



Slopes in eigenvarieties for definite unitary groups

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Slopes in eigenvarieties for definite unitary groups

A dissertation presented

by

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to

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in partial fulfillment of the requirements

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Slopes in eigenvarieties for definite unitary groups

Abstract

We generalize bounds of Liu-Wan-Xiao for slopes in eigencurves for definite unitary groups of rank 2, which formed the core of their proof of the Coleman-Mazur-Buzzard-Kilford conjecture about the decomposition of the eigencurve over the boundary of weight space, to eigenvarieties for definite unitary groups of any rank. We show that for a definite unitary group of rank n , the Newton polygon of the characteristic power series of the U_p Hecke operator has exact polynomial growth rate $x^{1+\frac{2}{n(n-1)}}$, with constant proportional to the distance of the weight from the boundary of weight space. This improves a previous lower bound of $x^{1+\frac{1}{2^n-n-1}}$ of Chenevier (which applied only to the center of weight space). The proof goes through the classification of forms associated to principal series representations. We also give a consequence for the geometry of these eigenvarieties over the boundary of weight space.

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1. INTRODUCTION

1.1. Background and statement of main theorem. The study of eigenvarieties began with the seminal work of Hida in the 1980s and Coleman and Mazur in the 1990s, culminating in Coleman-Mazur's construction in [10] of the modular eigencurve, a rigid analytic space parametrizing p -adic modular Hecke eigenforms. The eigencurve admits a natural projection map to *weight space*, a rigid analytic space parametrizing possible weights of p -adic modular forms, as well as a map to \mathbb{G}_m corresponding to the modular forms' U_p -eigenvalues. Since then, further work and generalizations by numerous authors have resulted in a massive collection of "eigenvarieties" for p -adic automorphic forms on various groups. Particularly relevant for our purposes are the papers of Buzzard [4], [5], Chenevier [9], and Bellaïche-Chenevier [2], in which eigenvarieties are constructed for p -adic automorphic forms on definite unitary groups of all dimensions.

The geometry of these eigenvarieties appears to be complicated. To discuss it, we need to introduce some notation. Let p be a prime. Let $q = 4$ if $p = 2$ and $q = p$ otherwise. We write v for the p -adic valuation and $|\cdot|$ for the p -adic norm, normalized so that $v(p) = 1$ and $|p| = p^{-1}$. A *weight* of a p -adic modular form is a continuous character of \mathbb{Z}_p^\times , and the *weight space* for modular forms is the rigid analytic space \mathcal{W} such that for any affinoid \mathbb{Q}_p -algebra A , $\mathcal{W}(A)$ is the set of continuous characters $\mathbb{Z}_p^\times \rightarrow A^\times$. The T -coordinate of an A -point $w \in \mathcal{W}(A)$ is the value $T(w) = w(\exp(q)) - 1$; the space \mathcal{W} turns out to be a disjoint union of $\varphi(q)$ open unit discs with parameter T . For $r \in (0, 1)$, we write $\mathcal{W}_{>r}$ for the subset of \mathcal{W} where $|T| > r$.

We fix a tame level and let \mathcal{Z} be the corresponding eigencurve. We let $w : \mathcal{Z} \rightarrow \mathcal{W}$ be the map taking an eigenform to its weight, $a_p : \mathcal{Z} \rightarrow \mathbb{G}_m$ be the map taking an eigenform to its U_p -eigenvalue, and $\mathcal{Z}_{>r}$ be the preimage of $\mathcal{W}_{>r}$ in \mathcal{Z} . One thread of approach to analyzing eigenvarieties has been to restrict analysis to the part of

the eigenvariety lying over the “boundary” of weight space, in this case $\mathcal{Z}_{>r}$ for r sufficiently close to 1, where it is expected to be simpler. In particular, the following folklore conjecture arose from a question of Coleman and Mazur [10], was suggested by a computation of Buzzard and Kilford [6], and is given below in the form stated by Liu, Wan, and Xiao [16].

Conjecture 1.1.1 (Coleman-Mazur-Buzzard-Kilford, as stated by Liu-Wan-Xiao [16]). *When $r \in (0, 1)$ is sufficiently close to 1^- , the following statements hold.*

- (1) *The space $\mathcal{Z}_{>r}$ is a disjoint union of (countably infinitely many) connected components Z_1, Z_2, \dots , such that the weight map $w : Z_n \rightarrow \mathcal{W}_{>r}$ is finite and flat for each n .*
- (2) *There exist nonnegative rational numbers $\alpha_1, \alpha_2, \dots \in \mathbb{Q}$ in non-decreasing order and tending to infinity such that, for each n and each point $z \in Z_n$, we have*

$$|a_p(z)| = |T(w(z))|^{\alpha_n}.$$

- (3) *The sequence $\alpha_1, \alpha_2, \dots$ is a disjoint union of finitely many arithmetic progressions, counted with multiplicity (at least when the indices are large enough).*

Note that Part 2 of Conjecture 1.1.1 implies that as one approaches the boundary, the slope $v(a_p(z))$ approaches 0 in proportion to $v(T(w(z)))$.

Following various explicit computations, Liu, Wan, and Xiao [16], building on the work of Wan, Xiao, and Zhang [24], proved the equivalent version of this conjecture for automorphic forms on definite quaternion algebras over \mathbb{Q} . The essence of their work, from which their result follows naturally, is their strong upper and lower bounds on the Newton polygon of the characteristic power series of the U_p -operator. For consistency with future discussion, we state their bounds (with some imprecision, to avoid unnecessary detail) for rank-2 definite unitary groups over \mathbb{Q} , for which the analysis is exactly the same.

Let G be an algebraic group over \mathbb{Q} such that $G(\mathbb{R}) \cong U_2(\mathbb{R})$ and $G(\mathbb{Q}_p) \cong GL_2(\mathbb{Q}_p)$, $\mathcal{U} \subset G(\mathbb{A}_f)$ a compact open subgroup satisfying reasonable minor technical conditions, and $\mathcal{S}_w(G, \mathcal{U})$ the space of p -adic automorphic forms on G of weight w and level \mathcal{U} . The basic idea is that we expect the Newton polygon of $\det(I - XU_p | \mathcal{S}_w(G, \mathcal{U}))$ to be of shape approximately $y = Av(T(w))x^2$.

Theorem 1.1.2 (Liu-Wan-Xiao, Corollary 3.16 and Proposition 3.20 of [16]). *Fix \mathcal{U} .*

- (1) *There are constants $A_1, C > 0$ such that for all w such that $T(w) > \frac{1}{p}$, the Newton polygon of the power series $\det(I - XU_p | \mathcal{S}_w(G, \mathcal{U}))$ lies above the curve $y = (A_1x^2 - C)v(T(w))$.*
- (2) *Suppose that $w(a) = a^t\chi(a)$, where $t \in \mathbb{Z}_{\geq 0}$ and χ is a finite character of conductor c . Then there is a constant h such that the Newton polygon of $\det(I - XU_p | \mathcal{S}_w(G, \mathcal{U}))$ contains at least $h(t+1)p^{c-1}$ segments of slope at most $t+1$, hence passes below the point*

$$(h(t+1)p^{c-1}, h(t+1)^2p^{c-1}) = (x, A_2x^2v(T(w)))$$

for $x = h(t+1)p^{c-1}$ and a constant A_2 depending on h .

The lower bound, statement 1 above, has since been re-proven by Johansson and Newton in [15] using a more conceptual method.

In this paper, we generalize Theorem 1.1.2 to definite unitary groups of all dimensions. Let G be an algebraic group over \mathbb{Q} such that $G(\mathbb{R}) \cong U_n(\mathbb{R})$ and $G(\mathbb{Q}_p) \cong GL_n(\mathbb{Q}_p)$, and $\mathcal{U} \subset G(\mathbb{A}_f)$ a compact open subgroup satisfying reasonable minor technical conditions. The corresponding eigenvariety \mathcal{Z} is now a rigid analytic space of dimension $n-1$ lying over the space \mathcal{W} such that for any affinoid \mathbb{Q}_p -algebra A , $\mathcal{W}(A)$ is the set of continuous characters $(\mathbb{Z}_p^\times)^{n-1} \rightarrow A^\times$. This \mathcal{W} is a disjoint union of $\varphi(q)^{n-1}$ open unit polydiscs of dimension $n-1$ with parameters

T_1, \dots, T_{n-1} . Again let $\mathcal{S}_w(G, \mathcal{U})$ be the space of p -adic automorphic forms on G of weight w and level \mathcal{U} .

The basic idea of our main theorem is that we expect the Newton polygon of $\det(I - XU_p | \mathcal{S}_w(G, \mathcal{U}))$ to be of shape approximately $y = Av(T_i(w))x^{1+\frac{2}{n(n-1)}}$, assuming that all the $v(T_i(w))$ are not extremely different in size. Of course, this recovers the Liu-Wan-Xiao approximation $y = Av(T_1(w))x^2$ for $n = 2$.

As far as we know, there is little prior work on the shape of the Newton polygon of $\det(I - XU_p | \mathcal{S}_w(G, \mathcal{U}))$. The only prior result we have been able to find in the literature is the following lower bound of Chenevier, which has an exponentially smaller exponent and is valid only in the center of weight space.

Theorem 1.1.3 (Chenevier, Lemma 5.1.1 of [9]). *Fix \mathcal{U} . Suppose that for some $r < p^{-\frac{1}{p-1}}$, $|T_i(w)| < r$ for all i . Then there are constants $A_r, C_r > 0$ such that the Newton polygon of $\det(I - XU_p | \mathcal{S}_w(G, \mathcal{U}))$ lies above the curve*

$$y = A_r x^{1+\frac{1}{2^n-n-1}} - C_r x.$$

As late as 2015, Andreatta, Iovita, and Pilloni said in Section 1.2.2 of [1] that there were not even any conjectures about the shape of the Newton polygon in higher dimensions in the literature.

Our precise theorem is as follows.

Theorem 1.1.4. *Fix \mathcal{U} .*

- (1) *There are constants $A_1, C > 0$ such that for all w such that each $|T_i(w)| > \frac{1}{p}$, the Newton polygon of the power series $\det(I - XU_p | \mathcal{S}_w(G, \mathcal{U}))$ lies above the curve*

$$y = \left(A_1 x^{1+\frac{2}{n(n-1)}} - C \right) \min_i v(T_i(w)).$$

(2) Suppose that $w(a_1, \dots, a_{n-1}) = \prod_i a_i^{t_i} \chi_i(a_i)$, where $(t_1, \dots, t_{n-1}) \in (\mathbb{Z}_{\geq 0})^{n-1}$ with $t_1 \geq \dots \geq t_{n-1}$, and each χ_i is a finite character of conductor c_i , satisfying technical restrictions. Let $\chi_{(1)}, \dots, \chi_{(n-1)}$ be the characters $\chi_1, \dots, \chi_{n-1}$ reordered so that $\text{cond}(\chi_{(1)}) \leq \text{cond}(\chi_{(2)}) \leq \dots \leq \text{cond}(\chi_{(n-1)})$, let $c_{(i)} = \text{cond}(\chi_{(i)})$, and let $T_{(i)} = T(\chi_{(i)})$.

Then there is a constant h , a polynomial $d_{t_1, \dots, t_{n-1}}$ of total degree $\frac{n(n-1)}{2}$ in the t_i s, and a linear function $l(t_1, \dots, t_{n-1})$ such that the Newton polygon of $\det(I - XU_p | \mathcal{S}_w(G, \mathcal{U}))$ contains at least

$$hp^{c_{(1)}+2c_{(2)}+\dots+(n-1)c_{(n-1)}-\frac{n(n-1)}{2}} d_{t_1, \dots, t_{n-1}}$$

segments of slope at most $l(t_1, \dots, t_{n-1})$, hence passes below the point

$$\begin{aligned} & \left(hp^{c_{(1)}+2c_{(2)}+\dots+(n-1)c_{(n-1)}-\frac{n(n-1)}{2}} d_t, hp^{c_{(1)}+2c_{(2)}+\dots+(n-1)c_{(n-1)}-\frac{n(n-1)}{2}} d_t l(t) \right) \\ & = \left(x, A_2 \left(v(T_{(1)})^{\frac{2}{n(n-1)}} v(T_{(2)})^{\frac{2 \cdot 2}{n(n-1)}} \dots v(T_{(n-1)})^{\frac{2 \cdot (n-1)}{n(n-1)}} \right) x^{1+\frac{2}{n(n-1)}} \right) \end{aligned}$$

for $x = hp^{c_{(1)}+2c_{(2)}+\dots+(n-1)c_{(n-1)}-\frac{n(n-1)}{2}} d_t$ and a constant A_2 . Note that in particular,

$$v(T_{(1)})^{\frac{2}{n(n-1)}} v(T_{(2)})^{\frac{2 \cdot 2}{n(n-1)}} \dots v(T_{(n-1)})^{\frac{2 \cdot (n-1)}{n(n-1)}} \leq \max_i v(T_i).$$

We also leverage Theorem 1.1.4 to prove two statements that may be more geometrically satisfying. First, we prove the following alternative version of the upper bound which provides infinitely many upper bound points on the same Newton polygon.

Theorem 1.1.5. *Suppose that $w(a_1, \dots, a_{n-1}) = \prod_i a_i^{t_i} \chi_i(a_i)$, where $(t_1, \dots, t_{n-1}) \in (\mathbb{Z}_{\geq 0})^{n-1}$ with $t_1 \geq \dots \geq t_{n-1}$, and each χ_i is a finite character of conductor c_i , satisfying technical restrictions. Then there is a constant A_2 such that for every radius $r > 0$, there is a weight s such that $|T_i(w) - T_i(s)| < r$ for all i , $|T_i(\chi t)| = |T_i(s)|$ for all i , and the Newton polygon of $\det(I - XU_p | \mathcal{S}_w(G, \mathcal{U}))$ lies below an infinite*

sequence of points lying on the curve parametrized by

$$\left(x, A_2 x^{1+\frac{1}{\binom{n}{2}}} \cdot \left(v(T_{(1)}(s))^{\frac{2}{n(n-1)}} v(T_{(2)}(s))^{\frac{2 \cdot 2}{n(n-1)}} \cdots v(T_{(n-1)}(s))^{\frac{2 \cdot (n-1)}{n(n-1)}} \right) \right).$$

We then use the lower bound of Theorem 1.1.4 to prove the following (vaguely stated) decomposition result for the boundary of the eigenvariety.

Theorem 1.1.6. *Let $a_p : \mathcal{Z} \rightarrow \mathbb{G}_m$ be the map taking a point of \mathcal{Z} to its U_p -eigenvalue. For $\alpha \in \mathbb{R}_{\geq 0}$, let $\mathcal{Z}(\alpha)$ be the subset of points $z \in \mathcal{Z}$ such that $v(a_p(z)) = \alpha v(T_i(z))$. Then over certain open subsets of the weight polydisc boundary where $v(T_i)$ is much smaller than all the other $v(T_j)$ s, $\mathcal{Z}(\alpha)$ is disconnected from its complement in \mathcal{Z} .*

1.2. Proof outline. The proof of Part 1 of Theorem 1.1.4 is a direct application of the method of Johansson-Newton [15]. They construct families of automorphic forms extending *over* the boundary of weight space, to points in what can be viewed as an adic compactification of weight space, and show that the eigenvariety also extends to those points. Consequently, the matrix coefficients of U_p can be computed in an explicit basis for the space of forms over the “points at infinity” given by monomials in the matrix coefficients of the dimension- $\frac{n(n-1)}{2}$ maximal lower unipotent subgroup of $GL_n(\mathbb{Q}_p)$. Explicit bounds on those matrix coefficients arise directly from the proof of complete continuity of U_p .

The proof of Part 2 of Theorem 1.1.4 requires a detailed analysis of p -adic automorphic representations which may be of independent interest. As in the proof of Proposition 3.20 of [16] (Part 2 of Theorem 1.1.2), we would like to carry out the following steps:

- (1) Construct a subspace S of $\mathcal{S}_w(G, \mathcal{U})$ of dimension $hp^{c_{(1)}+2c_{(2)}+\cdots+(n-1)c_{(n-1)}-\frac{n(n-1)}{2}} dt$ which can be thought of as the space of “classical forms of weight w and minimal level”.

(2) Prove that U_p is injective on S , so that all eigenforms in S have finite slope.

(3) Prove that finite-slope eigenforms in S have slope bounded above by $l(t)$.

In general, it is a fact that a classical form f is finite-slope if and only if the local component $\pi_{f,p}$ at p of its associated automorphic representation is a principal series representation of $GL_n(\mathbb{Q}_p)$. For $n = 2$ as in Liu-Wan-Xiao, the upper bound then follows from the fact that $\pi_{f,p}$ is a principal series if the level of f equals the conductor of its central character, which can be checked (as in Loeffler-Weinstein [18]) by comparing its level to that of the new vectors in each of the three possible Bernstein-Zelevinsky classes of representations (principal series, special, and supercuspidal), those new vectors having been written down by Casselman [8]. The dimension of the space of such f is easy to count.

To detect when $\pi_{f,p}$ is a principal series for all n without brute-forcing through Bernstein-Zelevinsky classes, we use Roche's analysis of principal series types for GL_n ([20]). For a character χ of $T(\mathbb{Q}_p)$, Roche gives a subgroup $J \subset GL_n(\mathbb{Z}_p)$ and a character ρ of J such that an irreducible representation π of $GL_n(\mathbb{Q}_p)$ is a principal series associated to an unramified twist of χ if and only if π contains a vector on which J acts by ρ (which we will call a (J, ρ) -vector).

To proceed, we construct S so that for any eigenform $f \in S$, $\pi_{f,p}$ admits a nontrivial map from $\text{Ind}_J^{\text{Iw}_p} \rho$, hence contains a (J, ρ) -vector and is a principal series. In order to embed S inside $\mathcal{S}_w(G, \mathcal{U})$, we have to prove that $\text{Ind}_J^{\text{Iw}_p} \rho$ is irreducible using Mackey theory, where Iw_p is the subgroup of $GL_n(\mathbb{Z}_p)$ of matrices that are upper triangular mod p . The dimension of S is proportional to the product of $\dim \text{Ind}_J^{\text{Iw}_p} \chi$, which is a function of the valuations $v(T_i)$, and the dimension of the algebraic representation of GL_n of highest weight corresponding to the algebraic part of (T_1, \dots, T_{n-1}) , which is a polynomial of total degree $\frac{n(n-1)}{2}$ in the weight parameters by a combinatorial calculation. This covers Steps 1 and 2. In fact, we can even show the following side result.

Theorem 1.2.1. *S is precisely the space of finite-slope classical forms of weight w .*

To show this, we slightly refine the setup of the Bellaïche-Chenevier construction of the eigenvariety in order to precisely define the sense in which S is “minimal level”. Consequently, our upper bound is the best possible with existing methods, except possibly for the size of $l(t)$. Step 3 is relatively standard; we do it by constructing companion forms f^w of f for each $w \in S_n$ such that the slopes of all the companion forms sum to $l(t)$.

In Section 2, we describe the construction of the eigenvarieties we are interested in, primarily following Chenevier ([9]) and Bellaïche-Chenevier ([2]), adding some extra details in places of particular importance to us. For example, we give a slightly more general definition of local analyticity of p -adic automorphic forms which allows different radii of analyticity for different coordinates and prove that it works, which aids in proving Theorem 1.2.1.

In Section 3, we analyze the subspaces of classical automorphic forms of locally algebraic weights and the automorphic representations they generate, thus carrying out Steps 1 and 2 above, and proving Theorem 3.6.6, a precise version of Theorem 1.2.1.

In Section 4, we carry out Step 3 to prove Theorem 1.1.4 and Theorem 1.1.5.

Finally, in Section 5, we state and prove a precise version of Theorem 1.1.6 and discuss other geometric consequences of Theorem 1.1.4. Unlike in Liu-Wan-Xiao’s setting, for higher-dimensional eigenvarieties, the lower and upper bounds do not match at any point on the Newton polygon, and we cannot expect them to, because there exist (probably) non-classical forms of slopes smaller than some classical forms. As a result, we cannot prove the equivalent of Conjecture 1.1.1 for these higher-dimensional eigenvarieties. However, we can prove that certain boundary sections of the eigenvariety decompose into many disconnected components (with the caveat that we cannot verify that those sections are nonempty, although in fact we expect them to be everything).

2. BELLAÏCHE-CHENEVIER EIGENVARIETIES FOR DEFINITE UNITARY GROUPS

Let p be a prime. Let $q = 4$ if $p = 2$ and $q = p$ otherwise. In this section, we go through the construction of eigenvarieties for definite unitary groups in the language of Chenevier and Bellaïche-Chenevier. In Section 2.1, we define the groups and the spaces of p -adic automorphic forms we are interested in, notably including the spaces of classical forms whose interpolation was the original motivation for this construction. In Section 2.2, we describe the properties of the space of p -adic weights. In Section 2.3, we define certain coordinates on spaces of functions on the Iwahori, using a convenient hybrid of the language of Chenevier and Bellaïche-Chenevier. In Section 2.4, we introduce systematic notation for certain subgroups of Iw_p . In Section 2.5, we define the space of families of p -adic automorphic forms over weight space, along with subspaces of locally analytic families. In Section 2.6, we define U_p -operators and work through their various important properties in great detail. In Section 2.7, we define the desired eigenvarieties.

2.1. p -adic automorphic forms. Let E be an imaginary quadratic field over \mathbb{Q} , and D a central simple E -algebra of rank n^2 which has an involution $x \mapsto x^*$ extending the nontrivial automorphism of E over \mathbb{Q} (for example, D could be $GL_n(E)$). Let G/\mathbb{Q} be the group whose R -points, for a \mathbb{Q} -algebra R , are

$$G(R) = \{x \in D \otimes_{\mathbb{Q}} R \mid xx^* = 1\}.$$

$G(\mathbb{Q}_p)$ is isomorphic to $GL_n(\mathbb{Q}_p)$ if p is split in E (since then $E_p \cong \mathbb{Q}_p \oplus \mathbb{Q}_p$ with the involution switching factors) and to $U_n(\mathbb{Q}_p)$ if p is inert; we will assume that p is split in E and $G(\mathbb{Q}_p) \cong GL_n(\mathbb{Q}_p)$. Also, $G(\mathbb{R}) \cong U_{s,t}(\mathbb{R})$ for (s, t) the signature of $Q(x) = xx^*$; we will assume that $Q(x)$ has signature $(n, 0)$ or $(0, n)$, so that $G(\mathbb{R})$ is compact.

As usual, we write B and \overline{B} for the upper and lower triangular Borel subgroups of GL_n respectively, T for the diagonal torus, and N and \overline{N} for the upper and lower unipotent subgroups of GL_n respectively.

Write $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}, \mathbb{A}_f$ for the finite adeles of \mathbb{A} , and \mathbb{A}_f^p for the finite adeles trivial at p . Let \mathcal{U} be a compact open subgroup of $G(\mathbb{A}_f)$ of the form $\mathcal{U}_p \times \mathcal{U}^p$, where \mathcal{U}_p is a compact open subgroup of $G(\mathbb{Q}_p)$ (called the wild level structure) and \mathcal{U}^p a compact open subgroup of $G(\mathbb{A}_f^p)$ (called the tame level structure). We can now define V -valued automorphic forms for any \mathcal{U}_p -module V .

Definition 2.1.1. If V is a $k[\mathcal{U}_p]$ -module for any field k , write $V(G, \mathcal{U})$ for the k -vector space of maps

$$f : G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \rightarrow V$$

such that $f(xu) = u_p^{-1}f(x)$ for all $x \in G(\mathbb{Q}) \backslash G(\mathbb{A}_f)$ and $u \in \mathcal{U}$. Equivalently,

$$V(G, \mathcal{U}) = (\text{Hom}(G(\mathbb{Q}) \backslash G(\mathbb{A}_f), k) \otimes V)^{\mathcal{U}}$$

where the action of \mathcal{U} on $\text{Hom}(G(\mathbb{Q}) \backslash G(\mathbb{A}_f), k)$ is right translation and the action on V is through \mathcal{U}_p . For any subgroup $\mathcal{U}' \supseteq \mathcal{U}$ of $G(\mathbb{A}_f)$ to which the action of \mathcal{U} on V extends, $V(G, \mathcal{U})$ is a \mathcal{U}' -module with action $(uf)(x) = u_p f(xu)$.

We will frequently express examples using the following notation: if $B \subseteq H$ are groups, R is a ring, and $s : B \rightarrow R$ is a character, let

$$\text{Ind}_B^H s = \{f : H \rightarrow R \mid f(hb) = s(b)f(h) \text{ for all } h \in H, b \in B\},$$

and if P is a property of some functions $f \in \text{Ind}_B^H s$ which is invariant under left translation by H , let

$$\text{Ind}_B^{H,P} s = \{f \in \text{Ind}_B^H s \mid f \text{ has property } P\}.$$

Then $\text{Ind}_B^{H,P} s$ is an R -module with a (left) action of H given by $(hf)(x) = f(h^{-1}x)$ for all $h, x \in H$.

For example, if k is a field, $t = (t_1, \dots, t_n) \in \mathbb{Z}^n$, and we write $\text{diag}(d_1, \dots, d_n)$ for the diagonal matrix with entries d_1, \dots, d_n along the diagonal, we can interpret t as the character of the diagonal torus $T(k)$ of $GL_n(k)$ taking $\text{diag}(d_1, \dots, d_n)$ to $\prod_{i=1}^n d_i^{t_i}$, and thus as the character of the upper triangular Borel $B(k)$ obtained by reducing to $T(k)$ and applying t . In the event that $t_1 \geq \dots \geq t_n$, the k -vector space

$$\text{Ind}_{B(k)}^{GL_n(k), \text{alg}} t,$$

where *alg* stands for *algebraic* (i.e. $f : GL_n(k) \rightarrow k$ comes from an element of $k[GL_n]$), is the irreducible algebraic representation of GL_n over k of highest weight t (See Section 12.1.3 of [13] and Proposition 2.2.1 of [9]). We call this representation $S_t(k)$. Then $S_t(k)(G, \mathcal{U})$ is the space of classical p -adic automorphic forms on G of weight t and level \mathcal{U} with coefficients in k .

One way to picture $V(G, \mathcal{U})$ is as follows. By the generalized finiteness of class groups (see Theorem 5.1 of [3]), the set $G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / \mathcal{U}$ is finite. Fix double coset representatives $x_1, \dots, x_h \in G(\mathbb{A}_f)$. Then we have an isomorphism

$$V(G, \mathcal{U}) \xrightarrow{\sim} \bigoplus_{i=1}^h V^{x_i^{-1}G(\mathbb{Q})x_i \cap \mathcal{U}}$$

$$f \mapsto (f(x_1), \dots, f(x_h)).$$

Because $G(\mathbb{R}) \cong U_n(\mathbb{R})$ is compact, $G(\mathbb{Q})$ is discrete in $G(\mathbb{A}_f)$ (see e.g. Proposition 1.4 of [14] or Proposition 3.1.2 of [17]). Since in addition \mathcal{U} is compact, the group $x_i^{-1}G(\mathbb{Q})x_i \cap \mathcal{U}$ is always finite, and it is trivial if \mathcal{U}^p is sufficiently small. (For example, by Proposition 4.1.1 of [9], there is an integer e_n depending only on n such that $x_i^{-1}G(\mathbb{Q})x_i \cap \mathcal{U}$ is guaranteed to be trivial if the image of \mathcal{U}^p in $G(\mathbb{Q}_l)$ is

contained in $\Gamma(l) = \{g \in GL_n(\mathbb{Z}_p) \mid g \equiv 1 \pmod{l}\}$ for some prime $l \nmid e_n$. It is this fact that makes the construction of the eigenvariety for G so sleek.

When convenient, we will assume that \mathcal{U}^p is sufficiently small and thus $V(G, \mathcal{U}) \cong V^h$. As in Remark 2.14 of [16], this is not important to us, because the eigenvariety for any \mathcal{U}^p is a union of connected components of the eigenvariety for a sufficiently small subgroup of \mathcal{U}^p .

2.2. Weight space. A *weight* is a continuous character of $T(\mathbb{Z}_p) \cong (\mathbb{Z}_p^\times)^n$. Such a weight can be viewed as a character of $B(\mathbb{Z}_p)$ by reduction to $T(\mathbb{Z}_p)$. (In the introduction, we defined a weight instead to be a character of $(\mathbb{Z}_p^\times)^{n-1}$, that is, a character of $T(\mathbb{Z}_p)$ that is trivial on the last \mathbb{Z}_p^\times -factor. We will go back to restricting possible weights to the subset that is trivial on the last \mathbb{Z}_p^\times -factor whenever it is convenient, because any character of $T(\mathbb{Z}_p)$ can be twisted by a central character to one in this restricted subset, and central characters do not change spaces of automorphic forms in an interesting way.)

The *weight space* \mathscr{W}^n is the rigid analytic space over \mathbb{Q}_p such that for any affinoid \mathbb{Q}_p -algebra A , $\mathscr{W}^n(A)$ is the set of continuous characters $(\mathbb{Z}_p^\times)^n \rightarrow A^\times$. Let $\Delta^n = ((\mathbb{Z}/q\mathbb{Z})^\times)^n$. We have

$$(\mathbb{Z}_p^\times)^n \cong \Delta^n \times (1 + q\mathbb{Z}_p)^n$$

so an A -point of \mathscr{W}^n is determined by a character of Δ^n and a character of $(1 + q\mathbb{Z}_p)^n$. Furthermore, a character s of $(1 + q\mathbb{Z}_p)^n$ is determined by the values $T_i(s) = s(1, \dots, 1, \exp(q), 1, \dots, 1) - 1$ (where the i th entry is $\exp(q)$ and all the others are 1), since $\exp(q)$ topologically generates $1 + q\mathbb{Z}_p$. By Lemma 1 of [4], the coordinates $(T_1, \dots, T_n) \in A^n$ come from an A -point of \mathscr{W}^n precisely when they are topologically nilpotent. Thus \mathscr{W}^n can be pictured as a finite disjoint union of wide-open unit polydiscs with coordinates (T_1, \dots, T_n) , one for each tame character of Δ^n .

We use

$$[\cdot] : (\mathbb{Z}_p^\times)^n \rightarrow \mathbb{Z}_p[[\mathbb{Z}_p^\times]^n]$$

to denote the universal character of $(\mathbb{Z}_p^\times)^n$ and Λ^n to denote the Iwasawa algebra

$$\Lambda^n = \mathbb{Z}_p[[\mathbb{Z}_p^\times]^n] \cong \mathbb{Z}_p[\Delta^n] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[1 + q\mathbb{Z}_p]^n] \cong \mathbb{Z}_p[\Delta^n] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[T_1, \dots, T_n]]$$

where $T_i = [(1, \dots, 1, \exp(q), 1, \dots, 1)] - 1$ with the $\exp(q)$ in the i th position; then continuous homomorphisms $\chi : \Lambda^n \rightarrow A$ are in bijection with A -points of \mathscr{W}^n via $\chi \mapsto \chi \circ [\cdot]$.

Example 2.2.1 (dominant algebraic weights). If $t_1 \geq \dots \geq t_n$ are integers, the algebraic character $(d_1, \dots, d_n) \mapsto \prod_{i=1}^n d_i^{t_i}$ is a \mathbb{Q}_p -point of \mathscr{W}^n with T -coordinates

$$(\exp(t_1 q) - 1, \dots, \exp(t_n q) - 1)$$

which have valuations

$$v\left(\left(1 + t_i q + \frac{(t_i q)^2}{2!} + \dots\right) - 1\right) = v(t_i q).$$

If $\chi : \mathbb{Z}_p \rightarrow \mathbb{C}^\times$ is a smooth (i.e. finite-order) character, we will borrow the following slightly nonstandard definition of the conductor $\text{cond}(\chi)$ of χ from Section 3 of [20]: it is the least *positive* integer n such that $1 + p^n \mathbb{Z}_p \subset \ker(\chi)$. Thus the conductor of the trivial character is 1 but the conductor of any other character is the same as with the usual definition.

Example 2.2.2 (locally algebraic weights). If $t_1 \geq \dots \geq t_n$ are integers and χ_1, \dots, χ_n are finite-order characters $\mathbb{Z}_p^\times \rightarrow \mathbb{C}^\times$, the “locally algebraic” character

$$(d_1, \dots, d_n) \mapsto \prod_{i=1}^n \chi_i(d_i) d_i^{t_i}$$

is a \mathbb{C}_p -point of \mathcal{W}^n . If χ_i is nontrivial with conductor c_i , we have

$$v(T_i) = v(\chi_i(\exp(q)) \exp(t_i q) - 1) = \begin{cases} v(t_i q) & \text{if } p > 2 \text{ and } c_i = 1 \\ \frac{q}{p^{c_i-1}(p-1)} & \text{if } p > 2 \text{ and } c_i \geq 2 \\ v(t_i q) & \text{if } p = 2 \text{ and } c_i = 3 \\ \frac{q}{p^{c_i-1}(p-1)} = \frac{1}{2^{c_i-3}} & \text{if } p = 2 \text{ and } c_i \geq 4. \end{cases}$$

For completeness, we quickly prove the second case above; the others are similar.

The value $\chi_i(\exp(q)) = \chi_i(\exp(p))$ is a primitive p^{c_i} th root of unity, say $\zeta_{p^{c_i}}$. Let

$$f(X) = \frac{X^{p^{c_i}} - 1}{X^{p^{c_i-1}} - 1} = \prod_{a \in (\mathbb{Z}/p^{c_i}\mathbb{Z})^\times} (X - \zeta_{p^{c_i}}^a) = X^{p^{c_i} - p^{c_i-1}} + X^{p^{c_i} - 2p^{c_i-1}} + \dots + X^{p^{c_i-1}} + 1.$$

Then $f(1) = p = \prod_{a \in (\mathbb{Z}/p^{c_i}\mathbb{Z})^\times} (1 - \zeta_{p^{c_i}}^a)$. Each term in the product has the same valuation, since they are Galois conjugate, and there are $p^{c_i-1}(p-1)$ such terms.

So $v(\chi_i(\exp(q)) - 1) = \frac{1}{p^{c_i-2}(p-1)}$. The factor of $\exp(t_i q)$ has no effect since it is 1 (mod p).

In general, if A is a Banach \mathbb{Q}_p -algebra, we say that a character $s : \mathbb{Z}_p^\times \rightarrow A^\times$ is c -locally analytic if its restriction to $1 + p^c\mathbb{Z}_p$ is given by a convergent power series with coefficients in A . Every continuous character s is c -locally analytic for some c : let $T = s(\exp(q)) - 1$ and choose c such that $\|T^{p^c/q}\| < q^{-1}$. Then we have

$$s(z) = s\left(\exp(p^c)^{\frac{1}{p^c} \log z}\right) = [(1+T)^{p^c/q}]^{\frac{1}{p^c} \log z} = [1 + ((1+T)^{p^c/q} - 1)]^{\frac{1}{p^c} \log z}$$

if this converges. But by our choice of c , we have $\|(1+T)^{p^c/q} - 1\| < q^{-1}$, and if $z \in (1 + p^c\mathbb{Z}_p)$ then $\left|\frac{1}{p^c} \log z\right| \leq 1$. By Lemma 3.6.1 of [9], this expression is a convergent power series in z .

Naturally, if $s : (\mathbb{Z}_p^\times)^n \rightarrow A^\times$ is a character, we say that it is (c_1, \dots, c_n) -locally analytic if it is c_i -locally analytic in the i th factor.

If W is any open affinoid subset of \mathscr{W} , we use

$$[\cdot]_W : (\mathbb{Z}_p^\times)^n \rightarrow \mathcal{O}(W)^\times$$

to denote the universal character of $(\mathbb{Z}_p^\times)^n$ with coefficients in $\mathcal{O}(W)$. Note that $[\cdot]_W$ is (c_1, \dots, c_n) -locally analytic with c_i depending on $\max_{s \in W(\mathbb{C}_p)} |T_i(s)|$.

2.3. Coordinates on spaces of functions on Iw_p . If A is an affinoid \mathbb{Q}_p -algebra and $s : (\mathbb{Z}_p^\times)^n \rightarrow A$ is a weight, we can view any function $f \in \text{Ind}_{B(\mathbb{Z}_p)}^{Iw_p} s$ as a function on $\mathbb{Z}_p^{n(n-1)/2}$ by restricting f to the lower unipotent subgroup \overline{N} and applying the map

$$\mathbb{Z}_p^{n(n-1)/2} \rightarrow \overline{N}$$

$$\underline{z} = (z_{ij}) \mapsto \overline{N}(\underline{z}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ pz_{21} & 1 & 0 & \cdots & 0 \\ pz_{31} & pz_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ pz_{n1} & pz_{n2} & pz_{n3} & \cdots & 1 \end{pmatrix} \in \overline{N}.$$

We say that f is *continuous* if it is continuous as a function on $\mathbb{Z}_p^{n(n-1)/2}$ via $\underline{z} \mapsto \overline{N}(\underline{z})$. Then $\mathcal{S}_s := \text{Ind}_{B(\mathbb{Z}_p)}^{Iw_p, cts}(s)$, where *cts* stands for *continuous*, is an $A[Iw_p]$ -module. If $s^0 : (\mathbb{Q}_p^\times)^n \rightarrow A$ is the trivial extension of s from $(\mathbb{Z}_p^\times)^n$ to $(\mathbb{Q}_p^\times)^n$ (that is, we set $s^0(d) = 1$ for any $d \in (\mathbb{Q}_p^\times)^n$ whose entries are powers of p), \mathcal{S}_s is canonically isomorphic to $\text{Ind}_{B(\mathbb{Q}_p)}^{B(\mathbb{Q}_p)Iw_p, cts}(s^0)$ by restriction of functions from $B(\mathbb{Q}_p)Iw_p$ to Iw_p . Consequently it has an action by $B(\mathbb{Q}_p)Iw_p$.

It will be useful to write out the natural action of Iw_p on $f \in \mathcal{S}_s$ more explicitly in terms of the coordinates z_{ij} . To do this, we interpret them as Plücker coordinates on $\overline{N}(\underline{z})$. Recall that for any $1 \leq j \leq n$ and subset σ of $\{1, \dots, n\}$ with $\#\sigma = j$, the Plücker coordinate $Z_{j, \sigma}$ associated to (j, σ) is the algebraic function on GL_n given by

the determinant of the minor associated to the rows corresponding to σ and the first j columns.

Give \mathbb{Q}_p^n the standard basis e_1, \dots, e_n and interpret elements of \mathbb{Q}_p^n as *horizontal* vectors. Give $\wedge^j(\mathbb{Q}_p^n)$ the corresponding standard basis

$$\{e_\sigma = e_{k_1} \wedge \dots \wedge e_{k_j} \mid \sigma = \{k_1 < \dots < k_j\} \subset \{1, \dots, n\}\},$$

ordered lexicographically, and again interpret elements of $\wedge^j(\mathbb{Q}_p^n)$ as horizontal vectors. Let $1_j = \{1, \dots, j\}$. If $GL_n(\mathbb{Q}_p)$ acts on \mathbb{Q}_p^n by right multiplication of horizontal vectors (the transpose of the standard action), and $\iota_j : GL_n(\mathbb{Q}_p) \hookrightarrow GL(\wedge^j(\mathbb{Q}_p^n))$ gives the induced action of $GL_n(\mathbb{Q}_p)$ on $\wedge^j(\mathbb{Q}_p^n)$ (where again $GL(\wedge^j(\mathbb{Q}_p^n))$ acts on $\wedge^j(\mathbb{Q}_p^n)$ by right multiplication of horizontal vectors), then for $x \in GL_n(\mathbb{Q}_p)$, $Z_{j,\sigma}(x)$ is the coefficient of e_{1_j} in $e_\sigma \cdot \iota_j(x)$, or the entry of $\iota_j(x)$ in the σ th row and first column. If $b = (b_{ij}) \in B(\mathbb{Q}_p)$, $\iota_j(b)$ is also upper triangular, so the coefficient of e_{1_j} in $e_\sigma \cdot \iota_j(xb) = e_\sigma \cdot \iota_j(x) \cdot \iota_j(b)$ is $Z_{j,\sigma}(x)$ times the top left entry of $\iota_j(b)$, which is $b_{11} \cdots b_{jj} =: t_j(b)$. That is, we have

$$Z_{j,\sigma}(xb) = t_j(b)Z_{j,\sigma}(x).$$

So $Z_{j,\sigma}$ is invariant under right multiplication by N , and $Z_{j,\sigma/1} := Z_{j,\sigma}/Z_{j,1_j}$ is invariant under right multiplication by B . Let

$$Z_{j,\sigma}(u^{-1}x) = \sum_{\#\tau=j} a_{j,\sigma,\tau}(u)Z_{j,\tau}(x)$$

(note that $a_{j,\sigma,\tau}(u) \in \mathbb{Z}_p$, and if $u \in \text{Iw}_p$ then $a_{j,1_j,1_j}(u) \in \mathbb{Z}_p^\times$) so that

$$Z_{j,\sigma/1}(u^{-1}x) = \frac{a_{j,\sigma,1_j} + \sum_{\#\tau=j, \tau \neq 1_j} a_{j,\sigma,\tau}(u)Z_{j,\tau/1}(x)}{a_{j,1_j,1_j} + \sum_{\#\tau=j, \tau \neq 1_j} a_{j,1_j,\tau}(u)Z_{j,\tau/1}(x)}.$$

For $i \geq j$, let $\sigma_{ij} = \{1, \dots, j-1, i\}$ (so $\sigma_{jj} = 1_j$); then we can see that

$$Z_{j,1_j}(\overline{N}(\underline{z})) = 1,$$

$$Z_{j,\sigma_{ij}}(\overline{N}(\underline{z})) = pz_{ij}.$$

Thus z_{ij} , or technically pz_{ij} , is indeed a Plücker coordinate for $\overline{N}(\underline{z})$. Now using the Iwahori decomposition for Iw_p , let

$$u^{-1}\overline{N}(\underline{z}) = \overline{N}(\underline{uz})T(u, \underline{z})N(u, \underline{z})$$

where $T(u, \underline{z}) \in T$ and $N(u, \underline{z}) \in N$. So if $f \in \mathcal{S}_s$, we have

$$\begin{aligned} (uf)(\overline{N}(\underline{z})) &= f(u^{-1}\overline{N}(\underline{z})) = f(\overline{N}(\underline{uz})T(u, \underline{z})N(u, \underline{z})) \\ &= s(T(u, \underline{z}))f(\overline{N}(\underline{uz})). \end{aligned}$$

We wish to write \underline{uz} and $T(u, \underline{z})$ in terms of u and \underline{z} . But we have

$$\begin{aligned} Z_{j,\sigma_{ij}}(u^{-1}\overline{N}(\underline{z})) &= Z_{j,\sigma_{ij}}(\overline{N}(\underline{uz})T(u, \underline{z})N(u, \underline{z})) \\ &= Z_{j,\sigma_{ij}}(\overline{N}(\underline{uz}))t_j(T(u, \underline{z})) = p(uz)_{ij}t_j(T(u, \underline{z})). \end{aligned}$$

So in fact, setting $i = j$, we find

$$t_j(T(u, \underline{z})) = Z_{j,\sigma_{jj}}(u^{-1}\overline{N}(\underline{z})) = \sum_{\#\tau=j} a_{j,1_j,\tau}(u)Z_{j,\tau}(\overline{N}(\underline{z}))$$

where $Z_{j,\tau}(\overline{N}(\underline{z}))$ is by definition a polynomial in the variables $\{z_{kl}\}_{l \leq j, k > l}$ with coefficients in $p\mathbb{Z}_p$. Similarly,

$$p(uz)_{ij} = Z_{j,\sigma_{ij}/1}(u^{-1}\overline{N}(\underline{z})) = \frac{a_{j,\sigma,1_j} + \sum_{\#\tau=j,\tau \neq 1_j} a_{j,\sigma,\tau}(u)Z_{j,\tau/1}(\overline{N}(\underline{z}))}{a_{j,1_j,1_j} + \sum_{\#\tau=j,\tau \neq 1_j} a_{j,1_j,\tau}(u)Z_{j,\tau/1}(\overline{N}(\underline{z}))}$$

where $Z_{j,\tau/1}(\overline{N}(\underline{z}))$ is again a polynomial in the variables $\{z_{kl}\}_{l \leq j, k > l}$ with coefficients in $p\mathbb{Z}_p$.

2.4. Notation for subgroups of Iw_p . Since we will work with numerous subgroups of Iw_p , we will introduce some notation to identify them. If $\underline{c} = (c_{ij}) \in \mathbb{Z}_{\geq 0}^{n \times n}$ is any

$n \times n$ matrix of nonnegative integers, we will write

$$\Gamma(\underline{c}) = \{(x_{ij}) \in GL_n(\mathbb{Z}_p) \mid p^{c_{ij}} \mid (x_{ij} - \delta_{ij}) \text{ for all } i, j\}.$$

One can compute that $\Gamma(\underline{c})$ is a group precisely when $c_{ij} \leq c_{ik} + c_{kj}$ for all i, j, k . Note that this means that if $\Gamma(\underline{c})$ is a group, then so is $T(\mathbb{Z}_p)\Gamma(\underline{c})$. If we instead only have half a matrix of nonnegative integers $\underline{c} = (c_{ij})_{n \geq i > j \geq 1} \in \mathbb{Z}_{\geq 0}^{n(n-1)/2}$, we will write

$$\Gamma_1(\underline{c}) = \{(x_{ij}) \in \text{Iw}_p \mid v(x_{ij}) \geq c_{ij} \forall i > j; v(x_{ii} - 1) \geq \min\{c_{ij} \mid j < i\} \cup \{c_{ji} \mid j > i\} \forall i\}$$

$$\Gamma_0(\underline{c}) = \{(x_{ij}) \in \text{Iw}_p \mid v(x_{ij}) \geq c_{ij} \forall i > j\} = T(\mathbb{Z}_p)\Gamma_1(\underline{c}) \subset \text{Iw}_p.$$

Definition 2.4.1. We say that $\underline{c} = (c_{ij})_{n \geq i > j \geq 1} \in \mathbb{Z}_{\geq 0}^{n(n-1)/2}$ is *group-shaped* if $c_{ij} \leq c_{ik} + c_{kj}$ for all k , where we set c_{ab} to be 0 if $a \leq b$.

Thus $\Gamma_1(\underline{c})$ and $\Gamma_0(\underline{c})$ are subgroups whenever \underline{c} is group-shaped.

Definition 2.4.2. We call an $n(n-1)/2$ -tuple $\underline{c} = (c_{ij})_{n \geq i > j \geq 1} \in \mathbb{Z}_{\geq 0}^{n(n-1)/2}$ *compatible* with an n -tuple $(c_1, \dots, c_n) \in \mathbb{Z}_{\geq 0}^n$ if $c_i \leq \min\{c_{ij} \mid j < i\} \cup \{c_{ji} \mid j > i\} \forall i$. Equivalently, if we define $\underline{c}' \in \mathbb{Z}_{\geq 0}^{n \times n}$ by $c'_{ij} = c_{ij}$ for $i > j$, $c'_{ii} = c_i$, and $c'_{ij} = 0$ for $i < j$, then $\Gamma(\underline{c}')$ is a group.

Then we see that if $\chi = (\chi_1, \dots, \chi_n) : T(\mathbb{Z}_p) \rightarrow \mathbb{C}^\times$ is a character of $T(\mathbb{Z}_p)$, and $\underline{c} \in \mathbb{Z}_p^{n(n-1)/2}$, χ extends to a well-defined character of $T(\mathbb{Z}_p)\Gamma_1(\underline{c}) = \Gamma_0(\underline{c})$, trivial on $\Gamma_1(\underline{c})$, whenever \underline{c} is compatible with $(\text{cond}(\chi_1), \dots, \text{cond}(\chi_n))$.

In the calculations below, whenever we write $\Gamma(\underline{c})$, $\Gamma_0(\underline{c})$, or $\Gamma_1(\underline{c})$ for a matrix or half-matrix of nonnegative integers \underline{c} , we will implicitly assume that \underline{c} has been chosen so that it is in fact a group.

Depending on convenience, we may also overload the above notation in the following ways. First, if $\underline{r} = (r_{ij}) \in [0, 1]^{n \times n}$ is any $n \times n$ matrix of *real numbers* in $[0, 1]$,

we will write

$$\Gamma(\underline{r}) = \{(x_{ij}) \in GL_n(\mathbb{Z}_p) \mid |x_{ij} - \delta_{ij}| \leq r_{ij} \text{ for all } i, j\}.$$

Then $\Gamma(\underline{r})$ is a group precisely when $r_{ij} \geq r_{ik}r_{kj}$ for all i, j, k . We may define $\Gamma_1(\underline{r}), \Gamma_0(\underline{r})$ similarly. Second, if $c \in \mathbb{Z}_{>0}$ is a single integer, we will write

$$\Gamma(c) = \{(x_{ij}) \in GL_n(\mathbb{Z}_p) \mid v(x_{ij} - \delta_{ij}) \geq c \text{ for all } i, j\}.$$

This is always a group. We may define $\Gamma_1(c), \Gamma_0(c)$ similarly. Finally, if r is a single real number in $[0, 1]$, we will write $\Gamma(r), \Gamma_1(r), \Gamma_0(r)$ for the obvious final abuse of the same notation.

2.5. The sheaf of p -adic automorphic forms on weight space. If $\underline{c} = (c_{ij})_{n \geq i > j \geq 1} \in \mathbb{Z}_{>0}^{n(n-1)/2}$, we say that $f \in \mathcal{S}_s$ is \underline{c} -locally analytic if, for any $\underline{a} = (a_{ij}) \in \mathbb{Z}_p^{n(n-1)/2}$, the restriction of f to

$$B(\underline{a}, \underline{c}) = \{z = (z_{ij})_{n \geq i > j \geq 1} \in \mathbb{Z}_p^{n(n-1)/2} \mid z_{ij} \in a_{ij} + p^{c_{ij}}\mathbb{Z}_p \forall i, j\}$$

is given by a convergent power series in the variables z_{ij} with coefficients in A .

Definition 2.5.1. We call an $n(n-1)/2$ -tuple $\underline{c} = (c_{ij})_{n \geq i > j \geq 1} \in \mathbb{Z}_{\geq 0}^{n(n-1)/2}$ *analytic-shaped* if we have $c_{(j+1)j} = c_{(j+2)j} = \dots = c_{nj}$ for all j and $c_{nj} \geq c_{n(j+1)}$ for all j . (Note that if \underline{c} is analytic-shaped it is also group-shaped.) We call \underline{c} *compatible with* an n -tuple $(c_1, \dots, c_n) \in \mathbb{Z}_{\geq 0}^n$ if $c_j \leq \min_{l \leq j, k > l} c_{kl}$ for all j . That is, for each j_0 , all the entries of (c_{ij}) corresponding to matrix entries appearing in or to the left of the j_0 th column should be at least c_{j_0} .

Definition 2.5.2. If $\underline{c} \in \mathbb{Z}_{>0}^{n(n-1)/2}$ is analytic-shaped, we say that $s : (\mathbb{Z}_p^\times)^n \rightarrow A^\times$ is \underline{c} -locally analytic if there is (c_1, \dots, c_n) such that s is (c_1, \dots, c_n) -locally analytic and \underline{c} is compatible with (c_1, \dots, c_n) .

Proposition 2.5.3. *If s is (c_1, \dots, c_n) -locally analytic and $f \in \mathcal{S}_s$ is \underline{c} -locally analytic for \underline{c} analytic-shaped and compatible with (c_1, \dots, c_n) (so that s is \underline{c} -locally analytic), then uf is also \underline{c} -locally analytic for all $u \in \text{Iw}_p$.*

Proof. By the calculations in Section 2.3, we have $(uf)(\overline{N}(\underline{z})) = s(T(u, \underline{z}))f(\overline{N}(\underline{uz}))$ where

- $(uz)_{ij}$ is a power series in the variables $\{z_{kl}\}_{l \leq j, k > l}$;
- the j th diagonal entry of $T(u, \underline{z})$, or $\frac{t_j(T(u, \underline{z}))}{t_{j-1}(T(u, \underline{z}))}$, is also a power series in the variables $\{z_{kl}\}_{l \leq j, k > l}$.

So if we restrict to $\underline{z} \in B(\underline{a}, \underline{c})$, the coefficient $(uz)_{ij}$ ranges over a ball of the form $a'_{ij} + p^{\min_{l \leq j, k > l} c_{kl}} \mathbb{Z}_p$; since \underline{c} is analytic-shaped, we have $c_{ij} \leq \min_{l \leq j, k > l} c_{kl}$, and we conclude that \underline{uz} is also restricted to a ball of the form $B(\underline{a}', \underline{c})$. Thus $f(\overline{N}(\underline{uz}))$ is analytic for $\underline{z} \in B(\underline{a}, \underline{c})$. Similarly, $\frac{t_j(T(u, \underline{z}))}{t_{j-1}(T(u, \underline{z}))}$ ranges over a ball of the form $a''_{jj} + p^{\min_{l \leq j, k > l} c_{kl}} \mathbb{Z}_p$; since $c_j \leq \min_{l \leq j, k > l} c_{kl}$ and s_j is analytic on $a'_{jj} + p^{c_j} \mathbb{Z}_p$, we conclude that $s_j(T(u, \underline{z}))$ is analytic for $\underline{z} \in B(\underline{a}, \underline{c})$. Thus $(uf)(\overline{N}(\underline{z}))$ is analytic for $\underline{z} \in B(\underline{a}, \underline{c})$, as desired. \square

By Proposition 2.5.3, if s is \underline{c} -locally analytic with \underline{c} group-like, the space $\mathcal{S}_{s, \underline{c}} = \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, \underline{c}\text{-loc.an.}}(s)$, where $\underline{c}\text{-loc.an.}$ stands for \underline{c} -locally analytic, is well-defined and has an action by Iw_p .

We let $\mathcal{S} = \mathcal{S}_{[\cdot]} = \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, \text{cts}}([\cdot])$. If $\mathcal{U}_p = \text{Iw}_p$, we call

$$\mathcal{S}(G, \mathcal{U}) = \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, \text{cts}}([\cdot])(G, \mathcal{U})$$

the space of integral p -adic automorphic forms for G of level \mathcal{U} ; it has an action by $B(\mathbb{Q}_p)\mathcal{U}$. This gives a sheaf on \mathcal{W} whose fiber over s is

$$\mathcal{S}_s(G, \mathcal{U}) = \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, \text{cts}}(s)(G, \mathcal{U}).$$

Similarly, let $\mathcal{S}_{W,\underline{c}} = \mathcal{S}_{[\cdot]_W,\underline{c}} = \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p,\underline{c}\text{-loc.an.}}([\cdot]_W)$ (for any \underline{c} such that $[\cdot]_W$ is \underline{c} -locally analytic). If $\mathcal{U}_p = \text{Iw}_p$, we call

$$\mathcal{S}_{W,\underline{c}}(G, \mathcal{U}) = \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p,\underline{c}\text{-loc.an.}}([\cdot]_W)(G, \mathcal{U})$$

the space of \underline{c} -locally analytic p -adic automorphic forms for G of level \mathcal{U} ; this does *not* have an action by $B(\mathbb{Q}_p)$, as some elements of $B(\mathbb{Q}_p)$ do not preserve the radius of local analyticity, but we will see in the next section that it has an action by a certain submonoid.

2.6. The operators U_p^a . If H is any locally compact, totally disconnected topological group, we write $\mathcal{H}(H)$ for the k -algebra of compactly supported, locally constant k -valued functions on H with the convolution product

$$(\varphi_1 \star \varphi_2)(g) = \int_{h \in H} \varphi_1(h) \varphi_2(h^{-1}g) d\mu$$

where μ is a Haar measure on H . This algebra usually has no identity, but many idempotents. If K is a compact open subgroup of H , the idempotent $e_K = \frac{\mathbb{1}_K}{\mu(K)}$ projects $\mathcal{H}(H)$ onto the subalgebra $\mathcal{H}(H // K)$ of K, K -bi-invariant functions. If V is a smooth H -module, it is an $\mathcal{H}(H)$ -module via

$$\varphi(v) = \int_H \varphi(h)(hv) dh$$

and similarly V^K is an $\mathcal{H}(H // K)$ -module.

In the particular case $H = B(\mathbb{Q}_p)\mathcal{U}$, $V = \mathcal{S}_s(G, \mathcal{U})$, $K = \mathcal{U}$, we can rephrase this as follows. We sometimes write $[\mathcal{U}\zeta\mathcal{U}]$ for the element $\mathbb{1}_{\mathcal{U}\zeta\mathcal{U}}$ of $\mathcal{H}(G(\mathbb{A}_f) // \mathcal{U})$. If ζ_1, \dots, ζ_r are left \mathcal{U} -coset representatives of $\mathcal{U}\zeta\mathcal{U}$, so that

$$\mathcal{U}\zeta\mathcal{U} = \coprod_{i=1}^r \zeta_i\mathcal{U},$$

then for any $\varphi \in \mathcal{S}_s(G, \mathcal{U})$ and $x \in G(\mathbb{Q}) \backslash G(\mathbb{A}_f)$, we have

$$\begin{aligned} [\mathcal{U}\zeta\mathcal{U}](\varphi)(x) &= \int_{G(\mathbb{A}_f)} [\mathcal{U}\zeta\mathcal{U}](g) \cdot (g \cdot \varphi)(x) dg \\ &= \int_{\mathcal{U}\zeta\mathcal{U}} g_p \varphi(xg) dg = \sum_{i=1}^r (\zeta_i)_p \cdot \varphi(x\zeta_i). \end{aligned}$$

The following is Lemma 4.5.2 of [9], or Proposition 3.3.3 of [17].

Lemma 2.6.1. *Fix coset representatives x_1, \dots, x_h of $G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / \mathcal{U}$, and thus an isomorphism $\mathcal{S}_s(G, \mathcal{U}) \cong \mathcal{S}_s^h$. Then we have*

$$[\mathcal{U}\zeta\mathcal{U}](\varphi)(x_j) = \sum_{k=1}^r \sum_{i | \zeta_i \in x_j^{-1} G(\mathbb{Q}) x_k \mathcal{U}} (\zeta_i u_{ij}^{-1})_p \cdot \varphi(x_k)$$

for some $u_{ij} \in \mathcal{U}$. That is, the action of $[\mathcal{U}\zeta\mathcal{U}]$ on $\mathcal{S}_s(G, \mathcal{U})$ is of the form $\sum T_j \circ \sigma_j$, where the σ_j s are compositions of permutation operators on the entries of vectors in \mathcal{S}_s^h with projections onto one of the coordinates, and the T_j s are diagonal translations of \mathcal{S}_s^h by elements of $\mathcal{U}\zeta\mathcal{U}$.

Proof. Write $x_j \zeta_i$ in the form $d_{ij} x_{k_{ij}} u_{ij}$ where $d_{ij} \in G(\mathbb{Q})$ and $u_{ij} \in \mathcal{U}$. Then

$$\begin{aligned} [\mathcal{U}\zeta\mathcal{U}](\varphi)(x_j) &= \sum_{i=1}^r (\zeta_i)_p \cdot \varphi(x_j \zeta_i) \\ &= \sum_{i=1}^r (\zeta_i)_p \cdot \varphi(d_{ij} x_{k_{ij}} u_{ij}) = \sum_{i=1}^r (\zeta_i u_{ij}^{-1})_p \cdot \varphi(x_{k_{ij}}). \end{aligned}$$

The values of i for which $k_{ij} = k$ are those for which $\zeta_i = x_j^{-1} d x_k u$ for some $d \in G(\mathbb{Q})$ and $u \in \mathcal{U}$, that is, $\zeta_i \in x_j^{-1} G(\mathbb{Q}) x_k \mathcal{U}$. \square

If $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, we write

$$u^a = \text{diag}(p^{a_1}, \dots, p^{a_n})$$

and define the subgroup

$$\Sigma = \{u^a = \text{diag}(p^{a_1}, \dots, p^{a_n}) \mid a = (a_1, \dots, a_n) \in \mathbb{Z}^n\} \subset GL_n(\mathbb{Q}_p)$$

and its submonoids

$$\Sigma^- = \{u^a = \text{diag}(p^{a_1}, \dots, p^{a_n}) \mid a_1 \geq a_2 \geq \dots \geq a_n\} \subset \Sigma$$

$$\Sigma^{--} = \{u^a = \text{diag}(p^{a_1}, \dots, p^{a_n}) \mid a_1 > a_2 > \dots > a_n\} \subset \Sigma^-.$$

We will frequently choose ζ to be an element of Σ^- . Let

$$U_p^a = [\mathcal{U} \text{diag}(p^{a_1}, \dots, p^{a_n}) \mathcal{U}].$$

Proposition 2.6.2. *If $f \in \mathcal{S}_s$ and $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, u^a acts on f by $z_{ij} \mapsto p^{a_i - a_j} z_{ij}$.*

Proof. We have

$$\begin{aligned} f((u^a)^{-1} \overline{N}(z_{ij})) &= f \left(\begin{pmatrix} p^{-a_1} & \dots & 0 \\ 0 & \vdots & 0 \\ 0 & \dots & p^{-a_n} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ pz_{21} & 1 & 0 & \dots & 0 \\ pz_{31} & pz_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ pz_{n1} & pz_{n2} & pz_{n3} & \dots & 1 \end{pmatrix} \right) \\ &= f \begin{pmatrix} p^{-a_1} & 0 & 0 & \dots & 0 \\ p^{-a_2+1} z_{21} & p^{-a_2} & 0 & \dots & 0 \\ p^{-a_3+1} z_{31} & p^{-a_3+1} z_{32} & p^{-a_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p^{-a_n+1} z_{n1} & p^{-a_n+1} z_{n2} & p^{-a_n+1} z_{n3} & \dots & p^{-a_n} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= f \left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ p^{a_1-a_2+1}z_{21} & 1 & 0 & \cdots & 0 \\ p^{a_1-a_3+1}z_{31} & p^{a_2-a_3+1}z_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p^{a_1-a_n+1}z_{n1} & p^{a_2-a_n+1}z_{n2} & p^{a_3-a_n+1}z_{n3} & \cdots & 1 \end{pmatrix} \begin{pmatrix} p^{-a_1} & \cdots & 0 \\ 0 & \vdots & 0 \\ 0 & \cdots & p^{-a_n} \end{pmatrix} \right) \\
&= f(\overline{N}(p^{a_i-a_j}z_{ij}))s^0(u^a) = f(\overline{N}(p^{a_i-a_j}z_{ij})).
\end{aligned}$$

□

Corollary 2.6.3. *If $f \in \mathcal{S}_s$ is \underline{c} -locally analytic and $u^a \in \Sigma^-$, then $u^a f$ is also \underline{c} -locally analytic. So translation by $\text{Iw}_p u^a \text{Iw}_p$ preserves $\mathcal{S}_{s,\underline{c}}$ (and hence, by Lemma 2.6.1, U_p^a preserves $\mathcal{S}_{s,\underline{c}}(G, \mathcal{U})$).*

Proof. When $u^a \in \Sigma^-$, we have $a_i - a_j \geq 0$ for all $i > j$; thus if (z_{ij}) varies in a ball $B(\underline{a}, \underline{c})$, so does $(p^{a_i-a_j}z_{ij}) = (u^a z_{ij})$. □

Let $\underline{c}^0 \in \mathbb{Z}_{>0}^{n(n-1)/2}$ be minimal such that s is \underline{c}^0 -locally analytic.

Corollary 2.6.4. *If $f \in \mathcal{S}_s$ is \underline{c} -locally analytic and $u^a \in \Sigma^{--}$, then $u^a f$ is $\underline{c}^{--} := (\max\{c_{ij} - 1, c_{ij}^0\})$ -locally analytic. So translation by $\text{Iw}_p u^a \text{Iw}_p$ takes $\mathcal{S}_{s,\underline{c}}$ into $\mathcal{S}_{s,\underline{c}^{--}}$ (and hence, by Lemma 2.6.1, U_p^a takes $\mathcal{S}_{s,\underline{c}}(G, \mathcal{U})$ into $\mathcal{S}_{s,\underline{c}^{--}}(G, \mathcal{U})$).*

Proof. When $u^a \in \Sigma^{--}$, we have $a_i - a_j > 0$ for all $i > j$; thus if (z_{ij}) varies in a ball $B(\underline{a}, \underline{c})$, then $(p^{a_i-a_j}z_{ij}) = (u^a z_{ij})$ varies in a smaller ball $B(\underline{a}', \underline{c} + \underline{1})$. □

$\mathcal{S}_{s,\underline{c}}$ is an orthonormalizable A -module, for which we choose the following orthonormal basis: for each $\underline{a} \in \prod_{n \geq i > j \geq 1} \mathbb{Z}_p/p^{c_{ij}}\mathbb{Z}_p$, we choose the set of monomials $\prod_{n \geq i > j \geq 1} z_{ij}^{e_{ij}}$ as an orthonormal basis for the restriction of $\mathcal{S}_{s,\underline{c}}$ to $B(\underline{a}, \underline{c})$; then for $\mathcal{S}_{s,\underline{c}}$, we may choose as orthonormal basis the set of monomials $\prod_{n \geq i > j \geq 1} (z_{ij}^{\underline{a}})^{e_{ij}}$, with one copy for each $\underline{a} \in \prod_{n \geq i > j \geq 1} \mathbb{Z}_p/p^{c_{ij}}\mathbb{Z}_p$.

Corollary 2.6.5. *When $a \in \Sigma^{-}$, the operator of translation by u^a acts completely continuously on $\mathcal{S}_{s,\underline{c}}$, in the sense that it is a uniform limit of operators with finite-dimensional images. So by Lemma 2.6.1, U_p^a is completely continuous on $\mathcal{S}_{s,\underline{c}}(G, \mathcal{U})$.*

Proof. By Proposition 2.6.2, u^a scales $\prod_{n \geq i > j \geq 1} (z_{ij}^a)^{e_{ij}}$ by $\prod_{n \geq i > j \geq 1} p^{(a_i - a_j)e_{ij}}$, which goes to ∞ as any e_{ij} goes to ∞ . Furthermore, since the formulas in Section 2.3 all have integer coefficients, it is clear that translation by Iw_p is norm 1. \square

Since U_p^a is completely continuous on $\mathcal{S}_{s,\underline{c}}(G, \mathcal{U})$, for any k , the matrix of the action of U_p^a (in any basis) has a finite number of nonzero rows mod p^k . Suppose that this matrix has r_k rows that are zero mod p^k but nonzero mod p^{k+1} . Then for any $N \geq r_0 + r_1 + \cdots + r_k$, the coefficient of X^N in the characteristic power series

$$P_{s,\underline{c}}^a(X) = \det(1 - XU_p^a | \mathcal{S}_{s,\underline{c}}(G, \mathcal{U}))$$

of U_p^a acting on $\mathcal{S}_{s,\underline{c}}(G, \mathcal{U})$, being a linear combination of minors of size $N \geq r_0 + r_1 + \cdots + r_k$, is divisible by $r_1 + 2r_2 + \cdots + kr_k$. Since this lower bound grows faster than any linear function of N , $P_{s,\underline{c}}^a(X)$ is an entire function of X .

Proposition 2.6.6. *$P_{s,\underline{c}}^a(X)$ is independent of \underline{c} . (So we will henceforth call it $P_s^a(X)$.)*

Proof. This follows from applying Corollary 2 of Proposition 7 of [22] to the map $U_p^a : \mathcal{S}_{s,\underline{c}}(G, \mathcal{U}) \rightarrow \mathcal{S}_{s,\underline{c}^{--}}(G, \mathcal{U})$ from Corollary 2.6.4 and the obvious inclusion $\mathcal{S}_{s,\underline{c}^{--}}(G, \mathcal{U}) \hookrightarrow \mathcal{S}_{s,\underline{c}}(G, \mathcal{U})$. \square

Let U_p^Σ be the subring of $\mathcal{H}(G(\mathbb{A}_f) // \mathcal{U})$ generated by the elements U_p^a for $a \in \Sigma^-$. By Proposition 6.4.1 of [2], the map from $k[\Sigma]$ to U_p^Σ sending u^a to $U_p^b(U_p^c)^{-1}$ where u^b, u^c are any elements of Σ^- such that $u^a = u^b(u^c)^{-1}$ is a well-defined isomorphism of rings. So, in particular, U_p^Σ is abelian. Let \mathcal{H} be a subalgebra of $\mathcal{H}(G(\mathbb{A}_f) // \mathcal{U})$ given by the product of $\mathbb{Z}[U_p^\Sigma]$ at p and some commutative subalgebra of $\mathcal{H}(G(\mathbb{A}_f^p) // \mathcal{U}^p)$ away from p .

We write u_i for the image of $\text{diag}(1, \dots, 1, p, 1, \dots, 1) \in k[\Sigma]$ in U_p^Σ . If $f \in \mathcal{S}_{s,c}(G, \mathcal{U})$ is a simultaneous eigenvector for \mathcal{H} , let $u_i(f) = \lambda_i f$. We call these the λ -values associated to f .

Unless otherwise specified, we will generally set \mathcal{U} to be a compact open subgroup of $G(\mathbb{A}_f)$ given by the product of Iw_p at p and a fixed tame level structure away from p chosen so that $x^{-1}G(\mathbb{Q})x \cap U_0(p) = 1$ for all x . Call this subgroup $U_0(p)$. (Note that for the same reason as in Proposition 3.1.2 below, our choice of Iw_p as the wild level structure does not actually affect $P_s^a(X)$.)

2.7. The eigenvariety. Given our setup so far, the eigenvariety is easy to define. For a given $u^a \in \Sigma^{--}$, let \mathcal{Z}^a be the subvariety of $\mathcal{W} \times \mathbb{G}_m$ which, in any subset $W \times \mathbb{G}_m$ where $W \subset \mathcal{W}$ is open affinoid, is cut out by the characteristic power series $P_W^a(X)$ of U_p^a acting on $\mathcal{S}_W(G, U_0(p))$. Let $w : \mathcal{Z}^a \rightarrow \mathcal{W}$ be the first projection (weight) map, and $s^a : \mathcal{Z}^a \rightarrow \mathbb{G}_m$ the *inverse* of the second projection (slope) map. Then for any point $z \in \mathcal{Z}^a$, $s^a(z)$ is a nonzero eigenvalue of U_p^a acting on $\mathcal{S}_{w(z)}(G, U_0(p))$, and for any $w \in \mathcal{W}$, all nonzero eigenvalues of U_p^a acting on $\mathcal{S}_w(G, U_0(p))$ can be found in the fiber of \mathcal{Z}^a over w . We call \mathcal{Z}^a the spectral variety associated to U_p^a .

It is convenient to fix a particular choice of $u^a \in \Sigma^{--}$; we will choose $a = (n-1, n-2, \dots, 1, 0)$. From now on, we will write $U_p = U_p^{(n-1, n-2, \dots, 1, 0)}$ and $\mathcal{Z} = \mathcal{Z}^{(n-1, n-2, \dots, 1, 0)}$. We call an eigenform $f \in \mathcal{S}_w(G, U_0(p))$ finite-slope if $U_p f \neq 0$ (i.e. the slope of the U_p -eigenvalue is finite, and f appears on the eigenvariety), and infinite-slope otherwise.

Since \mathcal{H} is commutative, we can construct the space \mathcal{D} whose points correspond to systems of eigenvalues of all Hecke operators in \mathcal{H} , including in particular all U_p^a s simultaneously, by simply taking \mathcal{D} to be the finite cover of \mathcal{Z} which, over an affinoid $W \subset \mathcal{W}$, is given by the MaxSpec of the image of $\mathcal{H} \otimes \Lambda^n$ in the endomorphism ring of $\mathcal{S}_W(G, U_0(p))$. Then \mathcal{D} inherits the weight map $w : \mathcal{Z}^a \rightarrow \mathcal{W}$ and each slope map $s^a : \mathcal{D} \rightarrow \mathbb{G}_m$. Note that $\mathcal{D} \rightarrow \mathcal{Z}^a$ is degree 1 away from multiple roots of

$P_W^a(X)$, hence degree 1 away from a Zariski-closed subset of \mathcal{W} of lower dimension. So in general, the bounds and geometric properties we get for \mathcal{Z}^a will also apply to \mathcal{D} . For most of this paper, we will focus on the properties of \mathcal{Z}^a for a fixed a .

For additional details on properties of \mathcal{Z}^a and \mathcal{D} and their proofs, see [9] or [5].

3. LOCALLY ALGEBRAIC WEIGHTS

In this section, we analyze classical automorphic forms of locally algebraic weights and their associated automorphic representations. In Section 3.1, we define these spaces of classical forms and check their basic properties, including that they embed into the infinite-dimensional spaces of Section 2.1. In Section 3.2, we reproduce Bellaïche-Chenevier’s slope criterion guaranteeing that a given form is classical, phrased to work for locally algebraic weights instead of just algebraic weights; while this is not directly needed for our purposes, it is useful to give a sense of where classical forms fit in among the world of all p -adic automorphic forms. In Section 3.3, we explain the standard translation between classical forms and automorphic representations. In Section 3.4, we analyze certain Iwahori subrepresentations that may appear in the local component at p of such an automorphic representation, including a particularly important irreducible subrepresentation. In Section 3.5, we apply the work of Roche to a calculation of Hecke eigenvalues in ramified principal series. In Section 3.6, we identify a subspace of the classical forms whose associated automorphic representations have ramified principal series as their local components at p , and compute their U_p -eigenvalues in terms of the parameters of the corresponding principal series.

3.1. p -adic automorphic forms of locally algebraic weights. In Section 2.1, we defined classical forms of algebraic weights via the algebraic representation $S_t(k)$ of $GL_n(\mathbb{Q}_p)$. This construction may be generalized to locally algebraic weights as follows. Let $\chi = \chi_1 \cdots \chi_n$ be a finite character of $(\mathbb{Z}_p^\times)^n$. Then $t\chi$ is a locally

algebraic character of $(\mathbb{Z}_p^\times)^n$, in the sense that it is algebraic upon restriction to $\prod_{i=1}^n (a_i + p^{m_i} \mathbb{Z}_p)$ for some choice of m_i s and any nonzero a_i s. Similarly to earlier notation, for a positive integer c , let

$$B(\underline{a}, c) = \{z = (z_{ij})_{n \geq i > j \geq 1} \in \mathbb{Z}_p^{n(n-1)/2} \mid z_{ij} \in a_{ij} + p^c \mathbb{Z}_p \forall i, j\}$$

Then there are two equivalent definitions of the space

$$S_{t\chi, c} = \operatorname{Ind}_{B(\mathbb{Z}_p)}^{\operatorname{Iw}_p, c\text{-loc.alg.}} (t\chi)$$

where *c-loc.alg.* stands for *c-locally algebraic*. The first is through the usual induction operator above, as follows. We say that $f \in \operatorname{Ind}_{B(\mathbb{Z}_p)}^{\operatorname{Iw}_p} (t\chi)$ is *c-locally algebraic* if it has an algebraic extension to $B(\underline{a}, c)$ for all $\underline{a} \in \mathbb{Z}_p^{n(n-1)/2}$ of degree bounded as follows: writing f as a polynomial in the variables $Z_{i,k/1}$ as in Section 2.3, we require that for each fixed i , the degree of f as a polynomial in all the variables $Z_{i,k/1}$ should be at most $t_i - t_{i+1} =: m_i$. As in Proposition 2.5.3, one can see using the formulas in Section 2.3 that assuming $\operatorname{cond}(\chi_i) \leq c$ for all i , this condition is invariant under right translation by Iw_p .

The second definition, coming from the perspective of Loeffler (Section 2.5 of [17]), is

$$\left(\operatorname{Ind}_{B(\mathbb{Z}_p)}^{\operatorname{Iw}_p, \text{alg}} t \right) \otimes \left(\operatorname{Ind}_{B(\mathbb{Z}_p)/B(\mathbb{Z}_p) \cap \Gamma(c)}^{\operatorname{Iw}_p / \Gamma(c)} \chi \right).$$

Note that $\Gamma(c)$ is normal in Iw_p because it is the kernel of the reduction map from Iw_p to the corresponding group with coefficients in $\mathbb{Z}_p/p^c \mathbb{Z}_p$.

Except for an annoying technical distinction which we will discuss at the end of this subsection, the space $\operatorname{Ind}_{B(\mathbb{Z}_p)}^{\operatorname{Iw}_p, \text{alg}} t$ is the same (as an Iw_p -representation) as the space $S_t(k)$ defined in Section 2.1, since Iw_p is Zariski-dense in GL_n . Let $d_t = \dim \operatorname{Ind}_{B(\mathbb{Z}_p)}^{\operatorname{Iw}_p, \text{alg}} t$.

Proposition 3.1.1. *The natural map*

$$\left(\begin{array}{c} \text{Iw}_p, \text{alg.} \\ \text{Ind} \\ B(\mathbb{Z}_p) \end{array} t \right) \otimes \left(\begin{array}{c} \text{Iw}_p / \Gamma(c) \\ \text{Ind} \\ B(\mathbb{Z}_p) / B(\mathbb{Z}_p) \cap \Gamma(c) \end{array} \chi \right) \rightarrow \begin{array}{c} \text{Iw}_p, c\text{-loc. alg.} \\ \text{Ind} \\ B \end{array} (t\chi)$$

$$f \otimes g \mapsto fg$$

is an isomorphism.

Proof. To construct an inverse, let $\varphi \in \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, c\text{-loc. alg.}} \chi$. Let $\varphi_{\text{alg}} : \text{Iw}_p \rightarrow \mathbb{C}$ be defined by

$$\varphi_{\text{alg}}(b\bar{n}) = t(b)\varphi'(\bar{n})$$

for all $b \in B, \bar{n} \in \bar{N} \cap \text{Iw}_p$, where φ' is the unique algebraic extension of $\varphi|_{\bar{N} \cap \Gamma(c)}$ to $\bar{N} \cap \text{Iw}_p$. Let $\varphi_{\text{sm}} : \text{Iw}_p / \text{Iw}_p \cap \Gamma(c) \rightarrow \mathbb{C}$ be defined by

$$\varphi_{\text{sm}}(\bar{b}\bar{n}) = \chi(b)(\varphi/\varphi')(\bar{n})$$

where b, \bar{n} are any lifts of $\bar{b} \in B/B \cap \Gamma(c), \bar{n} \in (\bar{N} \cap \text{Iw}_p) / (\bar{N} \cap \Gamma(c))$. This suffices to prove surjectivity.

Injectivity follows from dimension counting: both sides have dimension $d_t p^{c \binom{n}{2}}$. \square

Remark 1. There is a simple isomorphism of Iw_p -representations

$$\begin{array}{c} \text{Iw}_p / \Gamma(c) \\ \text{Ind} \\ B(\mathbb{Z}_p) / B(\mathbb{Z}_p) \cap \Gamma(c) \end{array} \chi \xrightarrow{\sim} \begin{array}{c} \text{Iw}_p \\ \text{Ind} \\ \Gamma_0(c) \end{array} \chi$$

so we could just as easily have phrased this section in terms of $\text{Ind}_{\Gamma_0(c)}^{\text{Iw}_p} \chi$. For now, we have no particular reason to do this, but it may be more convenient for future work.

We call

$$S_{t\chi, c}(G, \mathcal{U}) = \begin{array}{c} \text{Iw}_p, c\text{-loc. alg.} \\ \text{Ind} \\ B \end{array} (t\chi)(G, \mathcal{U})$$

the space of classical p -adic automorphic forms on G of weight $t\chi$, radius c , and level \mathcal{U} . By the definitions, it embeds into $\mathcal{S}_{t\chi}(G, \mathcal{U})$, and we call its image a classical

subspace of $\mathcal{S}_{t\chi}(G, \mathcal{U})$. The following proposition is a quick generalization of part 4 of Lemma 4 of [4].

Proposition 3.1.2. *For any positive integers c, d , and e with $d \leq e$ and $c+d-e \geq 1$, we have a natural vector space isomorphism*

$$S_{t\chi, c}(G, \mathcal{U}^p \Gamma_0(d)) \cong S_{t\chi, c+d-e}(G, \mathcal{U}^p \Gamma_0(e)).$$

such that systems of \mathcal{H} -eigenvalues (where \mathcal{H} is obtained with respect to $\mathcal{U}^p \Gamma_0(d)$) on the left go to identical systems of \mathcal{H} -eigenvalues on the right (where \mathcal{H} is obtained with respect to $\mathcal{U}^p \Gamma_0(e)$).

Proof. For the purposes of this proposition, let $X = G(\mathbb{Q}) \backslash G(\mathbb{A}_f)$. The left-hand side is the subset of (*)

$$(\mathrm{Hom}(X, \mathbb{C}_p) \otimes \mathrm{Ind}_B^{\mathrm{Iw}_p, c\text{-loc. alg.}} (t\chi))^{\Gamma_0(e)}$$

that remains invariant under a set of coset representatives A for $\Gamma_0(e) \backslash \Gamma_0(d)$. This subset has a map by restriction of the second factor to

$$(\mathrm{Hom}(X, \mathbb{C}_p) \otimes \mathcal{O})^{\Gamma_0(e)}$$

where \mathcal{O} is the space of functions on $B((p^{e-d}\mathbb{Z}_p)^{n(n-1)/2}, c)$ that are algebraic on each ball $B(\underline{a}, c)$. The map is an isomorphism: if $\varphi \in (\mathrm{Hom}(X, \mathbb{C}_p) \otimes \mathcal{O})^{\Gamma_0(e)}$, its inverse ψ may be defined by

$$\psi(x)(z) = \varphi(xa^{-1})(\overline{N}^{-1}(\overline{N}(z)a^{-1})) \text{ for } a \in A \text{ such that } za^{-1} \in B((p^{e-d}\mathbb{Z}_p)^{n(n-1)/2}, c).$$

In $\overline{N}(z)a^{-1}$, a should be interpreted as a coset representative for $\Gamma_0(e-d+1) \backslash \mathrm{Iw}_p$.

Note that this inverse depends on the choice of coset representatives A . Now $B((p^{e-d}\mathbb{Z}_p)^{n(n-1)/2}, c)$ is isomorphic to $B(\mathbb{Z}_p^{n(n-1)/2}, c+d-e)$ via multiplication by p^{d-e} , so $(\mathrm{Hom}(X, \mathbb{C}_p) \otimes \mathcal{O})^{\Gamma_0(e)}$ is the desired right-hand side.

To check that the Hecke operator action is preserved, it suffices to note that the Hecke operator action on the left-hand side can be calculated on its inclusion into (*). \square

Corollary 3.1.3. *For all positive integers c and group-like $\underline{d} \in \mathbb{Z}_{\geq 0}^{n(n-1)/2}$, we have a vector space embedding*

$$S_{t_X, c}(G, \mathcal{U}^p \Gamma_0(\underline{d})) \hookrightarrow \mathcal{S}_{t_X}(G, U_0(p))$$

preserving systems of \mathcal{H} -eigenvalues.

Proof. Let $d = \max d_{ij}$. Then we have an embedding

$$S_{t_X, c}(G, \mathcal{U}^p \Gamma_0(\underline{d})) \hookrightarrow S_{t_X, c}(G, \mathcal{U}^p \Gamma_0(d)).$$

By Proposition 3.1.2, we have an isomorphism

$$S_{t_X, c}(G, \mathcal{U}^p \Gamma_0(d)) \cong S_{t_X, c+d-1}(G, \mathcal{U}^p \Gamma_0(1)) = S_{t_X, c+d-1}(G, \mathcal{U}^p \text{Iw}).$$

The space on the right certainly embeds into $\mathcal{S}_{t_X}(G, \mathcal{U}^p \text{Iw}_p) = \mathcal{S}_{t_X}(G, U_0(p))$ as discussed above. \square

For future reference, it will be important to note the following distinction between the space $S_t(G, 1, U_0(p))$ defined above and the space $S_t(k)(G, U_0(p))$ of classical algebraic automorphic forms defined in Section 2.1, which is that they are identical except for the normalization of the action of the U_p -operator. This is because, as in the beginning of Section 2.3, the action of u^a on $S_t = \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, \text{alg}} t = \text{Ind}_{B(\mathbb{Q}_p)}^{B(\mathbb{Q}_p) \text{Iw}_p, \text{alg}} t^0$ implicitly arises from the extension of t to $t^0 : (\mathbb{Q}_p^\times)^n \rightarrow \mathbb{C}$ where $t^0(u^a) = 1$, whereas the action of u^a on $S_t(k)$ arises from the algebraic character $t : (\mathbb{Q}_p^\times)^n \rightarrow \mathbb{C}$, for which we can compute $t(u^a) = p^{\sum_i a_i t_i}$. Thus we have

$$U_p^a | S_t(G, 1, U_0(p)) = p^{\sum_i a_i t_i} U_p^a | S_t(k)(G, U_0(p)).$$

3.2. A classicality theorem following Bellaïche-Chenevier. This is essentially Proposition 7.3.5 of [2]. We will just summarize the proof with modifications so that it also works for locally algebraic weights.

Theorem 3.2.1. *Let $f \in \mathcal{S}_{t\chi}(G, \mathcal{U})$ where $t\chi = (t_1\chi_1, \dots, t_n\chi_n)$, in which the t_i are integers such that $t_1 \geq \dots \geq t_n$ and the χ_i are finite, such that f is an eigenform for all operators $U_p^{(a_1, \dots, a_n)}$. Let $\lambda_1, \dots, \lambda_{n-1}$ be the λ -values associated to f as defined at the end of Section 2.6. If*

$$v(\lambda_1 \lambda_2 \cdots \lambda_i) < t_i - t_{i+1} + 1$$

for all $i = 1, \dots, n-1$, then f is classical (i.e. lies in the image of $S_{t\chi, c}(G, \mathcal{U})$ for any c such that this is well-defined).

Proof. Let $V = \mathbb{Q}_p v \oplus \mathbb{Q}_p R$ be a finite-dimensional vector space generated by a nonzero vector v and a lattice R . In some basis whose first vector is v , we define

$$P = \begin{pmatrix} a & B \\ 0 & D \end{pmatrix}_{\mathbb{Q}_p} \in GL_{\mathbb{Q}_p}(V)$$

$$\bar{N} = \begin{pmatrix} 1 & 0 \\ C & I_R \end{pmatrix}_{\mathbb{Z}_p}, J = \begin{pmatrix} a & B \\ pC & D \end{pmatrix}_{\mathbb{Z}_p} \in GL_{\mathbb{Z}_p}(V)$$

so that

$$J = (\bar{N} \cap J) \times (P \cap J) = \begin{pmatrix} 1 & 0 \\ pC & I_R \end{pmatrix}_{\mathbb{Z}_p} \times \begin{pmatrix} a & B \\ 0 & D \end{pmatrix}_{\mathbb{Z}_p}.$$

We have an isomorphism $\alpha : \bar{N} \cap J \rightarrow R$ given by

$$\alpha \begin{pmatrix} 1 & 0 \\ pC & I_r \end{pmatrix} = C.$$

We also define

$$\mathfrak{U}^- = \begin{pmatrix} p^k & 0 \\ 0 & p^{\leq k} GL_{\mathbb{Z}_p}(R) \end{pmatrix}_{\mathbb{Z}_p}, \mathfrak{U}^{--} = \begin{pmatrix} p^k & 0 \\ 0 & p^{\leq k-1} GL_{\mathbb{Z}_p}(R) \end{pmatrix}_{\mathbb{Z}_p}.$$

We define $\mathfrak{M} = \langle \mathfrak{U}^-, J \rangle$. Let $\chi : P \rightarrow \mathbb{Q}_p^\times$ be the character of P acting on $\mathbb{Q}_p v$. We have a $\mathbb{C}_p[H]$ -equivariant isomorphism

$$\mathrm{Sym}^m(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^\vee \rightarrow \mathrm{Ind}_P^{H,alg}(\chi^m)$$

$$\varphi \mapsto (h \mapsto \varphi(h(e))).$$

We get a natural \mathfrak{M} -equivariant map

$$\mathrm{Ind}_P^{H,alg}(\chi^m) \rightarrow \mathrm{Ind}_P^{JP,an}(\chi^m)$$

by restriction. Let $\delta : \mathfrak{M} \rightarrow \mathbb{C}_p^\times$ be the character such that $\delta(J) = 1$ and $\delta(u) = p^a$ if

$$u = \begin{pmatrix} p^a & 0 \\ 0 & U \end{pmatrix} \in \mathfrak{U}^-.$$

Let e_1, \dots, e_n be the standard basis of \mathbb{Q}_p^n . Let $V_i = \wedge^i(\mathbb{Q}_p^n)$, $v_i = e_1 \wedge \dots \wedge e_i$, $m_i = t_i - t_{i+1}$ if $i < n$ and $m_n = t_n$, and R_i be the \mathbb{Z}_p -span of the elements $e_{j_1} \wedge \dots \wedge e_{j_i}$ with $j_1 < \dots < j_i$ and $(j_1, \dots, j_i) \neq (1, \dots, i)$. Then for $i = 1, \dots, n$, we get

$$H_i, P_i, \chi_i, \overline{N}_i, J_i, \alpha_i, \mathfrak{U}_i^-, \mathfrak{U}_i^{--}, \mathfrak{M}_i, \delta_i$$

as defined above.

Write $S_i(\mathbb{C}_p)^\vee$ for the space $\mathrm{Ind}_{P_i}^{H_i,alg}(\chi_i^{m_i})$ viewed as a representation of $G(\mathbb{Q}_p)$ via $\wedge^i : G(\mathbb{Q}_p) \rightarrow H_i$. Write $\mathcal{S}_i(m_i)$ for the space $\mathrm{Ind}_{P_i}^{H_i,an}(\chi_i^{m_i}) \otimes \delta_i^{m_i}$ viewed as a representation of \mathfrak{M} via \wedge^i . We have surjections

$$\bigotimes_{i=1}^m S_i(\mathbb{C}_p)^\vee \rightarrow S_t(\mathbb{C}_p)^\vee$$

$$\widehat{\bigotimes}_{i=1}^m \mathcal{S}_i(m_i) \rightarrow \mathcal{S}_t$$

which are equivariant with respect to $G(\mathbb{Q}_p)$ and \mathfrak{M} respectively, both given by the formula

$$(f_1, \dots, f_m) \mapsto (g \mapsto \prod_{i=1}^m f_i(\wedge^i(g))).$$

Let

$$\begin{aligned} Q &= \mathcal{S}_t / (S_t(\mathbb{C}_p)^\vee \otimes \delta_t) \otimes \left(\begin{array}{c} \text{Iw}_p / \text{Iw}_p \cap \Gamma(c) \\ \text{Ind} \\ B(\mathbb{Z}_p) / B(\mathbb{Z}_p) \cap \Gamma(c) \end{array} \chi \right), \\ Q' &= \widehat{\bigotimes}_{i=1}^m \mathcal{S}_i(m_i) / \left(\bigotimes_{i=1}^m S_i(\mathbb{C}_p)^\vee \otimes \delta_i^{m_i} \right) \otimes \left(\begin{array}{c} \text{Iw}_p / \text{Iw}_p \cap \Gamma(c) \\ \text{Ind} \\ B(\mathbb{Z}_p) / B(\mathbb{Z}_p) \cap \Gamma(c) \end{array} \chi \right), \\ Q'_i &= \left(\widehat{\bigotimes}_{j \neq i} \mathcal{S}_j(m_j) \right) \otimes (\mathcal{S}_i(m_i) / S_i(\mathbb{C}_p)^\vee \otimes \delta_i^{m_i}) \otimes \left(\begin{array}{c} \text{Iw}_p / \text{Iw}_p \cap \Gamma(c) \\ \text{Ind} \\ B(\mathbb{Z}_p) / B(\mathbb{Z}_p) \cap \Gamma(c) \end{array} \chi \right). \end{aligned}$$

Then we have a surjection

$$Q'(G, \mathcal{U}) \twoheadrightarrow Q(G, \mathcal{U})$$

and an injection

$$Q'(G, \mathcal{U}) \hookrightarrow \prod_{i=1}^n Q'_i(G, \mathcal{U}).$$

We wish to show that if $w \in Q(G, \mathcal{U})$ satisfies the hypotheses of the theorem, that is, $u_i(w) = \lambda_i w$ with the λ_i satisfying the given inequalities, then $w = 0$. We can instead check this claim for $w' \in Q'(G, \mathcal{U})$ satisfying the same condition, and for this it suffices to check that the image w'_i of w' vanishes in $Q'_i(G, \mathcal{U})$ for each i . Let $U_i = \frac{\prod_{j=1}^i u_j}{p^{m_i+1}}$, so that w'_i has U_i -eigenvalue $\frac{\lambda_1 \lambda_2 \dots \lambda_i}{p^{m_i+1}} w$, which has norm > 1 . Thus it suffices to check that U_i has norm ≤ 1 on $Q'_i(G, \mathcal{U})$, which follows from the claim that any element of the form

$$\frac{g \left(\prod_{j=1}^i u_j \right) g'}{p^{m_i+1}}$$

for $g, g' \in \text{Iw}_p$ has norm ≤ 1 on Q'_i . This follows from Lemma 7.3.6 of [2]. \square

3.3. Automorphic representations associated to automorphic forms of locally algebraic weights. Fix an isomorphism $\iota_p : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Let $f \in \mathcal{S}_{t\chi}(G, U_0(p))$

be a p -adic automorphic form coming from some classical subspace $S_{t,\chi,c}(G, U_0(p))$. Let $W = \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, c\text{-loc. alg.}}(t\chi)$, so that f is a function $G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \rightarrow W$. Following the proof of Proposition 3.8.1 of [17], let $W = W^{sm,c}(\chi) \otimes S_t(\mathbb{C})$, where, as in Section 3.1,

$$W^{sm,c}(\chi) = \text{Ind}_{B(\mathbb{Z}_p)/B(\mathbb{Z}_p) \cap \Gamma(c)}^{\text{Iw}_p / \text{Iw}_p \cap \Gamma(c)} \chi,$$

$$S_t(\mathbb{C}) = \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, \text{alg}} t,$$

and let ρ_{sm}, ρ_{alg} denote the actions of Iw_p on $W^{sm,c}(\chi) \otimes S_t(\mathbb{C})$ given by acting on only the first factor and only the second factor respectively. Then we can define a function $f_\infty : G(\mathbb{A}) \rightarrow W$ by $f_\infty(g) = \rho_{alg}(g_\infty^{-1} \iota_p(g_p)) f(g_f)$ which satisfies the relation

$$f_\infty(gu) = \rho_{sm}(u_p)^{-1} \rho_{alg}(u_\infty)^{-1} f_\infty(g)$$

for all $u \in G(\mathbb{R})U_0(p)$. Equivalently, f_∞ can be viewed as the function

$$f_\infty^\vee : (W^{sm,c}(\chi)^\vee \otimes S_t(\mathbb{C})^\vee) \times G(\mathbb{Q}) \backslash G(\mathbb{A}) \rightarrow \mathbb{C}$$

$$(\varphi, x) \mapsto \varphi(f_\infty(x))$$

which satisfies

$$f_\infty^\vee(\varphi, xu) = \varphi(f_\infty(xu)) = \varphi(\rho_{sm}(u_p)^{-1} f_\infty(x)) = f_\infty^\vee(u_p \varphi, x)$$

for all $u \in U_0(p)$. Thus for each $\varphi \in W^{sm,c}(\chi)^\vee \otimes S_t(\mathbb{C})^\vee$, the function $f_\infty^\vee(\varphi, \cdot)$ is an element of $C(G(\mathbb{Q}) \backslash G(\mathbb{A}), \mathbb{C})$ which generates under right translation by Iw_p a representation containing an irreducible component of $W^{sm,c}(\chi)^\vee$. The right translates of $f_\infty^\vee(\varphi, \cdot)$ under $G(\mathbb{A})$ generate an automorphic representation π_f of $G(\mathbb{A})$ which decomposes as a tensor product $\bigotimes'_p \pi_{f,p}$. We are interested in describing the structure of $\pi_{f,p}$.

Note that this process is reversible, in that given $\psi \in C(G(\mathbb{Q}) \backslash G(\mathbb{A}), \mathbb{C})$ which generates a representation containing an irreducible component of $W^{sm,c}(\chi)^\vee$ under right translation by Iw_p , we get a unique $f_\psi \in S_{t\chi,c}(G, U_0(p))$.

3.4. Structure of $W^{sm,c}(\chi)$. We are interested in the representation

$$W^{sm,c}(\chi) = \text{Ind}_{B(\mathbb{Z}_p)/B(\mathbb{Z}_p) \cap \Gamma(c)}^{Iw_p/\Gamma(c)} \chi$$

of Iw_p . Note that there is an obvious embedding $W^{sm,c}(\chi) \hookrightarrow W^{sm,c+1}(\chi)$ which takes $f \in W^{sm,c}(\chi)$ to the composition of f with the reduction map $Iw_p/\Gamma(c+1) \rightarrow Iw_p/\Gamma(c)$.

Let J be the compact open subgroup of $GL_n(\mathbb{Q}_p)$ corresponding to χ defined in Section 3 of [20]; we have $J = \Gamma(\underline{c})$ where $c_{ii} = 0$, $c_{ij} = \lfloor \text{cond}(\chi_i \chi_j^{-1})/2 \rfloor$ if $i < j$, and $c_{ij} = \lfloor [\text{cond}(\chi_i \chi_j^{-1}) + 1]/2 \rfloor$ if $i > j$. Then χ extends to a character of J which we will also call χ ; it is defined by the equation $\chi(j^- j j^+) = \chi(j)$ when $j^- \in J \cap \overline{N}(\mathbb{Z}_p)$, $j \in T(\mathbb{Z}_p)$, and $j^+ \in J \cap N(\mathbb{Z}_p)$.

Now note that $W^{sm,c}(\chi)$ contains the vector

$$f(\bar{x}) = \begin{cases} \chi(j)\chi(b) & \text{if } \bar{x} = \bar{j}b \text{ with } j \in J \text{ and } b \in B(\mathbb{Z}_p), \\ 0 & \text{otherwise.} \end{cases}$$

Note furthermore that for any $j \in J$ and $\bar{x} \in Iw_p/\Gamma(c)$, we have

$$(jf)(\bar{x}) = f(j^{-1}\bar{x}) = \chi(j^{-1})f(\bar{x}) = \chi^{-1}(j)f(\bar{x})$$

so that f is (J, χ^{-1}) -isotypic.

Proposition 3.4.1. *Assume $\chi = (\chi_1, \dots, \chi_n)$ satisfies*

- (1) *for all $i \neq j$, $\text{cond}(\chi_i \chi_j^{-1}) = \max(\text{cond}(\chi_i), \text{cond}(\chi_j))$; and*
- (2) *for all $i \neq j$ with $i, j \neq n$, $\text{cond}(\chi_i) < 2 \text{cond}(\chi_j)$.*

Then $U^{sm}(\chi) := \text{Ind}_J^{Iw_p} \chi$ is irreducible.

Proof. Recall that $J = \Gamma(\underline{c})$ where $c_{ij} = \lfloor [\text{cond}(\chi_i \chi_j^{-1}) + 1]/2 \rfloor$ if $i > j$, or $\lfloor [\text{cond}(\chi_i \chi_j^{-1})]/2 \rfloor$ if $i < j$. By Mackey's criterion, it is necessary and sufficient to show that for any $s \in \text{Iw}_p \setminus J$, the characters χ and $\chi^s : j \mapsto \chi(sjs^{-1})$ are not identically equal on $J \cap s^{-1}Js$. If $s \in \text{Iw}_p \setminus J$, let $t = s^{-1}$. Then there is a pair $i \neq j$ such that t_{ji} is not divisible by $p^{c_{ji}}$. Among all such $i \neq j$, choose a pair such that either

—among the integers $c_{kj} + c_{jk} - v(t_{jk})$, $1 \leq k \leq n$, $k \neq j$, $c_{ij} + c_{ji} - v(t_{ji})$ is the unique maximal one;

—or, if this is not possible, among the integers $c_{kj} + c_{jk} - v(t_{jk})$, $1 \leq k \leq n$, $k \neq j$, $c_{ij} + c_{ji} - v(t_{ji})$ is maximal and i is minimal such that this is the case.

Let $x \in J$ be the identity except for the ij th entry; let $x_{ij} = b$. Note that we must have $p^{c_{ij}}|b$. We will show that we can choose b such that $sxt \in J$ and $1 = \chi(x) \neq \chi(sxt)$, and hence $\chi(x) \neq \chi^s(x)$, as desired.

The matrix xt is the same as t except for the i th row, which is

$$(t_{i1} + bt_{j1}, \dots, t_{in} + bt_{jn}).$$

The kk th entry of sxt is

$$s_{k1}t_{1k} + \dots + s_{ki}(t_{ik} + bt_{jk}) + \dots + s_{kn}t_{nk} = s_{k1}t_{1k} + \dots + s_{kn}t_{nk} + bs_{ki}t_{jk} = 1 + bs_{ki}t_{jk}.$$

Because of condition 1, one can check that for all $j \in J$, we have $\chi(j) = \chi_1(j_{11}) \cdots \cdots \chi_n(j_{nn})$. So we wish to choose b such that

$$\chi_1(1 + bs_{1i}t_{j1}) \cdots \chi_i(1 + bs_{ii}t_{ji}) \cdots \chi_j(1 + bs_{ji}t_{jj}) \cdots \chi_n(1 + bs_{ni}t_{jn}) \neq 1.$$

Note that for all $k \neq i, j$, we have

$$v(s_{ki}) + c_{ij} + c_{ji} - v(t_{ji}) > c_{jk} + c_{kj} - v(t_{jk}).$$

This is just because we chose i, j such that $c_{ij} + c_{ji} - v(t_{ji}) \geq c_{jk} + c_{kj} - v(t_{jk})$, and such that if equality holds then $k > i$, in which case $v(s_{ki}) \geq 1$ since $s \in \text{Iw}_p$. So if we choose b such that $v(b) = c_{ij} + c_{ji} - v(t_{ji}) - 1 \geq c_{ij}$, then we have

$$v(bs_{ki}t_{jk}) \geq c_{jk} + c_{kj} = \text{cond}(\chi_k\chi_j^{-1})$$

for all $k \neq i, j$, hence $\chi_k(1 + bs_{ki}t_{jk}) = 1$. Then we have

$$\begin{aligned} & \chi_1(1 + bs_{1i}t_{j1}) \cdots \chi_i(1 + bs_{ii}t_{ji}) \cdots \chi_j(1 + bs_{ji}t_{jj}) \cdots \chi_n(1 + bs_{ni}t_{jn}) \\ &= \chi_i(1 + bs_{ii}t_{ji})\chi_j(1 + bs_{ji}t_{jj}) = \chi_i(1 + bs_{ii}t_{ji})\chi_j \left(1 + b \left(\sum_{k \neq i} s_{ki}t_{jk} \right) \right) \end{aligned}$$

since $v(bs_{ki}t_{jk}) \geq \text{cond}(\chi_j)$, but this is

$$\chi_i(1 + bs_{ii}t_{ji})\chi_j(1 - bs_{ii}t_{ji})$$

since $\sum_k s_{ki}t_{jk} = \sum_k t_{jk}s_{ki} = (ts)_{ji} = 0$, and this can be rewritten as

$$\frac{\chi_i}{\chi_j}(1 + bs_{ii}t_{ji})\chi_j(1 - b^2s_{ii}^2t_{ji}^2) = \frac{\chi_i}{\chi_j}(1 + bs_{ii}t_{ji})$$

because if $i > j$ then $v(b^2) \geq 2c_{ij} \geq c_{ij} + c_{ji}$ and if $i < j$ then $v(b^2) \geq 2c_{ij} \geq c_{ij} + c_{ji} - 1$ and $v(t_{ji}) \geq 1$. But since $v(b) < c_{ij} + c_{ji} - v(t_{ji})$, we have $v(bs_{ii}t_{ji}) < \text{cond}(\chi_i\chi_j^{-1})$, so we can choose b to make $\frac{\chi_i}{\chi_j}(1 + bs_{ii}t_{ji}) \neq 1$.

Finally, we verify that for this choice of b , we actually have $sxt \in J$. The kl th entry of sbt is

$$s_{k1}t_{1l} + \cdots + s_{ki}(t_{il} + bt_{jl}) + \cdots + s_{kn}t_{nk} = \delta_{kl} + bs_{ki}t_{jl}.$$

We have

$$\begin{aligned} v(bs_{ki}t_{jl}) &= c_{ij} + c_{ji} - v(t_{ji}) - 1 + v(s_{ki}) + v(t_{jl}) \\ &= c_{ij} + c_{ji} - v(t_{ji}) - (c_{lj} + c_{jl} - v(t_{jl})) + c_{lj} + c_{jl} - 1 + v(s_{ki}) \end{aligned}$$

$$\geq c_{lj} + c_{jl} - 1 + v(s_{ki}) \geq c_{kl}$$

by condition 2. □

Remark 2. We do not believe that either condition 1 or condition 2 of Proposition 3.4.1 should be strictly necessary. Notably, most of the proof of Proposition 3.4.1 can be easily rephrased to avoid references to condition 1. Our only sticking point is the calculation of $\chi(j)$ in terms of j_{11}, \dots, j_{nn} .

We call χ “simple” if it satisfies the conditions of Proposition 3.4.1. By Frobenius reciprocity, we conclude that if χ is simple, $W^{sm,c}(\chi)$ contains $U^{sm}(\chi)$. Also note that if the conductors of the nontrivial components of $\chi = (\chi_1, \dots, \chi_{n-1}, 1)$ are, in order from least to greatest, $c_{(1)} \leq c_{(2)} \leq \dots \leq c_{(n-1)}$, then $\text{rank}(U^{sm}(\chi))$ is the index

$$p^{c_{(1)}+2c_{(2)}+\dots+(n-1)c_{(n-1)}-\frac{n(n-1)}{2}} =: p^{j(\chi)}$$

of J in Iw_p . So if $\text{cond}(\chi_i) = c$ for all $i \neq n$, then $W^{sm,c}(\chi)$ and $U^{sm}(\chi)$ have the same dimension and must actually be isomorphic.

3.5. Hecke eigenvalues of ramified principal series. The representations we are interested in will turn out to be ramified principal series of $GL_n(\mathbb{Q}_p)$, so we now cover the properties of these that we will need. To harmonize with the literature, for this section only, we will use different conventions from the rest of the paper. If $\chi = (\chi_1, \dots, \chi_n) : (\mathbb{Q}_p^\times)^n \rightarrow \mathbb{C}$ is a smooth character of $T(\mathbb{Q}_p) = (\mathbb{Q}_p^\times)^n$, we will write

$$i_{B(\mathbb{Q}_p)}^{GL_n(\mathbb{Q}_p)} \chi = \{f : GL_n(\mathbb{Q}_p) \rightarrow \mathbb{C} \mid f(bg) = \chi(b)f(g) \forall g \in GL_n(\mathbb{Q}_p) \text{ and } b \in B(\mathbb{Q}_p)\}$$

for the representation of $GL_n(\mathbb{Q}_p)$ with the given underlying vector space and the right translation action of $GL_n(\mathbb{Q}_p)$. We let $\delta^{1/2} : (\mathbb{Q}_p^\times)^n \rightarrow \mathbb{C}$ be the modulus character

$$\delta^{1/2} := (|\cdot|^{(n-1)/2}, |\cdot|^{(n-3)/2}, \dots, |\cdot|^{(1-n)/2}).$$

Then we define

$$\pi(\chi) := \pi(\chi_1, \dots, \chi_n) := i_{B(\mathbb{Q}_p)}^{GL_n(\mathbb{Q}_p)}(\chi\delta^{1/2}).$$

The representation $\pi(\chi)$ is called the normalized parabolic induction of χ . Assume that for all $i \neq j$, we have $\chi_i(p) \neq \chi_j(p)p$. Let $J = \Gamma(\underline{c})$ be the subgroup defined at the beginning of Section 3.4. Let $\mathcal{H}(GL_n(\mathbb{Q}_p) // J, \chi)$ be the subspace of $\mathcal{H}(GL_n(\mathbb{Q}_p))$ generated by the functions $\varphi : GL_n(\mathbb{Q}_p) \rightarrow \mathbb{C}$ satisfying $\varphi(j_1 x j_2) = \chi(j_1)^{-1} \varphi(x) \chi(j_2)^{-1}$ for all $j_1, j_2 \in J$ and $x \in GL_n(\mathbb{Q}_p)$.

Lemma 3.5.1. *The (J, χ) -isotypic piece of $\pi(\chi)$ is 1-dimensional.*

Proof. By Theorem 6.3 of [20], $\mathcal{H}(GL_n(\mathbb{Q}_p) // J, \chi)$ is abelian. (To be precise, the theorem gives an isomorphism between $\mathcal{H}(GL_n(\mathbb{Q}_p) // J, \chi)$ and $\mathcal{H}(W_\chi^0, S_\chi^0) \otimes \mathbb{C}[\Omega_\chi]$, where by our assumption that $\chi_i(p) \neq \chi_j(p)p$ for $i \neq j$, we have $W_\chi^0 = S_\chi^0 = 1$, $\mathcal{H}(W_\chi^0, S_\chi^0) = \mathbb{C}$, and $\Omega_\chi = \mathbb{Z}^n$.) Thus the (J, χ) -isotypic piece of $\pi(\chi)$ decomposes as a representation of $\mathcal{H}(GL_n(\mathbb{Q}_p) // J, \chi)$ into 1-dimensional pieces. But by Theorem 9.2 of [20], because $\pi(\chi)$ is irreducible, the (J, χ) -isotypic piece of $\pi(\chi)$ is irreducible as a representation of $\mathcal{H}(GL_n(\mathbb{Q}_p) // J, \chi)$. Thus it is itself 1-dimensional. \square

Lemma 3.5.2. *If $a = (a_1, \dots, a_n)$ is such that $a_1 \geq a_2 \geq \dots \geq a_n$, the action of the element $[Ju^aJ]$ of $\mathcal{H}(GL_n(\mathbb{Q}_p) // J, \chi)$ corresponding to $u^a = \text{diag}(p^{a_1}, \dots, p^{a_n})$ on the (J, χ) -isotypic piece of $\pi(\chi)$ is multiplication by*

$$\chi(u^a) = \chi_1(p^{a_1}) \cdots \chi_n(p^{a_n}).$$

Proof. The (J, χ) -isotypic piece is generated by

$$f(g) = \begin{cases} (\chi\delta)(b)\chi(j) & \text{if } g = bj \text{ with } b \in B(\mathbb{Q}_p) \text{ and } j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

This is just because this function f satisfies the (J, χ) -isotypic condition by construction, and is well-defined because $(\chi\delta)(b) = \chi(b)$ for any $b \in B \cap J$. We claim that

(*)

$$f(ju^a) = \chi(j)\chi(u^a)\delta(u^a) = \chi(j)\chi(u^a)\delta(u^a)f(1) \text{ for any } j \in J.$$

The lemma follows from this, because if $Ju^aJ = \coprod_{i=1}^r j_i u^a J$, then

$$([Ju^aJ]f)(1) = \sum_{i=1}^r \chi(j_i)^{-1} f(j_i u^a) = \sum_{i=1}^r \chi(u^a)\delta(u^a)f(1) = \chi(u^a)f(1)$$

because $r = \delta(u^a)^{-1}$ (reason: the same calculation as in Proposition 2.6.2 shows that the index of J in $[(u^a)^{-1}Ju^a]J$ is

$$p^{\sum_{i < j} (a_i - a_j)} = p^{(n-1)a_1 + (n-3)a_2 + \dots + (1-n)a_n}.$$

For (*), first write $j = j^- j^0 j^+$ where $j^- \in J \cap \overline{N}(\mathbb{Z}_p)$, $j^0 \in T(\mathbb{Z}_p)$, and $j^+ \in J \cap N(\mathbb{Z}_p)$. Then we have $\chi(j) = \chi(j^0)$. Let $j_1^+ = j^0 j^+ (j^0)^{-1}$; then $j_1^+ \in J \cap N(\mathbb{Z}_p)$ as well, and $j = j^- j_1^+ j^0$. Use Lemmas 3.1 and 3.2 of [20] to write $j^- j_1^+ = j_2^+ j_2^- c$, where $j_2^+ \in J \cap N(\mathbb{Z}_p)$, $j_2^- \in J \cap \overline{N}(\mathbb{Z}_p)$, and $c \in T(\mathbb{Z}_p)$ is a correction torus element in the kernel of χ . Then we have

$$ju^a = j_2^+ j_2^- c j^0 u^a = u^a [(u^a)^{-1} j_2^+ u^a] [(u^a)^{-1} j_2^- u^a] (j^0 c).$$

We have $(u^a)^{-1} j_2^+ u^a \in N(\mathbb{Q}_p)$, and by the same calculation as in Proposition 2.6.2, we have $(u^a)^{-1} j_2^- u^a \in J \cap \overline{N}(\mathbb{Z}_p)$. Therefore

$$\begin{aligned} f(ju^a) &= f(u^a [(u^a)^{-1} j_2^+ u^a] [(u^a)^{-1} j_2^- u^a] (j^0 c)) \\ &= (\chi\delta)(u^a [(u^a)^{-1} j_2^+ u^a]) \chi([(u^a)^{-1} j_2^- u^a] (j^0 c)) = (\chi\delta)(u^a) \chi(j^0) = (\chi\delta)(u^a) \chi(j) \end{aligned}$$

as desired. □

3.6. Structure of automorphic representations of locally algebraic weights.

Let $f \in S_{t_\chi, c}(G, U_0(p))$ be a classical eigenform, and let $\pi_{f,p}$ be an irreducible component of the local component at p of the automorphic representation π_f associated

to f in Section 3.3. We first verify a standard fundamental fact about the structure of $\pi_{f,p}$ for those f associated to points on the eigenvariety \mathcal{D} .

Proposition 3.6.1. *f may be associated to a classical point x on \mathcal{D} (equivalently, f is finite-slope) if and only if $\pi_{f,p}$ has nonzero Jacquet module with respect to B , or equivalently is a subquotient of a principal series.*

To show this, we use the following proposition of Casselman in [7] on canonical liftings. Recall the submonoid Σ^{--} from Section 2.6.

Proposition 3.6.2 (Casselman, Proposition 4.1.4). *Let (π, V) be an admissible representation of $GL_n(\mathbb{Q}_p)$, $P = MN$ a parabolic subgroup with Levi factorization, and $K_0 = \overline{N}_0 M_0 N_0$ a compact open subgroup with Iwahori factorization. If $u^a \in \Sigma^{--}$, then the projection from V^{K_0} to $V_N^{M_0}$ given by $[K_0 u^a K_0]$ is a surjection. If $u^a N_1 (u^a)^{-1} \subseteq N_0$, where N_1 is a compact subgroup of N such that $V^{K_0} \cap V(N) \subseteq V(N_1)$, then the projection is an isomorphism.*

Also see Sections 3 and 4 of [7] for more basic information about Jacquet modules.

Proof of Proposition 3.6.1. We apply Proposition 3.6.2 with $P = B$ and $M = T$.

Suppose first that $\pi_{f,p} = (\pi, V)$ has nonzero Jacquet module. Let $v \in V_N$ be a nonzero vector and let M_0 be a compact open subgroup of M fixing v . Let K_0 be a compact open subgroup of G such that $K_0 \cap M = M_0$. By the proposition of Casselman, $[K_0 u^a K_0]V$ surjects onto $V_N^{M_0} \neq 0$, so is itself nonzero. Thus $[K_0 u^a K_0]$ has some nonzero eigenvalue, corresponding to an eigenvector in $\pi_{f,p}$ which must be the image of an eigenform in $S_{t_x, c}(G, U_0(p))$ by the procedure of Section 3.3.

Now suppose in the other direction that $\pi = \pi_{f_x}$, so contains a vector $\text{im}(f_x)$ with nonzero Hecke eigenvalue for $[K_0 u^a K_0]$ for some compact open subgroup K_0 and all $a \in \Sigma^-$. Choose a compact subgroup N_1 of N such that $V^{K_0} \cap V(N) \subseteq V(N_1)$. We claim that for sufficiently large powers $(u^a)^k$ of u^a , we must have $(u^a)^k N_1 (u^a)^{-k} \subseteq N_0$;

this is just the effect of conjugation by $(u^a)^k$ on the ij th entry of N_1 is scaling by $p^{k(a_i - a_j)}$, and $k(a_i - a_j)$ becomes arbitrarily large as k does. Then $[K_0(u^a)^k K_0]V \cong V_N^{M_0}$, and we must have $V_N^{M_0} \neq 0$. \square

Now assume χ_n is trivial and let $c_0 = \max_{1 \leq i < n} \text{cond}(\chi_i)$. Let $(W^{sm, c_0}(\chi))^\perp$ be the complement of $W^{sm, c_0}(\chi)$ in $W^{sm, c}(\chi)$. We now observe that as c goes to infinity, almost all eigenforms in $S_{t\chi, c}(G, U_0(p))$ are infinite-slope.

Proposition 3.6.3. *Suppose that f is an eigenform in $((W^{sm, c_0}(\chi))^\perp \otimes S_t)(G, U_0(p)) \subset S_{t\chi, c}(G, U_0(p))$. Then $U_p f = 0$, f is not associated to a point on the eigenvariety, and $\pi_{f, p}$ is not a subquotient of a principal series.*

Proof. By Proposition 2.6.6, in order for $U_p f$ to be nonzero, f must lie in $\mathcal{S}_{t\chi, c_0}(G, U_0(p))$. But the intersection of $\mathcal{S}_{t\chi, c_0}(G, U_0(p))$ with $((W^{sm, c_0}(\chi))^\perp \otimes S_t)(G, U_0(p))$ is trivial. \square

We now find a subspace of $S_{t\chi, c}(G, U_0(p))$ in which all forms are finite-slope.

Proposition 3.6.4. *Suppose that f is an eigenform in $(U^{sm}(\chi) \otimes S_t)(G, U_0(p))$. Then $\pi_{f, p}$ is a subquotient of a principal series, in particular one of the form $\pi(\psi_1, \dots, \psi_n)$ where $\psi_i : \mathbb{Q}_p^\times \rightarrow \mathbb{C}$ are characters of \mathbb{Q}_p^\times whose restrictions to \mathbb{Z}_p^\times are the same as χ_1, \dots, χ_n in some order.*

Proof. We know that $\pi_{f, p}$ is an irreducible subrepresentation of $C(GL_n(\mathbb{Q}_p), \mathbb{C})$ whose restriction to $\Gamma_0(p)$ admits a nontrivial homomorphism from $(U^{sm}(\chi))^\vee$. By Frobenius reciprocity, the restriction of $\pi_{f, p}$ to J admits a nontrivial homomorphism from the representation of J given by χ ; that is, it contains a (J, χ) -isotypic vector. By Theorem 7.7 of [20], $\pi_{f, p}$ is a subquotient of $\pi(\psi_1, \dots, \psi_n)$. \square

Remark 3. If χ is simple, one can also prove Proposition 3.6.4 by noting that if $\pi_{f, p}$ admits a nontrivial homomorphism from the irreducible $(U^{sm}(\chi))^\vee$, it must in fact

contain all of $(U^{sm}(\chi))^\vee$, in particular the (J, χ) -isotypic vector. We are grateful to Jessica Fintzen for pointing out the more general proof above.

By Propositions 3.6.4 and 3.6.1, U_p is injective on the space $(U^{sm}(\chi) \otimes S_t)(G, U_0(p))$. Now assume that χ is simple; then by the end of Section 3.4, $(U^{sm}(\chi) \otimes S_t)(G, U_0(p))$ is a subspace of $S_{t\chi, c}(G, U_0(p))$. This means that the linear operator U_p on $S_{t\chi, c}(G, U_0(p))$ has rank at least

$$\text{rank}((U^{sm}(\chi) \otimes S_t)(G, U_0(p))) = hd_t p^{j(\chi)}$$

where, as before, $d_t = \dim \text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, \text{alg.}} t$, $h = \#(G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / U_0(p))$, and $j(\chi)$ is defined as at the end of Section 3.4.

Remark 4. Even if χ is not simple, if $f \in (U^{sm}(\chi) \otimes S_t)(G, U_0(p))$, one can associate to it the following $f' \in S_{t\chi, c}(G, U_0(p))$: $\pi_{f, p}$ has an irreducible component which is a principal series $\pi(\psi)$ (assuming χ is such that $\pi(\psi)$ is irreducible). This principal series contains the $(\Gamma_0(c), \chi)$ -vector f given by

$$f(x) = \begin{cases} (\psi\delta)(b)\chi(\gamma) & \text{if } x = \gamma b \text{ with } \gamma \in \Gamma_0(c) \text{ and } b \in B(\mathbb{Q}_p), \\ 0 & \text{otherwise.} \end{cases}$$

A preimage f' of this vector in $C(G(\mathbb{Q}) \backslash G(\mathbb{A}), \mathbb{C})$ lies in $S_{t\chi, c}(G, U_0(p))$. However, it is not obvious to us that $f \mapsto f'$ is one-to-one, and hence not obvious that it gives rise to an appropriate lower bound on the rank of U_p acting on $S_{t\chi, c}(G, U_0(p))$. We suspect this issue is easy to resolve, but we have not resolved it.

In fact, we can extend Proposition 3.6.3 to show that U_p has rank *exactly* $\text{rank}((U^{sm}(\chi) \otimes S_t)(G, U_0(p)))$ when χ is simple.

Proposition 3.6.5. *Suppose that χ is simple and f is an eigenform in $(U^{sm}(\chi)^\perp \otimes S_t)(G, U_0(p)) \subset S_{t\chi, c}(G, U_0(p))$. Then $U_p f = 0$, f is not associated to a point on the eigenvariety, and $\pi_{f, p}$ is not a subquotient of a principal series.*

Proof. Let $c_i = \text{cond}(\chi_i)$, and first assume that $c_1 \geq \dots \geq c_{n-1}$. Then the tuple $\underline{c}^0 \in \mathbb{Z}_{>0}^{n(n-1)/2}$ associated to χ defined immediately before Corollary 2.6.4 satisfies $c_{ij}^0 = c_i$ for all $i > j$. We claim that the intersection of $\mathcal{S}_{t_X, \underline{c}^0}(G, U_0(p))$ with $S_{t_X, c}(G, U_0(p))$ is precisely $(U^{sm}(\chi) \otimes S_t)(G, U_0(p))$.

To show that $(U^{sm}(\chi) \otimes S_t)(G, U_0(p))$ is contained in $\mathcal{S}_{t_X, \underline{c}^0}(G, U_0(p))$, it suffices to note that $U^{sm}(\chi) \otimes S_t$ is contained in $\mathcal{S}_{t_X, \underline{c}^0}$. This is because $f \otimes \varphi \in U^{sm}(\chi) \otimes S_t$ is clearly contained in $\mathcal{S}_{t_X, \underline{c}^0}$ for the vector $f \in U^{sm}(\chi)$ defined at the beginning of Section 3.4 and any $\varphi \in S_t$, and $U^{sm}(\chi) \otimes S_t$ is irreducible.

To show that $(U^{sm}(\chi) \otimes S_t)(G, U_0(p))$ exhausts $\mathcal{S}_{t_X, \underline{c}^0}(G, U_0(p)) \cap S_{t_X, c}(G, U_0(p))$, we simply note that the latter space also has dimension $hd_t p^{j(\chi)}$, since as a vector space it is h copies of the locally algebraic vector subspace of $\mathcal{S}_{t_X, \underline{c}^0}$. By Proposition 2.6.6, in order for $U_p f$ to be nonzero, f must lie in $\mathcal{S}_{t_X, \underline{c}^0}(G, U_0(p))$; this completes the proof.

If the c_i are not in decreasing order, by the beginning of Section 4.2, the finite-slope subspace of $S_{t_X, c}(G, U_0(p))$ has the same dimension as that of $S_{t_X^w, c}(G, U_0(p))$ where χ^w is χ with the components rearranged so that the c_i are in decreasing order. This completes the argument for all χ simple. \square

The combination of Propositions 3.6.4 and 3.6.5 gives us the following precise version of Theorem 1.2.1.

Theorem 3.6.6. *If χ is simple, then the finite-slope classical subspace of $\mathcal{S}_{t_X, c}(G, U_0(p))$ is precisely $(U^{sm}(\chi)^\perp \otimes S_t)(G, U_0(p))$.*

In the following, for convenience, we will sometimes refer to the algebraic weight $(t_1, \dots, t_{n-1}, 0)$, $t_1 \geq \dots \geq t_{n-1}$, by its successive differences $m_1 = t_1 - t_2$, $m_2 = t_2 - t_3$, ..., $m_{n-1} = t_{n-1}$.

Proposition 3.6.7. *Suppose that χ is simple and $f = f_x$ is an eigenform in $(U^{sm}(\chi) \otimes S_t)(G, U_0(p)) \subset S_{t_X, c}(G, U_0(p))$. Suppose that we have $\pi_{f, p} = \pi(\psi_1, \dots, \psi_n)$ (note that*

this is an equality because when χ is simple, $\pi(\psi_1, \dots, \psi_n)$ is irreducible). The λ -values associated to x as in Section 2.6 satisfy

$$\lambda_i = p^{(n-1)/2-i+1-m_n-m_{n-1}-\dots-m_{n-i+1}}\psi_i(p).$$

Proof. We are given that for all $u^a \in \Sigma^-$, we have $U_p^a f = \lambda_1^{a_1} \dots \lambda_n^{a_n} f$. Since any eigenvector of $U_p^a = [U_0(p)u^a U_0(p)]$ is also an eigenvector of $[Ju^a J]$, we can calculate its eigenvalue using $[Ju^a J]$ instead. Let

$$Ju^a J = \prod_{i=1}^r \zeta_i J.$$

Then for any $\varphi \in U^{sm}(\chi) \otimes S_t$, we have

$$\begin{aligned} (U_p^a f)_\infty^\vee(\varphi, x) &= \varphi(\rho_{alg}(x_\infty^{-1} \iota_p(x_p)))(U_p^a f)(x_f) \\ &= \delta^{1/2}(u^a) p^{-\sum a_i t_i} \varphi \left(\rho_{alg}(x_\infty^{-1} \iota_p(x_p(\zeta_i)_p)) \sum_{i=1}^r \rho_{sm}((\zeta_i)_p) f(x_{\zeta_i}) \right). \end{aligned}$$

Choose $\varphi = \varphi_{sm} \otimes \varphi_{alg}$ so that φ_{sm} is a (J, χ) -isotypic vector in $U^{sm}(\chi)$. Then by definition

$$\varphi(\rho_{sm}((\zeta_i)_p) f(x_{\zeta_i})) = \psi((\zeta_i)_p) \varphi(f(x_{\zeta_i}))$$

so

$$\begin{aligned} (U_p^a f)_\infty^\vee(\varphi, x) &= \delta^{1/2}(u^a) p^{-\sum a_i t_i} \sum_{i=1}^r \psi((\zeta_i)_p) \varphi(\rho_{alg}(x_\infty^{-1} \iota_p(x_p(\zeta_i)_p)) f(x_{\zeta_i})) \\ &= \delta^{1/2}(u^a) p^{-\sum a_i t_i} \sum_{i=1}^r \psi((\zeta_i)_p) f_\infty^\vee(\varphi, x_{\zeta_i}). \end{aligned}$$

That is, we have

$$\sum_{i=1}^r \psi((\zeta_i)_p) f_\infty^\vee(\varphi, x_{\zeta_i}) = \delta^{-1/2}(u^a) p^{\sum a_i t_i} \lambda_1^{a_1} \dots \lambda_n^{a_n} f_\infty^\vee(\varphi, x).$$

So the image of $f_\infty^\vee(\varphi, \cdot)$ in $\pi_{f,p}$ is a J -new vector ($\mathcal{H}(J, \psi)$ -module). By Lemma 3.5.2, we have

$$\delta^{-1/2}(u^a) p^{\sum a_i t_i} \lambda_1^{a_1} \cdots \lambda_n^{a_n} = \psi_1(p^{a_1}) \cdots \psi_n(p^{a_n}).$$

The proposition follows. □

One application of this structure theory is the following comparison theorem between Chenevier's and Emerton's eigenvarieties. This is similar to Proposition 3.10.3 of [17].

Proposition 3.6.8. *There is a natural isomorphism from \mathcal{D} as constructed in Section 2 to Emerton's eigenvariety, the space $E(0, \mathcal{U}^p)$ in Definition 0.6 of [12].*

Proof. By Proposition 3.6.1, over a locally algebraic weight, the automorphic representations that appear as classical points of \mathcal{D} are precisely those whose p -parts have nonzero Jacquet module. But Emerton's eigenvariety also has this property built into its construction. In particular, by Proposition 2.3.3(iii) of [12], $E(0, \mathcal{U}^p)$ has a map to the space \hat{T} of locally analytic characters of $T(\mathbb{Q}_p)$ such that points in the fiber over a character χ of $T(\mathbb{Q}_p)$ correspond to χ -eigenspaces of Emerton's locally analytic Jacquet functor $J_B(H^0(\mathcal{U}^p)_{\mathbb{Q}_p\text{-loc.an.}})$. For χ locally algebraic, by Section 0.13 of [11], this is just the usual Jacquet module of $H^0(\mathcal{U}^p)_{\mathbb{Q}_p\text{-loc.an.}}$. So under the natural map $E(0, \mathcal{U}^p) \rightarrow \mathcal{W}$ given by composing $E(0, \mathcal{U}^p) \rightarrow \hat{T}$ with the projection $\hat{T} \rightarrow \mathcal{W}$, the classical points in the fiber over a locally algebraic $w \in \mathcal{W}$ indeed correspond to automorphic representations of weight w whose p -parts have nonzero Jacquet module. But Proposition 7.2.8 of [2] says that a space satisfying this property for automorphic forms on G is unique up to unique isomorphism. □

4. BOUNDS ON THE NEWTON POLYGON

In this section, we prove Theorem 1.1.4. We prove Part 1 in Section 4.1 and Part 2 in Section 4.2. In Section 4.3, we prove a modified version of Part 2 which generates infinitely many upper bound points for the same Newton polygon.

Fix a character of Δ^n , and thus a particular polydisc in \mathscr{W}^n . Over the subset of this polydisc where $T_n = 0$, we have

$$\det(I - XU_p) = \sum_{N \geq 0} c_N(T_1, \dots, T_{n-1}) X^N \in \mathbb{Z}_p[[T_1, \dots, T_{n-1}]][[X]]$$

with $c_0(T_1, \dots, T_{n-1}) = 1$.

4.1. A lower bound on the Newton polygon. The following is Part 1 of Theorem 1.1.4.

Theorem 4.1.1. *There are constants A_1, C (depending on n, p , and h) such that for all T_1, \dots, T_{n-1} such that all $|T_j| > \frac{1}{p}$, the Newton polygon of $\sum_{N \geq 0} c_N(T_1, \dots, T_{n-1}) X^N$ lies above the points*

$$\left(x, \left(A_1 x^{1 + \frac{2}{n(n-1)}} - C \right) \cdot \min v(T_j) \right)$$

for all x .

Proof. We use the language of [15]. Fix an index a , and restrict to the subset $|T_a| \geq |T_j|$ for all $j \neq a$. Let R° be the T_a -adic completion of

$$\mathbb{Z}_p[[T_1, \dots, T_{n-1}]] \left[\frac{p}{T_a}, \frac{T_1}{T_a}, \dots, \frac{T_{n-1}}{T_a} \right]$$

and let $R = R^\circ[1/T_a]$. Give R the norm $|r| = \inf\{p^{-n} \mid r \in T_a^n R_\eta^\circ\}$. Let $[\cdot]_R : (\mathbb{Z}_p^\times)^n \rightarrow R^\times$ be the universal character with values in R . Let \mathcal{D} be the continuous R -dual of $\text{Ind}_{B(\mathbb{Z}_p)}^{\text{Iw}_p, \text{cts}} [\cdot]_R$.

\mathcal{D} is orthonormalizable with the following norm: choose topological generators $\bar{n} = (\bar{n}_1, \dots, \bar{n}_{n(n-1)/2})$ for \bar{N} , for example the matrix coefficients $pz_{21}, pz_{31}, pz_{32}, \dots, pz_{n(n-1)}$ of Section 2.3. Let $\bar{\mathbf{n}}_i \in \mathcal{D}$ be the Dirac distribution at \bar{n}_i on \bar{N} . For $\eta = (\eta_1, \dots, \eta_{n(n-1)/2}) \in \mathbb{Z}_{\geq 0}^{n(n-1)/2}$, write $\bar{\mathbf{n}}^\eta := \prod_{i=1}^{n(n-1)/2} \bar{\mathbf{n}}_i^{\eta_i}$ and $|\eta| = \sum_{i=1}^{n(n-1)/2} \eta_i$ for short. Then $\{\bar{\mathbf{n}}^\eta\}_{\eta \in \mathbb{Z}^{n(n-1)/2}}$ is a basis for \mathcal{D} , and the norm is

$$\left\| \sum_{\eta} d_{\eta} \bar{\mathbf{n}}^{\eta} \right\|_r = \sup_{\eta} |d_{\eta}| r^{|\eta|}.$$

Let \mathcal{D}^r be the completion of \mathcal{D} with respect to this norm. By Corollary 4.1.5 of [15], $\sum_{N \geq 0} c_N(T_1, \dots, T_{n-1}) X^N$ can be computed by the action of U_p on the space $\mathcal{D}^{1/p}(G, U_0(p))$.

By Section 3.3 of [15], \mathcal{D}^r has a potential orthonormal basis given by the elements $e_{r,\eta} := T_a^{-n(r, T_a, \eta)} \bar{\mathbf{n}}^\eta$, where

$$n(r, T_a, \eta) = \left\lfloor \frac{|\eta| \log_p r}{\log_p |T_a|} \right\rfloor,$$

and correspondingly $\mathcal{D}^r(G, U_0(p))$ has a potential orthonormal basis given by the elements

$$e_{r,\eta}^t := (0, \dots, 0, e_{r,\eta}, 0, \dots, 0) \subset \bigoplus_{t=1}^h \mathcal{D}^r \cong \mathcal{D}^r(G, U_0(p))$$

where the $e_{r,\eta}$ is in the t th position. By Lemma 6.2.1 of [15], we have

$$U_p(e_{r,\eta}^t) = \sum_{u,\mu} a_{\mu}^u e_{r,\mu}^u$$

with

$$|a_{\mu}^u| \leq |T_a|^{n(r, T_a, \mu) - n(r^{1/p}, T_a, \mu)}.$$

We have $n(p^{-1}, T_a, \mu) = |\mu|$ and $n(p^{-1/p}, T_a, \mu) = \lfloor |\mu|/p \rfloor$. So whenever $|\mu| = N$, every matrix entry of U_p in the row $e_{r,\mu}^u$ has coefficient a_{μ}^u divisible by $|T_a|^{N - \lfloor N/p \rfloor}$.

There are

$$h \binom{N + n(n-1)/2 - 1}{n(n-1)/2 - 1}$$

choices of u and μ such that $|\mu| = N$, and hence that many rows which we can guarantee are divisible by $T_a^{N - \lfloor N/p \rfloor}$ (not counting rows which we can guarantee are divisible by higher powers of T_a). We conclude that NP $(\sum_{N \geq 0} c_N(T_1, \dots, T_{n-1})X^N)$ passes above the point

$$\left(h \sum_{N=0}^M \binom{N + n(n-1)/2 - 1}{n(n-1)/2 - 1}, h \sum_{N=0}^M \binom{N + n(n-1)/2 - 1}{n(n-1)/2 - 1} (N - \lfloor N/p \rfloor) v(T_a) \right)$$

for every integer $M \geq 0$. Since the x -coordinate of the above expression is a polynomial in M of degree $n(n-1)/2$ and the y -coordinate is $v(T_a)$ times a polynomial in M of degree $n(n-1)/2 + 1$, the claim follows. \square

4.2. Systems of eigenvalues associated to classical points. A “refined principal series” is a principal series representation π of $GL_n(\mathbb{Q}_p)$ together with an ordered sequence of characters $(\psi_1, \dots, \psi_n) : (\mathbb{Q}_p^\times)^n \rightarrow \mathbb{C}^\times$ such that $\pi \cong \pi(\psi_1, \dots, \psi_n)$. So there are $n!$ possible refinements of each π . The language comes from Galois representation theory. Assume χ is simple. From our setup so far, it is easy to see that an eigenform $f \in (U^{sm}(\chi) \otimes S_t)(G, U_0(p))$ is naturally associated to a particular refined principal series: the principal series $\pi_{f,p}$, together with, if f has λ -values $\lambda_1, \dots, \lambda_n$, the ordered sequence $(\psi_1, \dots, \psi_n) : (\mathbb{Q}_p^\times)^n \rightarrow \mathbb{C}^\times$ such that $\pi \cong \pi(\psi_1, \dots, \psi_n)$ and $\lambda_i = p^{(n-1)/2 - i + 1 - m_n - m_{n-1} - \dots - m_{n-i+1}} \psi_i(p)$. Also note that this refined principal series depends only on the point x on \mathcal{D} that f is associated to.

For a character $\chi : (\mathbb{Z}_p^\times)^n \rightarrow \mathbb{C}^\times$ or $\psi : (\mathbb{Q}_p^\times)^n \rightarrow \mathbb{C}^\times$, and for any $w \in S_n$, we write $\chi^w = (\chi_{w(1)}, \dots, \chi_{w(n)})$, and ψ^w similarly.

Now note that if $f_x \in (U^{sm}(\chi) \otimes S_t)(G, U_0(p)) \subset S_{t\chi,c}(G, U_0(p))$ is an eigenform associated to a point x on \mathcal{D} with associated refined principal series $(\pi(\psi), \psi^{\text{id}})$, then the refined principal series $(\pi(\psi), \psi^w)$ is also associated to a point x^w on \mathcal{D} and a form

$f_x^w \in (U^{sm}(\chi^w) \otimes S_t)(G, U_0(p)) \subset S_{t\chi^w}(G, c, U_0(p))$ (arising from the unique (J, χ^w) -vector in $\pi(\psi)$). The forms f_x^w are called companion forms of f_x . Having defined these companion forms, it is straightforward to show that the slopes appearing in $(U^{sm}(\chi) \otimes S_t)(G, U_0(p))$ are not only finite but bounded above by a linear function of t , as follows.

Proposition 4.2.1. *If $f \in (U^{sm}(\chi) \otimes S_t)(G, U_0(p)) \subset S_{t\chi,c}(G, U_0(p))$ is a U_p^a -eigenform with eigenvalue a_p^{id} , and each companion form f^w has U_p^a -eigenvalue a_p^w , then we have*

$$\sum_{w \in S_n} v(a_p^w) = l^a(t)$$

where $l^a(t)$ is a linear function of t_1, \dots, t_n .

In particular, let $l^{(n-1, n-2, \dots, 0)}(t) = l(t)$. Then for each w , the Newton polygon of

$$\sum_{N \geq 0} c_N(T_1(t\chi^w), \dots, T_{n-1}(t\chi^w))X^N$$

contains $hp^{j(\chi)}d_t$ slopes of size at most $l(t)$, hence in particular passes below the point

$$(hp^{j(\chi)}d_t, hp^{j(\chi)}d_t l(t)).$$

Proof. Let $\pi_{f,p} = \pi(\psi_1, \dots, \psi_n)$. By Proposition 3.6.7, we have

$$\prod_i \lambda_i = p^{-(nm_n + (n-1)m_{n-1} + \dots + m_1)} \prod_i \psi_i(p).$$

The λ -values of x^w are given by

$$\lambda_i^w = p^{(n-1)/2 - i + 1 - m_n - m_{n-1} - \dots - m_{n-i+1}} \psi_{w(i)}(p)$$

for each $w \in S_n$. Then for $a = (a_1, \dots, a_n)$, the U_p^a -eigenvalue associated to x^w is

$$\prod_i (\lambda_i^w)^{a_{n-i+1}} = \prod_i p^{a_{n-i+1}[(n-1)/2 - i + 1 - m_n - m_{n-1} - \dots - m_{n-i+1}]} \psi_{w(i)}(p)^{a_{n-i+1}}$$

so the product of the U_p^a -eigenvalues associated to all the x^w s is

$$\begin{aligned} & p^{(n-1)! \sum_i a_{n-i+1} [(n-1)/2 - i + 1 - m_n - m_{n-1} - \dots - m_{n-i+1}]} \left(\prod_i \psi_i(p) \right)^{(n-1)! \sum_i a_i} \\ &= p^{(n-1)! \sum_i a_{n-i+1} [(n-1)/2 - i + 1 - m_n - m_{n-1} - \dots - m_{n-i+1}]} \left(p^{nm_n + (n-1)m_{n-1} + \dots + m_1} \prod_i \lambda_i \right)^{(n-1)! \sum_i a_i}. \end{aligned}$$

But $\prod_i \lambda_i$ is the eigenvalue associated to the operator $U_p^{(1,1,\dots,1)}$, which is just right translation by the central matrix $\text{diag}(p, p, \dots, p)$, which preserves f , so $\prod_i \lambda_i = 1$. So the sum of the valuations of the U_p^a -eigenvalues associated to the companion points is

$$\begin{aligned} & (n-1)! \sum_i a_{n-i+1} [(n-1)/2 - i + 1 - m_n - m_{n-1} - \dots - m_{n-i+1}] \\ & \quad + (nm_n + (n-1)m_{n-1} + \dots + m_1)(n-1)! \sum_i a_i \\ &= (n-1)! \left(\sum_i a_{n-i+1} ((n-1)/2 - i + 1) - \sum_j m_j (a_1 + \dots + a_j) + \sum_j j m_j \left(\sum_i a_i \right) \right). \end{aligned}$$

Defining $l^a(t)$ to be this last expression, we find that $\sum_{w \in S_n} v(a_p^w) = l^a(t)$ as desired.

The conclusion that each individual $v(a_p^w)$ is bounded above by $l^a(t)$ follows because all the a_p^w s are algebraic integers. \square

Let $c_i = \text{cond}(\chi_i)$, let $\chi_{(1)}, \dots, \chi_{(n-1)}$ be the characters $\chi_1, \dots, \chi_{n-1}$ reordered so that $\text{cond}(\chi_{(1)}) \leq \text{cond}(\chi_{(2)}) \leq \dots \leq \text{cond}(\chi_{(n-1)})$, let $c_{(i)} = \text{cond}(\chi_{(i)})$, and let $T_{(i)} = T(\chi_{(i)})$. To get from Proposition 4.2.1 to the statement of Theorem 1.1.4, we just need to check that for all t and χ , $(hp^{j(\chi)} d_t, hp^{j(\chi)} d_t l(t))$ lies below the curve

$$y = A_2 \left(v(T_{(1)})^{\frac{2}{n(n-1)}} v(T_{(2)})^{\frac{2 \cdot 2}{n(n-1)}} \dots v(T_{(n-1)})^{\frac{2 \cdot (n-1)}{n(n-1)}} \right) x^{1 + \frac{2}{n(n-1)}}$$

for a fixed constant A_2 (depending only on n , p , and h). Note that by the formula stated in Example 2.2.2, we have

$$v(T_{(i)}) = v(T(\chi_{(i)})) = Ap^{-c_{(i)}}$$

for a constant A (depending on p). Thus we have

$$p^{j(x)} = p^{c(1)+2c(2)+\dots+(n-1)c(n-1)-\frac{n(n-1)}{2}} = A'v(T_{(1)})^{-1}v(T_{(2)})^{-2}\dots v(T_{(n-1)})^{-(n-1)}.$$

Next we check the size of d_t .

Proposition 4.2.2. *The dimension d_t is a polynomial of total degree $\frac{n(n-1)}{2}$ in m_1, \dots, m_{n-1} .*

Proof. By Corollary 14.9 of [19], $\text{Ind}_{B(\mathbb{Z}_p)}^{GL_n(\mathbb{Z}_p), \text{alg.}} t$ has a basis indexed by chains in the poset described in Section 14.2 of [19]. For a subset σ of $\{1, \dots, n\}$, let $f(\sigma) = \sum_{k \notin \sigma} (n+1-k)$. We claim that when you take one step down the poset, $f(\sigma)$ goes down by 1. This is because, if σ is one step below τ , there are two possibilities. The first is that $|\tau| = |\sigma|$ and there is some i for which $\sigma_i = \tau_i - 1$ and $\sigma_j = \tau_j$ for all j with $j \neq i$; in this case the complements σ^c and τ^c are the same except for $\sigma_i \in \tau^c$ and $\sigma_i + 1 = \tau_i \in \sigma^c$, which contribute $n - \sigma_i$ and $n - \sigma_i - 1$ to the sums $f(\sigma)$ and $f(\tau)$, so $f(\sigma) = f(\tau) - 1$. The second is that $|\sigma| = |\tau| + 1$ and σ contains n and τ does not, so again $f(\sigma) = f(\tau) - 1$.

So a maximal chain in this poset starts with $\{n\}$, which has f -value $2 + \dots + n = \frac{n(n+1)}{2} - 1$, and ends with $\{1, 2, \dots, n-1\}$, which has f -value 1; its length is therefore $\frac{n(n+1)}{2} - 1$. A leading term of $d_{m_1, \dots, m_{n-1}, 0}$ comes from distributing m_1, \dots, m_{n-1} among corresponding variables in a maximal chain. So it is a product $\prod \binom{m_i + c_i}{c_i}$ where the $c_i + 1$ s sum to $\frac{n(n+1)}{2} - 1$; that is, the c_i s sum to $\frac{n(n+1)}{2} - 1 - (n-1) = \frac{n(n-1)}{2}$. \square

Fix $\epsilon > 0$ and assume that for all $i \neq j$, we have $m_i \geq \epsilon m_j$. Then we can find some A_ϵ such that $l(t) \leq A_\epsilon d_t^{\frac{2}{n(n-1)}}$ for all such m_1, \dots, m_{n-1} . So if we let $x = hp^{j(x)}d_t$ and $y = hp^{j(x)}d_t l(t)$, we have

$$\begin{aligned} y &= (hp^{j(x)}d_t)^{1+\frac{2}{n(n-1)}} (hp^{j(x)}d_t)^{-\frac{2}{n(n-1)}} l(t) \\ &= Ax^{1+\frac{2}{n(n-1)}} (p^{j(x)})^{-\frac{2}{n(n-1)}} d_t^{-\frac{2}{n(n-1)}} l(t) \end{aligned}$$

$$\begin{aligned}
&\leq AA_\epsilon x^{1+\frac{2}{n(n-1)}} \left(v(T_{(1)})^{-1} v(T_{(2)})^{-2} \cdots v(T_{(n-1)})^{-(n-1)} \right)^{-\frac{2}{n(n-1)}} \\
&= A' \left(v(T_{(1)})^{\frac{2}{n(n-1)}} v(T_{(2)})^{\frac{2 \cdot 2}{n(n-1)}} \cdots v(T_{(n-1)})^{\frac{2 \cdot (n-1)}{n(n-1)}} \right) x^{1+\frac{2}{n(n-1)}}
\end{aligned}$$

as desired. This proves Part 2 of Theorem 1.1.4 for all $t\chi$ such that $m_i \geq \epsilon m_j$ for all $i \neq j$ and χ is simple.

4.3. Combining upper bound points. We show that Theorem 1.1.5 is a natural consequence of Part 2 of Theorem 1.1.4. First we need the following lemma of Wan, which is stated in [23] with \mathbb{Z}_p -coefficients but works identically with $\mathcal{O}_{\mathbb{C}_p}$ -coefficients.

Lemma 4.3.1 (Wan 1998). *Let $Q_1(X), Q_2(X)$ be two elements in $\mathcal{O}_{\mathbb{C}_p}[[X]]$ with $Q_1(0) = Q_2(0) = 1$. Let $N_i(x)$ be the function on $\mathbb{R}_{\geq 0}$ whose graph is the Newton polygon of $Q_i(X)$. Assume that $\nu(x)$ is a strictly increasing continuous function on $\mathbb{R}_{\geq 0}$ such that $\nu(0) \leq 0$, $N_i(x) \geq x\nu(x)$ for $1 \leq i \leq 2$ and $x \geq 1$, and $\lim_{x \rightarrow \infty} \nu(x) = \infty$. Assume further that the function $x\nu^{-1}(x)$ is increasing on $\mathbb{R}_{> 0}$, where $\nu^{-1}(x)$ denotes the inverse function of $\nu(x)$ defined at least on $\mathbb{R}_{\geq 0}$. For $x \geq 0$, we define the integer-valued increasing function $m_\nu(x) = \lfloor x\nu^{-1}(x) \rfloor$. If the congruence*

$$Q_1(X) \equiv Q_2(X) \pmod{p^{m_\nu(\alpha)+1}}$$

holds for some $\alpha \geq 0$, then the two Newton polygons $N_i(x)$ coincide for all the sides with slopes at most α .

Proof of Theorem 1.1.5. By Corollary 4.2.1, $\text{NP}(t\chi)$ passes below the point

$$(hp^{j(x)}d_t, hp^{j(x)}d_t l(t)).$$

Note that the slope of $\text{NP}(t\chi)$ at x -coordinate $hp^{j(x)}d_t$ is at most $l(t)$. We may apply Lemma 4.3.1 with $\nu(x) = A_1 x^{\frac{2}{n(n-1)}} \min_i v(T(\chi_i))$, so that

$$m_\nu(x) \asymp \frac{x^{1+\frac{n(n-1)}{2}}}{(\min_i v(T(\chi_i)))^{\frac{n(n-1)}{2}}}.$$

Let $t_i^{(1)} = t_i + (n-i)p^{m_\nu(l(t))+1}$. By Lemma 4.3.1 applied to $P(X, t\chi)$ and $P(X, t^{(1)}\chi)$, we find that $\text{NP}(t^{(1)}\chi)$ also passes below this point. However, by Corollary 4.2.1, $\text{NP}(t^{(1)}\chi)$ also passes below

$$(hp^{j(\chi)}d_{t^{(1)}}, hp^{j(\chi)}d_{t^{(1)}}l(t^{(1)})).$$

Repeating this, we find a sequence $t = t^{(0)}, t^{(1)}, t^{(2)}, \dots$ of dominant algebraic weights such that $\text{NP}(t^{(k)}\chi)$ passes below

$$(hp^{j(\chi)}d_{t^{(0)}}, hp^{j(\chi)}d_{t^{(0)}}l(t^{(0)})), \dots, (hp^{j(\chi)}d_{t^{(k)}}, hp^{j(\chi)}d_{t^{(k)}}l(t^{(k)})).$$

Evidently the $t^{(k)}$ approach a limit t^∞ , and $\text{NP}(t^\infty\chi)$ passes below

$$(hp^{j(\chi)}d_{t^{(k)}}, hp^{j(\chi)}d_{t^{(k)}}l(t^{(k)}))$$

for all k . The result follows as in the end of Section 4.2. (Note that since $m_i^{(k)} = m_i^{(k-1)} + p^{m_\nu(l(t^{(k-1)}))+1}$, if $m_i^{(k-1)} \geq \epsilon m_j^{(k-1)}$ for all $i \neq j$, the same is true for the $m_i^{(k)}$.) \square

5. GEOMETRY OF THE EIGENVARIETY OVER THE BOUNDARY OF WEIGHT SPACE

Fix an index a , and let $\mathscr{W}_{<\nu}$ be the subset of characters w such that $v(T_a(w)) < \nu$ and $v(T_a(w)) < \nu v(T_j(w))$ for all $j \neq a$ (so in particular $v(T_a) = \min_i v(T_i)$). Let $\mathscr{Z}_{<\nu}$ be the preimage of $\mathscr{W}_{<\nu}$ in the eigencurve \mathscr{Z} . For any real number α , let $X(<\alpha)$ be the subset of \mathscr{Z} of points x for which $v(a_p(x)) < \alpha v(T_a(w(x)))$, and define $X(=\alpha)$, $X(>\alpha)$ similarly.

As in the previous section, fix a polydisc in \mathscr{W} . For $T = (T_1, \dots, T_{n-1})$ in the polydisc and $m = (m_1, \dots, m_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$, write $T^m = T_1^{m_1} \dots T_{n-1}^{m_{n-1}}$ for short. Let $\det(1 - XU_p) = \sum_{N \geq 0} c_N(T)X^N$, where $c_N(T) = \sum_{m=(m_1, \dots, m_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}} b_{N,m}T^m \in \mathbb{Z}_p[[T_1, \dots, T_{n-1}]]$. Let $y = \text{NP}(T)(x)$ be the Newton polygon of $\sum_{N \geq 0} c_N(T)X^N$.

For the following theorem, the only input we need is a lower bound for $y = \text{NP}(T)(x)$ of the form $y = v(T_a)f(x)$ where $f(x)$ is a convex function, which we have (with $f(x) = A_1x^{1+\frac{2}{n(n-1)}}$) from Part 1 of Theorem 1.1.4.

Theorem 5.0.2. *For every $\alpha \in \mathbb{R}_{\geq 0}$, there is some valuation $\nu(\alpha) > 0$ such that $X(=\alpha)_{<\nu(\alpha)}$ is disconnected from its complement in $\mathcal{L}_{<\nu(\alpha)}$.*

Proof. Let $d(\alpha, T)$ be the number of slopes in $y = \text{NP}(T)(x)$ of value strictly less than $\alpha v(T_a)$ (so the dimension of $\mathcal{S}_T(G, U_0(p))^{<\alpha v(T_a)}$). Assume $v(T_a) < 1$.

We claim that the point $(d(\alpha, T), \text{NP}(T)(d(\alpha, T)))$ lies inside the region bounded by the line $y = \alpha v(T_a)x$ and the function $y = v(T_a)f(x)$. It lies below $y = \alpha v(T_a)x$ because all slopes of $\text{NP}(T)$ up to $d(\alpha, T)$ are less than $\alpha v(T_a)$. It lies above $y = v(T_a)f(x)$ because this is a lower bound for $y = \text{NP}(T)(x)$.

This region lies inside the box whose lower left corner is $(0, 0)$ and whose upper right corner is $(d(\alpha), \alpha d(\alpha)v(T_a))$, where $d(\alpha)$ is the nonzero solution to $\alpha x = f(x)$.

We have $(d(\alpha, T), \text{NP}(T)(d(\alpha, T))) = (j, v(c_j(T)))$ for some j . This is a vertex of $y = \text{NP}(T)(x)$. The vertex immediately preceding it is of the form $(i, v(c_i(T)))$ for some i . The slope between the two is

$$\frac{v(c_j(T)) - v(c_i(T))}{j - i}.$$

This is the largest slope of $y = \text{NP}(T)(x)$ less than $\alpha v(T_a)$. We have $1 \leq j - i \leq d(\alpha)$.

But $c_j(T) = \sum_{m \geq 0} b_{j,m} T^m$ is a sum of terms $b_{j,m} T^m$ where $v(b_{j,m})$ is an integer and $v(T^m) = m_1 v(T_1) + \cdots + m_{n-1} v(T_{n-1})$. Thus $v(c_j(T)) = \mu_j + \lambda_j^1 v(T_1) + \cdots + \lambda_j^{n-1} v(T_{n-1})$ where μ_j, λ_j^k are integers in the range $[0, \alpha d(\alpha)]$ (since $v(c_j(T)) \leq \alpha d(\alpha)v(T_a)$). Similarly $v(c_i(T)) = \mu_i + \lambda_i^1 v(T_1) + \cdots + \lambda_i^{n-1} v(T_{n-1})$ where $\mu_i, \lambda_i^k \in [0, \alpha d(\alpha)]$ as well.

Assume that $v(T_a) < \frac{1}{\alpha d(\alpha)}$, so that $\alpha d(\alpha)v(T_a) < 1$, and furthermore that $v(T_a) < \frac{1}{\alpha d(\alpha)}v(T_j)$ for all $j \neq a$. Then in order to have $v(c_i(T)), v(c_j(T)) \leq \alpha d(\alpha)v(T_a)$, we must have $\mu_i = \mu_j = 0$ and $\lambda^k = 0$ for all $k \neq i$.

So the largest slope of $y = \text{NP}(T)(x)$ less than $\alpha v(T_a)$ is of the form $\frac{\lambda_j - \lambda_i}{j - i}v(T_a)$, where $\lambda_j - \lambda_i \in [0, \alpha d(\alpha)]$ and $j - i \in [1, d(\alpha)]$. This is a finite, discrete set of points. So the ratio of the largest slope of $y = \text{NP}(T)(x)$ less than $\alpha v(T_a)$ to $v(T_a)$ is bounded away from α independently of T_a .

Setting $\nu(\alpha) < \frac{1}{\alpha d(\alpha)}$, we conclude that $X(< \alpha)_{< \nu(\alpha)}$ is disconnected from its complement in $\mathcal{Z}_{< \nu(\alpha)}$.

This argument goes through exactly the same way if $X(< \alpha)$ is replaced by $X(\leq \alpha)$: either the smallest slope greater than α is at least $\alpha + 1$, or, if not, the next endpoint is again trapped in a box whose area is at most linear in $v(T)$, and the same argument applies. So we can choose $\nu(\alpha)$ such that $X(= \alpha)_{< \nu(\alpha)}$ is disconnected from its complement in $\mathcal{Z}_{< \nu(\alpha)}$. \square

As Liu-Wan-Xiao do in Theorem 3.19 of [16], we can also use Part 1 of Theorem 1.1.4 to give a simple proof of the fact that the ordinary part of \mathcal{Z} is finite and flat over \mathcal{W} and disconnected from its complement.

Theorem 5.0.3. *$X(= 0)$ is finite and flat over \mathcal{W} and is a union of connected components of \mathcal{Z} .*

Proof. The proof of Theorem 3.19 of [16] goes through almost word-for-word. By Part 1 of Theorem 1.1.4, there is some maximal N such that $c_N(T_1, \dots, T_{n-1})$ is a unit in $\mathbb{Z}_p[[T_1, \dots, T_{n-1}]]$, or equivalently, the constant term of $c_N(T_1, \dots, T_{n-1})$ is a unit in \mathbb{Z}_p . Then for each (T_1, \dots, T_{n-1}) , the Newton polygon of $\sum_{n=0}^{\infty} c_N(T_1, \dots, T_{n-1})X^N$ starts with N segments of slope 0 followed by a segment of slope at least $\max(1, B \min_j v(T_j))$ for some constant B . Since $\max(1, B \min_j v(T_j))$ is uniformly

bounded away from 0 over any affinoid subdomain, $X(= 0)$ is disconnected from its complement, and it is finite and flat of degree N . \square

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