2} \\
&= (\gamma\bar{m}_1)\bar{r}_1^2 + \bar{r}_3 + \gamma\bar{r}_3 \pmod{M_2} \\
&= (\gamma\bar{m}_1 - \bar{m}_1)\bar{r}_1^2 + \bar{r}_3 + \gamma\bar{r}_3 \pmod{M_2} \\
&= -\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3 \pmod{M_2} \\
&= \bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3 \pmod{M_2}; \\
\bar{v}_3 &= 2\bar{m}_3 \pmod{M_3} \\
&= (\gamma\bar{m}_2)\bar{r}_1^4 + (\gamma\bar{m}_1)\bar{r}_3^2 - \bar{m}_2(\gamma\bar{r}_1)^4 - \bar{m}_1(\gamma\bar{r}_3)^2 \pmod{M_3} \\
&= (\gamma\bar{m}_2)\bar{r}_1^4 + (\gamma\bar{m}_1)\bar{r}_3^2 \pmod{M_3} \\
&= (\gamma\bar{m}_2 + \bar{m}_2)\bar{r}_1^4 + (\gamma\bar{m}_1 + \bar{m}_1)\bar{r}_3^2 \pmod{M_3} \\
&= (\gamma\bar{r}_3)\bar{r}_1^4 + (\gamma\bar{r}_1)\bar{r}_3^2 \pmod{M_3} \\
&= \bar{r}_1\left((\gamma\bar{r}_3)(\bar{r}_3 + \gamma\bar{r}_3) + \bar{r}_3^2\right) \pmod{2M_3} \\
&= \bar{r}_1(\bar{r}_3^2 + \bar{r}_3(\gamma\bar{r}_3) + (\gamma\bar{r}_3)^2) \pmod{M_3}; \\
\bar{v}_4 &= 2\bar{m}_4 \pmod{M_4} \\
&= (\gamma\bar{m}_3)\bar{r}_1^8 + (\gamma\bar{m}_2)\bar{r}_3^4 - \bar{m}_3(\gamma\bar{r}_1)^8 - \bar{m}_2(\gamma\bar{r}_3)^4 \pmod{M_4} \\
&= (\gamma\bar{m}_3)\bar{r}_1^8 + (\gamma\bar{m}_2)\bar{r}_3^4 \pmod{M_4} \\
&= (\gamma\bar{m}_3 + \bar{m}_3)\bar{r}_1^8 + (\gamma\bar{m}_2 + \bar{m}_2)\bar{r}_3^4 \pmod{M_4} \\
&= (\gamma\bar{r}_3)\bar{r}_3^4 \pmod{M_4}.
\end{aligned}
$$

These formulas, combined with Lemma 9.10, give the desired formulas. □

9.5 Lubin–Tate Theories

We will now establish the relationship between $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ and a specific height-4 Lubin–Tate theory.

Theorem 9.12. *There exists a height-4 Lubin–Tate theory E_4 such that the C_4-equivariant orientation $\mathrm{BP}^{(\!(C_4)\!)} \longrightarrow E_4$ factors through $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$:*

$$\begin{array}{ccc} \mathrm{BP}^{(\!(C_4)\!)} & \longrightarrow & E_4 \\ \downarrow & \nearrow & \\ \mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle & & \end{array}$$

Furthermore, after inverting the element

$$D_2 := N(\bar{v}_4)N(\bar{r}_3)N(\bar{r}_3^2 + \bar{r}_3(\gamma\bar{r}_3) + (\gamma\bar{r}_3)^2) \in \pi_{24\rho_4}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$$

and applying $K(4)$-localization, there is an equivalence

$$(L_{K(4)}D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)^{C_4} = E_4^{h\,\mathrm{Gal}(\mathbb{F}_{2^4}/\mathbb{F}_2)\ltimes C_{12}}.$$

Proof. Recall that the underlying homotopy group of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ is

$$\pi_*^u\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle = \mathbb{Z}[r_1, \gamma r_1, r_3, \gamma r_3].$$

Consider the map

$$\pi_* BP \longrightarrow \pi_*^u\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$$

that is induced by taking $\pi_*^u(-)$ of the composite map

$$\mathrm{BP}_{\mathbb{R}} \longrightarrow i_{C_2}^* \mathrm{BP}^{(\!(C_4)\!)} \longrightarrow i_{C_2}^* \mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle.$$

Using the formulas in Theorem 9.11, we see that $(2, v_1, v_2, v_3)$ forms a regular sequence in $\pi_*^u \mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$. Furthermore, after inverting the element D_2,

$$\pi_*^u D_2^{-1} \mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle/(2, v_1, v_2, v_3) = \mathbb{F}_2[\bar{r}_3^{\pm}].$$

After applying the $K(4)$-localization functor $L_{K(4)}(-) := F(EC_{4+}, L_{K(4)} i_e^*(-))$, the underlying coefficient ring of $L_{K(4)} D_2^{-1} \mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ is

$$\pi_*^u (L_{K(4)} D_2^{-1} \mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle) = \mathbb{Z}_2[[r_1, \gamma r_1, r_3 + \gamma r_3]][r_3^{\pm}].$$

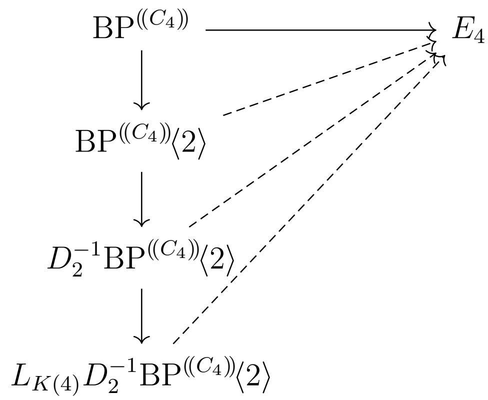
Let E_4 be the height-4 Lubin–Tate theory with coefficient ring

$$\pi_* E_4 = W(\mathbb{F}_{2^4})[[r_1, \gamma r_1, r_3 + \gamma r_3]][u^{\pm}]/(u^3 = r_3).$$

The previous discussion shows that the C_4-equivariant orientation

$$\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \longrightarrow E_4$$

factors through $L_{K(4)}D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$:



There is a C_3-action on E_4. The effect of this C_3-action on the homotopy groups $\pi_* E_4$ is as follows:

$$\begin{aligned}
\omega(u) &= \omega u \\
\omega(r_1) &= r_1 \\
\omega(\gamma r_1) &= \gamma r_1 \\
\omega(r_3) &= r_3 \\
\omega(\gamma r_3) &= \gamma r_3.
\end{aligned}$$

The spectra $E_4^{hC_3}$ has coefficient ring

$$\pi_* E_4^{hC_3} = W(\mathbb{F}_{2^4})[[r_1, \gamma r_1, r_3 + \gamma r_3]][r_3^{\pm}].$$

Therefore, it follows from this and our discussions above that

$$L_{K(4)}D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \simeq E_4^{h\,\mathrm{Gal}(\mathbb{F}_{2^4}/\mathbb{F}_2)\ltimes C_3}.$$

In particular, since the spectrum $L_{K(4)}D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ is cofree [39, Theorem 10.8], there is an equivalence

$$\begin{aligned}(L_{K(4)}D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)^{C_4} &\simeq (L_{K(4)}D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)^{hC_4} \\ &\simeq \left(E_4^{h\,\mathrm{Gal}(\mathbb{F}_{2^4}/\mathbb{F}_2)\ltimes C_3}\right)^{hC_4} \\ &\simeq E_4^{h\,\mathrm{Gal}(\mathbb{F}_{2^4}/\mathbb{F}_2)\ltimes C_{12}}.\end{aligned}$$

□

Theorem 9.13. *The spectrum* $D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ *is* 384*-periodic.*

Proof. This is a direct consequence of the discussion in [39, Section 9]. There are three periodicities for $D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$:

1. $S^{3\rho_4}\wedge D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \simeq D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$.

2. $S^{24-24\sigma}\wedge D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \simeq D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$.

3. $S^{32+32\sigma-32\lambda}\wedge D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \simeq D_2^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$.

The first periodicity is induced from $N(\bar{r}_3)$, which has been inverted. The second periodicity follows from the fact that $u_{8\sigma}$ is a permanent cycle in the slice spectral sequence of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ (see [39, Theorem 9.9]). For the third periodicity, note that that class $u_{32\sigma_2}$ is a permanent cycle in the C_2-slice spectral sequence of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$. Therefore, the norm

$$N(u_{32\sigma_2}) = \frac{u_{32\lambda}}{u_{32\sigma}}$$

is a permanent cycle in the C_4-slice spectral sequence. Combining these three periodicities produces the desired 384-periodicity:

$$\begin{aligned} & 32\cdot(3\rho_4)+8\cdot(24-24\sigma)+3\cdot(32+32\sigma-32\lambda) \\ = \; & 32\cdot(3+3\sigma+3\lambda)+24\cdot(8-8\sigma)+3\cdot(32+32\sigma-32\lambda) \\ = \; & 384. \end{aligned}$$

□

Remark 9.14. *The careful reader may worry about the choices present in the construction of* $\mathrm{BP}^{((C_4))}\langle 2\rangle$ *or the more general quotients of* $\mathrm{BP}^{((G))}$*. The terse answer is that the slice spectral sequence only cares about the indecomposables in the underlying homotopy ring of* $\mathrm{BP}^{((G))}$*, not the particular lifts. As a dramatic example of this, consider the class* $\bar{r}_3$ *for* C_{2^n}*. This is only well-defined modulo the ideal generated by* $\bar{r}_1$ *and its conjugates and the element* 2*. Consider now the differential on the class* $u_{2\sigma}^2$*:*

$$u_{2\sigma}^2 \mapsto N_{C_2}^{C_{2^n}}\bar{r}_3 a_\sigma^4 a_{3\bar{\rho}}.$$

Since multiplication by a_σ *annihilates the transfer, the norm is additive after being multiplied by* a_σ*. Moreover, the norm of* 2 *is killed by* a_σ *and the norm of* $\bar{r}_1$ *is killed by* a_σ^3*, so any possible indeterminacy in the definition of* $\bar{r}_3$ *results in the exact same differentials. Our computation applies to any form of* $\mathrm{BP}^{((C_4))}\langle 2\rangle$*.*

10 The slice spectral sequence of $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$

The C_4-equivariant refinement of $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ is

$$S^0[\bar{r}_1, \gamma\bar{r}_1] \longrightarrow \mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle.$$

(See [39, Section 5.3] for the definition of a refinement.) The proofs of the slice theorem and the reduction theorem in [39] apply to $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ as well, from which we deduce its slices:

$$\begin{cases} \bar{r}_1^i\gamma\bar{r}_1^i : S^{i\rho_4} \wedge H\underline{\mathbb{Z}}, & i \geq 0 \ \ (4i\text{-slice}), \\ \bar{r}_1^i\gamma\bar{r}_1^i(\bar{r}_1^j, \gamma\bar{r}_1^j) : C_{4+} \wedge_{C_2} S^{(2i+j)\rho_2} \wedge H\underline{\mathbb{Z}}, & i \geq 0, j \geq 1 \ \ (\text{induced } (4i+2j)\text{-slice}). \end{cases}$$

10.1 The C_2-slice spectral sequence

The C_2-spectrum $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ has no odd slice cells, and its $(2k)$-slice cells are indexed by the monomials

$$\{\bar{r}_1^i\gamma\bar{r}_1^j \,|\, i, j \geq 0,\, i+j = k\}.$$

Let $\bar{v}_i \in \pi^{C_2}_{i\rho_2}\mathrm{BP}_{\mathbb{R}}$ be the C_2-equivariant lifts of the classical v_i-generators for π_*BP. We can also regard them as elements in $\pi^{C_2}_{i\rho_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ via the map

$$\mathrm{BP}_{\mathbb{R}} \xrightarrow{i_L} i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)} \longrightarrow i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle.$$

In [40, Section 7], Hill, Hopkins, and Ravenel proved

$$\begin{aligned}
\bar{v}_1 \pmod{2} &= \bar{r}_1 + \gamma\bar{r}_1, \\
\bar{v}_2 \pmod{2, \bar{v}_1} &= \bar{r}_1^3, \\
\bar{v}_i \pmod{2, \bar{v}_1, \ldots, \bar{v}_{i-1}} &= 0, \;\; i \geq 3.
\end{aligned}$$

In C_2-SliceSS($\mathrm{BP}_{\mathbb{R}}$), all the differentials are known. They are determined by the differentials

$$d_{2^{i+1}-1}(u_{2^i\sigma_2}) = \bar{v}_i a_{\sigma_2}^{2^{i+1}-1}, \;\; i \geq 1,$$

and multiplicative structures. This, combined with the formulas above, implies that in C_2-SliceSS($i_{C_2}^* \mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle$), all the differentials are determined by

$$\begin{aligned}
d_3(u_{2\sigma_2}) &= \bar{v}_1 a_{\sigma_2}^3 = (\bar{r}_1 + \gamma\bar{r}_1)a_{\sigma_2}^3, \\
d_7(u_{4\sigma_2}) &= \bar{v}_2 a_{\sigma_2}^7 = \bar{r}_1^3 a_{\sigma_2}^7,
\end{aligned}$$

and multiplicative structures. The class $u_{8\sigma_2}$ is a permanent cycle.

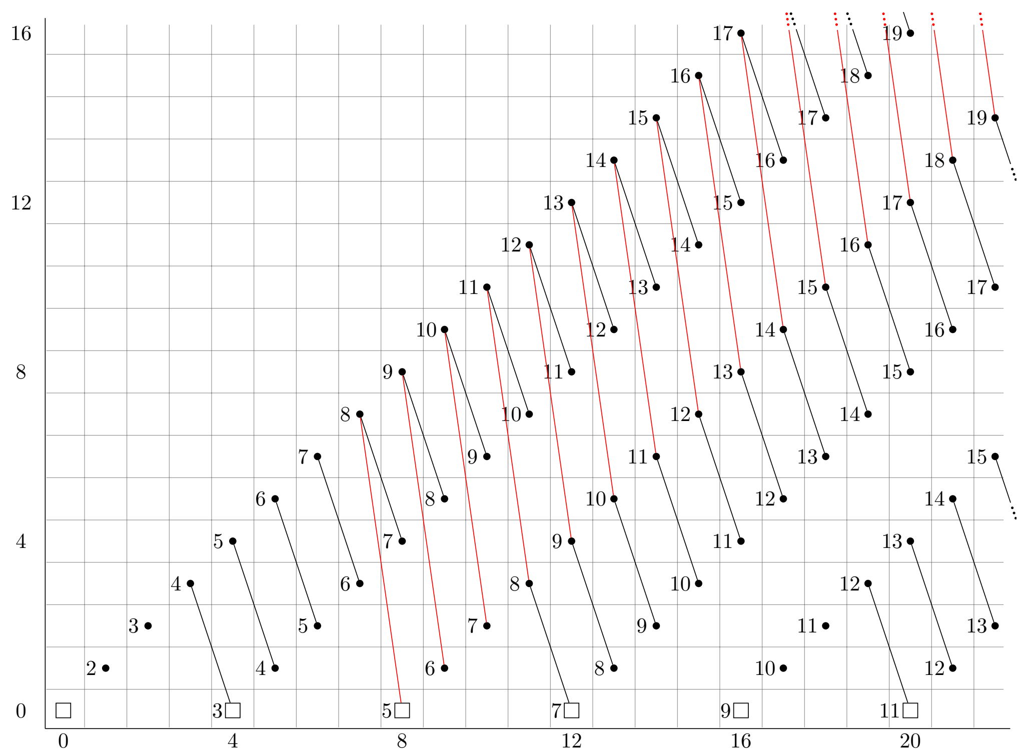


Figure 9: The C_2-slice spectral sequence for $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$.

10.2 Organizing the slices, d_3-differentials

We can organize the slices into the following table

$$
\begin{array}{c|c|c|c}
\bar{\mathfrak{d}}_1^0 & \bar{\mathfrak{d}}_1^1 & \bar{\mathfrak{d}}_1^2 & \cdots \\
\hline
\bar{\mathfrak{d}}_1^0 \bar{s}_1^1 & \bar{\mathfrak{d}}_1^1 \bar{s}_1^1 & \bar{\mathfrak{d}}_1^2 \bar{s}_1^1 & \cdots \\
\bar{\mathfrak{d}}_1^0 \bar{s}_1^2 & \bar{\mathfrak{d}}_1^1 \bar{s}_1^2 & \bar{\mathfrak{d}}_1^2 \bar{s}_1^2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\tag{10.1}
$$

where $\bar{\mathfrak{d}}_1 := N(\bar{r}_1)$, and $\bar{s}_1^i := \bar{r}_1^i(1+\gamma) = \bar{r}_1^i + \gamma\bar{r}_1^i$ (note that by an abuse of notation, $\bar{s}_1^i$ does *not* mean $(\bar{r}_1 + \gamma\bar{r}_1)^i$). The first row consists of non-induced slices and the rest of the rows are all induced slices. Also note that with the definition above, $res(\bar{\mathfrak{d}}_1) = \bar{r}_1\gamma\bar{r}_1$.

Theorem 10.1. $d_3(u_\lambda) = \bar{s}_1 a_\lambda a_{\sigma_2}$.

Proof. The restriction of u_λ is $res(u_\lambda) = u_{2\sigma_2}$. In the C_2-slice spectral sequence, the class $u_{2\sigma_2}$ supports a nonzero d_3-differential

$$d_3(u_{2\sigma_2}) = (\bar{r}_1 + \gamma\bar{r}_1)a_{3\sigma_2}.$$

Therefore, u_λ must support a differential of length at most 3. For degree reasons, this differential must be a d_3-differential. Natuality implies that

$$d_3(u_\lambda) = \bar{s}_1 a_\lambda a_{\sigma_2},$$

as desired. □

To organize the C_4-slices in table 10.1, we separate them into columns. Each column

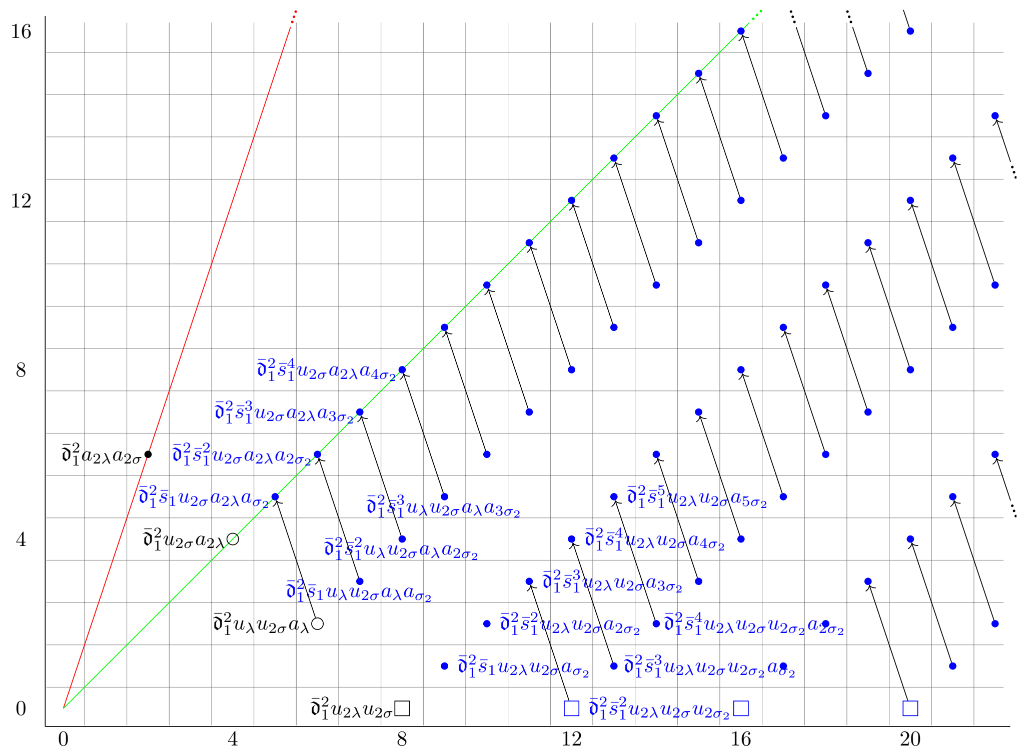


Figure 10: d_3-differentials within the column containing $\bar{\mathfrak{d}}_1^2$.

consists of one non-induced slice cell, $\bar{\mathfrak{d}}_1^i$, and all the induced slice cells of the form $\bar{\mathfrak{d}}_1^i\bar{s}_1^j$, where $j \geq 1$.

In light of Theorem 10.1, each column can be treated as an individual unit with respect to the d_3-differentials. More precisely, the leading terms of any of the d_3-differential are slices belonging to the same column. When drawing the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 1\rangle$, we first produce the E_2-page of each column individually, together with their d_3-differentials (See Figure 10 and 11). Afterwards, we combine the E_5-pages of every column all together into one whole spectral sequence.

Remark 10.2. Some classes support d_3-differentials with target the sum of two classes.

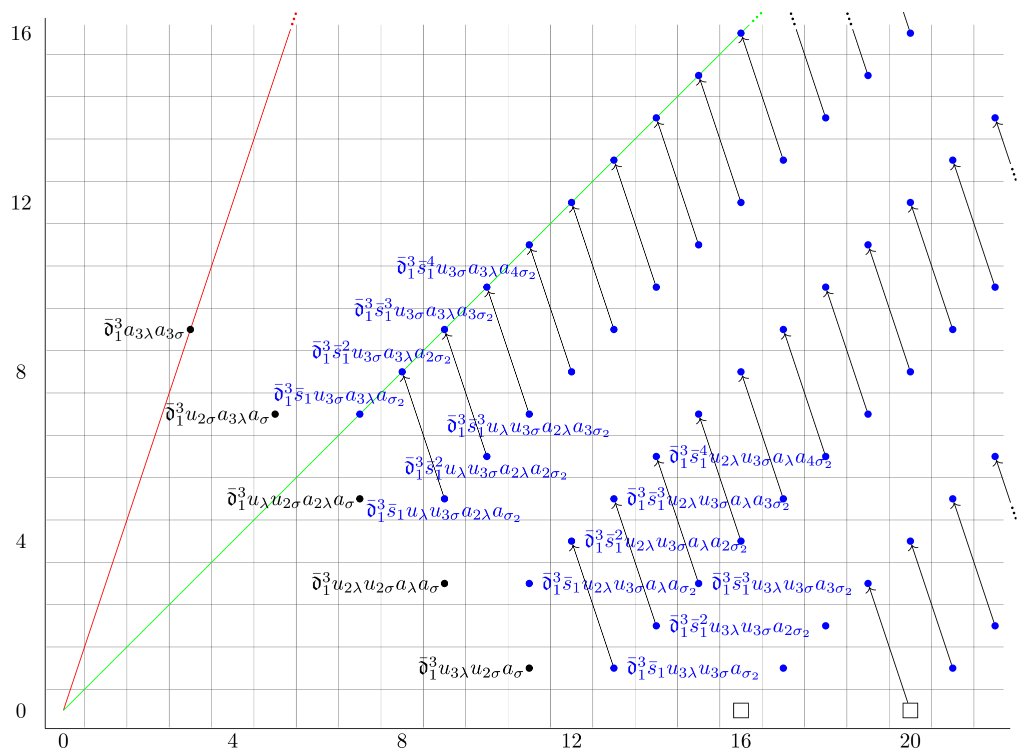


Figure 11: d_3-differentials within the column containing $\bar{\mathfrak{d}}_1^3$.

For example, the class $\bar{\mathfrak{d}}_1^2 \bar{s}_1^2 u_{2\lambda} u_{2\sigma} u_{2\sigma_2}$ in bidegree $(12, 0)$ supports the d_3-differential

$$d_3(\bar{\mathfrak{d}}_1^2 \bar{s}_1^2 u_{2\lambda} u_{2\sigma} u_{2\sigma_2}) = \bar{\mathfrak{d}}_1^2 \bar{s}_1^2 u_{2\lambda} u_{2\sigma} \cdot \bar{s}_1 a_{3\sigma_2} = \bar{\mathfrak{d}}_1^2 \bar{s}_1^3 u_{2\lambda} u_{2\sigma} a_{3\sigma_2} + \bar{\mathfrak{d}}_1^3 \bar{s}_1 u_{2\lambda} u_{2\sigma} a_\lambda a_{\sigma_2},$$

because $\bar{s}_1^2 \cdot \bar{s}_1 = \bar{s}_1^3 + \bar{\mathfrak{d}}_1 \bar{s}_1$. The first term is in the same column as the source, but the second term is not (it belongs to a slice cell in the next column).

This d_3-differential is introducing the relation $\bar{\mathfrak{d}}_1^2 \bar{s}_1^3 u_{2\lambda} u_{2\sigma} a_{3\sigma_2} = \bar{\mathfrak{d}}_1^3 \bar{s}_1 u_{2\lambda} u_{2\sigma} a_\lambda a_{\sigma_2}$ after the E_3-page. As a convention, when we are drawing the slice spectral sequence, we only kill the leading term of the target:

$$d_3(\bar{\mathfrak{d}}_1^2 \bar{s}_1^2 u_{2\lambda} u_{2\sigma} u_{2\sigma_2}) = \bar{\mathfrak{d}}_1^2 \bar{s}_1^3 u_{2\lambda} u_{2\sigma} a_{3\sigma_2}.$$

Note that both the source and the target are in the same column.

10.3 d_5-differentials

Theorem 10.3. $d_5(u_{2\sigma}) = \bar{\mathfrak{d}}_1 a_\lambda a_\sigma^3$.

Proof. This differential is given by Hill–Hopkins–Ravenel's slice differential theorem [39, Theorem 9.9]. □

In the integer graded slice spectral sequence, this d_5-differential produces all the d_5-differentials between the line of slope 1 and the line of slope 3 (by using the Leibniz rule).

Theorem 10.4. *The class* $\bar{\mathfrak{d}}_1^2 u_{2\lambda} u_{2\sigma}$ *at* $(8, 0)$ *supports the* d_5*-differential*

$$d_5(\bar{\mathfrak{d}}_1^2 u_{2\lambda} u_{2\sigma}) = \bar{\mathfrak{d}}_1^3 u_\lambda u_{2\sigma} a_{2\lambda} a_\sigma.$$

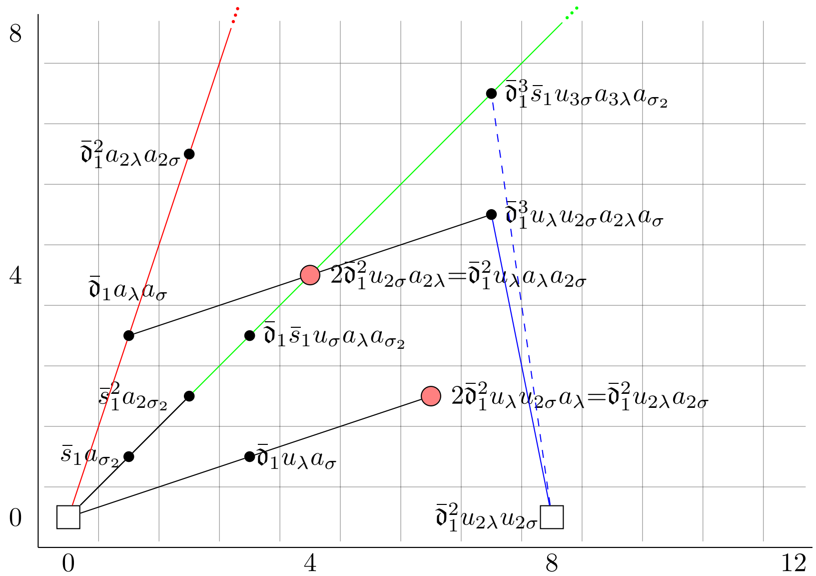


Figure 12: d_5-differential on $\bar{\mathfrak{d}}_1^2u_{2\lambda}u_{2\sigma}$.

Proof. The restriction of $\bar{\mathfrak{d}}_1^2u_{2\lambda}u_{2\sigma}$ is $res(\bar{\mathfrak{d}}_1^2u_{2\lambda}u_{2\sigma}) = \bar{r}_1^2\gamma\bar{r}_1^2u_{4\sigma_2}$, which supports the d_7-differential

$$d_7(\bar{r}_1^2\gamma\bar{r}_1^2u_{4\sigma_2}) = \bar{r}_1^5\gamma\bar{r}_1^2a_{7\sigma_2}$$

in C_2-SliceSS($\mathrm{BP}^{((C_4))}\langle 1\rangle$). This implies that the class $\bar{\mathfrak{d}}_1^2u_{2\lambda}u_{2\sigma}$ must support a differential of length at most 7 in C_4-SliceSS($\mathrm{BP}^{((C_4))}\langle 1\rangle$). The only possible targets are the classes $\bar{\mathfrak{d}}_1^3u_\lambda u_{2\sigma}a_{2\lambda}a_\sigma$ at $(7,5)$ and $\bar{\mathfrak{d}}_1^3\bar{s}_1u_{3\sigma}a_{3\lambda}a_{\sigma_2}$ at $(7,7)$ (see Figure 12).

To prove the desired d_5-differential, it suffices to show that the d_7-differential

$$d_7(\bar{\mathfrak{d}}_1^2u_{2\lambda}u_{2\sigma}) = \bar{\mathfrak{d}}_1^3\bar{s}_1u_{3\sigma}a_{3\lambda}a_{\sigma_2}$$

does not exist. For the sake of contradiction, suppose that this d_7-differential does occur. By natuality, this differential must be compatible with the restriction map. The left-hand side restricts to $\bar{r}_1^2\gamma\bar{r}_1^2 u_{4\sigma_2}$, but the right-hand side restricts to

$$res(\bar{\mathfrak{d}}_1^3\bar{s}_1 u_{3\sigma}a_{3\lambda}a_{\sigma_2}) = \bar{r}_1^3\gamma\bar{r}_1^3(\bar{r}_1+\gamma\bar{r}_1)a_{7\sigma_2} = 0 \neq \bar{r}_1^5\gamma\bar{r}_1^2 a_{7\sigma_2}$$

because the d_3-differential $d_3(u_{2\sigma_2}) = (\bar{r}_1+\gamma\bar{r}_1)a_{3\sigma_2}$ introduced the relation $(\bar{r}_1+\gamma\bar{r}_1)a_{3\sigma_2} = 0$. This is a contradiction. □

Corollary 10.5. $d_5(u_{2\lambda}) = \bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_\sigma$.

Proof. Using the Leibniz rule, we have

$$\begin{aligned} d_5(\bar{\mathfrak{d}}_1^2 u_{2\lambda}u_{2\sigma}) &= \bar{\mathfrak{d}}_1^2 u_{2\lambda} d_5(u_{2\sigma}) + \bar{\mathfrak{d}}_1^2 u_{2\sigma} d_5(u_{2\lambda}) \\ &= \bar{\mathfrak{d}}_1^2 u_{2\lambda}\cdot \bar{\mathfrak{d}}_1 a_\lambda a_{3\sigma} + \bar{\mathfrak{d}}_1^2 u_{2\sigma} d_5(u_{2\lambda}) \\ &= 0 + \bar{\mathfrak{d}}_1^2 u_{2\sigma} d_5(u_{2\lambda}) \\ &= \bar{\mathfrak{d}}_1^2 u_{2\sigma} d_5(u_{2\lambda}), \end{aligned}$$

where we have used the gold relation $u_\lambda a_{3\sigma} = 2u_{2\sigma}a_\lambda a_\sigma = 0$. Theorem 10.4 implies that $\bar{\mathfrak{d}}_1^2 u_{2\sigma} d_5(u_{2\lambda}) = \bar{\mathfrak{d}}_1^3 u_\lambda u_{2\sigma} a_{2\lambda} a_\sigma$. Rearranging, we obtain the equality

$$\bar{\mathfrak{d}}_1^2 u_{2\sigma}(d_5(u_{2\lambda}) - \bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_\sigma) = 0,$$

from which the desired differential follows (multiplication by $\bar{\mathfrak{d}}_1^2 u_{2\sigma}$ is faithful on the E_5-page). □

All the other d_5-differentials are obtained from Theorem 10.4 via multiplication with the classes

1. $\bar{\mathfrak{d}}_1 a_\lambda a_\sigma$ at $(1,3)$ (permanent cycle);

2. $\bar{\mathfrak{d}}_1 u_\lambda a_\sigma$ at $(3,1)$ (permanent cycle);

3. $\bar{\mathfrak{d}}_1^2 u_{2\sigma} a_{2\lambda}$ at $(4,4)$ ($d_5(\bar{\mathfrak{d}}_1^2 u_{2\sigma} a_{2\lambda}) = \bar{\mathfrak{d}}_1^3 a_{3\lambda} a_{3\sigma_2}$);

4. $\bar{\mathfrak{d}}_1^4 u_{4\lambda} u_{4\sigma}$ at $(16,0)$ (d_5-cycle).

and using the Leibniz rule (see Figure 13).

There is an alternative way to prove Corollary 10.5 by using the norm formulas. Start with the d_3-differential

$$d_3(u_{2\sigma_2}) = (\bar{r}_1 + \gamma\bar{r}_1)a_{3\sigma_2}$$

in the C_2-slice spectral sequence. The first formula of Theorem 9.8 predicts the d_3-differential

$$d_3\left(\frac{u_{2\lambda}}{u_{2\sigma}}\right) = tr(u_{2\sigma_2} \cdot (\bar{r}_1 + \gamma\bar{r}_1)a_{3\sigma_2}).$$

However, this prediction is void because the right-hand side is equal to 0:

$$tr(res(u_\lambda a_\lambda)(\bar{r}_1 + \gamma\bar{r}_1)a_{\sigma_2}) = u_\lambda a_\lambda tr(res(\bar{s}_1 a_{\sigma_2})) = u_\lambda a_\lambda \cdot 2\bar{s}_1 a_{\sigma_2} = 0.$$

This is due to the fact that $d_3(u_\lambda) \neq 0$, and so $u_{2\lambda}$ is a d_3-cycle.

The second formula, however, predicts the d_5-differential

$$d_5\left(a_\sigma \cdot \frac{u_{2\lambda}}{u_{2\sigma}}\right) = N_{C_2}^{C_4}(\bar{r}_1 + \gamma\bar{r}_1)a_{3\lambda}$$

in the C_4-slice spectral sequence. Using the Leibniz rule, this formula predicts the d_5-

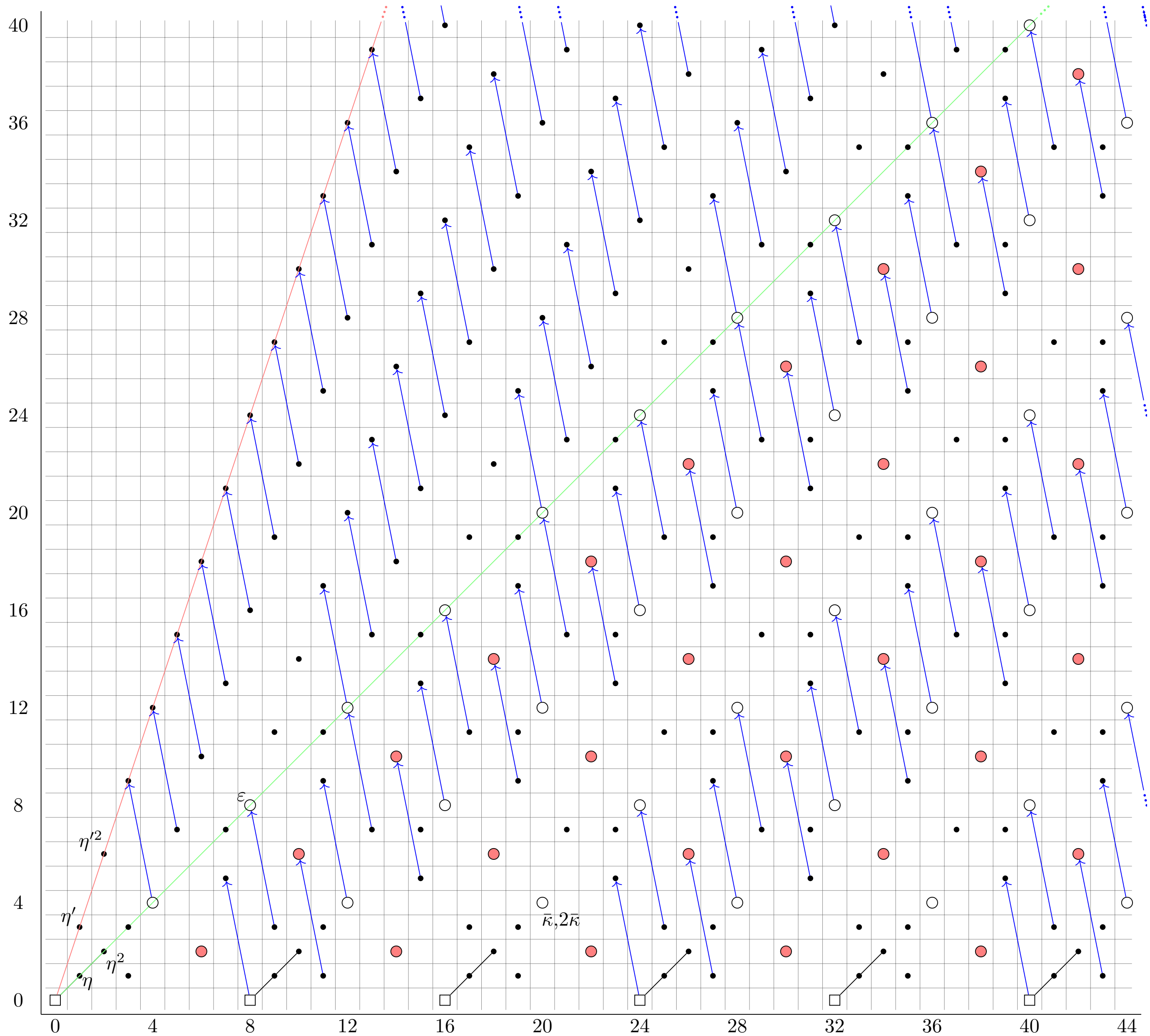


Figure 13: d_5-differentials in C_4-SliceSS$(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$.

differential

$$\begin{aligned} d_5(a_\sigma u_{2\lambda}) = d_5\left(u_{2\sigma}\left(a_\sigma \cdot \frac{u_{2\lambda}}{u_{2\sigma}}\right)\right) &= d_5(u_{2\sigma})\left(a_\sigma \cdot \frac{u_{2\lambda}}{u_{2\sigma}}\right) + u_{2\sigma} d_5\left(a_\sigma \cdot \frac{u_{2\lambda}}{u_{2\sigma}}\right) \\ &= (\bar{\mathfrak{d}}_1 a_\lambda a_{3\sigma})\left(a_\sigma \cdot \frac{u_{2\lambda}}{u_{2\sigma}}\right) + N_{C_2}^{C_4}(\bar{r}_1 + \gamma\bar{r}_1)u_{2\sigma}a_{3\lambda} \\ &= 0 + N_{C_2}^{C_4}(\bar{r}_1 + \gamma\bar{r}_1)u_{2\sigma}a_{3\lambda} \\ &= N_{C_2}^{C_4}(\bar{r}_1 + \gamma\bar{r}_1)u_{2\sigma}a_{3\lambda}. \end{aligned}$$

To compute $N_{C_2}^{C_4}(\bar{r}_1 + \gamma\bar{r}_1)$, note that

$$res(N_{C_2}^{C_4}(\bar{r}_1 + \gamma\bar{r}_1)) = (\bar{r}_1 + \gamma\bar{r}_1)(\gamma\bar{r}_1 - \bar{r}_1) = -(\bar{r}_1^2 - \gamma r_1^2) = -\bar{r}_1^2 u_\sigma^{-1}(1+\gamma).$$

Therefore, $N_{C_2}^{C_4}(\bar{r}_1 + \gamma\bar{r}_1) = -tr(\bar{r}_1^2 u_\sigma^{-1})$, and the target of the normed d_5-differential is

$$-tr(\bar{r}_1^2 u_{-\sigma})u_{2\sigma}a_{3\lambda} = -tr(\bar{r}_1^2 u_\sigma a_{6\sigma_2}) = tr(\bar{r}_1^2 u_\sigma a_{6\sigma_2}).$$

The last equality holds because multiplication by 2 kills transfer of classes with filtration at least 1.

To identity this target with a more familiar expression, we add $tr(\bar{r}_1\gamma\bar{r}_1 u_\sigma a_{6\sigma_2})$ to it and use the Frobenius relation:

$$\begin{aligned} tr(\bar{r}_1^2 u_\sigma a_{6\sigma_2}) + tr(\bar{r}_1\gamma\bar{r}_1 u_\sigma a_{6\sigma_2}) &= tr(\bar{r}_1(\bar{r}_1 + \gamma\bar{r}_1)u_\sigma a_{6\sigma_2}) \\ &= tr(\bar{r}_1 u_\sigma a_{\sigma_2} res(tr(\bar{r}_1 a_{\sigma_2})a_{2\lambda})) \\ &= tr(\bar{r}_1 u_\sigma a_{\sigma_2})tr(\bar{r}_1 a_{\sigma_2})a_{2\lambda} \\ &= 0. \end{aligned}$$

The last expression is 0 on the E_5-page because $d_3(u_\lambda) = \bar{s}_1 a_\lambda a_{\sigma_2} = tr(\bar{r}_1 a_{\sigma_2}) a_\lambda$. Therefore,

$$\begin{aligned}
tr(\bar{r}_1^2 u_\sigma a_{6\sigma_2}) &= tr(\bar{r}_1 \gamma \bar{r}_1 u_\sigma a_{6\sigma_2}) \\
&= tr(res(\bar{\mathfrak{d}}_1 u_{2\sigma} a_{3\lambda})) \quad (res(\bar{\mathfrak{d}}_1) = \bar{r}_1 \gamma \bar{r}_1 u_\sigma^{-1}) \\
&= 2\bar{\mathfrak{d}}_1 u_{2\sigma} a_{3\lambda} \\
&= \bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_{2\sigma} \quad (2u_{2\sigma} a_\lambda = u_\lambda a_{2\sigma}).
\end{aligned}$$

It follows that

$$a_\sigma(d_5(u_{2\lambda}) - \bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_\sigma) = 0,$$

and $d_5(u_{2\lambda}) = \bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_\sigma$, as desired.

10.4 d_7-differentials

Theorem 10.6. *The classes $2\bar{\mathfrak{d}}_1^2 u_{2\lambda} u_{2\sigma}$ at $(8,0)$ and $2\bar{\mathfrak{d}}_1^4 u_{2\lambda} u_{4\sigma} a_{2\lambda}$ at (12, 4) support the d_7-differentials*

$$\begin{aligned}
d_7(2\bar{\mathfrak{d}}_1^2 u_{2\lambda} u_{2\sigma}) &= \bar{\mathfrak{d}}_1^3 \bar{s}_1 u_{3\sigma} a_{3\lambda} a_{\sigma_2}, \\
d_7(2\bar{\mathfrak{d}}_1^4 u_{2\lambda} u_{4\sigma} a_{2\lambda}) &= \bar{\mathfrak{d}}_1^5 \bar{s}_1 u_{5\sigma} a_{5\lambda} a_{\sigma_2}.
\end{aligned}$$

Proof. Consider the d_7-differential

$$d_7(\bar{r}_1^2 \gamma \bar{r}_1^2 u_{4\sigma_2}) = \bar{r}_1^5 \gamma \bar{r}_1^2 a_{7\sigma_2} = \bar{r}_1^4 \gamma \bar{r}_1^3 a_{7\sigma_2}$$

in the C_2-slice spectral sequence (the last equality holds because $\bar{r}_1 = \gamma \bar{r}_1$ after the d_3-

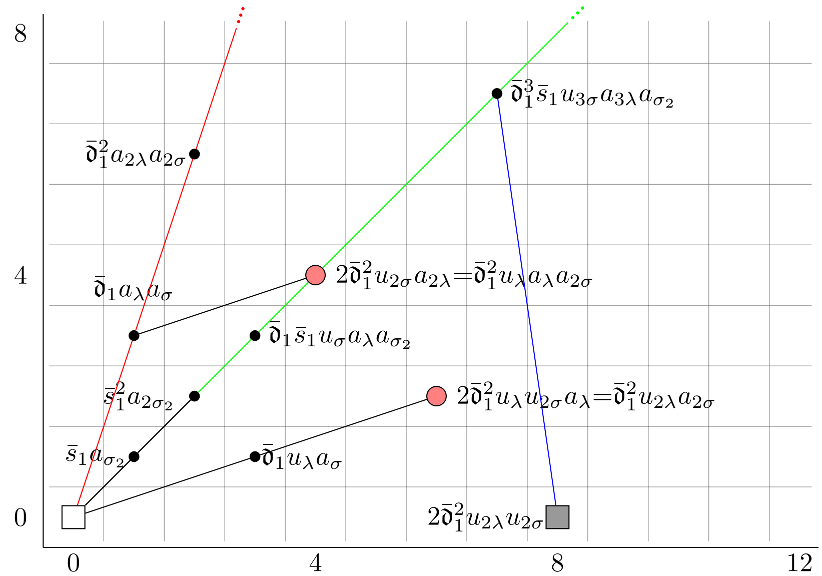


Figure 14: d_7-differential on $2\bar{\mathfrak{d}}_1^2 u_{2\lambda} u_{2\sigma}$.

differentials). The transfer of the target is

$$tr(\bar{r}_1^4 \gamma \bar{r}_1^3 a_{7\sigma_2}) = tr(res(\bar{\mathfrak{d}}_1^3 u_{3\sigma} a_{3\lambda}) \bar{r}_1 a_{\sigma_2}) = \bar{\mathfrak{d}}_1^3 \bar{s}_1 u_{3\sigma} a_{3\lambda} a_{\sigma_2}.$$

For degree reasons, this class must be killed by a differential of length exactly 7 (see Figure 14). Natuality implies that the source is

$$tr(\bar{r}_1^2 \gamma \bar{r}_1^2 u_{4\sigma_2}) = tr(res(\bar{\mathfrak{d}}_1^2 u_{2\sigma} u_{2\lambda})) = 2\bar{\mathfrak{d}}_1^2 u_{2\sigma} u_{2\lambda}.$$

The second differential is proved using the same method, by applying the transfer to the d_7-differential

$$d_7(\bar{r}_1^4 \gamma \bar{r}_1^4 u_{4\sigma_2} a_{4\sigma_2}) = \bar{r}_1^7 \gamma \bar{r}_1^4 a_{11\sigma_2} = \bar{r}_1^6 \gamma \bar{r}_1^5 a_{11\sigma_2}.$$

□

Corollary 10.7. *The classes $2u_{2\lambda}$ and $2u_{2\lambda}u_{2\sigma}$ support the d_7-differentials*

$$\begin{aligned} d_7(2u_{2\lambda}) &= \bar{\mathfrak{d}}_1\bar{s}_1u_\sigma a_{3\lambda}a_{\sigma_2}, \\ d_7(2u_{2\lambda}u_{2\sigma}) &= \bar{\mathfrak{d}}_1\bar{s}_1u_{3\sigma}a_{3\lambda}a_{\sigma_2}. \end{aligned}$$

Proof. Since the classes $\bar{\mathfrak{d}}_1$, $u_{4\sigma}$, and a_λ are permanent cycles, the second differential in Theorem 10.6 can be rewritten as

$$\bar{\mathfrak{d}}_1^4u_{4\sigma}a_{2\lambda}(d_7(2u_{2\lambda}) - \bar{\mathfrak{d}}_1\bar{s}_1u_\sigma a_{3\lambda}a_{\sigma_2}) = 0,$$

from which the first differential follows. The second differential is proven similarly by using the first differential in Theorem 10.6. □

Remark 10.8. Corollary 10.7 can also be proved by applying the transfer to the d_7-differential

$$d_7(u_{4\sigma_2}) = \bar{r}_1^3a_{7\sigma_2} = \bar{r}_1^2\gamma\bar{r}_1a_{7\sigma_2}$$

in the C_2-slice spectral sequence.

Remark 10.9. On the E_7-page of C_4-SliceSS($BP^{(\!(C_4)\!)}\langle 1\rangle$), there is more than one class at $(8, 0)$. They are

1. $2\bar{\mathfrak{d}}_1^2u_{2\lambda}u_{2\sigma} = tr(res(\bar{\mathfrak{d}}_1^2u_{2\lambda}u_{2\sigma})) = tr(\bar{r}_1^2\gamma\bar{r}_1^2u_{4\sigma_2})$;

2. $\bar{\mathfrak{d}}_1\bar{s}_1^2u_\lambda u_\sigma u_{2\sigma_2} = (\bar{\mathfrak{d}}_1u_\lambda u_\sigma)(\bar{s}_1^2u_{2\sigma_2}) = tr(res(\bar{\mathfrak{d}}_1u_\lambda u_\sigma)\bar{r}_1^2u_{2\sigma_2}) = tr(\bar{r}_1^3\gamma\bar{r}_1u_{4\sigma_2})$;

3. $\bar{s}_1^4u_{4\sigma_2} = tr(\bar{r}_1^4u_{4\sigma_2})$.

Except for class (1), the classes (2) and (3) are "grayed out" on the upper-left of Hill, Hopkins, and Ravenel's original computation of C_4-SliceSS($D_1^{-1}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$) [40, pg. 4].

On the E_2-page, there is more than one class at $(7,7)$ as well:

1. $\bar{\mathfrak{d}}_1^3 \bar{s}_1 u_{3\sigma} a_{3\lambda} a_{\sigma_2} = tr(\bar{r}_1^4 \gamma \bar{r}_1^3 a_{7\sigma_2})$;

2. $\bar{\mathfrak{d}}_1^2 \bar{s}_1^3 u_{2\sigma} a_{2\lambda} a_{3\sigma_2} = tr(\bar{r}_1^5 \gamma \bar{r}_1^2 a_{7\sigma_2})$;

3. $\bar{\mathfrak{d}}_1 \bar{s}_1^5 u_{\sigma} a_{\lambda} a_{5\sigma_2} = tr(\bar{r}_1^6 \gamma \bar{r}_1 a_{7\sigma_2})$;

4. $\bar{s}_1^7 a_{7\sigma_2} = tr(\bar{r}_1^7 a_{7\sigma_2})$.

Applying transfers to the following d_3-differentials in C_2-SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$) yields d_3-differentials in C_4-SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$):

1. $d_3(\bar{r}_1^6 u_{2\sigma_2} a_{4\sigma_2}) = \bar{r}_1^6(\bar{r}_1 + \gamma \bar{r}_1) a_{\sigma_2}^7$: transfer of this kills (3) + (4);

2. $d_3(\bar{r}_1^5 \gamma \bar{r}_1 u_{2\sigma_2} a_{4\sigma_2}) = \bar{r}_1^5 \gamma \bar{r}_1 (\bar{r}_1 + \gamma \bar{r}_1) a_{7\sigma_2}$: transfer of this kills (2) + (3);

3. $d_3(\bar{r}_1^4 \gamma \bar{r}_1^2 u_{2\sigma_2} a_{4\sigma_2}) = \bar{r}_1^4 \gamma \bar{r}_1^2 (\bar{r}_1 + \gamma \bar{r}_1) a_{7\sigma_2}$: transfer of this kills (1) + (2).

These d_3-differentials identified the four classes at $(7,7)$. The transfer argument in Theorem 10.6 shows that each of the three classes at $(8,0)$ supports a d_7-differential, all killing the single remaining class at $(7,7)$.

The proof of Hill–Hopkins–Ravenel's Periodicity theorem [39, Section 9] shows that the class $\bar{\mathfrak{d}}_1^8 u_{8\lambda} u_{8\sigma}$ at $(32,0)$ is a permanent cycle. For degree reasons, the following classes are also permanent cycles and survive to the E_∞-page:

1. $\eta = \bar{s}_1 a_{\sigma_2}$ at $(1,1)$;

2. $\eta^2 = \bar{s}_1^2 a_{2\sigma_2}$ at $(2,2)$;

3. $\eta' = \bar{\mathfrak{d}}_1 a_\lambda a_\sigma$ at $(1,3)$;

4. $\eta'^2 = \bar{\mathfrak{d}}_1^2 a_{2\lambda} a_{2\sigma}$ at $(2,6)$;

5. $\epsilon = \bar{\mathfrak{d}}_1^4 u_{4\sigma} a_{4\lambda}$ at $(8,8)$.

Their names come from the spherical classes that they detect in $\pi_*\mathbb{S}$ [40, Theorem 9.8] under the Hurewicz map

$$\pi_*\mathbb{S} \longrightarrow \pi_*\mathrm{BP}^{((C_4))}\langle 1\rangle^{C_4}.$$

Theorem 10.10. *The class $u_{4\lambda}a_\sigma$ supports the d_{13}-differential*

$$d_{13}(u_{4\lambda}a_\sigma) = \bar{\mathfrak{d}}_1^3 u_{4\sigma} a_\lambda^7.$$

Proof. Applying the second norm formula of Theorem 9.8 to the d_7-differential

$$d_7(u_{4\sigma_2}) = \bar{r}_1^3 a_{7\sigma_2}$$

in the C_2-slice spectral sequence predicts the d_{13}-differential

$$\begin{aligned} d_{13}(u_{4\lambda}a_\sigma) &= u_{4\sigma} N_{C_2}^{C_4}(\bar{r}_1^3 a_{7\sigma_2}) \\ &= \bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{7\lambda} \end{aligned}$$

in the C_4-slice spectral sequence. The target is not zero on the E_{13}-page because multiplying it by the permanent cycle $\bar{\mathfrak{d}}_1^5 u_{4\sigma} a_\lambda$ gives the nonzero class $\bar{\mathfrak{d}}_1^8 u_{8\sigma} a_{8\lambda}$ at $(16,16)$. Therefore, this d_{13}-differential exists. □

Multiplying the differential in Theorem 10.10 by the permanent cycle $\bar{\mathfrak{d}}_1^5 u_{4\sigma} a_\lambda$ produces a d_{13}-differential in the integer graded spectral sequence.

Corollary 10.11. *The class* $\bar{\mathfrak{d}}_1^5 u_{4\lambda} u_{4\sigma} a_\lambda a_\sigma$ *at* $(17,3)$ *supports the* d_{13}*-differential*

$$d_{13}(\bar{\mathfrak{d}}_1^5 u_{4\lambda} u_{4\sigma} a_\lambda a_\sigma) = \bar{\mathfrak{d}}_1^8 u_{8\sigma} a_{8\lambda} = \epsilon^2.$$

Theorem 10.12. *The class* $\bar{\mathfrak{d}}_1^4 u_{4\lambda} u_{4\sigma}$ *at* $(16,0)$ *supports the* d_7*-differential*

$$d_7(\bar{\mathfrak{d}}_1^4 u_{4\lambda} u_{4\sigma}) = \bar{\mathfrak{d}}_1^5 \bar{s}_1 u_{2\lambda} u_{5\sigma} a_{3\lambda} a_{\sigma_2}.$$

Proof. The class $\eta' = \bar{\mathfrak{d}}_1 a_\lambda a_\sigma$ is a permanent cycle. By Corollary 10.11, the class $\bar{\mathfrak{d}}_1^4 u_{4\lambda} u_{4\sigma}$ at $(16,0)$ must support a differential of length at most 13. For degree reasons, the only possible target is $\bar{\mathfrak{d}}_1^5 \bar{s}_1 u_{2\lambda} u_{5\sigma} a_{3\lambda} a_{\sigma_2}$. □

Corollary 10.13. *The class* $u_{4\lambda}$ *supports the* d_7*-differential*

$$d_7(u_{4\lambda}) = \bar{\mathfrak{d}}_1 \bar{s}_1 u_{2\lambda} u_\sigma a_{3\lambda} a_{\sigma_2}.$$

Proof. This follows directly from Theorem 10.12 because

$$\bar{\mathfrak{d}}_1^4 u_{4\sigma}(d_7(u_{4\lambda}) - \bar{\mathfrak{d}}_1 \bar{s}_1 u_{2\lambda} u_\sigma a_{3\lambda} a_{\sigma_2}) = 0$$

and multiplication by $\bar{\mathfrak{d}}_1^4 u_{4\sigma}$ is faithful on the E_7-page. □

Once we have proven the d_7-differentials in Theorem 10.6 and Theorem 10.12, all the other d_7-differentials are obtained via multiplication with the classes

1. $\bar{\mathfrak{d}}_1^4 u_{4\sigma} a_{4\lambda}$ at $(8,8)$ (permanent cycle);

2. $\bar{\mathfrak{d}}_1^4 u_{4\lambda} u_{4\sigma}$ at $(16,0)$ (Theorem 10.12);

3. $\bar{\mathfrak{d}}_1^8 u_{8\lambda} u_{8\sigma}$ at $(32,0)$ (d_7-cycle).

and using the Leibniz rule (see Figure 15).

10.5 Higher differentials

Fact 10.14. Multiplication by $\epsilon = \bar{\mathfrak{d}}_1^4 u_{4\sigma} a_{4\lambda}$ is injective on the E_2-page. The image of this multiplication map is the region defined by the inequalities

$$\begin{aligned} s &\geq 8, \\ 3(s-8) &\leq t-s-8. \end{aligned}$$

In other words, this region consists of classes with filtrations at least 8 and these classes are all on or below the ray of slope 3, starting at $(8,8)$. Starting from the E_5-page, all the classes in this region are divisible by ϵ. Therefore, when $r \geq 5$, multiplication by ϵ induces a surjective map from the whole E_r-page to this region.

Lemma 10.15. *Let $d_r(x) = y$ be a nontrivial differential in C_4-SliceSS($BP^{(\!(C_4)\!)}\langle 1\rangle$).*

1. *The class ϵx and ϵy both survive to the E_r-page, and $d_r(\epsilon x) = \epsilon y$.*

2. *If both x and y are divisible by ϵ on the E_2-page, then x/ϵ and y/ϵ both survive to the E_r-page, and $d_r(x/\epsilon) = y/\epsilon$.*

Proof. We will prove both statements by using induction on r, the length of the differential. Both claims are true in the base case when $r = 3$.

Now suppose that both statements hold for all differentials with length $k < r$. Given a nontrivial differential $d_r(x) = y$, we will first show that ϵy survives to the E_r-page.

If ϵy supports a differential, then y must support a differential as well. This is a contradiction because y is the target of a differential. Therefore if ϵy does not survive to

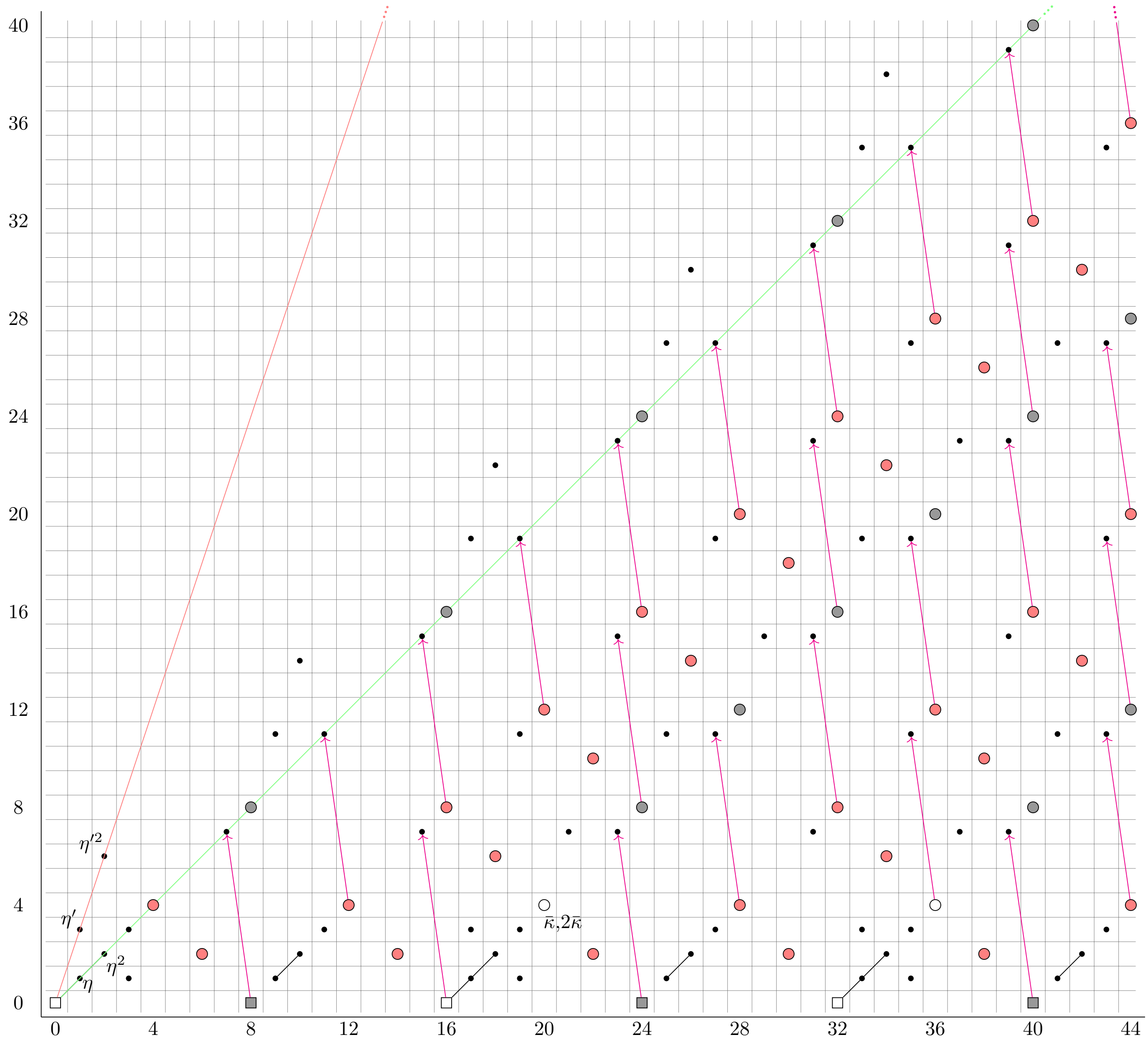
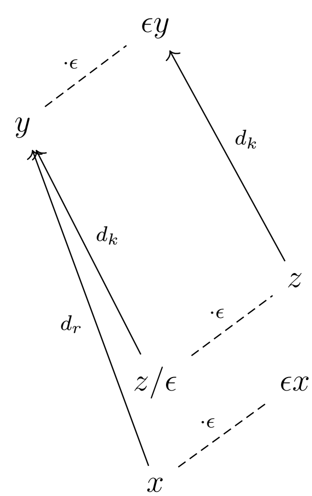


Figure 15: d_7-differentials in C_4-SliceSS($BP^{(\!(C_4)\!)}\langle 1\rangle$).

the E_r-page, it must be killed by a differential $d_k(z) = \epsilon y$ where $k < r$. By Fact 10.14, z is divisible by ϵ. The inductive hypothesis, applied to the differential $d_k(z) = \epsilon y$, shows that $d_k(z/\epsilon) = y$. This is a contradiction because $d_r(x) = y$ is a nontrivial d_r-differential. Therefore, ϵy survives to the E_r-page.



If ϵx does not survive to the E_r-page, then it must be killed by a shorter differential as well. This shorter differential introduces the relation $\epsilon x = 0$ on the E_r-page. However, the Leibniz rule, applied to the differential $d_r(x) = y$, shows that

$$d_r(\epsilon x) = \epsilon y \neq 0$$

on the E_r-page. This is a contradiction. It follows that ϵx survives to the E_r-page as well, and it supports the differential

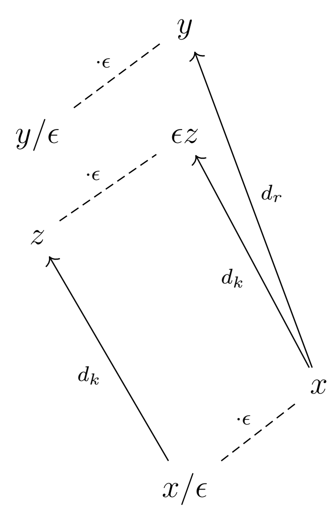
$$d_r(\epsilon x) = \epsilon y.$$

This proves (1).

To prove (2), note that if y/ϵ supports a differential of length smaller than r, then the induction hypothesis would imply that y also supports a differential of the same length.

Similarly, if y/ϵ is killed by a differential of length smaller than r, then the induction hypothesis would imply that y is also killed a by a differential of the same length. Both scenarios lead to contradictions. Therefore, y/ϵ survives to the E_r-page.

We will now show that x/ϵ survives to the E_r-page as well. Since x supports a d_r-differential, x/ϵ must also support a differential of length at most r. Suppose that $d_k(x/\epsilon) = z$, where $k < r$. The induction hypothesis, applied to this d_k-differential, implies the existence of the differential $d_k(x) = \epsilon z$. This is a contradiction.



It follows that x/ϵ survives to the E_r-page, and it supports a nontrivial d_r-differential. Since y/ϵ also survives to the E_r-page, the Leibniz rule shows that

$$d_r(x/\epsilon) = y/\epsilon,$$

as desired. □

Theorem 10.16. *Any class $x = \epsilon^2 a$ on the E_2-page of C_4-SliceSS$(\mathrm{BP}^{((C_4))}\langle 1\rangle)$ must die on or before the E_{13}-page.*

Proof. If the class a is a d_{13}-cycle, then x is a d_{13}-cycle as well. Since ϵ^2 is killed by a d_{13}-differential by Corollary 10.11, $\epsilon^2 a$ must be killed by a differential of length at most 13.

Now suppose that the class a is not a d_{13}-cycle and it supports the differential $d_r(a) = b$, where $r \leq 13$. Applying Lemma 10.15, we deduce that the class $x = \epsilon^2 a$ must support the nontrivial d_r-differential

$$d_r(\epsilon^2 a) = \epsilon^2 b,$$

and therefore cannot survive to the E_{13}-page. □

Theorem 10.17. *The class $\bar{\mathfrak{d}}_1^7 \bar{s}_1 u_{2\lambda} u_{7\sigma} a_{5\lambda} a_{\sigma_2}$ at $(19, 11)$ supports the d_{11}-differential*

$$d_{11}(\bar{\mathfrak{d}}_1^7 \bar{s}_1 u_{2\lambda} u_{7\sigma} a_{5\lambda} a_{\sigma_2}) = \bar{\mathfrak{d}}_1^{10} u_{8\sigma} a_{10\lambda} a_{2\sigma}.$$

Proof. The class $\bar{\mathfrak{d}}_1^{10} u_{8\sigma} a_{10\lambda} a_{2\sigma}$ at $(18, 22)$ is equal to

$$\bar{\mathfrak{d}}_1^{10} u_{8\sigma} a_{10\lambda} a_{2\sigma} = \epsilon^2 (\bar{\mathfrak{d}}_1^2 a_{2\lambda} a_{2\sigma}).$$

By Theorem 10.16, this class must die on or before the E_{13}-page. For degree reasons, the only possibility is for it to be killed by a d_{11}-differential coming from the class $\bar{\mathfrak{d}}_1^7 \bar{s}_1 u_{2\lambda} u_{7\sigma} a_{5\lambda} a_{\sigma_2}$. □

Corollary 10.18. *The class $\bar{s}_1 u_{2\lambda} u_{3\sigma} a_{\sigma_2}$ supports the d_{11}-differential*

$$d_{11}(\bar{s}_1 u_{2\lambda} u_{3\sigma} a_{\sigma_2}) = \bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{5\lambda} a_{2\sigma}.$$

Proof. The d_{11}-differential in Theorem 10.17 can be rewritten as

$$d_1^7 u_{4\sigma} a_{5\lambda} (d_{11}(\bar{s}_1 u_{2\lambda} u_{3\sigma} a_{\sigma_2}) - \bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{5\lambda} a_{2\sigma}) = 0.$$

Since multiplication by $\bar{\mathfrak{d}}_1^7 u_{4\sigma} a_{5\lambda}$ is injective on the E_{11}-page, the claim follows. □

Corollary 10.19. *The class $\bar{\mathfrak{d}}_1^6 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ at $(20, 4)$ is a permanent cycle that survives to the E_∞-page. In homotopy, it detects the class $\bar{\kappa} \in \pi_{20}\mathbb{S}$.*

Proof. For degree reasons, this class is a permanent cycle that survives to the E_∞-page. To show this detects $\bar{\kappa}$, consider the commutative diagram

$$\begin{array}{ccc} \pi_*\mathbb{S} & \longrightarrow & \pi_* \mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle^{C_4} \\ \downarrow & \searrow & \downarrow{\scriptstyle res} \\ \pi_* \mathrm{BP}_{\mathbb{R}}^{C_2} & \longrightarrow & \pi_*(i^*_{C_2} \mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle)^{C_2}, \end{array}$$

where the bottom horizontal map is the composition

$$\mathrm{BP}_{\mathbb{R}} \xrightarrow{\iota_L} i^*_{C_2} \mathrm{BP}^{(\!(C_4)\!)} \longrightarrow i^*_{C_2} \mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle.$$

It is proven in [47, Section 6] that $\bar{\kappa}$ is detected in the C_2-slice spectral sequence of $\mathrm{BP}_{\mathbb{R}}$ by the class $\bar{v}_2 u_{8\sigma_2} a_{4\sigma_2}$. Since $\bar{v}_2 = \bar{r}_1^3$, $\bar{\kappa}$ is detected in C_2-SliceSS$(i^*_{C_2} \mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle)$ by the class $\bar{r}_1^{12} u_{8\sigma_2} a_{4\sigma_2}$. This is exactly the restriction of the class $\bar{\mathfrak{d}}_1^6 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ because

$$res(\bar{\mathfrak{d}}_1^6 u_{4\lambda} u_{6\sigma} a_{2\lambda}) = \bar{r}_1^6 \gamma \bar{r}_1^6 u_{8\sigma_2} a_{4\sigma_2} = \bar{r}_1^{12} u_{8\sigma_2} a_{4\sigma_2}.$$

Therefore, $\bar{\kappa}$ is detected by $\bar{\mathfrak{d}}_1^6 u_{4\lambda} u_{6\sigma} a_{2\lambda}$, as desired. □

As shown in Figure 16, all the other d_{11}-differentials are obtained from the d_{11}-differential in Theorem 10.17 via multiplication with the permanent cycles ϵ, $\bar{\kappa}$, and $\bar{\mathfrak{d}}_1^8 u_{8\lambda} u_{8\sigma}$ (at $(32, 0)$).

Similarly, all the other d_{13}-differentials are obtained from Corollary 10.11 by using multiplicative structures with the classes η', ϵ, $\bar{\kappa}$, and $\bar{\mathfrak{d}}_1^8 u_{8\lambda} u_{8\sigma}$ (see Figure 17).

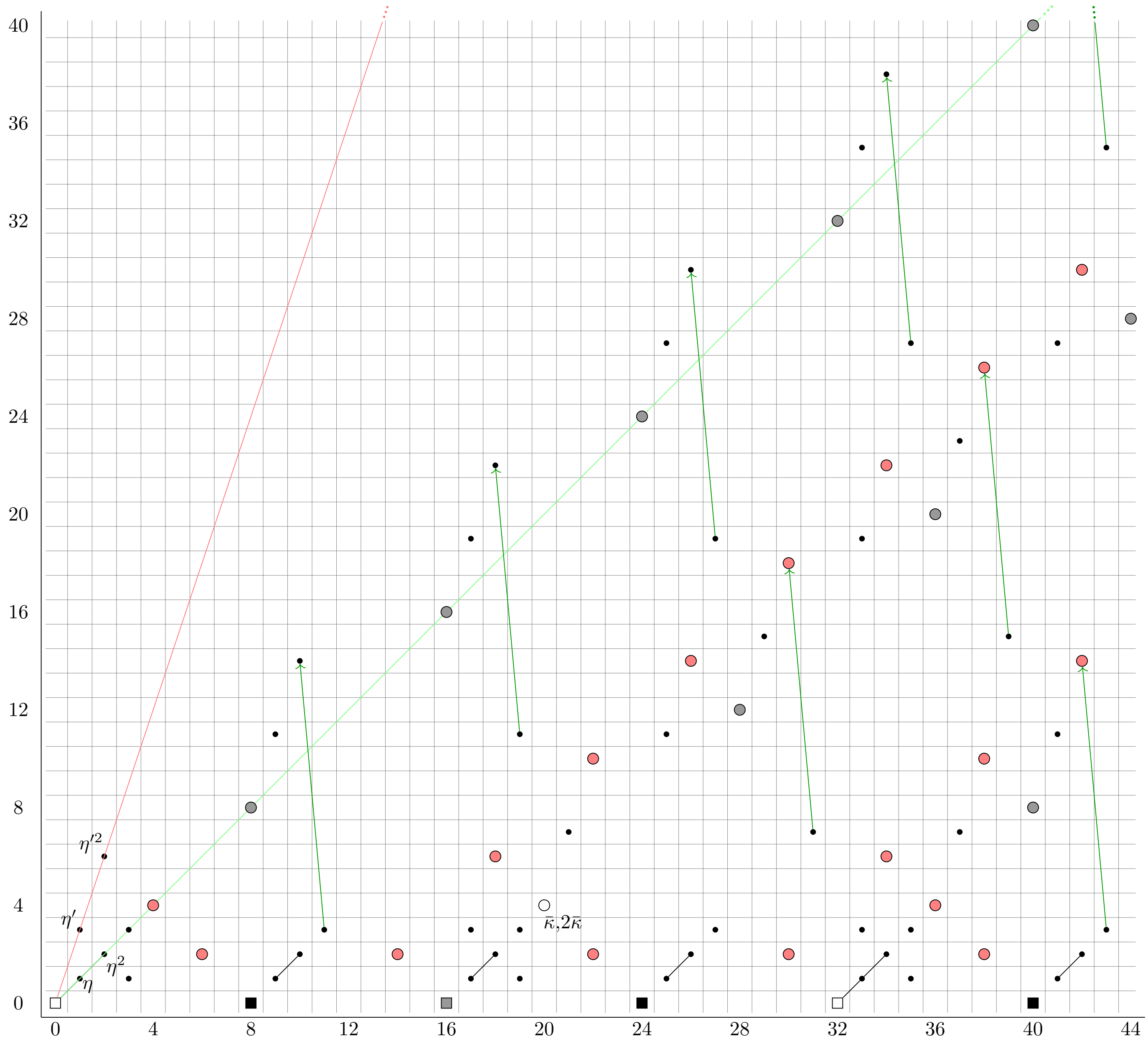


Figure 16: d_{11}-differentials in C_4-SliceSS($\mathrm{BP}^{((C_4))}\langle 1\rangle$).

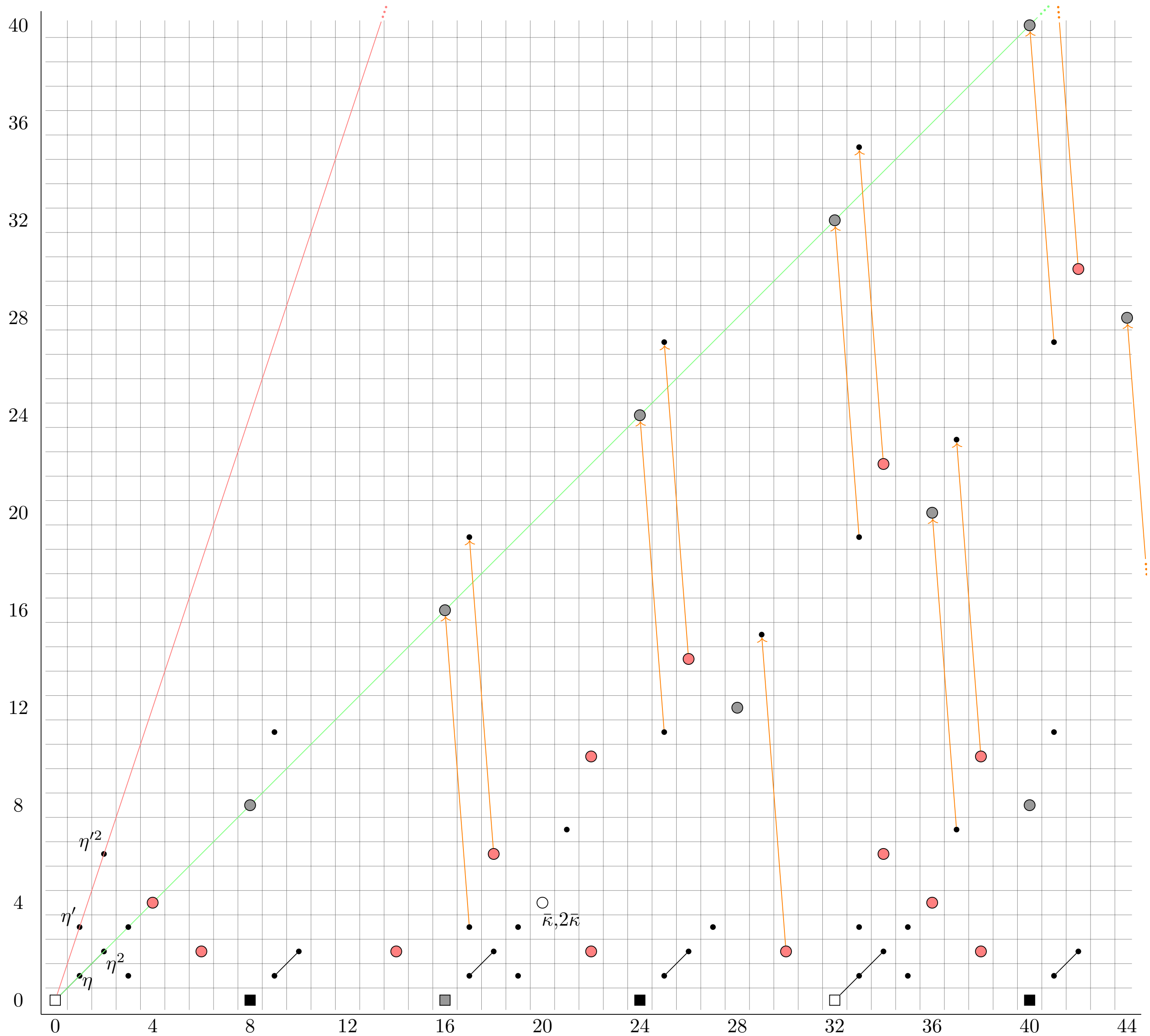


Figure 17: d_{13}-differentials in C_4-SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$).

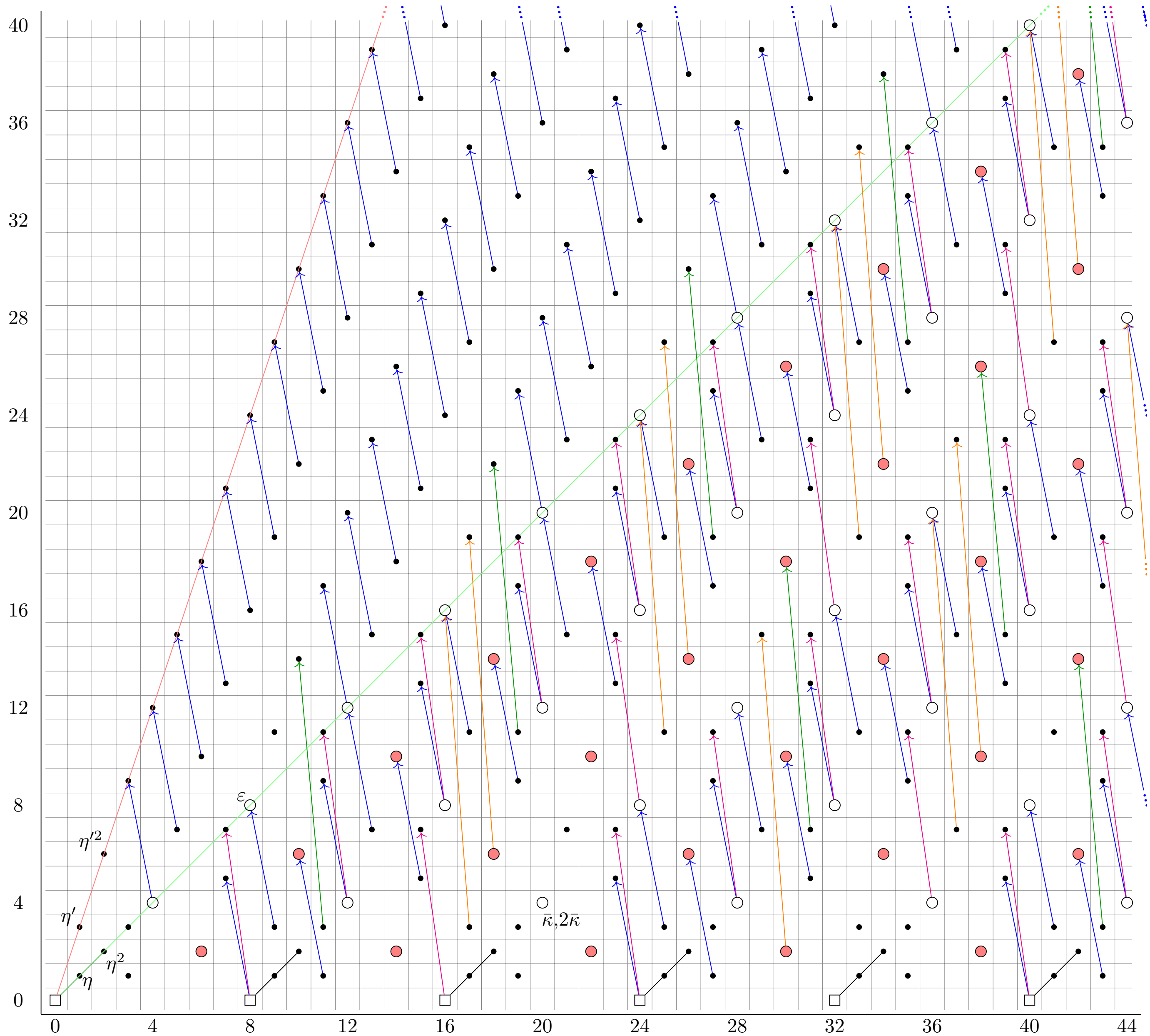


Figure 18: Differentials in C_4-SliceSS$(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$.

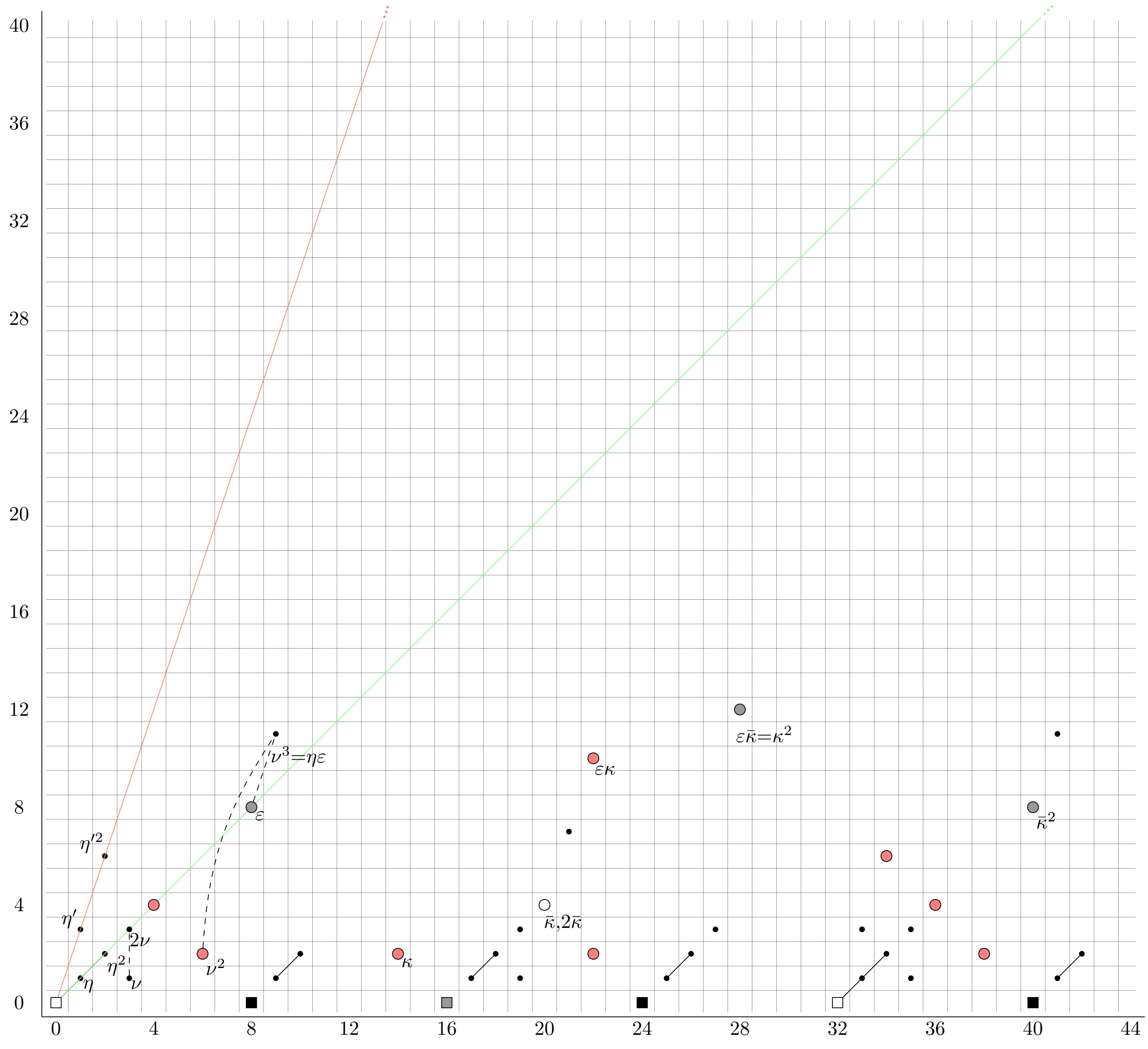


Figure 19: E_∞-page of C_4-SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$).

10.6 Summary of differentials

Differential	Formula	Proof
d_3	$d_3(u_\lambda) = \bar{s}_1 a_\lambda a_{\sigma_2}$ $d_3(u_{2\sigma_2}) = \bar{s}_1 a_{3\sigma_2}$	Theorem 10.1 (restriction)
d_5	$d_5(u_{2\sigma}) = \bar{\mathfrak{d}}_1 a_\lambda a_{3\sigma}$	[39, Theorem 9.9] (Slice Differentials Theorem)
d_5	$d_5(u_{2\lambda}) = \bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_\sigma$	Theorem 10.4 and Corollary 10.5 (restriction)
d_7	$d_7(2u_{2\lambda}) = \bar{\mathfrak{d}}_1 \bar{s}_1 u_\sigma a_{3\lambda} a_{\sigma_2}$ $d_7(2u_{2\lambda}u_{2\sigma}) = \bar{\mathfrak{d}}_1 \bar{s}_1 u_{3\sigma} a_{3\lambda} a_{\sigma_2}$	Theorem 10.6 and Corollary 10.7 (transfer)
d_7	$d_7(u_{4\lambda}) = \bar{\mathfrak{d}}_1 \bar{s}_1 u_{2\lambda} u_\sigma a_{3\lambda} a_{\sigma_2}$	Theorem 10.12 and Corollary 10.13 (norm)
d_{11}	$d_{11}(\bar{s}_1 u_{2\lambda} u_{3\sigma} a_{\sigma_2}) = \bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{5\lambda} a_{2\sigma}$	Theorem 10.17 and Corollary 10.18 (uses Theorem 10.16)
d_{13}	$d_{13}(u_{4\lambda} a_\sigma) = \bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{7\lambda}$	Theorem 10.10 and Corollary 10.11 (norm)

11 The slice filtration of $BP^{((C_4))}\langle 2\rangle$

The refinement of $BP^{((C_4))}\langle 2\rangle$ is

$$S^0[\bar{r}_1, \gamma\bar{r}_1, \bar{r}_3, \gamma\bar{r}_3] \longrightarrow BP^{((C_4))}\langle 2\rangle.$$

Its slices are the following:

$$\begin{cases} \bar{r}_3^i\gamma\bar{r}_3^i\bar{r}_1^k\gamma\bar{r}_1^k : S^{(3i+k)\rho_4} \wedge H\underline{\mathbb{Z}}, & i,k \geq 0 \\ \bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1^k\gamma\bar{r}_1^l + \bar{r}_3^j\gamma\bar{r}_3^i\bar{r}_1^l\gamma\bar{r}_1^k : C_{4+} \wedge_{C_2} S^{(3i+3j+k+l)\rho_2} \wedge H\underline{\mathbb{Z}}, & i \neq j \text{ or } k \neq l. \end{cases}$$

Similar to the slices of $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$, we organize the slices for $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ in order to facilitate our computation.

Consider the monomial $\bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1^k\gamma\bar{r}_1^l$. When $i = j$, this monomial can be written as $\bar{\mathfrak{d}}_3^i\bar{r}_1^k\gamma\bar{r}_1^l$. Fix an non-negative integer i. After the d_3-differentials, the classes of filtration ≥ 3 that are contributed by these slices cells are exactly the same as the classes on the E_5-page of C_4-SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$), truncated at the line $(t-s)+s = 12i$. For this reason, we will call this collection of slices $\bar{\mathfrak{d}}_3^i\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$. Figure 20 shows the truncation lines for the slices $\bar{\mathfrak{d}}_3^i\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ and Figures 21 and 22 illustrate the classes contributed by the slices in $\bar{\mathfrak{d}}_3\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ and $\bar{\mathfrak{d}}_3^2\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$.

When $i \neq j$, the monomial $\bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1^k\gamma\bar{r}_1^l$ contributes an induced slice of the form $\bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1^k\gamma\bar{r}_1^l+$ $\bar{r}_3^j\gamma\bar{r}_3^i\bar{r}_1^l\gamma\bar{r}_1^k$. By symmetry, let $i < j$. The d_3-differential $d_3(u_{2\sigma_2}) = \bar{s}_1 a_{\sigma_2}^3 = (\bar{r}_1 + \gamma\bar{r}_1)a_{\sigma_2}^3$ identifies $\bar{r}_1$ and $\gamma\bar{r}_1$ when the filtration is at least 3. By an abuse of notation, we can rewrite this slice cell as

$$\bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1^k\gamma\bar{r}_1^l + \bar{r}_3^j\gamma\bar{r}_3^i\bar{r}_1^k\gamma\bar{r}_1^l = (\bar{r}_3^i\gamma\bar{r}_3^i)(\bar{r}_3^{j-i} + \gamma\bar{r}_3^{j-i})\bar{r}_1^k\gamma\bar{r}_1^l = \bar{\mathfrak{d}}_3^i\bar{s}_3^{j-i}\bar{r}_1^k\gamma\bar{r}_1^l.$$

For a fixed pair $\{i,j\}$ with $i < j$, the classes of filtration ≥ 3 that are contributed by these slice cells after the d_3-differentials are exactly the same as the classes on the E_5-page of C_2-SliceSS($i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$), truncated at the line $(t-s)+s = 6i+6j$. For this reason, we will call this collection of slices $\bar{\mathfrak{d}}_3^i\bar{s}_3^j i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$. Figure 23 shows the truncation lines for

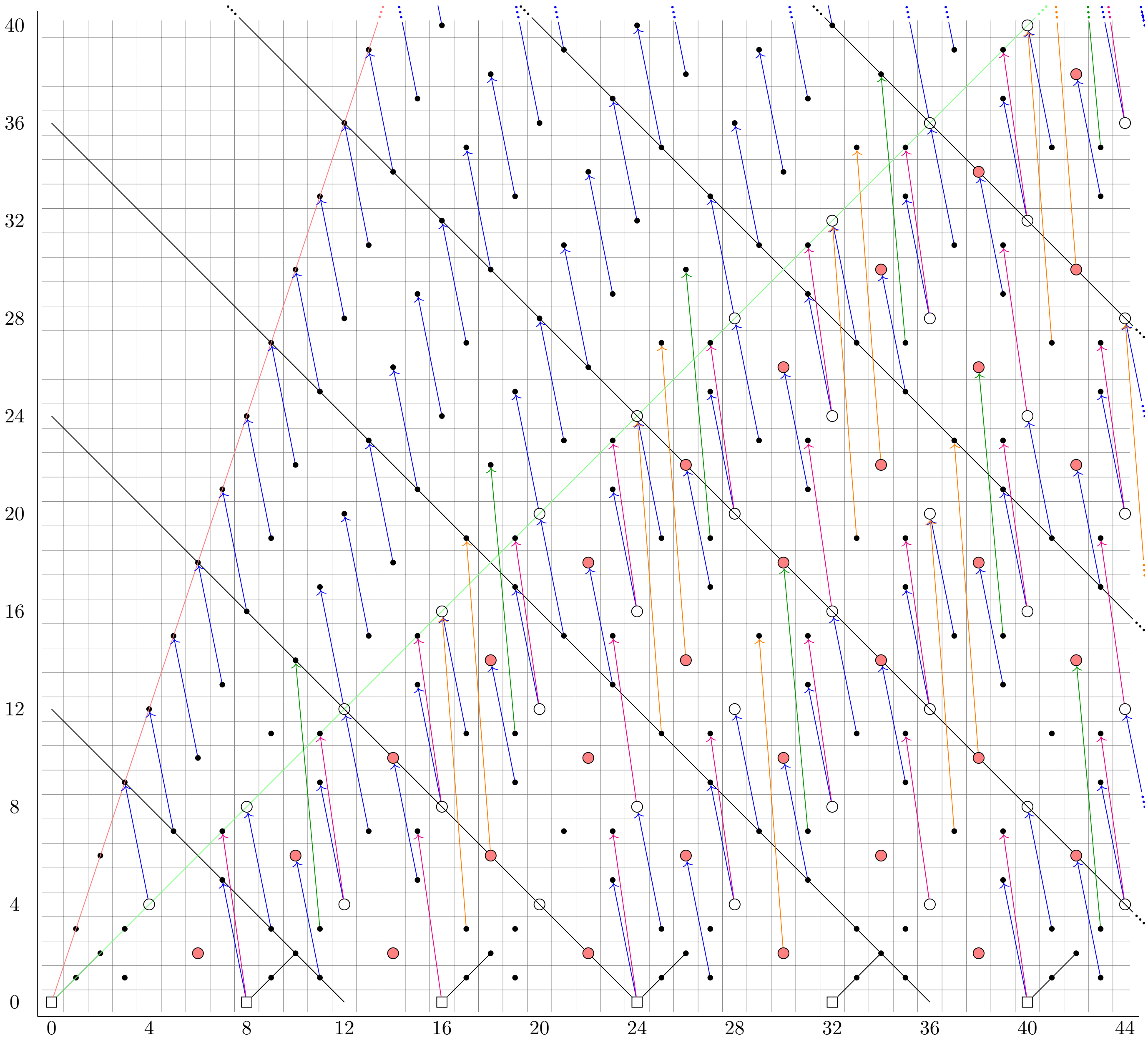


Figure 20: Truncations lines for the slices $\bar{\mathfrak{d}}_3^i \mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$.

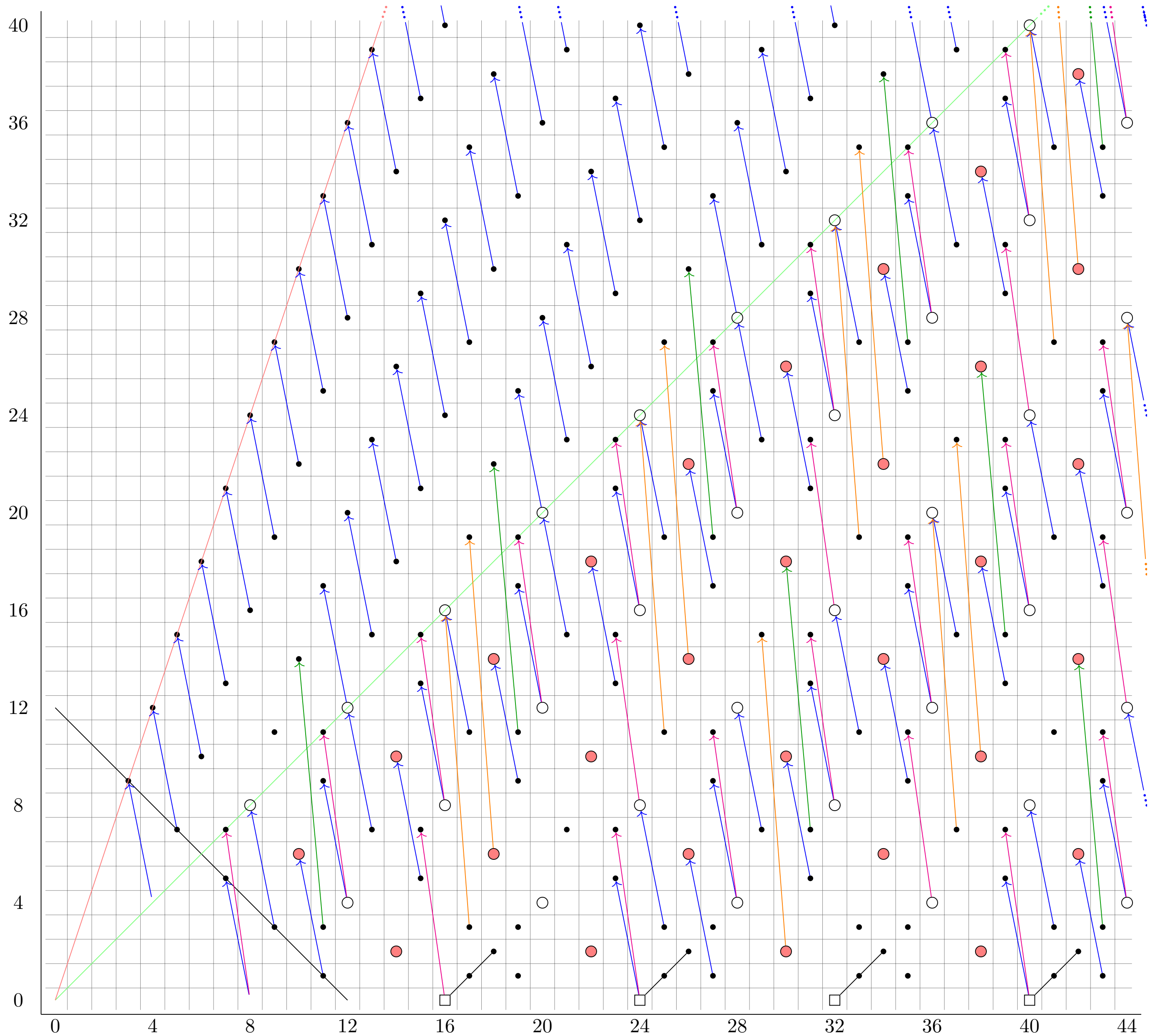


Figure 21: Classes contributed by the slices in $\bar{\mathfrak{d}}_3\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$.

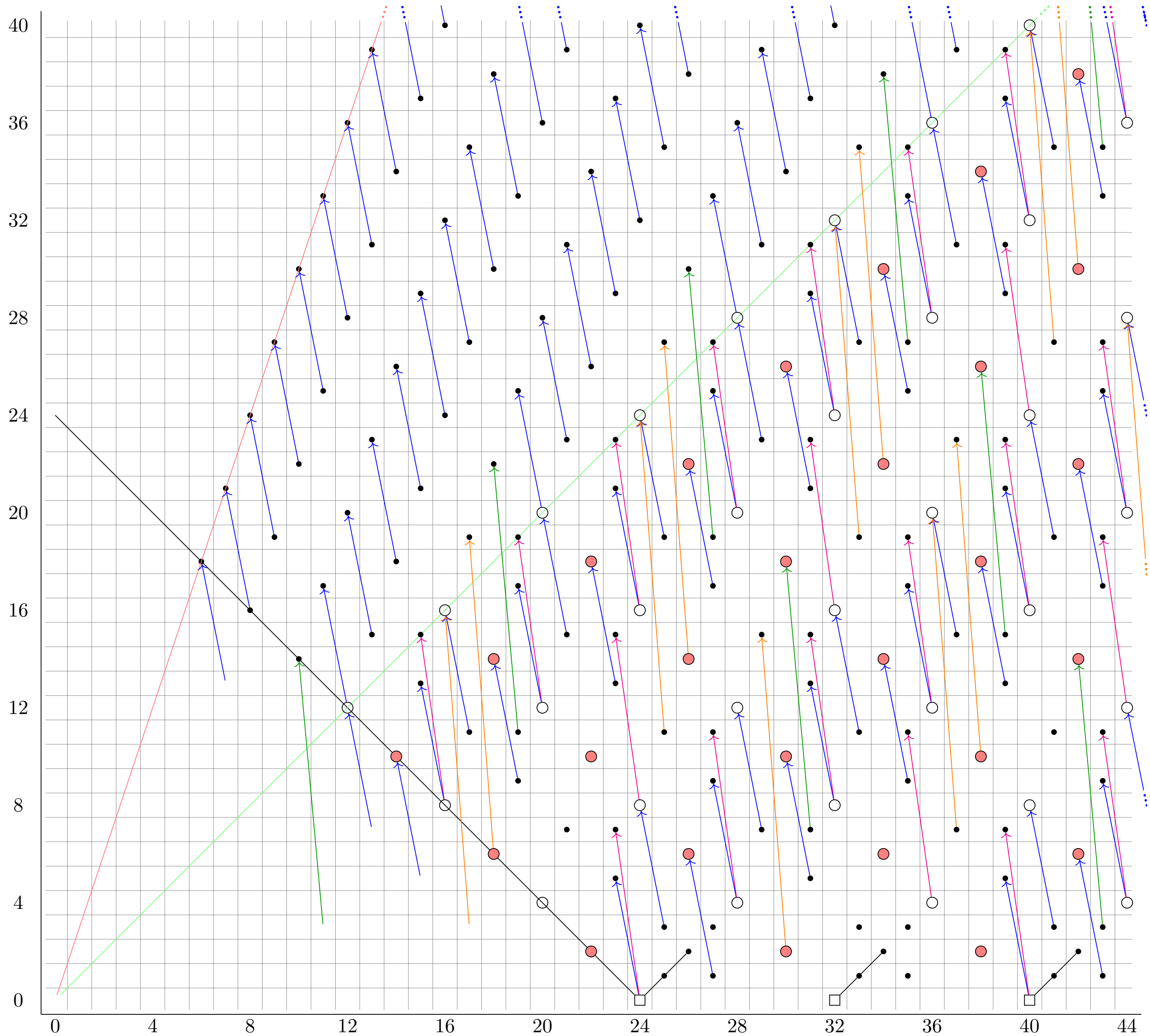


Figure 22: Classes contributed by the slices in $\bar{\mathfrak{d}}_3^2\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$.

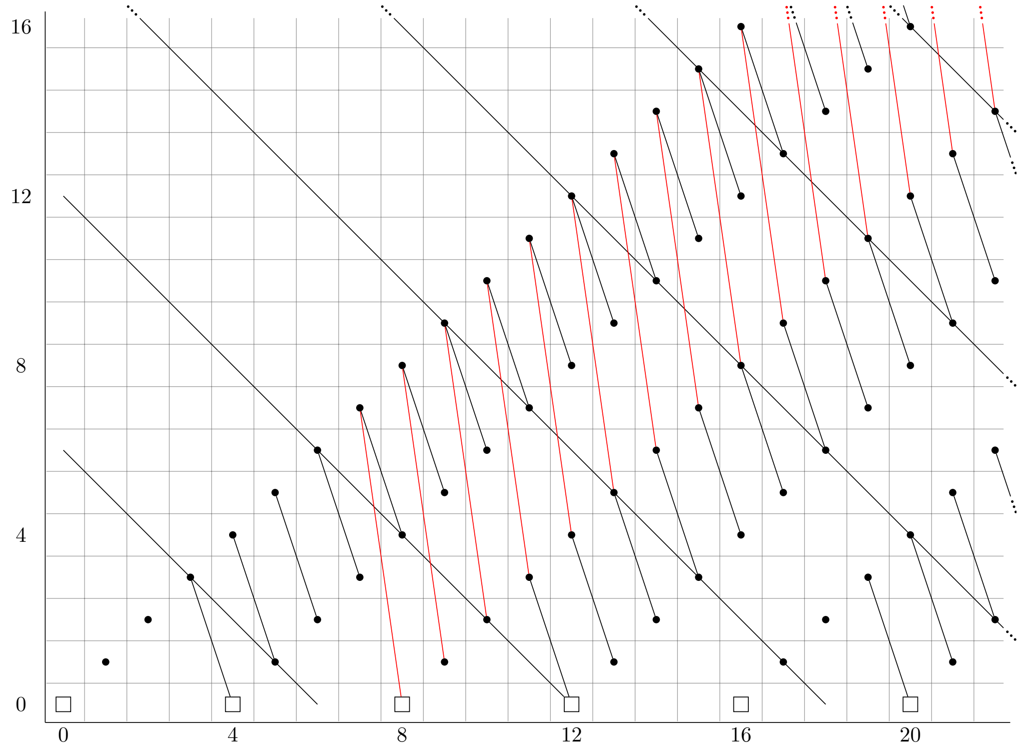


Figure 23: Truncation lines for the slices $\bar{\mathfrak{d}}_3^i \bar{s}_3^j i_{C_2}^* \mathrm{BP}^{((C_4))}\langle 1\rangle$.

these slices. Figures 24, 25, and 26 illustrate the classes contributed by some of these slices.

All the slices of $\mathrm{BP}^{((C_4))}\langle 2\rangle$ are organized into the following table, where the number inside the parenthesis indicates the truncation line. For convenience, we will refer to each of the collections on the top row as a $\mathrm{BP}^{((C_4))}\langle 1\rangle$**-truncation**, and each of the other collections as a $i_{C_2}^* \mathrm{BP}^{((C_4))}\langle 1\rangle$-truncation.

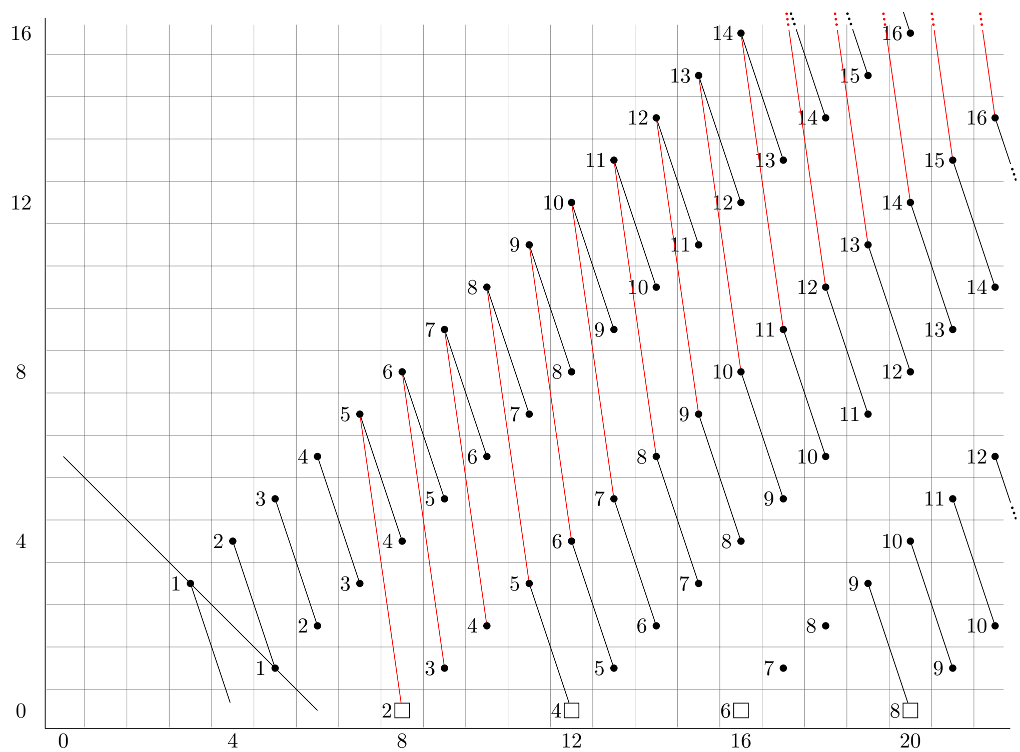


Figure 24: Classes contributed by the slices in $\bar{s}_3 i^*_{C_2} \mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$.

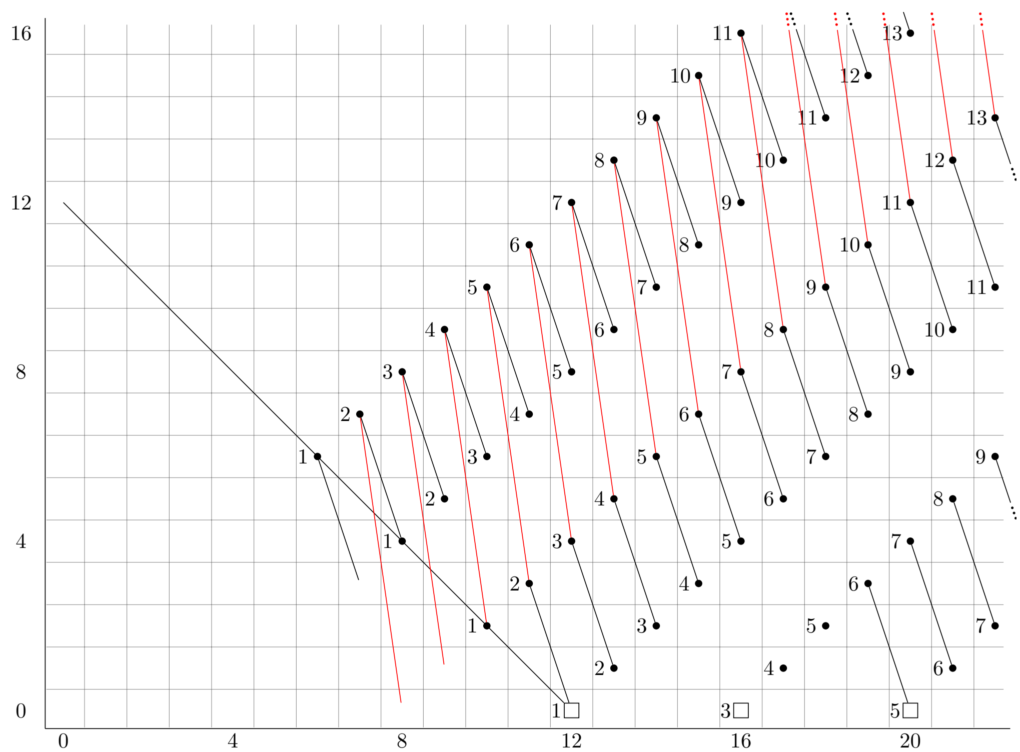


Figure 25: Classes contributed by the slices in $\bar{s}_3^2 i^*_{C_2} \mathrm{BP}^{((C_4))}\langle 1\rangle$.

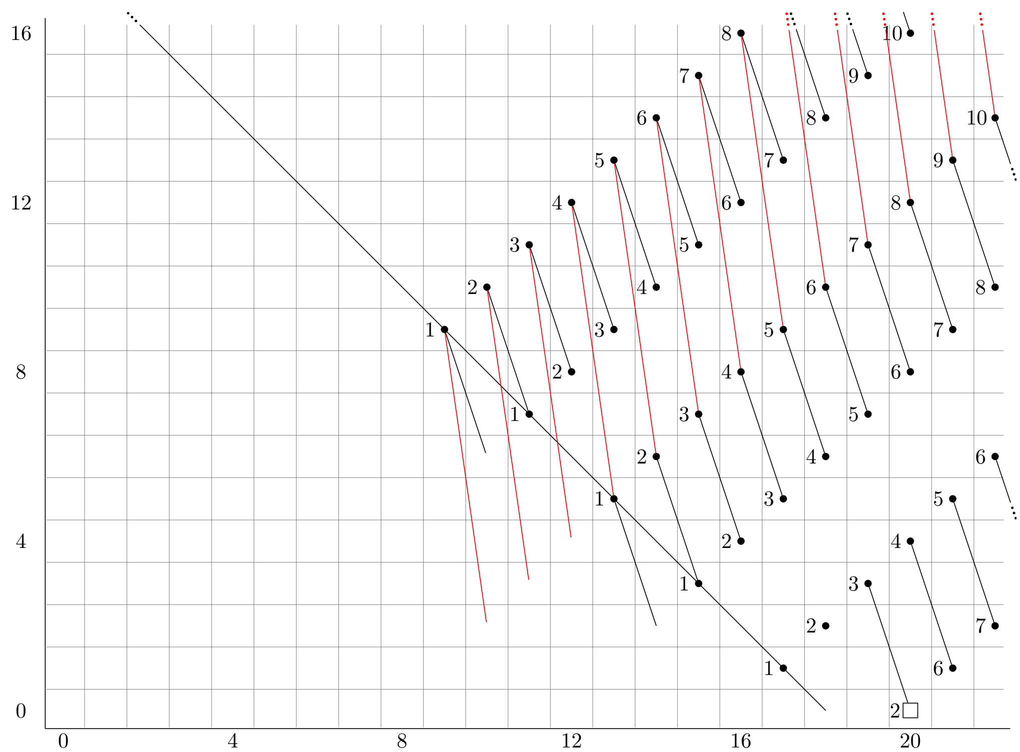


Figure 26: Classes contributed by the slices in $\bar{s}_3^3 i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ or $\bar{\mathfrak{d}}_3 \bar{s}_3 i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$.

$\bar{\mathfrak{d}}_3^0\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle - (0)$	$\bar{\mathfrak{d}}_3^1\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle - (12)$	$\bar{\mathfrak{d}}_3^2\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle - (24)$	$\cdots$
$\bar{\mathfrak{d}}_3^0\bar{s}_3^1 i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle - (6)$	$\bar{\mathfrak{d}}_3^1\bar{s}_3^1 i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle - (18)$	$\bar{\mathfrak{d}}_3^2\bar{s}_3^1 i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle - (30)$	$\cdots$
$\bar{\mathfrak{d}}_3^0\bar{s}_3^2 i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle - (12)$	$\bar{\mathfrak{d}}_3^1\bar{s}_3^2 i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle - (24)$	$\bar{\mathfrak{d}}_3^2\bar{s}_3^2 i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle - (36)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$

(11.1)

12 The C_2-slice spectral sequence of $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$

In this section, we will compute the C_2-slice spectral sequence for $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$. The composition map

$$\mathrm{BP}_{\mathbb{R}} \xrightarrow{i_L} i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)} \longrightarrow i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$$

induces a map

$$C_2\text{- SliceSS}(\mathrm{BP}_{\mathbb{R}}) \longrightarrow C_2\text{- SliceSS}(i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$$

of C_2-slice spectral sequences. The formulas in Theorem 9.11 translate the differentials in C_2- SliceSS$(\mathrm{BP}_{\mathbb{R}})$ to the following differentials in C_2- SliceSS$(i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$:

$$\begin{aligned}
d_3(u_{2\sigma_2}) &= (\bar{r}_1 + \gamma\bar{r}_1)a_{\sigma_2}^3 \\
d_7(u_{4\sigma_2}) &= (\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)a_{\sigma_2}^7 \\
d_{15}(u_{8\sigma_2}) &= \bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2)a_{\sigma_2}^{15} \\
d_{31}(u_{16\sigma_2}) &= \bar{r}_3^4\gamma\bar{r}_3 a_{\sigma_2}^{31}
\end{aligned}$$

The class $u_{32\sigma_2}$ is a permanent cycle.

On the E_2-page, the refinement

$$S^0[\bar{r}_1, \gamma\bar{r}_1, \bar{r}_3, \gamma\bar{r}_3] \longrightarrow i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$$

implies that all the slice cells are indexed by monomials of the form $\bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1^k\gamma\bar{r}_1^l$, where $i, j, k, l \geq 0$.

We now give a step-by-step description of the surviving classes after each differential:

1. After the d_3-differentials, the relation $\bar{r}_1 = \gamma\bar{r}_1$ is introduced for classes with filtrations ≥ 3. Therefore, the slice cells corresponding to these classes can be written as a sum of monomials from the set $\{\bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1^k \,|\, i, j, k \geq 0\}$.

2. After the d_7-differentials, the relation $\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3 = 0$ is introduced for classes with filtrations ≥ 7. Therefore given a class with filtration at least 7, depending on its bidegree, its corresponding slice cell can be written as a sum of monomials from the set $\{\bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1 \,|\, i, j \geq 0\}$, $\{\bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1^2 \,|\, i, j \geq 0\}$, or $\{\bar{r}_3^i\gamma\bar{r}_3^j\bar{r}_1^3, \bar{r}_3^{i+j}\gamma\bar{r}_3 \,|\, i, j \geq 0\}$.

3. After the d_{15}-differentials, the relation $\bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2) = 0$ is introduced for classes with filtrations ≥ 15. Given a class with filtration at least 15, depending on its bidegree, its corresponding slice cell can be written as a sum of monomials from the set $\{\bar{r}_3^{i+1}\gamma\bar{r}_3^1\bar{r}_1, \bar{r}_3^i\gamma\bar{r}_3^2\bar{r}_1, \,|\, i \geq 4\}$, $\{\bar{r}_3^{i+1}\gamma\bar{r}_3^1\bar{r}_1^2, \bar{r}_3^i\gamma\bar{r}_3^2\bar{r}_1^2, \,|\, i \geq 4\}$, or $\{\bar{r}_3^{i+1}\gamma\bar{r}_3^1\bar{r}_1^3, \bar{r}_3^i\gamma\bar{r}_3^2\bar{r}_1^3, \bar{r}_3^{i+2}\gamma\bar{r}_3 \,|\, i \geq 4\}$.

4. After the d_{31}-differentials, the relation $\bar{r}_3^4\gamma\bar{r}_3 = 0$ is introduced for classes with filtrations ≥ 31. Since all the classes with filtrations ≥ 31 have slice cells divisible by $\bar{r}_3^4\gamma\bar{r}_3$ on the E_{31}-page, they are all wiped out by the d_{31}-differentials. The spectral sequence collapses afterwards and there is a horizontal vanishing line with filtration 31 on the E_∞-page.

Example 12.1. Consider all the classes at $(31, 31)$. On the E_2-page, their names are of the form $x \cdot a_{31\sigma_2}$, where x is a sum of slice cells of the form $\{\bar{r}_3^i \gamma \bar{r}_3^j \bar{r}_1^k \gamma \bar{r}_1^l \,|\, 3i + 3j + k + l = 31\}$.

1. After the d_3-differentials, x can be written as a sum of slice cells from the set $\{\bar{r}_3^i \gamma \bar{r}_3^j \bar{r}_1^k \,|\, 3i + 3j + k = 31\}$.

2. After the d_7-differentials, x can be written as a sum of slice cells from the set $\{\bar{r}_3^i \gamma \bar{r}_3^j \bar{r}_1 \,|\, i + j = 10\}$.

3. After the d_{15}-differentials, x can be written as a sum of slice cells from the set $\{\bar{r}_3^9 \gamma \bar{r}_3 \bar{r}_1, \bar{r}_3^8 \gamma \bar{r}_3^2 \bar{r}_1\}$.

4. After the d_{31}-differentials, all the remaining classes are killed.

Example 12.2. Consider all the classes at $(33, 33)$. On the E_2-page, their names are of the form $x \cdot a_{33\sigma_2}$, where x is a sum of slice cells of the form $\{\bar{r}_3^i \gamma \bar{r}_3^j \bar{r}_1^k \gamma \bar{r}_1^l \,|\, 3i + 3j + k + l = 33\}$.

1. After the d_3-differentials, x can be written as a sum of slice cells from the set $\{\bar{r}_3^i \gamma \bar{r}_3^j \bar{r}_1^k \,|\, 3i + 3j + k = 33\}$.

2. After the d_7-differentials, x can be written as a sum of slice cells from the set $\{\bar{r}_3^i \gamma \bar{r}_3^j \bar{r}_1^3, \bar{r}_3^{10} \gamma \bar{r}_3 \,|\, i + j = 10\}$.

3. After the d_{15}-differentials, x can be written as a sum of slice cells from the set $\{\bar{r}_3^9 \gamma \bar{r}_3 \bar{r}_1^3, \bar{r}_3^8 \gamma \bar{r}_3^2 \bar{r}_1^3, \bar{r}_3^{10} \gamma \bar{r}_3\}$.

4. After the d_{31}-differentials, all the remaining classes are killed.

13 Induced differentials from $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$

In section 11, the slices of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$ are subdivided into collections of the form $\bar{\mathfrak{d}}_3^i\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ ($\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation) and $\bar{\mathfrak{d}}_3^i\bar{s}_3^j i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ ($i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation), where $i \geq 0$ and $j \geq 1$. On the E_2-page of the C_4-slice spectral sequence, the classes contributed by the slices in $\bar{\mathfrak{d}}_3^i\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ is a truncation of the E_2-page of C_4- $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$, and the classes contributed by the slices in $\bar{\mathfrak{d}}_3^i\bar{s}_3^j i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ is a truncation of the E_2-page of C_2- $\mathrm{SliceSS}(i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$.

Recall that in the computation of $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$, we have also divided the slices into collections (they are the columns in Table 10.1). The computation was simplified by treating each collection individually with respect to the d_3-differentials. After the d_3-differentials, we combined the E_5-pages of every collection together to form the E_5-page of $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$.

In light of this simplification for $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$, it is natural to expect that in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$, each collection can be treated individually with respect to differentials of lengths up to 13 (the longest differential in $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$). Knowing this will allow us to compute the E_{13}-page of each collection individually, and then combine them together to form the E_{13}-page of $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

Definition 13.1. A *predicted differential* is a differential whose leading terms for the source and the target belong to slices in the same collection and the position of that differential matches with a differential in C_4- $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$ or C_2- $\mathrm{SliceSS}(i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$.

For example, all of the differentials whose source and target are on or above the truncation lines in Figures 21, 22, 24, 25, 26 are predicted differentials.

Definition 13.2. An *interfering differential* is a differential whose source and target are in different collections.

Given the definitions above,

Theorem 13.3. *The collections can be treated individually with respect to differentials of lengths up to 13. More specifically:*

1. *For $3 \leq r \leq 11$, all the predicted d_r-differentials occur and there are no interfering d_r-differentials.*

2. *All the predicted d_{13}-differentials occur.*

13.1 d_3-differentials

The quotient map $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle \to \mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$ induces a map

$$\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle) \longrightarrow \mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$$

of C_4-slice spectral sequences. In $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle)$, the d_3-differentials are generated under multiplication by

$$d_3(u_\lambda) = \bar{s}_1 a_\lambda a_{\sigma_2}.$$

For natuality and degree reasons, the same differential occurs in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$ as well. Moreover, by considering the restriction map

$$C_4\text{-}\,\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle) \longrightarrow C_2\text{-}\,\mathrm{SliceSS}(i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle),$$

we deduce the d_3-differential

$$d_3(u_{2\sigma_2}) = \bar{s}_1 a_{3\sigma_2}$$

as well. (even though we are working with a C_4-slice spectral sequence, this differential applies to some of the classes in $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations because of our naming conventions). All the predicted differentials are generated by these two differentials. Afterwards, there are no more d_3-differentials by degree reasons.

13.2 d_5-differentials

In SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$), all the d_5-differentials are generated under multiplication by the differentials

$$d_5(u_{2\sigma}) = \bar{\mathfrak{d}}_1 a_\lambda a_{3\sigma}$$

and

$$d_5(u_{2\lambda}) = \bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_\sigma.$$

In SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$), the first differential still exists by Hill–Hopkins–Ravenel's Slice Differential Theorem [39, Theorem 9.9]. To prove that the second differential exists as well, consider again the map

$$\text{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle) \longrightarrow \text{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle).$$

For natuality reasons, $u_{2\lambda}$ must support a differential of length at most 5 in SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$). Since u_λ supports a nonzero d_3-differential, $u_{2\lambda}$ is a d_3-cycle. This implies that $u_{2\lambda}$ must support a d_5-differential whose target maps to $\bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_\sigma$ under the quotient map (which sends $\bar{r}_3$ and $\gamma\bar{r}_3$ to zero). It follows that the only possible target is $\bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_\sigma$, and the same d_5-differential on $u_{2\lambda}$ exists in SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$).

All the predicted d_5 differentials in SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$) are generated by these two differentials.

It remains to show that there are no interfering d_5-differentials. There are two cases to consider:

(1) *The source is in a* $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$*-truncation.* Every class in a $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation is in the image of the transfer map

$$C_2\text{- SliceSS}(i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle) \xrightarrow{tr} C_4\text{- SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle).$$

On the E_5-page of C_2- SliceSS$(i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$, every class is a d_5-cycle because there are no d_5-differentials. Therefore after applying the transfer map, all the images must be d_5-cycles as well.

(2) *The source is in a* $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$*-truncation.* If the source is in the image of the transfer, then by the same reasoning as above, it must be a d_5-cycle. If the source is not in the image of the transfer, then it can be written as $\bar{\mathfrak{d}}_3^i\bar{\mathfrak{d}}_1^j u_\lambda^a u_\sigma^b a_\lambda^c a_\sigma^d$ for some $i, j, a, b, c, d \geq 0$. The only possibilities are the blue classes in Figure 27. These classes might support d_5-differentials whose targets are classes in $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations. However, using the differentials $d_5(u_{2\lambda}) = \bar{\mathfrak{d}}_1 u_\lambda a_{2\lambda} a_\sigma$ and $d_5(u_{2\sigma}) = \bar{\mathfrak{d}}_1 a_\lambda a_{3\sigma}$, we can easily show that all of these classes are d_5-cycles.

13.3 d_7-differentials

In the slice spectral sequence for $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$, the d_7-differentials are generated under multiplicative structure by three differentials:

1. $d_7(2u_{2\lambda}) = \bar{\mathfrak{d}}_1 \bar{s}_1 u_\sigma a_{3\lambda} a_{\sigma_2}$;
2. $d_7(2u_{2\lambda}u_{2\sigma}) = \bar{\mathfrak{d}}_1 \bar{s}_1 u_{3\sigma} u_{3\lambda} a_{\sigma_2}$;
3. $d_7(u_{4\lambda}) = \bar{\mathfrak{d}}_1 \bar{s}_1 u_{2\lambda} u_{2\sigma} a_{3\lambda} a_{\sigma_2}$.

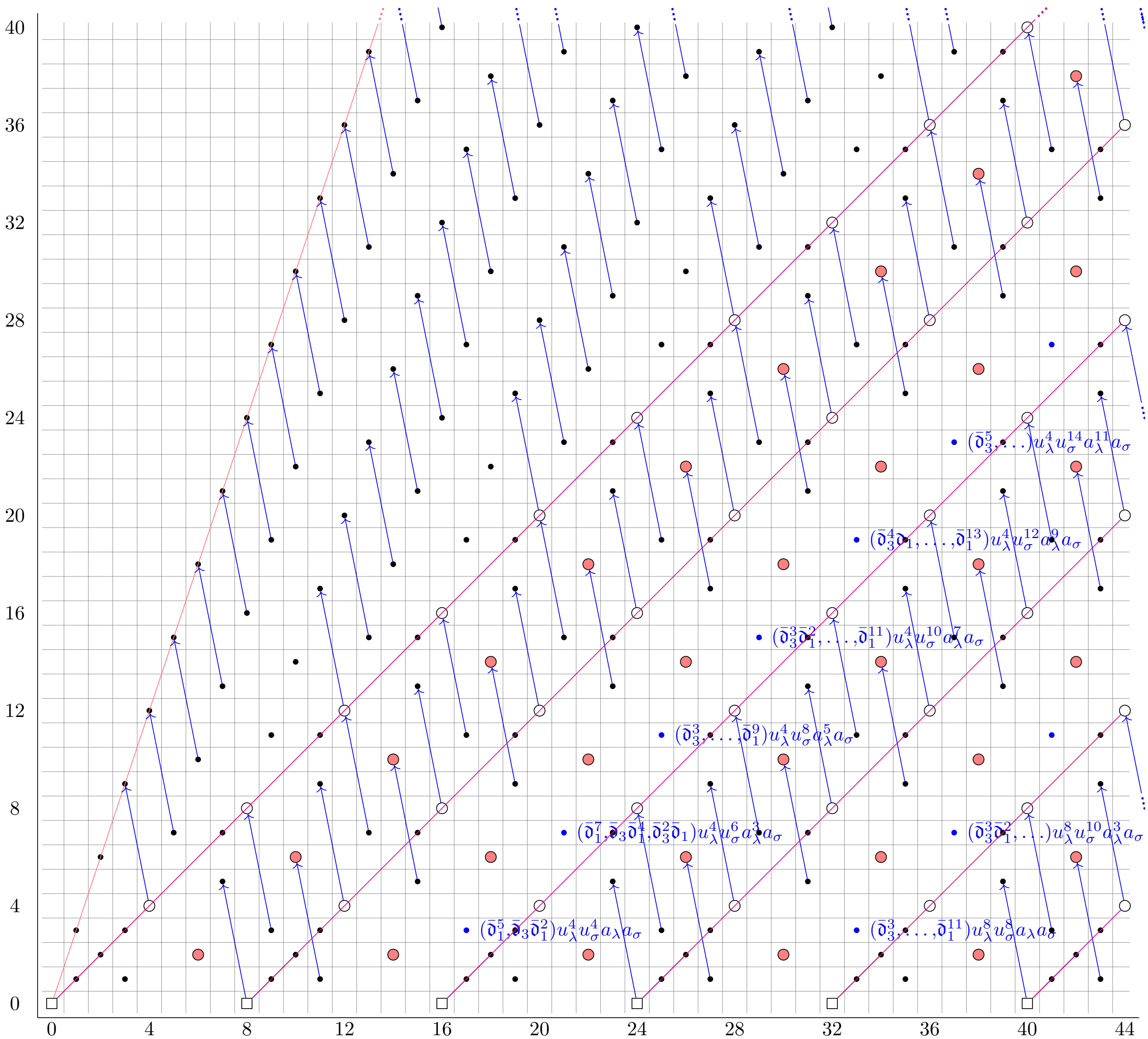


Figure 27: Possible sources in $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations that could support d_5-interfering differentials. The magenta lines indicate the locations of the classes in $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations.

Using the natuality of the quotient map

$$\text{SliceSS}(\text{BP}^{(\!(C_4)\!)}\langle 2\rangle) \longrightarrow \text{SliceSS}(\text{BP}^{(\!(C_4)\!)}\langle 1\rangle),$$

we deduce that the classes $2u_{2\lambda}$, $2u_{2\lambda}u_{2\sigma}$, and $u_{4\lambda}$ must all support differentials of length at most 7 in SliceSS($\text{BP}^{(\!(C_4)\!)}\langle 2\rangle$). The formulas for the d_5-differentials on $u_{2\lambda}$ and $u_{2\sigma}$ imply that all three classes above are d_5-cycles. Therefore, they must all support d_7-differentials. It follows by natuality that we have the exact same d_7-differentials in SliceSS($\text{BP}^{(\!(C_4)\!)}\langle 2\rangle$). These differentials generate the predicted d_7-differentials in all the $\text{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations.

All of the predicted d_7-differentials in $i_{C_2}^*\text{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations are obtained by using the transfer map

$$C_2\text{-}\,\text{SliceSS}(\text{BP}^{(\!(C_4)\!)}\langle 2\rangle) \xrightarrow{tr} C_4\text{-}\,\text{SliceSS}(\text{BP}^{(\!(C_4)\!)}\langle 2\rangle).$$

More precisely, the transfer map takes in a d_7-differential in $\text{BP}^{(\!(C_4)\!)}\langle 2\rangle$, which is generated by $d_7(u_{4\sigma_2}) = (\bar{r}_3+\gamma\bar{r}_3+\bar{r}_1^3)a_{7\sigma_2} = (\bar{s}_3+\bar{r}_1^3)a_{7\sigma_2}$, and produces a corresponding d_7-differential in a $i_{C_2}^*\text{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation.

Note that in the C_2-slice spectral sequence for $\text{BP}^{(\!(C_4)\!)}\langle 1\rangle$, the d_7-differentials are generated by $d_7(u_{4\sigma_2}) = \bar{r}_1^3 a_{7\sigma_2}$, whereas in the C_2-slice spectral sequence for $\text{BP}^{(\!(C_4)\!)}\langle 2\rangle$, they are generated by $d_7(u_{4\sigma_2}) = (\bar{s}_3 + \bar{r}_1^3)a_{7\sigma_2}$. The readers should be warned that strictly speaking, the d_7-differentials are *not* appearing independently within each $i_{C_2}^*\text{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations, but rather identifying classes between different $i_{C_2}^*\text{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations. The exact formulas for this identification will be discussed in Section 14. Nevertheless, since the leading terms are independent, the d_7-differentials do occur independently within each $i_{C_2}^*\text{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation.

It remains to prove that there are no interfering d_7-differentials. There are two cases to consider.

(1) *The source is in the image of the transfer* (in other words, the source is produced by an induced slice cell). Denote the source by $tr(x)$, where x is a class in the C_2-slice spectral sequence. If $tr(x)$ supports a d_7-differential in the C_4-slice spectral sequence, then natuality of the transfer map implies that in the C_2-slice spectral sequence, x must support a differential of length at most 7. This means that x either supports a d_3-differential or a d_7-differential.

If x supports a d_3-differential $d_3(x) = y$, then since the transfer map is faithful on the E_3-page, applying the transfer to this d_3-differential yields the nontrivial d_3-differential

$$d_3(tr(x)) = tr(y)$$

in the C_4-slice spectral sequence. This is a contradiction to the assumption that $d_7(tr(x)) \neq 0$.

Therefore, x must support a d_7-differential $d_7(x) = y$ in the C_2-slice spectral sequence. Applying the transfer map to this d_7-differential gives $d_7(tr(x)) = tr(y)$, which must be the d_7-differential on $tr(x)$ by natuality. However, this will not be an interfering d_7-differential because it is a predicted d_7-differential that is obtained via the transfer.

Example 13.4. In Figure 28, there is a possibility for a d_7-interfering differential with source a class at $(11, 3)$ coming from a $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation (it is supposed to support a predicted d_{11}-differential), and the target a class at $(10, 10)$ coming from $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations (a pink class).

The two possible sources at $(11, 3)$ are $\bar{\mathfrak{d}}_3\bar{s}_1u_{2\lambda}u_{3\sigma}a_\lambda a_{\sigma_2} = tr(\bar{r}_3\gamma\bar{r}_3\bar{r}_1u_{4\sigma_2}a_{3\sigma_2})$ and $\bar{\mathfrak{d}}_1^3\bar{s}_1u_{2\lambda}u_{3\sigma}a_\lambda a_{\sigma_2} = tr(\bar{r}_1^4\gamma\bar{r}_1^3u_{4\sigma_2}a_{3\sigma_2})$. By the discussion above, if any of these two classes support a d_7-differential hitting a class in the image of the transfer, then this differential must be obtained by applying the transfer map to a d_7-differential in the C_2-slice spectral sequence.

In the C_2-slice spectral sequence, the relevant differentials are the following:

$$\begin{aligned} d_7(\bar{r}_3\gamma\bar{r}_3\bar{r}_1 u_{4\sigma_2}a_{3\sigma_2}) &= \bar{r}_3\gamma\bar{r}_3\bar{r}_1(\bar{r}_3+\gamma\bar{r}_3+\bar{r}_1^3)a_{11\sigma_2} \\ d_7(\bar{r}_1^4\gamma\bar{r}_1^3 u_{4\sigma_2}a_{3\sigma_2}) &= \bar{r}_1^4\gamma\bar{r}_1^3(\bar{r}_3+\gamma\bar{r}_3+\bar{r}_1^3)a_{11\sigma_2}. \end{aligned}$$

The transfer of the targets are $(\bar{\mathfrak{d}}_3\bar{s}_3\bar{s}_1+\bar{\mathfrak{d}}_3\bar{s}_1^4)a_{3\lambda}a_{4\sigma_2}$ and $(\bar{\mathfrak{d}}_1^3\bar{s}_3\bar{s}_1+\bar{\mathfrak{d}}_1^3\bar{s}_1^4)a_{3\lambda}a_{4\sigma_2}$, respectively. They are both 0 on the E_7-page because they are targets of d_3-differentials. It follows that the d_7-interfering differentials do not occur at $(11,3)$. The same argument also shows that there are no d_7-interfering differentials with sources at $(19,11)$, $(23,15)$, $(31,7)$, $\ldots$.

(2) *The source is not in the image of the transfer* (in other words, the source is produced by a regular, non-induced slice cell). As shown in Figure 28, for degree reasons, there are possible interfering d_7-differentials with sources at

1. $(20,4)$, $(28,12)$, $(36,20)$, $\ldots$;
2. $(32,0)$, $(40,8)$, $(48,16)$, $\ldots$;
3. $(52,4)$, $(60,12)$, $(68,20)$, $\ldots$;

   $\ldots$.

To prove that these d_7-differentials do not exist, it suffices to prove that all the classes at $(20,4)$ are d_7-cycles. Once we prove this, all the other possible sources above will be d_7-cycles as well by multiplicative reasons.

The quotient map $\text{SliceSS}(BP^{((C_4))}\langle 2\rangle) \longrightarrow \text{SliceSS}(BP^{((C_4))}\langle 1\rangle)$ shows that both classes at $(11,3)$, $\bar{\mathfrak{d}}_1^3\bar{s}_1 u_{2\lambda}u_{3\sigma}a_\lambda a_{\sigma_2}$ and $\bar{\mathfrak{d}}_3\bar{s}_1 u_{2\lambda}u_{3\sigma}a_\lambda a_{\sigma_2}$, must support nontrivial d_{11}-differentials.

Multiplication by the permanent cycles at $(8,8)$ ($\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1 u_{4\sigma}a_{4\lambda}$ and $\bar{\mathfrak{d}}_1^4 u_{4\sigma}a_{4\lambda}$) implies that all three classes at $(19,11)$ (coming from the slice cells $\bar{\mathfrak{d}}_3^2\bar{\mathfrak{d}}_1\bar{s}_1$, $\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^4\bar{s}_1$, and $\bar{\mathfrak{d}}_1^7\bar{s}_1$) must support nontrivial d_{11}-differentials. In fact, these are the predicted d_{11}-differentials.

There are three classes at $(20,4)$: $\bar{\mathfrak{d}}_3^2 u_{6\sigma}u_{4\lambda}a_{2\lambda}$, $\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3 u_{6\sigma}u_{4\lambda}a_{2\lambda}$, and $\bar{\mathfrak{d}}_1^6 u_{6\sigma}u_{4\lambda}a_{2\lambda}$. If any of these classes supports a nontrivial d_7-differential, the target would be a classes at $(19,11)$, which, as we have shown in the previous paragraph, supports a nontrivial d_{11}-differential. This is a contradiction because something killed on the d_7-page becomes trivial on the d_{11}-page, and cannot support a nontrivial d_{11}-differential.

13.4 d_{11}-differentials

For degree reasons, there are no possible d_9-differentials. The next possible differentials are the d_{11}-differentials.

In the slice spectral sequence for $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$, all the d_{11}-differentials are generated by the single d_{11}-differential

$$d_{11}(\bar{s}_1 u_{2\lambda}u_{3\sigma}a_{\sigma_2}) = \bar{\mathfrak{d}}_1^3 u_{4\sigma}a_{5\lambda}a_{2\sigma}$$

under multiplication. Using the quotient map

$$\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle) \longrightarrow \mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle),$$

we deduce that the class $\bar{s}_1 u_{2\lambda}u_{3\sigma}a_{\sigma_2}$ must support a differential of length at most 11 in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

Our knowledge of the earlier differentials implies that this class is a d_r-cycle for $r \leq 10$, and hence it must support a d_{11}-differential. Furthermore, the formula of the d_{11}-differential

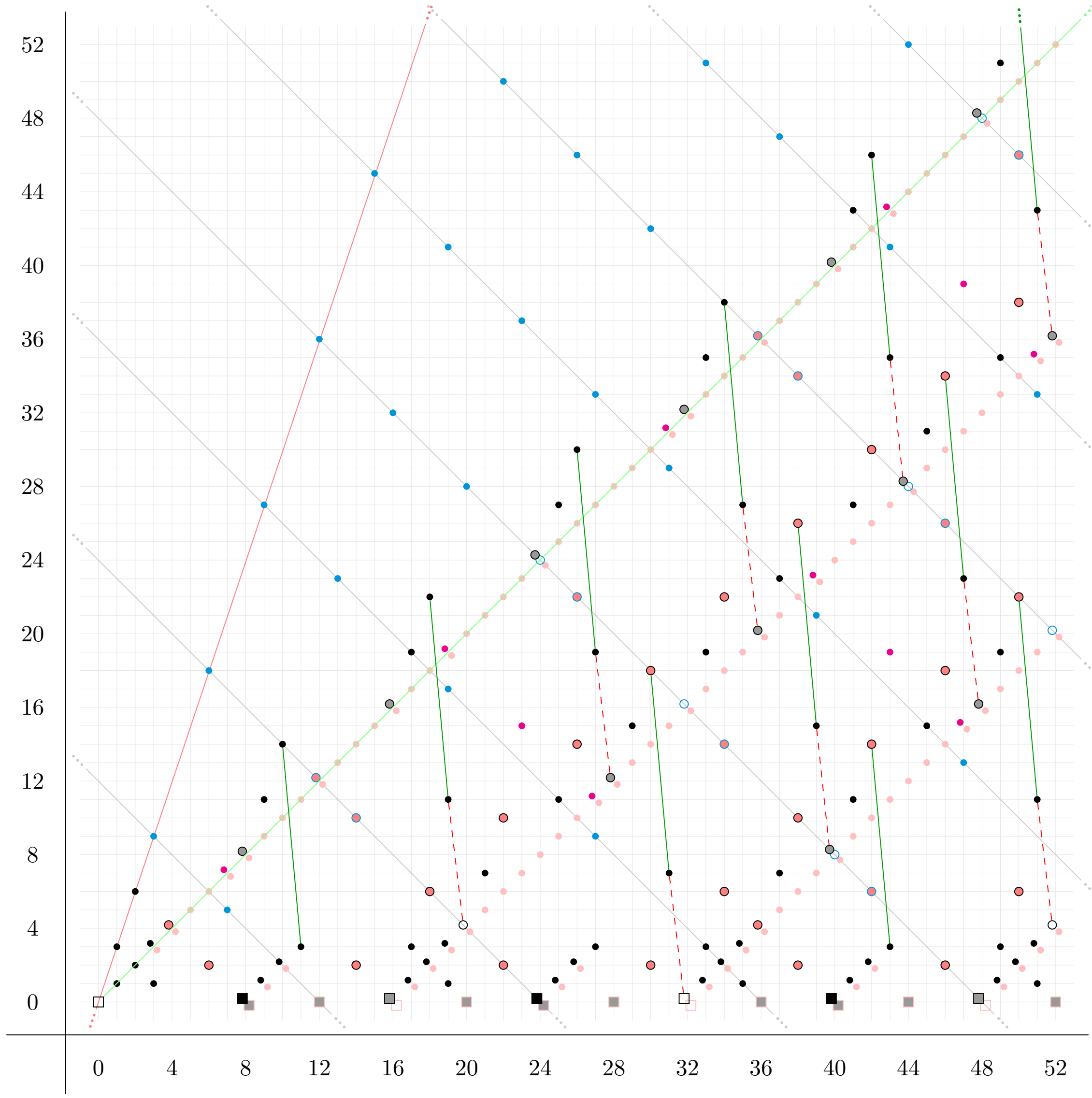


Figure 28: The dashed red lines are the possible d_7-interfering differentials. The cyan classes are the d_5-truncation classes, the magenta classes are the d_7-truncation classes, and the pink classes are classes in $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations after the predicted d_7-differentials. The green differentials are the predicted d_{11}-differentials which all occur.

is of the form

$$d_{11}(\bar{s}_1 u_{2\lambda} u_{3\sigma} a_{\sigma_2}) = \bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{5\lambda} a_{2\sigma} + \cdots ,$$

where "$\cdots$" indicates terms that go to 0 under the quotient map (which sends $\bar{r}_3, \gamma\bar{r}_3 \mapsto 0$). All of the predicted d_{11}-differentials are obtained using this d_{11}-differential under multiplication.

Similar to situation of the d_7-differentials, strictly speaking, the d_{11}-differentials do not necessarily occur within each $BP^{(\!(C_4)\!)}\langle 1\rangle$-truncation. For instance, in the formula above, the "$\cdots$" could be $\bar{\mathfrak{d}}_3 u_{4\sigma} a_{5\lambda} a_{2\sigma}$. If this happens, the d_{11}-differential would be identifying the two classes, $\bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{5\lambda} a_{2\sigma}$ and $\bar{\mathfrak{d}}_3 u_{4\sigma} a_{5\lambda} a_{2\sigma}$, which are located in different $BP^{(\!(C_4)\!)}\langle 1\rangle$-truncations. Given this, we can kill off the leading term and assume that the rest of the terms remain. This will give us the same distribution of classes after the d_{11}-differentials and will not affect later computations.

It remains to show that there are no d_{11}-interfering differentials. Figure 29 shows all the possible d_{11}-interfering differentials. We will prove that none of them exist.

(1) Blue differentials. These differentials have sources at

- $\{(27, 11), (39, 23), (51, 35), \ldots\}$;
- $\{(35, 3), (47, 15), (59, 27), \ldots\}$;
- $\{(55, 7), (67, 19), (79, 31), \ldots\}$;
- $\{(75, 11), (87, 23), (99, 35), \ldots\}$;
- $\cdots$.

The sources of these differentials are in the image of the transfer map. Their pre-images in the C_2-slice spectral sequence are all d_{11}-cycles (more specifically, they all support differentials of length at least 15). Therefore, their images under the transfer map cannot support nontrivial d_{11}-differentials.

(2) Gray differentials. These differentials have sources at

- $\{(43, 19), (67, 43), (91, 67), \ldots\}$;
- $\{(63, 23), (87, 47), (111, 71), \ldots\}$;
- $\{(59, 3), (83, 27), (107, 51), \ldots\}$;
- $\{(79, 7), (103, 31), (127, 55), \ldots\}$;
- $\{(99, 11), (123, 35), (147, 59), \ldots\}$;
- $\{(119, 15), (143, 39), (167, 63), \ldots\}$;
- $\{(139, 19), (163, 43), (187, 67), \ldots\}$;
- $\cdots$.

Each of the sources is a d_7-truncation class. If any of these differentials exist, we will obtain a contradiction when we multiply this differential by the classes at $(8, 8)$ (either $\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1 u_{4\sigma}a_{4\lambda}$ or $\bar{\mathfrak{d}}_1^4 u_{4\sigma}a_{4\lambda}$).

For example, suppose the class at $(43, 19)$ supports a nontrivial d_{11}-differential. The target (a class at $(42, 30)$), when multiplied by the class $\bar{\mathfrak{d}}_1^4 u_{4\sigma}a_{4\lambda}$, is a nonzero class at $(50, 38)$. The source, however, becomes 0. This is a contradiction.

(3) Black differentials. These differentials have sources at

- $\{(18, 6), (26, 14), (34, 22), \ldots\}$;
- $\{(30, 2), (38, 10), (46, 18), \ldots\}$;

- $\{(50,6),(58,14),(66,22),\ldots\}$;
- $\{(62,2),(70,10),(78,18),\ldots\}$;
- $\cdots$.

It suffices to show that all of the classes in the first set are d_{11}-cycles. Once we have proven this, multiplication by the class $u_{8\lambda}u_{8\sigma}$ (d_{11}-cycle), the three classes at $(20,4)$ ($(\bar{\mathfrak{d}}_3^2, \bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3, \bar{\mathfrak{d}}_1^6)u_{4\lambda}u_{6\sigma}a_{2\lambda}$, all d_{11}-cycles), and the two permanent cycles at $(8,8)$ ($(\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1, \bar{\mathfrak{d}}_1^4)u_{4\sigma}a_{4\lambda}$) will show that all the other classes are d_{11}-cycles as well.

Now, for the first set, the names of the classes at each of the possible sources are as follows:

- $(18,6)$: $2(\bar{\mathfrak{d}}_3^2, \bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3, \bar{\mathfrak{d}}_1^6)u_{3\lambda}u_{6\sigma}a_{3\lambda} = (\bar{\mathfrak{d}}_3^2, \bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3, \bar{\mathfrak{d}}_1^6)u_{4\lambda}u_{4\sigma}a_{2\lambda}a_{2\sigma}$
- $(26,14)$: $2(\bar{\mathfrak{d}}_3^3\bar{\mathfrak{d}}_1, \ldots, \bar{\mathfrak{d}}_1^{10})u_{3\lambda}u_{10\sigma}a_{7\lambda} = (\bar{\mathfrak{d}}_3^3\bar{\mathfrak{d}}_1, \ldots, \bar{\mathfrak{d}}_1^{10})u_{4\lambda}u_{8\sigma}a_{6\lambda}a_{2\sigma}$
- $(34,22)$: $2(\bar{\mathfrak{d}}_3^4\bar{\mathfrak{d}}_1^2, \ldots, \bar{\mathfrak{d}}_1^{14})u_{3\lambda}u_{14\sigma}a_{11\lambda} = (\bar{\mathfrak{d}}_3^4\bar{\mathfrak{d}}_1^2, \ldots, \bar{\mathfrak{d}}_1^{14})u_{4\lambda}u_{12\sigma}a_{10\lambda}a_{2\sigma}$
- $\cdots$.

The names can all be written as products of the following d_{11}-cycles: $\bar{\mathfrak{d}}_1$, $\bar{\mathfrak{d}}_3$, a_λ, a_σ, $u_{4\lambda}a_\sigma$ (supports d_{13}-differential), and $u_{4\sigma}$ (supports d_{13}-differential). Therefore, there are no d_{11}-interfering differentials in this case.

(4) Red differentials. These differentials have sources at

- $\{(14,2),(22,10),(30,18),\ldots\}$;
- $\{(34,6),(42,14),(50,22),\ldots\}$;
- $\{(46,2),(54,10),(62,18),\ldots\}$;

- $\{(66,6),(74,14),(82,22),\ldots\}$;

- $\cdots$.

Similar to (3), it suffices to show that all of the classes in the first set are d_{11}-cycles. Afterwards, all the other classes can be proven to be d_{11}-cycles via multiplication by the class $u_{8\lambda}u_{8\sigma}$, the classes at $(20,4)$, and the classes at $(8,8)$ (all of which are d_{11}-cycles).

Now, for the first set, the classes at $(22,10)$, $(30,18)$, $(38,26)$, $\ldots$ are all d_{11}-cycles because they can be written as products of classes at $(20,4)$ and classes at $(2,6)$, $(10,14)$, $(18,22)$, $\ldots$ (all of which are d_{11}-cycles). Afterwards, we deduce that the classes at $(14,2)$ are d_{11}-cycles as well because if they are not, then multiplying the d_{11}-differential by the classes at $(8,8)$ would produce a nontrivial d_{11}-differential on the classes at $(22,10)$. This is a contradiction because we have just proven that all the classes at $(22,10)$ are d_{11}-cycles.

13.5 Predicted d_{13}-differentials

In the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 1\rangle$, all the d_{13}-differentials are generated by $d_{13}(u_{4\lambda}a_\sigma) = \bar{\mathfrak{d}}_1^3 u_{4\sigma}a_{7\lambda}$ under multiplication. This differential was proven by applying the norm formula (see Theorem 9.8, Theorem 10.10 and Corollary 10.11). In fact, we can also prove this differential in SliceSS($BP^{(\!(C_4)\!)}\langle 2\rangle$) by using the norm formula, and we will do so in Section 15 when we discuss the norm in depth.

Alternatively, we can analyze the quotient map

$$\mathrm{SliceSS}(BP^{(\!(C_4)\!)}\langle 2\rangle) \longrightarrow \mathrm{SliceSS}(BP^{(\!(C_4)\!)}\langle 1\rangle)$$

again. Since $u_{4\lambda}a_\sigma$ supports a d_{13}-differential in SliceSS($BP^{(\!(C_4)\!)}\langle 1\rangle$), it must support a differential of length at most 13 in SliceSS($BP^{(\!(C_4)\!)}\langle 2\rangle$). Our knowledge of the earlier differ-

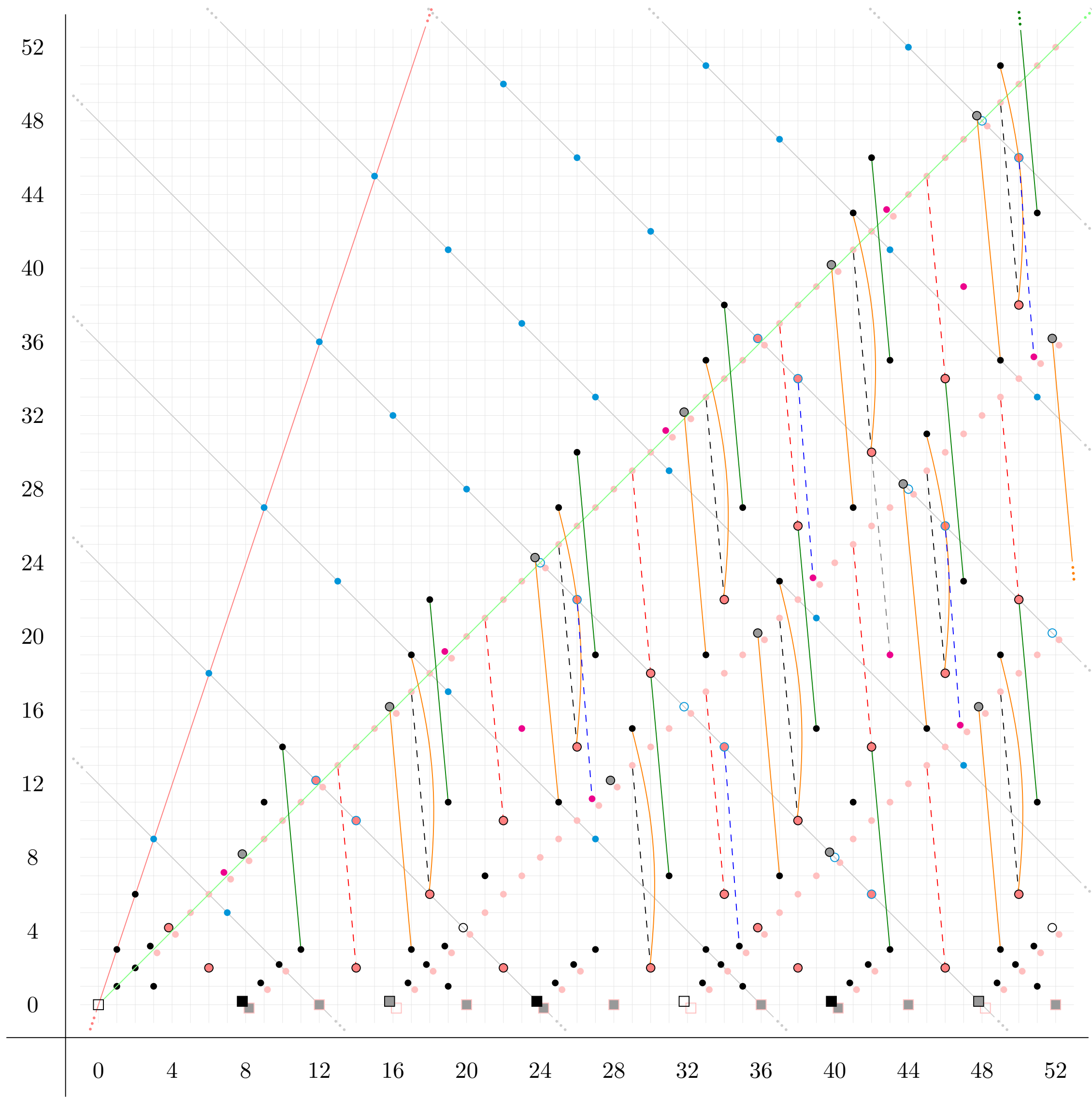


Figure 29: The dashed lines are the possible d_{11}-interfering differentials. The cyan classes are the d_5-truncation classes, the magenta classes are the d_7-truncation classes, and the pink classes are classes in $i^*_{C_2}\mathrm{BP}^{((C_4))}\langle 1\rangle$-truncations after the predicted d_7-differentials. The green differentials are the predicted d_{11}-differentials and the orange differentials are the predicted d_{13}-differentials.

entials implies that $u_{4\lambda}a_\sigma$ must be a d_{11}-cycle, and hence must support a d_{13}-differential. More specifically, we can deduce this fact by analyzing the class $\bar{\mathfrak{d}}_3^3 u_{4\lambda}u_{8\sigma}a_{5\lambda}a_\sigma$ at $(25, 11)$. If $u_{4\lambda}a_\sigma$ supports a d_r-differential of length $r < 13$, then $\bar{\mathfrak{d}}_3^3 u_{4\lambda}u_{8\sigma}a_{5\lambda}a_\sigma$ must support a d_r-differential as well, which is impossible by degree reasons.

Since the d_{13}-differential on $u_{4\lambda}a_\sigma$ respects natuality under the quotient map, it must be of the form

$$d_{13}(u_{4\lambda}a_\sigma) = \bar{\mathfrak{d}}_1^3 u_{4\sigma}a_{7\lambda} + \cdots,$$

where "$\cdots$" denote terms that go to 0 under the quotient map sending $\bar{r}_3, \gamma\bar{r}_3 \mapsto 0$ (in particular, it could contain $\bar{\mathfrak{d}}_3 u_{4\sigma}a_{7\lambda}$, as we will see in Section 15). All the predicted d_{13}-differentials are generated by this differential under multiplication.

Similar to the cases for d_7 and d_{11}-differentials, the readers should be warned that the d_{13}-differentials are not necessarily occurring within each $BP^{(\!(C_4)\!)}\langle 1\rangle$-truncation. The above formula identifies the leading term, $\bar{\mathfrak{d}}_1^3 u_{4\sigma}a_{7\lambda}$, with the rest of the terms (possibly none). Therefore, we can kill off the leading term and assume that the rest of the terms remain.

13.6 E_{13}-page of SliceSS$(BP^{(\!(C_4)\!)}\langle 2\rangle)$

Figure 30 shows the E_{13}-page of SliceSS$(BP^{(\!(C_4)\!)}\langle 2\rangle)$ with the predicted d_{13}-differentials already taken out. The truncation classes are color coded as follows:

1. Cyan classes: d_5-truncation classes;
2. Magenta classes: d_7-truncation classes;
3. Green classes: d_{11}-truncation classes;
4. Orange classes: d_{13}-truncation classes;

5. Pink classes: $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes.

14 Higher differentials I: d_{13} and d_{15}-differentials

In this section, we prove all the d_{13} and d_{15}-differentials in the slice spectral sequence of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$, as well as differentials between $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes.

14.1 d_{13}-differentials

Proposition 14.1. *The class $u_{4\sigma}$ supports the d_{13}-differential*

$$d_{13}(u_{4\sigma}) = \bar{\mathfrak{d}}_3 a_{3\lambda} a_{7\sigma}.$$

Proof. This is an immediate application of Hill–Hopkins–Ravenel's Slice Differential Theorem [39, Theorem 9.9]. □

The d_{13}-differential in Proposition 14.1 generates all the d_{13}-differentials between the line of slope 1 and the line of slope 3 under multiplication (see Figure 31).

Proposition 14.2. *The class $\bar{\mathfrak{d}}_3^2 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ at $(20, 4)$ supports the d_{13}-differential*

$$d_{13}(\bar{\mathfrak{d}}_3^2 u_{4\lambda} u_{6\sigma} a_{2\lambda}) = \bar{\mathfrak{d}}_3^3 u_{\lambda} u_{8\sigma} a_{8\lambda} a_{\sigma}.$$

Proof. We will prove this differential by using the restriction map

$$res: C_4\text{-}\operatorname{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle) \longrightarrow C_2\text{-}\operatorname{SliceSS}(i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle).$$

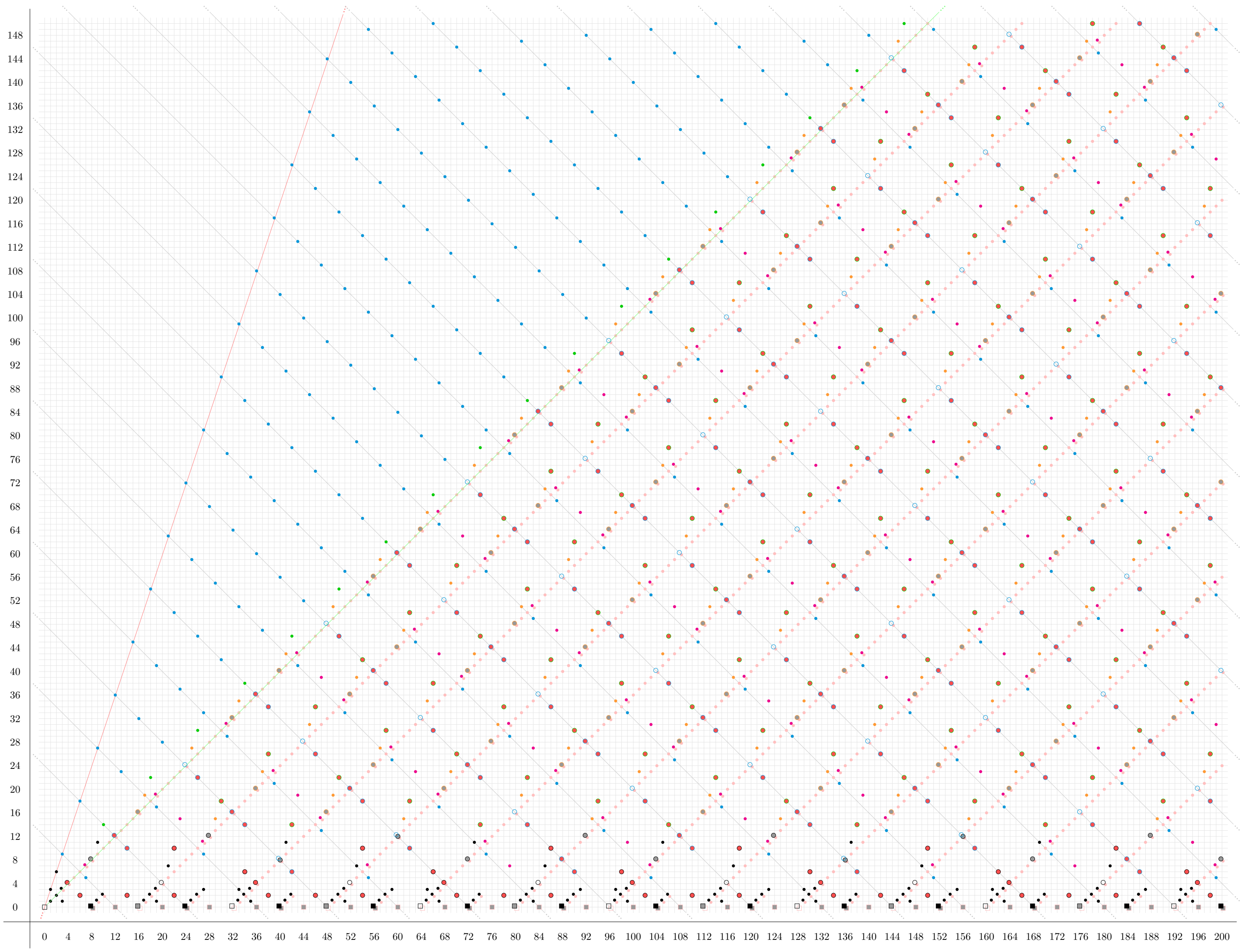


Figure 30: The E_{13}-page of SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$) with the predicted d_{13}-differentials already taken out. The cyan classes are d_5-truncation classes, the magenta classes are d_7-truncation classes, the green classes are d_{11}-truncation classes, the orange classes are d_{13}-truncation classes, and the pink classes are $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes.

The restriction of $\bar{\mathfrak{d}}_3^2 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ is $\bar{r}_3^2 \gamma \bar{r}_3^2 u_{8\sigma_2} a_{4\sigma_2}$. In the C_2-slice spectral sequence, this class supports the d_{15}-differential

$$\begin{aligned} d_{15}(\bar{r}_3^2 \gamma \bar{r}_3^2 u_{8\sigma_2} a_{4\sigma_2}) &= \bar{r}_3^2 \gamma \bar{r}_3^2 \bar{r}_1 (\bar{r}_3^2 + \bar{r}_3 \gamma \bar{r}_3 + \gamma \bar{r}_3^2) a_{19\sigma_2} \\ &= \bar{r}_3^3 \gamma \bar{r}_3^3 \bar{r}_1 a_{19\sigma_2} + \bar{r}_3^2 \gamma \bar{r}_3^2 (\bar{r}_3^2 + \gamma \bar{r}_3^2) \bar{r}_1 a_{19\sigma_2}. \end{aligned}$$

This implies that in the C_4-slice spectral sequence, $\bar{\mathfrak{d}}_3^2 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ must support a differential of length at most 15.

If $d_{15}(\bar{\mathfrak{d}}_3^2 u_{4\lambda} u_{6\sigma} a_{2\lambda}) = x$, then by natuality,

$$res(x) = \bar{r}_3^3 \gamma \bar{r}_3^3 \bar{r}_1 a_{19\sigma_2} + \bar{r}_3^2 \gamma \bar{r}_3^2 (\bar{r}_3^2 + \gamma \bar{r}_3^2) \bar{r}_1 a_{19\sigma_2}.$$

This is impossible because while the class $\bar{r}_3^2 \gamma \bar{r}_3^2 (\bar{r}_3^2 + \gamma \bar{r}_3^2) \bar{r}_1 a_{19\sigma_2}$ has a pre-image on the E_{15}-page ($\bar{\mathfrak{d}}_3^2 \bar{s}_3^2 \bar{r}_1 a_{6\lambda} a_{7\sigma_2}$), the class $\bar{r}_3^3 \gamma \bar{r}_3^3 \bar{r}_1 a_{19\sigma_2}$ does not. The closest thing to its possible pre-image is $\bar{\mathfrak{d}}_3^3 \bar{s}_1 a_{9\lambda} a_{\sigma_2}$, which restrictions to 0 in the C_2-slice spectral sequence because it is killed by a d_3-differential.

Therefore, the class $\bar{\mathfrak{d}}_3^2 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ must support a d_{13}-differential. There is one possible target, which is the class $\bar{\mathfrak{d}}_3^3 u_{\lambda} u_{8\sigma} a_{8\lambda} a_{\sigma}$ at $(19, 17)$. This proves the desired differential. □

Consider the following classes:

1. $\bar{\mathfrak{d}}_3 u_{\lambda} u_{2\sigma} a_{2\lambda} a_{\sigma}$ at $(7, 5)$. This class is a permanent cycle by degree reasons.

2. $\bar{\mathfrak{d}}_3^3 u_{\lambda} u_{8\sigma} a_{8\lambda} a_{\sigma}$ at $(19, 17)$. This class is a permanent cycle (it is the target of the d_{13}-differential in Proposition 14.2).

3. $\bar{\mathfrak{d}}_3^3 u_{8\sigma} a_{9\lambda} a_{\sigma}$ at $(17, 19)$. This class is a permanent cycle by degree reasons.

4. $\bar{\mathfrak{d}}_3^4 u_{12\sigma} a_{12\lambda}$ at $(24, 24)$. This class supports the d_{13}-differential $d_{13}(\bar{\mathfrak{d}}_3^4 u_{12\sigma} a_{12\lambda}) = \bar{\mathfrak{d}}_3^5 u_{8\sigma} a_{15\lambda} a_{7\sigma}$.

Using the Leibniz rule on the differential in Proposition 14.2 with the classes above produces all the d_{13}-differentials under the line of slope 1 (see Figure 31).

14.2 Differentials between $i_{C_2}^* \mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle$-truncation classes

Using the restriction map

$$res : C_4\text{-}\operatorname{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2 \rangle) \longrightarrow C_2\text{-}\operatorname{SliceSS}(i_{C_2}^* \mathrm{BP}^{(\!(C_4)\!)}\langle 2 \rangle)$$

and the transfer map

$$tr : C_2\text{-}\operatorname{SliceSS}(i_{C_2}^* \mathrm{BP}^{(\!(C_4)\!)}\langle 2 \rangle) \longrightarrow C_4\text{-}\operatorname{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2 \rangle),$$

we can prove all the d_3, d_7, d_{15}, and d_{31}-differentials between $i_{C_2}^* \mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle$-truncation classes (pink classes).

The general argument goes as follows: suppose $tr(a)$ and $tr(b)$ are two $i_{C_2}^* \mathrm{BP}^{(\!(C_4)\!)}\langle 1 \rangle$-truncation classes on the E_r-page, and $d_r(a) = b$ in $C_2\text{-}\operatorname{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2 \rangle)$. We want to prove the differential $d_r(tr(a)) = tr(b)$ in $C_4\text{-}\operatorname{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2 \rangle)$. Since $d_r(a) = b$, $tr(b)$ must be killed by a differential of length at most r (natuality). Moreover, if $res(tr(a))$ and $res(tr(b))$ are both nonzero on the E_r-page, $d_r(a) = b$ implies $d_r(res(tr(a))) = res(tr(b))$. By natuality again, $tr(a)$ must support a differential of length at most r. In all the cases of interest, either our complete knowledge of all the shorter differentials (when $r = 3$, 7, and 15) or degree reasons will deduce our desired differential.

Convention 14.3. From now on, we will only specify the bidegrees and the name of their

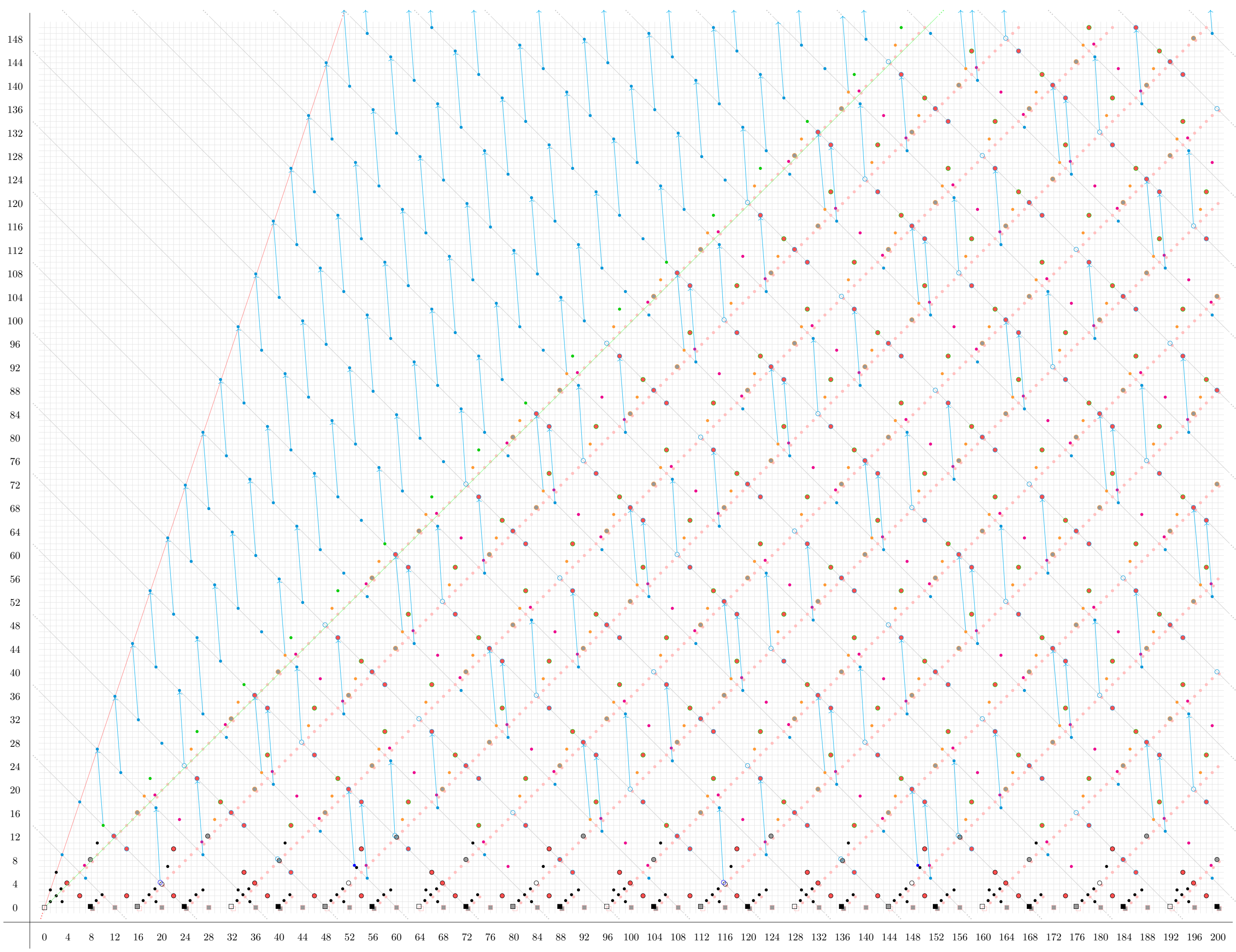


Figure 31: d_{13}-differentials in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

slice cells for $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes. This reduces cluttering of notations. For example, instead writing out the full name $\bar{s}_3^6\bar{r}_1^2u_{8\sigma_2}a_{12\sigma_2}$, we will write $\bar{s}_3^6\bar{r}_1^2$ at $(28,12)$ instead. It is unnecessary to write down their full names for computations and our convention improves the readability of our formulas.

Example 14.4. On the E_{15}-page, there are three classes at $(28,12)$ ($\bar{s}_3^6\bar{r}_1^2$, $\bar{\mathfrak{d}}_3\bar{s}_3^4\bar{r}_1^2$, $\bar{\mathfrak{d}}_3^2\bar{s}_3^2\bar{r}_1^2$) and five classes at $(27,27)$ ($\bar{s}_3^9$, $\bar{\mathfrak{d}}_3\bar{s}_3^7$, $\bar{\mathfrak{d}}_3^2\bar{s}_3^5$, $\bar{\mathfrak{d}}_3^3\bar{s}_3^3$, $\bar{\mathfrak{d}}_3^4\bar{s}_3$) coming from $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncations. In the C_2-slice spectral sequence, there are d_{15}-differentials

$$\begin{aligned}
d_{15}(\bar{s}_3^6\bar{r}_1^2) &= \bar{s}_3^6\bar{r}_1^2\cdot\bar{r}_1(\bar{\mathfrak{d}}_3+\bar{s}_3^2) = \bar{s}_3^6\bar{s}_3(\bar{\mathfrak{d}}_3+\bar{s}_3^2)\\
&= \bar{\mathfrak{d}}_3\bar{s}_3^6\bar{s}_3 + (\bar{\mathfrak{d}}_3\bar{s}_3^6\bar{s}_3+\bar{s}_3^6\bar{s}_3^3) = \bar{s}_3^6\bar{s}_3^3 = \bar{s}_3^9+\bar{\mathfrak{d}}_3^3\bar{s}_3^3,\\
d_{15}(\bar{\mathfrak{d}}_3\bar{s}_3^4\bar{r}_1^2) &= \bar{\mathfrak{d}}_3\bar{s}_3^4\bar{r}_1^2\cdot\bar{r}_1(\bar{\mathfrak{d}}_3+\bar{s}_3^2) = \bar{\mathfrak{d}}_3\bar{s}_3^4\bar{s}_3(\bar{\mathfrak{d}}_3+\bar{s}_3^2) = \bar{\mathfrak{d}}_3\bar{s}_3^4\cdot\bar{s}_3^3 = \bar{\mathfrak{d}}_3\bar{s}_3^7+\bar{\mathfrak{d}}_3^4\bar{s}_3,\\
d_{15}(\bar{\mathfrak{d}}_3^2\bar{s}_3^2\bar{r}_1^2) &= \bar{\mathfrak{d}}_3^2\bar{s}_3^2\bar{r}_1^2\cdot\bar{r}_1(\bar{\mathfrak{d}}_3+\bar{s}_3^2) = \bar{\mathfrak{d}}_3^2\bar{s}_3^2\bar{s}_3(\bar{\mathfrak{d}}_3+\bar{s}_3^2) = \bar{\mathfrak{d}}_3^2\bar{s}_3^2\cdot\bar{s}_3^3 = \bar{\mathfrak{d}}_3^2\bar{s}_3^5+\bar{\mathfrak{d}}_3^4\bar{s}_3.
\end{aligned}$$

This implies that the three classes $\bar{s}_3^6\bar{r}_1^2$, $\bar{\mathfrak{d}}_3\bar{s}_3^4\bar{r}_1^2$, $\bar{\mathfrak{d}}_3^2\bar{s}_3^2\bar{r}_1^2$ all support differentials of length at most 15 in the C_4-slice spectral sequence. Since we have complete knowledge of all the shorter differentials, the d_{15}-differentials above must occur.

Alternatively, we can use the transfer. The first differential can be rewritten as

$$d_{15}(tr(\bar{r}_3^6\bar{r}_1^2)) = tr(\bar{r}_3^9+\bar{r}_3^6\gamma\bar{r}_3^3).$$

In the C_2-slice spectral sequence, we have the d_{15}-differential

$$\begin{aligned} d_{15}(\bar{r}_3^6\bar{r}_1^2) &= \bar{r}_3^6\bar{r}_1^2 \cdot \bar{r}_1(\bar{\mathfrak{d}}_3 + \bar{s}_3^2) \\ &= \bar{r}_3^6\bar{s}_3(\bar{\mathfrak{d}}_3 + \bar{s}_3^2) \\ &= \bar{\mathfrak{d}}_3\bar{s}_3\bar{r}_3^6 + (\bar{\mathfrak{d}}_3\bar{s}_3\bar{r}_3^6 + \bar{r}_3^6\bar{s}_3^3) \\ &= \bar{r}_3^6\bar{s}_3^3 \\ &= \bar{r}_3^9 + \bar{r}_3^6\gamma\bar{r}_3^3. \end{aligned}$$

Applying the transfer shows that the class $tr(\bar{r}_3^9 + \bar{r}_3^6\gamma\bar{r}_3^3)$ must be killed by a differential of length at most 15. Our knowledge of the previous differentials again proves the desired differential. The other two differentials above can be proved in the same way by using the transfer.

The formulas in Section 12 describe explicitly the surviving $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes on each page. The d_3-differentials introduce the relation $\bar{r}_1 = \gamma\bar{r}_1$ for $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes with filtrations at least 3. After the d_3-differentials, their slice cells can all be written as

$$\bar{\mathfrak{d}}_3^i\bar{s}_3^j\bar{r}_1^k,$$

where $j > 0$.

The d_7-differentials introduce the relation $\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3 = 0$ for classes with filtrations at least 7. In other words, $\bar{r}_1^3 = \bar{s}_3$ for $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes with filtrations at least 7. After the d_7-differentials, their corresponding slice cells can all be written as

$$\bar{\mathfrak{d}}_3^i\bar{s}_3^j\bar{r}_1^k,$$

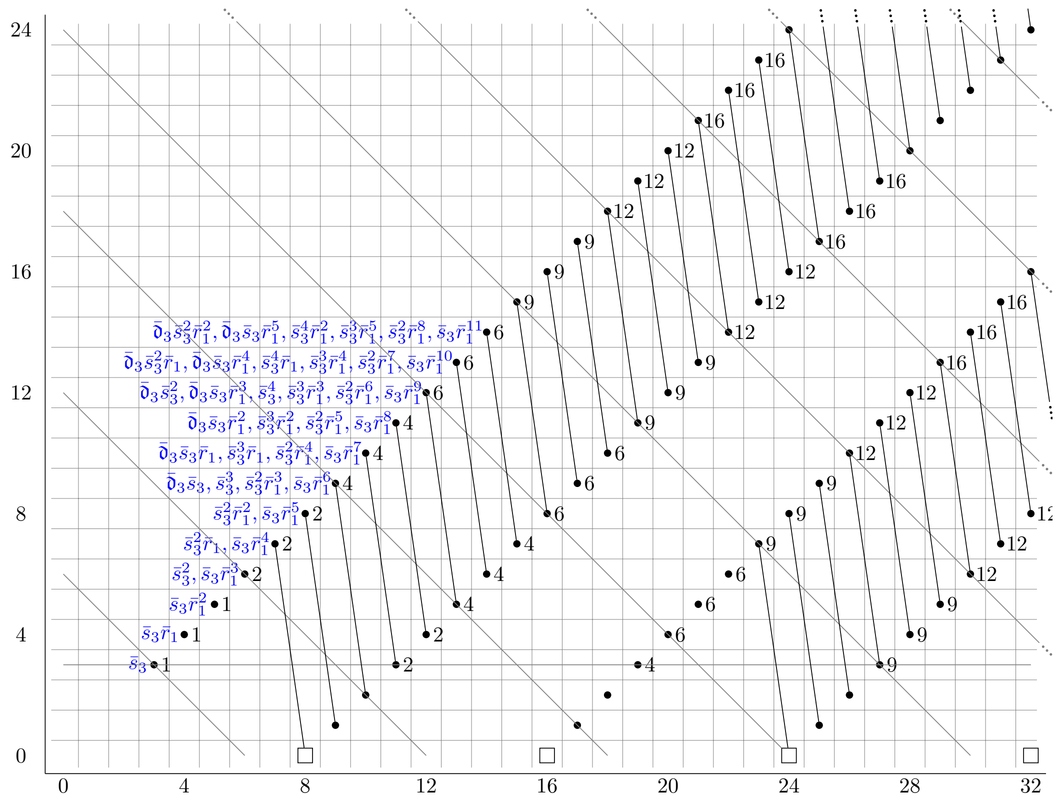


Figure 32: d_7-differentials between $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes.

where $j > 0$ and $0 \leq k \leq 2$. Figure 32 shows the d_7-differentials between $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes.

Proposition 14.5. *After the d_{15}-differentials between $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes, the following relations hold for the classes in filtrations at least 15:*

1. $\bar{s}_3^3\bar{r}_1 = \bar{s}_3^3\bar{r}_1^2 = 0$;
2. $\bar{s}_3^{6m+1} = \bar{\mathfrak{d}}_3^{3m}\bar{s}_3$ *for all* $m \geq 0$;
3. $\bar{s}_3^{6m+2} = \bar{\mathfrak{d}}_3^{3m}\bar{s}_3^2$ *for all* $m \geq 0$;

4. $\bar{s}_3^{6m+3} = \bar{\mathfrak{d}}_3^{3m}\bar{s}_3^3$ *for all* $m \geq 0$*;*

5. $\bar{s}_3^{6m+4} = \bar{\mathfrak{d}}_3^{3m+1}\bar{s}_3^2$ *for all* $m \geq 0$*;*

6. $\bar{s}_3^{6m+5} = \bar{\mathfrak{d}}_3^{3m+2}\bar{s}_3$ *for all* $m \geq 0$*;*

7. $\bar{s}_3^{6m} = 2\bar{\mathfrak{d}}_3^{3m}$ *for all* $m \geq 1$*.*

Proof. The d_{15}-differential in the C_2-slice spectral sequence multiplies the slice cell of the source by $\bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2) = \bar{r}_1(\bar{\mathfrak{d}}_3 + \bar{s}_3)$.

(1) We have the equality

$$\bar{r}_3^3 + \gamma\bar{r}_3^3 = (\bar{r}_3 + \gamma\bar{r}_3)(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2) = \bar{s}_3(\bar{\mathfrak{d}}_3 + \bar{s}_3^2).$$

Therefore,

$$\begin{aligned} res(\bar{s}_3^3\bar{r}_1) &= (\bar{r}_3^3 + \gamma\bar{r}_3^3)\bar{r}_1 \\ &= (\bar{r}_3 + \gamma\bar{r}_3)\cdot\bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2) \\ &= res(tr(\bar{r}_3))\cdot\bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2). \end{aligned}$$

Consider the class $\bar{r}_3 u_{8\sigma_2}$ in the C_2-spectral sequence. It supports the d_{15}-differential

$$\begin{aligned} d_{15}(\bar{r}_3 u_{8\sigma_2}) &= \bar{r}_3\cdot\bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2)a_{15\sigma_2} \\ &= (\bar{r}_3^3\bar{r}_1 + \bar{r}_3^2\gamma\bar{r}_3\bar{r}_1 + \bar{r}_3\gamma\bar{r}_3^2\bar{r}_1)a_{15\sigma_2} \\ &= (\bar{r}_3^3\bar{r}_1 + res(\bar{\mathfrak{d}}_3\bar{s}_3\bar{r}_1))a_{15\sigma_2}. \end{aligned}$$

Applying the transfer to this d_{15}-differential and using natuality implies the d_{15}-differential

$$d_{15}(\bar{s}_3) = tr(\bar{r}_3^3 \bar{r}_1 a_{15\sigma_2}) + tr(res(\bar{\mathfrak{d}}_3 \bar{s}_3 \bar{r}_1 a_{15\sigma_2})) = \bar{s}_3^3 \bar{r}_1$$

in the C_4-slice spectral sequence. Therefore, $\bar{s}_3^3 \bar{r}_1 = 0$ after the d_{15}-differentials.

For $\bar{s}_3^3 \bar{r}_1^2$, the proof is exactly the same. The exact same argument as above shows the d_{15}-differential

$$d_{15}(\bar{s}_3 \bar{r}_1) = \bar{s}_3^3 \bar{r}_1^2$$

in the C_4-slice spectral sequence.

(2) The statement holds trivially when $m = 0$. When $m \geq 1$, we have the equality $\bar{r}_3^{6m+1} + \bar{r}_3^{6m-2} \gamma \bar{r}_3^3 = \bar{r}_3^{6m-2}(\bar{r}_3^3 + \gamma \bar{r}_3^3)$. This implies the d_{15}-differential

$$d_{15}(\bar{r}_3^{6m-2} \bar{r}_1^2 u_{8\sigma_2}) = (\bar{r}_3^{6m+1} + \bar{r}_3^{6m-2} \gamma \bar{r}_3^3) a_{15\sigma_2}$$

in the C_2-slice spectral sequence. Applying the transfer and using natuality, we obtain the d_{15}-differential

$$d_{15}(\bar{s}_3^{6m-2} \bar{r}_1^2) = tr(\bar{r}_3^{6m+1}) + tr(\bar{r}_3^{6m-2} \gamma \bar{r}_3^3) = \bar{s}_3^{6m+1} + \bar{\mathfrak{d}}_3^3 \bar{s}_3^{6m-5}$$

in the C_4-slice spectral sequence. This produces the relation

$$\bar{s}_3^{6m+1} = \bar{\mathfrak{d}}_3^3 \bar{s}_3^{6m-5}$$

for all $m \geq 1$. Induction on m proves the desired equality.

(3) The statement holds trivially when $m = 0$. When $m \geq 1$, we have the equality

$$\bar{r}_3^{6m+2} + \bar{r}_3^{6m-1}\gamma\bar{r}_3^3 = \bar{r}_3^{6m-1}(\bar{r}_3^3 + \gamma\bar{r}_3^3).$$

This implies the d_{15}-differential

$$d_{15}(\bar{r}_3^{6m-1}\bar{r}_1^2 u_{8\sigma_2}) = (\bar{r}_3^{6m+2} + \bar{r}_3^{6m-1}\gamma\bar{r}_3^3)a_{15\sigma_2}$$

in the C_2-slice spectral sequence. Applying the transfer and using natuality produces the d_{15}-differential

$$d_{15}(\bar{s}_3^{6m-1}\bar{r}_1^2) = tr(\bar{r}_3^{6m+2}) + tr(\bar{r}_3^{6m-1}\gamma\bar{r}_3^3) = \bar{s}_3^{6m+2} + \bar{\mathfrak{d}}_3^3\bar{s}_3^{6m-4}$$

in the C_4-slice spectral sequence. Induction on m proves the desired equality.

(4) The statement holds trivially when $m = 0$. When $m \geq 1$, we have the equality

$$\bar{r}_3^{6m+3} + \bar{r}_3^{6m}\gamma\bar{r}_3^3 = \bar{r}_3^{6m}(\bar{r}_3^3 + \gamma\bar{r}_3^3).$$

This implies the d_{15}-differential

$$d_{15}(\bar{r}_3^{6m}\bar{r}_1^2 u_{8\sigma_2}) = (\bar{r}_3^{6m+3} + \bar{r}_3^{6m}\gamma\bar{r}_3^3)a_{15\sigma_2}$$

in the C_2-slice spectral sequence. Applying the transfer and using natuality produces the d_{15}-differential

$$d_{15}(\bar{s}_3^{6m}\bar{r}_1^2) = tr(\bar{r}_3^{6m+3}) + tr(\bar{r}_3^{6m}\gamma\bar{r}_3^3) = \bar{s}_3^{6m+3} + \bar{\mathfrak{d}}_3^3\bar{s}_3^{6m-3}$$

in the C_4-slice spectral sequence. Induction on m proves the desired equality.

(5) We have the equality

$$\bar{r}_3^{6m+4} + \bar{r}_3^{6m+1}\gamma\bar{r}_3^3 = \bar{r}_3^{6m+1}(\bar{r}_3^3 + \gamma\bar{r}_3^3).$$

This implies the d_{15}-differential

$$d_{15}(\bar{r}_3^{6m+1}\bar{r}_1^2 u_{8\sigma_2}) = (\bar{r}_3^{6m+4} + \bar{r}_3^{6m+1}\gamma\bar{r}_3^3)a_{15\sigma_2}$$

in the C_2-slice spectral sequence. Applying the transfer and using natuality produces the d_{15}-differential

$$d_{15}(\bar{s}_3^{6m+1}\bar{r}_1^2) = tr(\bar{r}_3^{6m+4}) + tr(\bar{r}_3^{6m+1}\gamma\bar{r}_3^3)$$

in the C_4-slice spectral sequence. When $m = 0$, the target is $\bar{s}_3^4 + \bar{\mathfrak{d}}_3\bar{s}_3^2$, from which we get the relation $\bar{s}_3^4 = \bar{\mathfrak{d}}_3\bar{s}_3^2$. For $m \geq 1$, the target is $\bar{s}_3^{6m+4} + \bar{\mathfrak{d}}_3^3\bar{s}_3^{6m-2}$, from which we get the relation $\bar{s}_3^{6m+4} = \bar{\mathfrak{d}}_3^3\bar{s}_3^{6m-2}$. Induction on m proves the desired equality.

(6) We have the equality

$$\bar{r}_3^{6m+5} + \bar{r}_3^{6m+2}\gamma\bar{r}_3^3 = \bar{r}_3^{6m+2}(\bar{r}_3^3 + \gamma\bar{r}_3^3).$$

This implies the d_{15}-differential

$$d_{15}(\bar{r}_3^{6m+2}\bar{r}_1^2 u_{8\sigma_2}) = (\bar{r}_3^{6m+5} + \bar{r}_3^{6m+2}\gamma\bar{r}_3^3)a_{15\sigma_2}$$

in the C_2-slice spectral sequence. Applying the transfer and using natuality produces the

d_{15}-differential

$$d_{15}(\bar{s}_3^{6m+2}\bar{r}_1^2) = tr(\bar{r}_3^{6m+5}) + tr(\bar{r}_3^{6m+2}\gamma\bar{r}_3^3)$$

in the C_4-slice spectral sequence. When $m = 0$, the target is $\bar{s}_3^5 + \bar{\mathfrak{d}}_3^2\bar{s}_3$, from which we get the relation $\bar{s}_3^5 = \bar{\mathfrak{d}}_3^2\bar{s}_3$. For $m \geq 1$, the target is $\bar{s}_3^{6m+5} + \bar{\mathfrak{d}}_3^3\bar{s}_3^{6m-1}$, from which we get the relation $\bar{s}_3^{6m+5} = \bar{\mathfrak{d}}_3^3\bar{s}_3^{6m-1}$. Induction on m proves the desired equality.

(7) Since

$$\bar{r}_3^{6m} + \bar{r}_3^{6m-3}\gamma\bar{r}_3^3 = \bar{r}_3^{6m-3}(\bar{r}_3^3 + \gamma\bar{r}_3^3),$$

there is the d_{15}-differential

$$d_{15}(\bar{r}_3^{6m-3}\bar{r}_1^2 u_{8\sigma_2}) = \bar{r}_3^{6m-3}(\bar{r}_3^3 + \gamma\bar{r}_3^3)a_{15\sigma_2}$$

in the C_2-slice spectral sequence. Applying the transfer and using natuality produces the d_{15}-differential

$$d_{15}(\bar{s}_3^{6m-3}\bar{r}_1^2) = tr(\bar{r}_3^{6m}) + tr(\bar{r}_3^{6m-3}\gamma\bar{r}_3^3) = \bar{s}_3^{6m} + tr(\bar{r}_3^{6m-3}\gamma\bar{r}_3^3)$$

in the C_4-slice spectral sequence.

We will now use induction on m. When $m = 1$, the target is $\bar{s}_3^6 + 2\bar{\mathfrak{d}}_3^3$, from which we deduce $\bar{s}_3^6 = 2\bar{\mathfrak{d}}_3^3$. When $m > 1$, the target is $\bar{s}_3^{6m} + \bar{\mathfrak{d}}_3^3\bar{s}_3^{6m-6}$, from which we deduce $\bar{s}_3^{6m} = \bar{\mathfrak{d}}_3^3\bar{s}_3^{6m-6}$. Induction on m shows that $\bar{s}_3^{6m} = 2\bar{\mathfrak{d}}_3^{3m}$.

□

Warning 14.6. The class $\bar{s}_3^3$ is not 0 after the d_{15}-differentials between $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes. In particular, the classes $\bar{\mathfrak{d}}_3\bar{s}_3^3$ at $(15, 15)$, $\bar{\mathfrak{d}}_3^2\bar{s}_3^3$ at $(21, 21)$, $\bar{\mathfrak{d}}_3^3\bar{s}_3^3$ at $(27, 27)$, ... are not targets of d_{15}-differentials with sources coming from $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-

truncation classes. However, some of these classes ($\bar{\mathfrak{d}}_3^3\bar{s}_3^3 \in (27,27)$ and $\bar{\mathfrak{d}}_3^7\bar{s}_3^3 \in (51,51)$, for example) are still targets of d_{15}-differentials with sources coming from $\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes. We will discuss this in the next subsection.

Figures 33 and 34 illustrate the d_{15}-differentials between $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes.

14.3 All the other d_{15}-differentials and some d_{31}-differentials.

We will now prove the rest of the d_{15}-differentials (see Figure 35).

Proposition 14.7. *The class $2\bar{\mathfrak{d}}_3^2 u_{4\lambda}u_{6\sigma}a_{2\lambda}$ at $(20,4)$ supports the d_{15}-differential*

$$d_{15}(2\bar{\mathfrak{d}}_3^2 u_{4\lambda}u_{6\sigma}a_{2\lambda}) = \bar{\mathfrak{d}}_3^3\bar{s}_1 a_{9\lambda}a_{\sigma_2}.$$

(Under our naming convention, the target is abbreviated as $\bar{\mathfrak{d}}_3^3\bar{s}_1$ at $(19,19)$).

Proof. In the C_2-slice spectral sequence, the restriction of the class $\bar{\mathfrak{d}}_3^2 u_{4\lambda}u_{6\sigma}a_{2\lambda}$ at $(20,4)$ supports the d_{15}-differential

$$\begin{aligned} d_{15}(res(\bar{\mathfrak{d}}_3^2 u_{4\lambda}u_{6\sigma}a_{2\lambda})) &= \bar{r}_3^2\gamma\bar{r}_3^2\cdot\bar{r}_1(\bar{r}_3^2+\bar{r}_3\gamma\bar{r}_3+\gamma\bar{r}_3^2) \\ &= \bar{\mathfrak{d}}_3^2\bar{r}_1(\bar{\mathfrak{d}}_3+\bar{s}_3^2) \\ &= \bar{\mathfrak{d}}_3^3\bar{r}_1+\bar{\mathfrak{d}}_3^2\bar{s}_3^2\bar{r}_1 \\ &= \bar{\mathfrak{d}}_3^3\bar{r}_1+res(tr(\bar{\mathfrak{d}}_3^2\bar{r}_3^2\bar{r}_1)). \end{aligned}$$

Applying the transfer map shows that the class

$$tr(\bar{\mathfrak{d}}_3^3\bar{r}_1)+tr(res(tr(\bar{\mathfrak{d}}_3^2\bar{r}_3^2\bar{r}_1))) = \bar{\mathfrak{d}}_3^3\bar{s}_1+2\cdot tr(\bar{\mathfrak{d}}_3^2\bar{r}_3^2\bar{r}_1) = \bar{\mathfrak{d}}_3^3\bar{s}_1$$

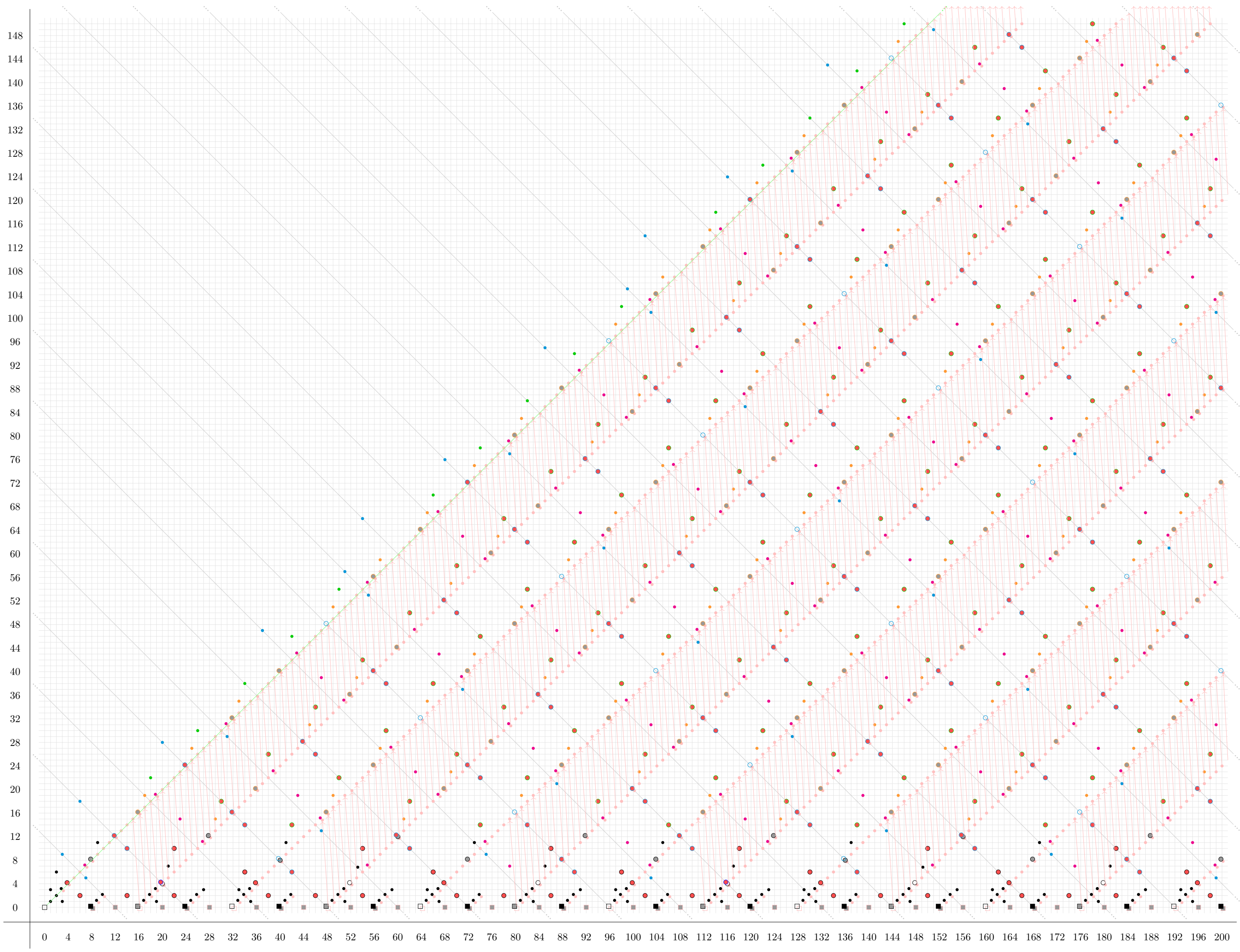


Figure 33: d_{15}-differentials between $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes.

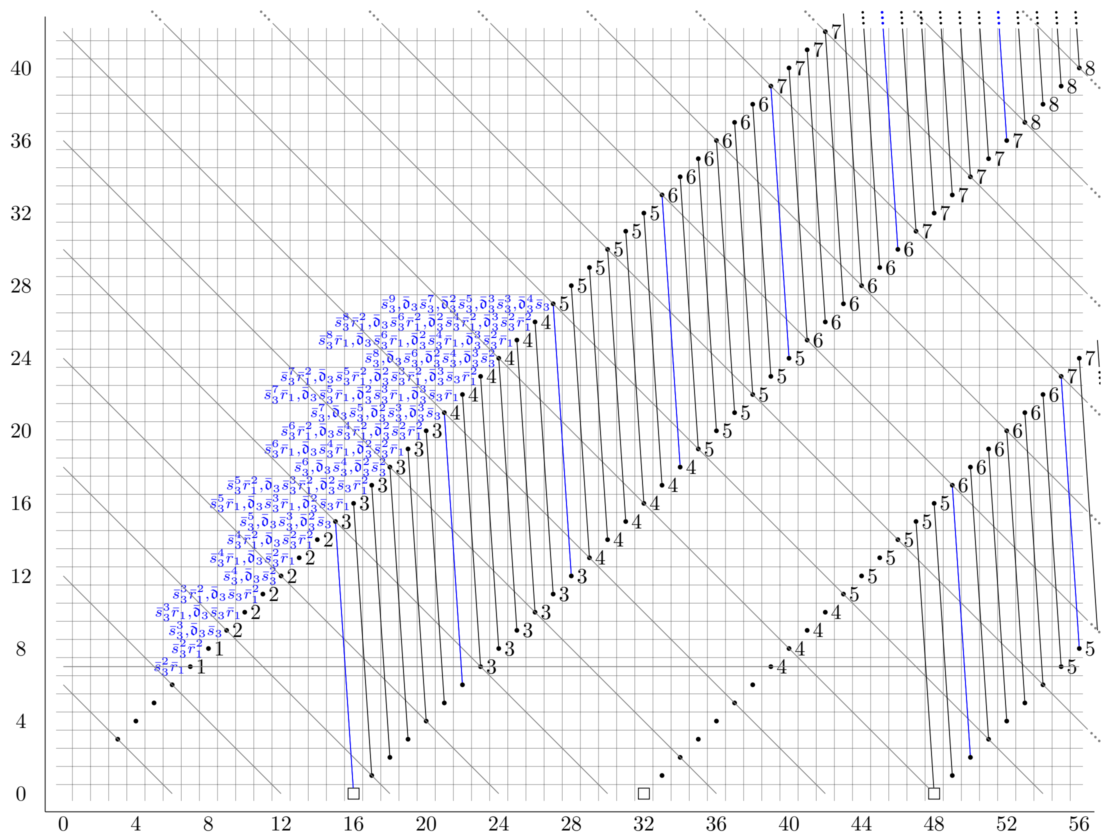


Figure 34: d_{15}-differentials between $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes. The targets of the blue differentials have two surviving classes instead of one.

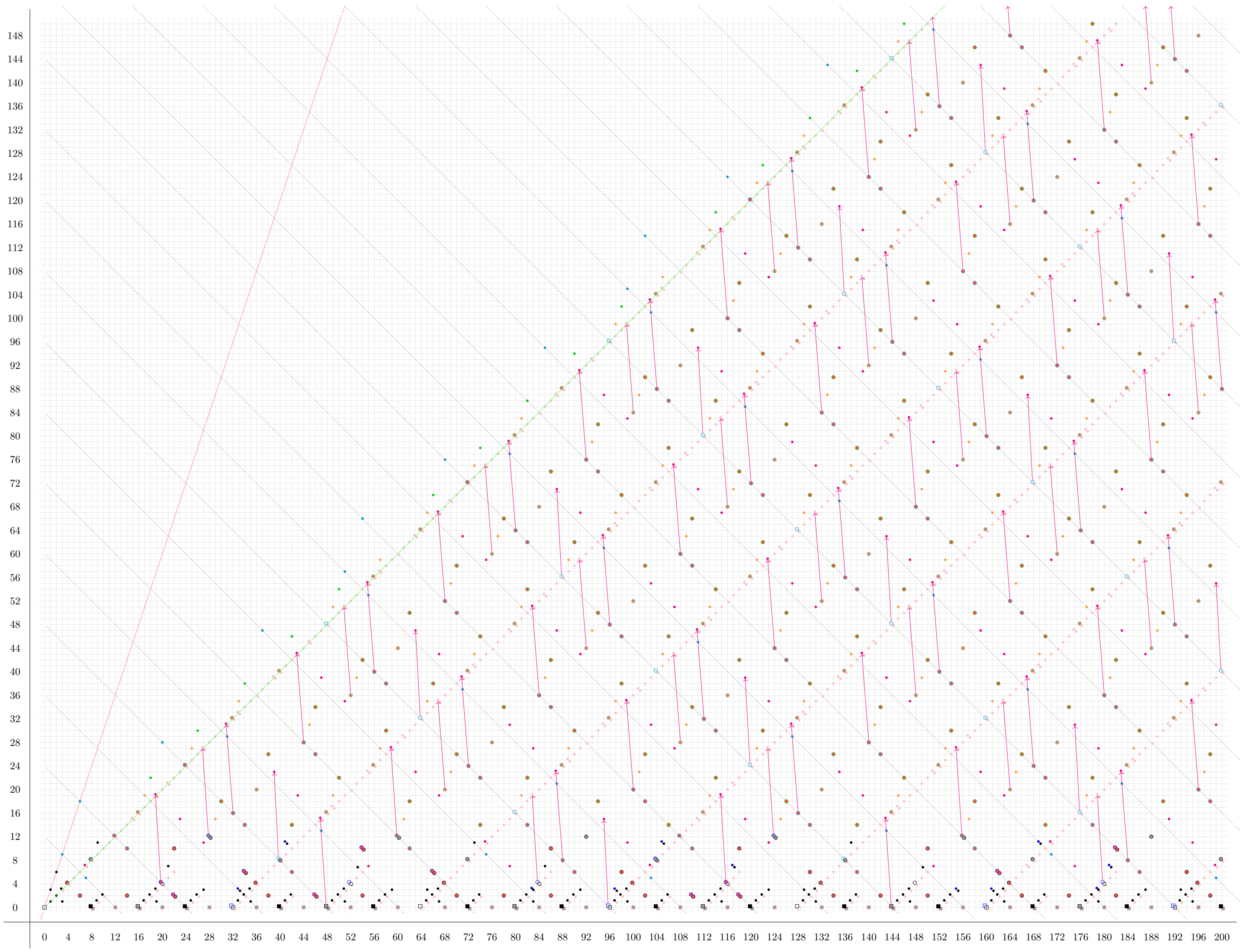


Figure 35: The rest of the d_{15}-differentials.

must be killed by a differential of length at most 15. By degree reasons, the differential must of length 15, and the source must be

$$tr(res(\bar{\mathfrak{d}}_3^2 u_{4\lambda} u_{6\sigma} a_{2\lambda})) = 2\bar{\mathfrak{d}}_3^2 u_{4\lambda} u_{6\sigma} a_{2\lambda}$$

by natuality. This proves the desired differential. □

Using the exact same method as the proof of Proposition 14.7, we can prove d_{15}-differentials on the following classes:

1. $\{(20,4),(32,16),(44,28),\ldots\}$, $\{(116,4),(128,16),(140,28),\ldots\}$, ….
2. $\{(48,0),(60,12),(72,24),\ldots\}$, $\{(144,0),(156,12),(168,24),\ldots\}$, ….
3. $\{(88,8),(100,20),(112,32),\ldots\}$, $\{(184,8),(196,20),(208,32),\ldots\}$, ….

Remark 14.8. The restrictions of the classes $\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ and $\bar{\mathfrak{d}}_1^6 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ at $(20,4)$ support the following d_{15}-differentials in the C_2-spectral sequence:

$$\begin{aligned} d_{15}(\bar{r}_3\gamma\bar{r}_3\bar{r}_1^6) &= \bar{r}_3\gamma\bar{r}_3\bar{r}_1^6 \cdot \bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2) = \bar{\mathfrak{d}}_3\bar{s}_3^2 \cdot \bar{r}_1(\bar{s}_3^2 + \bar{\mathfrak{d}}_3) = (\bar{\mathfrak{d}}_3^2\bar{s}_3^2 + \bar{\mathfrak{d}}_3\bar{s}_3^4)\bar{r}_1, \\ d_{15}(\bar{r}_1^{12}) &= \bar{r}_1^{12} \cdot \bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2) = \bar{s}_3^4 \cdot \bar{r}_1(\bar{s}_3^2 + \bar{\mathfrak{d}}_3) = (\bar{\mathfrak{d}}_3\bar{s}_3^4 + \bar{\mathfrak{d}}_3^2\bar{s}_3^2 + \bar{s}_3^6)\bar{r}_1. \end{aligned}$$

(In the formulas above, we used the relation $\bar{r}_1^3 = \bar{r}_3 + \gamma\bar{r}_3 = \bar{s}_3$.) By natuality and degree reasons, there exist d_{15}-differentials

$$d_{15}(\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3) = (\bar{\mathfrak{d}}_3^2\bar{s}_3^2 + \bar{\mathfrak{d}}_3\bar{s}_3^4)\bar{r}_1$$

and

$$d_{15}(\bar{\mathfrak{d}}_1^6) = (\bar{\mathfrak{d}}_3\bar{s}_3^4 + \bar{\mathfrak{d}}_3^2\bar{s}_3^2 + \bar{s}_3^6)\bar{r}_1$$

in the C_4-slice spectral sequence.

We also have the following d_{15}-differentials on the classes $\bar{\mathfrak{d}}_3\bar{s}_3^2$ and $\bar{s}_3^4$ at $(20, 4)$:

$$\begin{aligned} d_{15}(\bar{\mathfrak{d}}_3\bar{s}_3^2) &= (\bar{\mathfrak{d}}_3^2\bar{s}_3^2 + \bar{\mathfrak{d}}_3\bar{s}_3^4)\bar{r}_1, \\ d_{15}(\bar{s}_3^4) &= (\bar{\mathfrak{d}}_3\bar{s}_3^4 + \bar{\mathfrak{d}}_3^2\bar{s}_3^2 + \bar{s}_3^6)\bar{r}_1. \end{aligned}$$

After the d_{15}-differentials, the surviving classes at $(20, 4)$ are $2\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3$, $2\bar{\mathfrak{d}}_1^6$, $\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3 + \bar{\mathfrak{d}}_3\bar{s}_3^2$, and $\bar{\mathfrak{d}}_1^6 + \bar{s}_3^4$. There is a slight subtlety here because from the way we are organizing the d_{15}-differentials, the surviving leading terms should be $\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3$, $2\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^3$, $\bar{\mathfrak{d}}_1^6$, and $2\bar{\mathfrak{d}}_1^6$. Our presentation of the surviving classes will not affect later computations.

Proposition 14.9. *The class* $\bar{\mathfrak{d}}_3^3\bar{\mathfrak{d}}_1 u_{4\lambda}u_{10\sigma}a_{6\lambda}$ *at* $(28, 12)$ *supports the* d_{15}*-differential*

$$d_{15}(\bar{\mathfrak{d}}_3^3\bar{\mathfrak{d}}_1 u_{4\lambda}u_{10\sigma}a_{6\lambda}) = \bar{\mathfrak{d}}_3^3\bar{s}_3^3 a_{9\lambda}a_{9\sigma_2}.$$

Proof. The restriction of the class $\bar{\mathfrak{d}}_3^3\bar{\mathfrak{d}}_1 u_{4\lambda}u_{10\sigma}a_{6\lambda}$ supports the d_{15}-differential

$$\begin{aligned} d_{15}(res(\bar{\mathfrak{d}}_3^3\bar{\mathfrak{d}}_1 u_{4\lambda}a_{10\sigma}a_{6\lambda})) &= \bar{\mathfrak{d}}_3^3\bar{r}_1^2 \cdot \bar{r}_1(\bar{\mathfrak{d}}_3 + \bar{s}_3^2) \\ &= \bar{\mathfrak{d}}_3^3\bar{s}_3(\bar{\mathfrak{d}}_3 + \bar{s}_3^2) \\ &= \bar{\mathfrak{d}}_3^4\bar{s}_3 + \bar{\mathfrak{d}}_3^4\bar{s}_3 + \bar{\mathfrak{d}}_3^3\bar{s}_3^3 \\ &= \bar{\mathfrak{d}}_3^3\bar{s}_3^3 \end{aligned}$$

in the C_2-slice spectral sequence. This implies that the class $\bar{\mathfrak{d}}_3^3\bar{\mathfrak{d}}_1 u_{4\lambda}u_{10\sigma}a_{6\lambda}$ must support a differential of length at most 15 in the C_4-slice spectral sequence. For degree reasons, it must support a d_{15}-differential, and the target must be $\bar{\mathfrak{d}}_3^3\bar{s}_3^3$ by natuality. □

The proof of Proposition 14.9 can be used to prove d_{15}-differentials on the following classes:

1. $\{(28,12),(52,36),(76,60),\ldots\}$, $\{(124,12),(148,36),(172,60),\ldots\}$, $\ldots$;

2. $\{(68,20),(92,44),(116,68),\ldots\}$, $\{(164,20),(188,44),(212,68),\ldots\}$, $\ldots$;

3. $\{(84,4),(108,28),(132,52),\ldots\}$, $\{(180,4),(204,28),(228,52),\ldots\}$, $\ldots$.

Proposition 14.10. *The class* $\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda}$ *at* $(40,8)$ *supports the* d_{15}*-differential*

$$d_{15}(\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda}) = \bar{\mathfrak{d}}_3^5 \bar{s}_1 u_{4\lambda}u_{15\sigma}a_{11\lambda}a_{\sigma_2}.$$

Proof. In the C_2-slice spectral sequence, the restriction of $\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda}$ supports the d_{31}-differential

$$d_{31}(res(\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda})) = \bar{\mathfrak{d}}_3^4 \cdot (\bar{\mathfrak{d}}_3 \bar{r}_3^3) = \bar{\mathfrak{d}}_3^5 \bar{r}_3^3.$$

This implies that the class $\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda}$ must support a differential of length at most 31 in the C_4-slice spectral sequence. By degree reasons, the target can either be at $(39,39)$ (d_{31}-differential) or at $(39,23)$ (d_{15}-differential).

There are two classes at $(39,39)$ — $\bar{\mathfrak{d}}_3^6 \bar{s}_3$ and $\bar{\mathfrak{d}}_3^5 \bar{s}_3^3$. Since neither class restricts to $\bar{\mathfrak{d}}_3^5 \bar{r}_3^3$, the target cannot be at $(39,39)$ by natuality. The only possibility left is the class $\bar{\mathfrak{d}}_3^5 \bar{s}_1$ at $(39,23)$. This is the desired differential. □

Proposition 14.11. *The class* $2\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda}$ *at* $(40,8)$ *supports the* d_{31}*-differential*

$$d_{31}(2\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda}) = \bar{\mathfrak{d}}_3^5 \bar{s}_3^3 a_{39\sigma_2}.$$

Proof. As in the proof of Proposition 14.10, we have the differential

$$d_{31}(res(\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda})) = \bar{\mathfrak{d}}_3^4 \cdot (\bar{\mathfrak{d}}_3 \bar{r}_3^3) = \bar{\mathfrak{d}}_3^5 \bar{r}_3^3$$

in the C_2-spectral sequence. Applying the transfer to the target of this differential shows that the class $tr(\bar{\mathfrak{d}}_3^5\bar{r}_3^3) = \bar{\mathfrak{d}}_3^5\bar{s}_3^3$ must be killed in the C_4-slice spectral sequence by a differential of length at most 31.

For degree reasons, the target can only be killed by a differential of length 31. Therefore, by natuality, the source must be

$$tr(res(\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda})) = 2\bar{\mathfrak{d}}_3^4 u_{8\lambda}u_{12\sigma}a_{4\lambda}.$$

□

The proofs of Proposition 14.10 and 14.11 can be used to prove d_{15} and d_{31}-differentials on the following classes:

1. $\{(40,8),(64,32),(88,56),\ldots\}$, $\{(232,8),(256,32),(280,56),\ldots\}$, $\ldots$;
2. $\{(96,0),(120,24),(144,48),\ldots\}$, $\{(288,0),(312,24),(336,48),\ldots\}$, $\ldots$;
3. $\{(176,16),(200,40),(224,64),\ldots\}$, $\{(368,16),(392,40),(416,64),\ldots\}$, $\ldots$.

Remark 14.12. All the other classes at $(40,8)$ support d_{31}-differentials hitting the same target, $\bar{\mathfrak{d}}_3^6\bar{s}_3$. In the formulas below, we use the relation $\bar{r}_3^3 = \gamma\bar{r}_3^3$, which is produced by the d_{15}-differentials, as well as Proposition 14.5.

1. $d_{31}(res(\bar{\mathfrak{d}}_3^3\bar{\mathfrak{d}}_1^3)) = d_{31}(res(\bar{\mathfrak{d}}_3^3\bar{s}_3^2)) = \bar{\mathfrak{d}}_3^3\bar{s}_3^2\cdot(\bar{\mathfrak{d}}_3\bar{r}_3^3) = \bar{\mathfrak{d}}_3^4(\bar{r}_3^5 + \bar{r}_3^3\gamma\bar{r}_3^2) = \bar{\mathfrak{d}}_3^4\bar{s}_3^5 = \bar{\mathfrak{d}}_3^6\bar{s}_3$.
2. $d_{31}(res(\bar{\mathfrak{d}}_3^2\bar{\mathfrak{d}}_1^6)) = d_{31}(res(\bar{\mathfrak{d}}_3^2\bar{s}_3^4)) = \bar{\mathfrak{d}}_3^2\bar{s}_3^4\cdot(\bar{\mathfrak{d}}_3\bar{r}_3^3) = \bar{\mathfrak{d}}_3^3(\bar{r}_3^7 + \bar{r}_3^3\gamma\bar{r}_3^4) = \bar{\mathfrak{d}}_3^3\bar{s}_3^7 = \bar{\mathfrak{d}}_3^6\bar{s}_3$.
3. $d_{31}(res(\bar{\mathfrak{d}}_3\bar{\mathfrak{d}}_1^9)) = d_{31}(res(\bar{\mathfrak{d}}_3(\bar{s}_3)^6)) = \bar{\mathfrak{d}}_3\bar{s}_3^4\bar{s}_3^2\cdot(\bar{\mathfrak{d}}_3\bar{r}_3^3) = \bar{\mathfrak{d}}_3(\bar{\mathfrak{d}}_3^2\bar{s}_3^2 + \bar{s}_3^6)\cdot(\bar{\mathfrak{d}}_3\bar{r}_3^3) = \bar{\mathfrak{d}}_3^4\bar{s}_3^2\bar{r}_3^3 + \bar{\mathfrak{d}}_3^2\bar{s}_3^6\bar{r}_3^3 = \bar{\mathfrak{d}}_3^4\bar{s}_3^5 + \bar{\mathfrak{d}}_3^2\bar{s}_3^9 = \bar{\mathfrak{d}}_3^6\bar{s}_3 + 0 = \bar{\mathfrak{d}}_3^6\bar{s}_3$.
4. $d_{31}(res(\bar{\mathfrak{d}}_1^{12})) = d_{31}(res(\bar{s}_3^8)) = \bar{s}_3^8(\bar{\mathfrak{d}}_3\bar{r}_3^3) = \bar{\mathfrak{d}}_3(\bar{r}_3^{11} + \bar{r}_3^3\gamma\bar{r}_3^8) = \bar{\mathfrak{d}}_3\bar{s}_3^{11} = \bar{\mathfrak{d}}_3^6\bar{s}_3$.

This, combined with Proposition 14.11, shows that both remaining $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes ($\bar{\mathfrak{d}}_3^6\bar{s}_3$ and $\bar{\mathfrak{d}}_3^5\bar{s}_3^3$) at $(39,39)$ are killed by d_{31}-differentials. The same phenomenon occurs at the following bidegrees as well:

1. $\{(39,39),(63,63),(87,87),\ldots\}$, $\{(231,39),(255,63),(279,87),\ldots\}$, $\ldots$;
2. $\{(95,31),(119,55),(143,79),\ldots\}$, $\{(287,31),(311,55),(335,79),\ldots\}$, $\ldots$;
3. $\{(175,47),(199,71),(223,95),\ldots\}$, $\{(367,47),(391,71),(415,95),\ldots\}$, $\ldots$.

Now we turn to prove the nonexistence of d_{15}-differentials on classes.

Proposition 14.13. *There is no d_{15}-differential on the class $\bar{\mathfrak{d}}_3^4\bar{\mathfrak{d}}_1^2u_{4\lambda}u_{14\sigma}a_{10\lambda}$ at $(36,20)$. In other words, the class $\bar{\mathfrak{d}}_3^4\bar{\mathfrak{d}}_1^2u_{4\lambda}u_{14\sigma}a_{10\lambda}$ is the leading term of a d_{15}-cycle.*

Proof. The restriction of $\bar{\mathfrak{d}}_3^4\bar{\mathfrak{d}}_1^2u_{4\lambda}u_{14\sigma}a_{10\lambda}$ supports the d_{15}-differential

$$\begin{aligned} d_{15}(res(\bar{\mathfrak{d}}_3^4\bar{\mathfrak{d}}_1^2u_{4\lambda}u_{14\sigma}a_{10\lambda})) &= \bar{\mathfrak{d}}_3^4\bar{r}_1^4\cdot\bar{r}_1(\bar{\mathfrak{d}}_3+\bar{s}_3^2) \\ &= \bar{\mathfrak{d}}_3^4\bar{s}_3\bar{r}_1^2(\bar{\mathfrak{d}}_3+\bar{s}_3^2) \\ &= \bar{\mathfrak{d}}_3^5\bar{s}_3\bar{r}_1^2+(\bar{\mathfrak{d}}_3^5\bar{s}_3\bar{r}_1^2+\bar{\mathfrak{d}}_3^4\bar{s}_3^3\bar{r}_1^2) \\ &= \bar{\mathfrak{d}}_3^4\bar{s}_3^3\bar{r}_1^2. \end{aligned}$$

The class $\bar{\mathfrak{d}}_3^4\bar{s}_3^3\bar{r}_1^2$ is also killed by the d_{15}-differential supported by the $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation class $\bar{\mathfrak{d}}_3^4\bar{s}_3\bar{r}_1$:

$$\begin{aligned} d_{15}(\bar{\mathfrak{d}}_3^4\bar{s}_3\bar{r}_1) &= \bar{\mathfrak{d}}_3^4\bar{s}_3\bar{r}_1\cdot\bar{r}_1(\bar{\mathfrak{d}}_3+\bar{s}_3^2) \\ &= \bar{\mathfrak{d}}_3^5\bar{s}_3\bar{r}_1^2+(\bar{\mathfrak{d}}_3^5\bar{s}_3\bar{r}_1^2+\bar{\mathfrak{d}}_3^4\bar{s}_3^3\bar{r}_1^2) \\ &= \bar{\mathfrak{d}}_3^4\bar{s}_3^3\bar{r}_1^2. \end{aligned}$$

Technically, after these two d_{15}-differentials, the surviving class is $\bar{\mathfrak{d}}_3^4\bar{\mathfrak{d}}_1^2 + \bar{\mathfrak{d}}_3^4\bar{s}_3\bar{r}_1$. However, since the second differential has already been accounted for, there is no more d_{15}-differential on the class $\bar{\mathfrak{d}}_3^4\bar{\mathfrak{d}}_1^2 u_{4\lambda}u_{14\sigma}a_{10\lambda}$. □

The proof of Proposition 14.13 can be used to show that there are no d_{15}-differentials on the following classes:

1. $\{(36, 20), (60, 44), (84, 68), \ldots\}$, $\{(132, 20), (156, 44), (180, 68), \ldots\}$, $\ldots$;
2. $\{(52, 4), (76, 28), (100, 52), \ldots\}$, $\{(148, 4), (172, 28), (196, 52), \ldots\}$, $\ldots$;
3. $\{(92, 12), (116, 36), (140, 60), \ldots\}$, $\{(188, 12), (212, 36), (236, 60), \ldots\}$, $\ldots$.

Proposition 14.14. *There is no d_{15}-differential on the class $\bar{s}_3^3\bar{s}_1 u_{4\lambda}u_{9\sigma}a_{5\lambda}a_{\sigma_2}$ at $(27, 11)$.*

Proof. The class $\bar{s}_3^3\bar{s}_1 u_{4\lambda}u_{9\sigma}a_{5\lambda}a_{\sigma_2}$ is in the image of the transfer because $\bar{\mathfrak{d}}_3^3\bar{s}_1 = tr(\bar{\mathfrak{d}}_3^3\bar{r}_1)$. In the C_2-slice spectral sequence, the class $\bar{\mathfrak{d}}_3^3\bar{r}_1$ supports the d_{15}-differential

$$d_{15}(\bar{\mathfrak{d}}_3^3\bar{r}_1) = \bar{\mathfrak{d}}_3^3\bar{r}_1 \cdot \bar{r}_1(\bar{\mathfrak{d}}_3 + \bar{s}_3^2) = \bar{\mathfrak{d}}_3^4\bar{r}_1^2 + \bar{\mathfrak{d}}_3^3\bar{s}_3^2\bar{r}_1^2.$$

If the class $\bar{\mathfrak{d}}_3^3\bar{s}_1$ does support a d_{15}-differential, then by natuality, the target must be

$$tr(\bar{\mathfrak{d}}_3^4\bar{r}_1^2 + \bar{\mathfrak{d}}_3^3\bar{s}_3^2\bar{r}_1^2) = tr(\bar{\mathfrak{d}}_3^4\bar{r}_1^2) + tr(\bar{\mathfrak{d}}_3^3\bar{s}_3^2\bar{r}_1^2) = \bar{\mathfrak{d}}_3^4\bar{s}_1^2 + 0 = \bar{\mathfrak{d}}_3^4\bar{s}_1^2.$$

This is impossible because the class $\bar{\mathfrak{d}}_3^4\bar{s}_1^2$ is killed by a d_3-differential and no longer exists on the d_{15}-page (the only class left at $(26, 26)$ is $\bar{\mathfrak{d}}_3^3\bar{s}_3^2$). □

The proof of Proposition 14.14 can be used to show that there are no d_{15}-differentials on the following classes:

1. $\{(27, 11), (39, 23), (51, 35), \ldots\}$, $\{(123, 11), (135, 23), (147, 35), \ldots\}$, $\ldots$;

2. $\{(55,7),(67,19),(79,31),\ldots\}$, $\{(151,7),(163,19),(175,31),\ldots\}$, $\ldots$;

3. $\{(83,3),(95,15),(107,27),\ldots\}$, $\{(179,3),(191,15),(203,27),\ldots\}$, $\ldots$.

15 Higher Differentials II: the Norm

In this section, we will use Theorem 9.8 to "norm up" differentials in C_2-SliceSS($BP^{(\!(C_4)\!)}\langle 2\rangle$) to differentials in C_4-SliceSS($BP^{(\!(C_4)\!)}\langle 2\rangle$).

Recall that in the C_2-slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2\rangle$, all the differentials are generated under multiplication by the following differentials:

$$\begin{aligned} d_3(u_{2\sigma_2}) &= \bar{v}_1 a_{3\sigma_2} = (\bar{r}_1 + \gamma\bar{r}_1)a_{3\sigma_2} \\ d_7(u_{4\sigma_2}) &= \bar{v}_2 a_{7\sigma_2} = (\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)a_{7\sigma_2} \\ d_{15}(u_{8\sigma_2}) &= \bar{v}_3 a_{15\sigma_2} = \bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2)a_{15\sigma_2} \\ d_{31}(u_{16\sigma_2}) &= \bar{v}_4 a_{31\sigma_2} = \bar{r}_3^4\gamma\bar{r}_3 a_{31\sigma_2} \end{aligned}$$

Theorem 15.1. *In the C_4-slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2\rangle$, the class $u_{2\lambda}a_\sigma$ supports the d_5-differential*

$$d_5(u_{2\lambda}a_\sigma) = 2\bar{\mathfrak{d}}_1 u_{2\sigma}a_{3\lambda}.$$

Proof. Applying Theorem 9.8 to the d_3-differential $d_3(u_{2\sigma_2}) = (\bar{r}_1 + \gamma\bar{r}_1)a_{3\sigma_2}$ predicts the d_5-differential

$$d_5\left(\frac{u_{2\lambda}}{u_{2\sigma}}a_\sigma\right) = N(\bar{r}_1 + \gamma\bar{r}_1)a_{3\lambda}.$$

If this differential exists, then applying the Leibniz rule yields the differential

$$\begin{aligned} d_5(u_{2\lambda}a_\sigma) = d_5\left(u_{2\sigma}\cdot\frac{u_{2\lambda}a_\sigma}{u_{2\sigma}}\right) &= d_5(u_{2\sigma})\cdot\frac{u_{2\lambda}a_\sigma}{u_{2\sigma}} + u_{2\sigma}\cdot d_5\left(\frac{u_{2\lambda}a_\sigma}{u_{2\sigma}}\right) \\ &= \bar{\mathfrak{d}}_1 a_\lambda a_\sigma^3\cdot\frac{u_{2\lambda}a_\sigma}{u_{2\sigma}} + u_{2\sigma}\cdot N(\bar{r}_1+\gamma\bar{r}_1)a_{3\lambda} \\ &= 0 + N(\bar{r}_1+\gamma\bar{r}_1)u_{2\sigma}a_{3\lambda}\ \ (u_\lambda a_{3\sigma} = 2u_{2\sigma}a_\lambda a_\sigma = 0) \\ &= N(\bar{r}_1+\gamma\bar{r}_1)u_{2\sigma}a_{3\lambda} \end{aligned}$$

In fact, the existence of these two d_5-differentials are equivalent, and it suffices to prove the d_5-differential on $u_{2\lambda}a_\sigma$.

Since u_λ supports a d_3-differential, $u_{2\lambda}$ is a d_3-cycle and the class $u_{2\lambda}a_\sigma$ survives to the E_5-page. To identify the target, note that

$$\begin{aligned} res(N(\bar{r}_1+\gamma\bar{r}_1)) &= (\bar{r}_1+\gamma\bar{r}_1)(\gamma\bar{r}_1-\bar{r}_1) \\ &= -(\bar{r}_1^2-\gamma\bar{r}_1^2) \\ &= -\bar{r}_1^2u_{-\sigma}(1+\gamma) \end{aligned}$$

This implies that $N(\bar{r}_1+\gamma\bar{r}_1) = -tr(\bar{r}_1^2u_{-\sigma})$. Note that $res(u_\sigma^{-1}) = 1$. We need to include this term because $N(\bar{r}_1+\gamma\bar{r}_1)$ is in degree $1+\sigma+\lambda$. If we apply the transfer to $\bar{r}_1^2$, which is in degree $2+2\sigma_2$, we would obtain something in degree $2+\lambda$. This does not match the degree of $N(\bar{r}_1+\gamma\bar{r}_1)$. Applying the transfer to $\bar{r}_1^2u_\sigma^{-1}$ yields matching degrees.

The target of the predicted differential is

$$-tr(\bar{r}_1^2u_\sigma^{-1})u_{2\sigma}a_{3\lambda} = -tr(\bar{r}_1^2u_\sigma a_{6\sigma_2}) = tr(\bar{r}_1^2u_\sigma a_{6\sigma_2}).$$

To identify this with $2\bar{\mathfrak{d}}_1 u_{2\sigma} a_{3\lambda}$, consider the equality

$$\begin{aligned} tr(\bar{r}_1^2 u_\sigma a_{6\sigma_2}) + tr(\bar{r}_1 \gamma \bar{r}_1 u_\sigma a_{6\sigma_2}) &= tr(\bar{r}_1(\bar{r}_1 + \gamma \bar{r}_1) u_\sigma a_{6\sigma_2}) \\ &= tr(\bar{r}_1 u_\sigma a_{\sigma_2} res(tr(\bar{r}_1 a_{\sigma_2}) a_{2\lambda})) \\ &= tr(\bar{r}_1 u_\sigma a_{\sigma_2}) tr(\bar{r}_1 a_{\sigma_2}) a_{2\lambda}. \end{aligned}$$

The last expression is 0 because $d_3(u_\lambda) = \bar{s}_1 a_\lambda a_{\sigma_2} = tr(\bar{r}_1 a_{\sigma_2}) a_\lambda$. It follows that

$$\begin{aligned} tr(\bar{r}_1^2 u_\sigma a_{6\sigma_2}) &= -tr(\bar{r}_1 \gamma \bar{r}_1 u_\sigma a_{6\sigma_2}) \\ &= tr(\bar{r}_1 \gamma \bar{r}_1 u_\sigma a_{6\sigma_2}) \\ &= tr(res(\bar{\mathfrak{d}}_1 u_{2\sigma} a_{3\lambda})) \\ &= 2\bar{\mathfrak{d}}_1 u_{2\sigma} a_{3\lambda}. \end{aligned}$$

This class is not zero on the E_5-page. Therefore, the d_5-differential on $u_{2\lambda} a_\sigma$ exists. □

Remark 15.2. In the integer graded spectral sequence, the normed d_5-differential can be seen on the class $\bar{\mathfrak{d}}_1^3 u_{2\lambda} u_{2\sigma} a_\lambda a_\sigma$ at $(9, 3)$:

$$d_5(\bar{\mathfrak{d}}_1^3 u_{2\lambda} u_{2\sigma} a_\lambda a_\sigma) = 2\bar{\mathfrak{d}}_1^4 u_{4\lambda} a_{4\lambda}.$$

This is the product of the differential in Theorem 15.1 and $\bar{\mathfrak{d}}_1^3 u_{2\sigma} a_\lambda$. An alternative, perhaps easier way to identify the target is to note that

$$\begin{aligned}
\bar{\mathfrak{d}}_1^3 u_{2\sigma} a_\lambda tr(\bar{r}_1^2 u_\sigma a_{6\sigma_2}) &= tr(res(\bar{\mathfrak{d}}_1^3 u_{2\sigma} a_\lambda)\bar{r}_1^2 u_\sigma a_{6\sigma_2}) \\
&= tr(\bar{r}_1^5 \gamma \bar{r}_1^3 a_{8\sigma_2}) \\
&= tr(\bar{r}_1^4 \gamma \bar{r}_1^4 a_{8\sigma_2}) \ \ (\bar{r}_1 = \gamma \bar{r}_1 \text{ after the } d_3\text{-differentials}) \\
&= tr(res(\bar{\mathfrak{d}}_1^4 u_{4\sigma} a_{4\lambda})) \\
&= 2\bar{\mathfrak{d}}_1^4 u_{4\sigma} a_\lambda.
\end{aligned}$$

Theorem 15.3. *In the C_4-slice spectral sequence of* $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$*, the class $u_{4\lambda} a_\sigma$ supports the d_{13}-differential*

$$d_{13}(u_{4\lambda} a_\sigma) = \bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{7\lambda} + tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2}).$$

Proof. Applying Theorem 9.8 to the d_7-differential $d_7(u_{4\sigma_2}) = (\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)a_{7\sigma_2}$ predicts the d_{13}-differential

$$d_{13}\left(\frac{u_{4\lambda}}{u_{4\sigma}} a_\sigma\right) = N(\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)a_{7\lambda}.$$

Since $d_{13}(u_{4\sigma}) = \bar{\mathfrak{d}}_3 a_{3\lambda} a_{7\sigma}$, multiplying the source of this differential by $u_{4\sigma}$ gives

$$\begin{aligned}
d_{13}(u_{4\lambda} a_\sigma) = d_{13}\left(u_{4\sigma} \cdot \frac{u_{4\lambda}}{u_{4\sigma}} a_\sigma\right) &= d_{13}(u_{4\sigma}) \cdot \frac{u_{4\lambda}}{u_{4\sigma}} a_\sigma + u_{4\sigma} \cdot d_{13}\left(\frac{u_{4\lambda}}{u_{4\sigma}} a_\sigma\right) \\
&= \bar{\mathfrak{d}}_3 a_{3\lambda} a_{7\sigma} \cdot \frac{u_{4\lambda}}{u_{4\sigma}} a_\sigma + u_{4\sigma} \cdot N(\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)a_{7\lambda} \\
&= 0 + u_{4\sigma} \cdot N(\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)a_{7\lambda} \ \ (u_\lambda a_{3\sigma} = 2u_{2\sigma} a_\lambda a_\sigma = 0) \\
&= N(\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)u_{4\sigma} a_{7\lambda}.
\end{aligned}$$

The existence of these two differentials are equivalent. To prove that the differential on $u_{4\lambda} a_\sigma$ exists, it suffices to show that the predicted target is not zero on the E_{13}-page.

The restriction of $N(\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)$ is

$$\begin{aligned}
res(N(\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)) &= (\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)(\gamma\bar{r}_1^3 + \gamma\bar{r}_3 - \bar{r}_3) \\
&= \bar{r}_1^3\gamma\bar{r}_1^3 + (\gamma\bar{r}_3\bar{r}_1^3 + \bar{r}_3\gamma\bar{r}_1^3) + (\gamma\bar{r}_3\gamma\bar{r}_1^3 - \bar{r}_3\bar{r}_1^3) - (\bar{r}_3^2 - \gamma\bar{r}_3^2) \\
&= res(\bar{\mathfrak{d}}_1^3) + \gamma\bar{r}_3\bar{r}_1^3 u_{-3\sigma}(1+\gamma) + \bar{r}_3\gamma\bar{r}_1^3 u_{-3\sigma}(1+\gamma) - \bar{r}_3^2 u_{-3\sigma}(1+\gamma)
\end{aligned}$$

The predicted target of the differential on $u_{4\lambda}a_\sigma$ is

$$\begin{aligned}
N(\bar{r}_1^3 + \bar{r}_3 + \gamma\bar{r}_3)u_{4\sigma}a_{7\lambda} &= \bar{\mathfrak{d}}_1^3 u_{4\sigma}a_{7\lambda} + tr(\gamma\bar{r}_3\bar{r}_1^3 u_\sigma a_{14\sigma_2}) + tr(\gamma\bar{r}_3\gamma\bar{r}_1^3 u_\sigma a_{14\sigma_2}) - tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2}) \\
&= \bar{\mathfrak{d}}_1^3 u_{4\sigma}a_{7\lambda} + tr(\gamma\bar{r}_3\bar{r}_1^3 u_\sigma a_{14\sigma_2}) + tr(\gamma\bar{r}_3\bar{r}_1^3 u_\sigma a_{14\sigma_2}) + tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2}) \\
&= \bar{\mathfrak{d}}_1^3 u_{4\sigma}a_{7\lambda} + 2tr(\gamma\bar{r}_3\bar{r}_1^3 u_\sigma a_{14\sigma_2}) + tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2}) \\
&= \bar{\mathfrak{d}}_1^3 u_{4\sigma}a_{7\lambda} + tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2})
\end{aligned}$$

To show that this is not zero, we multiply the predicted differential on $u_{4\lambda}a_\sigma$ by $\bar{\mathfrak{d}}_1^9 u_{8\sigma}a_{5\lambda}$ (and use the Leibniz rule) to bring it to the integer-graded part of the slice spectral sequence:

$$d_{13}(\bar{\mathfrak{d}}_1^9 u_{4\lambda}u_{8\sigma}a_{5\lambda}a_\sigma) = \bar{\mathfrak{d}}_1^9 u_{8\sigma}a_{5\lambda} \cdot (\bar{\mathfrak{d}}_1^3 u_{4\sigma}a_{7\lambda} + tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2})).$$

The source of this new predicted differential is at $(25, 11)$ and the target is at $(24, 24)$. Once we verify that the target of this new differential is not zero on the E_{13}-page, we can then conclude that the target of the original differential on $u_{4\lambda}a_\sigma$ is also not zero on the

E_{13}-page. Indeed,

$$
\begin{aligned}
& \bar{\mathfrak{d}}_1^9 u_{8\sigma} a_{5\lambda} \cdot (\bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{7\lambda} + tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2})) \\
= \;& \bar{\mathfrak{d}}_1^{12} u_{12\sigma} a_{12\lambda} + tr(res(\bar{\mathfrak{d}}_1^9 u_{8\sigma} a_{5\lambda}) \bar{r}_3^2 u_\sigma a_{14\sigma_2}) \\
= \;& \bar{\mathfrak{d}}_1^{12} u_{12\sigma} a_{12\lambda} + tr(\bar{r}_3^2 \bar{r}_1^9 \gamma \bar{r}_1^9 a_{24\sigma_2}) \\
= \;& \bar{\mathfrak{d}}_1^{12} u_{12\sigma} a_{12\lambda} + tr(\bar{r}_3^2 \bar{r}_1^{18} a_{24\sigma_2}) \\
= \;& \bar{\mathfrak{d}}_1^{12} u_{12\sigma} a_{12\lambda} + tr(\bar{r}_3^2 (\bar{r}_3 + \gamma\bar{r}_3)^6 a_{24\sigma_2}) \;\; (\text{in the } C_2\text{-spectral sequence, } \bar{r}_1^3 = \bar{r}_3 + \gamma\bar{r}_3) \\
= \;& \bar{\mathfrak{d}}_1^{12} u_{12\sigma} a_{12\lambda} + tr(\bar{r}_3^2 (\bar{r}_3^6 + \bar{r}_3^4 \gamma \bar{r}_3^2 + \bar{r}_3^2 \gamma \bar{r}_3^4 + \gamma \bar{r}_3^6) a_{24\sigma_2}) \\
= \;& \bar{\mathfrak{d}}_1^{12} u_{12\sigma} a_{12\lambda} + tr(\bar{r}_3^4 \gamma \bar{r}_3^4 a_{24\sigma_2}) + tr(\bar{r}_3^8 a_{24\sigma_2}) + tr((\bar{r}_3^6 \gamma \bar{r}_3^2 + \bar{r}_3^2 \gamma \bar{r}_3^6) a_{24\sigma_2}) \\
= \;& \bar{\mathfrak{d}}_1^{12} u_{12\sigma} a_{12\lambda} + tr(res(\bar{\mathfrak{d}}_3^4 u_{12\sigma} a_{12\lambda})) + tr(\bar{r}_3^8 a_{24\sigma_2}) + tr(res(\bar{\mathfrak{d}}_3^2 u_{6\sigma} a_{6\lambda} tr(\bar{r}_3^4 a_{12\sigma_2})) \\
= \;& \bar{\mathfrak{d}}_1^{12} u_{12\sigma} a_{12\lambda} + 2\bar{\mathfrak{d}}_3^4 u_{12\sigma} a_{12\lambda} + \bar{s}_3^8 + 2\bar{\mathfrak{d}}_3^2 u_{6\sigma} a_{6\lambda} tr(\bar{r}_3^4 a_{12\sigma_2}) \\
= \;& \bar{\mathfrak{d}}_1^{12} u_{12\sigma} a_{12\lambda} + 2\bar{\mathfrak{d}}_3^4 u_{12\sigma} a_{12\lambda} + \bar{s}_3^8,
\end{aligned}
$$

which is not zero on the E_{13}-page. Therefore, the normed d_{13}-differential on $u_{4\lambda} a_\sigma$ exists.

□

Remark 15.4. The term $tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2})$ in the expression of the target can also be rewritten as

$$
\begin{aligned}
tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2}) &= tr(\bar{r}_3(\bar{r}_3 + \gamma\bar{r}_3) u_\sigma a_{14\sigma_2} + \bar{r}_3 \gamma \bar{r}_3 u_\sigma a_{14\sigma_2}) \\
&= tr(res(\bar{\mathfrak{d}}_3 u_{4\sigma} a_{7\lambda})) + tr(res(tr(\bar{r}_3 a_{3\sigma_2}) a_{4\lambda}) \bar{r}_3 u_\sigma a_{3\sigma_2}) \\
&= 2\bar{\mathfrak{d}}_3 u_{4\sigma} a_{7\lambda} + tr(\bar{r}_3 u_\sigma a_{3\sigma_2}) tr(\bar{r}_3 a_{3\sigma_2}) a_{4\lambda}.
\end{aligned}
$$

Intuitively, this is saying that after the d_{13}-differentials,

$$\bar{\mathfrak{d}}_1^3 = 2\bar{\mathfrak{d}}_3 + \text{terms in } i_{C_2}^* \mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle\text{-truncations.}$$

This intuition will be useful in the proof of the next theorem.

Theorem 15.5. *In the C_4-slice spectral sequence of $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$, the class $u_{8\lambda}a_\sigma$ supports the d_{29}-differential*

$$d_{29}(u_{8\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^2\bar{\mathfrak{d}}_1 u_{8\sigma}a_{15\lambda} + tr(\bar{r}_3^4\bar{r}_1^2 u_\sigma a_{15\lambda}).$$

Proof. Applying Theorem 9.8 to the d_{15}-differential $d_{15}(u_{8\sigma_2}) = \bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2)a_{15\sigma_2}$ predicts the d_{29}-differential

$$d_{29}\left(\frac{u_{8\lambda}}{u_{8\sigma}}a_\sigma\right) = N(\bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2))a_{15\lambda}.$$

If this d_{29}-differential exists, multiplying it by $u_{8\sigma}$ (a permanent cycle) yields the d_{29}-differential

$$d_{29}(u_{8\lambda}a_\sigma) = N(\bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2))a_{15\lambda}.$$

In fact, the existence of these two differentials are equivalent, and it suffices to show that the second differential exists. We will identify its target and show that it is not zero on the E_{29}-page.

The restriction of $N(\bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2))$ is

$$\begin{aligned} res(N(\bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2))) &= \bar{r}_1(\bar{r}_3^2 + \bar{r}_3\gamma\bar{r}_3 + \gamma\bar{r}_3^2)\cdot\gamma\bar{r}_1(\gamma\bar{r}_3^2 - \bar{r}_3\gamma\bar{r}_3 + \bar{r}_3^2) \\ &= (\bar{r}_3\gamma\bar{r}_3)^2\bar{r}_1\gamma\bar{r}_1 + (\bar{r}_3^4 + \gamma\bar{r}_3^4)\bar{r}_1\gamma\bar{r}_1 \\ &= res(\bar{\mathfrak{d}}_3^2\bar{\mathfrak{d}}_1) + \bar{r}_3^4\bar{r}_1\gamma\bar{r}_1 u_{-7\sigma}(1+\gamma). \end{aligned}$$

Therefore, the target of the normed differential on $u_{8\lambda}a_\sigma$ is

$$\bar{\mathfrak{d}}_3^2\bar{\mathfrak{d}}_1 u_{8\sigma}a_{15\lambda} + tr(\bar{r}_3^4\bar{r}_1\gamma\bar{r}_1 u_{-7\sigma})u_{8\sigma}a_{15\lambda} = \bar{\mathfrak{d}}_3^2\bar{\mathfrak{d}}_1 u_{8\sigma}a_{15\lambda} + tr(\bar{r}_3^4\bar{r}_1^2 u_\sigma a_{30\sigma_2}).$$

To show that this target is not zero, we will multiply it by $\bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1^2 u_{16\sigma}a_{9\lambda}$ to bring it to the integer graded part of the spectral sequence. After this multiplication, the predicted d_{29}-differential becomes

$$d_{29}(\bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{16\sigma}a_{9\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^7\bar{\mathfrak{d}}_1^3 u_{24\sigma}a_{24\lambda} + \bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1^2 u_{16\sigma}a_{9\lambda} tr(\bar{r}_3^4\bar{r}_1^2 u_\sigma a_{30\sigma_2}).$$

The source of this differential is at $(49, 19)$ and the target is at $(48, 48)$ (see Figure 36). Once we verify that the target of this new differential is not zero on the E_{29}-page, we can then conclude that the original target is also not zero on the E_{29}-page, and the normed differential exists.

The new target is equal to

$$\begin{aligned}
& \bar{\mathfrak{d}}_3^7\bar{\mathfrak{d}}_1^3 u_{24\sigma}a_{24\lambda} + \bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1^2 u_{16\sigma}a_{9\lambda} tr(\bar{r}_3^4\bar{r}_1^2 u_\sigma a_{30\sigma_2}) \\
= \; & \bar{\mathfrak{d}}_3^7\bar{\mathfrak{d}}_1^3 u_{24\sigma}a_{24\lambda} + tr((\bar{r}_3\gamma\bar{r}_3)^5\bar{r}_3^4\bar{r}_1^4\gamma\bar{r}_1^2 a_{48\sigma_2}) \\
= \; & \bar{\mathfrak{d}}_3^7\bar{\mathfrak{d}}_1^3 u_{24\sigma}a_{24\lambda} + tr((\bar{r}_3\gamma\bar{r}_3)^5\bar{r}_3^4\bar{r}_1^6 a_{48\sigma_2}) \\
= \; & \bar{\mathfrak{d}}_3^7\bar{\mathfrak{d}}_1^3 u_{24\sigma}a_{24\lambda} + tr((\bar{r}_3\gamma\bar{r}_3)^5\bar{r}_3^4(\bar{r}_3+\gamma\bar{r}_3)^2 a_{48\sigma_2}) \;\; \text{(in the } C_2\text{-spectral sequence, } \bar{r}_1^3 = \bar{r}_3+\gamma\bar{r}_3) \\
= \; & \bar{\mathfrak{d}}_3^7\bar{\mathfrak{d}}_1^3 u_{24\sigma}a_{24\lambda} + tr((\bar{r}_3\gamma\bar{r}_3)^5\bar{r}_3^6 a_{18\sigma_2}) + tr((\bar{r}_3\gamma\bar{r}_3)^5\bar{r}_3^4\gamma\bar{r}_3^2 a_{48\sigma_2}) \\
= \; & \bar{\mathfrak{d}}_3^7\bar{\mathfrak{d}}_1^3 u_{24\sigma}a_{24\lambda} + \bar{\mathfrak{d}}_3^5\bar{s}_3^6 + \bar{\mathfrak{d}}_3^7\bar{s}_3^2.
\end{aligned}$$

To further simplify the target, take the d_{13}-differential on $u_{4\lambda}a_\sigma$ and multiply it by

$\bar{\mathfrak{d}}_3^7 u_{20\sigma} a_{17\lambda}$. This produces the d_{13}-differential

$$\begin{aligned} d_{13}(\bar{\mathfrak{d}}_3^7 u_{4\lambda} u_{20\sigma} a_{17\lambda} a_\sigma) &= \bar{\mathfrak{d}}_3^7 u_{20\sigma} a_{17\lambda} (\bar{\mathfrak{d}}_1^3 u_{4\sigma} a_{7\lambda} + tr(\bar{r}_3^2 u_\sigma a_{14\sigma_2})) \\ &= \bar{\mathfrak{d}}_3^7 \bar{\mathfrak{d}}_1^3 u_{24\sigma} a_{24\lambda} + tr((\bar{r}_3 \gamma \bar{r}_3)^7 \bar{r}_3^2 a_{48\sigma_2}) \\ &= \bar{\mathfrak{d}}_3^7 \bar{\mathfrak{d}}_1^3 u_{24\sigma} a_{24\lambda} + \bar{\mathfrak{d}}_3^7 \bar{s}_3^2 \end{aligned}$$

This is a differential with source at $(49, 35)$ and target $(48, 48)$, and it introduces the relation $\bar{\mathfrak{d}}_3^7 \bar{\mathfrak{d}}_1^3 u_{24\sigma} a_{24\lambda} + \bar{\mathfrak{d}}_3^7 \bar{s}_3^2 = 0$ after the E_{13}-page. Therefore, the target of our d_{29}-differential is $\bar{\mathfrak{d}}_3^5 \bar{s}_3^6 = tr((\bar{r}_3 \gamma \bar{r}_3)^5 \bar{s}_3^6 a_{48\sigma_2})$.

We will now show that $\bar{\mathfrak{d}}_3^5 \bar{s}_3^6 \neq 0$ on the E_{29}-page. Recall that in the C_2-slice spectral sequence, we have the d_{15}-differential

$$\begin{aligned} d_{15}((\bar{r}_3 \gamma \bar{r}_3)^5 \bar{r}_3^3 \bar{r}_1^2 u_{8\sigma_2} a_{33\sigma_2}) &= (\bar{r}_3 \gamma \bar{r}_3)^5 \bar{r}_3^3 \bar{r}_1^2 \cdot \bar{r}_1 (\bar{r}_3^2 + \bar{r}_3 \gamma \bar{r}_3 + \gamma \bar{r}_3^2) a_{48\sigma_2} \\ &= (\bar{r}_3 \gamma \bar{r}_3)^5 \bar{r}_3^3 \cdot (\bar{r}_3 + \gamma \bar{r}_3)(\bar{r}_3^2 + \bar{r}_3 \gamma \bar{r}_3 + \gamma \bar{r}_3^2) a_{48\sigma_2} \\ &= (\bar{r}_3 \gamma \bar{r}_3)^5 \bar{r}_3^3 (\bar{r}_3^3 + \gamma \bar{r}_3^3) a_{48\sigma_2} \\ &= (\bar{r}_3 \gamma \bar{r}_3)^5 \bar{r}_3^6 a_{48\sigma_2} + (\bar{r}_3 \gamma \bar{r}_3)^8 a_{48\sigma_2} \end{aligned}$$

Applying the transfer to this differential yields the d_{15}-differential

$$d_{15}(\bar{\mathfrak{d}}_3^5 \bar{s}_3^3 \bar{r}_1^2) = \bar{\mathfrak{d}}_3^5 \bar{s}_3^6 + 2\bar{\mathfrak{d}}_3^8 u_{24\sigma} a_{24\lambda}$$

in the C_4-spectral sequence (cf. Proposition 14.5). Therefore, the target of our normed d_{29}-differential can be identified with the class $2\bar{\mathfrak{d}}_3^8 u_{24\sigma} a_{24\lambda}$ on the E_{29}-page, which is not zero. This completes the proof of the theorem. □

Theorem 15.6. *In the C_4-slice spectral sequence of* $\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$*, the class $u_{16\lambda} a_\sigma$ supports* *the d_{61}-differential*

$$d_{61}(u_{16\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^5 u_{16\sigma}a_{31\lambda}.$$

Proof. Applying Theorem 9.8 to the d_{31}-differential $d_{31}(u_{16\sigma_2}) = \bar{r}_3^4\gamma\bar{r}_3 a_{31\sigma_2}$ predicts the d_{61}-differential

$$d_{61}\left(\frac{u_{16\lambda}}{u_{16\sigma}}a_\sigma\right) = N(\bar{r}_3^4\gamma\bar{r}_3)a_{31\lambda} = \bar{\mathfrak{d}}_3^5 a_{31\lambda}.$$

To show that this differential exists, it suffices to show that the target is not 0 on the E_{61}-page. Multiplying this differential by the permanent cycle $u_{16\sigma}$ gives

$$d_{61}(u_{16\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^5 u_{16\sigma}a_{31\lambda}.$$

If the target of this new differential is not zero on the E_{61}-page, then the original target will also not be zero, and both d_{61}-differentials will exist.

Now, we will multiply the predicted d_{61}-differential on $u_{16\lambda}a_\sigma$ by $\bar{\mathfrak{d}}_3^{11}u_{32\sigma}a_{17\lambda}$ to move it to the integer graded spectral sequence:

$$d_{61}(\bar{\mathfrak{d}}_3^{11}u_{16\lambda}u_{32\sigma}a_{17\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^{16}u_{48\sigma}a_{48\lambda}.$$

This is a predicted d_{61}-differential with source at $(97, 65)$ and target at $(96, 96)$. It suffices to show that the target of this differential is no zero on the E_{61}-page.

Theorem 15.5 shows that there is a d_{29}-differential

$$d_{29}(\bar{\mathfrak{d}}_3^{13}\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{40\sigma}a_{33\lambda}a_\sigma) = 2\bar{\mathfrak{d}}_3^{16}u_{48\sigma}a_{48\lambda}.$$

For degree reasons, if the class $\bar{\mathfrak{d}}_3^{16}u_{48\sigma}a_{48\lambda}$ is zero on the E_{61}-page, then the only possibility is for it to be killed by a d_{31}-differential (see Figure 36). This is impossible by considering the class $\bar{\mathfrak{d}}_3^{17} u_{48\sigma} a_{51\lambda} a_{3\sigma} = \bar{\mathfrak{d}}_3^{16} u_{48\sigma} a_{48\lambda} \cdot \bar{\mathfrak{d}}_3 a_{3\lambda} a_{3\sigma}$ at $(99, 105)$. If the class $\bar{\mathfrak{d}}_3^{16} u_{48\sigma} a_{48\lambda}$ is indeed zero after the d_{31}-differentials, then $\bar{\mathfrak{d}}_3^{17} u_{48\sigma} a_{51\lambda} a_{3\sigma}$ will also be zero after the E_{31}-page (this is because $\bar{\mathfrak{d}}_3 a_{3\lambda} a_{3\sigma}$ is a permanent cycle). However, by degree reasons, $\bar{\mathfrak{d}}_3^{17} u_{48\sigma} a_{51\lambda} a_{3\sigma}$ cannot be killed by a differential of length at most 31.

Therefore, the class $\bar{\mathfrak{d}}_3^{16} u_{48\sigma} a_{48\lambda}$ is not zero on the E_{61}-page and the normed d_{61}-differential on $u_{8\lambda} a_\sigma$ exists. □

16 Higher Differentials III: The Vanishing Theorem

16.1 α and α^2

Let α be the class $\bar{\mathfrak{d}}_3^{8} u_{24\sigma} u_{24\lambda}$ at $(48, 48)$. Theorem 15.5 and Theorem 15.6 show that

1. The class α is a permanent cycle that survives to the E_∞-page;
2. The class $\alpha^2 = \bar{\mathfrak{d}}_3^{16} u_{48\sigma} a_{48\lambda}$ at $(96, 96)$ is killed by the d_{61}-differential

$$d_{61}(\bar{\mathfrak{d}}_3^{11} u_{16\lambda} u_{32\sigma} a_{17\lambda} a_\sigma) = \alpha^2.$$

In the C_4-slice spectral sequence of $BP^{((C_4))}\langle 2\rangle$, α will be playing the role of ϵ in C_4- SliceSS($BP^{((C_4))}\langle 1\rangle$). The following lemma is the higher height analogue of Lemma 10.15 and is proven using the exact same method.

Lemma 16.1. *Let $d_r(x) = y$ be a nontrivial differential in C_4-SliceSS($BP^{((C_4))}\langle 2\rangle$).*

1. *The class αx and αy both survive to the E_r-page, and $d_r(\alpha x) = \alpha y$.*
2. *If both x and y are divisible by α on the E_2-page, then x/α and y/α both survive to the E_r-page, and $d_r(x/\alpha) = y/\alpha$.*

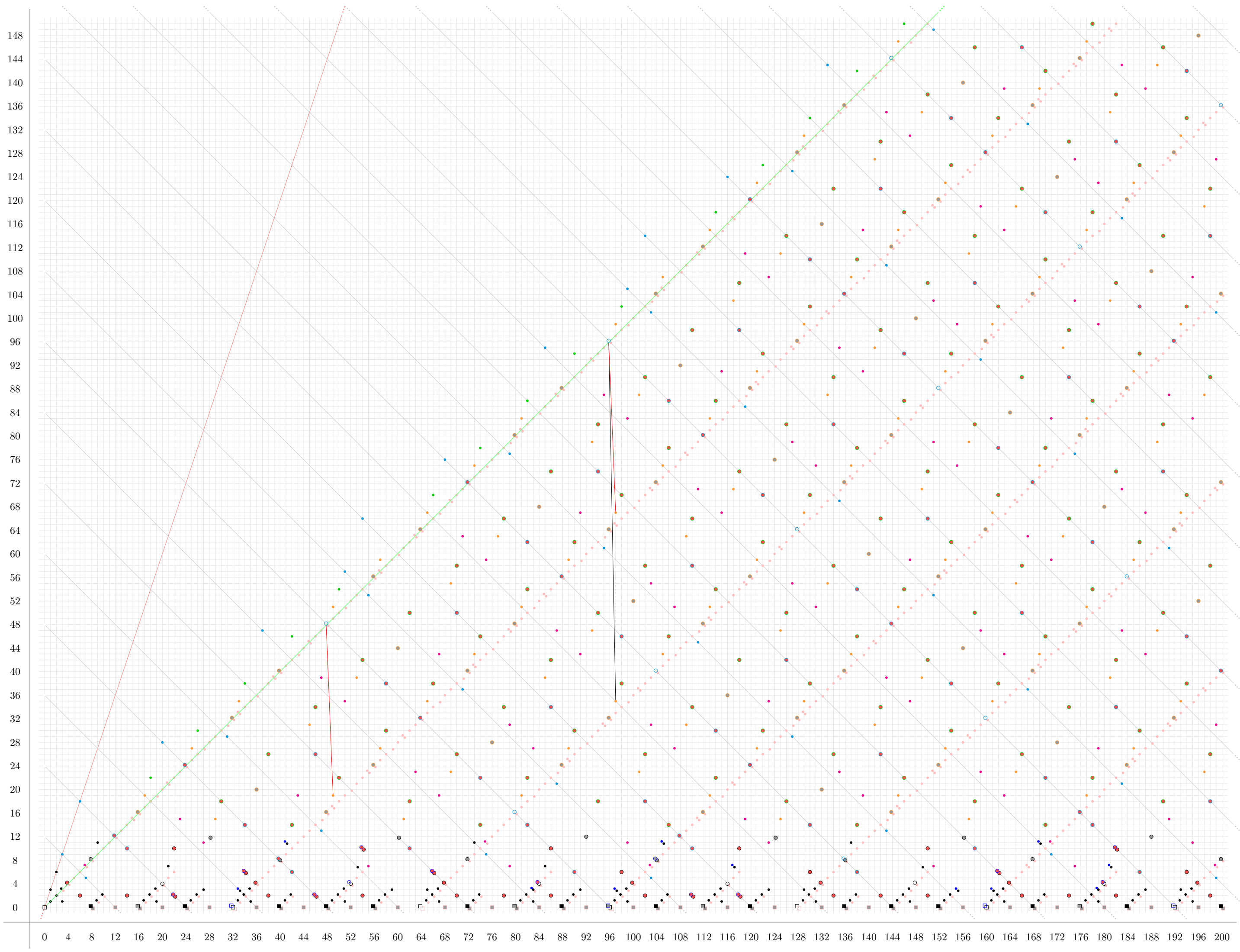


Figure 36: Normed d_{29} and d_{61}-differentials.

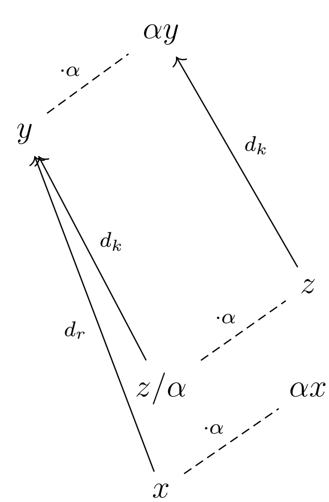
Proof. We will prove both statements by using induction on r, the length of the differential. Both claims are true when $r \leq 15$.

Now, suppose that both statements hold for all differentials of length smaller than r. Given a nontrivial differential $d_r(x) = y$, we will first show that αy survives to the E_r-page.

If αy supports a differential, then y must support a differential as well. This is a contradiction because y is the target of a differential. Therefore if αy does not survive to the E_r-page, it must be killed by a differential $d_k(z) = \alpha y$, where $k < r$.

We claim that z is divisible by α. If $k \leq 15$, then this is true because we have characterized completely all the differentials of length ≤ 15, and in all the cases z will be divisible by α. If $k > 15$, then k will be divisible by α as well because it is a class on or under the line of slope 1 with filtration at least 48, and all such classes are divisible by α starting from the E_{16}-page.

The inductive hypothesis, applied to the differential $d_k(z) = \alpha y$, shows that $d_k(z/\alpha) = y$. This is a contradiction because $d_r(x) = y$ is a nontrivial d_r-differential. Therefore, αy survives to the E_r-page.



If αx does not survive to the E_r-page, then it must be killed by a shorter differential.

This shorter differential will introduce the relation $\epsilon x = 0$ on the E_r-page. However, the Leibniz rule, applied to the differential $d_r(x) = y$, shows that

$$d_r(\alpha x) = \alpha y \neq 0$$

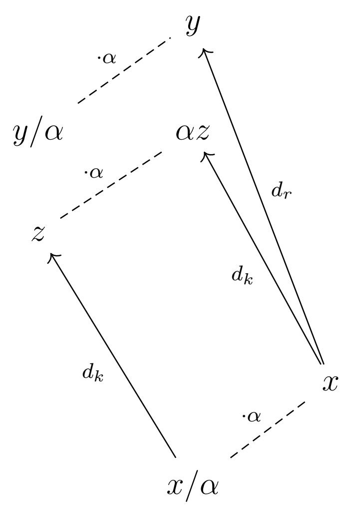
on the E_r-page. This is a contradiction. Therefore, αx must survive to the E_r-page as well, and it supports the differential

$$d_r(\alpha x) = \alpha y.$$

This proves (1).

To prove (2), note that if y/α supports a differential of length smaller than r, then the induction hypothesis would imply that y also supports a differential of the same length. Similarly, if y/α is killed by a differential of length smaller than r, then the induction hypothesis would imply that y is also killed a by a differential of the same length. Both scenarios lead to contradictions. Therefore, y/α survives to the E_r-page.

We will now show that x/α also survives to the E_r-page. Since x supports a d_r-differential, x/α must also support a differential of length at most r. Suppose that $d_k(x/\alpha) = z$, where $k < r$. The induction hypothesis, applied to this d_k-differential, implies the existence of the differential $d_k(x) = \alpha z$. This is a contradiction because $d_r(x) = y$.



It follows that x/α survives to the E_r-page, and it supports a nontrivial d_r-differential. Since y/α also survives to the E_r-page, the Leibniz rule shows that

$$d_r(x/\alpha) = y/\alpha,$$

as desired. □

Theorem 16.2 (Vanishing Theorem)**.** *Any class of the form $\alpha^2 x$ on the E_2-page of C_4-*$\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$ *must die on or before the E_{61}-page.*

Proof. If x is a d_{61}-cycle, then the class $\alpha^2 x$ is a d_{61}-cycle as well. Since α^2 is killed by a d_{61}-differential, $\alpha^2 x$ must also be killed by a differential of length at most 61.

Now suppose that the class x is not a d_{61}-cycle and it supports the differential $d_r(x) = y$, where $r \leq 61$. Applying Lemma 16.1(1), we deduce that the class $\alpha^2 x$ must support the nontrivial d_r-differential

$$d_r(\alpha^2 x) = \alpha^2 y.$$

Therefore, it cannot survive past the E_{61}-page. □

16.2 Important permanent cycles

Proposition 16.3. *The following classes are permanent cycles that survive to the E_∞-page of C_4-SliceSS($\mathrm{BP}^{((C_4))}\langle 2\rangle$).*

- $\eta' := \bar{\mathfrak{d}}_1 a_\lambda a_\sigma$ *at* $(1, 3)$;
- $\xi := \bar{\mathfrak{d}}_3 a_{3\lambda} a_{3\sigma}$ *at* $(3, 9)$;
- $\epsilon' :=$ *at* $(8, 8)$;
- $\beta := \bar{\mathfrak{d}}_3^3 u_{8\sigma} a_{9\lambda} a_\sigma$ *at* $(17, 19)$.

Proof. All the classes are clearly permanent cycles. It is also immediately clear that η' and ξ survive to the E_∞-page.

Suppose that ϵ' is killed by a differential. For degree reasons, the length of that differential must be 7. This implies that $\alpha \cdot \epsilon'$ at $(56, 56)$ must also be killed on or before the E_7-page. This is impossible for degree reasons. This shows that ϵ' survives to the E_∞-page.

Now, suppose that β is killed by a differential. For degree reasons, the length of that differential must be 17. This implies that $\alpha \cdot \beta$ at $(65, 67)$ must also be killed on or before the E_{17}-page. This is again impossible because of degree reasons. □

In Figure 37, we have drawn some multiplications by ξ (red structure lines) and β (blue structure lines). These multiplications will be useful later when we prove long differentials that cross the vanishing line of slope 1.

Proposition 16.4. *Let* $\gamma := \bar{\mathfrak{d}}_3^2 u_{4\lambda} u_{6\sigma} a_{2\lambda}$ *at* $(20, 4)$. *Then*

1. $d_{13}(\gamma) = \bar{\mathfrak{d}}_3^3 u_\lambda u_{8\sigma} a_{8\lambda} a_\sigma$ $(d_{13}(20, 4) = (19, 17))$;

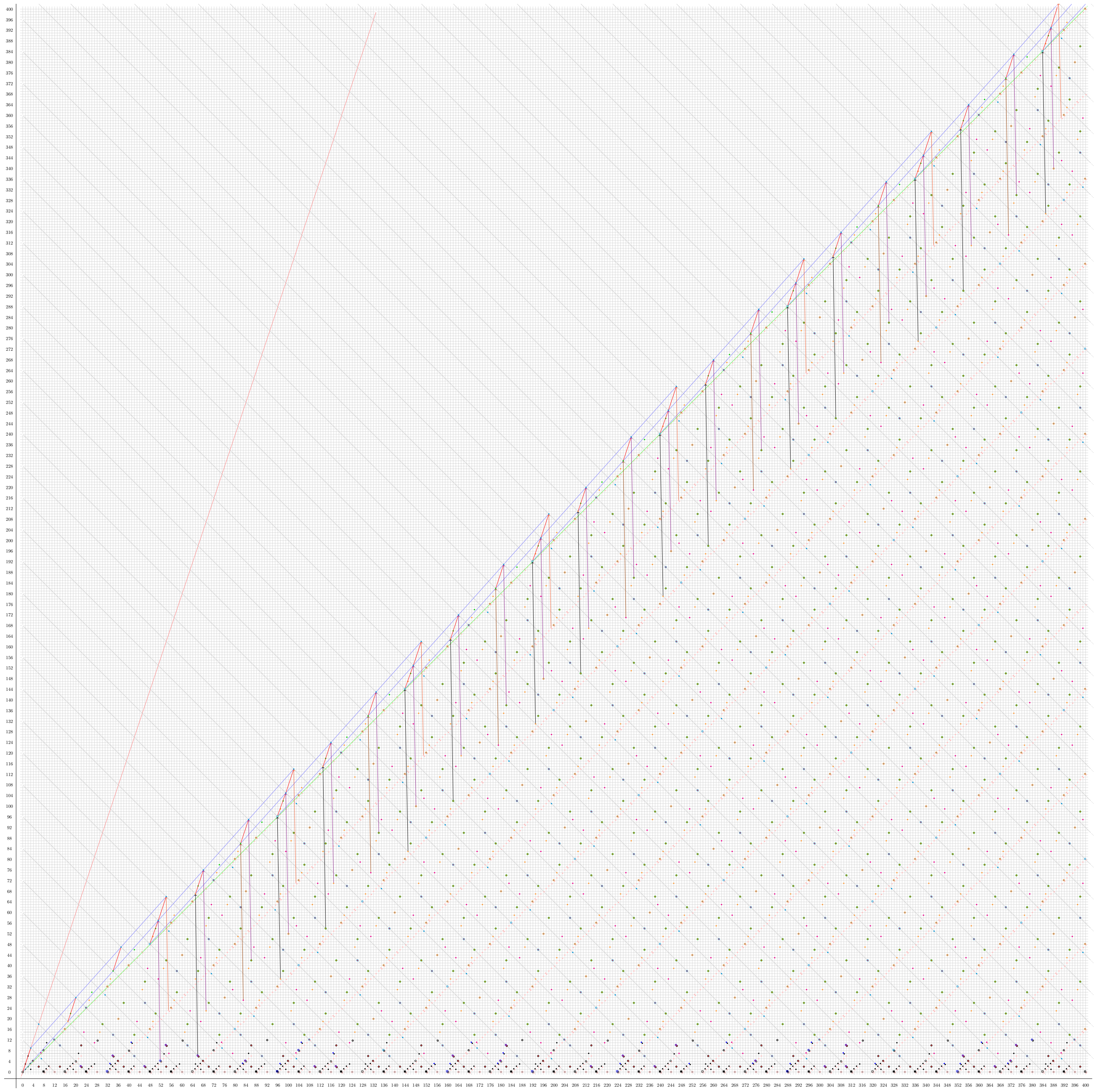


Figure 37: E_{16}-page of SliceSS($BP^{((C_4))}\langle 2\rangle$). The differentials shown are the long differentials that cross the vanishing line of slope 1. The black differentials are the d_{61}-differentials; the sienna differentials are the d_{59}-differentials; the plum differentials are the d_{53}-differentials; and the red-orange differentials are the d_{43}-differentials.

2. $d_{15}(\gamma^2) = \bar{\mathfrak{d}}_3^5 \bar{s}_1 u_{4\lambda} u_{15\sigma} a_{11\lambda} a_{\sigma_2}$ *(*$d_{15}(40, 8) = (39, 23)$*);*

3. $d_{31}(\gamma^4) = \bar{\mathfrak{d}}_3^9 \bar{s}_3^3 = tr(\bar{r}_3^{12}\gamma\bar{r}_3^9 u_{16\sigma_2} a_{47\sigma_2})$ *(*$d_{31}(80, 16) = (79, 47)$*);*

4. The class $\gamma^8 = \bar{\mathfrak{d}}_3^{16} u_{32\lambda} u_{48\sigma} a_{16\lambda}$ *at* $(160, 32)$ *is a permanent cycle.*

Proof. (1) and (2) are Proposition 14.2 and Propostion 14.10, respectively.

To prove (3), note that $\gamma^4 \cdot \beta$ is the class $\bar{\mathfrak{d}}_3^{11} u_{16\lambda} u_{32\sigma} a_{17\lambda} a_\sigma$ at $(97, 35)$, which supports the d_{61}-differential killing α^2. Therefore γ^4 must support a differential of length at most 61. The possible targets are the following classes:

- $\bar{\mathfrak{d}}_3^9 \bar{s}_3^3$ at $(79, 47)$;
- $\bar{\mathfrak{d}}_3^{10} \bar{s}_3$ at $(79, 47)$;
- $\bar{\mathfrak{d}}_3^{13} u_\lambda u_{38\sigma} a_{38\lambda} a_\sigma$ at $(79, 77)$;
- $\bar{\mathfrak{d}}_3^{12} \bar{s}_3^2 \bar{r}_1$ at $(79, 79)$.

We know all the d_{31}-differentials between $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes. In particular, the class $\bar{\mathfrak{d}}_3^{10}\bar{s}_3$ at $(79, 47)$ supports the d_{31}-differential

$$d_{31}(\bar{\mathfrak{d}}_3^{10}\bar{s}_3) = \bar{\mathfrak{d}}_3^{12}\bar{s}_3^2 \ \ (d_{31}(79, 47) = (78, 78)),$$

and the class $\bar{\mathfrak{d}}_3^{12}\bar{s}_3^2\bar{r}_1$ at $(79, 79)$ is killed by the d_{31}-differential

$$d_{31}(\bar{\mathfrak{d}}_3^{10}\bar{s}_3\bar{r}_1) = \bar{\mathfrak{d}}_3^{12}\bar{s}_3^2\bar{r}_1 \ \ (d_{31}(80, 48) = (79, 79)).$$

Now, consider the class $\bar{\mathfrak{d}}_3^{10}\bar{\mathfrak{d}}_1^2 u_{8\lambda} u_{32\sigma} a_{24\lambda}$ at $(80, 48)$. The product of this class with

$\beta = \bar{\mathfrak{d}}_3^3 u_{8\sigma} a_{9\lambda} a_\sigma$ is the class

$$\bar{\mathfrak{d}}_3^{10} \bar{\mathfrak{d}}_1^2 u_{8\lambda} u_{32\sigma} a_{24\lambda} \cdot \bar{\mathfrak{d}}_3^3 u_{8\sigma} a_{9\lambda} a_\sigma = \bar{\mathfrak{d}}_3^{13} \bar{\mathfrak{d}}_1^2 u_{8\lambda} u_{40\sigma} a_{33\lambda} a_\sigma$$

at $(97, 67)$. This class supports the d_{29}-differential

$$d_{29}(\bar{\mathfrak{d}}_3^{13} \bar{\mathfrak{d}}_1^2 u_{8\lambda} u_{40\sigma} a_{33\lambda} a_\sigma) = 2\bar{\mathfrak{d}}_3^{16} u_{48\sigma} a_{48\lambda} \ \ (d_{29}(97, 67) = 2(96, 96)).$$

Therefore, the class $\bar{\mathfrak{d}}_3^{10} \bar{\mathfrak{d}}_1^2 u_{8\lambda} u_{32\sigma} a_{24\lambda}$ must support a differential of length at most 29. The only possibility is the class $\bar{\mathfrak{d}}_3^{13} u_\lambda u_{38\sigma} a_{38\lambda} a_\sigma$ at $(79, 77)$.

It follows that the only possibility for the target of the differential supported by γ^4 is the class $\bar{\mathfrak{d}}_3^9 \bar{s}_3^3$ at $(79, 47)$. The differential will be the d_{31}-differential that we claimed. This proves (3).

For (4), since the classes $\bar{\mathfrak{d}}_3^{16}$, $u_{32\lambda}$, $u_{48\sigma}$, and $a_{16\lambda}$ are all permanent cycles, their product is a permanent cycle as well. □

Proposition 16.4 shows that whenever we have proved a d_r-differential of length $r < 31$, we can multiply that differential by α and γ^4 and use Lemma 16.1 to deduce more d_r-differentials. If the length of the differential is $r \geq 31$, then we can multiply that differential by $(48, 48)$ and $(160, 32)$ and use Lemma 16.1 to produce more d_r-differentials.

16.3 Long Differentials Crossing the Line of Slope 1

Theorem 16.5. *The following differentials exist:*

1. $d_{61}(\overline{\mathfrak{d}}_3^{11} u_{16\lambda} u_{32\sigma} a_{17\lambda} a_\sigma) = \alpha^2$ *(*$d_{61}(97, 35) = (96, 96)$*);*

2. $d_{61}(2\overline{\mathfrak{d}}_3^{14} u_{15\lambda} u_{42\sigma} a_{27\lambda}) = \alpha^2 \beta$ *(*$d_{61}(2(114, 54)) = (113, 115)$*);*

3. $d_{59}(\overline{\mathfrak{d}}_3^{17}\overline{s}_1 u_{14\lambda}u_{51\sigma}a_{37\lambda}a_{\sigma_2}) = \alpha^2\beta^2$ *(*$d_{59}(131, 75) = (130, 134)$*);*

4. $d_{53}(\overline{\mathfrak{d}}_3^{20}\overline{\mathfrak{d}}_1^{2}u_{12\lambda}u_{62\sigma}a_{50\lambda}) = \alpha^2\beta^3$ *(*$d_{53}(148, 100) = (147, 153)$*);*

5. $d_{53}(\overline{\mathfrak{d}}_3^{23}\overline{\mathfrak{d}}_1^{2}u_{12\lambda}u_{70\sigma}a_{59\lambda}a_{\sigma}) = \alpha^2\beta^4$ *(*$d_{53}(165, 119) = (164, 172)$*);*

6. $d_{53}(2\overline{\mathfrak{d}}_3^{26}\overline{\mathfrak{d}}_1^{2}u_{11\lambda}u_{80\sigma}a_{69\lambda}) = \alpha^2\beta^5$ *(*$d_{53}(2(182, 138)) = (181, 191)$*);*

7. $d_{43}(\overline{\mathfrak{d}}_3^{29}\overline{s}_3^{3}) = \alpha^2\beta^6$ *(*$d_{43}(199, 167) = (198, 210)$*).*

Proof. Since β is a permanent cycle, the classes $\alpha^2\beta^i$ $(1 \leq i \leq 6)$ are all killed on or before the E_{61}-page by differentials of decreasing length. More precisely, if a d_r-differential kills $\alpha^2\beta^i$ and a $d_{r'}$-differential kills $\alpha^2\beta^{i'}$ with $0 \leq i < i' \leq 6$, then $r \geq r'$.

Consider the class $\alpha^2\beta^6$. The shortest differential that can kill this class is a d_{43}-differential. Therefore, all the differentials killing the class $\alpha^2\beta^i$ for $1 \leq i \leq 6$ must all be of length at least 43 and at most 61.

(1) follows directly from Theorem 15.6.

For (2), the only differential that can kill $\alpha^2\beta$ that's of length $43 \leq r \leq 61$ is the claimed d_{61}-differential.

For (3), the only differential that can kill $\alpha^2\beta^2$ that's of length $43 \leq r \leq 61$ is the claimed d_{59}-differential.

For (4), the only differential that can kill $\alpha^2\beta^3$ that's of length $43 \leq r \leq 59$ is the claimed d_{53}-differential.

For (5), the only differential that can kill $\alpha^2\beta^4$ that's of length $43 \leq r \leq 53$ is the claimed d_{53}-differential.

For (6), the only differential that can kill $\alpha^2\beta^5$ that's of length $43 \leq r \leq 53$ is the claimed d_{53}-differential.

Lastly, for (7), the only differential that can kill $\alpha^2\beta^6$ that's of length $43 \le r \le 53$ is the claimed d_{43}-differential. $\square$

Applying Lemma 16.1 to the differentials in Theorem 16.5, we obtain all the other long differentials crossing the line of slope 1. These differentials are shown in Figure 37.

17 Higher differentials IV: Everything until the E_{29}-page

17.1 d_{21}-differentials

Proposition 17.1. *The following d_{21}-differentials exist:*

$$\begin{aligned} d_{21}(\bar{\mathfrak{d}}_3^{21} u_{9\lambda} u_{62\sigma} a_{54\lambda} a_\sigma) &= 2\bar{\mathfrak{d}}_3^{22}\bar{\mathfrak{d}}_1^2 u_{3\lambda} u_{68\sigma} a_{65\lambda} \ \ (d_{21}(143,109) = 2(142,130)), \\ d_{21}(\bar{\mathfrak{d}}_3^{25} u_{9\lambda} u_{74\sigma} a_{66\lambda} a_\sigma) &= 2\bar{\mathfrak{d}}_3^{26}\bar{\mathfrak{d}}_1^2 u_{3\lambda} u_{80\sigma} a_{77\lambda} \ \ (d_{21}(167,133) = 2(166,154)). \end{aligned}$$

Proof. The Vanishing Theorem (Theorem 18.1) shows that the class $2(142,30)$ must die on or before the E_{61}-page. For degree reasons, the only possibility is for it to be killed. The only possibilities for the source of the differential are the following classes:

1. $\bar{\mathfrak{d}}_3^{21} u_{9\lambda} u_{62\sigma} a_{54\lambda} a_\sigma$ at $(143,109)$;

2. $\bar{\mathfrak{d}}_3^{18} \bar{s}_3$ at $(143,79)$;

3. $\bar{\mathfrak{d}}_3^{17} \bar{s}_3^3$ at $(143,79)$.

Class (2) is killed by the class $\bar{\mathfrak{d}}_3^{15}\bar{s}_3^2$ at $(144,48)$ by a d_{31}-differential (this is a d_{31}-differential between $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes). Class (3) is killed by the class $2\bar{\mathfrak{d}}_3^{16} u_{24\lambda} u_{48\sigma} a_{24\lambda}$ at $(144,48)$ via a d_{31}-differential (see the discussion after Proposition 14.11).

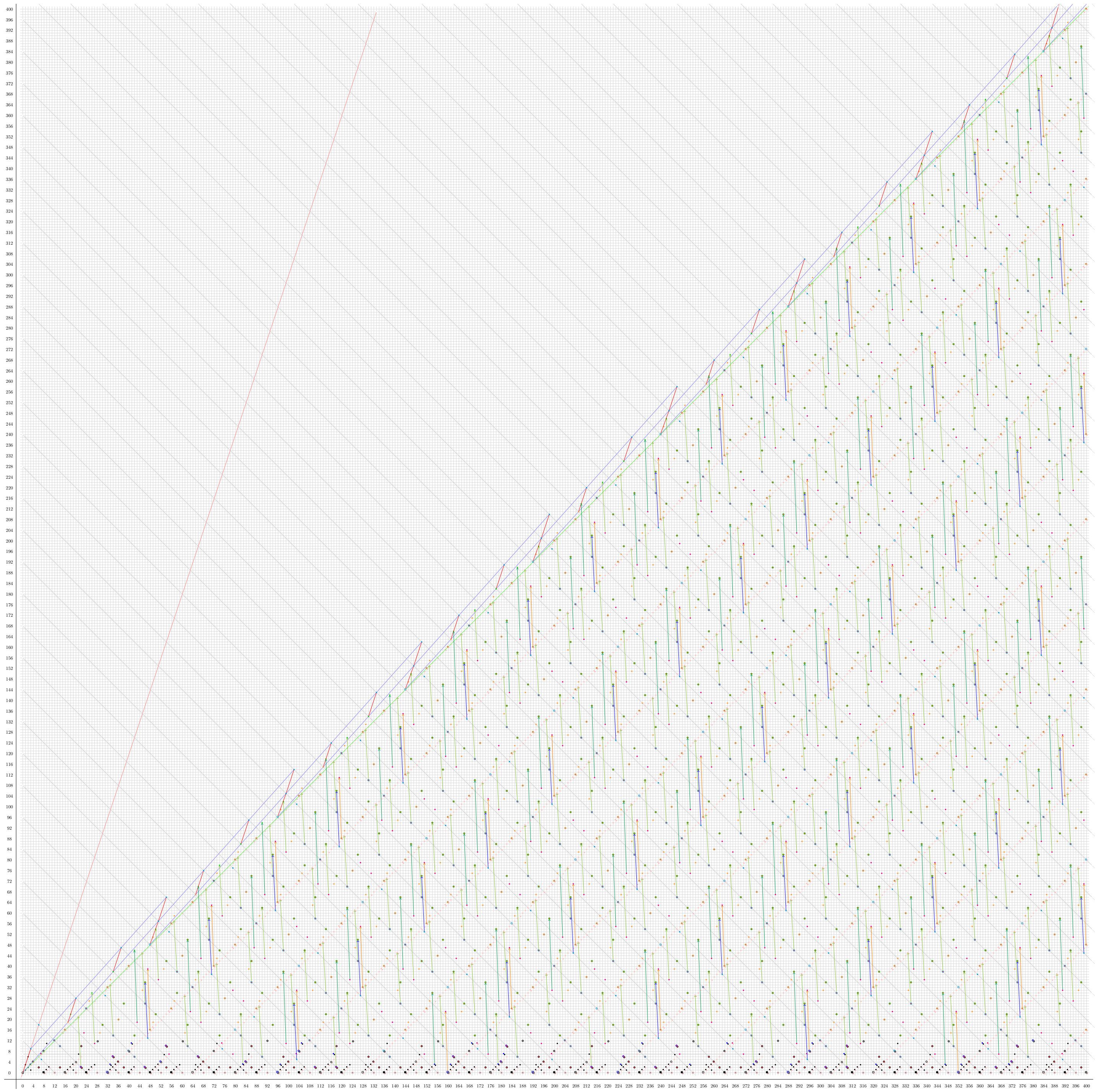


Figure 38: d_r-differentials of lengths $15 < r < 29$. The d_{19}-differentials are shown in lime-green, the d_{21}-differentials are shown in blue, the d_{23}-differentials are shown in orange, and the d_{27}-differentials are shown in forest-green.

Therefore, the only possibility for the source is class (1), and we deduce our desired d_{21}-differential. □

All the other d_{21}-differentials are obtained from the differentials in Proposition 17.1 by using product structures with the classes α and γ^4 (see the discussion after Propositon 16.4). These differentials are the blue differentials in Figure 38.

17.2 d_{23}-differentials

Proposition 17.2. *The following d_{23}-differentials exist:*

$$\begin{aligned} d_{23}(\bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1 u_{8\lambda}u_{16\sigma}a_{8\lambda}) &= \bar{\mathfrak{d}}_3^7\bar{s}_1 u_{2\lambda}u_{21\sigma}a_{19\lambda}a_{\sigma_2} \ \ (d_{23}(48,16)=(47,39)), \\ d_{23}(\bar{\mathfrak{d}}_3^9\bar{\mathfrak{d}}_1 u_{8\lambda}u_{28\sigma}a_{20\lambda}) &= \bar{\mathfrak{d}}_3^{11}\bar{s}_1 u_{2\lambda}u_{33\sigma}a_{31\lambda}a_{\sigma_2} \ \ (d_{23}(72,40)=(71,63)). \end{aligned}$$

Proof. We will prove the first d_{23}-differential. The proof of the second d_{23}-differential is exactly the same. The restriction of the class $\bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1 u_{8\lambda}u_{16\sigma}a_{8\lambda}$ is $\bar{r}_3^5\gamma\bar{r}_3^5\bar{r}_1\gamma\bar{r}_1 u_{16\sigma_2}a_{16\sigma_2}$ in the C_2-slice spectral sequence. It supports the d_{31}-differential

$$d_{31}(\bar{r}_3^5\gamma\bar{r}_3^5\bar{r}_1\gamma\bar{r}_1 u_{16\sigma_2}a_{16\sigma_2}) = \bar{r}_3^5\gamma\bar{r}_3^5\bar{r}_1^2\cdot(\bar{r}_3^4\gamma\bar{r}_3)a_{47\sigma_2} = \bar{r}_3^9\gamma\bar{r}_3^6\bar{r}_1^2 a_{47\sigma_2}.$$

This implies that in the C_4-spectral sequence, the class $\bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1 u_{8\lambda}u_{16\sigma}a_{8\lambda}$ must support a differential of length at most 31. There are two possible choices for the target:

1. $\bar{\mathfrak{d}}_3^7\bar{s}_3\bar{r}_1^2$ at (47, 47) (d_{31}-differential);

2. $\bar{\mathfrak{d}}_3^7\bar{s}_1 u_{2\lambda}u_{21\sigma}a_{19\lambda}a_{\sigma_2}$ at $(47,39)$ (d_{23}-differential).

Class (1) is impossible for natuality reasons because the class $\bar{\mathfrak{d}}_3^7\bar{s}_3\bar{r}_1^2$ does not restrict to $\bar{r}_3^9\gamma\bar{r}_3^6\bar{r}_1^2 a_{47\sigma_2}$. Therefore, the target must be class (2) and we deduce the desired d_{23}-differential. □

All the other d_{23}-differentials are obtained from the differentials in Proposition 17.2 by using product structures with the classes α and γ^4 (see the discussion after Propositon 16.4). These differentials are the orange differentials in Figure 38.

17.3 d_{19} and d_{27}-differentials

Lemma 17.3. *The following d_{29}-differentials exist:*

1. $d_{29}(2\bar{\mathfrak{d}}_3^3\bar{\mathfrak{d}}_1 u_{7\lambda}u_{10\sigma}a_{3\lambda}) = \bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1^2 u_{16\sigma}a_{17\lambda}a_\sigma$ $(d_{29}(2(34,6)) = (33,35))$;
2. $d_{29}(\bar{\mathfrak{d}}_3^2\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{8\sigma}) = \bar{\mathfrak{d}}_3^5 u_\lambda u_{14\sigma}a_{14\lambda}a_\sigma$ $(d_{29}((32,0)) = (31,29))$;
3. $d_{29}(\bar{\mathfrak{d}}_3^{22}\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{68\sigma}a_{60\lambda}) = \bar{\mathfrak{d}}_3^{25} u_\lambda u_{74\sigma}a_{74\lambda}a_\sigma$ $(d_{29}((152,120)) = (151,149))$;
4. $d_{29}(\bar{\mathfrak{d}}_3^9\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{28\sigma}a_{21\lambda}a_\sigma) = 2\bar{\mathfrak{d}}_3^{12} u_{36\sigma}a_{36\lambda}$ $(d_{29}((73,43)) = 2(72,72))$;
5. $d_{29}(\bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{16\sigma}a_{9\lambda}a_\sigma) = 2\bar{\mathfrak{d}}_3^8 u_{24\sigma}a_{24\lambda}$ $(d_{29}((49,19)) = 2(48,48))$.

Proof. (1): This follows directly from Theorem 15.5.

(2): If we multiply the class $\bar{\mathfrak{d}}_3^2\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{8\sigma}$ at $(32,0)$ by β and use Theorem 15.5, we deduce the d_{29}-differential

$$d_{29}(\bar{\mathfrak{d}}_3^5\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{16\sigma}a_{9\lambda}a_\sigma) = 2\bar{\mathfrak{d}}_3^8 u_{24\sigma}a_{24\lambda}\ (d_{29}(49,19) = 2(48,48)).$$

Therefore, the class $\bar{\mathfrak{d}}_3^2\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{8\sigma}$ at $(32,0)$ must support a differential of length at most 29. For degree reasons, the only possible target is the class at $(31,29)$.

(3): Consider the class $\overline{\mathfrak{d}}_3^{22}\overline{\mathfrak{d}}_1^2 u_{8\lambda}u_{68\sigma}a_{60\lambda}$ at $(152, 120)$. We will show that this class must support a differential of length at most 29. Once we have shown this, the only possible target will be the class $\overline{\mathfrak{d}}_3^{25} u_\lambda u_{74\sigma}a_{74\lambda}a_\sigma$ at $(151, 149)$.

By Theorem 15.5, we have the d_{29}-differential.

$$d_{29}(\overline{\mathfrak{d}}_3^{24}\overline{\mathfrak{d}}_1 u_{8\lambda}u_{72\sigma}a_{65\lambda}a_\sigma) = \overline{\mathfrak{d}}_3^{26}\overline{\mathfrak{d}}_1^2 u_{80\sigma}a_{80\lambda} \;\; (d_{29}(161, 131) = (160, 160)).$$

If we multiply the target of this differential by ϵ', we get the class $2\overline{\mathfrak{d}}_3^{28} u_{84\sigma}a_{84\lambda}$ at $(168, 168)$. This class must be killed by a differential of length at most 29. For degree reasons, the only possibility is the d_{29}-differential

$$d_{29}(\overline{\mathfrak{d}}_3^{25}\overline{\mathfrak{d}}_1^2 u_{8\lambda}u_{76\sigma}a_{69\lambda}a_\sigma) = 2\overline{\mathfrak{d}}_3^{28} u_{84\sigma}a_{84\lambda} \;\; (d_{29}(169, 139) = 2(168, 168)).$$

The source of this differential is equal to $\beta \cdot (152, 120)$. Therefore, the class $\overline{\mathfrak{d}}_3^{22}\overline{\mathfrak{d}}_1^2 u_{8\lambda}u_{68\sigma}a_{60\lambda}$ at $(152, 120)$ must support a differential of length 29, as desired.

(4): By Theorem 15.5, we have the d_{29}-differential

$$d_{29}(\overline{\mathfrak{d}}_3^{8}\overline{\mathfrak{d}}_1 u_{8\lambda}u_{24\sigma}a_{17\lambda}a_\sigma) = \overline{\mathfrak{d}}_3^{10}\overline{\mathfrak{d}}_1^2 u_{32\sigma}a_{32\lambda} \;\; (d_{29}(65, 35) = (64, 64)).$$

If we multiply the target of this differential by ϵ', we get the class $2\overline{\mathfrak{d}}_3^{12} u_{36\sigma}a_{36\lambda}$ at $(72, 72)$. Therefore, this class at $(72, 72)$ must be killed by a differential of length at most 29. The only possibility is the d_{29}-differential that we claimed.

(5): This differential is proven in the proof of Theorem 15.5. □

Lemma 17.4. *The following d_{35}-differentials exist:*

$$d_{35}(\overline{\mathfrak{d}}_3^{17}\overline{s}_1 u_{12\lambda}u_{51\sigma}a_{39\lambda}a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{20}u_{3\lambda}u_{60\sigma}a_{57\lambda} \ \ (d_{35}(127,79) = 2(126,114)),$$

$$d_{35}(\overline{\mathfrak{d}}_3^{21}\overline{s}_1 u_{12\lambda}u_{63\sigma}a_{51\lambda}a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{24}u_{3\lambda}u_{72\sigma}a_{69\lambda} \ \ (d_{35}(151,103) = 2(150,138)).$$

Proof. Consider the target of the first differential. By Theorem 18.1, it must be killed on or before the E_{61}-page. The possibilities for the sources are

1. $\bar{\mathfrak{d}}_3^{17}\bar{s}_3^3$ at $(127,95)$;

2. $\bar{\mathfrak{d}}_3^{18}\bar{s}_3$ at $(127,95)$;

3. $\overline{\mathfrak{d}}_3^{17}\overline{s}_1 u_{12\lambda}u_{51\sigma}a_{39\lambda}a_{\sigma_2}$ at $(127,79)$;

4. $\bar{\mathfrak{d}}_3^{15}\bar{s}_3\bar{r}_1^2$ at $(127,63)$;

5. $\overline{\mathfrak{d}}_3^{15}\overline{s}_1 u_{18\lambda}u_{45\sigma}a_{27\lambda}a_{\sigma_2}$ at $(127,55)$;

(1) is impossible because by Proposition 16.4, the class $\bar{\mathfrak{d}}_3^{17}\bar{s}_3^3$ at $(127,95)$ is the target of the d_{31}-differential

$$d_{31}(\overline{\mathfrak{d}}_3^{16}u_{16\lambda}u_{48\sigma}a_{32\lambda}) = \bar{\mathfrak{d}}_3^{17}\bar{s}_3^3 \ \ (d_{31}(128,64) = (127,95)).$$

(2) is impossible because $\bar{\mathfrak{d}}_3^{18}\bar{s}_3$ at $(127,95)$ supports the d_{31}-differential

$$d_{31}(\bar{\mathfrak{d}}_3^{18}\bar{s}_3) = \bar{\mathfrak{d}}_3^{20}\bar{s}_3^2 \ \ (d_{31}(127,95) = (126,126)).$$

This is a d_{31}-differential between $i^*_{C_2}\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes.

(4) is impossible because $\bar{\mathfrak{d}}_3^{15}\bar{s}_3\bar{r}_1^2$ at $(127, 63)$ is the target of the d_{31}-differential

$$d_{31}(\bar{\mathfrak{d}}_3^{12}\bar{s}_3^2\bar{r}_1^2) = \bar{\mathfrak{d}}_3^{15}\bar{s}_3\bar{r}_1^2 \ \ (d_{31}(128, 32) = (127, 63))$$

between $i_{C_2}^*\mathrm{BP}^{(\!(C_4)\!)}\langle 1\rangle$-truncation classes.

(5) is impossible because by Proposition 17.2, this class is the target of the d_{23}-differential

$$d_{23}(\bar{\mathfrak{d}}_3^{13}\bar{\mathfrak{d}}_1 u_{24\lambda}u_{40\sigma}a_{16\lambda}) = \bar{\mathfrak{d}}_3^{15}\bar{s}_1 u_{18\lambda}u_{45\sigma}a_{27\lambda}a_{\sigma_2} \ \ (d_{23}(128, 32) = (127, 63))$$

It follows that the only possibility for the source is (3), and we deduce our claimed d_{35}-differential.

The second differential is proven in the exact same way, except that we just need the extra fact that the class $\bar{\mathfrak{d}}_3^{21}\bar{s}_3^3$ at $(151, 119)$ supports a d_{43}-differential (Theorem 16.5). □

Theorem 17.5. *The following differentials exist:*

1. $d_{19}(2\bar{\mathfrak{d}}_3^{30}u_{5\lambda}u_{90\sigma}a_{85\lambda}) = \bar{\mathfrak{d}}_3^{30}\bar{s}_3^3 \ \ (d_{19}(2(190, 170)) = (189, 189))$;
   $d_{19}(2\bar{\mathfrak{d}}_3^{26}u_{5\lambda}u_{78\sigma}a_{73\lambda}) = \bar{\mathfrak{d}}_3^{26}\bar{s}_3^3 \ \ (d_{19}(166, 146) = (165, 165))$;

2. $d_{19}(\bar{\mathfrak{d}}_3^{31}\bar{s}_1 u_{4\lambda}u_{93\sigma}a_{89\lambda}a_{\sigma_2}) = \bar{\mathfrak{d}}_3^{32}\bar{\mathfrak{d}}_1^2 u_{96\sigma}a_{98\lambda}a_{2\sigma} \ \ (d_{19}(195, 179) = (194, 198))$;
   $d_{19}(\bar{\mathfrak{d}}_3^{27}\bar{s}_1 u_{4\lambda}u_{81\sigma}a_{77\lambda}a_{\sigma_2}) = \bar{\mathfrak{d}}_3^{28}\bar{\mathfrak{d}}_1^2 u_{84\sigma}a_{86\lambda}a_{2\sigma} \ \ (d_{19}(171, 155) = (170, 174))$;

3. $d_{19}(2\bar{\mathfrak{d}}_3^{32}u_{5\lambda}u_{96\sigma}a_{91\lambda}) = \bar{\mathfrak{d}}_3^{32}\bar{s}_3^3 \ \ (d_{19}(2(202, 182)) = (201, 201))$;
   $d_{19}(2\bar{\mathfrak{d}}_3^{28}u_{5\lambda}u_{84\sigma}a_{79\lambda}) = \bar{\mathfrak{d}}_3^{28}\bar{s}_3^3 \ \ (d_{19}(2(178, 158)) = (177, 177))$;

4. $d_{19}(2\bar{\mathfrak{d}}_3^{30}u_{13\lambda}u_{90\sigma}a_{77\lambda}) = \bar{\mathfrak{d}}_3^{30}\bar{s}_3^3 \ \ (d_{19}(2(206, 154)) = (205, 173))$;
   $d_{19}(2\bar{\mathfrak{d}}_3^{26}u_{13\lambda}u_{78\sigma}a_{65\lambda}) = \bar{\mathfrak{d}}_3^{26}\bar{s}_3^3 \ \ (d_{19}(2(182, 130)) = (181, 149))$;

5. $d_{19}(\bar{\mathfrak{d}}_3^{31}\bar{s}_1 u_{12\lambda}u_{93\sigma}a_{81\lambda}a_{\sigma_2}) = 2\bar{\mathfrak{d}}_3^{32}\bar{\mathfrak{d}}_1^2 u_{7\lambda}u_{98\sigma}a_{91\lambda} \ \ (d_{19}(211, 163) = 2(210, 182))$;
   $d_{19}(\bar{\mathfrak{d}}_3^{27}\bar{s}_1 u_{12\lambda}u_{81\sigma}a_{69\lambda}a_{\sigma_2}) = 2\bar{\mathfrak{d}}_3^{28}\bar{\mathfrak{d}}_1^2 u_{7\lambda}u_{86\sigma}a_{79\lambda} \ \ (d_{19}(187, 139) = 2(186, 158))$;

6. $d_{19}(2\overline{\mathfrak{d}}_3^{32}u_{13\lambda}u_{96\sigma}a_{83\lambda}) = \overline{\mathfrak{d}}_3^{32}\overline{s}_3^3$ $(d_{19}(2(218,166)) = (217,185))$;
   $d_{19}(2\overline{\mathfrak{d}}_3^{28}u_{13\lambda}u_{84\sigma}a_{71\lambda}) = \overline{\mathfrak{d}}_3^{28}\overline{s}_3^3$ $(d_{19}(2(194,142)) = (193,161))$;

7. $d_{27}(\overline{\mathfrak{d}}_3^{31}\overline{s}_1u_{10\lambda}u_{93\sigma}a_{83\lambda}a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{33}\overline{\mathfrak{d}}_1u_{3\lambda}u_{100\sigma}a_{97\lambda}$ $(d_{27}(207,167) = 2(206,194))$;
   $d_{27}(\overline{\mathfrak{d}}_3^{27}\overline{s}_1u_{10\lambda}u_{81\sigma}a_{71\lambda}a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{29}\overline{\mathfrak{d}}_1u_{3\lambda}u_{88\sigma}a_{85\lambda}$ $(d_{27}(183,143) = 2(182,170))$;

8. $d_{27}(\overline{\mathfrak{d}}_3^{33}\overline{s}_1u_{6\lambda}u_{99\sigma}a_{93\lambda}a_{\sigma_2}) = \overline{\mathfrak{d}}_3^{35}\overline{\mathfrak{d}}_1u_{104\sigma}a_{106\lambda}a_{2\sigma}$ $(d_{27}(211,187) = (210,214))$;
   $d_{27}(\overline{\mathfrak{d}}_3^{29}\overline{s}_1u_{6\lambda}u_{87\sigma}a_{81\lambda}a_{\sigma_2}) = \overline{\mathfrak{d}}_3^{31}\overline{\mathfrak{d}}_1u_{92\sigma}a_{94\lambda}a_{2\sigma}$ $(d_{27}(187,163) = (186,190))$.

Proof. (1): Consider the class $2\overline{\mathfrak{d}}_3^{30}u_{5\lambda}u_{90\sigma}a_{85\lambda}$ at $(190,170)$. By Theorem 18.1, this class must die on or before the E_{61}-page. If this class supports a differential, then we are done.

If it is the target of a differential, then for degree reasons, the only possibility for the source is the class $\overline{\mathfrak{d}}_3^{25}\overline{s}_3^3$ at $(191,127)$. However, by Proposition 14.11 and the discussion afterwards, this class is the target of the d_{31}-differential

$$d_{31}(2\overline{\mathfrak{d}}_3^{24}u_{24\lambda}u_{72\sigma}a_{48\lambda}) = \overline{\mathfrak{d}}_3^{25}\overline{s}_3^3 \ \ (d_{31}(2(192,96)) = (191,127)).$$

Therefore, this class cannot be the target of a differential. This proves the first differential in (1).

The second differential in (1) is proven by the exact same method.

(2): To prove the first differential in (2), consider the class $\overline{\mathfrak{d}}_3^{32}\overline{\mathfrak{d}}_1^2u_{96\sigma}a_{98\lambda}a_{2\sigma}$ at $(194,198)$. By Theorem 18.1, this class must die on or before the E_{61}-page. For degree reasons, the only possible source is the class $\overline{\mathfrak{d}}_3^{31}\overline{s}_1u_{4\lambda}u_{93\sigma}a_{89\lambda}a_{\sigma_2}$ at $(195,179)$. This proves the first differential in (2). The second differential is proven in the exact same way.

(5): For the first differential, consider the class $2\overline{\mathfrak{d}}_3^{32}\overline{\mathfrak{d}}_1^2u_{7\lambda}u_{98\sigma}a_{91\lambda}$ at $2(210,182)$. By Theorem 18.1, this class must die on or before the E_{61}-page. If this class is the source of a differential, the target must be the class $\overline{\mathfrak{d}}_3^{35}u_{104\sigma}a_{105\lambda}a_{\sigma}$ at $(209,211)$. This is impossible because we have proven in Theorem 16.5 that the class at $(209, 211)$ is the target of a d_{61}-differential.

Therefore, this class must be killed by a differential of length at most 61. For degree reasons, the only possible source is the class $\overline{\mathfrak{d}}_3^{31}\overline{s}_1 u_{12\lambda}u_{93\sigma}a_{81\lambda}a_{\sigma_2}$ at $(211, 163)$. This proves the first differential in (5). The second differential is proven in the exact same way.

(3): For the first differential, consider the class $\overline{\mathfrak{d}}_3^{32}\overline{s}_3^3$ at $(201, 201)$. By Theorem 18.1, this class must be killed on or before the E_{61}-page. For degree reasons, the only possible sources are the following classes:

- $2\overline{\mathfrak{d}}_3^{32}u_{5\lambda}u_{96\sigma}a_{91\lambda}$ at $(202, 182)$;
- $2\overline{\mathfrak{d}}_3^{31}\overline{\mathfrak{d}}_1 u_{7\lambda}u_{94\sigma}a_{87\lambda}$ at $(202, 174)$;
- $2\overline{\mathfrak{d}}_3^{28}\overline{\mathfrak{d}}_1^2 u_{15\lambda}u_{86\sigma}a_{71\lambda}$ at $(202, 142)$.

If the class $2\overline{\mathfrak{d}}_3^{31}\overline{\mathfrak{d}}_1 u_{7\lambda}u_{94\sigma}a_{87\lambda}$ at $(202, 174)$ is the source, then the differential will be a d_{27}-differential. However, by Theorem 15.5 the class $2\overline{\mathfrak{d}}_3^{30}u_{7\lambda}u_{90\sigma}a_{83\lambda}$ at $(194, 166)$ support the d_{29}-differential

$$d_{29}(2\overline{\mathfrak{d}}_3^{30}u_{7\lambda}u_{90\sigma}a_{83\lambda}) = \overline{\mathfrak{d}}_3^{32}\overline{\mathfrak{d}}_1 u_{96\sigma}a_{97\lambda}a_\sigma \ \ (d_{29}(2(194, 166)) = (193, 195)).$$

Since $2(194, 166) \cdot \epsilon' = 2(202, 174)$, this is a contradiction.

The class $2\overline{\mathfrak{d}}_3^{28}\overline{\mathfrak{d}}_1^2 u_{15\lambda}u_{86\sigma}a_{71\lambda}$ at $(202, 142)$ cannot be the source either because it is the target of the d_{19}-differential

$$d_{19}(\overline{\mathfrak{d}}_3^{27}\overline{s}_1 u_{20\lambda}u_{81\sigma}a_{61\lambda}a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{28}\overline{\mathfrak{d}}_1^2 u_{15\lambda}u_{86\sigma}a_{71\lambda} \ \ (d_{19}(203, 123) = 2(202, 142)).$$

This d_{19}-differential can be deduced from the second differential of (2) by using multiplicative structures with the classes α and γ^4.

Therefore, the only possibility for the source is the class $2\overline{\mathfrak{d}}_3^{32} u_{5\lambda} u_{96\sigma} a_{91\lambda}$ at $(202, 182)$. This proves the first d_{19}-differential.

For the second differential, consider the class $\overline{\mathfrak{d}}_3^{28} \overline{s}_3^3$ at $(177, 177)$. By Theorem 18.1, it must be killed by a differential of length at most 61. The possible sources are the following classes:

- $2\overline{\mathfrak{d}}_3^{28} u_{5\lambda} u_{84\sigma} a_{79\lambda}$ at $(178, 158)$;
- $2\overline{\mathfrak{d}}_3^{27} \overline{\mathfrak{d}}_1 u_{7\lambda} u_{82\sigma} a_{75\lambda}$ at $(178, 150)$;
- $2\overline{\mathfrak{d}}_3^{24} \overline{\mathfrak{d}}_1^2 u_{15\lambda} u_{74\sigma} a_{59\lambda}$ at $(178, 118)$.

By using Lemma 17.3 (1) and multiplicative structures with α, the class $2\overline{\mathfrak{d}}_3^{28} u_{5\lambda} u_{84\sigma} a_{79\lambda}$ at $(178, 150)$ support a d_{29}-differential, and therefore cannot be the source. The class $2\,\overline{\mathfrak{d}}_3^{24} \overline{\mathfrak{d}}_1^2 u_{15\lambda} u_{74\sigma} a_{59\lambda}$ at $(178, 118)$ is the target of the d_{19}-differential

$$d_{19}(\overline{\mathfrak{d}}_3^{23} \overline{s}_1 u_{20\lambda} u_{69\sigma} a_{49\lambda} a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{24} \overline{\mathfrak{d}}_1^2 u_{15\lambda} u_{74\sigma} a_{59\lambda} \;\; (d_{19}(179, 99) = 2(178, 118)).$$

This d_{19}-differential can be deduced from the first differential of (2) by using multiplicative structures with the classes α and γ^4.

Therefore, there is only one possible source left, and this leads to the desired d_{19}-differential.

(4): To prove the first differential in (4), we will first multiply the source by γ^4 and prove the d_{19}-differential

$$d_{19}(2\overline{\mathfrak{d}}_3^{38} u_{29\lambda} u_{114\sigma} a_{85\lambda}) = \overline{\mathfrak{d}}_3^{38} \overline{s}_3^3 \;\; (d_{19}(2(286, 170)) = (285, 189)).$$

Once we have proven this, we can immediately deduce the first differential.

Consider the class $\overline{\mathfrak{d}}_3^{38}\overline{s}_3^3$ at $(285, 189)$. By Theorem 18.1, this class must die on or before the E_{61}-page. For degree reasons, it cannot support a differential (because the length of that differential must be $15 < r \leq 61$). Therefore, it must be the target of a differential. For degree reasons, the possible sources are the following classes:

- $2\overline{\mathfrak{d}}_3^{38}u_{29\lambda}u_{114\sigma}a_{85\lambda}$ at $(286, 170)$;
- $2\overline{\mathfrak{d}}_3^{36}u_{35\lambda}u_{108\sigma}a_{73\lambda}$ at $(286, 146)$.

Using the first differential in Lemma 17.4 and multiplicative structures with γ^8, we deduce that the class $2\overline{\mathfrak{d}}_3^{36}u_{35\lambda}u_{108\sigma}a_{73\lambda}$ at $(286, 146)$ is the target of the d_{35}-differential

$$d_{35}(\overline{\mathfrak{d}}_3^{33}\overline{s}_1u_{44\lambda}u_{99\sigma}a_{55\lambda}a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{36}u_{35\lambda}u_{108\sigma}a_{73\lambda} \;\; (d_{35}(287, 111) = 2(286, 146)).$$

Therefore, the only possible source left is the class $2\overline{\mathfrak{d}}_3^{38}u_{29\lambda}u_{114\sigma}a_{85\lambda}$ at $(286, 170)$. This proves our desired differential.

The proof of the second differential is exactly the same (except near the end we use the second differential in Lemma 17.4 and multiplicative structures with γ^8 and α).

(7): For the first differential, consider the class $2\overline{\mathfrak{d}}_3^{33}\overline{\mathfrak{d}}_1u_{3\lambda}u_{100\sigma}a_{97\lambda}$ at $(206, 194)$. By Theorem 18.1 and degree reasons, this class must be killed on or before the E_{61}-page. The only possibilities for the source are the following classes:

- $\overline{\mathfrak{d}}_3^{31}\overline{s}_1u_{10\lambda}u_{93\sigma}a_{83\lambda}a_{\sigma_2}$ at $(207, 167)$;
- $\overline{\mathfrak{d}}_3^{29}u_{17\lambda}u_{86\sigma}a_{70\lambda}a_{\sigma}$ at $(207, 141)$.

Using Lemma 17.3 (2) and multiplicative structures with α and γ^4, we deduce that the class $\overline{\mathfrak{d}}_3^{29} u_{17\lambda} u_{86\sigma} a_{70\lambda} a_\sigma$ at $(207, 141)$ is the target of the d_{29}-differential

$$d_{29}(\overline{\mathfrak{d}}_3^{26} \overline{\mathfrak{d}}_1^{2} u_{24\lambda} u_{80\sigma} a_{56\lambda}) = \overline{\mathfrak{d}}_3^{29} u_{17\lambda} u_{86\sigma} a_{70\lambda} a_\sigma \ \ (d_{29}(208, 112) = (207, 141)).$$

Therefore, the source must be the class $\overline{\mathfrak{d}}_3^{31} \overline{s}_1 u_{10\lambda} u_{93\sigma} a_{83\lambda} a_{\sigma_2}$ at $(207, 167)$.

The second differential is proven in the exact same way, except near the end we use Lemma 17.3 (3) and multiplicative structures with α and γ^4 to deduce a d_{29}-differential.

(8): To prove the first differential, consider the class $\overline{\mathfrak{d}}_3^{35} \overline{\mathfrak{d}}_1 u_{104\sigma} a_{106\lambda} a_{2\sigma}$ at $(210, 214)$. By Theorem 18.1, it must be killed on or before the E_{61}-page. The only possibilities for the sources are the following classes:

- $\overline{\mathfrak{d}}_3^{33} \overline{s}_1 u_{6\lambda} u_{99\sigma} a_{93\lambda} a_{\sigma_2}$ at $(211, 187)$;
- $\overline{\mathfrak{d}}_3^{31} \overline{s}_1 u_{12\lambda} u_{93\sigma} a_{81\lambda} a_{\sigma_2}$ at $(211, 163)$.

By (5), the class $(211, 163)$ supports the d_{19}-differential

$$d_{19}(\overline{\mathfrak{d}}_3^{31} \overline{s}_1 u_{12\lambda} u_{93\sigma} a_{81\lambda} a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{32} \overline{\mathfrak{d}}_1^{2} u_{7\lambda} u_{98\sigma} a_{91\lambda} \ \ (d_{19}(211, 163) = 2(210, 182)).$$

Therefore, the only possibility for the source is the class $\overline{\mathfrak{d}}_3^{33} \overline{s}_1 u_{6\lambda} u_{99\sigma} a_{93\lambda} a_{\sigma_2}$ at $(211, 187)$. This proves our desired differential.

The proof of the second differential is exactly the same.

(6): For the first differential, consider the class $\overline{\mathfrak{d}}_3^{32} \overline{s}_3^{3}$ at $(217, 185)$. By Theorem 18.1, this class must die on or before the E_{61}-page. If this class supports a differential, then the only possible target is the class $2\overline{\mathfrak{d}}_3^{36} u_{108\sigma} a_{108\lambda}$ at $(216, 216)$. This is impossible because this class at $(216, 216)$ is the target of the d_{29}-differential

$$d_{29}(\overline{\mathfrak{d}}_3^{33}\overline{\mathfrak{d}}_1^{2}u_{8\lambda}u_{100\sigma}a_{93\lambda}a_{\sigma}) = 2\overline{\mathfrak{d}}_3^{36}u_{108\sigma}a_{108\lambda} \;\; (d_{29}(217, 187) = 2(216, 216)).$$

We can deduce this differential from Lemma 17.3 (4) and multiplicative structures with α.

Therefore, this class must be killed by a differential of length at most 61. The only possibilities for the source are the following classes:

- $2\overline{\mathfrak{d}}_3^{32}u_{13\lambda}u_{96\sigma}a_{83\lambda}$ at $(218, 166)$;
- $2\overline{\mathfrak{d}}_3^{31}\overline{\mathfrak{d}}_1u_{15\lambda}u_{94\sigma}a_{79\lambda}$ at $(218, 158)$;
- $2\overline{\mathfrak{d}}_3^{28}\overline{\mathfrak{d}}_1^{2}u_{23\lambda}u_{86\sigma}a_{63\lambda}$ at $(218, 126)$.

The class $2\overline{\mathfrak{d}}_3^{31}\overline{\mathfrak{d}}_1u_{15\lambda}u_{94\sigma}a_{79\lambda}$ at $(218, 158)$ is the target of the d_{27}-differential

$$d_{27}(\overline{\mathfrak{d}}_3^{29}\overline{s}_1u_{22\lambda}u_{87\sigma}a_{65\lambda}a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{31}\overline{\mathfrak{d}}_1u_{15\lambda}u_{94\sigma}a_{79\lambda} \;\; (d_{27}(219, 131) = 2(218, 158)).$$

We can deduce this differential from the second differential in (8) and multiplication with α and γ^4.

The class $2\overline{\mathfrak{d}}_3^{28}\overline{\mathfrak{d}}_1^{2}u_{23\lambda}u_{86\sigma}a_{63\lambda}$ at $(218, 126)$ is the target of the d_{19}-differential

$$d_{19}(\overline{\mathfrak{d}}_3^{27}\overline{s}_1u_{28\lambda}u_{81\sigma}a_{53\lambda}a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{28}\overline{\mathfrak{d}}_1^{2}u_{23\lambda}u_{86\sigma}a_{63\lambda} \;\; (d_{19}(219, 107) = 2(218, 126)).$$

We can deduce this differential from the second differential in (5) and multiplication with α and γ^4.

It follows that the only possibility left for the source is the class $2\overline{\mathfrak{d}}_3^{32}u_{13\lambda}u_{96\sigma}a_{83\lambda}$ at $(218, 166)$, as desired.

The proof of the second differential is exactly the same. □

All the other d_{19}- and d_{27}-differentials are obtained from the differentials in Proposition 17.5 by using product structures with the classes α and γ^4 (see the discussion after Propositon 16.4). These differentials are the lime-green differentials and the forest-green differentials in Figure 38, respectively.

18 Higher differentials V: d_{29}-differentials and d_{31}-differentials

18.1 d_{29}-differentials

Theorem 18.1. *The following d_{29}-differentials exist:*

1. $d_{29}(\bar{\mathfrak{d}}_3^{26}\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{80\sigma}a_{72\lambda}) = \bar{\mathfrak{d}}_3^{29} u_\lambda u_{86\sigma}a_{86\lambda}a_\sigma$ $(d_{29}(176,144) = (175,173))$;
2. $d_{29}(\bar{\mathfrak{d}}_3^{27} u_{8\lambda}u_{80\sigma}a_{73\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^{29}\bar{\mathfrak{d}}_1 u_{88\sigma}a_{88\lambda}$ $(d_{29}(177,147) = (176,176))$;
3. $d_{29}(2\bar{\mathfrak{d}}_3^{27}\bar{\mathfrak{d}}_1 u_{7\lambda}u_{82\sigma}a_{75\lambda}) = \bar{\mathfrak{d}}_3^{29}\bar{\mathfrak{d}}_1^2 u_{88\sigma}a_{89\lambda}a_\sigma$ $(d_{29}(2(178,150)) = (177,179))$;
4. $d_{29}(\bar{\mathfrak{d}}_3^{28}\bar{\mathfrak{d}}_1 u_{8\lambda}u_{84\sigma}a_{77\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^{30}\bar{\mathfrak{d}}_1^2 u_{92\sigma}a_{92\lambda}$ $(d_{29}(185,155) = (184,184))$;
5. $d_{29}(\bar{\mathfrak{d}}_3^{29}\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{88\sigma}a_{81\lambda}a_\sigma) = 2\bar{\mathfrak{d}}_3^{32} u_{96\sigma}a_{96\lambda}$ $(d_{29}(193,163) = 2(192,192))$;
6. $d_{29}(2\bar{\mathfrak{d}}_3^{30} u_{7\lambda}u_{90\sigma}a_{83\lambda}) = \bar{\mathfrak{d}}_3^{32}\bar{\mathfrak{d}}_1 u_{96\sigma}a_{97\lambda}a_\sigma$ $(d_{29}(2(194,166)) = (193,195))$;
7. $d_{29}(\bar{\mathfrak{d}}_3^{30}\bar{\mathfrak{d}}_1^2 u_{8\lambda}u_{92\sigma}a_{84\lambda}) = \bar{\mathfrak{d}}_3^{33} u_\lambda u_{98\sigma}a_{98\lambda}a_\sigma$ $(d_{29}(200,168) = (199,197)$;
8. $d_{29}(\bar{\mathfrak{d}}_3^{31} u_{8\lambda}u_{92\sigma}a_{85\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^{33}\bar{\mathfrak{d}}_1 u_{100\sigma}a_{100\lambda}$ $(d_{29}(201,171) = (200,200))$;
9. $d_{29}(2\bar{\mathfrak{d}}_3^{31}\bar{\mathfrak{d}}_1 u_{7\lambda}u_{94\sigma}a_{87\lambda}) = \bar{\mathfrak{d}}_3^{33}\bar{\mathfrak{d}}_1^2 u_{100\sigma}a_{101\lambda}a_\sigma$ $(d_{29}(2(202,174)) = (201,203))$;
10. $d_{29}(\bar{\mathfrak{d}}_3^{32}\bar{\mathfrak{d}}_1 u_{8\lambda}u_{96\sigma}a_{89\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^{34}\bar{\mathfrak{d}}_1^2 u_{104\sigma}a_{104\lambda}$ $(d_{29}(209,179) = (208,208))$;

11. $d_{29}(\overline{\mathfrak{d}}_3^{33}\overline{\mathfrak{d}}_1^2 u_{8\lambda}u_{100\sigma}a_{93\lambda}a_\sigma) = 2\overline{\mathfrak{d}}_3^{36} u_{108\sigma}a_{108\lambda}$ $(d_{29}(217,187) = 2(216,216))$;

12. $d_{29}(2\overline{\mathfrak{d}}_3^{34} u_{7\lambda}u_{102\sigma}a_{95\lambda}) = \overline{\mathfrak{d}}_3^{36}\overline{\mathfrak{d}}_1 u_{108\sigma}a_{109\lambda}a_\sigma$ $(d_{29}(2(218,190)) = (217,219))$;

13. $d_{29}(2\overline{\mathfrak{d}}_3^{28} u_{11\lambda}u_{84\sigma}a_{73\lambda}) = \overline{\mathfrak{d}}_3^{30}\overline{\mathfrak{d}}_1 u_{4\lambda}u_{90\sigma}a_{87\lambda}a_\sigma$ $(d_{29}(2(190,146)) = (189,175))$;

14. $d_{29}(2\overline{\mathfrak{d}}_3^{29}\overline{\mathfrak{d}}_1 u_{11\lambda}u_{88\sigma}a_{77\lambda}) = \overline{\mathfrak{d}}_3^{31}\overline{\mathfrak{d}}_1^2 u_{4\lambda}u_{94\sigma}a_{91\lambda}a_\sigma$ $(d_{29}(2(198,154)) = (197,183))$;

15. $d_{29}(\overline{\mathfrak{d}}_3^{30}\overline{\mathfrak{d}}_1 u_{12\lambda}u_{90\sigma}a_{79\lambda}a_\sigma) = \overline{\mathfrak{d}}_3^{32}\overline{\mathfrak{d}}_1^2 u_{4\lambda}u_{98\sigma}a_{94\lambda}$ $(d_{29}(205,159) = (204,188))$;

16. $d_{29}(2\overline{\mathfrak{d}}_3^{32} u_{11\lambda}u_{96\sigma}a_{85\lambda}) = \overline{\mathfrak{d}}_3^{34}\overline{\mathfrak{d}}_1 u_{4\lambda}u_{102\sigma}a_{99\lambda}a_\sigma$ $(d_{29}(2(214,170)) = (213,199))$;

17. $d_{29}(2\overline{\mathfrak{d}}_3^{33}\overline{\mathfrak{d}}_1 u_{11\lambda}u_{100\sigma}a_{89\lambda}) = \overline{\mathfrak{d}}_3^{35}\overline{\mathfrak{d}}_1^2 u_{4\lambda}u_{106\sigma}a_{103\lambda}a_\sigma$ $(d_{29}(2(222,178)) = (221,207))$;

18. $d_{29}(\overline{\mathfrak{d}}_3^{34}\overline{\mathfrak{d}}_1 u_{12\lambda}u_{102\sigma}a_{91\lambda}a_\sigma) = \overline{\mathfrak{d}}_3^{36}\overline{\mathfrak{d}}_1^2 u_{4\lambda}u_{110\sigma}a_{106\lambda}$ $(d_{29}(229,183) = (228,212))$.

Proof. The differentials (2), (3), (5), (10) are immediate from Theorem 15.5.

(1): Consider the class $\overline{\mathfrak{d}}_3^{26}\overline{\mathfrak{d}}_1^2 u_{8\lambda}u_{80\sigma}a_{72\lambda}$ at $(176,144)$. If we multiply this class by β, we get the class $(193,163)$, which supports the d_{29}-differential (5). Therefore, the class $\overline{\mathfrak{d}}_3^{26}\overline{\mathfrak{d}}_1^2 u_{8\lambda}u_{80\sigma}a_{72\lambda}$ at $(176,144)$ must support a differential of length at most 29. The only possibility is the d_{29}-differential that we claimed.

(4): This differential follows from (2) via multiplication by ϵ'.

(6): Consider the class $\overline{\mathfrak{d}}_3^{32}\overline{\mathfrak{d}}_1 u_{96\sigma}a_{97\lambda}a_\sigma$ at $(193,195)$. By Theorem , this class must be killed by a differential of length at most 61. By degree reasons, the only possibility is the d_{29}-differential we claimed.

(8): Consider the class $\overline{\mathfrak{d}}_3^{31} u_{8\lambda}u_{92\sigma}a_{85\lambda}a_\sigma$ at $(201,171)$. If we multiply this class by ϵ', we get the class $\overline{\mathfrak{d}}_3^{32}\overline{\mathfrak{d}}_1 u_{8\lambda}u_{96\sigma}a_{89\lambda}a_\sigma$ at $(209,179)$, which supports differential (10). Therefore, the class $\overline{\mathfrak{d}}_3^{31} u_{8\lambda} u_{92\sigma} a_{85\lambda} a_\sigma$ at $(201, 171)$ must support a differential of length at most 29. The only possibility is the differential that we claimed.

(9): This differential follows form (6) via multiplication by ϵ'.

(11): This differential follows from (10) via multiplication by ϵ'.

(7): Consider the class $\overline{\mathfrak{d}}_3^{30} \overline{\mathfrak{d}}_1^2 u_{8\lambda} u_{92\sigma} a_{84\lambda}$ at $(200, 168)$. If we multiply this class by β, we get the class $\overline{\mathfrak{d}}_3^{33} \overline{\mathfrak{d}}_1^2 u_{8\lambda} u_{100\sigma} a_{93\lambda} a_\sigma$ at $(217, 187)$, which supports differential (11). Therefore, the class $\overline{\mathfrak{d}}_3^{30} \overline{\mathfrak{d}}_1^2 u_{8\lambda} u_{92\sigma} a_{84\lambda}$ at $(200, 168)$ must support a differential of length at most 29. The only possibility is the differential that we claimed.

(12): Consider the class $\overline{\mathfrak{d}}_3^{36} \overline{\mathfrak{d}}_1 u_{108\sigma} a_{109\lambda} a_\sigma$ at $(217, 219)$. By Theorem , this class must be killed by a differential of length at most 61. For degree reasons, the only possible differential is the d_{29}-differential that we claimed.

(13): Consider the class $\overline{\mathfrak{d}}_3^{30} \overline{\mathfrak{d}}_1 u_{4\lambda} u_{90\sigma} a_{87\lambda} a_\sigma$ at $(189, 175)$. By Theorem , this class must die on or before the E_{61}-page. For degree reasons, the only possible way for this to happen is for the claimed d_{29}-differential to exist.

(14): This differential follows from (13) via multiplication by ϵ' (alternatively, we can also use Theorem).

(15): Consider the class $\overline{\mathfrak{d}}_3^{30} \overline{\mathfrak{d}}_1 u_{12\lambda} u_{90\sigma} a_{79\lambda} a_\sigma$ at $(205, 159)$. If we multiply this class by ϵ', we get the class $\overline{\mathfrak{d}}_3^{31} \overline{\mathfrak{d}}_1^2 u_{12\lambda} u_{94\sigma} a_{83\lambda} a_\sigma$ at $(213, 167)$. This class support a d_{53}-differential by Theorem 16.5. Therefore, the class $\overline{\mathfrak{d}}_3^{30} \overline{\mathfrak{d}}_1 u_{12\lambda} u_{90\sigma} a_{79\lambda} a_\sigma$ at $(205, 159)$ must support a differential of length at most 53. For degree reasons, this implies the d_{29}-differential that we claimed.

(16): Consider the class $\overline{\mathfrak{d}}_3^{34} \overline{\mathfrak{d}}_1 u_{4\lambda} u_{102\sigma} a_{99\lambda} a_\sigma$ at $(213, 199)$. By Theorem 18.1, this class must die on or before the E_{61}-page. For degree reasons, the only way for this to happen is for the claimed d_{29}-differential to exist.

(17): This differential follows from (16) via multiplication by ϵ'.

(18): Consider the class $\bar{\mathfrak{d}}_3^{34}\bar{\mathfrak{d}}_1 u_{12\lambda}u_{102\sigma}a_{91\lambda}a_\sigma$ at $(229, 183)$. If we multiply this class by η', we get the class $2\bar{\mathfrak{d}}_3^{34}\bar{\mathfrak{d}}_1^2 u_{11\lambda}u_{104\sigma}a_{93\lambda}$ at $(230, 186)$. By Theorem 16.5, this class supports a d_{53}-differential. Therefore, the class $\bar{\mathfrak{d}}_3^{34}\bar{\mathfrak{d}}_1 u_{12\lambda}u_{102\sigma}a_{91\lambda}a_\sigma$ at $(229, 183)$ must support a differential of length at most 53. For degree reasons, we deduce the claimed d_{29}-differential.

□

Theorem 18.1, combined with using multiplicative structures on the classes α and γ^4, produces all the d_{29}-differentials in SliceSS($BP^{((C_4))}\langle 2\rangle$) (see our discussion after Proposition 16.4). These differentials are shown in Figure 39.

18.2 d_{31}-differentials

Almost all of the d_{31}-differentials are induced d_{31}-differentials from $i^*_{C_2}BP^{((C_4))}\langle 1\rangle$-truncation classes, and they can be proven by using the transfer and the restriction map (see Section 14.2).

The rest of the d_{31}-differentials follows from Proposition 14.11 (and the discussion afterwards), Proposition 16.4, and multiplication with the following classes:

- α at $(48, 48)$ (permanent cycle);
- γ^8 at $(160, 32)$ (permanent cycle);
- $\bar{\mathfrak{d}}_3^{12} u_{16\lambda}u_{36\sigma}a_{20\lambda}$ at $(104, 40)$ (d_{31}-cycle).

The d_{31}-differentials are shown in Figure 40.

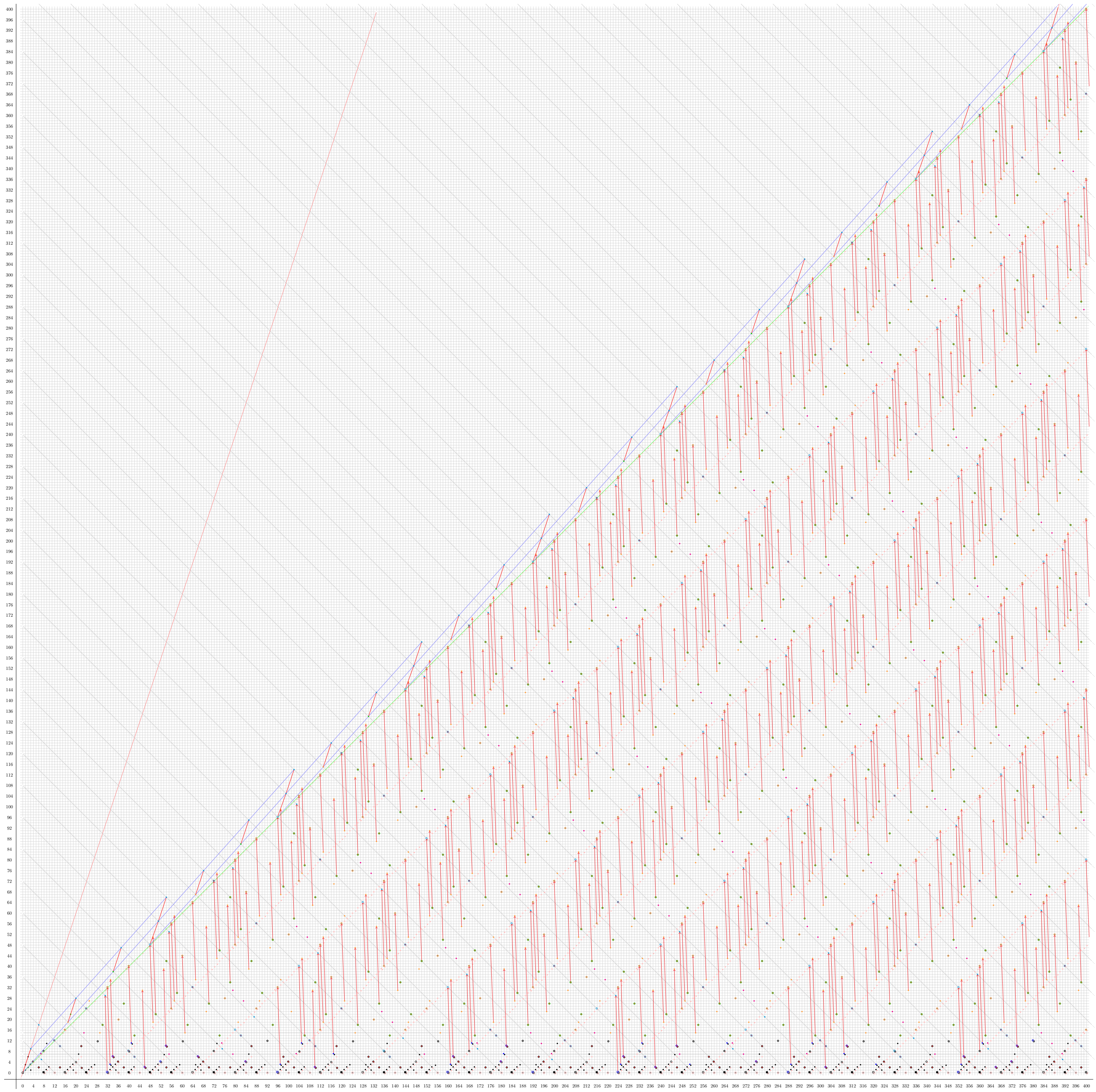


Figure 39: d_{29}-differentials in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

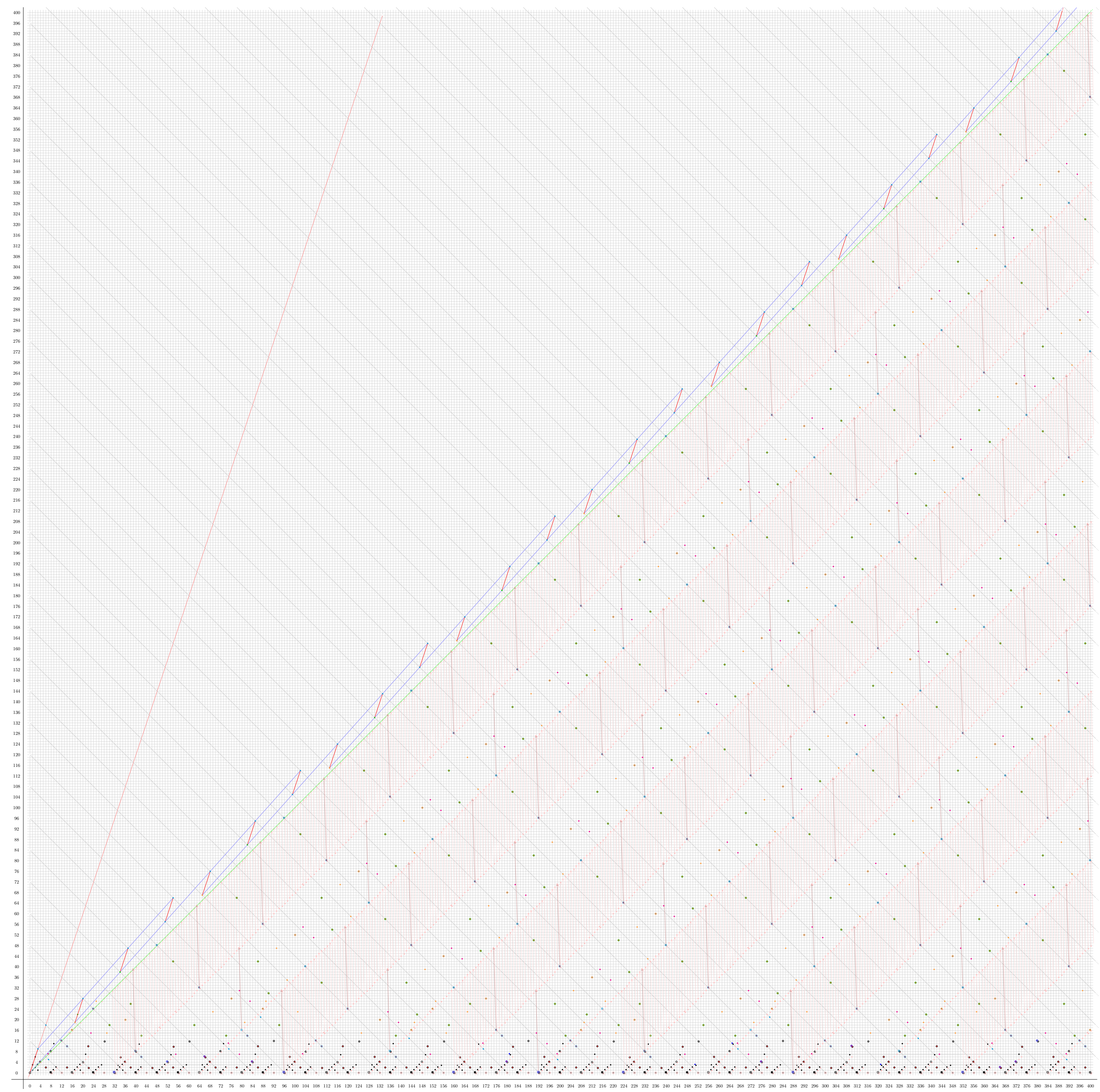


Figure 40: d_{31}-differentials in SliceSS($BP^{((C_4))}\langle 2\rangle$).

19 Higher differentials VI: d_{35} to d_{61}-differentials

Proposition 19.1. *The following d_{35}-differentials exist:*

1. $d_{35}(\bar{\mathfrak{d}}_3^{17}\bar{s}_1 u_{12\lambda}u_{51\sigma}a_{39\lambda}a_{\sigma_2}) = 2\bar{\mathfrak{d}}_3^{20}u_{3\lambda}u_{60\sigma}a_{57\lambda}$ $(d_{35}(127,79) = 2(126,114))$;

2. $d_{35}(\bar{\mathfrak{d}}_3^{21}\bar{s}_1 u_{12\lambda}u_{63\sigma}a_{51\lambda}a_{\sigma_2}) = 2\bar{\mathfrak{d}}_3^{24}u_{3\lambda}u_{72\sigma}a_{69\lambda}$ $(d_{35}(151,103) = 2(150,138))$;

3. $d_{35}(\bar{\mathfrak{d}}_3^{21}\bar{s}_1 u_{28\lambda}u_{63\sigma}a_{35\lambda}a_{\sigma_2}) = 2\bar{\mathfrak{d}}_3^{24}u_{19\lambda}u_{72\sigma}a_{53\lambda}$ $(d_{35}(183,71) = 2(182,106))$;

4. $d_{35}(\bar{\mathfrak{d}}_3^{25}\bar{s}_1 u_{28\lambda}u_{75\sigma}a_{47\lambda}a_{\sigma_2}) = 2\bar{\mathfrak{d}}_3^{28}u_{19\lambda}u_{84\sigma}a_{65\lambda}$ $(d_{35}(207,95) = 2(206,130))$.

Proof. (1) and (2) are proven in Lemma 17.4. For (3), consider the class $2\bar{\mathfrak{d}}_3^{24}u_{19\lambda}u_{72\sigma}a_{53\lambda}$ at $(182,106)$. By Theorem 18.1, this class must die on or before the E_{61}-page. For degree reasons, the only way this can happen is for this class to be killed by the class $\bar{\mathfrak{d}}_3^{21}\bar{s}_1 u_{28\lambda}u_{63\sigma}a_{35\lambda}a_{\sigma_2}$ at $(183,71)$. This proves the desired d_{35}-differential.

For (4), first consider the class $2\bar{\mathfrak{d}}_3^{30}\bar{\mathfrak{d}}_1^{2}u_{11\lambda}u_{92\sigma}a_{81\lambda}$ at $(206,162)$. By Theorem 18.1, this class must die on or before the E_{61}-page. For degree reasons, this class must be killed by the class $\bar{\mathfrak{d}}_3^{25}\bar{s}_3^{3}$ at $(207,111)$.

Now, consider the class $2\bar{\mathfrak{d}}_3^{28}u_{19\lambda}u_{84\sigma}a_{65\lambda}$ at $(206,130)$. By Theorem 18.1 again, this class must die on or before the E_{61}-page. For degree reasons, this class must be killed by one of the following classes:

- $\bar{\mathfrak{d}}_3^{25}\bar{s}_3^{3}$ at $(207,111)$;
- $\bar{\mathfrak{d}}_3^{25}\bar{s}_1 u_{28\lambda}u_{75\sigma}a_{47\lambda}a_{\sigma_2}$ at $(207,95)$.

Since we have already shown in the previous paragraph the class $\bar{\mathfrak{d}}_3^{25}\bar{s}_3^{3}$ at $(207,111)$ supports a d_{51}-differential, the source must be the class $\bar{\mathfrak{d}}_3^{25}\bar{s}_1 u_{28\lambda}u_{75\sigma}a_{47\lambda}a_{\sigma_2}$ at $(207,95)$. This proves the desired d_{35}-differential. □

Using product structures with α and γ^8, the d_{35}-differentials in Proposition 19.1 produce all the other d_{35}-differentials. They are shown in Figure 41.

The d_{43}-differentials follow from Theorem 16.5 (7) and using product structures with α. They are shown in Figure 42.

Proposition 19.2. *The following d_{51}-differentials exist:*

1. $d_{51}(\overline{\mathfrak{d}}_3^{25}\overline{s}_3^3) = 2\overline{\mathfrak{d}}_3^{30}\overline{\mathfrak{d}}_1^2 u_{11\lambda}u_{92\sigma}a_{81\lambda}$ $(d_{51}(207, 111) = 2(206, 162))$;

2. $d_{51}(\overline{\mathfrak{d}}_3^{29}\overline{s}_3^3) = 2\overline{\mathfrak{d}}_3^{34}\overline{\mathfrak{d}}_1^2 u_{27\lambda}u_{104\sigma}a_{77\lambda}$ $(d_{51}(263, 103) = 2(262, 154))$.

Proof. For (1), consider the class $2\overline{\mathfrak{d}}_3^{30}\overline{\mathfrak{d}}_1^2 u_{11\lambda}u_{92\sigma}a_{81\lambda}$ at $(206, 162)$. By Theorem 18.1, this class must die on or before the E_{61}-page. For degree reasons, the only way for this to happen is for the claimed d_{51}-differential to exist.

For (2), consider the class $2\overline{\mathfrak{d}}_3^{34}\overline{\mathfrak{d}}_1^2 u_{27\lambda}u_{104\sigma}a_{77\lambda}$ at $(262, 154)$. By Theorem 18.1 again, this class must die on or before the E_{61}-page. There are two possibilities:

- This class supports a d_{61}-differential and kills the class $\overline{\mathfrak{d}}_3^{39}\overline{\mathfrak{d}}_1^2 u_{12\lambda}u_{118\sigma}a_{107\lambda}a_\sigma$ at $(261, 215)$;

- This class is killed by a d_{51}-differential coming from the class $d_{51}(\overline{\mathfrak{d}}_3^{29}\overline{s}_3^3)$ at $(263, 103)$.

The first case is impossible because by Theorem 16.5 (5) and multiplication with α, the class $\overline{\mathfrak{d}}_3^{39}\overline{\mathfrak{d}}_1^2 u_{12\lambda}u_{118\sigma}a_{107\lambda}a_\sigma$ at $(261, 215)$ supports a d_{53}-differential. It follows that the claimed d_{51}-differential exists. □

Using product structures with α and γ^8, the d_{51}-differentials in Proposition 19.2 produce all the other d_{51}-differentials. They are shown in Figure 43.

All the d_{53}-differentials are obtained from the d_{53}-differentials in Theorem 16.5 and using product structures with α. They are shown in Figure 44

Proposition 19.3. *The following d_{55}-differentials exist:*

1. $d_{55}(\overline{\mathfrak{d}}_3^{24}\overline{\mathfrak{d}}_1^2 u_{28\lambda}u_{74\sigma}a_{46\lambda}) = \overline{\mathfrak{d}}_3^{29}\overline{s}_1 u_{14\lambda}u_{87\sigma}a_{73\lambda}a_{\sigma_2}$ $(d_{55}(204,92) = (203,147))$;

2. $d_{55}(\overline{\mathfrak{d}}_3^{28}\overline{\mathfrak{d}}_1^2 u_{44\lambda}u_{86\sigma}a_{42\lambda}) = \overline{\mathfrak{d}}_3^{33}\overline{s}_1 u_{30\lambda}u_{99\sigma}a_{69\lambda}a_{\sigma_2}$ $(d_{55}(260,84) = (259,139))$.

Proof. For (1), consider the class $\overline{\mathfrak{d}}_3^{29}\overline{s}_1 u_{14\lambda}u_{87\sigma}a_{73\lambda}a_{\sigma_2}$ at $(203,147)$. By Theorem 18.1, this class must die on or before the E_{61}-page. For degree reasons, the only way this can happen is for the claimed d_{55}-differential to exist.

For (2), consider the class $\overline{\mathfrak{d}}_3^{33}\overline{s}_1 u_{30\lambda}u_{99\sigma}a_{69\lambda}a_{\sigma_2}$ at $(259,139)$. By Theorem 18.1, this class must die on or before the E_{61}-page. For degree reasons, there are two possibilities:

- This class supports a d_{59}-differential to kill the class $2\overline{\mathfrak{d}}_3^{38}u_{15\lambda}u_{114\sigma}a_{99\lambda}$ at $(258,198)$;
- this class is killed by a d_{55}-differential from the class $\overline{\mathfrak{d}}_3^{28}\overline{\mathfrak{d}}_1^2 u_{44\lambda}u_{86\sigma}a_{42\lambda}$ at $(260,84)$.

The first case is impossible because by Theorem 16.5 (2) and multiplication with α, the class $2\overline{\mathfrak{d}}_3^{38}u_{15\lambda}u_{114\sigma}a_{99\lambda}$ at $(258,198)$ supports a d_{61}-differential. Therefore, the second possibility must occur, and we get our desired d_{55}-differential. □

Using product structures with α and γ^8, the d_{55}-differentials in Proposition 19.3 produce all the other d_{55}-differentials. They are shown in Figure 45.

Proposition 19.4. *The following d_{59}-differentials exist:*

1. $d_{59}(\overline{\mathfrak{d}}_3^{17}\overline{s}_1 u_{14\lambda}u_{51\sigma}a_{37\lambda}a_{\sigma_2}) = \overline{\mathfrak{d}}_3^{22}u_{64\sigma}a_{66\lambda}a_{2\sigma}$ $(d_{59}(131,75) = (130,134))$;

2. $d_{59}(\overline{\mathfrak{d}}_3^{21}\overline{s}_1 u_{30\lambda}u_{63\sigma}a_{33\lambda}a_{\sigma_2}) = 2\overline{\mathfrak{d}}_3^{26}u_{15\lambda}u_{78\sigma}a_{63\lambda}$ $(d_{59}(187,67) = 2(186,126))$.

Proof. (1) is Theorem 16.5 (3). To prove (2), consider the class $2\overline{\mathfrak{d}}_3^{26}u_{15\lambda}u_{78\sigma}a_{63\lambda}$ at $(186,126)$. By Theorem 18.1, this class must die on or before the E_{61}-page. For degree reasons, the only way this can happen is for the claimed d_{59}-differential to exist. □

Using product structures with α and γ^8, the d_{59}-differentials in Proposition 19.4 produce all the other d_{59}-differentials. They are shown in Figure 46.

Proposition 19.5. *The following d_{61}-differentials exist:*

1. $d_{61}(2\bar{\mathfrak{d}}_3^{26}u_{31\lambda}u_{78\sigma}a_{47\lambda}) = \bar{\mathfrak{d}}_3^{31}u_{16\lambda}u_{92\sigma}a_{77\lambda}a_\sigma$ $(d_{61}(2(218,194)) = (217,155))$;

2. $d_{61}(\bar{\mathfrak{d}}_3^{27}\bar{\mathfrak{d}}_1^{2}u_{28\lambda}u_{82\sigma}a_{55\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^{32}\bar{\mathfrak{d}}_1^{2}u_{12\lambda}u_{98\sigma}a_{86\lambda}$ $(d_{61}(221,111) = (220,172))$;

3. $d_{61}(2\bar{\mathfrak{d}}_3^{30}\bar{\mathfrak{d}}_1^{2}u_{27\lambda}u_{92\sigma}a_{65\lambda}) = \bar{\mathfrak{d}}_3^{35}\bar{\mathfrak{d}}_1^{2}u_{12\lambda}u_{106\sigma}a_{95\lambda}a_\sigma$ $(d_{61}(2(238,130)) = (237,191))$;

4. $d_{61}(\bar{\mathfrak{d}}_3^{31}u_{32\lambda}u_{92\sigma}a_{61\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^{36}u_{16\lambda}u_{108\sigma}a_{92\lambda}$ $(d_{61}(249,123) = (248,184))$;

5. $d_{61}(2\bar{\mathfrak{d}}_3^{30}u_{47\lambda}u_{90\sigma}a_{43\lambda}) = \bar{\mathfrak{d}}_3^{35}u_{32\lambda}u_{104\sigma}a_{73\lambda}a_\sigma$ $(d_{61}(2(274,86)) = (273,147))$;

6. $d_{61}(\bar{\mathfrak{d}}_3^{31}\bar{\mathfrak{d}}_1^{2}u_{44\lambda}u_{94\sigma}a_{51\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^{36}\bar{\mathfrak{d}}_1^{2}u_{28\lambda}u_{110\sigma}a_{82\lambda}$ $(d_{61}(277,103) = (276,164))$;

7. $d_{61}(2\bar{\mathfrak{d}}_3^{34}\bar{\mathfrak{d}}_1^{2}u_{43\lambda}u_{104\sigma}a_{61\lambda}) = \bar{\mathfrak{d}}_3^{39}\bar{\mathfrak{d}}_1^{2}u_{28\lambda}u_{118\sigma}a_{91\lambda}a_\sigma$ $(d_{61}(2(294,122)) = (293,183))$;

8. $d_{61}(\bar{\mathfrak{d}}_3^{35}u_{48\lambda}u_{104\sigma}a_{57\lambda}a_\sigma) = \bar{\mathfrak{d}}_3^{40}u_{32\lambda}u_{120\sigma}a_{88\lambda}$ $(d_{61}(305,115) = (304,176))$.

Proof. All of these differentials are proven by using the same method: we first consider the target, which, by Theorem 18.1, must die on or before the E_{61}-page. Once we know this, then for degree reasons, we deduce the claimed d_{61}-differential. □

Using product structures with α and γ^8, the d_{61}-differentials in Proposition 19.5 produce all the other d_{61}-differentials. They are shown in Figure 47.

20 Summary of Differentials

In this section, we summarize all the differentials in the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2\rangle$ (Figure 48 shows all the differentials from d_{13} to d_{61}). To better organize the differen-

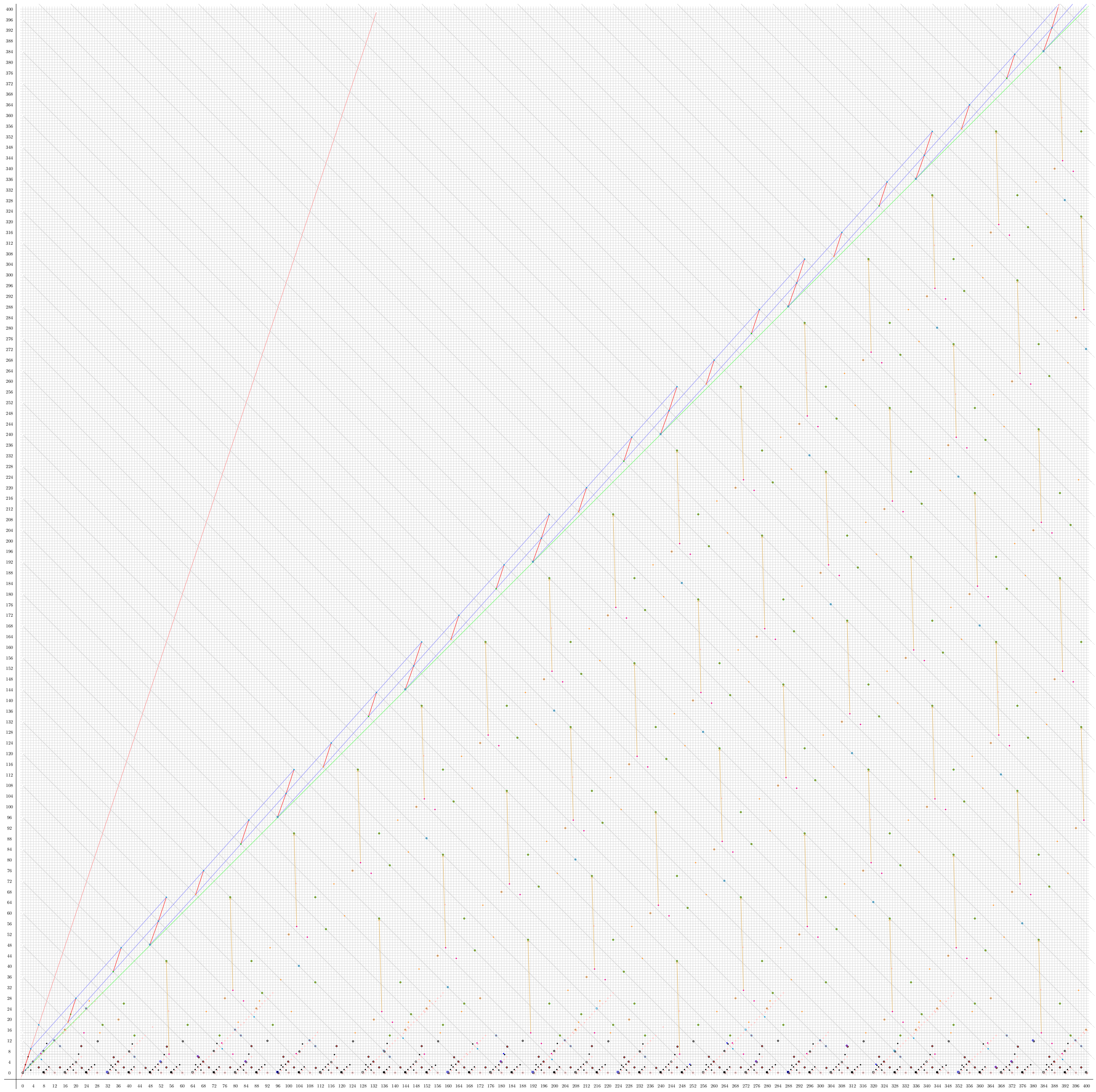


Figure 41: d_{35}-differentials in SliceSS$(BP^{(\!(C_4)\!)}\langle 2\rangle)$.

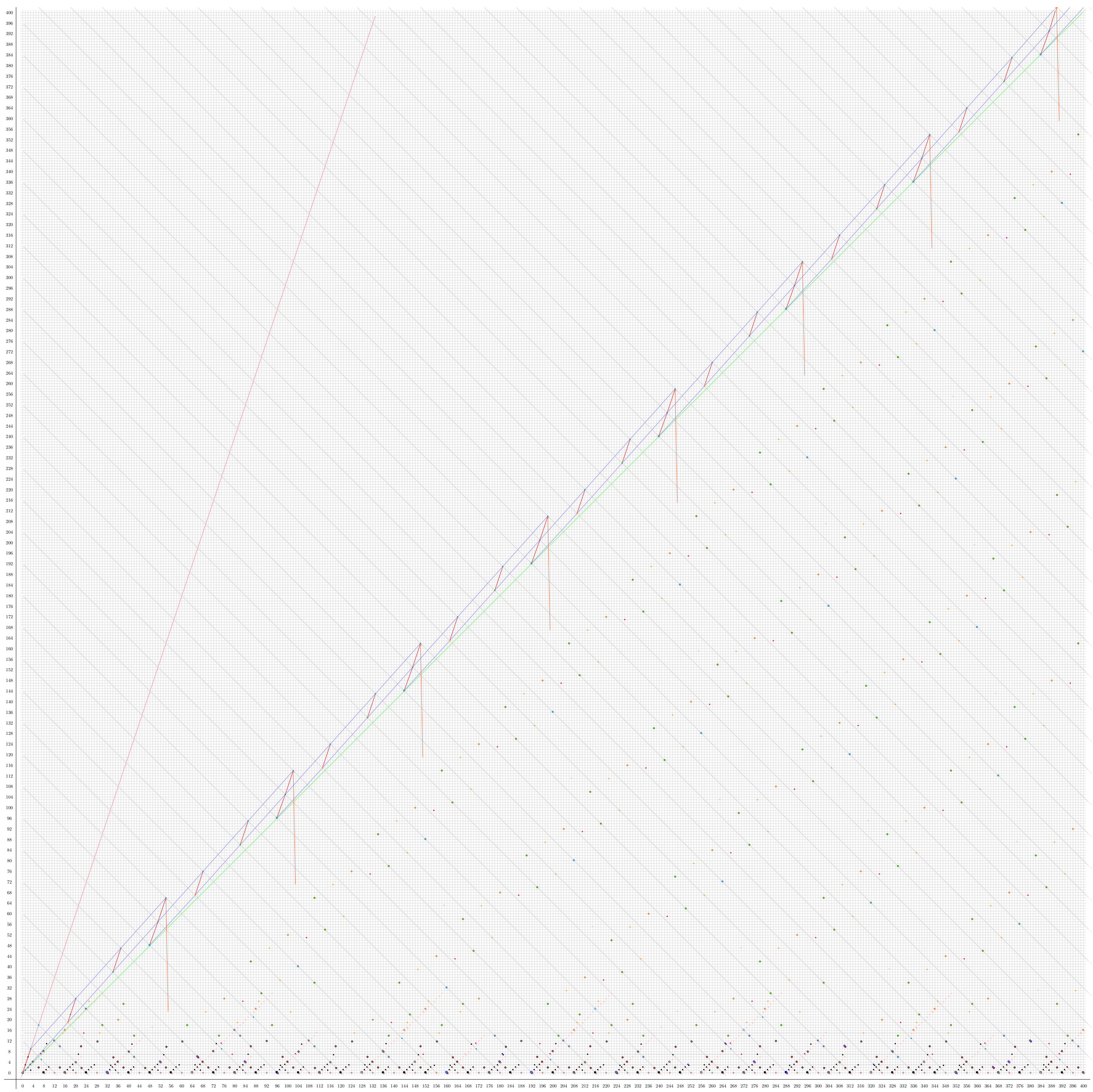


Figure 42: d_{43}-differentials in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

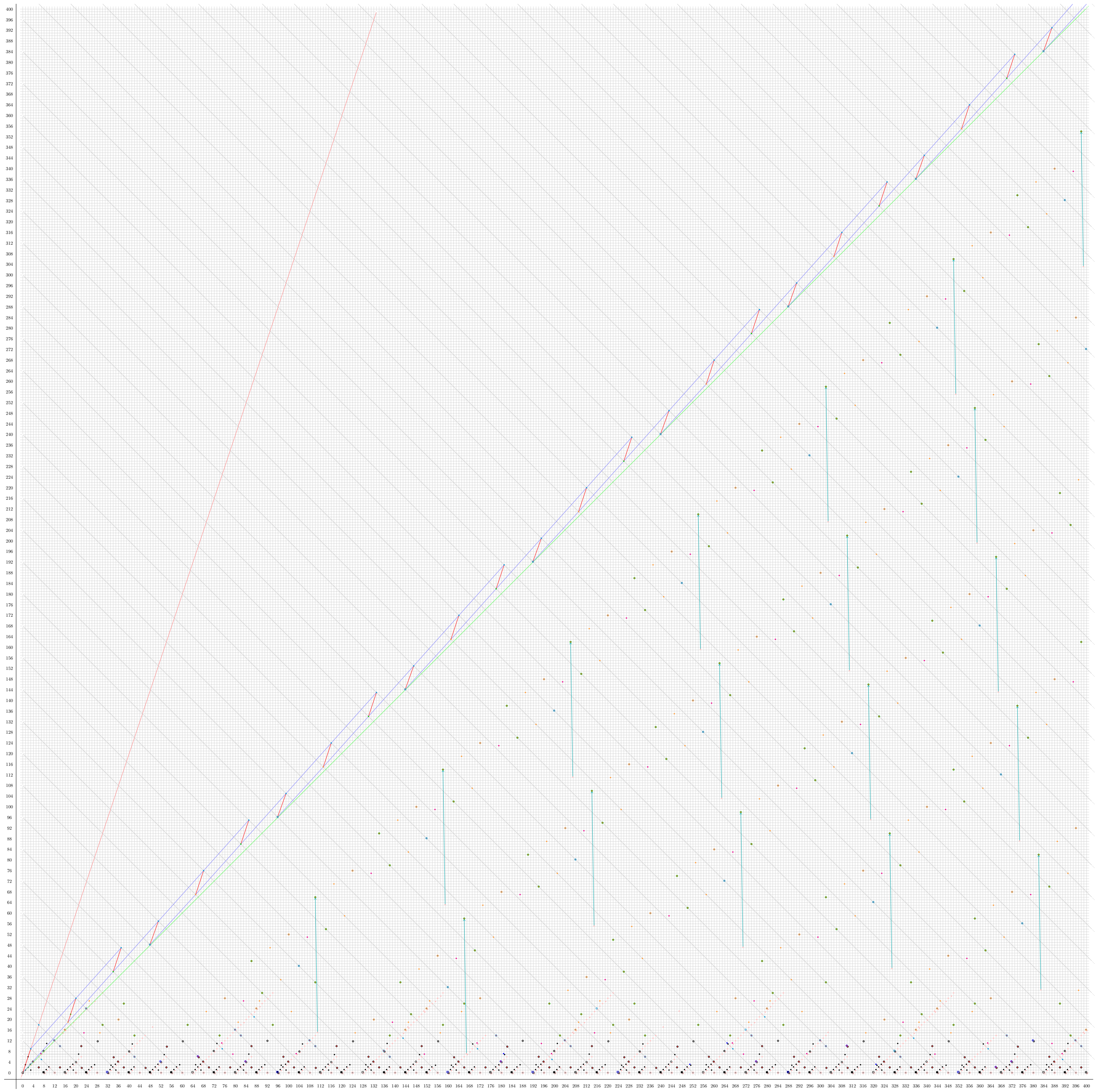


Figure 43: d_{51}-differentials in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

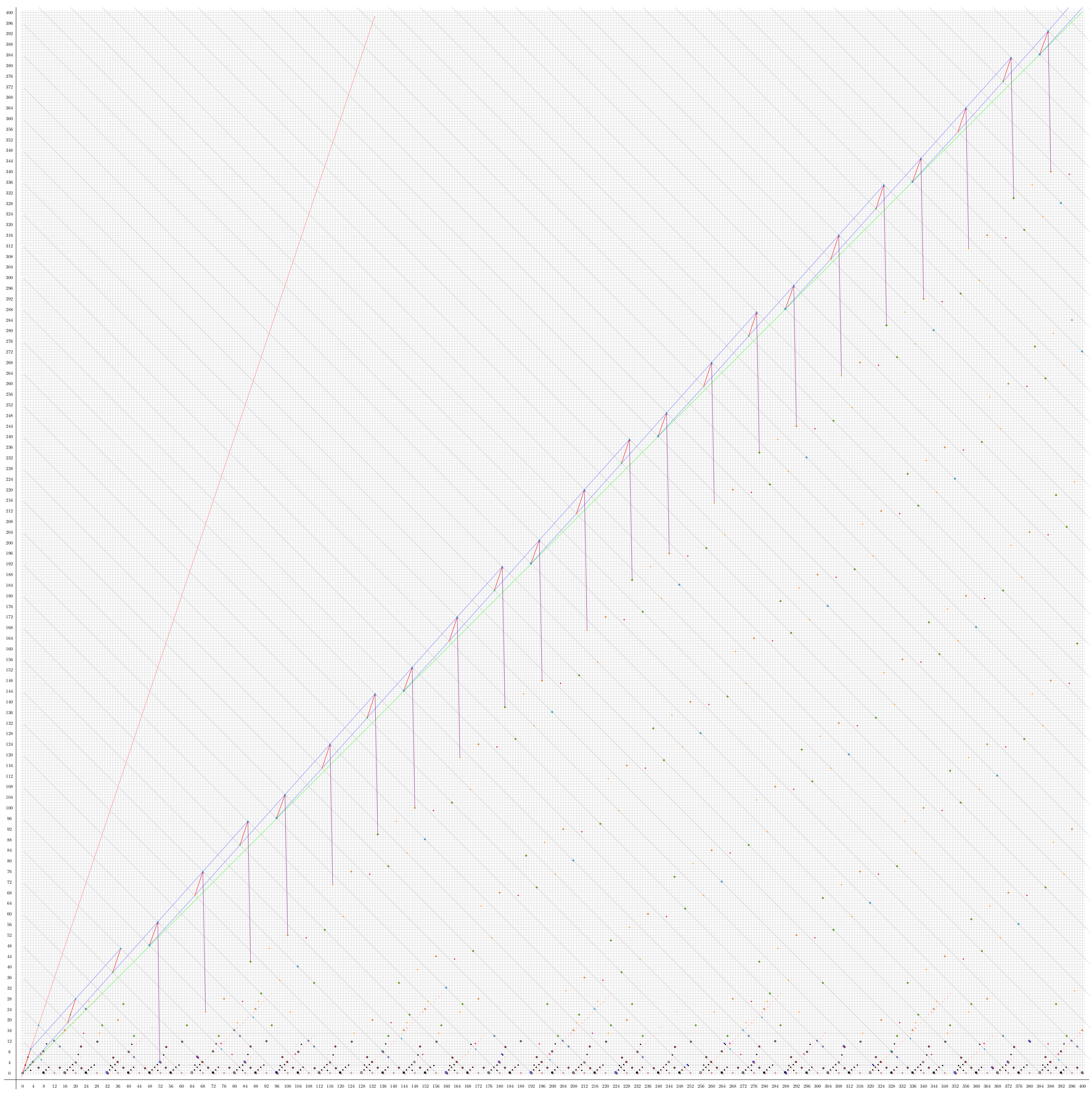


Figure 44: d_{53}-differentials in SliceSS($BP^{((C_4))}\langle 2\rangle$).

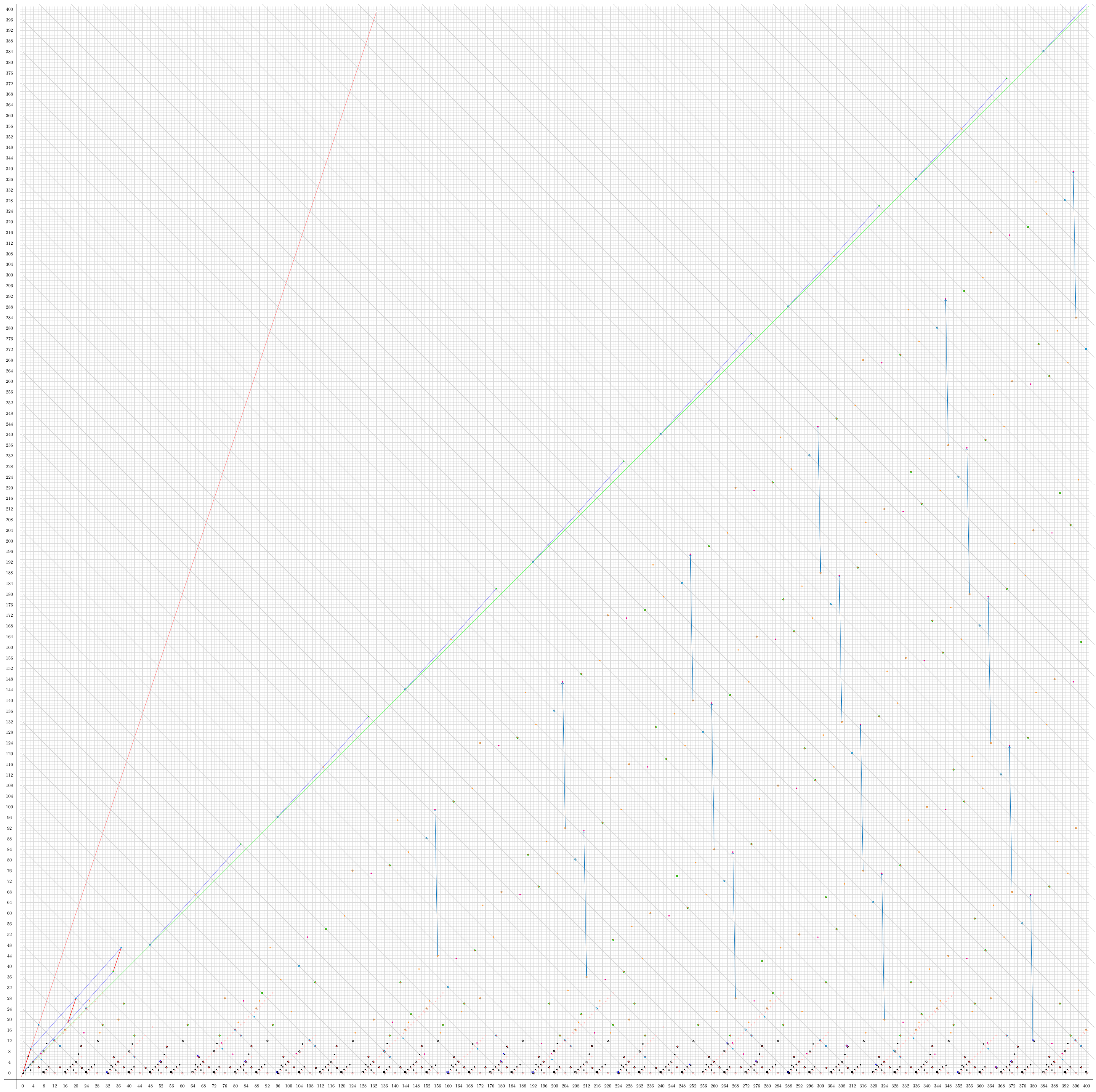


Figure 45: d_{55}-differentials in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

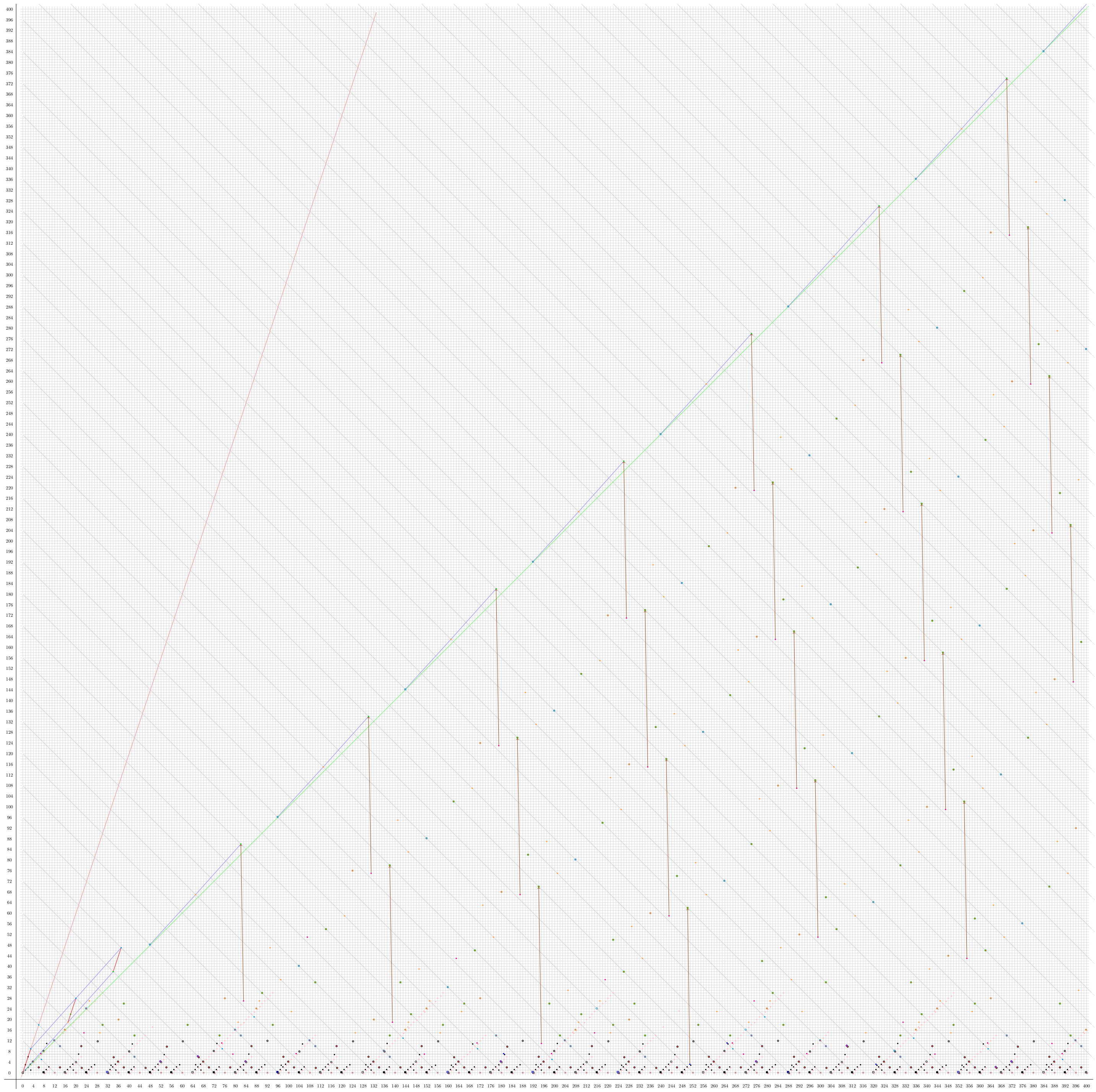


Figure 46: d_{59}-differentials in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

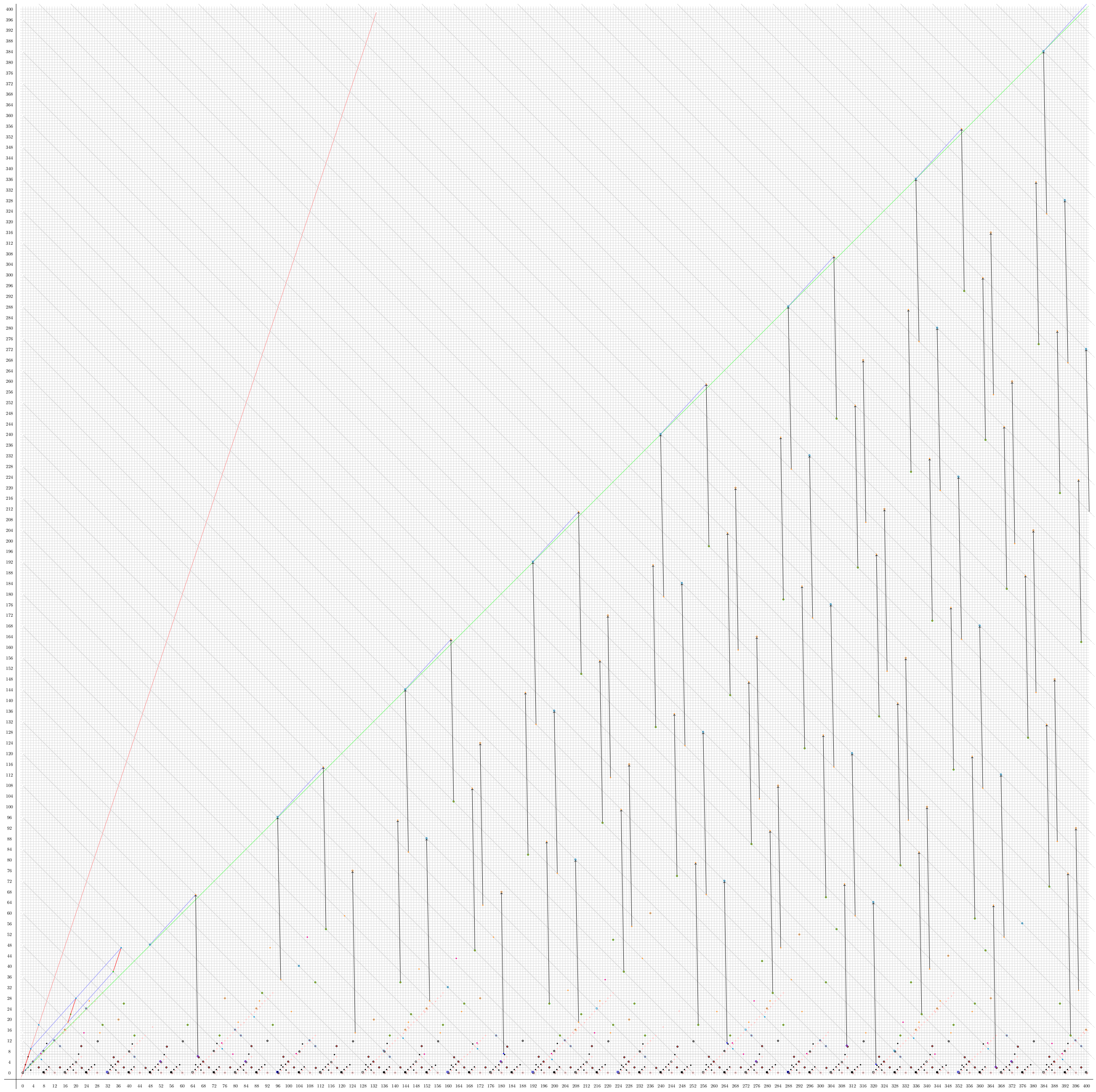


Figure 47: d_{61}-differentials in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

tials visually, Figure 49 shows the d_{13} to d_{15}-differentials; Figure 50 shows the d_{19} to d_{31}-differentials; and Figure 51 shows the d_{35} to d_{61}-differentials. The E_∞-page is shown in Figure 52. We observe that there is a horizontal vanishing line at filtration 60.

Differential	Proof
d_3, d_5, d_7, d_{11}	Induced differentials from C_4- SliceSS$(\mathrm{BP}^{((C_4))}\langle 1\rangle)$ (Section 13)
d_{13}	Induced differentials from C_4- SliceSS$(\mathrm{BP}^{((C_4))}\langle 1\rangle)$ (Section 13.5) Norm Formula (Theorem 15.3) Hill–Hopkins–Ravenel Slice Differential Theorem (Proposition 14.1) Restriction map (Proposition 14.2)
d_{15}	Induced differentials from C_2- SliceSS $\mathrm{BP}^{((C_4))}\langle 2\rangle$ (Section 14.2) Restriction-transfer (Section 14.3)
d_{19}	Proven in Proposition 17.5 Proven together with d_{27}-differentials Uses the Vanishing Theorem (Theorem 18.1) and some d_{29} and d_{35}-differentials
d_{21}	Proven in Proposition 17.1. Uses the Vanishing Theorem (Theorem 18.1)
d_{23}	Proven in Proposition 17.2. Uses the restriction map
d_{27}	Proven in Proposition 17.5 Proven together with d_{19}-differentials Uses the Vanishing Theorem (Theorem 18.1) and some d_{29} and d_{35}-differentials
d_{29}	Proven in Theorem 18.1 Uses the norm Formula (Theorem 15.5) and the Vanishing Theorem (Theorem 18.1)
d_{31}	Induced differentials from C_2- SliceSS$(\mathrm{BP}^{((C_4))}\langle 2\rangle)$ (Section 14.2); Proposition 14.11; and Section 18.2
d_{35}	Proven in Proposition 19.1. Uses the Vanishing Theorem (Theorem 18.1)
d_{43}	Proven in Theorem 16.5. Uses the Vanishing Theorem (Theorem 18.1)
d_{51}	Proven in Proposition 19.2. Uses the Vanishing Theorem (Theorem 18.1)
d_{53}	Proven in Theorem 16.5. Uses the Vanishing Theorem (Theorem 18.1)
d_{55}	Proven in Proposition 19.3. Uses the Vanishing Theorem (Theorem 18.1)
d_{59}	Proven in Proposition 19.4. Uses the Vanishing Theorem (Theorem 18.1)
d_{61}	Proven in Proposition 19.5 Uses the norm formula (Theorem 15.6) and the Vanishing Theorem (Theorem 18.1)

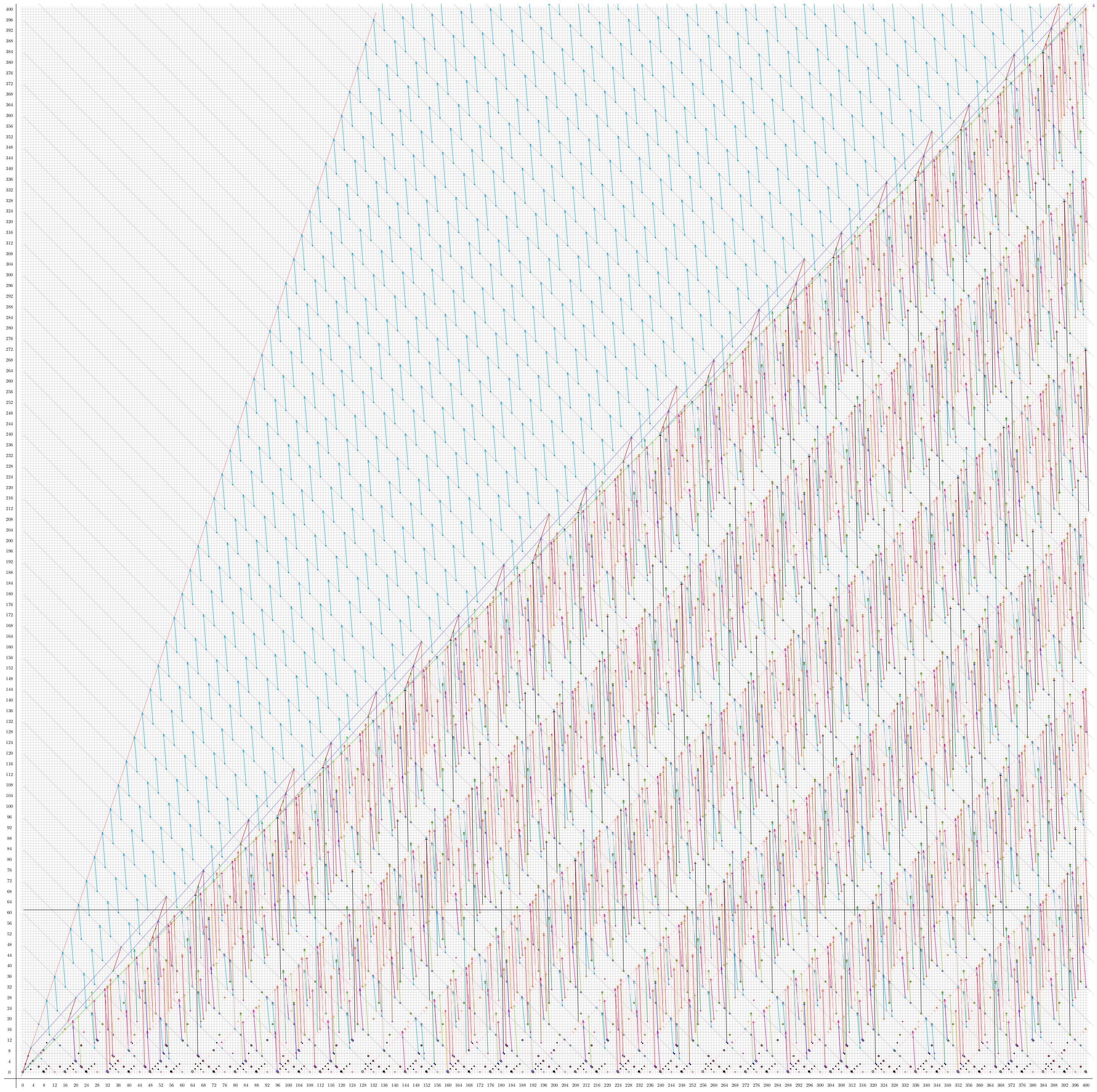


Figure 48: d_{13} to d_{61}-differentials in SliceSS($BP^{(\!(C_4)\!)}\langle 2\rangle$).

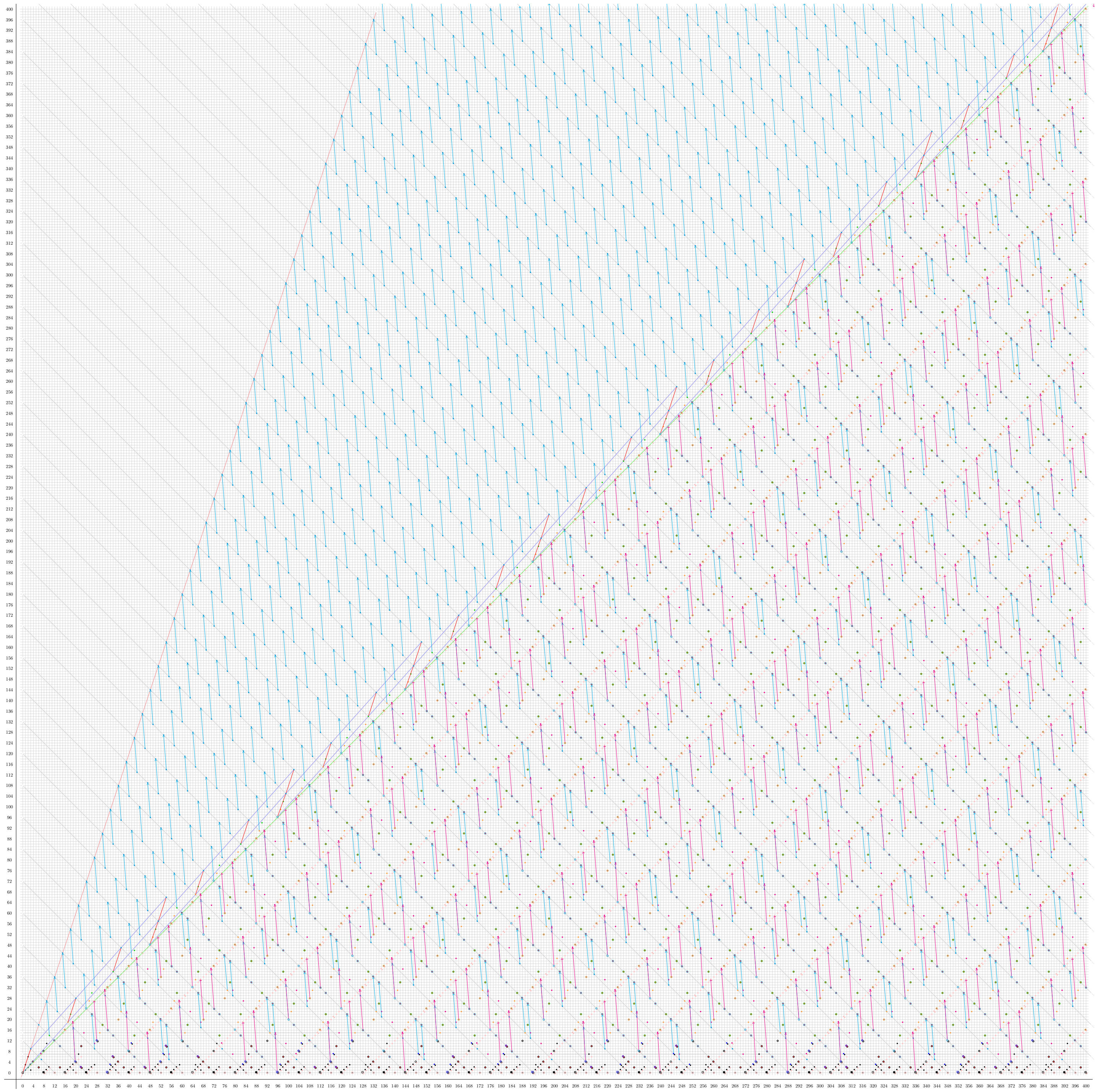


Figure 49: d_{13} to d_{15}-differentials in $\mathrm{SliceSS}(\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle)$.

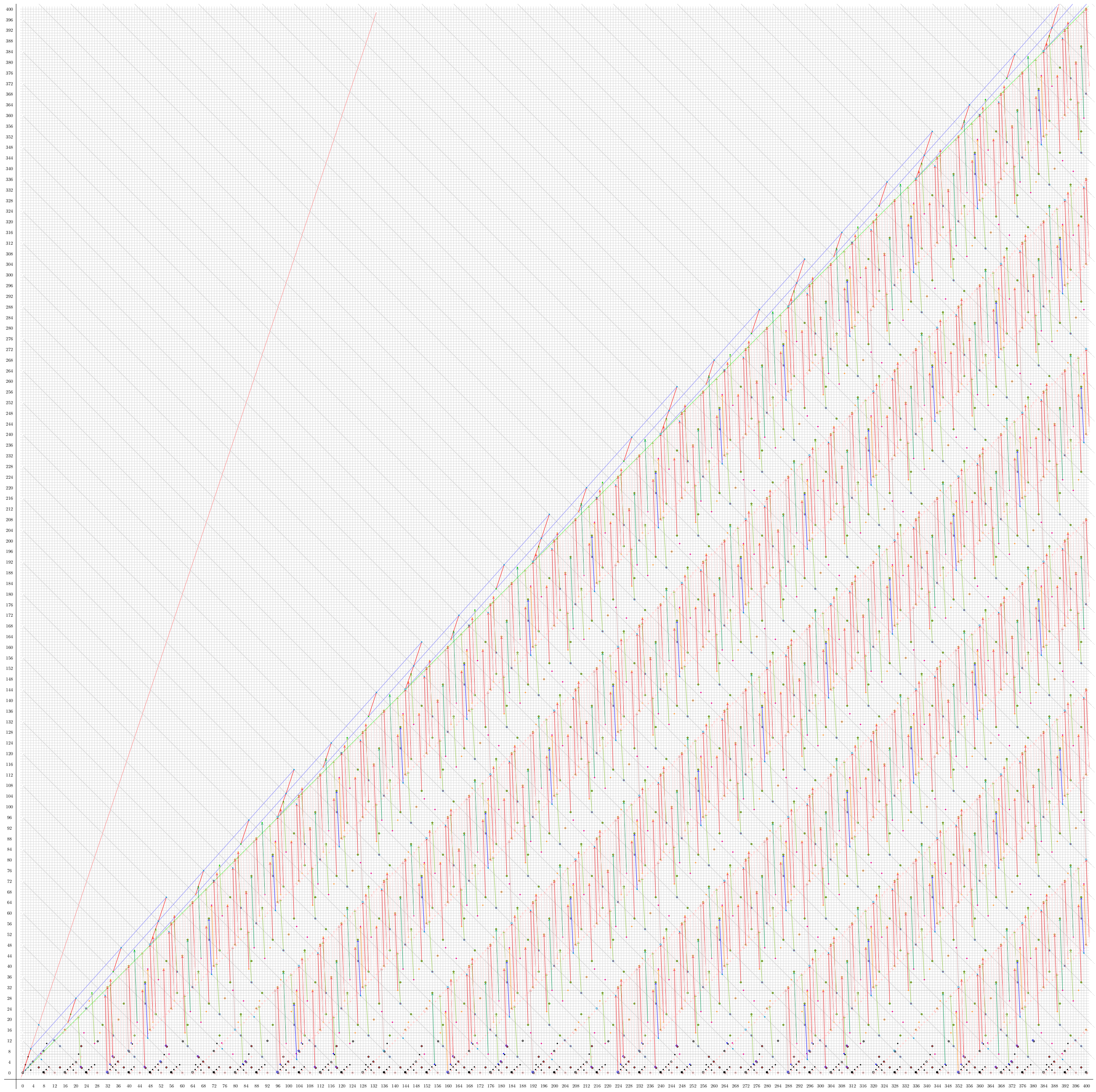


Figure 50: d_{19} to d_{31}-differentials in SliceSS($\mathrm{BP}^{(\!(C_4)\!)}\langle 2\rangle$).

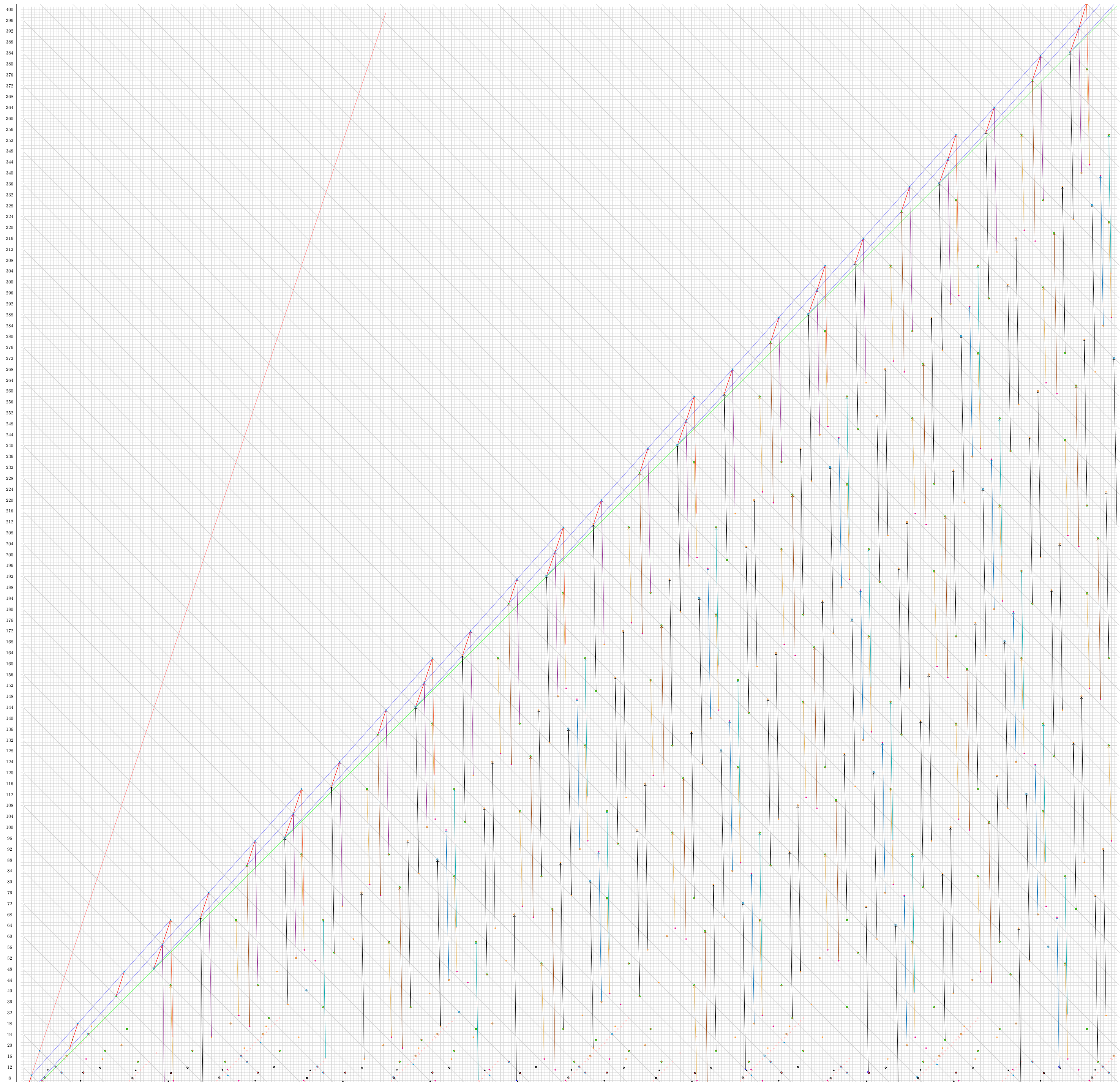


Figure 51: d_{35} to d_{61}-differentials in SliceSS($BP^{(\!(C_4)\!)}\langle 2\rangle$).

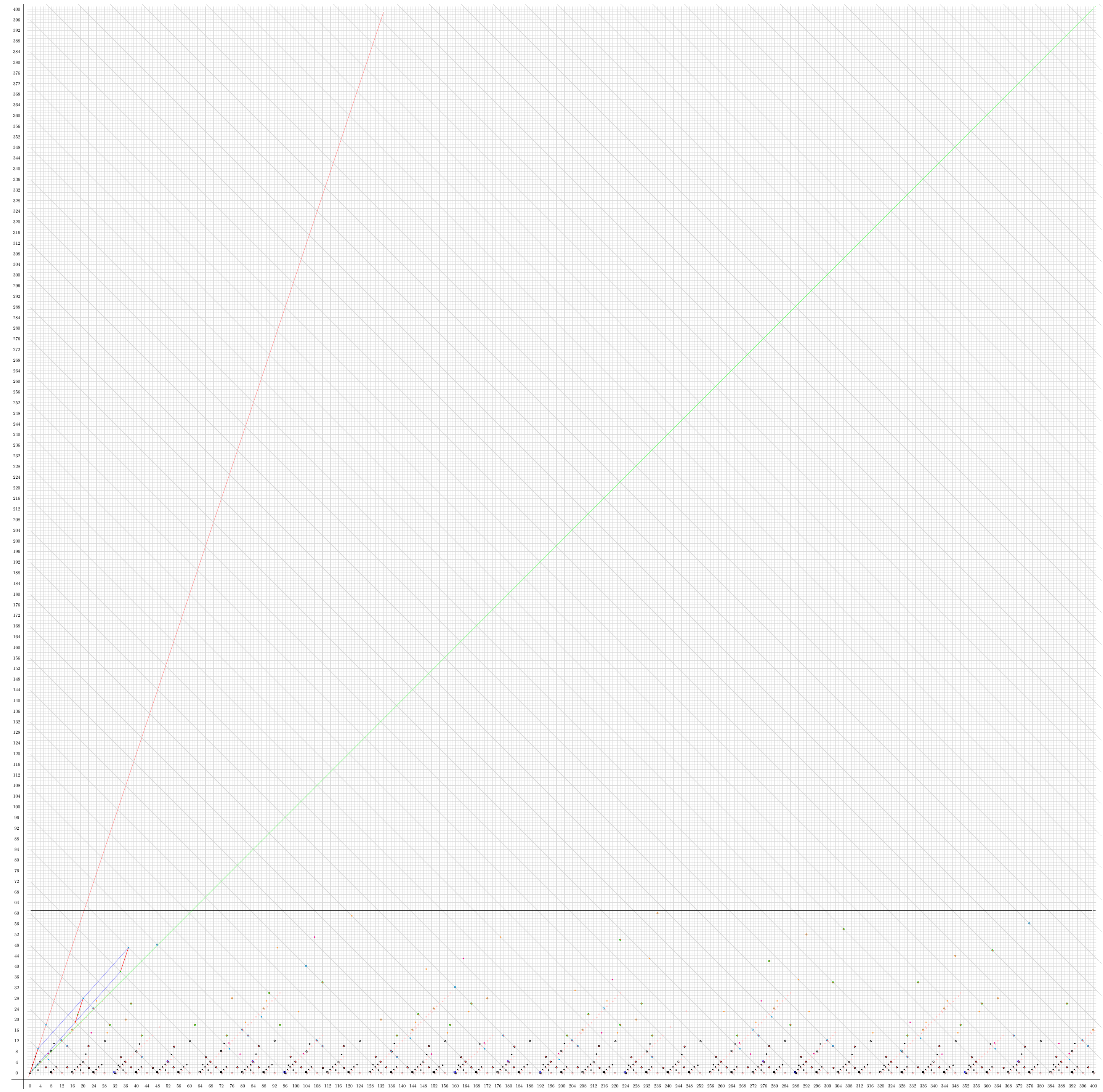


Figure 52: E_∞-page of SliceSS($BP^{((C_4))}\langle 2\rangle$).

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