Bootstrapping High-Energy States in Conformal Field Theories

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Bootstrapping High-Energy States in Conformal Field Theories

A dissertation presented by Baurzhan Mukhametzhanov to The Department of Physics in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the subject of Physics

Harvard University Cambridge, Massachusetts May 2019
Bootstrapping High-Energy States in Conformal Field Theories

ABSTRACT

We analyze the operator spectrum of Conformal Field Theories at large conformal dimensions $\Delta$ with and without global symmetries. General constraints of crossing symmetry and unitarity allow us to extract a number of universal properties of heavy operator spectrum perturbatively in $\frac{1}{\Delta}$.

First, we consider four-point correlators and solve crossing equations in the deep Euclidean regime. Large scaling dimension $\Delta$ tails of the weighted spectral density of primary operators of given spin in one channel are matched to the Euclidean OPE data in the other channel. Subleading $\frac{1}{\Delta}$ tails are systematically captured by including more operators in the Euclidean OPE in the dual channel. We use dispersion relations for conformal partial waves in the complex $\Delta$ plane, the Lorentzian inversion formula and complex tauberian theorems to derive this result. We make predictions for the 3d Ising model.

Second, we apply the methods of tauberian theory to modular invariance in 2d CFTs. We derive lower and upper bounds on the number of operators within a given energy interval. At high energies we rigorously derive the Cardy formula for the mi-
crocanonical entropy together with optimal error estimates for various widths of the averaging energy shell. We identify a new universal contribution to the microcanonical entropy controlled by the central charge and the width of the shell. We derive an upper bound on the spacings between Virasoro primaries. Analogous results are obtained in holographic 2d CFTs.

Third, we consider unitary CFTs with continuous global symmetries in $d > 2$. We consider a state created by the lightest operator of large charge $Q \gg 1$ and analyze the correlator of two light charged operators in this state. Assuming the correlator admits a well-defined large $Q$ expansion and, relatedly, that the macroscopic (thermodynamic) limit of the correlator exists, we find that the crossing equations admit a consistent truncation, where only a finite number $N$ of Regge trajectories contribute at leading nontrivial order. We classify all such solutions to the crossing. For one Regge trajectory $N = 1$, the solution is unique and given by the effective field theory of a Goldstone mode.
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Citations to Previously Published Work

- The Chapter 1 has appeared in the following paper:

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TO MY MOM
Crossing equations express associativity of the operator product expansion (OPE) \([1, 2]\). They are nonperturbative consistency conditions on the Conformal Field Theory (CFT) data (spectrum of local operators and their three-point functions). Extracting physical information from crossing equations is not an easy task. But in the last decade, starting with a seminal paper \([3]\), significant progress
in this direction was achieved both numerically and analytically, for a review and references see e.g. [4–6].

Most analytic computations become possible when there is an expansion parameter in the problem (see however [7, 8]). One such parameter is spin $J$ [9–11]. Large spin expansion arises from solving crossing equations in the vicinity of a light cone. In this case it is possible to isolate families of operators that are dual to each other on both sides of the crossing equation and match their spectral data. Other examples include expansions in small coupling constant [12, 13] and large central charge (large $N$) [14].

There is yet another universal quantum number in the problem, namely the scaling dimension $\Delta$. It is natural to ask if it is possible to solve crossing equations by expanding in $\frac{1}{\Delta}$. In the present thesis we explore this idea in various contexts: four-point function crossing, modular invariance of 2d CFT partition functions and heavy-light-light-heavy crossing in CFTs with global symmetries. Let us summarize our results in each case.

0.1 Analytic Euclidean Bootstrap

In Chapter [1] we consider the crossing equation for a four-point function of scalar primary operators in a unitary CFT. The idea is that light operators in one
channel map to some averaged property of the high energy (scaling dimension) OPE data in the crossed channel. This allows us to derive a universal high-energy asymptotic of the integrated spectral density of operators.

Let us introduce a weighted spectral density $\rho_{J}^{\text{OPE}}(\Delta)$ of primary operators of given spin

$$\rho_{J}^{\text{OPE}}(\Delta) \equiv \sum_{k} \frac{p_{\Delta,J}}{K_{J,\Delta}} \delta(\Delta - \Delta_k),$$

$$K_{J,\Delta} = \frac{\Gamma(\Delta - 1)}{2\pi^2} \frac{\Gamma(\Delta + J)\Gamma(\Delta + J - 1)}{\Gamma(\Delta - \frac{d}{2}) \Gamma(\Delta + J) \Gamma(\Delta + J - 1)},$$

(1)

where $p_{\Delta,J}$ stands for the squares of OPE coefficients with the standard normalization for conformal blocks. Then we derive that in any unitary CFT the following asymptotic of the integrated spectral density holds

$$\int_{0}^{\Delta} d\Delta' \, \rho_{J}^{\text{OPE}}(\Delta') \sim f_{J} \Delta^4(\delta_{\phi} - \frac{1}{2}), \quad \delta_{\phi} > \frac{1}{2}, \quad (\Delta \to \infty),$$

$$f_{J} = [1 + (-1)^J] \pi^2 \frac{2^{2J+d-4\delta_{\phi}+2} \Gamma(J + \frac{d}{2})}{\Gamma(J + 1) \Gamma(\Delta_{\phi})^2 \Gamma(\delta_{\phi})^2}, \quad \Delta_{\phi} = \frac{d - 2}{2} + \delta_{\phi}.$$

(2)

For $\delta_{\phi} < \frac{1}{2}$ we have to consider higher moments of $\rho_{J}^{\text{OPE}}(\Delta)$ to which similar formulas apply.\footnote{Strictly speaking, we derived the formula only for $J > 1$. However, we observed in a few simple examples that it holds down to $J = 0$. It would be interesting to systematically under-}
sion relations for conformal partial waves $c_J(\Delta)$. Lorentzian inversion formula \[15\] allows us to interpret the dispersion relations as crossing for the four-point function. Finally, to extract the asymptotics \[2\] we use the tools of tauberian theory \[16, 17\].

0.2 MODULAR INVARIANCE OF 2d CFTs

Having developed the methods of tauberian theory for extracting the asymptotics of spectral densities, in Chapter \[2\] we apply these tools to modular invariance of 2d CFT partition functions. We derive a set of rigorous results about the microcanonical entropy $S_\delta(\Delta)$. These concern either all operators present in the theory, or only Virasoro primaries in CFTs with $c > 1$ for heavy operators $\Delta \to \infty$ at fixed central charge $c$ or in the holographic regime $\Delta \sim c \to \infty$. E.g. for the spectral density $\rho(\Delta)$ of all operators, both primaries and descendants, we derive the asymptotic of the microcanonical entropy

$$S_\delta(\Delta) = \log \int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \rho(\Delta') = 2\pi \sqrt{\frac{c\Delta}{3}} + \frac{1}{4} \log \left( \frac{c\delta^4}{3\Delta^3} \right) + s(\delta, \Delta), \quad \Delta \to \infty,$$

stand the status of \[3\] for $J = 0, 1$ in a generic CFT. To do that one should include arc contributions in the Lorentzian inversion formula.
where depending on the size of the averaging energy shell $\delta$ we show that  

$$
\delta \sim \Delta^\alpha : \quad s(\delta, \Delta) = \log \left( \frac{\sinh \left( \frac{\pi \sqrt{3} \delta}{\sqrt{\Delta}} \right)}{\pi \sqrt{3} \delta} \right) + O \left( \Delta^{-\alpha} \right), \quad 0 < \alpha \leq \frac{1}{2},
$$

$$
\delta = O(1) : \quad s_-(\delta) \leq s(\delta, \Delta) \leq s_+(\delta), \quad \delta > \delta_{gap} = \frac{\sqrt{3}}{\pi} \approx 0.55 .
$$

(4)

for some $s_\pm(\delta) = O(1)$ that we will specify in Chapter 2.

The first two terms in the RHS of 3 are the Cardy formula and the leading log correction to it discussed for example in [18, 19]. The results for $s(\delta, \Delta)$ are new to the best of our knowledge. In particular, we see that for $\delta \sim \Delta^\alpha$ there is yet another universal correction\(^2\) to the microcanonical entropy that is controlled by the central charge $c$ and the width of the energy shell $\delta$ and given by the first line in 4. Note that for any $\alpha > 0$ the error decays at large $\Delta$ and the non-decaying contribution to the entropy is fully captured by the $\frac{1}{4} \log \left( \frac{\delta^4}{3\Delta^3} \right)$ term. For $\delta = O(1)$ the the lower bound $s_-(\delta)$ diverges logarithmically as we approach $\delta \to \delta_{gap}$. The interpretation of this is that the asymptotic 3 is only applicable for $\delta > \delta_{gap}$ for which the leading behavior of the microcanonical entropy $S_\delta(\Delta)$ takes the form 4. Note that the lower bound implies that there

\(^2\)By $a \sim b$ we mean $\lim a/b = \text{const} \neq 0$ in the corresponding limit.

\(^3\)It dominates over the error term $O(\Delta^{-\alpha})$ only for $\alpha > \frac{1}{4}$. 

5
have to be operators in an energy shell of the size \( \delta > \delta_{\text{gap}} \). This is of course trivial when we include descendants, but we will see in Chapter 2 the same result holds for the spectral density of Virasoro primaries giving a non-trivial upper bound on the spacings between primary operators.

0.3 Large Charge Bootstrap

In Chapter 3 we consider CFTs in \( d > 2 \) with continuous global symmetries. The spectrum of these CFTs contains operators charged under these symmetries. For simplicity, we focus on the case of \( U(1) \). We denote the lightest operator of charge \( Q \) as \( \mathcal{O}_Q \), its dimension being \( \Delta_Q \). We are interested in the limit when \( Q \) becomes large.

One of the simplest nontrivial examples of this type is given by the \( O(2) \) Wilson-Fischer CFT in \( d = 3 \). This theory has \( U(1) \times \mathbb{Z}_2 \) global symmetry and is common in Nature, see, e.g., [20]. It is commonly defined as the IR fixed point of the flow generated by the \( (\phi^\dagger \phi)^2 \) deformation of the free complex scalar theory in the UV. In a recent paper [21] it was argued that the large \( Q \) subsector of this theory is described by a conformally invariant effective field theory (EFT) Lagrangian of a Goldstone boson. In particular, the authors of [21] predicted the spectrum of operators \( \Delta \) with dimensions slightly above \( \Delta_Q \), namely
the operators with $\Delta - \Delta_Q \sim O(1)$ in the large $Q$ limit. This approach was further developed in [22], where the correlation functions of light charged operators in the background of the heavy state were computed.

These results are the starting point for our analysis. We would like to understand how universal they are and what assumptions would go into their derivation in generic CFTs. Therefore, we study a crossing equation for heavy-heavy-light-light operators in an abstract CFT with a global symmetry. We take the heavy state to be $O_Q$, the lightest operator with a given large charge.

Under certain assumptions described in Chapter 3 we classify the leading order solutions to the crossing equation. The solutions for scaling dimensions as functions of spin are given by the roots of a certain polynomial that we describe in detail below. We show that if only one $N = 1$ Regge trajectory contributes at the first non-trivial order, then the Goldstone EFT is the unique solution. For $N \geq 2$ there are many possibilities. Some of them correspond to adding extra particles. Other solutions do not come from any weakly coupled EFT Lagrangian.
Perturbative expansion at large scaling dimension naturally arises when analyzing crossing equations in the deep Euclidean regime. This question was first addressed in [23, 24]. The basic idea is very simple: light operators in one channel map to some cumulative property of the high energy (scaling dimension) OPE data tails in the other channel. In this way a universal high-energy asymptotic
of the integrated spectral density of operators was derived in [23]. In this chapter we develop this idea further.

Let us briefly review the results of [23]. Consider a four-point function of identical scalar primary operators \( \phi(x) \). Let us introduce an integrated spectral density of operators that appear in the OPE of two \( \phi \)'s

\[
F(E) \equiv \int_0^E dE' f(E'),
\]

\[
f(E) \equiv \sum_k \rho_k \delta(E - E_k),
\]

(1.1)

where the sum is over all states present in the theory, both primaries and descendants of arbitrary spin. \( E_k \) stands for the scaling dimension of the exchanged state. The coefficients \( \rho_k \) could be read off the OPE expansion of the correlator (see [23] for additional details) and are essentially given by the squares of the three-point functions. Unitarity implies that \( \rho_k \geq 0 \). It was shown in [23] that \( F(E) \) has a universal asymptotic \footnote{Notation \( a \sim b \) stands for \( \frac{a}{b} \to 1 \) in the corresponding limit.}

\[
F(E) \sim \frac{E^{2\Delta_\phi}}{\Gamma(2\Delta_\phi + 1)}, \quad (E \to \infty).
\]

(1.2)
This rigorous result follows from unitarity and the leading contribution of the unit operator in the crossed channel Euclidean OPE via the so-called Hardy-Littlewood tauberian theorem.

There are two natural questions regarding Eq. 1.2. First, is it possible to write a formula similar to Eq. 1.2 for primary operators of given spin only? In other words, can we disentangle the contribution of operators of different spin as well as of primaries and descendants. Second, can we systematically compute corrections to Eq. 1.2 by including contributions of extra operators in the crossed channel? The answer to both questions is affirmative and is the subject of the present paper.

Let us introduce a weighted spectral density $\rho_{\mathcal{OPE}}^{\Delta}(\Delta)$ of primary operators of given spin

$$\rho_{\mathcal{OPE}}^{\Delta}(\Delta) \equiv \sum_k \frac{p_{\Delta_k,J}}{K_{J,\Delta_k}} \delta(\Delta - \Delta_k),$$

$$K_{J,\Delta} = \frac{\Gamma(\Delta - 1)}{2\pi^2 \Gamma(\Delta - \frac{d}{2}) \Gamma(\Delta + J) \Gamma(\Delta + J - 1)},$$

where $p_{\Delta,J}$ stands for the squares of OPE coefficients with the standard normalization for conformal blocks.

Note that we have an additional factor $\frac{1}{K_{J,\Delta}}$ compared to the standard normalization of the three-point functions. It is, of course, a matter of choice how
to normalize three-point functions. However, we will find that there is a canonical choice dictated by the behavior of conformal partial waves at large complex $\Delta$, which leads to 1.3.

The generalization of 1.2 to primaries of given spin takes the form

$$\int_0^\Delta d\Delta' \rho_{j}^{OPE}(\Delta') \sim f_j \frac{\Delta^4(\delta_\phi - \frac{1}{2})}{4(\delta_\phi - \frac{1}{2})}, \quad \delta_\phi > \frac{1}{2}, \quad (\Delta \to \infty),$$

$$f_j = [1 + (-1)^d\pi^2] \frac{2^{2J+d-4\delta_\phi+2}\Gamma(J+\frac{d}{2})}{\Gamma(J+1)\Gamma(\Delta_\phi)\Gamma(\delta_\phi)^2}, \quad \Delta_\phi = \frac{d-2}{2} + \delta_\phi. \quad (1.4)$$

For $\delta_\phi < \frac{1}{2}$ we have to consider higher moments of $\rho_{j}^{OPE}(\Delta)$ to which similar formulas apply. This will be discussed in details in section 4.

At large $\Delta$ (and fixed $J$): $\frac{1}{K_{J,\Delta}} \sim \pi 4^{\Delta - J - 1} \Delta^{1-d/2}$. Therefore, the contribution of heavy operators in 1.3 is exponentially enhanced. What 1.4 roughly states is that after we multiply three-point couplings by this universal exponentially growing pre-factor, they behave polynomially in $\Delta$. The actual power is controlled by the Euclidean OPE in the dual channel. The asymptotic behavior 1.4 is completely rigorous and holds in any unitary CFT.

---

2For $\delta_\phi = \frac{1}{2}$ we have $\int_0^\Delta d\Delta' \rho_{j}^{OPE}(\Delta') \sim f_j \log \Delta$.

3Strictly speaking, we derived the formula 1.4 only for $J > 1$. However, we observed in a few simple examples that it holds down to $J = 0$. It would be interesting to systematically understand the status of 1.4 for $J = 0,1$ in a generic CFT. To do that one should include arc contributions in the Lorentzian inversion formula.
A second natural question to ask about 1.2 and 1.4 is regarding the corrections to this leading behavior. This question was briefly addressed in [23] where it was noticed that application of Hardy-Littlewood theorem in a real domain leads only to very weak logarithmic bounds on the correction to the integrated spectral density. In this paper we point out that the situation changes if we note that the OPE expansion is valid in a complex domain of the corresponding cross ratios. In this case one can apply more powerful complex tauberian theorems to the problem at hand [16, 25, 17, 26]. As a result one can develop a systematic $\frac{1}{\Delta}$ expansion for certain moments of the integrated spectral density. We discuss corrections to 1.2 and the corresponding tauberian theorem in section 1.

In sections 2-3 we develop CFT dispersion relations for conformal partial waves $c_{J}(\Delta)$. They are meromorphic polynomially bounded functions of $\Delta$ which encode the OPE data in the structure of their singularities: they have poles at the dimensions of operators appearing in the OPE with the residues given by squares of the OPE coefficients as well as an infinite set of kinematic poles. By the standard argument we write Cauchy integral in the complex $\Delta$ plane and deform the contour. This provides us with the desired dispersion relation: $c_{J}(\Delta)$ at some complex $\Delta$ is related to an integral of the weighted spectral density.
with an appropriate kernel plus a contribution of kinematic poles of $c_J(\Delta)$. We argue that at large $\Delta$ away from the real axis both $c_J(\Delta)$ and the contribution of kinematic poles can be computed by the OPE in the crossed channel via the Lorentzian inversion formula [27]. Kinematic poles produce terms of two types at large $\Delta$. Universal terms computable by the Euclidean OPE. And non-universal terms (not computable by the OPE) that are mapped to contributions of individual operators in the weighted spectral density [1.3] and, therefore, required for consistency.

In section 4 we use CFT dispersion relations to derive asymptotics of the integrated weighted spectral density of the type [1.4]. The crucial mathematical result that we use is a so-called complex tauberian theorem for Stieltjes transform. Dispersion relations hold for complex $\Delta$ and the corresponding complex tauberian theorem leads to a systematic $\frac{1}{\Delta}$ expansion for moments of the weighted spectral density. The main result of this analysis is the formula (5.11) for moments of the weighted spectral density defined in (5.10). It systematically maps $\frac{1}{\Delta}$ expansion of the weighted spectral density moments in one channel to the Euclidean OPE in the crossed channel.

In section 5 we proceed by studying large $\Delta$ expansion in a series of examples. To our knowledge (and surprise) this question was never addressed in the ex-
isting literature. The two basic CFT examples we consider are generalized free
fields and the 2d Ising model. In all cases we find that the corrections to 1.4 are
power-like and that relations 1.4 work extremely well already for small $\Delta$. We
also make predictions for the 3d Ising model.

Finally, we point out that our analysis of dispersion relations is very general
and might be useful beyond CFTs. In section 6 we discuss an application to
meromorphic scattering amplitudes. As an example, we study Veneziano am-
plitude and again find perfect agreement with our predictions.

We prove complex tauberian theorems for Laplace and Stieltjes transforms in
Appendix A.

As a historical remark, let us mention that a similar analysis appeared in the
context of hadronic scattering amplitudes in the late 60’s. It was found that ex-
perimental data for the pion-nucleon scattering exhibits a set of resonances at
low energies and Regge behavior at high energies. Dolen, Horn and Schmid used
analyticity of the scattering amplitude to derive the so-called finite energy sum
rules (FESR) which showed that resonances at low energy and Regge behavior
are dual to each other [28] (see [29–31] for earlier works), namely one should not
add them up to avoid double counting. This was an example of crossing con-
firmed by the experimental data. Inspired by this observation very soon after
Veneziano wrote down the celebrated amplitude \[32\]. We review this reasoning in section 6. We show that for meromorphic amplitudes a rigorous way to use FESR is via complex tauberian theorems. Our analysis grew out of an attempt to understand FESR for meromorphic amplitudes.

1.1 Euclidean Crossing and Tauberian Theorems For Laplace Transform

In this section we analyze crossing equations in the Euclidean kinematics. We start by reviewing the argument of \[23\] and then slightly generalize it. Consider a four-point function of identical scalar primary operators

\[
\langle \phi(x_4)\phi(x_3)\phi(x_2)\phi(x_1) \rangle = \frac{\mathcal{G}(z, \bar{z})}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}},
\]

\[
u = \frac{x_{12}^{2}x_{34}^{2}}{x_{13}^{2}x_{24}^{2}} = z\bar{z}, \quad u = \frac{x_{14}^{2}x_{23}^{2}}{x_{13}^{2}x_{24}^{2}} = (1 - z)(1 - \bar{z}). \tag{1.5}
\]

If we set \(x_1 = 0, x_4 = \infty\) the correlation function above becomes the radial quantization matrix element \(\langle \phi|\phi(x_3)\phi(x_2)|\phi\rangle\). We also set \(\vec{x}_2 = r_2\vec{n}_2\) and \(\vec{x}_3 = r_3\vec{n}_3\) in terms of the coordinates on the plane

\[
ds_{\text{Rad}}^2 = dr^2 + r^2d\Omega_{d-1}^2. \tag{1.6}
\]
and \( \vec{n}_i \) are unit vectors.

Consider the OPE expansion of \( \mathcal{G}(z, \bar{z}) \) in the \( \phi(x_1) \times \phi(x_2) \) channel (s-channel). It takes the following form

\[
\mathcal{G}(z, \bar{z}) = \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2 \sum_{n=0}^{\infty} e^{-E_n \beta} \langle \mathcal{O}, n, \vec{n}_3 | \mathcal{O}, n, \vec{n}_2 \rangle, \quad E_n = \Delta_{\mathcal{O}} + n. \tag{1.7}
\]

The first sum in C.23 is over primaries, while the second one is over descendants. Squares of the three-point functions between primaries are \( f_{\phi\phi\mathcal{O}}^2 \), and the states \( |\mathcal{O}, n, \vec{n} \rangle \) are \( n \)-th level descendants of \( |\mathcal{O} \rangle \) properly contracted with \( \vec{n} \). We also introduced \( r = e^\tau \), which is the standard time coordinate on the cylinder and \( \beta = \tau_3 - \tau_2 \) for the time difference.

In the conformal frame above the cross ratios take the form

\[
u = (1 - z)(1 - \bar{z}) = 1 + \left( \frac{r_2}{r_3} \right) - 2 \left( \frac{r_2}{r_3} \right) \cos \alpha, \tag{1.8}
\]

where \( \alpha \) is the angle between \( \vec{n}_2 \) and \( \vec{n}_3 \). From 1.8 it follows that \( z = e^{-\tau} e^{i\alpha} \), \( \bar{z} = e^{-\tau} e^{-i\alpha} \).

Upon setting \( \vec{n}_2 = \vec{n}_3 \), or \( \alpha = 0 \), we get the following expansion for the corre-
\[ \mathcal{L}(\beta) = \int_{0}^{\infty} dE \, f(E) e^{-E\beta}, \]
\[ f(E) = \sum_{k} \rho_{k} \delta(E - E_{k}), \quad \rho_{k} \geq 0, \tag{1.9} \]

where \( \rho_{k} \) are positive coefficients that can be computed using C.23. We can easily compute the \( \beta \to 0 \) limit of the correlator using the Euclidean OPE in the dual channel \( \phi(x_{2}) \times \phi(x_{3}) \) \( (t\text{-channel}) \). The leading contribution comes from the unit operator. We therefore get\(^4\)

\[ \mathcal{L}(\beta) = \beta^{-2\Delta_{\phi}} \left[ 1 + O(\beta^{\Delta_{\phi}}) \right], \quad \beta \to 0 \tag{1.12} \]

where the corrections come both from expanding to higher orders the contribution of the unit operator \( \left( \frac{u}{v} \right)^{\Delta_{\phi}} = \left( \frac{e^{-\beta}}{1 - e^{-\beta}} \right)^{2\Delta_{\phi}} \), as well as from heavier operators.

As explained in [23, 3.83] fixes the high energy behavior of the integrated

\(^{4}\) Throughout the paper we often write \( O(x) \) to estimate the magnitude of different quantities. Recall that

\[ f(x) = O(g(x)), \quad x \to \infty \quad (x \to a) \tag{1.10} \]

iff there exist numbers \( M, x_{0} \) \( (M, \delta) \) s.t.

\[ |f(x)| < M|g(x)|, \quad \forall \, x > x_{0} \quad (\forall \, |x - a| < \delta) \tag{1.11} \]
spectral density $F(E)$ to be

$$F(E) \equiv \int_0^E dE' f(E') ,$$

$$F(E) = \frac{E^{2\Delta_\phi}}{\Gamma(2\Delta_\phi + 1)} \left( 1 + O \left( \frac{1}{\log E} \right) \right) .$$

This result, which crucially relies on the positivity of $f(E)$, is known as Hardy-Littlewood tauberian theorem. In general, a class of theorems which relate asymptotics of two different methods of integration (or summation) ($\mathcal{L}(\beta)$ and $F(E)$ in the present case) are called tauberian theorems. The result (1.13) follows from (1.12) for real $\beta$. A common feature of real tauberian theorems is that corrections to the leading asymptotic are only logarithmically suppressed, as is the case in (1.13).

This situation changes if the condition (1.12) is valid in a complex domain. In this case the corrections are more constrained, as we will shortly explain. An intuitive reason for weaker bounds in real tauberian theorems is that in a complex plane one can have two integral transforms with different analytic properties, which have the same asymptotic on a real line. Then the remainder term in a

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5For a detailed discussion of real tauberian theorems see, for example, [16]. In particular, Chapter VII for the discussion of the remainders. See also appendix E in [33] for an elementary proof of the leading asymptotic in (1.13).
real tauberian theorem would be dictated by the integral with the worst analytic properties.

1.1.1 Complex Tauberian Theorem

In the formulas above we kept $\beta$ real. In the context of CFTs, however, we know that the $t$-channel OPE expansion is valid not only along the real line $z = \bar{z} = e^{-\beta}$, but in the complex domain $|\beta| \ll 1$. A natural question is if this stronger condition leads to stronger tauberian theorems that are relevant for CFTs. It is indeed the case as we describe below.

In what follows it will be useful to introduce a notion of Cauchy moments

$$F_m(E) \equiv \frac{1}{(m-1)!} \int_0^E dE' (E - E')^{m-1} f(E')$$

$$= \int_0^E dE_{m-1} \int_0^{E_{m-1}} dE_{m-2} \ldots \int_0^{E_1} dE_0 f(E_0). \quad (1.14)$$

These are obtained by a repeated integration, which is equivalent to the first line via integration by parts.

Imagine also that we know the OPE expansion of the correlator up to an arbi-
trary order in $\beta$, namely

$$\mathcal{L}(\beta) = \int_0^\infty dE \ f(E)e^{-E\beta} = \frac{1}{\beta^{2\Delta_\phi}} \sum_{\Delta_i} c_{\Delta_i} \beta^{\Delta_i} + ... \ \beta \to 0,$$  \quad (1.15)$$

where we can imagine re-expanding the usual $t$-channel OPE to an arbitrary high order in $\beta$. We can integrate $m$ times by parts under the $E$ integral to get

$$\mathcal{L}(\beta) = \beta^m \int_0^\infty dE \ F_m(E)e^{-E\beta},$$  \quad (1.16)$$

where we used $F_m(0) = 0$. It is then possible to prove the following statement:

**Claim:** Given the expansion \ref{1.15} is valid in the complex domain $|\beta| \ll 1$ and $f(E)$ is a positive density, the Cauchy moments \ref{1.14} $m \geq 1$ satisfy

$$F_m(E) = E^{2\Delta_\phi} \left( \sum_{\Delta_i < m} \frac{c_{\Delta_i} E^{m-\Delta_i-1}}{\Gamma(2\Delta_\phi - \Delta_i + m)} + O\left(\frac{1}{E}\right) \right).$$  \quad (1.17)$$

The formula \ref{1.17} constitutes the statement of the complex tauberian theorem for Laplace transform, which we prove in appendix A.1.

This result therefore holds in a generic unitary CFT. We review the proof of \ref{1.17} in appendix A.1. It is a particular example of more general complex tauberian theorems proved in ???. It is also easy to see that \ref{1.17} cannot be improved.
The basic ingredient that makes possible is the validity of the asymptotic behavior in the complex $\beta$-plane as we approach $\beta = 0$ (and, of course, positivity of $f(E)$).

The improvement compared to the real tauberian theorem is two-fold. First, by considering higher order $m$ Cauchy moments we can probe sub-leading operators in the $t$-channel OPE. Note that taking higher moments does not increase the error which always stays the same $O(E^{2\Delta - 1})$! Second, the remainder is suppressed by a power $\frac{1}{E}$ instead of the logarithm $\frac{1}{\log E}$.

Intuitively, repeated integration in enhances smooth power-like tails and leaves intact oscillating pieces of the type $\sin cE$. What tells us that this naive picture is actually universal and the $t$-channel Euclidean OPE is encoded in the Cauchy moments of the $s$-channel OPE data. All the non-universal pieces, in particular the ones that encode the discreteness of the spectrum, enter only in the remainder term $O(E^{2\Delta - 1})$. Analytic properties of correlation functions guarantee that these non-universal pieces are not enhanced upon a repeated integration.

Let us illustrate the discussion above with a couple of simple examples.
1.1.2 Example 1

Let us demonstrate that the estimate of the remainder in the real tauberian theorem 1.13 cannot be improved. Consider a positive spectral density $f(E)$

$$f(E) = 1 + \sin[(\log E)^2] \geq 0, \quad E \geq 1. \quad (1.18)$$

One can check that for real $\beta$ we have

$$L(\beta) = \frac{1}{\beta} (1 + \beta(c_0 - 1) + ...) = \frac{1}{\beta} (1 + O(\beta)),$$

$$F(E) = E \left(1 - \frac{1}{2} \frac{\cos[(\log E)^2]}{\log E} + ...\right) = E \left(1 + O\left(\frac{1}{\log E}\right)\right), \quad (1.19)$$

where the explicit form of $c_0$ can be found in appendix A.1, where we discuss the evaluation of this integral. The result 1.19 shows that the estimate of the remainder term in 1.13 is optimal.

Next, one can check that the asymptotic 1.19 for $L(\beta)$ does not hold in the vicinity of $\beta = 0$ in the complex plane. We have not found analytically the leading behavior of the integral above in the complex domain, but observed numerically that it is qualitatively consistent with the following simple model. Consider a function $\beta^{2-1} \cos(\log \beta)^2 L(\beta)$, where $L(\beta)$ is a function of slow variation
(namely \( \lim_{\beta \to 0} \frac{L(\lambda \beta)}{L(\beta)} = 1 \) for \( \lambda > 0 \)). The asymptotic behavior of this function depends on \( \arg[\beta] \) and is given by \( \beta^{\pi - 2\arg[\beta] - 1} L(|\beta|) \). In particular, for imaginary \( \beta \) the leading power becomes \( \frac{1}{\beta} \). This is the reason why we get a weaker bound on the remainder in \( F(E) \).

### 1.1.3 Example 2

Consider now a simple example where the complex tauberian theorem \[1.17\] is applicable. We consider the spectral density to be

\[
f(E) = \sum_{n=0}^{\infty} \delta(E - n) .
\] (1.20)

We can explicitly compute the Laplace transform

\[
\mathcal{L}(\beta) = \frac{1}{1 - e^{-\beta}} = \frac{1}{\beta} \left( 1 + \frac{1}{2} \beta + \frac{\beta}{12} \beta^2 + ... \right) .
\] (1.21)

and Cauchy moments

\[
F_m(E) = E \left( \frac{E^{m-1}}{\Gamma(m+1)} + \frac{E^{m-2}}{2\Gamma(m)} + \frac{E^{m-3}}{12\Gamma(m-1)} + ... + O \left( \frac{1}{E} \right) \right) .
\] (1.22)
The error term in this example is a function of the fractional part of $E$, namely $E - \lfloor E \rfloor$. Since $0 \leq E - \lfloor E \rfloor < 1$, it is indeed $O(1)$ for any $E$. This is an example of 1.17 with $\Delta_\phi = \frac{1}{2}$.

1.2 Dispersion Relations for Conformal Partial Waves

In this section we derive dispersion relations for conformal partial waves in the complex $\Delta$ plane. These dispersion relations allow us to study separately the contribution of primary operators of given spin in the $s$-channel. We then analyze these dispersion relations in the limit $|\Delta| \gg 1$. We find that the large $\Delta$ behavior away from the real axis of the conformal partial waves is controlled by the $t$-channel OPE data. In section 4 we will use these dispersion relations together with complex tauberian theorems to arrive at our final result (5.11).

1.2.1 Conformal Partial Waves and Lorentzian Inversion Formula

Consider a four-point correlator of identical scalar primary operators $\phi$ in $d \geq 2$ dimensions

$$\langle \phi(0)\phi(z, \bar{z})\phi(1)\phi(\infty) \rangle = (z\bar{z})^{-\Delta_\phi}G(z, \bar{z}),$$

(1.23)

\footnote{In a sense, $s$-channel operators of dimension $\Delta$ probe $t$-channel distances $\frac{1}{\Delta}$.}
where, as usual, we used conformal invariance to put four points in a plane. The
relation to the conformal cross ratios is
\[
\begin{align*}
    u &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z \bar{z}, \\
    v &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1 - z)(1 - \bar{z}).
\end{align*}
\] (1.24)

The OPE expansions in different channels are given by
\[
\begin{align*}
    s - \text{channel} : & \quad G(z, \bar{z}) = \sum_{\Delta, J} p_{\Delta, J} G_{\Delta, J}(z, \bar{z}) \\
    t - \text{channel} : & \quad G(z, \bar{z}) = \left[\frac{z \bar{z}}{(1 - z)(1 - \bar{z})}\right]^\Delta \sum_{\Delta, J} p_{\Delta, J} G_{\Delta, J}(1 - z, 1 - \bar{z}) \tag{1.25}
\end{align*}
\]
where \(p_{\Delta, J} \geq 0\) are squares of the OPE coefficients. We choose the normalization
of conformal blocks as
\[
G_{\Delta, J}(z, \bar{z}) \sim z^{\frac{\Delta - J}{2}} \bar{z}^{\frac{\Delta + J}{2}}, \quad z \ll \bar{z} \ll 1 \tag{1.26}
\]
\footnote{We mostly follow the conventions of [27], except for conformal blocks and three-
point functions, which we write with a more conventional ordering of dimension and spin
\(G_{\Delta, J}, p_{\Delta, J}\), but keep the same normalization for them as in [27].}

25
In two and four dimensions the exact expressions are known and given by hyper-
geometric functions

\[
G_{\Delta,J}(z,\bar{z}) = \frac{1}{1 + \delta_{J,0}} \left[ k_{\Delta+J}(z)k_{\Delta-J}(\bar{z}) + k_{\Delta-J}(z)k_{\Delta+J}(\bar{z}) \right], \quad d = 2,
\]

\[
G_{\Delta,J}(z,\bar{z}) = \frac{z\bar{z}}{z - \bar{z}} \left[ k_{\Delta+J}(z)k_{\Delta-J-2}(\bar{z}) - k_{\Delta-J-2}(z)k_{\Delta+J}(\bar{z}) \right], \quad d = 4,
\]

\[
k_\alpha(x) = x^{\alpha/2} F(\alpha/2, \alpha/2, \alpha, x).
\]

(1.27)

Alternatively, we can expand the four-point function \[1.23\] into an orthogonal basis of eigenfunctions \(F_{\Delta,J}\) of the Casimir operator. The single-valued functions \(F_{\Delta,J}\), also called partial waves, are given by a linear combination of conformal block plus “shadow”

\[
F_{\Delta,J} = K_{J,\Delta} G_{\Delta,J} + K_{J,d-\Delta} G_{d-\Delta,J}
\]

(1.28)

where we defined following \[27\]

\[
K_{J,\Delta} = \frac{\Gamma(\Delta - 1)}{\Gamma(\Delta - d/2)} \kappa_{J+\Delta}, \quad \kappa_\beta = \frac{\Gamma\left(\frac{\beta}{2}\right)^4}{2\pi^2 \Gamma(\beta - 1) \Gamma(\beta)}.
\]

(1.29)

A complete set of square-integrable functions\[8\] is spanned by all \(F_{\Delta,J}\) with inte-

\[8\]See e.g. appendix A in \[34\].
ger spins $J$ and complex dimensions $\Delta = \frac{d}{2} + i\nu$, where $\nu$ is real and $\nu \geq 0$.

Therefore, we can expand the four-point function \[1.23\] as

\[G(z, \bar{z}) = \sum_{J=0}^{\infty} \int_{d/2}^{d/2+i\infty} \frac{d\Delta}{2\pi i} c_J(\Delta) F_{\Delta,J}(z, \bar{z}) = \sum_{J=0}^{\infty} \int_{d/2-i\infty}^{d/2+i\infty} \frac{d\Delta}{2\pi i} c_J(\Delta) K_{J,\Delta} G_{\Delta,J}(z, \bar{z}) \] \[1.30\]

Due to the shadow symmetry of the partial wave $F_{d-\Delta,J} = F_{\Delta,J}$ the partial wave coefficients $c_J(\Delta)$ are also shadow symmetric.

\[c_J(\Delta) = c_J(d - \Delta). \quad (1.31)\]

To relate the partial wave decomposition \[1.30\] to the $s$-channel OPE expansion \[A.6\], we can deform the contour in \[1.30\] to the real axis. The OPE expansion is reproduced if $c_J(\Delta)$ has poles at dimensions of operators appearing in the $s$-channel OPE with residues related to OPE coefficients. The precise rela-

\[9\text{More precisely, one should also add non-normalizable modes, coming from}\text{s-channel}\text{scalar operators with } \Delta \leq d/2, \text{as described in appendix B in } [34]. \text{Here, we will be interested in s-channel operators with } \Delta > d/2, \text{so we ignore these contributions.}\]

\[10\text{Our } c_J(\Delta) \text{ is related to } c(J, \Delta) \text{ in } [27] \text{ by } c_J(\Delta) \equiv \frac{c(J, \Delta)}{K_{J,\Delta}}.\]
tion was given in [27]

\[
p_{\Delta,J}^{\Delta,J} = -\text{Res}_{\Delta' \to \Delta} \begin{cases} 
  c_J(\Delta'), & \Delta \neq \Delta_n \\
  c_J(\Delta') - r_{J,\Delta'} \frac{K_{\Delta'+1-d,J+1-d-1}}{K_{J,\Delta'}} c_{\Delta'+1-d}(J + d - 1), & \Delta = \Delta_n
\end{cases}
\]

(1.32)

where we introduced

\[
\Delta_n = J + d + n, \quad n = 1, 3, 5, \ldots
\]

(1.33)

and \( r_{J,\Delta} \) is defined by \((x = \Delta - J - d + 2)\)

\[
r_{J,\Delta} = \frac{\Gamma(\Delta - 1)\Gamma(\Delta + 2 - d) \Gamma\left(J + \frac{d-2}{2}\right) \Gamma\left(J + \frac{d}{2}\right) \Gamma(2 - x)\Gamma\left(\frac{x}{2}\right)^2}{\Gamma(\Delta - \frac{d}{2}) \Gamma\left(\Delta - \frac{d-x}{2}\right) \Gamma(J + 1)\Gamma(J + d - 2) \Gamma(x)\Gamma\left(\frac{2-x}{2}\right)^2}
\]

(1.34)

The reason there is an extra term in (1.32) when \( \Delta = \Delta_n \) is that the conformal blocks in (1.30) have poles on the real \( \Delta \) axis. Their contributions are cancelled by extra poles of \( c_J(\Delta) \), which do not correspond to operators appearing in the OPE. These unphysical extra poles are explicitly subtracted in the second line of (1.32), as explained in [27].

\[11\text{For non-identical external operators there will be poles for all integer } n.\]
Since the functions $F_{\Delta,J}$ form an orthogonal basis, one can invert \ref{eq:1.30} and obtain the Euclidean inversion formula. Further, it is possible to deform the contour of integration in $z, \bar{z}$ to Lorentzian kinematics, which entails Caron-Huot’s inversion formula \cite{27}. In the case of identical external operators it is given by

$$c_J(\Delta) = \frac{1}{2} \delta_{J,\text{even}} \frac{\Gamma\left(\Delta - \frac{d}{2}\right)}{\Gamma(\Delta - 1)} \int_0^1 dzd\bar{z} \frac{\mu(z, \bar{z})}{2} G_{J+d-1,\Delta+1-d}(z, \bar{z}) \, \text{dDisc} \, G(z, \bar{z})$$

(1.35)

where $\mu(z, \bar{z})$ is the orthogonality measure of partial waves $F_{\Delta,J}$

$$\mu(z, \bar{z}) = \frac{1}{(z\bar{z})^2} \frac{|z - \bar{z}|^{d-2}}{z\bar{z}}$$

(1.36)

The double-discontinuity is defined \cite{27} by

$$\text{dDisc} \, G(z, \bar{z}) = G_{\text{eucl}}(\rho, \bar{\rho}) - \frac{1}{2} G(\rho, \bar{\rho} - i0) - \frac{1}{2} G(\rho, \bar{\rho} + i0) ,$$

$$G_{\text{eucl}}(\rho, \bar{\rho}) = G(\rho, 1/\bar{\rho}), \quad z = \frac{4\rho}{(1 + \rho)^2} .$$

(1.37)

In a generic CFT the derivation of \ref{eq:1.35} applies only to $J > 1$.

To recapitulate, $c_J(\Delta)$ are meromorphic shadow symmetric functions. They

\footnote{Recall that we write $G_{\Delta,J}$ for conformal blocks instead of the convention in \cite{27} $G_{J,\Delta}$.}
have poles at the positions of physical operators appearing in the $s$-channel OPE as well as a series of extra (kinematical) poles as dictated by (1.32). For $J > 1$ $c_J(\Delta)$ could be computed from the double discontinuity of the correlator using the inversion formula (1.35). In the next subsection we will show that a direct consequence of (1.35) is polynomial boundedness of $c_J(\Delta)$ at large $\Delta$. This allows us to write and study dispersion relations for $c_J(\Delta)$.

### 1.2.2 Polynomial Boundedness

We will be interested in the behavior of $c_J(\Delta)$ at large $\Delta$ in the complex plane and will observe that this limit in (1.35) is controlled by the Euclidean OPE expansion of the correlator in the $t$-channel $\Lambda_6$.

The simplest example of the use of (1.35) is to consider the unit operator in the $t$-channel

$$G(z, \bar{z}) = \left(\frac{z\bar{z}}{(1 - z)(1 - \bar{z})}\right)^{\Delta_\phi} + \ldots$$  \hspace{1cm} (1.38)$$

It gives a contribution to the double-discontinuity

$$\text{dDisc } G(z, \bar{z}) = 2\sin^2(\pi\Delta_\phi) \left(\frac{z\bar{z}}{(1 - z)(1 - \bar{z})}\right)^{\Delta_\phi}.$$  \hspace{1cm} (1.39)$$
We plug $1.39$ into the inversion formula $1.35$. The corresponding partial wave coefficients are given by

$$c_{J}^{\text{GFF}}(\Delta) = [1 + (-1)^{J}]\pi^{2} \frac{\Gamma(J + \frac{d}{2}) \Gamma(\frac{d}{2} - \Delta_{\phi})^{2}}{\Gamma(J + 1) \Gamma(\Delta_{\phi})^{2}} \times \frac{\Gamma(\Delta + J) \Gamma\left(\frac{-\Delta + 2\Delta_{\phi} + J}{2}\right)}{\Gamma\left(\frac{\Delta + J}{2}\right) \Gamma\left(\frac{\Delta + J}{2} - \Delta_{\phi} + \frac{d}{2}\right)} \times \frac{\Gamma(\tilde{\Delta} + J) \Gamma\left(\frac{-\tilde{\Delta} + 2\Delta_{\phi} + J}{2}\right)}{\Gamma\left(\frac{\tilde{\Delta} + J}{2}\right) \Gamma\left(\frac{\tilde{\Delta} + J}{2} - \Delta_{\phi} + \frac{d}{2}\right)}$$

(1.40)

where $\tilde{\Delta} \equiv d - \Delta$ is the shadow transform of $\Delta$. Incidentally, this is also the exact answer for the generalized free field theory (GFF). Indeed, the first two terms in the GFF 4-point function $G_{\text{GFF}} = 1 + (z\bar{z})^{\Delta_{\phi}} + \left[\frac{(z\bar{z})^{\Delta_{\phi}}}{(1 - z)(1 - \bar{z})}\right]^{\Delta_{\phi}}$ do not contribute to the double discontinuity.

The partial wave $1.40$ has simple poles at $\Delta = 2\Delta_{\phi} + J + 2n$ corresponding to double-trace operators in GFF, $13$ at $\Delta = \Delta_{n} 1.33$ and at their shadows. The poles at $\Delta = \Delta_{n}$ cancel in the physical combination $1.32$, as expected.

In $d = 2, 4$ the conformal blocks are known explicitly $1.27$ and the integrals in the inversion formula $1.35$ can be taken explicitly $14$. For general $d$ we simply guessed the formula $1.40$ by requiring that unphysical poles at $\Delta = \Delta_{n}$ cancel in the combination $1.32$ and that the residues reproduce correct 3-point functions.

$13$These are physical only in the theory of generalized free fields. In a generic CFT they are spurious and their proper treatment is the subject of the analytic Lorentzian bootstrap.

$14$This can be done using Euler type integral representation of the hypergeometric function.
of GFF [35].

Let us understand how $c_J(\Delta)$ behaves at large $\Delta$. In the upper half-plane we have from \[1.40\]

$$c_J(i\Delta) = d_J \Delta^{4\delta_\phi-3} + \ldots, \quad \Delta \to +\infty, \quad |\arg \Delta| < \frac{\pi}{2}, \quad (1.41)$$

where we introduced

$$\delta_\phi = \Delta_\phi - \frac{d-2}{2} \geq 0, \quad d_J = [1 + (-1)^J] \pi \frac{2^{2J+d-4\delta_\phi+1} \Gamma(J + \frac{d}{2}) \Gamma(\frac{d}{2} - \Delta_\phi)^2}{\Gamma(J+1) \Gamma(\Delta_\phi)^2}. \quad (1.42)$$

The bound on $\delta_\phi$ is the usual unitarity bound. In the lower-half plane the asymptotic can be obtained from \[1.41\] by shadow symmetry. The crucial observation is that including heavier operators in the $t$-channel would give rise to suppressed contributions in \[1.39\]. This is because they come with higher powers of $(1 - z)$.

Equivalently, we can expand in $\frac{1-z}{z}$ in the $t$-channel. Then extra powers of $\frac{1-z}{z}$ result in decreasing $\Delta_\phi$ in \[1.39\] and, therefore, a suppressed contribution in \[1.41\].

This argument is rigorous for scalar operators in the $t$-channel, for which $1 - z \quad (15)\text{See formula (43) in that paper. To translate to our normalization of 3-point functions, one has to divide their formula (43) by a factor $C_\Delta, C_{\Delta_1}$. Further, for identical operators one has to add a permutation term in their formula (37), which leads to an extra factor of 2 in (43).}
and $1 - \zbar$ come in the same powers. For spinning operators it is slightly less obvious and we will discuss it further in section 3.

The power-like expansion $1.41$ breaks down close to the real axis. One way to see it is to observe that there are nonperturbative corrections to $1.41$ coming from the expansion of $\Gamma$-functions in $1.40$. These non-perturbative corrections become negligible for large $\text{Im}(\Delta)$. We will assume that this is completely general and the power-like expansion of conformal partial waves, that we get by inserting the $t$-channel expansion into the inversion formula, is valid in the complex plane as soon as $\text{Im}(\Delta) \gtrsim |\Delta|^{\epsilon}$ for any positive fixed $\epsilon$.

Below, when analyzing dispersion relations, we will also need the asymptotic behavior of $1.40$ at large $J$ and fixed $\Delta$. We have from $1.40$

$$c_J(\Delta) = [1 + (-1)^J] \frac{2^{2J + d - 4\delta_{\phi}} + 1 \Gamma \left( \frac{d}{2} - \Delta_{\phi} \right)^2}{\Gamma(\Delta_{\phi})^2} J^{4\delta_{\phi} + \frac{d}{2} - 4} + \ldots, \quad J \to \infty .$$

(1.43)

As above, including heavier operators leads to terms in $1.43$ which are suppressed by further powers of $\frac{1}{J}$.

Let us emphasize that the large spin expansion of $1.43$ is different from the usual analytic Lorentzian bootstrap discussions. The latter corresponds to keeping the twist $\Delta - J$ fixed while taking the large spin limit. Here we are exploring
the unphysical regime of large $J$ and fixed $\Delta$. The claim is that this limit is controlled by the Euclidean rather than Lorentzian OPE.

1.2.3 Dispersion Relations

Having a meromorphic and polynomially bounded function $c_J(\Delta)$, it is natural to write down a dispersion relation. Consider a Cauchy integral

$$c_J(\Delta) = \oint \frac{d\Delta'}{2\pi i} \frac{c_J(\Delta')}{\Delta' - \Delta},$$

where the contour goes around $\Delta' = \Delta$ counterclockwise. This dispersion relation was briefly considered in [36], see section 2.5.1 in that paper.

We can deform the contour as indicated on the figure [1.1]. The arcs at infinity can be dropped if $c_J(\Delta) \to 0$ as $|\Delta| \to \infty$. Otherwise, we can write a dispersion relation with subtractions by taking an appropriate number of derivatives of

$$\frac{1}{N!} \partial^N_{\Delta} c_J(\Delta) = \oint \frac{d\Delta'}{2\pi i} \frac{c_J(\Delta')}{(\Delta' - \Delta)^{N+1}},$$

(1.45)

Since $c_J(\Delta) \sim \Delta^{4d_\phi - 3}$ as $\Delta \to \infty$, the contribution of arcs vanishes if we choose...
Figure 1.1: Dispersion relations in $\Delta$ plane. We consider the contour integral \[1.44\] and deform the contour in the usual way.

\[ N = \begin{cases} 
[4\delta_{\phi}] - 2, & \delta_{\phi} \geq \frac{3}{4} \\
0, & 0 \leq \delta_{\phi} < \frac{3}{4} 
\end{cases} \]  

(1.46)

For simplicity, let us first describe the dispersion relation \[1.44\] for $0 \leq \delta_{\phi} < \frac{3}{4}$, from which it will be trivial to generalize to the dispersion relation with subtractions \[1.45\] for $\delta_{\phi} \geq \frac{3}{4}$.

For $0 \leq \delta_{\phi} < \frac{3}{4}$ we can use the equation \[1.44\]. Deforming the contour in \[1.44\] to the real axis, dropping the arcs at infinity and using the shadow symmetry
we obtain

\[ c_J(d/2 + i\nu) = \int_0^\infty d\nu' \frac{2\nu'}{\nu'^2 + \nu^2} \rho_J(d/2 + \nu') \tag{1.47} \]

where we shifted variables as \( \Delta = \frac{d}{2} + i\nu, \Delta' = \frac{d}{2} + \nu' \) and introduced a spectral density

\[ \rho_J(\Delta) = -\sum_{\Delta_{\text{pole}}} \delta(\Delta - \Delta_{\text{pole}}) \text{Res}_{\Delta \to \Delta_{\text{pole}}} c_J(\Delta) \tag{1.48} \]

where the sum is over all poles of the partial wave coefficient \( c_J(\Delta) \). The partial wave coefficient, however, includes not only the OPE poles, but also extra poles \( 1.33 \) at \( \Delta_n = J + d + n \), while the OPE data is encoded in the combinations \( 1.32 \).

Thus, we can relate \( 2.13 \) to the spectral density of OPE coefficients \( \rho_J^{\text{OPE}}(\Delta) \) by

\[ \rho_J(\Delta) = \rho_J^{\text{OPE}}(\Delta) - \rho_J^{\text{extra}}(\Delta), \tag{1.49} \]

where we defined

\[ \rho_J^{\text{OPE}}(\Delta) \equiv \sum_{\Delta'} \frac{p_{\Delta',\Delta}}{K_{J,\Delta'}} \delta(\Delta - \Delta'), \]

\[ \rho_J^{\text{extra}}(\Delta) \equiv \sum_{n=0}^\infty \delta_{n,\text{odd}} \delta(\Delta - \Delta_n) \frac{K_{J+n+1,J+d-1}}{K_{J,J+d+n}} \text{Res}_{\Delta \to \Delta_n} (r_{J,\Delta}) c_{J+n+1}(J + d) \tag{1.50} \]
and the sum in $\rho_j^{OPE}$ is only over primary operators of spin $J$ appearing in the OPE. We also used that $K_{J+1,J+d+1}$ and $c_{J+n+1}(J + d - 1)$ are non-singular, so that $\text{Res}$ acts only on $r_{J,\Delta}$. Inserting 1.49 into 1.47 we arrive at the desired dispersion relation

$$\int_0^\infty d\nu' \rho_j^{OPE}(d/2 + \nu') \frac{2\nu'}{\nu'^2 + \nu^2} = c_J(d/2 + iv) + \text{extra} \quad (1.51)$$

where we defined

$$\text{extra} = \sum_{n=0}^{\infty} \delta_{n,\text{odd}} \frac{2(J + n) + d}{(J + \frac{d}{2} + n)^2 + \nu^2} \frac{K_{J+n+1,J+d-1}}{K_{J,J+d+n}} \text{Res}_{\Delta \to \Delta_n} (r_{J,\Delta}) c_{J+n+1}(J + d - 1) \quad (1.52)$$

and using definitions 1.29, 1.34 we can also compute

$$\frac{K_{J+n+1,J+d-1}}{K_{J,J+d+n}} \text{Res}_{\Delta \to \Delta_n} (r_{J,\Delta}) = \frac{(-1)^{n+1}}{n+1} \left[ \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\Gamma \left( -\frac{n}{2} \right) \Gamma(n+1)} \right]^2 \frac{(J + 1)_{n+1}}{(J + \frac{d}{2})_{n+1}} \quad (1.53)$$

where the Pochhammer symbol is $(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$. For the remainder of this section we will be preoccupied with studying the dispersion relation 1.51. It can be considered as a reformulation of $s = t$ crossing. Indeed, the LHS of 1.51 con-
tains $s$-channel OPE data of primary operators with fixed spin $J$. The kernel $\frac{2\nu'}{\nu'^2 + \nu^2}$ is centered around operators with dimension $\Delta' \sim \frac{d}{2} + \nu$. The RHS of (1.51) can be thought of as $t$-channel data if we use $t$-channel OPE and the inversion formula (1.35) to compute $c_{J}(\Delta)$ entering the RHS of (1.51).

As it is usual with the crossing equations, we cannot solve (1.51) for general values of parameters $(\nu, J)$. So let us consider a limit when only a few light operators dominate in one of the channels to make predictions for the other channel. We will take $\nu \gg 1$. In this limit the RHS of (1.51) will be dominated by light operators in the $t$-channel. This is reminiscent of the fact that large momentum corresponds to short distances in Fourier transforms. We now explain how it happens and find the asymptotic of the RHS of (1.51) at large $\nu$.

The asymptotic $\nu \gg 1$ of the first term in the RHS of (1.51) was already found in (1.41) and was indeed controlled by the identity operator in the $t$-channel. The asymptotic of the second term in the RHS of (1.51) is more subtle and we will study it in the next subsection.

For completeness let us also write down the most general dispersion relation that involves $N$ subtractions. Starting from (1.45) and going through the same
steps we have

\[
\int_{0}^{\infty} dv' \rho_{J}^{\text{OPE}}(d/2 + \nu') \left[ \frac{e^{\frac{\pi i}{N} N}}{(\nu' - i\nu)^{N+1}} + \text{c.c.} \right] = \frac{1}{N!} \partial_{\nu}^{N} c_{J}(d/2 + i\nu) + \\
+ \sum_{n=0}^{\infty} \left[ \frac{e^{\frac{\pi i}{N} N}}{(\Delta_{n} - \frac{d}{2} - i\nu)^{N+1}} + \text{c.c.} \right] \\
\times \text{Res}_{\Delta \rightarrow \Delta_{n}}(r_{J,\Delta}) \frac{K_{J+n+1,J+d-n-1}}{K_{J,J+d+n}} c_{J+n+1}(J + d - 1) .
\]

(1.54)

This equation can also be obtained by taking $N \nu$-derivatives of (1.51), but unlike (1.51) it is valid for any $\delta_{\phi}$ if we chose $N$ as in (1.46).

1.2.4 Contribution of Extra Poles

To make the dispersion relations (1.51), (1.54) useful we need to say something about the contribution of extra poles (1.52). Here we compute the large $n$ tails of the sum (1.52). These tails are fixed by the Euclidean $t$-channel OPE, since the large $n$ asymptotics of $c_{J+n+1}(J + d - 1)$ is controlled by the large spin expansion (1.43). The tails generate generic non-integer powers in the large $\nu$ expansion. In contrast, any fixed $n$ term in (1.52) is non-universal. It generates terms of the type $\frac{1}{\nu^{2k}}$ at large $\nu$. These are mapped to the contribution of individual operators in the $s$-channel.

Let us see how this works in details. We write the contribution of extra poles
as

\[
\text{extra} = \sum_{n=0}^{\infty} \delta_{n,\text{odd}} E_n \left( \frac{1}{\nu_n + i\nu} + \frac{1}{\nu_n - i\nu} \right), \tag{1.55}
\]

where

\[
\Delta_n = \frac{d}{2} + \nu_n = J + d + n,
\]

\[
E_n = \frac{1}{n+1} \left[ \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\Gamma \left( -\frac{n}{2} \right) \Gamma(n+1)} \right]^2 \frac{(J+1)_{n+1}}{(J + \frac{d}{2})_{n+1}} c_{J+n+1}(J + d - 1). \tag{1.56}
\]

We are interested in the large \(n\) tails of the sum, so we expand each term as

\[
E_n = \sum_{\chi} n^{\gamma_{\chi} - 1} \sum_{j=0}^{\infty} \frac{e_{j,\chi}^{(n)}}{n^j} \equiv \sum_{\chi} E_{n,\chi}, \tag{1.57}
\]

where we also sum over contributions of primary operators \(\chi\) in the \(t\)-channel to the inversion formula \[1.35\] for \(c_{J+n+1}(J + d - 1)\). For example, using \[1.43\] for the unit operator we have

\[
\chi = \hat{1}: \quad \gamma_1 = 4\delta_\phi - 2, \quad e_0^{(1)} = 2^{-\gamma_1 + J + d + 1} \frac{\Gamma \left( J + \frac{d}{2} \right) \Gamma \left( \frac{d}{2} - \Delta_\phi \right)}{\Gamma(J + 1) \Gamma(\Delta_\phi)} \tag{1.58}
\]
Let us compute the contribution of a single $t$-channel primary operator $\chi$

$$\text{extra}(\chi) \equiv \sum_{n=0}^{\infty} \delta_{n,\text{odd}} E_n^{(\chi)} \left( \frac{1}{\nu_n + i\nu} + \frac{1}{\nu_n - i\nu} \right) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (2k + 1)^{\gamma_{x-1-j}} \sum_{n=0}^{\infty} \delta_{n,\text{odd}} E_n^{(\chi)} \left( \frac{1}{\nu_n + i\nu} + \frac{1}{\nu_n - i\nu} \right) + c.c. =$$

$$= \sum_{j=0}^{\infty} e_j^{(\chi)} \sum_{k=0}^{\infty} (2k + 1)^{\gamma_{x-1-j}} \int_0^\infty dt e^{-t(\nu_{2k+1} + i\nu)} + c.c. =$$

$$= 2 \sum_{j=0}^{\infty} e_j^{(\chi)} \int_0^\infty dt \cos(t\nu) e^{-t(\nu_{2k+1} + i\nu)} \sum_{k=0}^{\infty} (2k + 1)^{\gamma_{x-1-j}} e^{-2tk}$$

(1.59)

The sum over $k$ is given by the so called Lerch transcendent

$$\sum_{k=0}^{\infty} (2k + 1)^{\gamma_{x-1-j}} e^{-2tk} = 2^{\gamma_{x-1-j}} \Phi(e^{-2t}, -\gamma_x + 1 + j, 1/2)$$

(1.60)

where by definition

$$\Phi(z, s, a) \equiv \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s}$$

(1.61)

For large $\nu$ the integral in (1.59) localizes to $t = 0$. Lerch transcendant has a useful expansion around this point

$$\Phi(z, s, a) = z^{-a} \left( \Gamma(1-s)(-\log z)^{s-1} + \sum_{k=0}^{\infty} \zeta(s-k,a) \frac{\log z^k}{k!} \right)$$

(1.62)
where $\zeta(s, a) = \sum_{k=0}^{\infty} (a + k)^{-s}$ is Hurwitz zeta function. Using this expansion we get

$$
\text{extra}(\chi) = \sum_{j=0}^{\infty} e_j^{(\chi)} \Gamma(\gamma\chi - j) \left( J + \frac{d}{2} \right)^{-\gamma\chi - 1 - j} \mathcal{F}_{j-\gamma\chi} \left( \frac{\nu}{J + \frac{d}{2}} \right) + 
$$

$$
+ \sum_{k=0}^{\infty} e_k^{(\chi)} \frac{(-2)^k}{k!} \left( J + \frac{d}{2} \right)^{-(k+1)} \mathcal{F}_k \left( \frac{\nu}{J + \frac{d}{2}} \right) \tag{1.63}
$$

where we defined

$$
e_k^{(\chi)} = \sum_{j=0}^{\infty} e_j^{(\chi)} 2^{\gamma\chi - j} \zeta(-\gamma\chi + 1 + j - k, 1/2)
$$

$$
\mathcal{F}_{s-1}(\nu) = \int_0^{\infty} dt \cos(t\nu) e^{-t}s-1 = \frac{\Gamma(s) \cos(s \arctan \nu)}{(1 + \nu^2)^{s/2}} \tag{1.64}
$$

The function $\mathcal{F}_s(\nu)$ can be expanded at large $\nu$ by changing the integration variable in \ref{1.64} to $x = t\nu$ and expanding the exponent. The result is

$$
\mathcal{F}_{s-1}(\nu) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(n + s) \cos \left( \frac{\pi}{2} (n + s) \right) \frac{1}{\nu^{s+n}} \tag{1.65}
$$

The expansion \ref{1.65} makes it clear that the first line in \ref{1.63} gives the universal tails controlled by the $t$-channel which are generically non-integer powers of $\frac{1}{\nu}$. 42
This happens because heavier operators in the $t$-channel have smaller $\gamma_\chi$ and therefore give suppressed contributions in $\ref{1.63}$. The second line in $\ref{1.63}$ makes it clear that each operator contributes to all integer powers of $\frac{1}{\nu}$. Using $\ref{1.65}$ we can write an expansion at large $\nu$

$$\text{extra}(\chi) =$$
$$\nu^{\gamma_\chi - 1} \sum_{n=0}^{\infty} \frac{(-1)^n}{\nu^n} \left( J + \frac{d}{2} \right)^n \Gamma(n - \gamma_\chi + 1) \cos \frac{\pi}{2} (n - \gamma_\chi + 1) \sum_{j=0}^{n} e_j^{(\chi)} \frac{(-1)^j \Gamma(\gamma_\chi - j)}{(J + \frac{d}{2})^j (n - j)!}$$

+ (even integer powers) ,

(1.66)

where $e_j^{(\chi)}$ and $\gamma_\chi$ are defined by the expansion $\ref{1.57}$. Note that due to shadow symmetry we expect all the odd $n$ powers in the sum to cancel. In all concrete computations it is indeed what happens. Let us quote the leading universal term in $\ref{1.66}$ from the contribution of the unit operator $\ref{1.58}$, which we will use later

$$\text{extra}(\hat{1}) = -\frac{1}{\cos 2\pi \delta_\phi} d_J \nu^{4\delta_\phi - 3} + ... ,$$

(1.67)

where $d_J$ is given by $\ref{1.42}$

\text{16This can be explicitly checked in } d = 2, 4 \text{ by computations outlined in section 3.
For $\delta_\phi \geq \frac{3}{4}$, when we have to use the dispersion relation with subtractions, we can obtain the asymptotics of the sum on the RHS of 1.54 by taking $N \nu$-derivatives of 1.66.

The $t$-channel information 1.66, though non-trivial, is impossible to interpret in terms of individual primary operators and three-point functions in the $s$-channel. The kernel $\frac{2\nu'}{\nu'^2+\nu^2}$ in 1.51 is centered around $\nu$, but has tails going to arbitrarily large $\nu'$. However, it turns out that for the large $\nu$ asymptotic it is possible to get rid of this tail and replace the kernel by the indicator function

$$\theta(0 \leq \nu' \leq \nu) = \begin{cases} 1, & 0 \leq \nu' \leq \nu \\ 0, & \text{otherwise} \end{cases}$$

(1.68)

using a certain tauberian theorem. This will give us a more direct probe of the spectral data in the $s$-channel and will be the topic of section 4.

1.3 Adding Extra Operators to The Large $\Delta$ Expansion

So far we have explicitly discussed only the contribution of the unit operator in the $t$-channel. Let us briefly generalize the discussion to an arbitrary operator in the $t$-channel.

The basic feature of the large $\nu$ expansion is that non-analytic tails $\frac{1}{\nu'^2}$ with
generically non-integer $\alpha$ are controlled by the $t$-channel OPE and are, thus, computable. The analytic terms $\frac{1}{\nu \pi}$ with integer $k$, on the other hand, come from individual operators in the $s$-channel and are non-universal.

Let us consider the problem of adding an extra operator in the $t$-channel. It is convenient not to distinguish between the primaries and descendants. The contribution of a state with quantum numbers $(h, \bar{h})$ to the correlator is given by

$$G(z, \bar{z}) = \left(\frac{z\bar{z}}{(1 - z)(1 - \bar{z})}\right)^{\Delta_{\phi}} (1 - z)^h (1 - \bar{z})^{\bar{h}}.$$ (1.69)

Its contribution to the double discontinuity is given by

$$\text{dDisc } G(z, \bar{z}) = 2 \sin^2(\pi(\Delta_{\phi} - \bar{h})) \left(\frac{z\bar{z}}{(1 - z)(1 - \bar{z})}\right)^{\Delta_{\phi}} (1 - z)^h (1 - \bar{z})^{\bar{h}}. \quad (1.70)$$

We would like to plug this into the inversion formula and study the result in the large $\Delta$ limit. To be able to compute integrals in the inversion formula explicitly

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it will be convenient to expand in $\frac{1-z}{z}$ instead of $1-z$

$$(1-z)^h = \left(\frac{1-z}{z}\right)^h \left(1 + \frac{1-z}{z}\right)^{-h} = 
= \left(\frac{1-z}{z}\right)^h \sum_{n=0}^{\infty} \frac{\Gamma(1-h)}{\Gamma(n+1)\Gamma(1-h-n)} \left(\frac{1-z}{z}\right)^n. \quad (1.71)$$

We will see that this expansion translates into $\frac{1}{\Delta}$ expansion.

Let us consider the case $d = 2$. The case of $d = 4$ is similar. We believe the same picture persists in all $d$, even though we have not proved that. In the inversion formula we are interested in the following integral

$$I_{p_0}^{p_1}(h) = \int_0^1 \frac{dz}{z^2} k(h)(z)z^{p_0}(1-z)^{p_1} = \frac{\Gamma(p_0 + \frac{h}{2} - 1)\Gamma(p_1 + 1)}{\Gamma(p_0 + p_1 + \frac{h}{2})} \times
\times _3F_2 \left( \frac{h}{2}, \frac{h}{2}, p_0 + \frac{h}{2} - 1; h, p_0 + p_1 + \frac{h}{2}; 1 \right). \quad (1.72)$$

In terms of $I_{p_1}(h)$ we get the following contribution to the partial wave from

$$c_J^{h,\tilde{h}}(\Delta) = \sin^2(\pi(\Delta_{p} - \bar{h}))) \left[ I_{h-\Delta_{p}}^{\Delta_{p}}(\Delta + J)I_{\bar{h}-\Delta_{p}}^{\Delta_{p}}(\bar{\Delta} + J) + (\Delta \to \bar{\Delta}) \right] \quad (1.73)$$

where $\tilde{\Delta} = 2 - \Delta$ is the shadow of $\Delta$ in 2d. We are interested in the large $h$
limit of $I_{p_0}^{p_0}(h)$. For this purpose it is convenient to introduce a simpler integral

$$I_{p}(h) = \int_{0}^{1} \frac{dz}{z^2} k_h(z) \left( \frac{z}{1-z} \right)^p = \frac{\Gamma(h)\Gamma(h/2 + p - 1)\Gamma(1-p)^2}{\Gamma(h/2)^2\Gamma(h/2 - p + 1)}.$$  \hfill (1.74)

In terms of this simpler integral and using (1.71) we can write

$$I_{p_0}^{p_0}(h) = \sum_{n=0}^{\infty} I_{-p_1-n}(h) \frac{(-1)^n \Gamma(p_0 + p_1 + n)}{\Gamma(p_0 + p_1)\Gamma(n + 1)}.$$  \hfill (1.75)

Expanding (1.74) at large $h$ the formula (1.75) provides us with an expansion at large $h$ for $I_{p_0}^{p_0}(h)$. Now we can readily compute the large $\Delta$ expansion of $c_{J,\bar{J}}^{h,\bar{h}}(\Delta)$ (1.73). The leading term takes the form

$$c_{J,\bar{J}}^{h,\bar{h}}(1 + i\nu) = \pi 4^{2+h+J-2\Delta_\phi} \nu^{4\Delta_\phi - 3 - 2(h+\bar{h})} \cos(\pi(h - \bar{h})) \frac{\Gamma(1 + h - \Delta_\phi)^2}{\Gamma(\Delta_\phi - \bar{h})^2} + \ldots .$$  \hfill (1.76)

Given a primary operator in the $t$-channel with dimension $\Delta_\chi$ and spin $J_\chi$ its $t$-channel conformal block involves terms $h = \frac{\Delta_\chi + J_\chi}{2}$ and $\bar{h} = \frac{\Delta_\chi + J_\chi}{2}$ together with corrections which are integer powers $(1 - z)^n(1 - \bar{z})^m$.

One can wonder about the convergence of the $t$-channel OPE after we applied the inversion formula. If one naively expands (1.76) at large $\Delta_\chi$ one gets that it diverges very quickly $e^{\Delta_\chi \log \Delta_\chi}$. This however only signifies that the large $\nu$ and
large $\Delta_\chi$ limits do not commute. Taking the full answer \ref{1.73} (instead of the first term in \ref{1.75}) one can check that it behaves at large $\Delta_\chi$ like a power. The situation is similar to the one in the context of the large spin expansion \cite{37,38}.

From the discussion above the contribution of an operator $\chi$ to the large $\nu$ expansion takes the form

$$c_{J,J}^{h,h}(d/2 + i\nu) = \nu^{4\delta_\chi - 3 - 2\Delta_\chi} \sum_{n=0}^{\infty} \frac{\alpha_n^{h,h}}{\nu^{2n}}, \quad (1.77)$$

where all coefficients are computable using the Lorentzian inversion formula. Moreover, the contribution of heavier operators $\chi$ in the $t$-channel is suppressed at large $\nu$, as can be seen from \ref{1.77}.

1.3.1 LARGE $\nu$ SUMMARY

Based on the discussion in the previous subsection we end up with the following dispersion relation at large $\nu$

$$\int_0^\infty dv' \rho^{OPE}_J \left( \frac{d}{2} + v' \right) \frac{2v'v}{v'^2 + v^2} = \sum_{\chi} \sum_{n=0}^{\infty} \alpha_n^{(\chi)} \nu^{-\delta_\chi - 2n} + \sum_{k=1}^{\infty} a_k \nu^{-2k+1} \quad (1.78)$$

where $\chi$ are $t$-channel operators contributing to the RHS of the dispersion relation \ref{1.51} via the inversion formula. The numbers $\alpha_n^{(\chi)}$ and $\delta_\chi$ are computable
using $t$-channel OPE, while $a_k$ receive contributions from all operators in the $t$-channel and, therefore, are non-universal. The same conclusion holds for dispersion relations with subtractions. We would like to use (1.78) to make some predictions about the asymptotic behavior of the spectral density itself. This is the subject of tauberian theorems. We discuss this in the next section.

Collecting formulas (1.41), (1.67) we find the leading contribution of the unit operator to the RHS of the dispersion relation

$$
\chi = \hat{1}: \quad \alpha_0^{(i)} = d_J \left( 1 - \frac{1}{\cos 2\pi \delta \phi} \right), \quad \delta_{\hat{1}} = -4\delta \phi + 2 \quad (1.79)
$$

where $d_J$ is defined in (1.42). From (1.77) we conclude that in 2d the powers $\delta_\chi$ are related to the operators $\chi$ in the $t$-channel OPE as follows

$$
\delta_\chi = -4\delta \phi + 2 + 2\Delta_\chi \quad (1.80)
$$

One can check that the same formula holds in 4d. Note that the LHS of (1.78) has an extra factor of $\nu$ compared to (1.51). This is the source of an extra $+1$ in (3.54).
1.4 Asymptotics of Spectral Densities

We would like to use dispersion relations to make predictions about the integrated weighted spectral density. This is done via a complex tauberian theorem. It will be again useful to introduce Cauchy moments

\[ F_m^J(\nu) \equiv \frac{1}{(m-1)!} \int_0^{\nu} (\nu - \nu')^{m-1} \rho_{J}^{\text{OPE}}(d/2 + \nu') . \tag{1.81} \]

These are also computed by the repeated integration of the weighted spectral density \( \rho_{J}^{\text{OPE}}(\nu) \) as in \(1.14\). Now let us consider the dispersion relation \(1.78\)

\[ \int_0^{\infty} d\nu' \rho_{J}^{\text{OPE}}(d/2 + \nu') \frac{2\nu'\nu}{\nu'^2 + \nu^2} = R(\nu) = \sum_i \alpha_i \nu^{-\delta_i} + \sum_{k=1}^{\infty} a_k \nu^{-2k+1} \tag{1.82} \]

where we simply use the notation \( \delta_i \) with the index \( i \) containing information about both \( \chi \) and \( n \) in \(1.78\) (which receives contribution from the \( n \)-th level descendants). We also arrange the powers such that \( \delta_{i+1} > \delta_i \). As we discussed above, there are two types of terms: computable using the \( t \)-channel OPE (first sum on the RHS of \(1.82\)) and terms that are sensitive to the details of the spectrum (second sum on the RHS of \(1.82\)).

To go from dispersion relations to statements about the Cauchy moments, we
need to understand what is the region of validity of $1.82$ in the complex $\nu$-plane, or, equivalently, in the $\Delta$-plane. Recall the relation between the two: $\Delta = \frac{d}{2} + i\nu$.

As we discussed in section 2, the large $\Delta$ (or $\nu$) expansion breaks down close to the real axis, where partial waves have poles at the locations of primary operators. In general, we do not know what exactly this region is. However, on general grounds we expect that the smooth polynomial behavior emerges as soon as $\text{Im}[\Delta] \gg 1$. This is also what we observed in concrete examples and from plugging separate $t$-channel operators in the inversion formula. In all these cases the corrections are suppressed by $e^{-\text{Im}[\Delta]}$. In particular, we assume that $1.82$ is valid for $|\Delta| \gg 1$ and $\text{Im}[\Delta] \gtrsim |\Delta|^\epsilon$ where $\epsilon$ is an arbitrary but fixed number.

Given that $1.82$ is valid in the complex $\nu$ region described above, the Cauchy moments have the following asymptotic at large $\nu$ (see appendix A.2 for a proof):

$$F_m^J(\nu) = \sum_i \alpha_i \beta_m(\delta_i) \nu^{m-\delta_i-1} + \sum_{k=1}^m b_k \frac{\nu^{m-k}}{(m-k)!} + O(\nu^{-\delta_i-1}),$$

$$\beta_m(\delta) = \frac{\cos \frac{\pi \delta}{2}}{\pi} \frac{\Gamma(-\delta)}{\Gamma(m-\delta)},$$  \hfill (1.83)

where $b_k$ are computable if $R(\nu)$ is known exactly, but cannot be computed based solely on the large $\nu$ data. The error term is defined by the smallest $\delta_i$.
The formula \( \text{1.83} \) constitutes the statement of the complex tauberian theorem for Stieltjes transform, which we prove in appendix A.2.

Note that \( \beta_m(\delta) \) has poles at even integer \( \delta \). The prescription in this case is to keep the regular piece in \( \beta_m(\delta) \nu^{m-\delta-1} \) which produces \( \nu^{m-\delta-1} \log \nu \) with computable coefficient and throw away the pole (see appendix A.2).

Heuristically, the coefficients of the first sum in \( \text{1.83} \) can be found by taking a naive power-law ansatz for \( \rho_j^{\text{OPE}}(\nu) \), as in appendix A.2, and computing Cauchy moments. The proof of \( \text{1.83} \), however, is much more subtle. In particular, it is crucial that we consider \( m \)-th Cauchy moment \( \text{1.81} \) in order to capture, roughly speaking, \( m \) subleading terms in \( \text{1.83} \). Again, the key ingredients are positivity of \( \rho_j^{\text{OPE}} \) and validity of the large \( \nu \) expansion in the complex domain.

Finally, notice that in \( \text{1.82} \) we used dispersion relations without subtractions. One can check that subtractions do not affect the result (see appendix A.2).

We would like to stress that \( \text{1.83} \) is a rather non-trivial consequence of \( \text{1.82} \). For example, consider the first Cauchy moment \( F_1^J \). For a discrete operator spectrum it is a discontinuous function with a “staircase” shape (e.g. see fig.6). It is remarkable that features of this staircase are captured by a smooth func-

\footnote{Strictly speaking, in appendix A.2 we prove \( \text{1.83} \) with the error estimate being \( O(\nu^{-\delta_1-1+\epsilon}) \), where \( \epsilon > 0 \) is arbitrarily small and fixed. Therefore, we leave a possibility of multiplying the error estimate by a function \( \Lambda(\nu) \) growing slower than a power. It will be implicit in what follows.}
tion on the RHS of (1.83).

1.4.1 **Leading Asymptotic**

Given the result (1.83) let us discuss the leading asymptotic for the integrated weighted spectral density. Recall that introducing \( \delta \phi = \Delta \phi - \frac{d-2}{2} \) the leading asymptotic in (1.83) comes from the unit operator (1.79). Setting \( m = 1 \) in (1.83) we get

\[
\int_{0}^{\nu} d\nu' \rho_{J}^{OPE}(d/2 + \nu') = f_{J} \frac{\nu^{4\delta_{\phi} - 2}}{4\delta - 2} + b_{1} + O(\nu^{4\delta_{\phi} - 3}) ,
\]

\[
f_{J} = [1 + (-1)^{J}] \pi^{2} 2^{2J+d-4\delta_{\phi}+2J} \frac{\Gamma(J + \frac{d}{2})}{\Gamma(J+1)\Gamma(\Delta \phi)^{2}\Gamma(\delta_{\phi})^{2}} , \tag{1.84}
\]

where we used (1.79). Note that \( \nu^{4\delta_{\phi} - 2} \) dominates over \( b_{1} \) only for \( \delta_{\phi} > \frac{1}{2} \), i.e. for operators with \( \Delta \phi > \frac{d-1}{2} \).

For \( \delta_{\phi} < \frac{1}{2} \) the constant term in (1.84) will dominate. Therefore, we consider the combination \( F_{J}^{2}(\nu) - \nu F_{1}^{J}(\nu) \) in which \( b_{1} \) cancels and get

\[
\int_{0}^{\nu} d\nu' \nu' \rho_{J}^{OPE}(d/2 + \nu') = f_{J} \frac{\nu^{4\delta_{\phi} - 1}}{4\delta_{\phi} - 1} + b_{2} + O(\nu^{4\delta_{\phi} - 2}) . \tag{1.85}
\]

Again, the first term dominates for \( \delta_{\phi} > \frac{1}{4} \), i.e. \( \Delta \phi > \frac{d-\frac{3}{2}}{2} \).

For \( 0 < \delta_{\phi} < \frac{1}{4} \) we can consider \( F_{J}^{3}(\nu) - \nu F_{2}^{J}(\nu) + \frac{\nu^{2}}{2} F_{1}^{J}(\nu) \) to remove \( b_{1}, b_{2} \)
and obtain

$$\int^\nu_0 dv' \nu'^2 \rho_j^{OPE}(d/2 + \nu') = f_j \frac{\nu^{4\delta_\phi}}{4\delta_\phi} + b_3 + O(\nu^{4\delta_\phi-1}) \, .$$  \hspace{1em} (1.86)$$

The choice of particular combinations of $F^J_m$ that we used to obtain 1.85, 1.86 will become clearer in the next subsection, when we will discuss how to systematically remove non-universal terms in 1.83 containing coefficients $b_k$. For $\delta_\phi = \frac{1}{2}$ the prescription in 1.84 is to keep the regular piece $\frac{\nu^{4\delta_\phi-2}}{4\delta_\phi-2} \to \log \nu$. Similarly for $\delta_\phi = \frac{1}{4}$ in 1.85.

To summarize, the leading asymptotic of the integrated weighted spectral density in any unitary CFT is given by formulas 1.84, 1.85, 1.86 depending on the scaling dimension of the external operator.

1.4.2 Systematic Corrections

There is still one last problem to be addressed in the formula 1.83. Indeed, in the large $\nu$ expansion 1.83 we encountered integer powers of $\nu$ that are not controlled by OPE. Therefore, it is more practical to take certain linear combinations of Cauchy moments 1.81 such that these integer powers cancel. Depending on how many integer powers we would like to cancel we can consider combina-
\[ G_{m,k}^J(\nu) = F_m^J(\nu) + p_1 \nu F_{m-1}^J(\nu) + \ldots + p_k \nu^k F_{m-k}^J(\nu) , \quad (1.87) \]

where \( k \) is the number of integer power terms that we want to remove. Using \( 1.83 \) we get the following set of equations for \( p_i \)

\[
\frac{1}{(m-1)!} + \frac{p_1}{(m-2)!} + \ldots + \frac{p_k}{(m-k-1)!} = 0, \]

\[
\ldots
\]

\[
\frac{1}{(m-k)!} + \frac{p_1}{(m-k-2)!} + \ldots + \frac{p_k}{(m-2k)!} = 0 , \quad (1.88)
\]

which ensure that terms \( \nu^{m-1}, \nu^{m-2}, \ldots, \nu^{m-k} \) cancel. The solution of the equations \( 1.88 \) takes the form

\[ p_j = (-1)^j \frac{k!}{j!(k-j)!} \frac{\Gamma(m-j)}{\Gamma(m)} . \quad (1.89) \]

In terms of the weighted spectral density the moments \( G_{m,k}^J(\nu) \) take the form

\[ G_{m,k}^J(\nu) = \frac{(-1)^k}{(m-1)!} \int_0^\nu d\nu' \, \nu^k(\nu - \nu')^{m-k-1} \rho_j^{OP} (d/2 + \nu') , \quad (1.90) \]
where $k$ is the number of integer power terms that we want to subtract. Note that the maximal error comes from the last term in \ref{1.87} since all $F_m^J$ have the same error term \ref{1.83}. Namely $G_{m,k}^J$ has a remainder term $O(\nu^{k+4\delta_0-3})$. Nevertheless, by taking an appropriate $m$ we can access as many terms in the $t$-channel OPE as we like.

In the discussion above for the leading asymptotic the combination $F_2^J(\nu) - \nu F_1^J(\nu)$ giving \ref{1.85} is nothing but $G_{2,1}^J(\nu)$. Similarly, the second moment \ref{1.86} is nothing but $G_{3,2}^J(\nu)$.

Further, we can access subleading terms in the $t$-channel OPE by considering higher $G_{m,k}^J$ moments. These have the error term $\delta G_{m,k}^J = O(\nu^{k+4\delta_0-3})$ and non-universal integer-power terms with the maximal power $\nu^{m-k-1}$. On the other hand, they enhance the OPE terms by a factor $\nu^{m-1}$. By taking $m$ to be large enough we can always extract arbitrary number of OPE controlled tails. In particular, given a term $\alpha_i\nu^{-\delta_i}$ in \ref{1.82} its contribution to $G_{m,k}^J$ is

$$G_{m,k}^J(\nu) = \sum_i \left( \frac{\cos \frac{\pi \delta_i}{2}}{\pi} \frac{\Gamma(1 - m)\Gamma(k - \delta_i)}{\Gamma(1 + k - m)\Gamma(m - \delta_i)} \right) \alpha_i\nu^{m-1-\delta_i}$$

$$+ b_{k+1}\nu^{m-k-1} + \cdots + b_m + O(\nu^{k-\delta_1-1}) , \quad m > k . \quad (1.91)$$

where the leading contribution comes from the unit operator \ref{1.79} and all $\alpha_i, \delta_i$
can be computed from the $t$-channel expansion.

Let us discuss some properties of the pre-factor \( \left( \frac{\cos \frac{\pi \delta_i}{2}}{\pi} \frac{\Gamma(1-m)\Gamma(k-\delta)}{\Gamma(1+k-m)\Gamma(m-\delta)} \right) \) which enters Eq. 1.91. For $m > k$ it has poles for even integer $\delta$'s

\[ k \leq \delta_p = 2p < m, \quad (1.92) \]

which correspond to the contribution of operators with dimensions $\Delta_p = 2\delta_p + p - 1$. As above, the right prescription is to throw away the pole and keep the finite term with $\log \nu$. Moreover, due to these poles the contribution of operators with dimensions close to $\Delta_p$ gets enhanced. Finally, at large $\delta$ we have

\[ \left( \frac{\cos \frac{\pi \delta}{2}}{\pi} \frac{\Gamma(1-m)\Gamma(k-\delta)}{\Gamma(m-\delta)\Gamma(1+k-m)} \right) \sim \frac{1}{\delta^{m-k}}. \]

Let us emphasize that even though we derived Eq. 1.83 for dispersion relations without subtractions Eq. 1.81, it holds for the most general case Eq. 1.54 as we show in appendix A.2.

Therefore, we can systematically access the $t$-channel OPE data by studying the moments of the $s$-channel weighted spectral density. This fact embodied in the formula Eq. 1.91 is the main result of our paper.
1.5 Examples

In this section we test the formula \( \ref{1.91} \) in Generalized Free Field theory (GFF) and 2d Ising. In particular, we will see that in these examples the large \( \Delta \) expansion will turn out to work well already for small values \( \Delta \). We will also make predictions for 3d Ising.

1.5.1 Generalized Free Field

The simplest example where we can test our claims is Generalized Free Field theory. Indeed, in this case \( c_J(\Delta) \) is explicitly known \( \ref{1.40} \). It is a meromorphic function with all the expected properties. One can write dispersion relations for it and check the corresponding complex tauberian theorems.

One slightly non-trivial fact in this case is that the large \( \nu \) expansion of extra contribution is simply related to \( c_{J}^{GFF}(d/2 + i\nu) \)

\[
\text{extra} = -\frac{1}{\cos 2\pi \delta_{\phi}} c_{J}^{GFF}(d/2 + i\nu) + (\text{even integer powers}) . \tag{1.93}
\]

We have not derived this result to all orders, but checked analytically first few terms in the large \( \nu \) expansion. Therefore, using \( \ref{1.93} \) we can easily make predic-
tions to an arbitrarily high order for the moments \[1.90\]

The first few terms in the expansion of \( c_{d/2 + i\nu}^{GFF} \) take the form

\[
c_{d/2 + i\nu}^{GFF} = \frac{d \nu^{4\delta \phi - 3}}{\nu^2} \left( 1 + \frac{\hat{\alpha}_1}{\nu^2} + \frac{\hat{\alpha}_2}{\nu^4} + \ldots \right),
\]

\[
\hat{\alpha}_1 = \frac{8}{3} \delta^3 - 8\delta^2 + \delta \left( 2J^2 + 2(d-2)J + \frac{d}{2}(d-4) + \frac{28}{3} \right)
- \frac{3}{2} J^2 + \frac{J}{2} (7 - 3d) + \frac{d}{8} (14 - 3d) - 4 ,
\]

and similarly for higher \( \hat{\alpha}_i \) which can be trivially computed by expanding \[1.40\].

Let us now plot a few moments to see how the formula \[1.91\] plays out. For concreteness we set \( d = 3, J = 0, \delta \phi = \frac{5}{8} \).

In the figure \[1.2\] we plot the result for the leading asymptotic of the integrated weighted spectral density. In the figure \[1.3\] we present the result for the difference between \( G_{1,0} \) and the fit multiplied by \( \sqrt{\nu} \). We see that the results are in perfect agreement with the formula \[1.91\].

Next, let us consider the moment that is sensitive to the subleading tail \( \hat{\alpha}_1 \) in \[1.94\]. One can check that the first moment in which we can access it is \( G_{5,2}^{J=0} \)

\[18\] In \( d = 2 \) we observed relations similar to \[1.93\] for generic operators. We have not tried to generalize \[1.93\] to arbitrary \( d \).
Figure 1.2: $G_{J=0}^{f=1,0}$ for GFF as a function of $\nu$. Parameters are chosen to be $d = 3$, $J = 0$, $\delta_\phi = \frac{5}{8}$. Based on 1.84 we expect the leading term to be $\frac{8\Gamma(-\frac{1}{4})^2}{\sqrt{\pi}} \sqrt{\nu} + b_1$. We fit the constant to be $b_1 \approx 79.4$. The asymptotic formula works very well down to $\nu = 0$.

Figure 1.3: Error term for $G_{J=0}^{f=1,0}$ as a function of $\nu$. We plot $\sqrt{\nu} \delta G_{J=0}^{f=1,0} = (G_{J=0}^{f=1,0} - \frac{8\Gamma(-\frac{1}{4})^2}{\sqrt{\pi}} \sqrt{\nu} - b_1) \times \sqrt{\nu}$ for GFF. Based on 1.84 we expect the difference between $G_{1,0}$ and the fit in 1.2 to be $O(\nu^{-1/2})$. 

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which takes the form

$$\frac{1}{\nu^2}G_{5,2}^{J=0}(\nu) = \frac{2048\Gamma \left(\frac{7}{4}\right)^2}{8505\sqrt{\pi}} \nu^{5/2} - \frac{608\Gamma \left(\frac{7}{4}\right)^2}{405\sqrt{\pi}} \nu^{1/2} + b_3 + O(\nu^{-1/2}). \quad (1.95)$$

We subtract the leading tail $\frac{2048\Gamma \left(\frac{7}{4}\right)^2}{8505\sqrt{\pi}} \nu^{5/2}$ from both sides to isolate the subleading term and plot the result in the figure 1.4.

Finally, let us check the error estimate in (1.95). To do this we plot the difference between $G_{5,2}^{J=0}$ and the fitting function in (1.95). We also multiply it by $\nu^{1/2}$ to make it clearer. The result is plotted in the figure 1.5.

Therefore, we see in this particular example that large $\Delta$ expansion works
Figure 1.5: Error estimate for $G_{5,2}^{J=0}$ for GFF as a function of $\nu$. Parameters are chosen to be $d = 3, J = 0, \delta_\phi = \frac{5}{8}$. We consider the difference between the LHS and the RHS in 1.95 which we also multiply by $\sqrt{\nu}$. We see that the result agrees with 1.95 very well already for $\Delta \sim 1$. Moreover, we can clearly access subleading terms in the $t$-channel OPE by considering proper moments of the weighted spectral density.

1.5.2 2d Ising Model

Let us consider the four-point function of $\langle \sigma \sigma \sigma \sigma \rangle$ in the 2d Ising model of scalars with dimension $\Delta_{\sigma} = \frac{1}{8}$. We have [39]

$$
\langle \sigma \sigma \sigma \sigma \rangle = \frac{G(\rho, \bar{\rho})}{(x_{12}^2 x_{34}^2)^{\frac{1}{8}}},
$$

$$
G(\rho, \bar{\rho}) = \frac{1 + \sqrt{\rho} \sqrt{\bar{\rho}}}{(1 - \rho^2)^{\frac{1}{4}}(1 - \bar{\rho}^2)^{\frac{1}{4}}},
$$

(1.96)
Since we are considering identical operators only even spins $J$ appear in the OPE. We have for conformal partial waves (see appendix B in [27])

\[
c_J(\Delta) = \frac{c_J \Delta}{\kappa J + \Delta} = I^0_{-\frac{3}{4}}(\Delta + J) I^0_{-\frac{1}{4}}(2 - \Delta + J) + I^\frac{1}{2}_{-\frac{3}{4}}(\Delta + J) I^\frac{1}{2}_{-\frac{1}{4}}(2 - \Delta + J) \\
- \frac{1}{\sqrt{2}} \left( I^\frac{1}{2}_{-\frac{3}{4}}(\Delta + J) I^0_{-\frac{1}{4}}(2 - \Delta + J) + I^0_{-\frac{1}{4}}(\Delta + J) I^\frac{3}{4}_{-\frac{1}{4}}(2 - \Delta + J) \right),
\]

(1.97)

where

\[
I^p_0(\beta) = \int_0^1 d\rho \frac{1 - \rho^2}{4\rho^2} k_\beta(\rho) \rho^p (1 - \rho^2)^p,
\]

\[
k_\beta(\rho) = (4\rho)^{\beta/2} \, _2F_1 \left( \frac{1}{2}, \frac{\beta}{2}, \frac{\beta + 1}{2}, \rho^2 \right).
\]

(1.98)

The partial wave $c_J(\Delta)$ is symmetric under the 2d shadow transform $\Delta \to 2 - \Delta$, as expected. Since $\delta_\sigma = \frac{1}{8} < \frac{3}{4}$ we consider dispersion relations without subtractions

\[
\int_0^\infty d\nu' \rho^J_{OPE}(1 + \nu') \frac{2\nu'}{\nu'^2 + \nu^2} = c_J(1 + i\nu) + \text{extra}.
\]

(1.99)

Let us compute the contributions of first few terms in the RHS of (1.99). The
three lightest operators in the $t$-channel OPE are

$$
\chi = \hat{1} : \quad \Delta = 0, \quad J = 0, \quad p_{0,0} = 1,
$$

$$
\chi = \varepsilon : \quad \Delta = 1, \quad J = 0, \quad p_{1,0} = \frac{1}{4},
$$

$$
\chi = T_{\mu\nu} : \quad \Delta = 2, \quad J = 2, \quad p_{2,2} = \frac{1}{64},
$$

(1.100)

We will set $J = 2$ in (1.99) for concreteness, but qualitatively same conclusions hold for any spin (including $J = 0$). We get the following contributions to the RHS of (1.99) from the first three operators

$$
c_{J=2}^{(1)}(1 + i\nu) = \sqrt{2} \frac{128\pi \Gamma\left(\frac{7}{8}\right)^2}{\Gamma\left(\frac{1}{8}\right)^2} \nu^{-5/2} \left(1 - \frac{333}{64} \frac{1}{\nu^2} + \frac{284565}{8192} \frac{1}{\nu^4} + \ldots\right),
$$

$$
c_{J=2}^{(e)}(1 + i\nu) = \sqrt{2} \frac{128\pi \Gamma\left(\frac{11}{8}\right)^2}{\Gamma\left(-\frac{3}{8}\right)^2} \nu^{-5/2} \left(1 + \frac{635}{64} \frac{1}{\nu^2} + \frac{284565}{8192} \frac{1}{\nu^4} + \ldots\right),
$$

$$
c_{J=2}^{(T)}(1 + i\nu) = \sqrt{2} \frac{\pi \Gamma\left(-\frac{1}{8}\right)^2}{\Gamma\left(-\frac{15}{8}\right)^2} \nu^{-5/2} \left(\frac{1}{\nu^4} + \ldots\right)
$$

(1.101)

Next, we evaluate tails coming from extra. We follow the procedure described at the end of section 2. The result is that extra terms contribute as follows
\[ extra(1) = -\frac{128\pi \Gamma\left(\frac{7}{8}\right)^2}{\Gamma\left(\frac{1}{8}\right)^2} \nu^{-5/2}\left(1 - \frac{333}{64} \frac{1}{\nu^2} + \frac{284565}{8192} \frac{1}{\nu^4} + \ldots\right), \]

\[ extra(\varepsilon) = \frac{128\pi \Gamma\left(\frac{1}{8}\right)^2}{\Gamma\left(-\frac{3}{8}\right)^2} \nu^{-5/2}\left(\frac{1}{\nu^2} - \frac{655}{64} \frac{1}{\nu^4} + \ldots\right), \]

\[ extra(T) = -\frac{\pi \Gamma\left(-\frac{1}{8}\right)^2}{\Gamma\left(-\frac{15}{8}\right)^2} \nu^{-5/2}\left(\frac{1}{\nu^4} + \ldots\right). \quad (1.102) \]

Adding together (1.101) and (1.102) we get the first three terms in the RHS of (1.99):

\[ \int_0^\infty d\nu' \rho_j^{\text{OPE}} (1 + \nu') \frac{2\nu' \nu}{\nu'^2 + \nu^2} = \frac{1}{\nu^{3/2}}\left(-4.92754 + \frac{99.773}{\nu^2} - \frac{936.518}{\nu^4} + \ldots\right) + \frac{a_1}{\nu} + \frac{a_2}{\nu^3} + \ldots, \]

\[ (1.103) \]

which defines \( \alpha_i, \delta_i \) of the first three terms in the RHS of (1.82). Below we present some plots for the moments (1.91).

We get the following prediction for the second moment

\[ \int_0^\nu d\nu' \nu'^2 \rho_j^{\text{OPE}} (1 + \nu') = \frac{512\sqrt{2\pi^2}}{\Gamma\left(\frac{1}{8}\right)^4} \nu^{1/2} + b_3 + O(\nu^{-1/2}). \quad (1.104) \]

\[ ^{19}\text{We present a numerical approximation of the coefficients not to clutter the text. Exact values are easily computable given the formulas in this section.} \]
The plots for the leading asymptotic and the remainder term of the second moment 1.104 are presented in the figures 1.6, 1.7.

At this point we would not be able to tell the difference between the 2d Ising model and a GFF field of the same dimension. Indeed, the leading asymptotic in 1.104 is controlled by the unit operator. To probe the difference between different theories we consider higher moments, that are sensitive to the subleading tails. For example, consider $G_{10,7}^{J=2}(\nu)$

$$\frac{1}{\nu^5} G_{10,7}^{J=2}(\nu) = -2.27979 \times 10^{-8} \nu^{5/2} - 1.4288 \times 10^{-6} \nu^{1/2} + O(1) . \quad (1.105)$$

The plots for the subleading and remainder terms in 1.105 are presented in...
Figure 1.7: Error estimate for the second moment. We consider the difference between the LHS and the smooth terms in the RHS of (1.104). We also multiply it by a factor $\nu^{1/2}$. The result is a highly oscillating function of approximately constant amplitude. This is consistent with the error estimate in (1.104).

Figure 1.8: Subleading tail in $G_{10,7}^{J=2}$ in the 2d Ising model. We plot $\frac{1}{\nu^5} G_{10,7}^{J=2}(\nu) + 2.2797910^{-8} \nu^{5/2}$ versus $\nu$. 

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Figure 1.9: Error estimate for $\frac{1}{\nu^2} G_{10.7}^{J=2}$ in the 2d Ising model. We multiply the difference between the LHS and the smooth terms in the RHS in (1.105) by $\sqrt{\nu}$. Again, we get that the error is consistent with the last term in (1.105).

The minimal moment sensitive to the subleading correction is $G_{7,4}^{J=2}(\nu)$. In this case the convergence in $\nu$ is slower. We believe that it might be related to the fact that in this case the corrections are enhanced due to their proximity to the poles (1.92). This problem does not arise for the moment $G_{10,7}$ above. Our prediction takes the form

$$\frac{1}{\nu^2} G_{7,4}^{J=2}(\nu) = \frac{\pi^2}{113400\sqrt{2}} \frac{\nu^{5/2}}{\Gamma\left(\frac{9}{8}\right)^4} + 0.0332694 \nu^{1/2} + b_5 + O(\nu^{-1/2}) \quad (1.106)$$

We subtract the leading tail from both sides of the equation (1.106) and plot the difference in the figure (1.110).
Figure 1.10: Subleading tail in $G_{7,4}^{J=2}$ in the 2d Ising model. We plot the difference $\frac{1}{\nu^2} G_{7,4}^{J=2}(\nu) - \frac{\pi^2}{113400\sqrt{2} \Gamma(\frac{3}{4}) \nu^{5/2}}$ versus $\nu$. We fit $b_5 \approx 0.06365$. Again the leading behavior exactly matches the prediction.

Figure 1.11: Error estimate for $\frac{1}{\nu^2} G_{7,4}^{J=2}$ in the 2d Ising model. We multiply the difference between the LHS and the RHS in 1.106 by $\sqrt{\nu}$. Again we get that the error is consistent with the expectation, though convergence in this case is slower.
Finally, we plot the difference between $\frac{1}{\nu^2}G_{7,4}$ and smooth terms in the RHS of \ref{fig:1.106} to estimate the error in the figure \ref{fig:1.11}.

The third term in the expansion \ref{fig:1.103}, which includes the contribution of the stress tensor, is also easily accessible. In particular, if we consider $\frac{1}{\nu^2}G_{12,6}^J$ moment from \ref{fig:1.91} we see that the three OPE terms from \ref{fig:1.103} contribute as $\nu^{9/2}$, $\nu^{5/2}$ and $\nu^{1/2}$ correspondingly.

\subsection*{3d Ising Model}

So far all explicit computations that we did with conformal blocks have been in $d = 2, 4$ where the expressions for them are explicitly known and relatively
simple. The only result that is valid in all dimensions is the formula \[ 1.40 \] for the partial waves of the generalized free field or, equivalently, the contribution of the unit operator in the \( t \)-channel.

Fortunately, this is all we need to make non-trivial predictions for the \( \langle \sigma \sigma \sigma \rangle \) correlator in the 3d Ising. Recall that \( \Delta_\sigma \simeq 0.51815 \) and therefore it falls into the category \( \delta_\sigma = \Delta_\sigma - \frac{1}{2} \simeq 0.01815 < \frac{1}{4} \) for which we can write the unsubtracted dispersion relation, but need to consider higher moments to match with the \( t \)-channel OPE. The lightest operator above the identity that contributes to the OPE is \( \varepsilon \) with dimension \( \Delta_\varepsilon \simeq 1.41 \). Therefore, we get the following structure in the RHS of the dispersion relation

\[
\int_0^\infty d\nu' \rho_j^{\text{OPE}} (3/2 + \nu') \frac{2\nu' \nu}{\nu'^2 + \nu^2} = d_J \left( 1 - \frac{1}{\cos 2\pi \delta_\sigma} \right) \nu^{4\delta_\sigma - 2} \left( 1 + \frac{\hat{\alpha}_1}{\nu^2} + O(\nu^{-2\Delta_\varepsilon}) \right) + \frac{a_1}{\nu} + \frac{a_2}{\nu^3} + \ldots ,
\]

\[
\hat{\alpha}_1 = 2J^2 \left( \delta_\sigma - \frac{3}{4} \right) + 2J \left( \delta_\sigma - \frac{1}{2} \right) + \frac{8}{3} \delta_\sigma^3 - 8\delta_\sigma^2 + \frac{47}{6} \delta_\sigma - \frac{17}{8} , \tag{1.107}
\]

where \( d_J \) is given in \[ 1.42 \]. Notice that due to \( \Delta_\varepsilon > 1 \) the first subleading universal term comes from the unit operator as well.

Therefore we can make the following predictions. The leading \( \nu \) asymptotic
can be extracted from the second moment \[1.86\]

\[
\int_0^\nu d\nu' \nu'^2 \rho^{\text{OPE}}_J(3/2 + \nu') = f_J \frac{\nu^{4\delta_\sigma}}{4\delta_\sigma} + b_3 + O(\nu^{4\delta_\sigma - 1}) , \tag{1.108}
\]

where \( f_J \) is defined in \[1.84\] (with \( \phi = \sigma \) in the present case) and \( b_3 \) is not computable in terms of the \( t \)-channel OPE.

We can also access the subleading term in \[1.107\] by, for example, considering \( G^J_{9,5} \) which takes the following form

\[
\nu^{-3} G_{9,5}(\nu) = -\frac{\Gamma(4\delta_\sigma + 3)}{6720\Gamma(4\delta_\sigma + 7)} f_J \nu^{4\delta_\sigma + 3} \left( 1 - \frac{(\delta_\sigma + \frac{3}{2})(\delta_\sigma + \frac{5}{2})}{(\delta_\sigma + \frac{1}{2})(\delta_\sigma + \frac{3}{2})} \hat{\alpha}_1 \right) + b_5 + O(\nu^{4\delta_\sigma - 1}) . \tag{1.109}
\]

where we computed the coefficients using \[1.91\].

Of course, in practice utility of \[1.108, 1.109\] depends on how large \( \nu \) has to be in order to observe the predicted behavior. Based on the examples above we believe that it should happen for small \( \nu \). It would be interesting to explore this question using the numerical bootstrap.
1.6 Meromorphic Scattering Amplitudes

In this section we apply the same ideas to meromorphic scattering amplitudes.

These arise for example in the tree-level string theory or large $N$ confining gauge theory\textsuperscript{40}. We can use analyticity and polynomial boundedness of the scattering amplitude to express it in terms of its discontinuity. This is achieved through the standard Cauchy argument. Imagine that for a given $t$ the amplitude is polynomially bounded, namely $A(s, t) \sim f(t)(-is)^{j(t)}$ at large $s$.\textsuperscript{41} Then we can write a subtracted dispersion relation

\[
\frac{1}{n!} \partial^n_s A(s, t) = \int \frac{ds'}{2\pi i} \frac{A(s', t)}{(s' - s)^{n+1}}, \quad n = [j(t)] + 1. \tag{1.110}
\]

Dropping the arcs at infinity we get

\[
\frac{1}{n!} \partial^n_s A(s, t) = \int_{4m^2}^\infty ds' \rho(s', t) \left( \frac{1}{(s' - s)^{n+1}} + \frac{(-1)^n}{(s' - u)^{n+1}} \right), \quad 
\rho(s', t) = \sum_{J=0}^\infty \sum_i \delta(s' - m_i^2)\lambda_{J,m_i}^2 P_J \left( 1 + \frac{2t}{m_i^2 - 4m^2} \right), \tag{1.111}
\]

where we used that the only singularities are simple poles at the positions of resonances and $m$ is the mass of external particles, which we consider to be identi-

\textsuperscript{20}Our choice of the phase will become clear below.
cal. Unitarity, therefore, implies that $\rho(s', t) \geq 0$ for $t \geq 0$ and $s > 4m^2$.

Let us assume now that at fixed $t$ and large $s$ the amplitude $A(s, t)$ admits a power-like expansion

$$A(\nu, t) = \sum_i f_i(t)\nu^{i(t)} + \ldots, \quad s = -\frac{t}{2} + i\nu$$

(1.112)

where we introduced $\nu$ variable to connect to the discussion of CFTs in earlier sections.

This is the usual expansion in terms of the Regge trajectories that one gets by closing the $J$ contour integral in the Froissart-Gribov representation $[27, 28]$. It is an asymptotic expansion that is valid away from the real axis. By going away from the real axis we get a cumulative effect from many resonances which produces a smooth power-like behavior $[112]$. If the only singularities are poles we will get only powers $s^{i(t)}$. If there are Regge cuts we might get some extra slowly growing factors $(\log s)^a$. These do not affect the discussion and we keep them implicit.

The expansion $[112]$ together with $[111]$ is exactly the same mathematical setup as we discussed in the previous sections. We can apply the same complex tauberian theorem as in section 4 to get an expansion of Cauchy moments of
the integrated spectral density

\[ F_m(\nu, t) \equiv \frac{1}{(m-1)!} \int_0^\nu d\nu' (\nu - \nu')^{m-1} \rho(\nu' - t/2, t). \]  \hspace{1cm} (1.113)

for which we get

\[ F_m(\nu, t) = - \sum_{i=0} f_i(t) \frac{\sin \frac{\pi j_i(t)}{2}}{\pi} \frac{\Gamma(j_i(t) + 1)}{\Gamma(m + j_i(t) + 1)} \nu^{m+j_i(t)} \]
\[ + \sum_{k=1}^m b_k \frac{\nu^{m-k}}{(m-k)!} + O(\nu^{j_0(t)}), \]  \hspace{1cm} (1.114)

where \(j_0(t)\) is the leading Regge trajectory, and we used validity of the Regge expansion for \(\text{Im}[s] > |s|^\epsilon\), where \(\epsilon\) is positive and fixed. As above we kept \(s^\epsilon\) factor in the error estimate implicit.

\subsection*{1.6.1 Veneziano Amplitude}

Consider as an example the Veneziano amplitude. We have

\[ A(s, t) = \frac{\Gamma(-s)\Gamma(-t)}{\Gamma(-s-t)} + \text{permutations}, \]  \hspace{1cm} (1.115)
where the external particles are taken to be massless $s + t + u = 0$. At large $s$ and fixed $t$ the amplitude admits an asymptotic expansion

$$A(s, t) = (-s)^t \Gamma(-t) \left(1 + \frac{\sin \pi s}{\sin \pi (s + t)}\right), \quad \text{Re}[s] < 0,$$

$$A(s, t) = s^t \Gamma(-t) \left(1 + \frac{\sin \pi (s + t)}{\sin \pi s}\right), \quad \text{Re}[s] > 0 ,$$

(1.116)

where we used the asymptotic expansion of the Gamma function (see appendix C in [41] for a thorough discussion). Away from the real axis, namely for $\text{arg}[s] \neq 0, \pi$, the oscillation terms in the brackets lead to exponentially suppressed corrections. Say, taking $s = s_0(1 + i\epsilon)$ the oscillating terms lead to the corrections of the type $e^{-\epsilon s_0}$. Neglecting those we get

$$A(s, t) = 2(-i s)^t \cos \frac{\pi t}{2} \Gamma(-t), \quad 0 < \text{arg}[s] < \pi ,$$

$$A(s, t) = 2(i s)^t \cos \frac{\pi t}{2} \Gamma(-t), \quad \pi < \text{arg}[s] < 2\pi .$$

(1.117)

In particular, plugging 1.117 in the Tauberian theorem 1.114 we get

$$\int_0^s ds' \rho(s', t) = \frac{(s + t/2)^{t+1}}{\Gamma(t + 2)} + \frac{(s + t/2)^t}{2\Gamma(t + 1)} + O(s^t) .$$

(1.118)

which we derived for $t > 0$, where the spectral density 1.111 is positive.
Figure 1.13: Integrated spectral density for the Veneziano amplitude. We plot $G_{1,0}$ for $t = 1.2$.

Figure 1.14: Error estimate in the Veneziano amplitude spectral density. We plot the difference $\frac{1}{\pi} \left( \int_0^s ds' \rho(s', t) - \frac{(s+t/2)^{t+1}}{1(t+2)} \right)$ for $t = 1.2$. The result is consistent with $G_{1,0}$. 
Figure 1.15: Subleading Correction for $G_{4,2}$ of the Veneziano amplitude. We subtract the leading $\nu^2$ tail from $\frac{1}{\nu^2+t}G_{4,2}$ and compare the subleading correction $\nu$ to the one predicted by (1.120). We set as above $t = 1.2.$

We plot (1.118) for the Veneziano amplitude in the figure (1.13). Similarly, we can consider the error in (1.118) which we plot in the figure (1.14).

To access the next-to-leading correction we can consider $G_{4,2}$ that is given by

$$G_{4,2}(\nu) = \frac{1}{3!} \int_0^\nu d\nu' \nu^2(\nu - \nu')(\nu' - t/2, t)$$  \hspace{1cm} (1.119)

for which we get the following prediction

$$\frac{1}{\nu^{2+t}}G_{4,2}(\nu) = \frac{(1 + t)(2 + t)}{6\Gamma(5 + t)}\nu^2 + \frac{t(1 + t)}{12\Gamma(4 + t)}\nu + O(1).$$  \hspace{1cm} (1.120)

We plot the first subleading term of (1.120) in the figure (1.15) and find perfect
agreement. The difference between the exact $G_{4,2}$ and the first two terms in the RHS of (1.120) is an oscillatory function of constant amplitude. We plot it in the figure 1.16.

1.6.2 Heuristic Derivation of Spectral Asymptotics

Let us start with a historical comment. Finite energy sum rules (FESR) [28–31] are consistency conditions imposed by analyticity of scattering amplitudes. In the case of usual dispersion relations one uses analyticity and polynomial boundedness of the scattering amplitude to express the amplitude through its discontinuity. FESR are closely related to the usual dispersion relations. In this case we consider an integral of the amplitude’s moment $K(s,t)A(s,t)$ over the contour
\[ \mathcal{C} \text{ in the complex plane (the larger blue contour this time is in the complex } s\text{-plane). The kernel } K(s, t) \text{ is chosen to be analytic inside } \mathcal{C} \text{ (we also choose it to be real on the real axis). Due to analyticity of the scattering amplitude and the kernel the integral vanishes} \]

\[ \oint_{\mathcal{C}} ds' K(s', t) A(s', t) = 0. \quad (1.121) \]

Let us (without loss of generality) further restrict our consideration to the scattering of four scalar identical particles. Permutation symmetry implies that \[ (1.121) \text{ is non-zero for the odd part of the kernel only } K(s, t) = -K(u, t). \] We can rewrite \[ (1.121) \] as follows

\[ S_n = \frac{1}{s_0^{n+1}} \int_0^{s_0} \frac{ds}{\pi} K(s, t) \text{Im}[A(s, t)] = \frac{1}{s_0^{n+1}} \frac{1}{2\pi i} \int_{\mathcal{C}'} ds K(s, t) A(s, t), \quad (1.122) \]

where \( s + t + u = 4m^2 \) and \( \mathcal{C}' \) stands for the integral over the arcs. Importantly, in this case we do not drop the contribution from the large arcs in \[ 1.1. \]

For application of these sum rules to the pion-nucleon scattering in QCD see \[ [28], \text{ where} \]

\[ K(s, t) = (s - u)^n. \quad (1.123) \]
In phenomenological applications one assumes that the LHS of 1.122 is dominated by a few low-energy resonances. One can then use the knowledge of the scattering amplitude \( \text{Im}[A(s, t)] \) to make predictions about the leading Regge asymptotic. Alternatively, one can use the knowledge of the Regge asymptotic to infer something about the properties of low-energy resonances. The basic point is that one should not add low-energy resonances contributions and the contributions of Regge poles. Adding them up would lead to a double counting as 1.122 clearly demonstrates. This duality between resonances in one channel and the Regge trajectory exchange in the other channel is also known as the Dolen-Horn-Schmid duality. It led to the Veneziano amplitude ?? and its better understanding was the original motivation of our analysis.

In the context of meromorphic amplitudes using the Regge asymptotic all the way to the real axis is not justified. Therefore we could not use the Regge limit to compute the integral over the arc.\textsuperscript{2} Instead, we should use the complex tauberian theorem as described above. This puts FESR for meromorphic amplitudes on a solid mathematical ground.

Let us however offer a non-rigorous intuitive explanation of the results that we obtained using complex tauberian theorems. Consider a FESR integral 1.122

\footnote{21\textsuperscript{2}Alternatively, FESR are derived from the the so-called superconvergence sum rules \textsuperscript{28}. However, this derivation suffers from exactly the same problem for meromorphic amplitudes.}
which after switching to the $\nu$ variable takes a form

$$
\int_0^{\nu_0} d\nu' \nu'^n \text{Im}[A(\nu', t)] = \frac{1}{2i} \int_{C'} d\nu' \nu'^n A(\nu', t). \tag{1.124}
$$

Let us rewrite the RHS of (1.124) as follows

$$
\frac{1}{2i} \int_{C'} d\nu' \nu'^n A(\nu', t) = \frac{1}{2i} \int_{C'} d\nu' \nu'^n A^{\text{Regge}}(\nu', t) + \frac{1}{2i} \int_{C'} d\nu' \nu'^n [A(\nu', t) - A^{\text{Regge}}(\nu', t)] ,
$$

(1.125)

where $A^{\text{Regge}}(\nu', t)$ is the power-like Regge asymptotic analytically continued all the way to the real axis, even though it is not a valid approximation of the amplitude in that region. The first term in the RHS of (1.125) is what produces $t$-channel predictions in the previous subsections. On the other hand, the $A(\nu', t) - A^{\text{Regge}}(\nu', t)$ term corresponds to an error estimate. The result of the theorem corresponds to an estimate $A(\nu', t) - A^{\text{Regge}}(\nu', t) = O(A^{\text{Regge}}(\nu', t))$ in the region of the complex plane close to the real axis. Integrating over the region where the Regge approximation is not valid (let us denote the size of this region $\Lambda \ll \nu_0$) we get an error estimate

$$
\int_0^{\nu_0} d\nu' \nu'^n \text{Im}[A(\nu', t)] = \frac{1}{2i} \int_{C'} d\nu' \nu'^n A^{\text{Regge}}(\nu', t) + O(\Lambda \nu_0^n A^{\text{Regge}}(\nu_0, t)). \tag{1.126}
$$
This is precisely the result of our theorems, where we chose $\Lambda = s_0^\epsilon$ where $\epsilon$ is some fixed but arbitrarily small number. In practice, say for the Veneziano amplitude, we find rather an estimate $A(\nu', t) - A^{\text{Regge}}(\nu', t) = O(e^{-c\text{Im}[\nu']} A^{\text{Regge}}(\nu', t))$.

An extra suppression factor $e^{-c\text{Im}[\nu']}\Lambda$ leads to the disappearance of an extra factor $\Lambda$ in (1.126) since the integral $\int_0^\Lambda da e^{-ca} = O(1)$ for order one number $c$ and $\Lambda \gg 1$. This is fully consistent with what we observed for the Veneziano amplitude. For CFTs we get an identical picture based on separate terms coming from the inversion formula. It is also what we get in the explicit examples of the 2d Ising and GFF. Assuming that this is a general phenomenon we would get $\Lambda(x) = \text{const}$ error estimates in the tauberian theorem of appendix A.2.

The argument above also illuminates what is special about the Cauchy moments. Indeed having a factor $(\nu_0 - \nu')^m$ inside the kernel $K(s, t)$ does not change the error estimate in (1.126). This is because effectively we have $\nu_0 - \nu' \simeq \text{Im}[\nu']$ in the relevant integration region and again $\int_0^\Lambda da a^m e^{-ca} = O(1)$. Strictly speaking since we have both the s- and the u-channel cuts we need to consider terms $(\nu_0 - \nu')^m(\nu_0 + \nu')^m$ instead or design an analytic kernel that is small on one of the cuts. The conclusion is, however, the same: we should set $\Lambda(x) = \text{const}$ in the estimates that we got from the complex tauberian theorems.
Modular Invariance, Tauberian Theorems, and Microcanonical Entropy

High energy estimates on various physical quantities are commonly stated locally even though they are only true on average. A few famous examples are: the Froissart bound on the growth of the cross section [42, 43], high frequency
expansion of conductivity at finite temperature [44, 45], high energy asymptotic of the electromagnetic current spectral density in the context of the so-called quark-hadron duality [46], and finally the Cardy formula for two-dimensional CFTs [47]. The latter is particularly interesting because of its importance for the problem of the black hole microstate counting [48–50, 10]. It is then a natural question to ask: how do these estimates depend on the details of the averaging?

Let us review the standard derivation of the Cardy formula. We consider a thermal partition function $Z(\beta)$ of a unitary 2d CFT on a Euclidean torus. The partition function is modular invariant $Z(\beta) = Z(\frac{4\pi^2}{\beta})$. This implies that the high-temperature limit $\beta \to 0$ of the partition function is captured by the contribution of the vacuum in the dual channel $Z(\beta) \sim e^{\frac{\pi^2 c}{3\beta^2}}$, where $c$ is the central charge. Using the standard thermodynamic formula $S(\beta) = (1 - \beta \partial_\beta) \log Z$ we can compute the entropy $S(\beta)$ at high temperatures

$$S(\beta) = \frac{2\pi^2 c}{3\beta^2} + ..., \quad \beta \to 0 . \quad (2.1)$$

In the $\beta \to 0$ limit the energy of the system $\langle \Delta \rangle = -\partial_\beta \log Z = \frac{\pi^2 c}{3\beta^2} + ...$ goes to infinity. The $\Delta \to \infty$ limit being the thermodynamic limit, see e.g. [51], one obtains from the usual thermodynamic arguments that the extensive part of the
entropy, which is given by 2.1, also correctly captures the leading behavior of the microcanonical entropy $S_\delta(\Delta)$ defined by

$$S_\delta(\Delta) \equiv \log \int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \rho(\Delta'),$$

$$\rho(\Delta) \equiv \sum_\sigma \delta(\Delta - \Delta_\sigma),$$

as soon as $\delta$ is large enough to include many energy levels. That is if we express the temperature as a function of the average energy $\beta = \pi \sqrt{c/3\langle \Delta \rangle}$ and plug it in 2.1 we arrive at the famous Cardy formula for the microcanonical entropy

$$S_\delta(\Delta) = 2\pi \sqrt{c\Delta/3} + ..., \quad \Delta \to \infty. \quad (2.3)$$

In discussions and applications of the Cardy formula the averaging width parameter $\delta$ is usually kept implicit. Moreover, the rigorous transition from 2.1 to 2.3 requires some extra work which is usually left to the reader.

Indeed, the spectral density $\rho(\Delta)$ is related to the partition function $Z(\beta)$ by the inverse Laplace transform. It is sometimes argued that this Laplace transform can be evaluated by a saddle point approximation from which the statement about $\rho(\Delta)$ and therefore $S_\delta(\Delta)$ can be made. A more accurate description of this procedure would be to say that one can easily find the crossing ker-
nel of the vacuum contribution \( e^{\frac{\pi^2 c}{3\beta}} \), or, in other words, a spectral density \( \rho_0(\Delta) \) that correctly reproduces the vacuum in the dual channel. The question then stays: what is the precise relation between the naive spectral density \( \rho_0(\Delta) \) and the actual physical density \( \rho(\Delta) \)? This relation cannot be too literal. Indeed, the former is a smooth function of \( \Delta \), whereas the latter is a sum of delta-functions. Once again the physical intuition is that they are related on average, but establishing this rigorously is a nontrivial task. The issue of making the argument precise becomes even more important if one considers “finite size” or \( \frac{1}{\Delta} \) corrections to the Cardy formula. The purpose of this work is to close this gap in the usual discussions of the Cardy formula and to develop further techniques that allow us to study \( \frac{1}{\Delta} \) corrections to it.

The physical question of going from the finite temperature partition function to the microcanonical entropy can be addressed in a mathematically rigorous way using the methods of tauberian theory \[16\], as explained in \[23, 33\]. From the conformal/modular bootstrap point of view tauberian theory provides a natural set of linear functionals with which we act on the crossing/modularity condition to derive optimal estimates on \( S_\delta(\Delta) \) or other spectral density averages.

As further noticed in \[52\] the optimal error estimates can be obtained using the so-called complex tauberian theorems, which exploit the fact that physi-
cal quantities of interest are very often analytic functions in a complex domain. This is indeed the case for the modularity condition of 2d CFTs. In this note we apply methods of tauberian theory to modular invariance in 2d CFTs and rigorously derive the Cardy formula and corrections to it, where we explicitly keep track of the dependence on $\delta$. Furthermore, combining these ideas with bounds on the the partition function of Hartman, Keller and Stoica (HKS) [53] we find lower and upper bounds on the number of operators within a given window of finite conformal dimensions $(\Delta - \delta, \Delta + \delta)$. Though true at finite $\Delta$, they are most revealing in the limit $\Delta \to \infty$.

2.1 Review of the Results

We consider a modular invariant partition function with zero angular potential and positive spectral density. We derive a set of rigorous results about $S_\delta(\Delta)$.

2.2 These concern either all operators present in the theory, or only Virasoro primaries in CFTs with $c > 1$.

- Let us first discuss densities of all operators, both primaries and descen-
dants. We derive a rigorous asymptotic for the microcanonical entropy

\[ S_\delta(\Delta) = \log \int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \rho(\Delta') = 2\pi \sqrt{\frac{c\Delta}{3}} + \frac{1}{4} \log \left( \frac{c\delta^4}{3\Delta^3} \right) + s(\delta, \Delta), \quad \Delta \to \infty, \]

(2.4)

where depending on the size of the averaging energy shell \( \delta \) we show that

\[ \delta \sim \Delta^\alpha : \quad s(\delta, \Delta) = \log \left( \frac{\sinh \left( \frac{\pi}{\sqrt{3}} \frac{\delta}{\sqrt{\Delta}} \right)}{\frac{\pi}{\sqrt{3}} \frac{\delta}{\sqrt{\Delta}}} \right) + O(\Delta^{-\alpha}), \quad 0 < \alpha \leq \frac{1}{2}, \]

\[ \delta = O(1) : \quad s_- (\delta) \leq s(\delta, \Delta) \leq s_+ (\delta), \quad \delta > \delta_{\text{gap}} = \frac{\sqrt{3}}{\pi} \approx 0.55. \]

(2.5)

The first two terms in the RHS of 2.4 are the Cardy formula 2.3 and the leading log correction to it discussed for example in [18, 19]. The results for \( s(\delta, \Delta) \) are new to the best of our knowledge. In particular, we see that for \( \delta \sim \Delta^\alpha \) there is yet another universal correction\(^1\) to the microcanonical entropy that is controlled by the central charge \( c \) and the width of the energy shell \( \delta \) and given by the first line in 2.5. Note that for any \( \alpha > 0 \) the error decays at large \( \Delta \) and the non-decaying contribution to the entropy is fully captured by the

\[ \frac{1}{4} \log \left( \frac{c\delta^4}{3\Delta^3} \right) \]

term. For \( \delta = O(1) \) the functions \( s_\pm (\delta) \) are plotted on 2.1. In par-

\(^1\)By \( a \sim b \) we mean \( \lim a/b = \text{const} \neq 0 \) in the corresponding limit.

\(^2\)It dominates over the error term \( O(\Delta^{-\alpha}) \) only for \( \alpha > \frac{1}{3} \).
Figure 2.1: On this figure we plot the upper (yellow) and lower (blue) bounds $s_\pm(\delta)$ on $s(\delta, \Delta)$. The vertical line is $\delta = \delta_{gap} = \sqrt{\frac{3}{\pi}}$, below which we do not have a lower bound. The divergence of $s_+(\delta)$ at $\delta = 0$ is spurious and is cancelled by $\log \delta$ in 2.4.

In particular, the lower bound diverges logarithmically as we approach $\delta \to \delta_{gap}$. The interpretation of this is that the asymptotic 2.4 is only applicable for $\delta > \delta_{gap}$ for which the leading behavior of the microcanonical entropy $S_\delta(\Delta)$ takes the form 2.4. Note that the lower bound implies that there have to be operators in an energy shell of the size $\delta > \delta_{gap}$.

For $\delta < \delta_{gap}$ we can only prove an upper bound on the microcanonical entropy which is given by 2.4 and $s_+(\delta)$ in 2.5. For a fixed $\delta = O(1)$ the function $s(\delta, \Delta)$ in general is not a constant and can oscillate as we change $\Delta$, but always between the values $s_\pm(\delta)$. In fact, we will explicitly see these oscillations in the

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3The precise form of the bounding curves can be found in section 4.
2d Ising model in section 7.

• For CFTs with \( c > 1 \) we can derive analogous formulae for Virasoro primary operators

\[
S^\text{Vir}_\delta (\Delta) = \log \int_{\Delta-\delta}^{\Delta+\delta} d\Delta' \rho^\text{Vir}(\Delta') = 2\pi \sqrt{\frac{c-1}{3} \Delta} - \frac{1}{4} \log \left( \frac{c-1}{48\delta^4 \Delta} \right) + s^\text{Vir}(\delta, \Delta),
\]

\[\delta \sim \Delta^\alpha : \quad s^\text{Vir}(\delta, \Delta) = \log \left( \frac{\sinh \left( \pi \sqrt{\frac{c-1}{3} \delta \sqrt{\Delta}} \right)}{\pi \sqrt{\frac{c-1}{3} \delta \sqrt{\Delta}}} \right) + O (\Delta^{-\alpha}) , \quad 0 < \alpha \leq \frac{1}{2},\]

\[\delta = O(1) : \quad s_-(\delta) \leq s^\text{Vir}(\delta, \Delta) \leq s_+(\delta) , \quad \delta > \delta_{\text{gap}} = \frac{\sqrt{3}}{\pi} \approx 0.55 .\]

where \( \Delta \to \infty \) and \( s_\pm(\delta) \) are the same as in 2.5. For finite width energy shells, or \( \alpha = 0 \), we can again write the lower and upper bounds on the entropy as in 2.5, as soon as \( \delta > \delta_{\text{gap}} \). A simple consequence of this result is an existence of maximal sparseness of Virasoro primaries. In other words, it follows that

• At large scaling dimensions \( \Delta \) the spacings between Virasoro primary operators in CFTs with \( c > 1 \) cannot be larger than \( 2\delta_{\text{gap}} = \frac{2\sqrt{3}}{\pi} \approx 1.1 \).

This bound is not necessarily optimal. Nevertheless, it is close to the optimal since there are many examples of theories with the spacings equal to 1.

\[\text{A famous example is the monster CFT}^4 \text{. The monster CFT is chiral with}\]

\[91\]
• We derive an asymptotic of the microcanonical entropy in holographic 2d CFTs in the limit $c \to \infty$ with $\Delta/c$ - fixed and $\Delta > \frac{\xi}{6}$

$$S_\delta(\Delta) = \log \int_{\Delta - \delta}^{\Delta + \delta} d\Delta' \rho(\Delta') = 2\pi \sqrt{\frac{c}{3} \left( \Delta + \delta - \frac{c}{12} \right)} - \frac{1}{2} \log c + O(1), \quad c \to \infty,$$

(2.7)

where $\delta \sim c^\alpha, 0 \leq \alpha < 1$ and $\delta > \delta_{\text{gap}}$. This relies on the sparseness condition of Hartman, Keller and Stoica (HKS) \cite{53} and extends their result \cite{53} for the microcanonical entropy which is \cite{2.7} with an extra constraint $\frac{1}{2} < \alpha < 1$.

As we will explain later on, an important ingredient in the derivation of $s_\pm(\delta)$ and $\delta_{\text{gap}}$ relies on the existence of functions $\phi_\pm(\Delta')$ with the following properties:

1) $\phi_+(\Delta')$ and $\phi_-(\Delta')$ bound the indicator function of the interval $(\Delta - \delta, \Delta + \delta)$ from above and below respectively;

2) Their Fourier transform has a bounded support.

We make an explicit choice of such functions to arrive at the particular value of $\delta_{\text{gap}}$ and the bounding curves in $s_\pm(\delta)$. Nevertheless, the method is completely

---

$(c_L, c_R) = (24, 0)$. However, for zero angular potential its partition function can be interpreted as the one of a non-chiral theory with $(c_L, c_R) = (12, 12)$, see e.g. \cite{56}. It, therefore, satisfies the modularity constraint imposed in the present work.

\footnote{See appendix A in their paper.}
general and we leave open the question of finding the functions $\phi_{\pm}(\Delta')$ giving optimal bounds.

- Above we stated our results at asymptotically high energies. They follow from more general bounds on the number of operators at finite $\Delta, c$, that we derive in section 4. Specifically, given the data about operators $\Delta \leq c/12$ we derive rigorous upper and lower bounds on the number of operators in a given window of scaling dimensions. We emphasize that all parameters can be kept finite. In particular, these bounds can be easily implemented numerically. For example, we can derive numerical bounds on the gap above the vacuum, though these turn out to be weaker than [57, 56]. On the other hand we can also bound a number of operators in any window of scaling dimensions at any $\Delta$ above the first excited state as well.

- We consider partition functions with a UV cutoff. We start by proving a generalized Ingham’s theorem:

**Theorem:** Consider a positive spectral density $\rho(\Delta)$, such that the partition function $Z(\beta) = \int_0^\infty d\Delta e^{-\beta(\Delta - \frac{c}{12})}\rho(\Delta)$ is modular invariant $Z(\beta) = Z(\frac{4\pi^2}{\beta})$.

Moreover, suppose that $Z(\beta) = e^{\frac{\Delta_1^2}{\beta}} \left(1 + O(e^{-\frac{\Delta_1}{\beta}})\right)$ with $\Delta_1 > 0$, when $|\beta| \to 0$.

---

6 Analogous bounds for the spectral density weighted by the squares of OPE coefficients in 1d CFTs were recently derived in [58].
for \( \text{Re}[\beta] > 0 \). Then the integrated spectral density satisfies

\[
F_\rho(\Delta) \equiv \int_0^\Delta d\Delta' \rho(\Delta') = \frac{1}{2\pi} \left( \frac{3}{c\Delta} \right)^{1/4} e^{2\pi\sqrt{\frac{2}{3}}[1 + O(\Delta^{-1/2})]} , \quad \Delta \to \infty .
\]

The RHS of (2.8) comes from the unit operator in the dual modular channel, which dominates the partition function at high temperatures. The average of the physical density of states in the LHS side of (2.8) is a discontinuous “staircase-like” function. It is approximated by a smooth function in the RHS of (2.8) with a bounded error term. The discontinuities of the LHS of (2.8) are hidden in the non-universal error term in the RHS. In particular, it does not make sense to write further smooth power suppressed terms in the RHS of (2.8). We will see it explicitly in the example of 2d Ising model that the error term is a highly oscillating function and cannot be approximated by a smooth function. This example will also demonstrate that the error estimate is optimal.

The asymptotic (2.4, 2.5) of the microcanonical entropy for energy shells \( \delta \sim \Delta^\alpha, 0 < \alpha < 1 \) follows directly from (2.8). Further, using this theorem we derive a

\[\text{94}\]
bound on the cutoff partition function at finite temperature:

\[
\int_0^\Delta d\Delta' \rho(\Delta') e^{-\beta(\Delta'-c/12)} = \int_0^\Delta d\Delta' \rho_0(\Delta') e^{-\beta(\Delta'-c/12)} + Z \left( \frac{4\pi^2}{\beta} \right) - e^{\frac{\pi^2c}{3\beta}} + O \left( \Delta^{-3/4} e^{2\pi \sqrt{\frac{\pi\Delta-3\Delta}} \beta} \right), \quad \beta > \pi \sqrt{\frac{c}{3\Delta}} , \quad (2.9)
\]

where \(\rho_0\) is the vacuum crossing kernel defined below. Depending on the temperature some operators in the dual channel in the RHS of (2.9) dominate over the error term and therefore are captured by the cutoff partition function in the LHS.

2.1.1 Related Works

The averaging procedure (2.8) was first pointed out in the context of CFTs in \[23\]. In the mathematical literature the asymptotic (2.8) without the error estimate is known as Ingham’s tauberian theorem for large Laplace transform \[59\]. For a nice exposition of this result see \[16\], Section IV.21. The relevance of Ingham’s theorem for Cardy formula was also emphasized in \[60\], Appendix C. We give a derivation of (2.8), which is different from the original proof \[59\]. The novelty of (2.8) is the error estimate which is absent in the Ingham’s theorem. In the proof

\[8\] And a similar bound for \(\beta < \pi \sqrt{\frac{c}{3\Delta}}\).
we use the methods of [17], Section 2.3, extensively discussed in [52]. In particular, the error estimate allows us to access subleading operators in the cutoff partition function 2.9.

2.2 Setup

Consider a unitary 2d CFT on a torus with the modular parameter \( \tau = \frac{1}{2\pi} (\theta + i\beta) \) and the coordinate on the torus \( z = \frac{1}{2\pi} (\phi + it_E) \) with standard identifications \( z \sim z + 1 \sim z + \tau \). In these conventions the spatial circle \( \phi \) has length \( 2\pi \) and the Euclidean time circle \( t_E \) has length \( \beta \). The partition function

\[
Z(\tau, \bar{\tau}) = \text{Tr} q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24}, \quad q = e^{2\pi i \tau} \tag{2.10}
\]

is invariant under the modular transformation \( \tau \rightarrow -1/\tau \). In what follows we restrict to zero angular potential \( \theta = 0 \) so that \( q = e^{-\beta} \). However, we consider complex \( \beta \) with \( \text{Re}[\beta] > 0 \). This is possible due to unitarity. In this case the modular invariance is expressed by

\[
Z(\beta) = Z \left( \frac{4\pi^2}{\beta} \right), \quad \text{Re}[\beta] > 0, \tag{2.11}
\]

9Unitarity implies that degeneracies of operators are positive. Therefore, for complex \( \beta \) the trace in 2.10 converges even better than for real \( \beta \) and, hence, finite.
or, equivalently,

\[
\int_0^\infty d\Delta \rho(\Delta) e^{-\beta(\Delta - c/12)} = \int_0^\infty d\Delta \rho(\Delta) e^{-\frac{4\pi^2}{\beta}(\Delta - c/12)},
\]

(2.12)

where the density of states is defined by

\[
\rho(\Delta) = \sum_\mathcal{O} \delta(\Delta - \Delta_\mathcal{O})
\]

(2.13)

and the sum is over all operators in the theory, both primaries and descendants.

We will be interested in exploring consequences of \(2.11\). \[10\]

In the high-temperature limit \(|\beta| \to 0\) the RHS of \(2.11\) is dominated by the unit operator

\[
Z(\beta) = e^{\frac{\pi^2}{4\beta}} \left[ 1 + O(e^{-\frac{4\pi^2}{\beta}\Delta_1}) \right],
\]

(2.14)

where \(\Delta_1\) is the first operator above the vacuum.

To write the asymptotic of spectral density it will convenient to introduce a “naive” spectral density \(\rho_0(\Delta)\) which correctly reproduces the contribution of

\[10\text{For some rational CFTs the solutions to }2.11\text{ were classified [61].}\]
the vacuum in the partition function. The correct expression takes the form

$$
\rho_0(\Delta) = \pi \sqrt{\frac{c}{3}} \frac{I_1(2\pi \sqrt{\frac{c}{3} (\Delta - \frac{c}{12})})}{\sqrt{\Delta - \frac{c}{12}}} \theta(\Delta - c/12) + \delta(\Delta - c/12)
$$

$$
= \left( \frac{c}{48\Delta^3} \right)^{1/4} e^{2\pi \sqrt{c\Delta/3}} \left[ 1 + O(\Delta^{-1/2}) \right] \theta(\Delta - c/12) + \delta(\Delta - c/12) \tag{2.15}
$$

where $\theta(x)$ is the Heaviside step function. This, of course, cannot be literally an approximation of the physical density of states $2.13$, as the latter is a sum of delta functions. The index “0 in the LHS of $2.15$ is reminding us of that. Nevertheless, the Laplace transform of $2.15$ coincides with the unit operator contribution into the partition function

$$
\int_0^\infty d\Delta \rho_0(\Delta) e^{-\beta(\Delta - c/12)} = e^{\frac{\pi^2}{3\beta}}. \tag{2.16}
$$

The function $\rho_0(\Delta)$ can be naturally called “crossing kernel” in analogy with $62$.

### 2.3 HKS Bound on Heavy Operators

An important result for obtaining bounds on the spectral density at finite $\Delta$ will be the bound of Hartman, Keller, Stoica (HKS bound) $53$ on the contribution of heavy operators into the partition function. We review its derivation in
this section.

We split the partition function as

\[ Z(\beta) = Z_L(\beta) + Z_H(\beta), \]

\[ Z_L(\beta) = \sum_{\Delta < \Delta_H} e^{-\beta(\Delta - c/12)}, \quad Z_H(\beta) = \sum_{\Delta \geq \Delta_H} e^{-\beta(\Delta - c/12)}. \]  \hspace{1cm} (2.17)

Modular invariance states that

\[ Z_L + Z_H = Z'_L + Z'_H, \]  \hspace{1cm} (2.18)

where by primes we denote the dual channel \( \beta' = \frac{4\pi^2}{\beta} \). Suppose \( \beta \geq 2\pi \). We would like to estimate \( Z_H \)

\[ Z_H = \sum_{\Delta \geq \Delta_H} e^{-(\beta - \beta')(\Delta - c/12)} e^{-\beta'(\Delta - c/12)} \leq e^{-(\beta - \beta')(\Delta_H - c/12)} Z'_H \]

\[ = e^{-(\beta - \beta')(\Delta_H - c/12)} (Z_H + Z_L - Z'_L). \]  \hspace{1cm} (2.19)

Now if \( \Delta_H > c/12 \) then (2.19) implies an upper bound on \( Z_H \)

\[ Z_H \leq e^{-(\beta - \beta')(\Delta_H - c/12)} \frac{Z_L - Z'_L}{1 - e^{-(\beta - \beta')(\Delta_H - c/12)}}, \quad \beta \geq 2\pi. \]  \hspace{1cm} (2.20)
This also implies a bound on $Z_H'$ via modular invariance

$$Z_H' = Z_H + Z_L - Z_L' \leq \frac{Z_L - Z_L'}{1 - e^{-(\beta - \beta') \left( \Delta_H - c/12 \right)}}, \quad \beta \geq 2\pi . \quad (2.21)$$

Exchanging $\beta$ and $\beta'$ in (2.21) we can turn it into a bound at high temperatures

$$Z_H \leq \frac{Z_L' - Z_L}{1 - e^{-(\beta' - \beta) \left( \Delta_H - c/12 \right)}}, \quad \beta \leq 2\pi . \quad (2.22)$$

Depending on the temperature the bound on the heavy operators is either (2.20) or (2.22). Everywhere we assume that $\Delta_H > c/12$.

Finally, (2.20), (2.22) lead to bounds on the full partition function

$$Z \leq \frac{1}{1 - e^{-(\beta - \beta') \left( \Delta_H - c/12 \right)}} \left[ Z_L - e^{-(\beta - \beta') \left( \Delta_H - c/12 \right)} Z_L' \right], \quad \beta \geq 2\pi ,$$
$$Z \leq \frac{1}{1 - e^{-(\beta' - \beta) \left( \Delta_H - c/12 \right)}} \left[ Z_L' - e^{-(\beta' - \beta) \left( \Delta_H - c/12 \right)} Z_L \right], \quad \beta \leq 2\pi . \quad (2.23)$$

Note that the bounds (2.20), (2.22) stay finite if we take $\beta \to 2\pi$. Indeed, $Z_L - Z_L'$ is zero and cancels the zero of the denominator. Whereas $\Delta_H$ is strictly above the BTZ threshold $\frac{c}{12}$. 

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2.4 Local Bound on the Number of Operators

We can use modular invariance together with the HKS bound to derive a local bound on the density of operators. To that end let us consider two functions $\phi_{\pm}(\Delta)$ such that

$$
\phi_- (\Delta') \leq \theta_{[\Delta - \delta, \Delta + \delta]}(\Delta') \leq \phi_+ (\Delta'),
$$

(2.24)

where $\theta_{[\Delta - \delta, \Delta + \delta]}(\Delta') = \theta (\Delta' \in [\Delta - \delta, \Delta + \delta])$.

We can multiply this inequality by $e^{-\beta \Delta'}$ and use $e^{\beta(\Delta - \delta)} e^{-\beta \Delta'} \theta_{[\Delta - \delta, \Delta + \delta]} \leq \theta_{[\Delta - \delta, \Delta + \delta]}$ to write

$$
e^{\beta (\Delta - \delta)} e^{-\beta \Delta'} \phi_- (\Delta') \leq \theta_{[\Delta - \delta, \Delta + \delta]}(\Delta') \leq e^{\beta (\Delta + \delta)} e^{-\beta \Delta'} \phi_+ (\Delta').
$$

(2.25)

Integrating both sides of (2.25) with the spectral density $\int_{0}^{\infty} dF(\Delta')$ we finally obtain an estimate

$$
e^{\beta (\Delta - \delta)} \int_{0}^{\infty} dF(\Delta') e^{-\beta \Delta'} \phi_- (\Delta') \leq \int_{\Delta - \delta}^{\Delta + \delta} dF(\Delta') \leq e^{\beta (\Delta + \delta)} \int_{0}^{\infty} dF(\Delta') e^{-\beta \Delta'} \phi_+ (\Delta').
$$

(2.26)

In the inequality above $\beta$ and $\delta$ are free parameters. We will fix $\beta$ below by
making the bound optimal.

Next the idea is to do the Fourier transform \( \phi_{\pm}(\Delta) = \int_{-\infty}^{\infty} dt \, \hat{\phi}_{\pm}(t)e^{-i\Delta t} \) which turns (2.26) into a bound in terms of the partition function

\[
e^{\beta(\Delta-\delta)} \int_{-\infty}^{\infty} dt \, \hat{\phi}_{-}(t) \mathcal{L}_\rho(\beta+it) \leq \int_{\Delta-\delta}^{\Delta+\delta} dF(\Delta') \leq e^{\beta(\Delta+\delta)} \int_{-\infty}^{\infty} dt \, \hat{\phi}_{+}(t) \mathcal{L}_\rho(\beta+it),
\]

(2.27)

where we introduced the Laplace transform \( \mathcal{L} \) of a density \( \rho \)

\[
\mathcal{L}_\rho(\beta) \equiv \int_0^{\infty} d\Delta \, \rho(\Delta) e^{-\beta\Delta}.
\]

(2.28)

As a next step we apply a modular transformation to \( \mathcal{L}(\beta + it) \) and separate the contribution of light and heavy operators in the dual channel. We write \( \mathcal{L}(\beta + it) = e^{-(\beta+it)c/12}Z(\beta + it) = e^{-(\beta+it)c/12}Z\left(\frac{4\pi^2}{\beta+it}\right) \) and split \( Z = Z_L + Z_H \). As in (2.16) we can rewrite \( e^{-(\beta+it)c/12}Z_L\left(\frac{4\pi^2}{\beta+it}\right) = \mathcal{L}_{\rho_0,L}(\beta + it), \) where the superscript \( \rho_0 \) refers to the fact that the Laplace transform is computed with the crossing
kernel rather than the density of actual physical operators. In this way we get

\[ e^{\beta(\Delta - \delta)} \left( \int_{-\infty}^{\infty} dt \, \hat{\phi}_-(t) L_{\rho_0, L} (\beta + it) - \left| \int_{-\infty}^{\infty} dt \, \hat{\phi}_-(t) e^{-(\beta + it)} \frac{\pi}{4} Z_H \left( \frac{4\pi^2}{\beta + it} \right) \right| \right) \]

\[ \leq \int_{\Delta - \delta}^{\Delta + \delta} dF(\Delta') \leq \]

\[ e^{\beta(\Delta + \delta)} \left( \int_{-\infty}^{\infty} dt \, \hat{\phi}_+(t) L_{\rho_0, L} (\beta + it) + \left| \int_{-\infty}^{\infty} dt \, \hat{\phi}_+(t) e^{-(\beta + it)} \frac{\pi}{4} Z_H \left( \frac{4\pi^2}{\beta + it} \right) \right| \right) . \]

(2.29)

We will see below that the light contribution produces the expected Cardy behavior, whereas the contribution of the heavy operators we can estimate using the HKS bound. First, we estimate \( \left| Z_H \left( \frac{4\pi^2}{\beta + it} \right) \right| \leq Z_H \left( \frac{4\pi^2\beta}{\beta^2 + t^2} \right) \) by removing phases. Then the RHS of the HKS bound \( 2.22 \) diverges exponentially as \( t \to \infty \) when applied to \( Z_H \left( \frac{4\pi^2\beta}{\beta^2 + t^2} \right) \). Therefore we require that \( \hat{\phi}_\pm(t) \) is decaying sufficiently rapidly at \( t \to \infty \) so that the integrals in \( 2.29 \) converge.

One simple choice is to take \( \hat{\phi}_\pm(t) \) with support in a bounded region \( t \in \)

\[ \text{\textsuperscript{11}} \]

Here it is implied that the crossing kernel \( \rho_0 \) is not only for the vacuum \( 2.16 \), but for all light operators entering \( Z_L \). Though in the large \( \Delta \) analysis below the vacuum contribution will be dominant.
[−Λ±, Λ±]. We then have

\[
\left| \int_{-\Lambda_-}^{\Lambda_-} dt \, \hat{\phi}_-(t) e^{-\beta t} \right| e^{-\beta c/12} Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + \Lambda_-^2} \right) \int_{-\Lambda_-}^{\Lambda_-} dt \, |\hat{\phi}_-(t)|, \\
\left| \int_{-\Lambda_+}^{\Lambda_+} dt \, \hat{\phi}_+(t) e^{-\beta t} \right| e^{-\beta c/12} Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + \Lambda_+^2} \right) \int_{-\Lambda_+}^{\Lambda_+} dt \, |\hat{\phi}_+(t)|, 
\]

(2.30)

where it was absolutely crucial that the theory under consideration is unitary.

The contribution of the heavy operators can be bounded using the HKS bound 2.22 or 2.20. Also rewriting the first term in 2.27 back in \( \Delta \)-space we have

\[
e^{\beta(\Delta-\delta)} \left( \int_0^\infty d\Delta' \rho_0(\Delta') e^{-\beta \Delta'} \hat{\phi}_-(\Delta') - e^{-\beta c/12} Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + \Lambda_-^2} \right) \int_{-\Lambda_-}^{\Lambda_-} dt \, |\hat{\phi}_-(t)| \right) \\
\leq \int_{\Delta-\delta}^{\Delta+\delta} dF(\Delta') \leq \\
e^{\beta(\Delta+\delta)} \left( \int_0^\infty d\Delta' \rho_0(\Delta') e^{-\beta \Delta'} \hat{\phi}_+(\Delta') + e^{-\beta c/12} Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + \Lambda_+^2} \right) \int_{-\Lambda_+}^{\Lambda_+} dt \, |\hat{\phi}_+(t)| \right). 
\]

(2.31)

We do not know what is the best choice of \( \phi_\pm(\Delta') \) within the class of functions with the Fourier transform of finite support and satisfying 2.24 that make
the bounds optimal. A simple and convenient choice is

\[
\phi_+ (\Delta') = \left( \frac{\sin \left( \frac{\Lambda_+ \delta}{4} \right)}{\frac{\Lambda_+ \delta}{4}} \right)^4 \left( \frac{\sin \left( \frac{\Lambda_+ (\Delta' - \Delta)}{4} \right)}{\frac{\Lambda_+ (\Delta' - \Delta)}{4}} \right)^4 ,
\]

\[
\phi_- (\Delta') = \left( \frac{\sin \left( \frac{\Lambda_- (\Delta' - \Delta)}{4} \right)}{\frac{\Lambda_- (\Delta' - \Delta)}{4}} \right)^4 \left( 1 - \left( \frac{\Delta' - \Delta}{\delta} \right)^2 \right)^2 .
\] (2.32)

Note that these functions indeed satisfy \[2.24\] and their Fourier transform has a bounded support. Moreover, for this particular choice we have \[\int_{-\Lambda_+}^{\Lambda_+} dt \left| \hat{\phi}_+ (t) \right| = 1.\] Similarly, for \(\frac{1}{\sqrt{\Lambda_+ \delta}} < \frac{1}{12}\) we have \[\int_{-\Lambda_-}^{\Lambda_-} dt \left| \hat{\phi}_- (t) \right| = 1.\] These are the values relevant for our finite \(\Delta\) results in the 2d Ising section.

### 2.4.1 Bounds at large \(\Delta\)

The bound \[2.31\] substantially simplifies in the limit \(\Delta \gg 1\). Below we will see that in this case the optimal choice is \(\beta = \pi \sqrt{\frac{c}{3 \Delta}} \ll 1\). Using the HKS bound we can show that the second terms in \[2.31\] proportional to \(Z_H\) are subleading for \(\Lambda_\pm < 2\pi\). Indeed we get

\[
e^{\beta \Delta} Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + \Lambda_\pm^2} \right) \sim e^{\beta \Delta} e^{\frac{\pi^2}{4} \left( \frac{\Lambda_\pm}{2\pi} \right)^2} \sim \rho_0 (\Delta)^{1 + \frac{1}{2} \left( \frac{\Lambda_\pm}{2\pi} \right)^2 - 1} ,
\] (2.33)
which will be subleading for \( \Lambda_\pm < 2\pi \) (we will see it momentarily below). Therefore we get the bound at large \( \Delta \)

\[
e^{\beta(\Delta-\delta)} \int_0^{\infty} d\Delta' \rho_0(\Delta')\phi_-(\Delta') e^{-\beta\Delta'} \\
\leq \int_{\Delta-\delta}^{\Delta+\delta} dF(\Delta') \leq \\
e^{\beta(\Delta+\delta)} \int_0^{\infty} d\Delta' \rho_0(\Delta')\phi_+(\Delta') e^{-\beta\Delta'} . \tag{2.34}
\]

The integrals can be computed by the saddle point approximation and give

\[
c_\pm \rho_0(\Delta) \leq \frac{1}{2\delta} \int_{\Delta-\delta}^{\Delta+\delta} dF(\Delta') \leq c_\pm \rho_0(\Delta) , \\
c_\pm = \frac{1}{2} \int_{-\infty}^{\infty} dx \phi_\pm(\Delta + \delta x) . \tag{2.35}
\]

We see that dropping the terms \( \Lambda.73 \) is indeed justified for \( \Lambda_\pm < 2\pi \). The explicit integration of \( \Lambda.32 \) gives

\[
c_+ = \frac{\pi}{3} \frac{(\delta\Lambda_+/4)^3}{[\sin(\delta\Lambda_+/4)]^4} , \quad c_- = \frac{4\pi}{3(\delta\Lambda_-)^3}[(\delta\Lambda_-)^2 - 12] . \tag{2.36}
\]
Note that for $\delta$ such that $c_- > 0$ we have to have at least one operator in the interval $[\Delta - \delta, \Delta + \delta]$ since in this case

$$
\int_{\Delta - \delta}^{\Delta + \delta} dF(\Delta') > 0. \tag{2.37}
$$

This happens if

$$
\delta^2 > \frac{12}{\Lambda_-^2} > \frac{3}{\pi^2} \equiv \delta_{\text{gap}}^2, \tag{2.38}
$$

where we also used the assumption $\Lambda_- < 2\pi$ to drop the term $A.73$. That is for the simple choice of functions $2.32$ we get $\delta_{\text{gap}}^2 = \frac{3}{\pi^2}$, which is to say that every modular invariant partition function has to have at least one operator within the window of size $2\delta_{\text{gap}} = \frac{2\sqrt{3}}{\pi} \approx 1.1$ at large $\Delta$. Of course, this is completely trivial in 2d CFTs due to the Virasoro descendants. However, in section 6 we will see that the same argument applies to Virasoro primaries as well provided $c > 1$ and with the same result. It is natural to conjecture that the maximum allowed spacing between Virasoro primary operators is in fact 1.

Similarly, keeping $\delta$ arbitrary we can optimize over $0 < \Lambda_\pm < 2\pi$ to get the
tightest possible bound \[ 2.35 \]. For the lower bound the result is

\[
\frac{F(\Delta + \delta) - F(\Delta - \delta)}{2\delta} \geq \frac{4\pi}{27} \rho_0(\Delta) \approx 0.46\rho_0(\Delta), \quad \delta \geq \frac{3}{\pi},
\]

\[
\frac{F(\Delta + \delta) - F(\Delta - \delta)}{2\delta} \geq \frac{2(\delta^2 - \delta_{\min}^2)}{3\delta^3} \rho_0(\Delta), \quad \delta < \frac{3}{\pi}.
\]

(2.39)

and for the upper bound

\[
\frac{F(\Delta + \delta) - F(\Delta - \delta)}{2\delta} \leq \frac{\pi}{3} \frac{(a_*/4)^3}{\sin(a_*/4)} \rho_0(\Delta) \approx 2.02\rho_0(\Delta), \quad \delta \geq \frac{a_*}{2\pi},
\]

\[
\frac{F(\Delta + \delta) - F(\Delta - \delta)}{2\delta} \leq \frac{\pi}{3} \frac{(\pi\delta/2)^3}{\sin(\pi\delta/2)} \rho_0(\Delta), \quad \delta < \frac{a_*}{2\pi},
\]

(2.40)

where \( a_* \) is the positive solution of the equation

\[
a_* = 3 \tan(a_*/4), \quad a_* \approx 3.38.
\]

(2.41)

This bounds the number of “outliers” and shows what is the maximal local deviation of the density of operators from the Cardy distribution. Note that ?? already imply Cardy formula in the sense of entropies

\[
S_\delta(\Delta) = \log \int_{\Delta-\delta}^{\Delta+\delta} dF(\Delta') = 2\pi \sqrt{\frac{c\Delta}{3}} + \frac{1}{4} \log \left( \frac{c}{48\Delta^3} \right) + \log 2\delta + s(\delta, \Delta),
\]

(2.42)
where \( s \) is of \( O(1) \) and can be bounded from \( \ref{2.39}, \ref{2.40} \). We find

\[
\log \frac{2(\delta^2 - \delta_{\text{min}}^2)}{3\delta^3} \leq s(\delta, \Delta) \leq \log \left( \frac{2}{3\delta \sin(\pi \delta/2)^4} \right), \quad 0 \leq \delta \leq \frac{a_*}{2\pi},
\]

\[
\log \frac{2(\delta^2 - \delta_{\text{min}}^2)}{3\delta^3} \leq s(\delta, \Delta) \leq \log \left( \frac{\pi (a_*/4)^3}{3 \sin(a_*/4)^4} \right), \quad \frac{a_*}{2\pi} \leq \delta \leq \frac{3}{\pi},
\]

\[-0.76 \approx \log \frac{4\pi}{27} \leq s(\delta, \Delta) \leq \log \left( \frac{\pi (a_*/4)^3}{3 \sin(a_*/4)^4} \right) \approx 0.70, \quad \delta \geq \frac{3}{\pi}. \quad (2.43)
\]

The formula \( \ref{2.42} \) is valid up to corrections suppressed at large \( \Delta \). The \( O(1) \) contribution \( s(\delta, \Delta) \) is generically an oscillating function of \( \Delta \). We will observe this explicitly in the 2d Ising model. The bounds \( \ref{2.43} \) are plotted in the \( \ref{2.1} \).

It would be interesting to find the optimal bounds on the local density of operators by a better choice of \( \phi_{\pm} \). To reiterate, in our argument these obey two defining properties: they satisfy \( \ref{2.24} \); they have a finite support in Fourier space \( \ref{A.73} \). Let us also emphasize that the bounds \( \ref{2.27}, \ref{2.31} \) are applicable at finite \( \Delta \) as well. In this case we should simply keep the terms \( \ref{2.30} \) which we can estimate using the HKS bound.

---

\(^{12}\)The latter can be slightly relaxed: it is sufficient to assume rapid decay at \( t \to \infty \) so that the integrals in \( \ref{2.27} \) converge after using the HKS bound. We have not explored this possibility.
2.5 Proof of the Theorem

In the previous section we investigated a local bound on the number of operators in a 2d CFT. In this section we derive a better bound for the case $\delta \gg 1$.

In particular we show that if $\Delta \gg 1$ then averaging $\rho(\Delta)$ over operators in the region $[\Delta - \delta, \Delta + \delta]$ with $\delta \sim \Delta^\alpha$ for some $\alpha > 0$ produces the fixed asymptotic identical to the one given by the crossing kernel $\rho_0(\Delta)$ with the controlled error $2.5$. As mentioned in the introduction it follows from the theorem $2.8$. We prove $2.8$ in this section which we repeat for convenience here

$$F(\Delta) \equiv \int_0^\Delta d\Delta' \rho(\Delta') = \frac{1}{2\pi} \left( \frac{3}{c\Delta} \right)^{1/4} e^{2\pi \sqrt{\frac{3}{c\Delta}}} \left[ 1 + O(\Delta^{-1/2}) \right], \quad \Delta \to \infty.$$ (2.44)

Few comments are in order. Note that by doing a naive inverse Laplace transform of the vacuum contribution, using the saddle point approximation, and integrating over $\Delta$ one would arrive at the correct estimate for $F(\Delta)$, namely $2.44$. Using the saddle point approximation to make a statement about $\rho(\Delta)$ itself however is not correct. It would be also incorrect to use the saddle point approximation to compute further corrections to $F(\Delta)$, beyond $2.44$.

Let us introduce the difference between the Laplace transform of the physical
density of states $\rho(\Delta)$ and the crossing kernel $\rho_0(\Delta)$

$$
\delta \mathcal{L}(\beta) = \mathcal{L}_\rho(\beta) - \mathcal{L}_{\rho_0}(\beta), \quad \delta \rho(\Delta) = \rho(\Delta) - \rho_0(\Delta) .
$$

(2.45)

The main idea is to apply a linear functional to the modular invariance equation 2.18 that produces the theta-function $\theta(\Delta - \Delta')$ that we want plus terms which we can easily estimate. A convenient choice of the functional is

$$
\frac{1}{2\pi i} \int_{\beta-i\Lambda}^{\beta+i\Lambda} \frac{dz}{z} \left[ \Lambda^2 + (z - \beta)^2 \right] e^{z\Delta} \delta \mathcal{L}(z) ,
$$

(2.46)

where the integration contour is the interval $C_+ = \{ \text{Re } z = \beta, -\Lambda < \text{Im } z < \Lambda \}$ as indicated in the figure 2.2. The parameters $\Lambda, \beta, \Delta$ are so far arbitrary in 2.46. The polynomial in the numerator of 2.46 is chosen to be such that it vanishes at the ends of the interval $C$, which will be helpful in estimates below.

On the one hand we can estimate 2.46 using modular invariance. Inserting the definition of the Laplace transform and swapping the order of integrations...
Figure 2.2: Integration contour in the complex temperature $z$-plane. We integrate the modular invariance equation 2.18 along the vertical segment $C_+$ to derive the bound on the integrated spectral density.

we have

$$\frac{1}{2\pi i} \int_{\beta - i\Lambda}^{\beta + i\Lambda} \frac{dz}{z} \frac{\Lambda^2 + (z - \beta)^2}{\Lambda^2 + \beta^2} e^{z\Delta} \delta \mathcal{L}(z) = \int_0^\infty d\Delta' \delta \rho(\Delta') G(\Delta - \Delta'),$$

$$G(\nu) = \frac{1}{2\pi i} \int_{\beta - i\Lambda}^{\beta + i\Lambda} \frac{dz}{z} \frac{\Lambda^2 + (z - \beta)^2}{\Lambda^2 + \beta^2} e^{-\nu z}, \quad \nu \equiv \Delta' - \Delta \tag{2.47}$$

Now the idea is to deform the contour $C_+$ in the last integral in 2.47 either to the left or to the right for $\Delta' < \Delta$ or $\Delta' > \Delta$ respectively in order to make the exponential factor $e^{(\Delta - \Delta')z}$ smaller. When we deform to the left we also pick up
the residue at \( z = 0 \). We have

\[
G(\nu) = \theta(-\nu) + \theta(\nu)G_+(\nu) + \theta(-\nu)G_-(\nu),
\]

where \( G_\pm(\nu) \) refer to the integrals over the arcs \( C_\pm \), see \( 2.2 \).

We can use \( 2.48 \) to rewrite the equation \( 2.47 \) as follows

\[
\int_0^\Delta d\Delta' \delta\rho(\Delta') = \frac{1}{2\pi i} \int_{\beta-i\Lambda}^{\beta+i\Lambda} \frac{dz}{z} \frac{\Lambda^2 + (z - \beta)^2}{\Lambda^2 + \beta^2} e^{z\Delta} \delta L(z) - \int_0^\infty d\Delta' \delta\rho(\Delta')[\theta(\Delta' - \Delta)G_+(\Delta - \Delta') + \theta(\Delta - \Delta')G_-(\Delta\Delta')]\]

In appendix B.1 we show that \( ^{13} \)

\[
|G_\pm(\nu)| \leq 2e^{-\beta\nu} \min[1, (\Lambda\nu)^{-2}] .
\]

\(^{13}\)The overall coefficient in this estimate is not optimal and can be improved, but it will be enough for our purposes.
Therefore we can bound \[2.49\] as follows

\[
\left| \int_0^\Delta d\Delta' \delta \rho(\Delta') \right| \leq \left| \frac{1}{2\pi i} \int_{\beta-i\Lambda}^{\beta+i\Lambda} \frac{dz}{z} \frac{\Lambda^2 + (z - \beta)^2}{\Lambda^2 + \beta^2} e^{z\Delta} \delta \mathcal{L}(z) \right|
\]
\[+ 2e^{\beta\Delta} \int_0^\infty dF(\Delta') e^{-\beta\Delta'} \min \left[ 1, \frac{1}{\Lambda^2(\Delta' - \Delta)^2} \right], \]
\[+ 2e^{\beta\Delta} \int_0^\infty dF_0(\Delta') e^{-\beta\Delta'} \min \left[ 1, \frac{1}{\Lambda^2(\Delta' - \Delta)^2} \right] \tag{2.51} \]

where we used the fact that \(|\delta \rho(\Delta')| \leq \rho(\Delta') + \rho_0(\Delta').\)

In the formula above \(\beta\) is an arbitrary parameter. We would like to choose it to optimize the bound. We will show below that in order to prove \[2.8\] the correct choice is to set

\[
\beta = \pi \sqrt{\frac{c}{3\Delta}}. \tag{2.52}
\]

Let us emphasize that the bound \[A.52\] is valid for finite \(\Delta\). In particular we can use the HKS bound to estimate the first term in the RHS of \[A.52\] and the local bound from the previous section to bound the second term. Below we investigate \[A.52\] in the large \(\Delta\) limit.
To estimate the third integral in the RHS of \( A.52 \) we use the asymptotic 2.15

\[
e^\beta \Delta \int_0^\infty d\Delta' \Delta'^{-3/4} e^{2\pi \sqrt{\Delta^3/3} - \beta \Delta'} \text{min}[1, (\Delta - \Delta')^{-2}] = O \left( \Delta^{-3/4} e^{2\pi \sqrt{\Delta^3/3}} \right). \tag{2.53}
\]

The saddle here is at \( \Delta' = \Delta \). Notice the importance of \( \text{min}[1, (\Delta - \Delta')^{-2}] \). The second argument suppresses the integral over fluctuations \( x = \Delta' - \Delta \) at large \( x \) to produce the correct prefactor in the RHS of 2.53. While the first argument cuts off the integral at small \( x \) and makes it convergent there.

To estimate the second integral in the RHS of \( A.52 \) we split it into three parts \( I_1, I_2, I_3 \)

\[
I_1 + I_2 + I_3 = \left( \int_{\Delta - \Delta^{3/8}}^{\Delta + \Delta^{3/8}} + \int_{\Delta - \Delta^{3/8}} + \int_{\Delta + \Delta^{3/8}}^\infty \right) d\Delta' \rho(\Delta') e^{\beta(\Delta - \Delta')} \text{min}[1, (\Delta - \Delta')^{-2}]
\]

\( \tag{2.54} \)

We would like to show that all three terms are of \( O \left( \Delta^{-3/4} e^{2\pi \sqrt{\Delta^3/3}} \right) \) separately.
For $I_1$ we have

$$I_1 = \int_0^{\Delta - \Delta^3/8} d\Delta' \rho(\Delta') e^{\beta(\Delta - \Delta')} (\Delta - \Delta')^{-2} \leq \Delta^{-3/4} e^{\beta \Delta} \int_0^{\Delta - \Delta^3/8} d\Delta' \rho(\Delta') e^{-\beta \Delta'} \leq \Delta^{-3/4} e^{\beta \Delta} L_\rho(\beta) = O(\Delta^{-3/4} e^{2\pi \sqrt{c/\Delta}}),$$

(2.55)

where we used monotonicity of $(\Delta - \Delta')^{-2}$ in the first line and $L_\rho(\beta) = O(e^{\pi^2/3})$

and 2.52 in the third line. In particular, 2.55 shows that we chose to split the integral as in 2.54 in order to produce the correct prefactor in 2.55 $(\Delta - \Delta')^{-2}|_{\Delta' = \Delta - \Delta^3/8} = \Delta^{-3/4}$. Similarly, $I_3$ is estimated to be of the same order

$$I_3 = \int_{\Delta + \Delta^3/8}^{\infty} d\Delta' \rho(\Delta') e^{\beta(\Delta - \Delta')} (\Delta - \Delta')^{-2} \leq \Delta^{-3/4} e^{\beta \Delta} L_\rho(\beta) = O(\Delta^{-3/4} e^{2\pi \sqrt{c/\Delta}}).$$

(2.56)

Finally, we need to estimate $I_2$. We will do so using a local bound from the previous section

$$F(\Delta + \delta) - F(\Delta - \delta) = O \left( \Delta^{-3/4} e^{2\pi \sqrt{c/\Delta}} \right).$$

(2.57)
We further split the integral $I_2$ into

$$I_2 = \left( \int_{\Delta - \Delta^3/8}^{\Delta - 1} d\Delta' \rho(\Delta') e^{\beta(\Delta - \Delta')} \right) + \left( \int_{\Delta - 1}^{\Delta + 1/8} d\Delta' \rho(\Delta') e^{\beta(\Delta - \Delta')} \right) + \left( \int_{\Delta + 1/8}^{\Delta + \Delta^3/8} d\Delta' \rho(\Delta') e^{\beta(\Delta - \Delta')} \right)$$

$$\min[1, (\Delta - \Delta')^{-2}]$$

$$= O\left( \Delta^{-3/4} e^{2\pi \sqrt{c^2} \Delta^3/8} \sum_{k=2}^{\Delta^3/8} e^{\beta k} \right) = O\left( \Delta^{-3/4} e^{2\pi \sqrt{\frac{c^2}{\Delta^3}}} \right)$$

(2.58)

To estimate $i_{1,2,3}$ we split the integrals into small windows of $\Delta'$ in each of which we can apply 2.57

$$i_1 = \int_{\Delta - \Delta^3/8}^{\Delta - 1} d\Delta' \rho(\Delta') e^{\beta(\Delta - \Delta')} (\Delta - \Delta')^{-2}$$

$$= \sum_{k=2}^{\Delta^3/8} \int_{\Delta - k}^{\Delta - k+1} d\Delta' \rho(\Delta') e^{\beta(\Delta - \Delta')} (\Delta - \Delta')^{-2}$$

$$\leq \sum_{k=2}^{\Delta^3/8} \frac{e^{\beta k}}{(k - 1)^2} \int_{\Delta - k}^{\Delta - k+1} d\Delta' \rho(\Delta') = \sum_{k=2}^{\Delta^3/8} \frac{e^{\beta k}}{(k - 1)^2} [F(\Delta - k + 1) - F(\Delta - k)]$$

$$= O\left( \Delta^{-3/4} e^{2\pi \sqrt{c^2} \Delta^3/8} \sum_{k=2}^{\Delta^3/8} e^{\beta k} \right) = O\left( \Delta^{-3/4} e^{2\pi \sqrt{\frac{c^2}{\Delta^3}}} \right)$$

(2.59)

where we used 2.57, 2.52. The integral $i_3$ is estimated in a similar fashion. Finally,

$$i_2 = \int_{\Delta - 1}^{\Delta + 1} d\Delta' \rho(\Delta') e^{\beta(\Delta - \Delta')} \leq e^\beta [F(\Delta + 1) - F(\Delta - 1)] = O\left( \Delta^{-3/4} e^{2\pi \sqrt{\frac{c^2}{\Delta^3}}} \right)$$

(2.60)
This finishes the estimate of \[2.54\].

The last step is to estimate the first term in the RHS of \[A.52\]. To this we need to use the modularity condition\[14\]

\[
|\delta \mathcal{L}(z)| = \left| e^{-\frac{\pi c}{4\pi^2}} Z_H \left( \frac{4\pi^2}{|z|^2} \right) \right| \leq e^{-\text{Re}[z] c/12} Z_H \left( \frac{4\pi^2 \text{Re}[z]}{|z|^2} \right),
\]

which we can estimate using the vacuum contribution in the dual channel. We get

\[
\left| \frac{1}{2\pi i} \int_{\beta-i\Lambda}^{\beta+i\Lambda} \frac{dz}{z} \frac{\Lambda^2 + (z - \beta)^2}{\Lambda^2 + \beta^2} e^{z\Delta} \delta \mathcal{L}(z) \right| \\
\leq \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} \frac{dt}{|\beta + it|} \frac{\Lambda^2 + t^2}{\Lambda^2 + \beta^2} e^{\beta(\Delta - \frac{c}{12})} Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + t^2} \right) \\
\leq \frac{2\pi \Lambda^3 e^{\beta(\Delta - \frac{c}{12})}}{\beta(\Lambda^2 + \beta^2)} Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + \Lambda^2} \right) = O \left( \Delta^{1/2} e^{\frac{\pi}{12} \sqrt{c/\pi} \left(1 + (\frac{\Lambda}{\beta})^2\right)} \right)
\]

where in the second line we used monotonicity of $Z_H$ and therefore assumed that $\Delta_H > \frac{c}{12}$. In the third line we estimated $Z_H$ using the vacuum contribution in the dual channel. Choosing $\Lambda < 2\pi$ we see that this term is sub-leading. This finishes the proof of \[2.44\].

\[\text{\ding{118}}\]

\[\text{\ding{118}}\]Here we imagine subtracting a finite number of light operators in $\delta \mathcal{L}$ below $\Delta_H > c/12$. This does not affect previous estimates since such light operators would contribute terms analogous to the third term in the RHS of \[A.52\] and would give exponentially small corrections to Cardy growth.
2.6 Virasoro Primaries

The analysis in previous sections can be readily generalized to the density of Virasoro primary operators. Let’s consider $c > 1$ so that there are infinitely many such operators. In this case Virasoro characters are simply related to the Dedekind function and the partition function takes the form, see e.g. [63],

$$Z(\beta) = |\eta(\tau)|^{-2} e^{\beta \frac{i}{12}} \left[ (1 - e^{-\beta})^2 + \sum_{n=1}^{\infty} d_n^{\text{Vir}} e^{-\beta \Delta_n} \right], \quad (2.63)$$

where $\tau = i\beta/2\pi$, $d_n^{\text{Vir}}$ is the degeneracy of a Virasoro primary $\Delta_n$ and the sum goes over all primaries except the vacuum $\Delta_n > 0$. Let’s define the density of Virasoro primaries

$$\rho^{\text{Vir}}(\Delta) = \sum_{n=1}^{\infty} d_n^{\text{Vir}} \delta(\Delta - \Delta_n). \quad (2.64)$$

The crossing kernel for the vacuum is given by

$$\rho^{\text{Vir}}_0(\Delta) = f(\Delta, 0) - 2f(\Delta, 1) + f(\Delta, 2),$$

$$f(\Delta, x) = 2\pi I_0 \left( 4\pi \sqrt{\left( \frac{c - 1}{12} - x \right) \left( \Delta - \frac{c - 1}{12} \right)} \right) \theta \left( \Delta - \frac{c - 1}{12} \right) - \delta(\Delta - 2). \quad (2.65)$$
so that it reproduces the vacuum contribution in the dual channel

\[ |\eta(\tau)|^{-2} e^{\frac{\beta - 1}{12}} \left( 1 - e^{-\beta} \right)^2 + \mathcal{L}_{\rho_0^{\text{Vir}}} = |\eta(\tau')|^{-2} e^{\frac{\beta' - 1}{12}} (1 - e^{-\beta'})^2, \]

\[ \beta' \equiv \frac{4\pi^2}{\beta}, \quad \tau' \equiv -\frac{1}{\tau}, \quad \mathcal{L}_\rho(\beta) = \int_0^\infty \Delta \rho(\Delta) e^{-\beta \Delta}. \quad (2.66) \]

2.6.1 **Local bounds on the number of Virasoro primaries**

We can derive bounds analogous to 2.27 2.31. Essentially the same argument gives \[15\]

\[ e^{\beta(\Delta - \delta)} \int_0^\infty \Delta' \rho_0^{\text{Vir}}(\Delta') e^{-\beta \Delta'} \phi_-(\Delta') \]

\[ - e^{\beta(\Delta - \delta - \frac{\epsilon - 1}{4\pi})} Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + \Lambda_+^2} \right) \int_{-\Lambda_-}^{\Lambda_-} dt \phi_-(t) \left| \eta \left( \frac{i(\beta + it)}{2\pi} \right) \right|^2 \]

\[ \leq \int_{\Delta - \delta}^{\Delta + \delta} d\Delta' \rho_0^{\text{Vir}}(\Delta') e^{-\beta \Delta'} \phi_-(\Delta') \]

\[ e^{\beta(\Delta + \delta)} \int_0^\infty \Delta' \rho_0^{\text{Vir}}(\Delta') e^{-\beta \Delta'} \phi_+ (\Delta') \]

\[ + e^{\beta(\Delta + \delta - \frac{\epsilon - 1}{4\pi})} Z_H \left( \frac{4\pi^2 \beta}{\beta^2 + \Lambda_+^2} \right) \int_{-\Lambda_+}^{\Lambda_+} dt \phi_+(t) \left| \eta \left( \frac{i(\beta + it)}{2\pi} \right) \right|^2. \quad (2.67) \]

\[15\] Here, as in 2.27, it is implied that the crossing kernel \(\rho_0^{\text{Vir}}\) is for all light operators entering \(Z_L\). But again in the large \(\Delta\) analysis below the vacuum contribution will be dominant.
The HKS bound for Virasoro primaries can also be derived and takes the form

\[ Z_H \leq \frac{\beta}{2\pi} e^{-(\beta - \beta')(\Delta_H - \frac{c-1}{12})} - \frac{Z_L - Z'_L}{1 - \frac{\beta}{2\pi} e^{-(\beta' - \beta)(\Delta_H - \frac{c-1}{12})}}, \quad \beta \geq 2\pi, \]

\[ Z_H \leq \frac{Z'_L - Z'_L}{1 - \frac{\beta'}{2\pi} e^{-(\beta' - \beta)(\Delta_H - \frac{c-1}{12})}}, \quad \beta \leq 2\pi. \tag{2.68} \]

where \( \Delta_H > \frac{c-1}{12} \) and we split the partition function into light and heavy contributions

\[ Z_L = |\eta(\tau)|^{-2} e^{\frac{\beta - 1}{2\pi}} \left[ (1 - e^{-\beta})^2 + \sum_{0<\Delta_n<\Delta_H} d_n^\text{Vir} e^{-\beta \Delta_n} \right], \]

\[ Z_H = |\eta(\tau)|^{-2} e^{\frac{\beta - 1}{2\pi}} \sum_{\Delta_n \geq \Delta_H} d_n^\text{Vir} e^{-\beta \Delta_n}. \tag{2.69} \]

The large \( \Delta \) analysis is identical to the section 4 and with essentially the same results. Namely we get

\[ \int_{\Delta - \delta}^{\Delta + \delta} dF(\Delta') > 0, \quad 2\delta > 2\delta_{\text{gap}} = 2\sqrt{\frac{3}{\pi^2}} \tag{2.70} \]

with the choice \[\boxed{2.32}\]. That is the gap between Virasoro primaries at large scaling dimensions must be no larger than \( 2\sqrt{\frac{3}{\pi^2}} \approx 1.1 \).

Repeating the rest of the argument from the section 4 we obtain the asymp-
totic of the microcanonical entropy for energy shells $\delta = O(1)$

$$S_{\delta}^{\text{Vir}}(\Delta) \equiv \log \int_{\Delta - \delta}^{\Delta + \delta} d\Delta' \rho_{\text{Vir}}^{\Delta'}(\Delta')$$

$$= 2\pi \sqrt{\frac{c - 1}{3} - \frac{1}{4} \log \left( \frac{c - 1}{3} - \Delta \right)} + \log(2\delta) + s_{\text{Vir}}^{\delta}(\delta, \Delta), \quad \Delta \to \infty, \quad (2.71)$$

where $s_{\text{Vir}}^{\delta}(\delta, \Delta)$ is again bounded as in 2.1.

**2.6.2 Cardy formula for Virasoro primaries**

The modular invariance dictates the behavior at high temperatures

$$\int_0^{\infty} d\Delta \rho_{\text{Vir}}^{\Delta}(\Delta) e^{-\beta(\Delta - \frac{c-1}{12})} = \frac{2\pi}{\beta} e^{\frac{4\pi^2}{\beta} - \frac{c-1}{12}} \left[ 1 + O\left( \max\left[ e^{-\frac{4\pi^2}{\beta}}, e^{-\frac{4\pi^2}{\beta} \Delta} \right] \right) \right], \quad \beta \to 0.$$  

(2.72)

Then the tauberian theorem similar to 2.8 takes the form

$$\int_0^{\Delta} d\Delta' \rho_{\text{Vir}}^{\Delta'}(\Delta') = \frac{1}{\pi} \left( \frac{3}{c - 1} \right)^{3/4} \Delta^{1/4} e^{2\pi \sqrt{\frac{c-1}{4} \Delta}} \left[ 1 + O(\Delta^{-1/2}) \right]. \quad (2.73)$$

Its proof is completely analogous to the proof of 2.8 given in the section 5. From here we derive that the microcanonical entropy has the asymptotic 2.71 with
for any $0 < \alpha \leq 1/2$.

### 2.7 Holographic CFTs

In this section we consider holographic 2d CFTs with a sparse spectrum in the limit $\Delta \sim c \to \infty$. The HKS sparseness condition states that $Z_L(\beta)$ is dominated by the vacuum state for $\beta > 2\pi$ and $c \to \infty$ in the sense that

$$\sum_{\Delta \leq \Delta_H} e^{-\beta \Delta} = O(1), \quad \beta > 2\pi, \quad c \to \infty . \quad (2.75)$$

We again start with and consider the limit

$$\Delta = c \left( \frac{1}{12} + \epsilon \right), \quad c \to \infty, \quad \epsilon-\text{fixed} . \quad (2.76)$$

In this limit the asymptotic of the vacuum crossing kernel is

$$\rho_0(\Delta) = \frac{1}{2 \cdot 3^{1/4}} e^{-1/2} e^{-3/4} e^{2\pi\epsilon\sqrt{\epsilon/3}} \theta(\epsilon) + \ldots . \quad (2.77)$$
To optimize the first term in Eq. 2.31 we choose

\[ \beta = \frac{\pi}{\sqrt{3}\epsilon}. \]  

As before we find that the second $Z_H$ term in Eq. 2.31 is suppressed if $\Lambda_{\pm} < 2\pi \sqrt{1 - \frac{1}{12\epsilon}}$. Therefore the bound Eq. 2.31 can be dominated by the first term only for $\epsilon > \frac{1}{12}$, i.e. for states with $\Delta > \frac{c}{6}$. In this case we drop the $Z_H$ terms, compute the first term in Eq. 2.31 by the saddle approximation and get

\[ e^{-\frac{\pi\delta}{\sqrt{3}}\rho_0(\Delta)c} \leq \frac{1}{2\delta} \int_{\Delta - \delta}^{\Delta + \delta} dF(\Delta') \leq e^{\frac{\pi\delta}{\sqrt{3}}\rho_0(\Delta)c_+}, \quad c \to \infty, \quad \epsilon \text{--fixed} \]

In Eq. 2.79 we tacitly assumed that the first term in Eq. 2.31 is dominated by the vacuum. For the RHS of Eq. 2.79 this relies on sparseness condition and we give more detail in appendix B.3. In particular, this means that we cannot compute the precise value of $c_+$ because it depends on the bound Eq. 2.75. On the other hand In the LHS of Eq. 2.79 we can simply drop operators above the vacuum since they give positive contribution.

The conclusion is that we have the asymptotic of the microcanonical entropy
of states with energy of $O(c)$

$$\log \int_{\Delta - \delta}^{\Delta + \delta} d\Delta' \rho(\Delta') = 2\pi \sqrt{\frac{c}{3} \left( \Delta - \frac{c}{12} \right)} - \frac{1}{2} \log c + O(1), \quad \Delta > \frac{c}{6}, \quad c \to \infty$$

(2.80)

for fixed $\delta > \delta_{\text{gap}} = \frac{\sqrt{7}}{\pi}$.

We can also consider large widths $\delta \sim c^\alpha, 0 < \alpha < 1$. We estimate by splitting into intervals of $O(1)$

$$\int_{\Delta - \delta}^{\Delta + \delta} d\Delta' \rho(\Delta') = \sum_{k=1}^{2\delta - 1} \int_{\Delta - \delta + k}^{\Delta - \delta + k + 1} d\Delta' \rho(\Delta')$$

(2.81)

and applying the bound $\underline{2.79}$ to each term. Both the upper and lower bounds are dominated by the largest exponent $k = 2\delta - 1$ and are estimated to be

$$O \left( \sum_{k=1}^{2\delta - 1} c^{-1/2} e^{2\pi c \sqrt{\frac{1}{3} \left( \epsilon + \frac{k-\delta}{c} \right)}} \right) = O \left( c^{-1/2} e^{2\pi c \sqrt{\frac{1}{3} \left( \epsilon + \frac{2}{c} \right)}} \right).$$

(2.82)

Therefore we have for the microcanonical entropy

$$S_\delta(\Delta) = \log \int_{\Delta - \delta}^{\Delta + \delta} d\Delta' \rho(\Delta') = 2\pi \sqrt{\frac{c}{3} \left( \Delta + \delta - \frac{c}{12} \right)} - \frac{1}{2} \log c + O(1), \quad c \to \infty ,$$

(2.83)
where $\delta \sim c^\alpha, 0 < \alpha < 1$. For $0 < \alpha \leq \frac{1}{2}$ only the first term in the expansion of the square root dominates the error. For $1/2 < \alpha < 1$ more terms in the expansion of the square root give a contribution. Essentially, the formula 2.83 states that the entropy is dominated by the states in an $O(1)$ window near the upper limit. In [53] it was derived that $S_\delta(\Delta) = 2\pi c\sqrt{\frac{T}{\alpha}} + O(c^\alpha), 1/2 < \alpha < 1$. The formula 2.83 extends their result to all $0 < \alpha < 1$ and computes corrections to it.

It would be interesting to reproduce our result for the microcanonical entropy 2.83 from the direct bulk computation. The leading contribution to the on-shell action is insensitive to the ensemble choice, however the state of the quantum fields in the black hole background changes which should be taken into account when computing the corrections to the leading Cardy formula, see e.g. [64, 65].

Note that the logarithmic correction to the microcanonical entropy 2.83 is completely universal. This feature of AdS black holes in AdS was observed in [19] (see section 5 in that paper) and is due to the fact that there are no translational zero modes in AdS. The situation is drastically different from flat space, where logarithmic corrections to the black hole entropy are sensitive to the low energy spectrum of the theory.
2.8 Accessing Subleading Operators

One way to access the subleading operators in the dual channel is the following.

Consider the modular condition written as

\[
\int_{0}^{\Delta} d\Delta' \, \delta\rho(\Delta') e^{-\beta(\Delta' - c/12)} + \int_{\Delta}^{\infty} d\Delta' \, \delta\rho(\Delta') e^{-\beta(\Delta' - c/12)} = Z \left( \frac{4\pi^2}{\beta} \right) - e^{\frac{x^2}{3\beta}},
\]

\[
\delta\rho(\Delta) \equiv \rho(\Delta) - \rho_0(\Delta),
\]

(2.84)

where we split the partition function into the contribution of light \( \Delta' \leq \Delta \) and heavy \( \Delta' > \Delta \) operators. Intuitively, it is clear that the partition is dominated by light (heavy) operators at small (high) temperatures. More precisely, we claim that at small temperatures modular invariance \( \textbf{2.84} \) can be written as

\[
\int_{0}^{\Delta} d\Delta' \, \delta\rho(\Delta') e^{-\beta(\Delta' - c/12)}
\]

\[
= Z \left( \frac{4\pi^2}{\beta} \right) - e^{\frac{x^2}{3\beta}} + O \left( \Delta^{-3/4} e^{2\pi \sqrt{3\Delta - \beta\Delta}} \right), \quad \beta \geq \pi \sqrt{\frac{c}{3\Delta}}, \quad (2.85)
\]
while at high temperatures

\[
\int_{\Delta}^{\infty} d\Delta' \, \delta \rho(\Delta') e^{-\beta(\Delta' - c/12)} = Z \left( \frac{4\pi^2}{\beta} \right) - e^{\frac{\pi^2 c}{3\beta}} + O \left( \Delta^{-3/4} e^{2\pi \sqrt{\frac{c}{3\beta}}} \right), \quad \beta \leq \pi \frac{c}{3\Delta}. \tag{2.86}
\]

Equivalently, at small (high) temperatures heavy (light) operators are suppressed

\[
\int_{\Delta}^{\infty} d\Delta' \, \delta \rho(\Delta') e^{-\beta(\Delta' - c/12)} = O \left( \Delta^{-3/4} e^{2\pi \sqrt{\frac{c}{3\beta}}} \right), \quad \beta \geq \pi \frac{c}{3\Delta},
\]

\[
\int_{0}^{\Delta} d\Delta' \, \delta \rho(\Delta') e^{-\beta(\Delta' - c/12)} = O \left( \Delta^{-3/4} e^{2\pi \sqrt{\frac{c}{3\beta}}} \right), \quad \beta \leq \pi \frac{c}{3\Delta}. \tag{2.87}
\]

Formulae 2.85, 2.86, 2.87 hold in the limit $\Delta \to \infty$. We derive them below. But first a few comments are in order. As we take $\beta \to \infty$ in 2.85 the LHS is dominated by a few light operators, while the RHS, i.e. the dual channel, receives contribution from a large number of heavy operators entering $Z(4\pi^2/\beta)$. The error term is exponentially small in this case. Similarly in 2.86 as we take $\beta \to 0$ an infinite number of heavy operators dominate the LHS, while a small number of light operators dominate in the RHS. Both cases are therefore consistent with the intuition that a light operator in one channel is reproduced by a large number of heavy operators in the dual channel. The most interesting case is the intermediate regime $\beta \sim \Delta^{-1/2}$ when both channels are dominated by light op-
operators in the following sense. In this case we can tune $\beta$ so that a finite number of light operators beyond the vacuum contribute in the RHS of 2.85. Their effect is then reflected in the density of “light” states $\Delta' < \Delta$ in the LHS of 2.85. This can be thought of as “non-perturbative corrections” to Cardy formula from operators beyond the vacuum and is discussed in more detail below.

Now let us derive 2.85, 2.87. Consider for example $\beta \geq \pi \sqrt{\frac{c}{3\Delta}}$ and the first estimate in 2.87. We write $\delta \rho(\Delta) = \partial_\Delta \delta F(\Delta)$ and integrate by parts to get

$$\int_\Delta^\infty d\Delta' \delta \rho(\Delta') e^{-\beta \Delta'} = -\delta F(\Delta) e^{-\beta \Delta} + \beta \int_\Delta^\infty d\Delta' \delta F(\Delta') e^{-\beta \Delta'},$$

$$\delta F(\Delta) \equiv F_\rho(\Delta) - F_{\rho_0}(\Delta). \tag{2.88}$$

We can estimate this using the error term in the Cardy formula 2.8. We get

$$-e^{-\beta \Delta} \delta F(\Delta) = O \left( \Delta^{-3/4} e^{2\pi \sqrt{\frac{c}{3\Delta}} - \beta \Delta} \right),$$

$$\beta \int_\Delta^\infty d\Delta' \delta F(\Delta') e^{-\beta \Delta'} = O \left( \beta \int_\Delta^\infty d\Delta' \Delta'^{-3/4} e^{2\pi \sqrt{\frac{c}{3\Delta'}} - \beta \Delta'} \right). \tag{2.89}$$

For $\beta > \pi \sqrt{\frac{c}{3\Delta}}$ the saddle point in the last integral is outside of the integration range and therefore it is dominated close to the lower limit $\Delta \to \infty$. As a result we get the first estimate in 2.87.

Similarly, the second estimate in 2.87 is obtained by integration by parts and
using Cardy formula \(2.8\).

The formulae \(2.85, 2.86\) allow us to probe subleading operators in the dual channel. In particular, one might hope to test \(2.85\) numerically for finite \(\Delta\). We will do so in the 2d Ising model in the next section. Let’s see what operators give contributions larger than the error term. Consider an operator with dimension \(\Delta^*\) in the RHS of \(2.85\). Its contribution to the partition function in the dual channel takes the form \(e^{\frac{\pi^2}{3\beta} - \frac{4\pi^2}{\beta}\Delta^*}\). The condition that it is greater than the error term is

\[
\Delta^* \leq \frac{c}{12} \left( \frac{\beta}{\pi \sqrt{\frac{\pi}{3\Delta}}} - 1 \right)^2.
\]  

(2.90)

In particular, \(A.10\) implies that we have to scale \(\beta \sim \Delta^{-1/2}\) if we would like to access a finite number of operators in the dual channel in the limit \(\Delta \to \infty\).

To summarize, the partition function \(2.85\) with the UV cut-off \(\Delta\) and temperature \(\beta > \pi \sqrt{\frac{\pi}{3\Delta}}\) allows to systematically probe the operators in the dual channel satisfying \(A.10\). We will test \(2.85\) numerically in the 2d Ising model in section 6.
Finally, the formulae similar to (2.87) for Virasoro primaries take the form

\[ \int_{\Delta}^{\infty} d\Delta \, \delta \rho(\Delta) e^{-\beta \Delta} = O \left( \Delta^{-1/4} e^{2\pi \sqrt{\frac{c-1}{3\Delta} - \beta \Delta}} \right), \quad \beta \geq \pi \sqrt{\frac{c-1}{3\Delta}}, \]

\[ \int_{0}^{\Delta} d\Delta \, \delta \rho(\Delta) e^{-\beta \Delta} = O \left( \Delta^{-1/4} e^{2\pi \sqrt{\frac{c-1}{3\Delta} - \beta \Delta}} \right), \quad \beta \leq \pi \sqrt{\frac{c-1}{3\Delta}}. \quad (2.91) \]

where \( \delta \rho(\Delta) = \rho(\Delta) - \rho_0(\Delta) \).

2.9 Example: 2d Ising

In this section we check our results in the 2d Ising model. In particular, we will see that the error estimates are optimal. The partition function is given by (2.92)

\[ Z(\beta) = \frac{1}{2} \left| \frac{\theta_2(\tau)}{\eta(\tau)} \right| + \frac{1}{2} \left| \frac{\theta_3(\tau)}{\eta(\tau)} \right| + \frac{1}{2} \left| \frac{\theta_4(\tau)}{\eta(\tau)} \right|, \quad (2.92) \]

where

\[ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad \theta_2(\tau) = 2q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n)^2, \]

\[ \theta_3(\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1/2})^2, \quad \theta_4(\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1/2})^2. \quad (2.93) \]
Figure 2.3: $F_{\rho}(\Delta)$ (blue) and its smooth approximation $F_{\rho_0}(\Delta)$ (orange). To check the error term in 2.8 we plot $(F_{\rho}(\Delta) - F_{\rho_0}(\Delta))\Delta^{3/4}e^{-2\pi\sqrt{\frac{c}{\Delta}}}$ (inside the box). It is oscillating with a constant amplitude, as predicted by 2.8.

and we restrict to zero angular potential $q = e^{-\beta}$ as before and the central charge is $c = \frac{1}{2}$. Expanding the partition function in $q$ we can find degeneracies of operators.

2.9.1 Unit operator

In the figure 2.3 we plot the leading order and the error term for the moment $F_{\rho}(\Delta) = \int_0^\Delta d\Delta' \rho(\Delta')$ and find perfect agreement with 2.44. In particular, it is clear from the figure 2.3 that the error estimate is, in fact, optimal.
2.9.2 \( \sigma \) OPERATOR

Now let’s see how the effect of the first operator above the vacuum \( \Delta_\sigma = \frac{1}{8} \) can be seen from the formula 2.85. According to A.10 for \( \sigma \) to give a contribution bigger than the error term and for \( \Delta_\epsilon = 1 \) be smaller than the error term we require

\[
\Delta_\sigma \leq \frac{c}{12} \left( \frac{\beta}{\pi \sqrt{\frac{3}{8}}} - 1 \right)^2 < \Delta_\epsilon.
\]  

(2.94)

Inserting the numerical values we find that \( \beta \) must be chosen in a window

\[
\frac{2.7\pi}{\sqrt{6\Delta}} \approx \frac{(1 + \sqrt{3})\pi}{\sqrt{6\Delta}} \leq \beta < \frac{(1 + 2\sqrt{6})\pi}{\sqrt{6\Delta}} \approx \frac{5.9\pi}{\sqrt{6\Delta}}.
\]  

(2.95)

We take \( \beta = \frac{4\pi}{\sqrt{6\Delta}} \). The formula 2.85 becomes

\[
\int_0^\Delta d\Delta' \ e^{-\beta\Delta'} \rho(\Delta') = \int_0^\Delta d\Delta' \ e^{-\beta\Delta'} \rho_0(\Delta') + \\
+ e^{-\beta/12} e^{\frac{3\pi}{\sqrt{6\Delta}}} e^{-\frac{4\pi^2}{3\Delta}} \Delta_\sigma + O \left( \Delta^{-3/4} e^{2\pi \sqrt{\frac{3}{8} - \beta \Delta}} \right).
\]  

(2.96)

Below we plot the contribution of \( \sigma \)-operator to 2.96 and find perfect agreement.

One can also plot the error term similarly to the figure 2.3.
Figure 2.4: The difference \( \int_{0}^{\Delta} d\Delta' e^{-\beta \Delta' \delta \rho(\Delta')} \) (blue) and \( e^{-\beta \delta e/12} e^{\frac{\Delta^2}{12} - \frac{\Delta^2}{2}} \) (orange).

2.9.3 Microcanonical Entropy

As discussed in the main text the microcanonical entropy \( S_{\delta}(\Delta) \) takes the universal form 2.4 at high energies \( \Delta \gg 1 \).

Here we explicitly plot the \( O(1) \) correction \( s(\delta, \Delta) \) to the leading behavior of the entropy in the 2d Ising model, see 2.5. In agreement with the general discussion we find that \( s(\delta, \Delta) \) is an oscillating function with oscillations satisfying general bounds 2.5. Note that, strictly speaking, the bound 2.5 was derived in the large \( \Delta \) limit and here we plot it at finite \( \Delta \). We present the finite \( \Delta \) version of the bound 2.31 on the 2.5 as well. Since for 2d Ising model vacuum is the only operator with \( \Delta < \frac{c}{12} \) we use only the vacuum contribution in \( Z_L \) that enters the HKS bound. We then use the HKS bound to estimate \( Z_H \).
2.10 Example: Monster CFT

Let us apply our bounds for the microcanonical entropy to monster CFT [54, 55]. Recall that it describes a chiral CFT with $c = 24$ and the partition function that takes the form

$$Z(q) = J(q) = \frac{1}{q} + 196884q + 21493760q^2 + \ldots .$$

(2.97)

In principle, nothing prevents us from deriving 2.74 for chiral CFTs. We do not do this here. Instead, at zero angular potential and without imposing invariance under $\tau \to \tau + 1$ we can interpret 2.97 as a partition function of a non-chiral
Figure 2.6: We plot $s_{\text{Vir}}(\delta, \Delta)$ (green) versus $\Delta$ as defined in 2.6 for the non-chiral Monster CFT partition function for $\delta = 2.4$ as a function of $\Delta$. The straight lines correspond to the upper $s_{\text{Vir}}(\delta)$ (orange) and lower $s_{\text{Vir}}(\delta)$ (blue) asymptotic bounds. We see that the actual microscopic entropy $s_{\text{Vir}}(\delta, \Delta)$ is oscillatory, however, the amplitude of oscillations lays well within the asymptotic bounds for $\Delta$'s greater $\sim 20$. Of course, as in the 2d Ising model the actual $\Delta$ bounds are weaker and are given by 2.67.

CFT with $c = 12$ that satisfies 2.18. Therefore we can apply the asymptotic 2.6 to it directly.

On 2.6 we see that for finite $\delta$ the difference between the actual microcanonical entropy and the large $\Delta$ expansion satisfies the expected bounds. We can also probe a subleading universal correction by taking $\delta = \Delta^\epsilon$. The result is presented in the figure 2.7.
Figure 2.7: We plot $s_{\text{Vir}}^{\delta}(\delta, \Delta)$ (blue) as defined in 2.6 for the non-chiral Monster CFT partition function for $\delta = \Delta^{0.4}$ as a function of $\Delta$. The orange line is given by the predicted universal correction $\log \frac{\sinh \pi \sqrt{c - \frac{1}{3} \Delta_0}}{\pi \sqrt{c - \frac{1}{3} \Delta_0}}$. In the box we plot $\left( s_{\text{Vir}}^{\Delta^{0.4}}(\Delta^{0.4}, \Delta) - \log \frac{\sinh \pi \sqrt{c - \frac{1}{3} \Delta_0}}{\pi \sqrt{c - \frac{1}{3} \Delta_0}} \right) \Delta^{0.4}$ to check that the non-universal difference between the two curves is consistent with 2.6.
In this chapter we consider CFTs in $d > 2$ with continuous global symmetries.

The spectrum of these CFTs contains operators charged under these symmetries. For simplicity, we focus on the case of $U(1)$. We denote the lightest operator of charge $Q$ as $\mathcal{O}_Q$, its dimension being $\Delta_Q$. We are interested in the limit when $Q$ becomes large.
One of the simplest nontrivial examples of this type is given by the $O(2)$ Wilson-Fischer CFT in $d = 3$. This theory has $U(1) \times Z_2$ global symmetry and is common in Nature, see, e.g., [20]. It is commonly defined as the IR fixed point of the flow generated by the $(\phi^\dagger \phi)^2$ deformation of the free complex scalar theory in the UV. In a recent paper [21] it was argued that the large $Q$ subsector of this theory is described by a conformally invariant effective field theory (EFT) Lagrangian of a Goldstone boson. In particular, the authors of [21] predicted the spectrum of operators $\Delta$ with dimensions slightly above $\Delta_Q$, namely the operators with $\Delta - \Delta_Q \sim O(1)$ in the large $Q$ limit. This approach was further developed in [22], where the correlation functions of light charged operators in the background of the heavy state were computed. Generalizations to systems with more symmetries were found in [22, 67–69]. Some of the EFT predictions have been tested using Monte-Carlo simulations in [70, 71].

These results are the starting point for our analysis. We would like to understand how universal they are and what assumptions would go into their derivation in generic CFTs. Therefore, we study a crossing equation for heavy-heavy-light-light operators in an abstract CFT with a global symmetry. We take the heavy state to be $O_Q$, the lightest operator with a given large charge. Notice that the large $Q$ limit is different from the more familiar large $N_c$ [72], or large
spin $J$ limits \cite{73,74}. In the latter cases one considers a fixed correlator and changes either parameters of the theory or cross ratios within the correlator. In the case of large $Q$ we analyze the limit of a family of correlators within one theory. Indeed, for every $Q$ the external operator is different. This leads to several peculiarities in the analysis of the crossing equation that we will discuss below. Nevertheless, we assume that correlation functions that involve $\mathcal{O}_Q$ admit a smooth large $Q$ limit, namely that we can build an expansion in inverse powers of $Q$ and think of $Q$ as a smooth parameter.

The essential simplification of the large $Q$ limit is that in a certain domain in the space of cross ratios, the dominant contributions to the four-point function in the heavy-light fusion channels come from a set of operators whose dimensions above that of the lightest large charge operator are of order 1 in the $Q$ scaling. These are the operators that are characterized by the effective field theory.

We begin by performing a detailed analysis of crossing for the four-point function computed from effective field theory in \cite{22}. This sets the stage for a more abstract analysis. One immediate feature of the large $Q$ limit is that in the $z$ conformal frame, the contribution of the descendants to the conformal blocks are suppressed by a power of $Q$. This greatly simplifies the problem. For exam-
ple, at leading order crossing is satisfied by a single operator! Further analysis, however, reveals two discomforting features:

a) $s$- and $u$-channel (heavy-light) OPEs do not have an overlapping region of convergence within EFT.\(^1\)

b) when EFT is applicable, the $t$-channel (light-light) OPE is dominated by unknown neutral heavy operators.

This looks like an impasse for any bootstrap analysis. There is, however, the third feature of the EFT result which allows us to make a further progress:

c) at leading nontrivial order only one Regge trajectory contributes to the $s$- and $u$- channel OPEs.

The presence of a single Regge trajectory in the conformal block decomposition of the EFT result is a direct consequence of having only one field, the Goldstone mode, in the EFT Lagrangian. A priori, it is not obvious that crossing equations admit solutions with only a finite number of Regge trajectories. Indeed, without taking the large $Q$ limit this would be impossible, and it is a nontrivial property of the conformal blocks in the large $Q$ limit.

It is, therefore, natural to consider a truncated ansatz for the correlation func-

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\(^1\)In the full CFT the $s$- and $u$- channels, of course, converge as always. However, in the overlapping region, the dominant operators in one of the channels are not the ones that are described by the EFT.
tion, where only a finite number of Regge trajectories contribute to the correlator at the first nontrivial order in the $s$- and $u$-channel OPEs. This seems to be a weak CFT version of what we mean by having an effective field theory description of the large $Q$ correlators. Moreover, we impose that this ansatz satisfies the following properties:

$a')$ Smooth matching of $s$- and $u$-channels at their common boundary of convergence.

$b')$ Existence of the macroscopic (thermodynamic) limit of the correlator.

In $a')$, instead of the $s = u$ crossing we impose analyticity of the correlation function at the boundary of convergence of the large $Q$ limit of each channel, namely that the two expansions should match smoothly. This argument is similar in spirit to the one used in $[75]$, where the asymptotic density of operators and three-point functions were found. In our case it becomes much more powerful, due to the crucial assumption, motivated by EFT considerations, that only $N$ Regge trajectories are present in the OPE at this order. This is a weak CFT version of a notion of having a finite number of “fields” in the “EFT.”

In $b')$, we note that in the absence of a controlled $t$-channel OPE, the short distance behavior of the correlator is controlled by the existence of what we will call a macroscopic limit. This limit was recently discussed in $[76]$ in the context
of the eigenstate thermalization hypothesis (ETH) ([77][79]) (for a review, see, e.g., [80]). Consider a CFT state on a cylinder \( \mathbb{R} \times S^{d-1} \), and take the radius of the sphere \( R \to \infty \) while keeping the correlators of light operators finite by appropriately scaling the parameters of the state, in this case its charge and energy. Equivalently, this is a combined limit in which the scaling dimension of the external operator is taken to infinity as we tune cross ratios appropriately.

The existence of such limits, which result in flat space correlators in a nontrivial background, seems to be a generic feature of any CFT. Typically, the energy and charge density of the state will remain fixed when the limit is taken such that the correlators remain finite. An important exception that we will discuss further in section 4 appears when there is a moduli space of vacua. We assume that such a limit exists. This type of limit seems to be a generic feature of any CFT, and indeed it exists for the case analyzed in [21, 22].

Furthermore, for generic heavy operators, that are not the lightest carrying some large charge, the physics of the macroscopic limit is expected to be thermal, and described by hydrodynamics at finite temperature. However, in the situation at hand, for the lightest operators with large charge, we expect a finite charge density configuration at zero temperature, associated with the quantum

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2One also needs to rescale the light operators appropriately. The nontrivial condition is that this can be done to keep all (2 heavy +n light)-point functions finite.
EFT, in the macroscopic flat space limit.

Under the assumptions $a', b', c$ we classify the leading order solutions to the crossing equation. The solutions for scaling dimensions as functions of spin are given by the roots of a certain polynomial that we describe in detail below. For $N = 1$ we show that the Goldstone EFT is the unique solution. For $N \geq 2$ there are many possibilities. Some of them correspond to adding extra particles. Other solutions do not come from any weakly coupled EFT Lagrangian. At present, we do not know which of the solutions are realized in CFTs and could be consistently promoted to a solution of the crossing equations higher orders in $\frac{1}{Q}$. We leave these questions for the future.

In section 1 we describe general features of the large $Q$ limit. In section 2 we describe the basic kinematics and the properties of conformal blocks in the large $Q$ limit. In section 3 we review the results of EFT for the spectrum and the four-point function. In section 4 we describe the macroscopic limits in a generic CFT. In section 5 we perform the bootstrap analysis of the four-point function. In section 6 we present some extensions by considering operators with spin, going to next order and doing an analog of the light-cone bootstrap in the macroscopic limit. We end with conclusions and future directions.
3.1 LARGE $Q$ LIMIT

We will be interested in the large charge $Q$ limit of correlators of the type

$$G_Q(x_i) \equiv \langle O_Q(0)O_{q_1}(x_1) \ldots O_{q_n}(x_n)O_{q_{n+1}}(1)O_{-Q}(\infty) \rangle,$$  \hspace{1cm} (3.1)

where we used conformal symmetry to fix the positions of three operators.

Note that the large $Q$ limit is taken not within a fixed correlator, but rather it is a limit of a family of different correlators which involve different operators $O_Q$. Charge $Q$ being a discrete quantum number, one might wonder to what extent this limit is well-defined. A somewhat similar situation arises in the discussion of the large spin $J$ limit \[73, 74\]. There, however, one can argue \[15\] that the CFT data is analytic in spin. A core assumption of the present work is that a similar analyticity exists in charge as well. Namely, we will treat operators with large $Q$ and their corresponding three-point functions as smooth functions of $Q$ that admit a large $Q$ expansion.

Imagine now a family of operators labeled by $Q$. The discreteness of the spectrum implies that

$$\lim_{Q \to \infty} \Delta_{\text{min}}(Q) \to \infty.$$  \hspace{1cm} (3.2)
Indeed, otherwise we would have an infinite number of operators with bounded dimensions $\Delta \leq \Delta^\ast$.

As we make $Q$ large, there are several possibilities. If the spectrum close to the lightest state is sparse, the correlator $3.3$ is dominated by the “vacuum” in each OPE channel together with excitations with energy of $O(1)$, namely

$$G^Q(z_i, \bar{z}_i) \sim \exp \left( - \sum_{i=1}^{n} \Delta_{\min}(Q + \sum_{k=1}^{i} q_k)|\tau_{i+1} - \tau_i| \right),$$  \hspace{1cm} (3.3)$$

where the proportionality coefficient is given by the corresponding three-point couplings. At this point we assume a type of large $Q$ clustering, namely that there is no extra large Euclidean time scale at which the picture $3.3$ breaks down. In principle, one could imagine a slightly heavier operator with an enhanced three-point function, which would lead to an extra factor of $Q^a \exp(-\beta\delta\tau)$ in comparison with $3.3$. Then for $\delta\tau \gg \log Q$ it would be suppressed, whereas for $\delta\tau \ll \log Q$ it would be dominant. We assume that this does not happen and the same lightest state dominates the correlator at large Euclidean times.

There are other possibilities that we do not consider. For example, one can imagine that there is a parametrically large degeneracy of operators close to the lightest state. This would be the case, for example, if the dual state were an extremal Reissner-Nordström black hole. It would be interesting to study this
possibility or rule it out.

For simplicity, let us set $n = 1$ and $q_1 = -q_2 = -q$, so we are considering a four-point function. We have

$$G^Q(z, \bar{z}) \sim \lambda^{2 \Delta_{\min}(Q-q)}(z\bar{z})^{\Delta_{\min}(Q-q)/2}.$$  \hfill (3.4)

The minimal energy state could be degenerate or carry spin. The large $Q$ expansion is dominated by the minimal energy state and $O(1)$ energy excitations around it. Note that operators that are parametrically heavier than $\Delta_{\min}$ are non-perturbatively suppressed by the factor $(z\bar{z})^{(\Delta_r-\Delta_{\min})/2}$. This picture is very similar to the usual saddle point approximation with $Q$ playing the role of $\frac{1}{\hbar} \to \infty$. Operators with scaling dimensions parametrically different in $Q$ correspond to different saddles, whereas the fluctuations around a given saddle are described by the operators which have parametrically the same scaling dimensions.

A state on the cylinder created by the lightest operator of charge $Q$ is characterized by the energy density $\epsilon$ and charge density $q$

$$\epsilon = \frac{\Delta Q}{R^d}, \quad q = \frac{Q}{R^{d-1}}.$$  \hfill (3.5)
where we used the fact that $E_{\text{cyl}} = \frac{\Delta Q}{R}$. As we take $Q$ to be large we can simultaneously take $R \to \infty$ so that $\epsilon$ is kept fixed. Generically, we expect that finite charge density $q$ states carry some fixed non-zero energy density as well. This implies that

$$\Delta_{\text{min}}(Q) \equiv \Delta Q \sim Q^{d/(d-1)}. \quad (3.6)$$

We expect \(3.6\) to hold in generic interacting CFTs. Alternatively, \(3.6\) is a consequence of a local relationship between charge and energy densities \([21]\). In the present work we mostly focus on the case \(3.6\), except for some parts of section 4.

### 3.2 Four-point Function Kinematics

In this section we review basic kinematics of the four-point correlator and set our conventions. We consider a four-point function of scalar operators

$$G(z, \bar{z}) \equiv \langle O_Q(0)O_{-q}(z, \bar{z})O_q(1)O_{-q}(\infty) \rangle,$$

$$u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2} = z\bar{z}, \quad v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2} = (1-z)(1-\bar{z}), \quad (3.7)$$

\(^3\)The situation is different in CFTs with a nontrivial moduli space of vacua. We discuss this in more detail in section 4.
where the charge $Q$ is very large and $\mathcal{O}_Q$ has the smallest dimension $\Delta_Q$ among operators with charge $Q$. As discussed in the previous section, as we take $Q \to \infty$, we have $\Delta_Q \to \infty$ as well. Therefore, \(3.7\) describes a heavy-light-light-heavy correlation function.\(^4\) It is instructive to analyze what happens to the conformal blocks in this limit.

The correlator $G(z, \bar{z})$ admits an expansion in terms of conformal blocks in three different channels

$$s - \text{channel: } \quad G(z, \bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_Q+\Delta_q)} \sum_{\mathcal{O}_{\Delta,J}} |\lambda_{Q,-q,\mathcal{O}_{\Delta,J}}|^2 g_{\Delta,J}^{\Delta_Q,-\Delta_q}(z, \bar{z}),$$

$$\Delta_{Q,q} = \Delta_Q - \Delta_q, \quad |z| < 1. \quad (3.8)$$

$$t - \text{channel: } \quad G(z, \bar{z}) = ([1 - z][1 - \bar{z}])^{-\Delta_q} \sum_{\mathcal{O}_{\Delta,J}} \lambda_{-q,q,\mathcal{O}_{\Delta,J}} \lambda_{Q,-Q,\mathcal{O}_{\Delta,J}} g_{0,J}^{0,0}(1 - z, 1 - \bar{z}), \quad |1 - z| < (3.9)$$

\(^4\)For a related discussion in $d = 2$, see \([81]\).
where the sum is over an infinite set of primary operators that appear in the corresponding OPE channel. The expressions for conformal blocks can be found, for example, in [72].

It is also convenient to define $g_q(z, \bar{z})$ as

$$g_q(z, \bar{z}) \equiv (zz)^{\Delta_q} G(z, \bar{z}) . \tag{3.11}$$

The correlation function is invariant if we send $q$ to $-q$ and exchange the locations of the two light operators, $\mathcal{O}_{-q} \leftrightarrow \mathcal{O}_q$. This is encoded in the crossing equation $s = u$

$$g_q(z, \bar{z}) = g_{-q} \left( \frac{1}{z}, \frac{1}{\bar{z}} \right) . \tag{3.12}$$

Of course, finding the most generic $G(z, \bar{z})$ consistent with the OPE and cross-
ing is an insurmountable task. The key simplification here is that there is a small parameter in the problem, namely $Q^{-1} \ll 1$. This allows us to find some universal features in the limit.

We will also need correlation functions on a cylinder $\mathbb{R} \times S^{d-1}$, which is conformally mapped to the plane $\mathbb{R}^d$ by $(\tau, n) \rightarrow (r = Re^{\tau}, n)$. In the conformal frame \ref{3.7}, when all operators lie in the same plane, we have a relation between cylinder coordinates and $z, \bar{z}$ plane coordinates

\begin{align*}
z &= e^{\tau + i\theta}, \quad z = e^{\tau - i\theta}, \\
z\bar{z} &= e^{2\tau}, \quad \frac{z + \bar{z}}{2\sqrt{z\bar{z}}} = \cos \theta \equiv x, \quad (3.13)
\end{align*}

where $\theta$ is the angle between two light operators in \ref{3.7}. Primary operators transform as

\begin{equation}
\mathcal{O}_{\text{cyl}}(\tau, n) = \left( \frac{r}{R} \right)^{\Delta_\mathcal{O}} \mathcal{O}(r, n).
\end{equation}

As a prerequisite for studying bootstrap equations, we review the structure of conformal blocks in the large charge limit.
3.2.1 Conformal Blocks in The Large $Q$ Limit

As is evident from 3.8, 3.9, 3.10, conformal blocks depend on $Q$ only in the $s$- and $u$-channels. To understand the structure of the blocks it is instructive to write them as a sum over descendants

$$\tilde{g}_{\Delta_{Q,q}}^{\Delta_{Q,q},J}(z, z) = \sum_{n=0}^{\infty} \sum_{j} a_{j,n}(zz) \frac{\Delta_{Q,q}^J}{C_{J}^{d/2 - 1}} \left( \frac{z + z}{2 \sqrt{zz}} \right),$$

(3.15)

and fix all the coefficients $a_{j,n}$ by solving the Casimir equation; here $C_{J}^{d/2 - 1}(x)$ are the usual Gegenbauer polynomials which become Legendre polynomials in $d = 3$. Say, for $n = 0$ we have $j = J$; for $n = 1$ we have $j = J + 1$ and $j = J - 1$, etc. An explicit solution for $a_{j,n}$’s in general case was found in [83]. Let us write explicitly the first nontrivial correction due to the level one descendants

$$\tilde{g}_{\Delta_{Q,q}}^{\Delta_{Q,q},J}(z, z) = (zz) \frac{\Delta_{Q,q}}{2} \left( C_{J}^{d/2 - 1} + \sqrt{zz} \left( a_{J+1,1} C_{J+1}^{d/2 - 1} + a_{J-1,1} C_{J-1}^{d/2 - 1} \right) + O(z\bar{z}) \right),$$

$$a_{J+1,1} = \frac{1}{2(d - 2 + 2J)} \frac{(J + 1)(\Delta_{Q,q} - \Delta_{Q,q} + J)^2}{\Delta_{Q,q} + J},$$

$$a_{J-1,1} = \frac{1}{2(d - 2 + 2J)} \frac{(J + d - 3)(\Delta_{Q,q} - \Delta_{Q,q} - J - d + 2)^2}{\Delta_{Q,q} - J - d + 2},$$

(3.16)
where for the sake of brevity we omitted the arguments of Gegenbauer polynomials which are the same as in (3.15). When $J = 0$, the term $C^{(\frac{d}{2} - 1)}_{J-1}$ is absent.

For our purposes we will also need the contribution of the level-two descendants in the case $J = 0$, which take the form

$$a_{2,2} = \frac{1}{4d(d-2)} \frac{(\Delta_{Q-q} - \Delta_{Q,q})^2(\Delta_{Q-q} - \Delta_{Q,q} + 2)^2}{\Delta_{Q-q}(\Delta_{Q-q} + 1)},$$
$$a_{0,2} = \frac{1}{4d} \frac{(\Delta_{Q-q} - \Delta_{Q,q})^2(\Delta_{Q-q} - \Delta_{Q,q} - d + 2)^2}{\Delta_{Q-q}(2\Delta_{Q-q} - d + 2)}.$$  \hspace{1cm} (3.17)

We would like to consider the limit of $Q \gg 1$ and fixed $d, J$. In the conformal bootstrap analysis of the large charge EFT, we will be interested in operators $\Delta_{Q-q}$ and $\Delta_Q$ belonging to the same family \[ \Delta_Q \sim Q^d. \] It is then clear from (3.16), (3.17) that the contribution of descendants is governed by the parameter

$$\frac{(\Delta_{Q-q} - \Delta_Q)^2}{\Delta_{Q-q}} \sim \frac{1}{\Delta_Q} \left( \frac{\partial \Delta_Q}{\partial Q} \right)^2 \sim Q^{-\frac{d}{d-1}} \rightarrow 0.$$  \hspace{1cm} (3.18)

Therefore, for $d > 2$ the expansion (3.15) is a controlled approximation of the conformal block in the large charge $Q$ limit. To leading order at large $Q$ only primary operators contribute. This simplifies our analysis in later sections.

One can use recursion relations for Gegenbauer polynomials to simplify (3.16).
in the large \( Q \) limit. The result for the first subleading correction takes the form

\[
g_{\Delta Q - \Delta Q, J}(z, \bar{z}) = (zz)^{\frac{\Delta Q - \Delta q}{2}} C_j^{(d-1)} \left( \frac{z + \bar{z}}{2\sqrt{zz}} \right) \left( 1 + \frac{q^2}{4\Delta Q} \left( \frac{\partial \Delta Q}{\partial Q} \right)^2 (z + \bar{z}) + \ldots \right)
\]

(3.19)

Curiously, the first correction in the parentheses does not depend on \( J \). We do not have an explanation for this fact beyond direct computation.

### 3.3 Effective Field Theory

The goal of this section is to provide the reader with the results for the operator spectrum \[21\] and correlation functions \[22\] at large charge and review the tools of effective field theory necessary to obtain them.

We consider a CFT with some global symmetry group \( G \) and assume that the CFT spectrum contains operators charged under this symmetry (which implies that there exist operators of arbitrarily large charge \( Q \), by repeated OPE contraction). For simplicity, we focus on the case \( G = U(1) \).

The essential idea of \[21\] is the following. Let us consider an operator of charge \( Q \), \( \mathcal{O}_Q \). By the operator/state correspondence this operator describes a state with charge density \( \rho \sim \frac{Q}{R^{d-1}} \) on the cylinder \( \mathbb{R} \times S^{d-1} \). Here \( R \) stands for the radius of the sphere. In the limit \( Q \gg 1 \) there is a large separation of UV and
IR scales $\rho \gg \frac{1}{R}$. One can view the state with charge $Q$ as spontaneously breaking the $U(1)$ symmetry (as well as some of the space-time symmetries \cite{22}). This leads to the existence of a massless Goldstone boson. At distances much bigger than the distance set by the charge density, this Goldstone mode is described by an EFT corresponding to a particular symmetry breaking pattern. The expansion parameter in the EFT is the ratio of UV and IR scales

$$\frac{\rho^{-1/(d-1)}}{R} \sim Q^{-1/(d-1)} \ll 1.$$  

The state with homogeneous charge density $\rho$ on the cylinder $\mathbb{R} \times S^{d-1}$ breaks the global symmetry group $SO(d + 1, 1) \times U(1)$ down to rotations of the sphere $SO(d)$ and a linear combination of $U(1)$ and time translations $\mathcal{H} = \mathcal{H} + \mu \hat{Q}$

$$SO(d + 1, 1) \times U(1) \rightarrow SO(d) \times \mathcal{H}'.$$  

(3.20)  

The corresponding effective Lagrangian can be obtained using the CCWZ construction \cite{84, 85}. It can be written in terms of a field $\chi(x)$, whose fluctuations around an appropriate saddle describe the Goldstone boson $\pi(x)$.

In particular, in three dimensions $d = 3$ we have $\mathbb{21}, \mathbb{22}$ (in the Euclidean space).  

$^5$We put a hat on the $U(1)$ generator $\hat{Q}$ to distinguish it from the c-number $Q$.  

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signature)

\[ S_E = - \int d^3 x \sqrt{g} \left( \frac{1}{12 \pi \alpha^2} |\partial \chi|^3 - \frac{\beta}{8 \pi \alpha} |\partial \chi| \left( \mathcal{R} + 2 \frac{(\partial_{\mu}|\partial \chi|)(\partial^{\mu}|\partial \chi|)}{|\partial \chi|^2} \right) + \ldots \right) + \\
+ i \rho \int d^3 x \sqrt{g} \dot{\chi}, \]

(3.21)

where \( |\partial \chi| \equiv (-g^{\mu\nu} \partial_{\mu} \chi \partial_{\nu} \chi)^{1/2} \) and \( \alpha, \beta, \gamma \) are undetermined coupling constants of the EFT. By ellipsis we denote higher order curvature couplings, which are suppressed by \( \frac{1}{Q} \) when we expand around the relevant saddle point. This action is Weyl invariant assuming that the metric has Weyl weight two and \( \chi \) has Weyl weight zero. The field \( \chi \) transforms by shifts under \( U(1) \), with the corresponding charge density being \( j^0(\tau, n) = \frac{\partial L}{\partial \dot{\chi}} \). The last term in (3.21) is the chemical potential which sets the charge density \( j^0(\tau, n) \) to a constant value \( \rho = \frac{Q}{4 \pi R^2} \). Note that this action is meaningful only when expanded around the saddle described below that gives the large charge state. Therefore it can be regarded as a tool for constructing the Goldstone action by giving a simpler re-

\[ \alpha = \frac{1}{\sqrt{2\pi c_1}}, \beta = -\frac{8\pi c_2}{\sqrt{2\pi c_1}}, \gamma = c_3. \]

This normalization will be more convenient for scaling dimensions and correlation functions.

The Lagrangian \( L \) here does not include the chemical potential.

---

\(^6\)In our convention the curvature of \( S^n \) is \( \mathcal{R} = \frac{n(n-1)}{R^2} \).

\(^7\)Our definitions of \( \alpha, \beta, \gamma \) are related to \( c_1, c_2, c_3 \) in \([22]\) by \( \alpha = \frac{1}{\sqrt{2\pi c_1}}, \beta = -\frac{8\pi c_2}{\sqrt{2\pi c_1}}, \gamma = c_3 \).

\(^8\)The Lagrangian \( L \) here does not include the chemical potential.
alization of the broken symmetries. In particular, the $\chi$ field is not meaningful near $\chi = 0$, and the above action is not meant to approximate the exact CFT in that regime.

To leading order one can use the first term in (3.21) to obtain the saddle-point. Assuming that the lowest energy state is homogeneous on $S^2$, the saddle-point is simply given by

$$\chi = -i \mu \tau + \chi_0 ,$$
$$\mu R = \alpha \sqrt{Q} + \frac{\beta}{2\sqrt{Q}} + O(Q^{-3/2}) ,$$

(3.22)

where $\mu, \chi_0$ are constants and $\mu$ is fixed by the eom at $\tau = \pm \infty$. Since $\chi$ transforms by shifts under $U(1)$, this solution indeed preserves $\mathcal{H}' = \mathcal{H} + \mu \hat{Q}$ in accordance with (3.20).

Expanding the action (3.21) around the saddle (3.22)

$$\chi(x) = -i \mu \tau + \frac{\alpha}{2} \frac{1}{\sqrt{\mu}} \pi(x)$$

(3.23)

---

Equivalently, we could have fixed $\mu$ by imposing $\langle Q|\hat{j^0}|Q\rangle = \frac{Q}{4\pi R^2}$. 

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we find

\[ S_E = \frac{\Delta Q}{R} (\tau_{\text{out}} - \tau_{\text{in}}) + S_\pi, \]

\[ S_\pi = \frac{1}{16\pi} \int d^3 x \, \sqrt{g} \left( \dot{\pi}^2 + \frac{1}{2} (\partial_i \pi)^2 \right) + \]

\[ + \frac{i}{96\pi \sqrt{\alpha} Q^{3/4}} \int d^3 x \, \sqrt{g} \left( \dot{\pi}^3 + \frac{3}{2} \dot{\pi} (\partial_i \pi)^2 \right) + O(Q^{-1}), \]

\[ \Delta Q = \frac{2}{3} \alpha Q^{3/2} + \beta \sqrt{Q} + C + O(Q^{-1/2}), \quad (3.24) \]

where \( E_Q = \frac{\Delta Q}{R} \) is the energy of the “vacuum” state in the large charge \( Q \) sector. The field \( \pi(x) \) is the Goldstone mode propagating at the speed of sound \( c_s^2 = \frac{1}{2} \). The quadratic part of the action \( S_\pi \) can be canonically quantized on the cylinder

\[ \pi(\tau, n) = \pi_0 + \pi_1 \tau + \sum_{J>0, m} \sqrt{\frac{4\pi}{\Omega_J}} \left( a_{Jm} Y_{Jm}(n) e^{-\Omega_J \tau} + a_{Jm}^\dagger Y_{Jm}^*(n) e^{\Omega_J \tau} \right), \]

\[ \Omega_J = \sqrt{\frac{J(J+1)}{2}}, \quad (3.25) \]

where \( \pi_0, \pi_1 \) are zero modes and canonical commutation relations are \([\pi_0, \pi_1] = 2\) and \([a_{Jm}, a_{Jm'}^\dagger] = \delta_{J,J'} \delta_{mm'}\). The Euclidean reality condition \( \pi(-\tau, n)^\dagger = \pi(\tau, n) \) implies \( \pi_0^\dagger = \pi_0, \pi_1^\dagger = -\pi_1 \). Thus, we can write zero modes \( \pi_0 = a^\dagger + a, \pi_1 = a^\dagger - a \) in terms of creation-annihilation operators with the latter acting on the vacuum
in the standard way. The free propagator

\[ D(\tau, x) \equiv \langle \pi(0, n_2)\pi(\tau, n_1) \rangle , \]  

(3.26)

of \( \pi \)'s is given by the solution to Green’s equation

\[ \left( \partial_\tau^2 + \frac{1}{2} \Delta_{S^2} \right) D(\tau, x) = -4\delta(\tau)\delta(1-x) , \]  

(3.27)

where \( x = n_1 n_2 = \cos \theta \) is the angle between two light operator insertions on \( S^2 \).

Again, notice a peculiar \( \frac{1}{2} \) which is a consequence of conformal symmetry. The explicit form of the solution to 3.27 is given by

\[ D(\tau, x) = -|\tau| + \sum_{J=1}^{\infty} \frac{2J + 1}{\Omega_J} e^{-\Omega_J|\tau|} P_J(x) . \]  

(3.28)

The expression 3.28 suggests that \( D(\tau, x) \) is non-analytic at \( \tau = 0 \). It is, however, manifest in 3.27 that \( D(\tau, x) \) is analytic everywhere except at \( \tau = 0, x = 1 \) where two operators collide.

We would like to use EFT to compute correlation functions of light operators \( O_q \) in the background of the state created by the heavy operator. Any light operator \( O_q \) with scaling dimension \( \Delta_q \) and charge \( q \), both of order \( O(1) \), can be
represented at low energies in terms of Godstone boson degrees of freedom\cite{22}

\[ \mathcal{O}_q = c_q |\partial \chi|^{\Delta_q} e^{i q \chi} + c_q^R \mathcal{R} |\partial \chi|^{\Delta_q-2} e^{i q \chi} + \ldots , \]  

(3.29)

where \( c_q \) and \( c_q^R \) are constants not fixed by EFT and by ellipsis we denote further curvature couplings which lead to corrections suppressed at large \( Q \). In practice, the expression for light operators\cite{3.29} should be expanded around the saddle\cite{3.23}

\[ \mathcal{O}_q(\tau, n) = c_q \mu^{\Delta_q} e^{i \mu \tau} \times \]

\[ \times \left( 1 + \frac{i q \alpha}{2 \sqrt{\mu}} \pi(\tau, n) + \frac{1}{2} \left( \frac{i q \alpha}{2 \sqrt{\mu}} \right)^2 \pi^2(\tau, n) + \frac{i \alpha}{2 \mu^{3/2}} \tilde{\pi}(\tau, n) + 2 \frac{c_q^R}{c_q} \frac{1}{\mu^2} + \ldots \right), \]  

(3.30)

where we only kept terms which contribute to the correlators below at the order relevant for us. In\cite{3.30} \( \pi^2(\tau, n) \) should be understood as a normal-ordered product.

Equations\cite{3.24}, \cite{3.30} provide us with a weakly coupled description of CFT in a state with large charge \( Q \). Canonical quantization of the Goldstone \( \pi \) gives the spectrum of operators in the charge \( Q \) sector, as was found in\cite{21}. Further, using the representation of light operators\cite{3.29}, \cite{3.30} one can systematically com-
pute correlators of the form

\[ \langle O_Q O_{q_1} \ldots O_{q_n} O_{-Q} \rangle = \int D\chi O_{q_1} \ldots O_{q_n} e^{-S_E}. \] \hspace{1cm} (3.31)

Now, we move on to describing the results of \cite{21,22} regarding the operator spectrum and correlation functions \ref{3.31}.

### 3.3.1 Operator Spectrum

Using the operator/state correspondence one finds that the lowest dimension operator with large charge \( Q \) has a scaling dimension \ref{3.24}

\[ \Delta_Q = \frac{2}{3} \alpha Q^{3/2} + \beta \sqrt{Q} + C + O(Q^{-1/2}) . \] \hspace{1cm} (3.32)

The coefficients of the first two terms depend on the UV theory. On the other hand, the third term of order \( O(1) \) is completely universal and given by \( C = -0.0937256 \ldots \). This is simply the Casimir energy of the Goldstone \( \pi \).

The spectrum of low-lying operators is parametrized by integers \( \vec{n} = (n_1, n_2, \ldots) \) and given by

\[ \Delta_{\vec{n}} = \Delta_Q + \sum_{J=1}^{\infty} n_J \Omega_J, \quad \Omega_J = \sqrt{\frac{J(J+1)}{2}} . \] \hspace{1cm} (3.33)
Each of the modes $\Omega_J$ corresponds to an excitation of the Goldstone boson $\pi$ with an angular momentum $J$ around the saddle 3.22. Excitations $\Omega_{J=1}$ are related to the descendants of primaries that appear in the $s$- and $u$-channel OPE. We will demonstrate this very explicitly shortly. Having $n_1$ modes $\Omega_{J=1}$ in 3.33 corresponds to the level $n_1$ descendant of $(0, n_2, n_3, \ldots)$ with dimension
\[
\Delta_Q^{(n_1, n_2, \ldots)} = \Delta_Q^{(0, n_2, \ldots)} + n_1.
\]
The modes $\Omega_{J>1}$ correspond to new primary operators of various spins $j \leq \sum_{J=1}^{\infty} n_J J$.

Further, using the CCWZ prescription 3.29, the authors in 22 computed three- and four-point correlations functions. The results are as follows.

### 3.3.2 Three-point Function

We consider a three-point function of two heavy and one light operator

\[
\langle O_Q(x_{in}) O_{-q}(x) O_{-(Q-q)}(x_{out}) \rangle = \frac{\lambda_{Q,-q,-(Q-q)}}{|x_{out} - x_{in}|^{\Delta_{Q-q} + \Delta_{Q-Q} - \Delta_q} |x_{out} - x|^{\Delta_{Q-q} - \Delta_{Q+Q} + \Delta_q} |x - x_{in}|^{\Delta_{Q-Q} - \Delta_{Q-q} + \Delta_q}}.
\]

(3.34)

To compute this three-point function in EFT, one has to slightly modify the path integral 3.31 to account for the extra charge $q$ in the final state. This is implemented by adding an extra term to the chemical potential $S_E \rightarrow S_E + iq \int \frac{dt}{4\pi} \chi(\tau_{out}, n)$. Using the prescription 3.23, 3.24, 3.29, 3.31 together with the
mentioned modification, the EFT computation for the three-point function on the cylinder gives

\[
\langle O_Q(\tau_{in})O_{-q}(\tau, n)O_{-(Q-q)}(\tau_{out}) \rangle_{cyl} = c_{-q} \mu^\Delta_q e^{-\Delta_Q(\tau_{out}-\tau_{in})} e^{q\mu(\tau_{out}-\tau)} \times \\
\times \left( 1 + \frac{(\alpha q)^2}{4\mu} \langle \pi(\tau, n) \rangle + \frac{(\alpha q)^4}{32\mu^2} \langle \pi(\tau, n) \rangle^2 - \frac{\alpha^2 q\Delta_q}{4\mu^2} \langle \pi(\tau, n) \rangle + \frac{2c_R}{\mu^2 c_q} + O(\mu^{-3}) \right),
\]

(3.35)

where it is assumed that \( \tau_{out} \to \infty, \tau_{in} \to -\infty \). Next, we can insert the expression for the propagator \( 3.28 \) into \( 3.35 \). The role of the integrals over \( n' \) is to project onto the zero mode in the propagator. Also changing large the \( \mu \) expansion to a large \( Q \) expansion via \( 3.22 \), we obtain

\[
\langle O_Q(\tau_{in})O_{-q}(\tau, n)O_{-(Q-q)}(\tau_{out}) \rangle_{cyl} = \\
= c_{-q} \mu^\Delta_q e^{-\Delta_Q(\tau_{out}-\tau_{in})} e^{q\mu(\tau_{out}-\tau)} \times \\
\times \left( 1 - \frac{\alpha q^2}{4\sqrt{Q}} (\tau_{out} - \tau) + \frac{\alpha^2 q^4}{32Q} (\tau_{out} - \tau)^2 - \frac{q\Delta_q}{4Q} + \frac{\beta}{2\alpha} \frac{\Delta_q}{Q} + \frac{2c_R}{c_{-q} \alpha^2 Q} + O(36^{1/2}) \right).
\]

(3.36)
Using the map from the cylinder to the plane, one can check that is a large $Q$ expansion of with $\lambda_{Q-q-(Q-q)}$ given by

$$
\lambda_{Q-q-(Q-q)} = c_q \alpha^\Delta \Delta \lambda \lambda \Delta \lambda (1 - \frac{qQ}{4Q} + \frac{\Delta}{2\alpha} Q + \frac{2cR}{\alpha^2 c_q Q} + O(Q^{-3/2})) .
$$

(3.37)

In particular, notice the leading universal scaling $\lambda_{Q-q-(Q-q)} \sim Q^{\Delta \lambda / 2}$, emphasized in [22].

3.3.3 FOUR-POINT FUNCTION

In a similar fashion one can compute the four-point function of two heavy and two light operators (note the term $\sim \frac{1}{D^2}$ which is missing in the formula (8.20) in [22])

$$
g_q(z, z) = c_q c_{q-q} \alpha^2 \Delta \lambda Q \lambda \Delta \lambda \lambda \Delta \lambda (1 - \frac{\beta}{2} \frac{qQ}{\sqrt{Q}} + \alpha \frac{q^2}{4 \sqrt{Q}} D(\tau, x))
+ \beta \frac{\beta}{\alpha Q} + \frac{\beta}{8} \frac{q^2}{Q} \tau^2 - \frac{\alpha \beta}{8} \frac{q^2}{Q} \tau D(\tau, x) - \frac{qQ}{2Q} \partial_x D(\tau, x) + \alpha^2 \frac{q^4}{32} D(\tau, x)^2 +
+ \frac{2}{\alpha^2 Q} \left( \frac{cR}{c_q} + \frac{cR}{c_{q-q}} \right) + O(Q^{-3/2}) ,
$$

(3.38)

where $g_q(z, z)$ was defined in [3.11] and the relation between the cylinder $(\tau, x)$ and plane $(z, z)$ coordinates is given in [3.13]. The overall prefactor $Q^{\Delta \lambda} e^{-aq \sqrt{Q}\tau}$
in 3.38 comes from evaluating the two light operators 3.29 on the saddle. The $D, \partial_\tau D, D^2$ terms are quantum corrections to the leading answer. The remaining terms in 3.38, constant and terms with explicit $\tau$’s, come from using 3.22 to convert $\mu$ into large $Q$ expansion, and the term in the third line comes from the curvature coupling in the light operator 3.29.

Let us discuss the structure of the formula 3.38 in more detail. First, it is manifestly $s = u$ crossing symmetric 3.12. Changing $\tau \rightarrow -\tau$, $q \rightarrow -q$ leaves invariant every term in 3.38 (this can be seen using 3.28). Second, the result 3.38 is analytic at non-coincident points, namely away from $\tau = 0, x = 1$. This follows from analyticity of the Goldstone propagator $D(\tau, x)$ defined by 3.27. Third, due to conformal invariance and unitarity, it should be possible to decompose 3.38 into a sum of conformal blocks with positive coefficients. Just from looking at 3.38 it is not obvious that it is the case. Of course, it is guaranteed by the conformal symmetry of the action 3.21, but it will prove instructive to explicitly see how this happens. To avoid overwhelming the reader with too many equations, let us first discuss the four-point function to the order $O(Q^{-1/2})$. 
3.3.4 **Four-point Function at $1/\sqrt{Q}$ Order**

At order $O(Q^{-1/2})$ the four-point function is given by the first line in (3.38). Using the propagator (3.28) it can be cast into the form

$$
g_q^{\text{EFT}}(z, \bar{z}) = c_q c_{-q} \alpha^{2\Delta_q} (zz)^{-2q\sqrt{Q}} \left( 1 + \frac{1}{4\sqrt{Q}} (-\beta q + \frac{\alpha q^2}{2}) \log(zz) \right) + \frac{\alpha}{4} \frac{q^2}{\sqrt{Q}} \sum_{J=1}^{\infty} \frac{2J+1}{\Omega_J} (zz)^{1/2\Omega_J} P_J(x) + O\left(O^{-1}\right),
$$

where the first line comes from the expansion of $(zz)^{1/2(\Delta_{Q_+\Delta_{Q_+}})}$ at large $Q$ with $\Delta_Q, \Delta_{Q-}$ being the dimensions of lightest operators in the sectors with charge $Q$ and $Q-q$ respectively, as given by (3.32).

The result (3.39) was derived for $zz < 1$ (equivalently $\tau < 0$). To obtain the EFT answer for $zz > 1$ (equivalently $\tau > 0$) one simply needs to substitute $z \rightarrow \frac{1}{z}, \bar{z} \rightarrow \frac{1}{\bar{z}}, q \rightarrow -q$ in (3.39), namely the full correlator takes the form

$$
g_q(z, \bar{z}) = \theta \left(1 - z\bar{z}\right) g_q^{\text{EFT}}(z, \bar{z}) + \theta \left(z\bar{z} - 1\right) g_{-q}^{\text{EFT}} \left(\frac{1}{z}, \frac{1}{\bar{z}}\right).
$$

The first term in (3.40) gives the $s$-channel expansion for $zz < 1$ and the second term gives the $u$-channel expansion for $zz > 1$. (This is somewhat reminiscent of the discussion in [86].) Indeed, in the $s$-channel formula (3.39) the leading term
is the contribution of the scalar with dimension $\Delta_{Q-q}$ and the term $J = 1$ is the contribution of the first descendant of $\Delta_{Q-q}$, in accordance with the form of the conformal block $3.16$. In particular, no new primary operators with dimension $\Delta_{Q-q} + 1$ appear. The terms with $J \geq 2$ are primary operators with spin $J$ and dimensions $\Delta_{Q-q} + \Omega_J$.

In the form $3.40$ the four-point function is manifestly expanded into conformal blocks and trivially satisfies $s = u$ crossing $z \to \frac{1}{z}, z \to \frac{1}{z}, q \to -q$. On the other hand, the reader may be puzzled by an apparent non-analyticity of the formula $3.40$ at $zz = 1$. However, as we reviewed in the previous subsection, the correlator is analytic away from $z = z = 1$ and the only singularity is at $z = z = 1$ when two operators collide. The $s$-channel OPE expansion in terms of EFT operators breaks down at $zz = 1$ and the $u$-channel expansion takes over at $|z| > 1$. As shown in $75$ the convergence of the $s$-channel OPE is optimal in terms of the so-called $\rho$-coordinate. The contribution of descendants, however, is not suppressed in the large $Q$ limit in the $\rho$-frame. This makes it unsuitable for the large $Q$ analysis.
3.3.5 Four-point Function at $\frac{1}{Q}$ Order

The main new feature at $\frac{1}{Q}$ order is the presence of an infinite number of operators of every spin in the OPE. Let us write down in detail the conformal block expansion of the correlator \(3.38\) at the order \(O(Q^{-1})\)

\[
g^\text{EFF}_q(z, z) = c_q c_{-q} \alpha^2 \Delta_q \left[(zz)^{\frac{1}{2}} P_1 + \frac{1}{4\sqrt{Q}} \left(-\beta q + \frac{\alpha q^2}{2}\right) \log(zz) + \frac{3\alpha}{4\sqrt{Q}} (zz)^{\frac{1}{2}} P_1 \right] + \frac{q^2}{4\sqrt{Q}} \frac{\alpha^2}{2} \delta_{\lambda_{Q, -q}, -(Q-q)} \delta_{J} q \left[\sum_{J=2}^{\infty} \left(2J + 1\right) \left(zz\right)^{\frac{1}{2}} \Omega_J P_J \right] + \frac{1}{32Q} \left(-\beta q + \frac{\alpha q^2}{2}\right)^2 \log^2(zz) + \frac{\alpha q^2}{16Q} \left(-\beta q + \frac{\alpha q^2}{2}\right) \log(zz) \sum_{J=1}^{\infty} \frac{2J + 1}{\Omega_J} \left(zz\right)^{\frac{1}{2}} \Omega_J P_J - \frac{3q^2}{4Q} \sqrt{zz} P_1 + \frac{\alpha^2 q^4}{32Q} \left(3\sqrt{zz} P_1\right)^2 + \frac{3\alpha^2 q^4}{16Q} \sum_{J=2}^{\infty} \frac{2J + 1}{\Omega_J} \left(zz\right)^{\frac{1}{2}} \Omega_J P_J \right] - \frac{3q\Delta_q}{2Q} \sqrt{zz} P_1 + \frac{\alpha^2 q^4}{32Q} \left(3\sqrt{zz} P_1\right)^2 + \frac{3\alpha^2 q^4}{16Q} \sum_{J=2}^{\infty} \frac{2J + 1}{\Omega_J} \left(zz\right)^{\frac{1}{2}} \Omega_J P_J \right] - \frac{q\Delta_q}{2Q} \delta_{\lambda_{Q, -q}, -(Q-q)} \delta_{J} q \left[\sum_{J=2}^{\infty} \left(2J + 1\right) \left(zz\right)^{\frac{1}{2}} \Omega_J P_J \right] + \frac{\alpha^2 q^4}{32Q} \left[\sum_{J=2}^{\infty} \frac{2J + 1}{\Omega_J} \left(zz\right)^{\frac{1}{2}} \Omega_J P_J \right] - \frac{2}{\alpha^2 Q} \left[\frac{c^R_q}{c_q} + \frac{c^R_{-q}}{c_{-q}}\right] + \frac{2}{\alpha^2 Q} \left[\frac{c^R_{Q, -q}}{c_q} + \frac{c^R_{-q, -(Q-q)}}{c_{-q}}\right] - \frac{3q^2}{4Q} \sqrt{zz} P_1 + \frac{\alpha^2 q^4}{32Q} \left(3\sqrt{zz} P_1\right)^2 + \frac{3\alpha^2 q^4}{16Q} \sum_{J=2}^{\infty} \frac{2J + 1}{\Omega_J} \left(zz\right)^{\frac{1}{2}} \Omega_J P_J \right] - \frac{q\Delta_q}{2Q} \delta_{\lambda_{Q, -q}, -(Q-q)} \delta_{J} q \left[\sum_{J=2}^{\infty} \left(2J + 1\right) \left(zz\right)^{\frac{1}{2}} \Omega_J P_J \right] + \frac{\alpha^2 q^4}{32Q} \left[\sum_{J=2}^{\infty} \frac{2J + 1}{\Omega_J} \left(zz\right)^{\frac{1}{2}} \Omega_J P_J \right] - \frac{2}{\alpha^2 Q} \left[\frac{c^R_q}{c_q} + \frac{c^R_{-q}}{c_{-q}}\right] + \frac{2}{\alpha^2 Q} \left[\frac{c^R_{Q, -q}}{c_q} + \frac{c^R_{-q, -(Q-q)}}{c_{-q}}\right],
\]  

(3.41)
where we indicated what is the interpretation of each term in s-channel conformal block expansion. Terms $\delta \lambda_{Q-q, -(Q-q)}$ and $\delta \lambda^J_{Q, -q, -(Q-q)}$ stand for the corrections to the three-point functions of the $\Delta_{Q-q}$ and $\Delta_{Q-q} + \Omega_J$ correspondingly.

The contribution of descendants in the fourth line of (3.41) is in perfect agreement with the conformal blocks (3.16, 3.17).

The $\frac{1}{Q}$ terms in (3.41) can be regrouped as in the second line in (3.38) to make crossing manifest

$$
\begin{align*}
\frac{\beta^2 q^2}{32Q} \log^2(zz) - \frac{\alpha \beta q^3}{16Q} \log(zz) \left( \frac{1}{2} \log(zz) + \sum_{J=1}^{\infty} \frac{2J+1}{\Omega_J} (zz)^{\frac{1}{2} \Omega_J} P_J \right) + \\
+ \frac{\alpha^2 q^4}{32Q} \left[ \frac{1}{4} \log^2(zz) + \log(zz) \sum_{J=1}^{\infty} \frac{2J+1}{\Omega_J} (zz)^{\frac{1}{2} \Omega_J} P_J + (3\sqrt{zz} P_1)^2 + \\
+ 6 \sum_{J=2}^{\infty} \frac{2J+1}{\Omega_J} (zz)^{\frac{1}{2} \Omega_J} P_J + \frac{\alpha^2 q^4}{32Q} \left( \sum_{J=2}^{\infty} \frac{2J+1}{\Omega_J} (zz)^{\frac{1}{2} \Omega_J} P_J \right)^2 \right] \\
- \frac{q \Delta_q}{2Q} \sum_{J=0}^{\infty} (2J+1)(zz)^{\frac{1}{2} \Omega_J} P_J + \frac{2}{\alpha^2 Q} \left( c^R_q - c^{-R}_q \right) \right). 
\end{align*}
$$

(3.42)

Thus, crossing invariant combinations are built from different types of corrections and there is a nontrivial interplay between them.

### 3.3.6 Short Distance Limit and Regime of Validity

It is instructive to write down a short distance expansion of the four-point function (3.38) in order to understand when the EFT approximation breaks down.
and to make connection with the macroscopic limit that we discuss in later sections. By short distances we mean the distance between the light operators becoming small. In terms of coordinates on the cylinder it corresponds to the region \( \tau, \theta \to 0 \). In the \((z, \bar{z})\) coordinates it corresponds to the region \( z, \bar{z} \to 1 \).

At small \( \tau, \theta \) one can approximate \( D(\tau, \theta) \) by a flat space propagator. Alternatively, one can compute it directly from (3.28). Indeed, the propagator \( D(\tau, \theta) \) has a singularity at \( \tau, \theta \to 0 \), which is given by the large \( J \) asymptotic of the sum (3.28) \( \sum_J e^{-\frac{\tau J}{\sqrt{2}}} P_J \). This is the generating function for Legendre polynomials. Thus, we have

\[
D(\tau, \theta) \approx \frac{2\sqrt{2}}{\sqrt{\frac{1}{2}\tau^2 + (\theta R)^2}} \approx \frac{4\sqrt{2}}{\sqrt{\frac{1}{2}(2 - z - \bar{z})^2 - (z - \bar{z})^2}}, \quad \tau, \theta \to 0, \quad z, \bar{z} \to 1.
\]

(3.43)

Inserting this into (3.38), we have at small \( \tau, \theta \)

\[
g_q(z, \bar{z}) = c_0 Q^{\Delta_q} \left( 1 + \frac{\alpha}{\sqrt{2}} \frac{q^2}{\sqrt{Q}} \frac{1}{\sqrt{\frac{1}{2}\tau^2 + \theta^2 R^2}} + \frac{\alpha^2}{4} \frac{q^4}{Q} \frac{1}{\left(\frac{1}{2}\tau^2 + \theta^2 R^2\right)^{3/2}} + \cdots \right).
\]

(3.44)

We see that the large \( Q \) expansion breaks down when \( \tau, \theta \sim \frac{1}{\sqrt{Q}} \). In particular,
the $t$-channel (light-light OPE) is not accessible within the EFT.

Let us now discuss in more details the regime of validity of EFT. It is supposed to be a good approximation when operator insertions are separated by distances much larger than the charge density scale. On the cylinder that means

$$(\tau_1 - \tau_2)^2 + \theta_{12}^2 R^2 \gg \frac{R^2}{Q} . \quad (3.45)$$

Similarly, the EFT breaks down close to the Lorentzian cone $\tau \to it$ and $t_{12} = \theta_{12} R$. Indeed, in the light-cone limit the $s$-channel expansion (heavy-light OPE) is dominated by double-twist operators, which simply reproduce the identity operator in the $t$-channel (light-light OPE) \cite{73, 74}. The whole EFT answer \textsuperscript{3.44} gives a subleading contribution in the light-cone limit. What really breaks down in the light-cone limit is the matching \textsuperscript{3.29} of the light operators onto the Goldstone boson degrees of freedom. In the light-cone limit the high energy modes of $O_q$ are excited and dominate the expansion \textsuperscript{3.29}.

An important lesson of this discussion is that there are contributions to the correlators which are not described by EFT and we need to make sure that they give a suppressed contribution in order for the EFT answer to be a reliable approximation. The leading EFT answer for the four-point function \textsuperscript{3.38} is
\[ \sim Q^{\Delta_q} e^{-\alpha q \sqrt{Q} \tau}. \]

So we have a situation of the type

\[ G(z, z) = f_{\text{non-EFT}}(z, z) + Q^{\Delta_q} e^{-\alpha q \sqrt{Q} \tau} f_{EFT}(z, z), \tag{3.46} \]

where \( f_{\text{non-EFT}}(z, z) \) is the contribution of operators not described by EFT. For example, the identity operator, stress-tensor, \( U(1) \) current in the \( t \)-channel or double-twist operators in the \( s \)-channel. Thus, EFT is a good approximation to the correlator only when the second term in \( 3.46 \) is exponentially large. Namely, EFT dominates the answer for the whole correlator at large \( Q \) in \( 3.46 \) when \( q \tau < 0 \).

### 3.3.7 Free Field Theories

Let us contrast the results of EFT with correlators in free field theories. In the theory of a free complex scalar the two-point function takes the form

\[ \langle \bar{\phi}(x) \phi(0) \rangle = \frac{1}{|x|^{d-2}}. \tag{3.47} \]

For the heavy operator we choose \( \mathcal{O}_Q = \frac{1}{(Q!)^{1/2}} \phi^Q \) and for the light operator we choose \( \mathcal{O}_{-q} = \frac{1}{(q!)^{1/2}} \bar{\phi}^q \), where the normalization factors are such that the two-point functions are normalized to one.
The correlation function then takes the following form

\[ \langle O_Q(0)O_{-q}(z, \bar{z})O_q(1)O_{-Q}(\infty) \rangle = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_q}} \times \]

\[ \sum_{n=0}^{q} Q^n q^n \prod_{k=0}^{q-1} \frac{(1 - \frac{k}{Q})(1 - \frac{k}{q})}{(n!)^2} \left( \frac{(1-z)(1-\bar{z})}{z\bar{z}} \right)^n \Delta_{\phi}, \]  

(3.48)

where \( \Delta_{\phi} = \frac{d-2}{2} \) and for \( n = 0 \) the product in the numerator is simply 1. Note that this expression is only valid for \( q > 0 \), so in a sense 3.48 should be multiplied by \( \theta(q) \). For \( q < 0 \) the answer is different and cannot be obtained by changing \( q \to -q \) in the expression above.

It is easy to take the large \( Q \) limit of the correlator 3.48. We get

\[ G_{q>0}(z, \bar{z}) = \frac{Q^q}{(z\bar{z})^{\Delta_{\phi}}} \frac{1}{\Gamma(q+1)} \left( 1 + \frac{q(1-q)}{2Q} + \frac{q^2}{Q} \left( \frac{z\bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_{\phi}} + O(Q^{-2}) \right). \]  

(3.49)

Note that we have the following identity

\[ \left( \frac{z\bar{z}}{(1-z)(1-\bar{z})} \right)^{\Delta_{\phi}} = \sum_{m=0}^{\infty} (z\bar{z})^{\Delta_{\phi} + \frac{m}{2}} C_{m}^{(\frac{d}{2}-1)} \left( \frac{z + \bar{z}}{2\sqrt{z\bar{z}}} \right). \]  

(3.50)

The absence of the descendant in 3.49 is due to the fact that in this case \( \Delta_{Q-q} - \Delta_{Q,q} = 0 \) (see 3.16). Thus, we see that, on the one hand, the structure of the
correlator is very similar to the one appearing in the EFT. On the other hand, if we ignored \( \theta(\pm q) \) and defined \( g_q(z, \bar{z}) \) according to 3.11 and used 3.49 for \( G(z, \bar{z}) \), the result for \( g_q(z, \bar{z}) \) would not be crossing symmetric. In the case of EFT, \( \theta(\pm q) \) did not arise.

For free charged fermions \( \psi_\alpha, \psi_\alpha \) in 3d we have a heavy charged charged operator that corresponds to a state with a Fermi sea. It has a scaling dimension \( \Delta_Q \sim Q^{3/2} \). The fluctuations around this state are, however, not described by a Goldstone boson EFT. In particular, correlators contain a non-analyticity in charges \( \theta(\pm q) \), similar to the free complex boson discussed above.

### 3.4 Macroscopic Limits of Correlators

Now we switch gears and discuss a seemingly unrelated subject. However, we will soon use it as an important input to our bootstrap analysis in the next section.

A heavy operator with a large global charge corresponds to a state with large energy and charge densities on the cylinder \( \mathbb{R} \times S^{d-1} \). The limit \( Q \to \infty \) with \( R \) fixed, where \( R \) is the radius of \( S^{d-1} \), results in both the energy and charge density going to infinity. In this context it is natural to consider a combined limit where both \( Q \to \infty \) and \( R \to \infty \) such that the correlation functions of
light operators (that is, whose dimensions held fixed in the limit) in this state remain finite, up to an overall rescaling of the light operators.

Equivalently, as \( Q \to \infty \), the background value of a light operator is given in terms of the three point function, \( \lambda_{Q,-q,-(Q-q)} \), where \(-q\) is the charge of the light operator and smoothness in \( Q \) is assumed to relate the spectrum at charge \( Q \) and \( Q - q \). The macroscopic limit corresponds to the scaling regime of cross ratios in a four-point function with two light operators, such that the ratio of the four-point function and the product of the above three-point functions is kept fixed.

This happens when the two light operators are brought sufficiently close together. The result are correlation functions of the flat space CFT in a nontrivial state. In various situations, the energy and/or charge density may remain fixed or be scaled to 0 in this limit. We call these type of limits macroscopic. It will turn out to be useful for solving bootstrap equations in the next section.

The limit \( R \to \infty \) when the energy density is kept fixed, also known as the thermodynamic limit, was recently discussed in the CFT context in [51]. We will consider this limit in our case as well, even though the correlators that we will get in this limit are not thermal. Rather, they are described by the Goldstone EFT. More generally, the limit with fixed energy density does not have
to coincide with the macroscopic limit (in which correlators are kept finite) for
every state, even though it is not clear to us what are all cases in which it fails.
Not very surprisingly, they fail to agree in the case of a free complex scalar field,
as we will review. It is also easy to show that it will not exist for BPS operators
with $\Delta \sim Q$ and, more generally, we expect it to fail for chiral ring operators in
CFTs with moduli. In this case the macroscopic limit has rather different prop-
erties, as we discuss below.

The basic idea is the following. Consider a CFT on the cylinder $\mathbb{R} \times S^{d-1}$

$$ds_{cyl}^2 = d\tau^2 + R^2 d\Omega_{d-1}^2 = \left(\frac{R}{r}\right)^2 (dr^2 + r^2 d\Omega_{d-1}^2) = \left(\frac{R}{r}\right)^2 ds_R^2, \quad \tau = R \log r .$$

(3.51)

The mapping of correlators to the plane takes the following form

$$\langle H | O(x) ... | H \rangle_{cyl} = \left(\frac{r}{R}\right)^{\Delta_H} \frac{\langle O_H(0) O(x) ... O_H(\infty) \rangle_{\mathbb{R}^d}}{\langle O_H(0) O_H(\infty) \rangle_{\mathbb{R}^d}} , \quad (3.52)$$

where we suppressed other operators and their conformal transformation factors.
A heavy operator insertion corresponds to a state on the cylinder with energy

$E = \frac{\Delta_H}{R}$ and, in general, some charge $Q$.

We can consider the limit $R \to \infty$, simultaneously with $\Delta_H \to \infty$ and $Q \to$
such that the energy density \(\epsilon\) and the charge density \(q\)

\[
\epsilon = \frac{\Delta_H}{R^d}, \quad q = \frac{Q}{R^{d-1}}.
\]

(3.53)

are kept fixed and non-zero. For this to be possible the scaling dimension \(\Delta_H\)

and the charge \(Q\) should be related as follows

\[
\Delta_H \sim Q^{\frac{d}{d-1}},
\]

(3.54)

as should be clear from 3.53. We, thus, first discuss the case 3.54 and later consider

the most general possibility.

In what follows, it will be useful for us to study correlation functions in the

macroscopic limit. To be more specific let us consider the following \((n + 3)\)-point

function on \(\mathbb{R}^d\)

\[
G(z_i, \bar{z}_i) \equiv \langle O_H(0)O_{L_1}(z_1, \bar{z}_1) \ldots O_{L_n}(z_n, \bar{z}_n)O_{L_{n+1}}(1)O_H^\dagger(\infty) \rangle,
\]

(3.55)

where for simplicity we put all the operators in one plane. The most generic

case is analogous.

We would like to describe the macroscopic limit above in the conformal in-
variant terms. Taking the radius of the sphere $R$ to infinity and keeping the distance between the light operators $L$ fixed is conformally equivalent to keeping the sphere intact and scale local operators toward each other. This becomes a limit in the space of cross ratios.

In terms of cross-ratios $z_i$, $\tilde{z}_i$ the physical distance between light operators is $L \sim R|1 - z_i|$, which corresponds to taking the limit

$$z_i = 1 - \frac{w_i}{\Delta_H^{1/d}}, \quad \tilde{z}_i = 1 - \frac{\tilde{w}_i}{\Delta_H^{1/d}}. \quad \text{(3.56)}$$

where we take $\Delta_H \to \infty$ and $w_i, \tilde{w}_i$ - fixed. In writing 3.56 we used that $R \sim \Delta_H^{\frac{1}{d}} \to \infty$, which follows from 3.53.

The statement that the macroscopic limit exists for the correlation functions, thus, becomes

$$G^\epsilon(w_i, \tilde{w}_i) \equiv \lim_{\Delta_H \to \infty} \Delta_H^{-\frac{1}{d} \sum_{i=1}^{n+1} \Delta L_i} G \left(1 - \frac{w_i}{\Delta_H^{1/d}}, 1 - \frac{\tilde{w}_i}{\Delta_H^{1/d}}\right). \quad \text{(3.57)}$$

The pre-factor $R^{-\sum_i \Delta L_i}$ in 3.57 is due to the conformal factor in 3.52.

Let us apply 3.57 to the case of one light operator, which corresponds to $n =
In this case the dependence on \((w_i, \bar{w}_i)\) trivializes and we get

\[
G^\epsilon = \lim_{\Delta H \to \infty} \Delta_H^{-\frac{\Delta L}{d}} c_{HH^L}.
\]  

(3.58)

The existence of the macroscopic limit, thus, immediately implies that \([51]\) (See comments after equation (9) in that paper.)

\[
c_{HH^L} \leq c_0 \Delta_H^{\frac{\Delta L}{d}},
\]  

(3.59)

for some constant \(c_0\) and any light operator \(O_L\). Only operators that saturate the bound \(3.59\) contribute to the macroscopic limit. Both the identity operator and the stress tensor saturate the bound \(3.59\). Indeed, for \(T_{\mu\nu}\) we have \(\Delta T_{\mu\nu} = d\) and \(c_{HH^T_{\mu\nu}} \sim \Delta_H\) so that \(3.59\) is saturated. For the conserved current we have \(c_{HH^J_{\mu}} \sim Q\) and it saturates the bound \(3.59\) when \(3.54\) holds.

Further, one can use the bound \(3.59\) to consider a contribution of light operators into the \((n + 3)\)-point function \(3.57\) in the light-light OPE channel. Each light operator \(L'\) contributes in the macroscopic limit \(3.56\sim ([1 - z][1 - \bar{z}])^{\frac{\Delta L'}{d}} c_{HH^L' L'} \sim \Delta_H^{-\frac{\Delta L'}{d}} c_{HH^L' L'}\). Thus, only operators that saturate the bound \(3.59\) contribute in the finite energy density limit. The presence of identity operator and stress-tensor implies that the \((n + 3)\)-point function \(G^\epsilon(w, \bar{w})\) is nontrivial,
For us the relevant example is $n = 1$, namely the case of the four-point function. We have for the macroscopic limit

$$G^e(w, \bar{w}) \equiv \lim_{\Delta_H \to \infty} \Delta_H^{-\frac{3}{2} \Delta_L} \ G \left( 1 - \frac{w}{\Delta_H^{1/d}}, 1 - \frac{\bar{w}}{\Delta_H^{1/d}} \right). \quad (3.60)$$

It is instructive to take the t-channel OPE expansion and see how each separate conformal block contributes in the macroscopic limit. We get

$$G^e(w, \bar{w}) = (w \bar{w})^{-\Delta_L} \sum_{\Delta, J} \lambda_{L,L,\Delta,J} \lambda_{H,H,\Delta,J} g_{\Delta,L}^{0,0} \left( \frac{w}{\Delta_H^{1/d}}, \frac{\bar{w}}{\Delta_H^{1/d}} \right). \quad (3.61)$$

Only primary operators that saturate the bound give a non-zero contribution to (3.61). Using it is also easy to understand that descendants decouple, since extra powers of $(1 - z)$ are suppressed by $\frac{1}{\Delta_H^{1/d}}$. The contribution of light operators in the t-channel, thus, takes the form

$$G^e(w, \bar{w}) = (w \bar{w})^{-\Delta_L} \sum_{\Delta, J} \lambda_{L,L,\Delta,J} \tilde{\lambda}_{H,H,\Delta,J} \left( \frac{w + w}{2 \sqrt{ww}} \right) + \ldots ,$$

$$\tilde{\lambda}_{\Delta, J} = \lim_{\Delta_H \to \infty} \lambda_{H,H,\Delta,J} \Delta_H^{-\frac{d}{2}} , \quad (3.62)$$

where the sum is over primary operators $\mathcal{O}_{\Delta,J}$ saturating the bound and by
ellipsis we denoted a potential contribution of operators whose dimensions scale with $\Delta_H$ as well. Note that we do not know the convergence properties of the $t$-channel OPE after taking the macroscopic limit. At the very least it should be a reliable asymptotic series for small $\bar{w}\bar{w}$.

3.4.1 Macroscopic Limit in the EFT

In EFT the relation 3.54 holds and, therefore, we can take the limit $R \to \infty$ with finite energy and charge densities described above. In three dimensions $d = 3$ the limit 3.56 becomes

$$z = 1 - \frac{w}{\sqrt{Q}}, \quad \bar{z} = 1 - \frac{w}{\sqrt{Q}} .$$

(3.63)

On the cylinder 3.13 in coordinates $\tau, \theta$ the limit 3.63 is equivalent to (recall that we are working in euclidian kinematics, where $w, \bar{w}$ are complex conjugates of each other)

$$\tau \approx -\frac{\text{Re} w}{\sqrt{Q}}, \quad \theta \approx \frac{\text{Im} w}{\sqrt{Q}} ,$$

(3.64)

up to corrections suppressed at large $Q$. To derive the macroscopic limit of the EFT four-point function 3.38, we simply insert 3.64 into the short-distance ex-
pansion \[3.44\] and obtain

\[
G^*(w, w) = \lim_{Q \to \infty} Q^{-\Delta_q} G \left( 1 - \frac{w}{\sqrt{Q}}, 1 - \frac{w}{\sqrt{Q}} \right) = \\
= c_0 Q^{\Delta_q} e^{\frac{\alpha}{2} q(w+w)} \left( 1 + \frac{\alpha q^2 \sqrt{2}}{\sqrt{\frac{1}{2} (w+w)^2 + (w-w)^2}} + \\
+ \frac{\alpha^2 q^4}{\frac{1}{2} (w+w)^2 + (w-w)^2} - \frac{2 \sqrt{2} q \Delta_q (w+w)}{\left[ \frac{1}{2} (w+w)^2 + (w-w)^2 \right]^{3/2}} + \ldots \right)
\]

In the macroscopic limit \[3.63\] the EFT regime of validity \[3.45\] becomes \(|w| \gg 1\).

In this regime the expansion \[3.65\] is a controlled approximation with corrections suppressed by inverse powers of \(w, w\).

Note, that when we take the macroscopic limit the structure is slightly different from the large \(Q\) limit, namely the contributions which were parametrically suppressed in the large \(Q\) limit could become of the same order in the macroscopic limit. This is essentially due to the fact that \(z^{\alpha(Q)} = (1 - \frac{w}{\alpha(Q)})^{\coa(Q)} \to e^{-cw}\) when \(\alpha(Q) \to \infty\). Therefore, for operators with different \(c\), say \(c_1\) and \(c_2\), the one with larger \(c\) is exponentially suppressed with respect to the one with smaller \(c\) in the large \(Q\) limit. While in the macroscopic limit they both become of the same order. Still, operators with larger \(c\) stay exponentially suppressed in limit \(|w| \gg 1\).
3.4.2 Other Limits

We can also imagine a situation when\ref{3.54} does not hold. A well-known example of this type is $\Delta(Q) \sim Q$, which is common in supersymmetric and free theories. In this case it is clear that the limit we described above does not exist.

Indeed, the conserved current $J_\mu$ would violate\ref{3.59}

$$c_{HH^\dagger J_\mu} \sim Q > c_0 \frac{\Delta}{\Delta H} \sim Q \frac{d-1}{d}.$$ \hfill (3.66)

More generally, if we imagine $\Delta(Q) \sim Q^\alpha$ then for $\alpha \geq \frac{d}{d-1}$ the bound\ref{3.59} is satisfied by the current and the thermodynamic limit described above might exist, whereas for $\alpha < \frac{d}{d-1}$ the bound\ref{3.59} is violated due to\ref{3.66}.

In situations when the thermodynamic limit does not exist we could imagine a different limit

$$G^\beta(w_i, \bar{w}_i) \equiv \lim_{\Delta H \to \infty} \Delta_H^{-\beta} \sum_{i=1}^{n+1} \Delta L_i \left(1 - \frac{w_i}{\Delta H} \beta, 1 - \frac{\bar{w}_i}{\Delta H} \beta\right)$$ \hfill (3.67)

for some $\beta$. The condition that such a limit exists implies that

$$c_{HH^\dagger L} \leq c_0 Q^\beta \Delta.$$ \hfill (3.68)
For large enough $\beta$ we expect that the limit exists and is trivial. The question then is what is $\beta$ for which the limit exists and is nontrivial. This is controlled by the operator, whose three-point function first saturates the bound 3.68.

The macroscopic limit of the type 3.67 exists in the free complex scalar theory, as we describe below, and more generally we expect that it is relevant for CFTs with moduli spaces.

3.4.3 Macroscopic Limit for the Free Complex Scalar

Let us again contrast EFT with the theory of a free complex scalar, discussed in section 3. In this case we have $\Delta_Q \sim Q$. The proper macroscopic limit in this case is a zero energy and charge density limit of the type 3.67 with $\beta = \frac{1}{d-2}$.

The operator that first saturates the bound 3.68 is the scalar $\bar{\phi}\phi$. The result for the correlator $3.48 \langle \phi^Q \bar{\phi}^q \phi^q \bar{\phi}^Q \rangle$ in this limit is given by

$$G(w, \bar{w}) = \lim_{Q \to \infty} Q^{-q} G \left(1 - \frac{w}{Q^{1/(d-2)}}, 1 - \frac{\bar{w}}{Q^{1/(d-2)}}\right) = \frac{L_q(- \bar{w}^{(d-2)/2})}{(w \bar{w})^{q(d-2)/2}},$$

(3.69)

where $L_n(x)$ is the Laguerre polynomial. In this limit both the energy and the charge density go to zero. Nevertheless, the limit is nontrivial. A simple computation shows that 3.69 coincides with the two-point function on the moduli
space \( \langle O_{-q}(w, \bar{w})O_q(0) \rangle_{\langle \phi \rangle = 1} \). Thus, in this case the macroscopic limit describes correlators on the moduli space.

### 3.5 Bootstrap at Large \( Q \)

In the previous section we reviewed two particular solutions to the large \( Q \) crossing: EFT and free theories. In this section we explore the structure of a general solution based on unitarity, crossing and the structure of the macroscopic limit.

We assume throughout that all operators that enter the large \( Q \) crossing belong to the families which have a smooth dependence on \( Q \).

#### 3.5.1 Crossing for the Vacuum

As we discussed in section 1, in the large \( Q \) limit at fixed \( z \) we expect the operator with minimal dimension to dominate. In the EFT, the leading contribution to the correlator was given by a single scalar operator which belonged to the same family \( \Delta_Q \) as the external state. In \( d \) dimensions the large \( Q \) asymptotic of the dimension is

\[
\Delta_Q = \frac{d-1}{d} \alpha Q^{d/(d-1)} + \ldots.
\]  

(3.70)
One might consider other possibilities as well. For instance, supersymmetric theories with BPS operators have $\Delta_Q \sim Q$. Here we focus on the case $\Delta_Q$, which is expected to hold in generic interacting CFTs. Moreover, we will focus on 3 dimensions, generalizing only some of the formulae to arbitrary $d$.

The crossing equation with a single scalar operator that dominates takes the form

$$g_q(z, \bar{z}) = |\lambda_{Q-q, -(q+q)}|^2 (zz)^{1/2(\Delta_{Q-q} + \Delta_Q)} + ...$$

$$= |\lambda_{Q,q, -(q+q)}|^2 \left( \frac{1}{\bar{z}z} \right)^{1/2(\Delta_{Q+q} - \Delta_Q)} + ... = g_{q-q} \left( \frac{1}{\bar{z}}, \frac{1}{\bar{z}} \right), \quad (3.71)$$

where we kept only the leading term of the conformal block. This equality implies that to leading order we have

$$\Delta_{Q-q} - \Delta_Q = \Delta_Q - \Delta_{Q+q}. \quad (3.72)$$

This is indeed true if all three operators belong to the same family $\Delta_Q$. Then $\Delta_Q$ becomes a continuity equation for $-q \frac{\partial \Delta_Q}{\partial Q}$ as a function of $Q$.

Similarly, (3.71) implies equality of the corresponding three-point functions to
leading order

$$|\lambda_{Q,-q,-(q-q)}|^2 = |\lambda_{Q,q,-(Q+q)}|^2 .$$ (3.73)

A simple consequence of this matching is that we cannot add a finite number of operators to \(3.71\) without spoiling crossing. Indeed, imagine that in \(3.71\) we had instead

$$g_q(z, \bar{z}) = |\lambda_{Q,-q,-(q-q)}|^2 (zz) \frac{1}{2} \left( \Delta_{Q-q} - \Delta_{Q} \right) \left( 1 + \sum_{i=1}^{N} (z\bar{z})^{\frac{1}{2}} C_{J_i}^{(\frac{d-1}{2})} \left( \frac{z + \bar{z}}{2\sqrt{z\bar{z}}} \right) \right) .$$ (3.74)

If the operators that enter the \(s\)- and \(u\)-channels depend on charge smoothly, then \(\delta_i\) are independent of \(q\). As a result crossing \(3.71\) implies that

$$\delta_i = 0 .$$ (3.75)

In other words, the vacuum could be degenerate and could carry spin, but all the excitations should be suppressed by \(\frac{1}{Q}\).

Another possibility is that \(N = \infty\) in \(3.74\). This case lies beyond the scope of the present work.
Assuming that macroscopic finite energy density limit exists, we immediately find using the leading answer for the four-point function that

$$\lim_{Q \to \infty} |\lambda_{Q,-q,-(Q-q)}|^2 \Delta_Q^{-2\Delta_q} = \text{const}. \quad (3.76)$$

This is indeed the case in the EFT as can be seen from after setting $d = 3$

$$\lambda_{Q,-q,-(Q-q)} \sim \Delta_Q^{\Delta_q/d} \sim Q^{\Delta_q/(d-1)}. \quad (3.77)$$

### 3.5.2 Crossing at Subleading Order

The strategy that we adopt is to approach $|z| = 1$ both from the $s$-channel and the $u$-channel and make sure that they match smoothly. The leading order correction appearing at order $\frac{1}{\sqrt{Q}}$ takes the form (compare with

$$f(\tau, \theta) = -|\tau| + 3e^{-|\tau|}P_1(x) + \sum_{J=0}^{\infty} c_J e^{-c_J |\tau|} P_J(x), \quad x = \cos \theta, \quad c_J > 0, \quad (3.78)$$

where the first term in $f$ is the correction to the scaling dimension of $\mathcal{O}_{Q-q}$ and the second term is the first descendant. These two terms are necessarily present due to the leading order scalar $\mathcal{O}_{Q-q}$. The sum over $J$ represents new primary operators appearing at this order. Analyticity at $|z| = 1$ which is
the same as analyticity at $\tau = 0$ is not manifest due to the non-analyticity of $|\tau| = \theta(\tau)\tau - \theta(-\tau)\tau$. If we formally compute $\partial^\alpha_\tau f(0, \theta)$, there will be terms that involve $\delta(\tau)$ and its derivatives. For the function to be analytic away from the point $(\tau, \theta) = (0, 0)$ where light operators collide, these terms should be set to zero. This condition leads to the following set of equations (independent smoothness conditions involve only odd derivatives of $f$)

$$\delta_{0,n} + 3P_1(x) + \sum_{J=0}^\infty c_J\epsilon_j^{2n+1}P_J(x) = 0, \quad x = \cos \theta \neq 1, \quad n = 0, 1, 2, \ldots, \quad (3.79)$$

where the sum should be understood as a limit of the regulated expression (alternatively, we can use any other regulator, see below).

In the case of the EFT $\epsilon_J = \sqrt{\frac{J(J+1)}{2}}, c_J = \frac{2J+1}{\epsilon_J}, J \geq 2$ one can easily check that (3.79) is satisfied using the generating functional for Legendre polynomials

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{J=0}^\infty P_J(x)t^J, \quad (3.80)$$

by application of a proper combination of $t\partial_t$ derivatives at $t = 1$. The singularity that appears on the RHS of (3.79) at $x = 1$ in EFT case is a linear combination of $\delta^{(n)}(1 - x), \delta^{(n-1)}(1 - x), \ldots$, as one can simply check by recalling the
identity

\[
\frac{1}{2} \sum_{J=0}^{\infty} (2J + 1) P_J(x) P_J(y) = \delta(x - y). \tag{3.81}
\]

and its derivatives at \( y = 1 \).

We would like to understand if the EFT solution is the unique solution to

\[
\text{3.79}
\]

given the form of the correction \[3.78\]. To do that it is useful to better understand the microscopic origin of \( \delta(1 - x) \) in \[3.81\]. As we will see, the precise form of \( c_J, \epsilon_J \) is related to the behavior of \( f(\tau, \theta) \) close to \( \tau = 0, x = 1 \).

Note, that this singularity has nothing to do with very short-distance \( t \)-channel regime, which we effectively collapsed to a point when taking \( Q \to \infty \) limit.

Rather, it is coming from the macroscopic limit region, as we will indeed see later. For now we assumed that we have one operator of each spin in \[3.78\] and will consider generalizations later.

The strategy we adopt is illustrated in the figure \[3.1\]. We consider the discontinuities of \( \tau \)-derivatives of \[3.78\] at \( \tau = \epsilon \to 0 \) and integrate both sides with a
Figure 3.1: The cross ratio $z$-plane. The region $|z| < 1$ is described by the $s$-channel OPE in the large $Q$ limit, whereas the region $|z| > 1$ is described by the $u$-channel OPE. We consider the integral of the derivatives of the correlation function along the drawn contours, namely we take the difference between the derivative evaluated slightly outside and slightly inside the circle. The smoothness of the correlator away from $z = 1$ implies that the difference in that region is of $O(\epsilon)$. On the other hand, we can use the OPE to see that the result should be $O(1)$. This means that the integral is controlled by the singularity close to $z = 1$. In this region the correlator is governed by its properties in the macroscopic limit. The shaded area denotes the short distance region relevant for this computation.
Legendre polynomial. We find

\[
\delta_{n,0}\delta_{J,0} + 3\delta_{J,1} + c_J \epsilon_J^{2n+1} = \\
- \frac{1}{4} (2J + 1) \lim_{\epsilon \to 0} \int_0^\pi d\theta \sin \theta \left[ \partial_\tau^{2n+1} f(\epsilon, \theta) - \partial_\tau^{2n+1} f(-\epsilon, \theta) \right] P_J(\cos \theta).
\]

(3.82)

Note that due to the smoothness of \(f(\tau, \theta)\) away from \(\tau = \theta = 0\) most of the integral produces a contribution of \(O(\epsilon)\), whereas the LHS of 3.82 is \(O(1)\). The sum rule 3.82, thus, makes it manifest that \(c_J, \epsilon_J\) are controlled by the behavior of \(f(\tau, \theta)\) near the singularity.

In principle, all the steps that we perform here could be repeated for a generic CFT. In this case instead of one operator of each spin on the LHS of 3.82 we have an infinite number of them. This structure makes the result much less powerful, see [75]. The benefit of the large Q limit is to make the number of Regge trajectories, that appear on the LHS of 3.82, finite.

The macroscopic limit discussed in section 4 provides us with a controllable short-distance approximation of the correlator and of the function \(f(\tau, \theta)\). We have

\[
G(z, \bar{z}) \sim 1 + \frac{1}{\sqrt{Q}} f(\tau, \theta) .
\]

(3.83)
Assuming that the macroscopic limit exists, we conclude that $f(\tau, \theta)$ can at most grow as $\sqrt{Q}$ in this limit

$$f \left( \tau = -\frac{\text{Re} \, w}{\sqrt{Q}}, \theta = \frac{\text{Im} \, w}{\sqrt{Q}} \right) \sim \sqrt{Q}.$$  

(3.84)

If the growth is slower than $\sqrt{Q}$, there will be no solution for $c_J, \epsilon_J$, as will become clear shortly. So it must grow exactly as $\sqrt{Q}$. This immediately implies that the leading term in $f(\tau, \theta)$ at small distances must be a homogeneous function of degree one

$$f(\lambda \tau, \lambda \theta) \approx \lambda^{-1} f(\tau, \theta), \quad \tau, \theta \ll 1.$$  

(3.85)

The most general solution of this homogeneity equation is

$$f(\tau, \theta) = \frac{1}{\sqrt{\tau^2 + \theta^2}} F(y) + \ldots, \quad y = \frac{\tau}{\sqrt{\tau^2 + \theta^2}}, \quad \tau, \theta \ll 1.$$  

(3.86)

where $F(y)$ is an arbitrary function regular on an interval $y \in [0, 1]$ and ...

stands for less singular terms. Indeed, the end points $y = 0$ and $y = 1$ correspond to $\tau = 0, \theta \neq 0$ and $\tau \neq 0, \theta = 0$ respectively. The correlator is regular

---

Footnote 10: In fact, it is regular in a larger domain, but this interval is all we need.
at these points and we can smoothly interpolate between them. Moreover, the function $F(y)$ is even $F(y) = F(-y)$ due to crossing. In the small $\tau$ and $\theta$ limit the two cross-ratios become $u \approx 1 + 2\tau, v \approx \tau^2 + \theta^2$.

Let us first consider $n = 0$ in (3.82). To evaluate the integral we can simply insert the expression (3.86) into (3.82). Any less singular term in (3.86) which we denoted by $+\ldots$ produces a contribution which is vanishing in the $\epsilon \to 0$ limit. To reproduce the LHS of (3.82) we can, therefore, use the short distance approximation $f(\tau, \theta) = \frac{1}{\sqrt{\tau^2 + \theta^2}} F(y)$. Plugging (3.86) into (3.82) we find for $n = 0$

$$\delta_{J,0} + 3\delta_{J,1} + c_J \epsilon_J = \frac{1}{2} (2J + 1) \int_0^1 dy \left( F(y) + y F'(y) - \frac{1}{y} F'(y) \right), \quad (3.87)$$

where we expanded

$$\sin \theta \, P_J(\cos \theta) = \sum_{m=0}^{\infty} \alpha_m \theta^{2m+1}, \quad (3.88)$$

switched to an integral over $y = \frac{\epsilon}{\sqrt{\epsilon^2 + \theta^2}}$, and took the limit $\epsilon \to 0$. The known coefficients $\alpha_m$ are polynomials in $\Omega_J^2 = \frac{J(J+1)}{2}$ of the maximal power $\alpha_m \sim (\Omega_J^2)^m$. For $n = 0$ only the leading term $m = 0$ contributes.

The integral in (3.87) is finite since $F(y)$ is a regular even function on the interval $[0, 1]$. In particular, the last term is finite since $F'(y) \sim y$ near $y = 0$. 194
Finally, we obtain

$$\delta_{J,0} + 3\delta_{J,1} + c_J \epsilon_J = \beta_{0}^{(0)}(2J + 1), \quad (3.89)$$

where $\beta_{0}^{(0)}$ is an unknown $J$-independent constant given by the integral in 3.87.

Next, we consider the case of $n = 1$ in 3.82 for regularity of $\partial^3 \tau f$. There are two changes compared to the case of $n = 0$. First, the terms with $m = 0$ and $m = 1$ in the expansion 3.88 contribute. Second, some of the less singular terms which we have not written explicitly in 3.86 could generate a finite contribution as well. Let us present the final result and then make a few comments

$$3\delta_{J,1} + c_J \epsilon_J^3 = -\frac{1}{4}(2J + 1) \lim_{\epsilon \to 0} \int_0^\pi d\theta \sin \theta \left[ \partial^3 \tau f(\epsilon, \theta) - \partial^3 \tau f(-\epsilon, \theta) \right] P_J(\cos \theta) =$$

$$= (2J + 1)(\beta_{0}^{(1)} + \beta_{1}^{(1)} \Omega_J^2), \quad \Omega_J^2 \equiv \frac{J(J + 1)}{2}. \quad (3.90)$$

The limit $\epsilon \to 0$ is somewhat subtler in this case. After inserting $\frac{1}{\sqrt{\tau^2 + \theta^2}} F(y)$ in the first line of 3.90 and changing integration variable to $y = \frac{\epsilon}{\sqrt{\epsilon^2 + \theta^2}}$, one finds the leading term of order $\frac{1}{\epsilon^2}$. It, however, should vanish if we are to obtain finite answer in the limit $\epsilon \to 0$. Indeed, one can check that the $y$-integral multiplying $\frac{1}{\epsilon^2}$ is zero for any $F(y)$. The finite contribution comes from two terms.
The term $\beta_{1}^{(1)}$ arises solely from the leading short distance asymptotic in $\beta_{1}^{(1)}$ multiplying the second term $m = 1$ in $\beta_{1}^{(1)}$. The term that involves $\beta_{0}^{(1)}$, on the other hand, receives a contribution from a less singular term in $\beta_{0}^{(1)}$ multiplying $m = 0$ term in $\beta_{0}^{(1)}$. The coefficients $\beta_{0}^{(1)}, \beta_{1}^{(1)}$ are related to the precise form of the function $F(y)$ and the subleading terms in $\beta_{0}^{(1)}$ and cannot be fixed. The dependence on $J$, however, is completely fixed in $\beta_{0}^{(1)}$.

Similar manipulations give equations for any $n = 0, 1, 2, \ldots$ in $\beta_{0}^{(1)}$. The result is simply

$$\delta_{n,0}\delta_{J,0} + 3\delta_{J,1} + c_{J}\epsilon^{2n+1} = (2J + 1)W_{n}(\Omega_{J}^{2}), \quad n = 0, 1, 2, \ldots , \quad (3.91)$$

where

$$W_{n}(\Omega_{J}^{2}) = \sum_{k=0}^{n} \beta_{k}^{(n)} \Omega_{J}^{2k} \quad (3.92)$$

is a polynomial of degree $n$ whose coefficients are arbitrary and related to the precise form of the function $F(y)$ and less singular terms in $\beta_{0}^{(1)}$. The self-consistency of $\beta_{0}^{(1)}$ requires that all the terms singular in $\epsilon$ integrate to zero. As in example above, the maximal power of the polynomial $W_{n}(\Omega_{J}^{2})$ is controlled by the lead-
ing asymptotic of \[3.86\] \[11\]

3.5.3 THE SMOOTHNESS CONDITIONS

Next, we would like to analyze in more details the equations \[3.91\]. So far we assumed that there is exactly one operator of each spin \[3.78\]. Let us start by relaxing this condition. We can generalize the ansatz \[3.78\] to have \(N\) Regge trajectories, i.e. \(N\) operators with each spin with squares of three-point functions and scaling dimensions \(c_{J,i}, \epsilon_{J,i}\) and \(i = 1, \ldots, N\). On general grounds we expect that the OPE data \(c_{J,i}, \epsilon_{J,i}\) is analytic in spin \(J\) for some \(J \geq J_0\). As was shown in \[15\], \(J_0 = 2\). For \(J = 0, 1\) the physical values of the OPE coefficients \((c_{ij}^{\text{phys}}, \Delta_{ij}^{\text{phys}})\) and the values that one obtains from the analytic expressions \((c_{ij}^{\text{analytic}}, \Delta_{ij}^{\text{analytic}})\) do not have to agree.

In this way natural generalizations of \[3.78\] take the following form

\[
f(\tau, \theta) = -|\tau| + 3xe^{-|\tau|} + \sum_{J=2}^{\infty} \sum_{i=1}^{N} \frac{2J + 1}{\epsilon_{J,i}} d_{J,i} e^{-\epsilon_{J,i}|\tau|} P_J(x) \\
+ \sum_{i=1}^{N_0} \frac{1}{E_{0,i}} D_{0,i} e^{-E_{0,i}|\tau|} + 3x \sum_{i=1}^{N_1} \frac{1}{E_{1,i}} D_{1,i} e^{-E_{1,i}|\tau|},
\]

(3.93)

where we parameterized the squares of three-point functions by \(d_{J,i}, D_{0,i}, D_{1,i}\).

\[11\] The structure of the equations is reminiscent of the Casimir trick used in the conformal bootstrap.
This ansatz consists of \( N \) Regge trajectories together with a finite number \( N_0 \) of scalar operators and \( N_1 \) spin one operators. As discussed above, \( \epsilon_{J,i} \) and \( d_{J,i} \) are analytic functions of \( J \) for \( \text{Re}[J] \geq 2 \). Unitarity implies that the squares of three-point functions are positive and scaling dimensions are real. Moreover, we assume that scaling dimensions \( \epsilon_{J,i}, E_{0,i}, E_{0,1} \) are also positive\(^{12}\)

\[
d_{J,i}, D_{1,i}, D_{0,i}, \epsilon_{J,i}, E_{0,i}, E_{0,1} > 0 .
\]

The case when some of the three-point functions or energies are zero reduces to the one with smaller \( N, N_0, N_1 \). We also assume that

\[
\epsilon_{J,i} \neq \epsilon_{J,j}, \quad E_{0,i} \neq E_{0,j}, \quad E_{1,i} \neq E_{1,j}
\]

for \( i \neq j \), since otherwise the solution is again equivalent to one with smaller \( N, N_0, N_1 \).

Let us first analyze the analytic part of (3.93) which is encoded in \( \epsilon_{J,i} \) and \( d_{J,i} \).

\(^{12}\)This positivity is not necessary and one can consider cases when this does not hold. However, we restrict our analysis to positive \( \epsilon_{J,i}, E_{0,i}, E_{0,1} \). This assumption is equivalent to clustering in the large \( Q \) limit that we discussed in section 1.
The corresponding generalization of equations 3.91 takes the form

$$\sum_{i=1}^{N} d_{J,i} \epsilon_{J,i}^2 = W_n(\Omega_j^2), \quad n = 0, 1, 2, \ldots, \quad J \geq 2. \quad (3.96)$$

Originally, these equations are written for integer spins $J = 2, 3, 4, \ldots$. However, analyticity in $J$ of both the LHS and the RHS implies that 3.96 should hold in the whole $J$ plane, due to the Carlson theorem.

For $J = 0, 1$ the smoothness conditions take the form

$$J = 0: \quad \delta_n,0 + \sum_{i=1}^{N_0} D_{0,i} E_{0,i}^{2n} = W_n(0), \quad n = 0, 1, \ldots,$$

$$J = 1: \quad 1 + \sum_{i=1}^{N_1} D_{1,i} E_{1,i}^{2n} = W_n(1), \quad n = 0, 1, \ldots. \quad (3.97)$$

Equations 3.96, 3.97 comprise the full set of smoothness conditions of the function 3.93. Let us describe their solutions. We start with the smoothness conditions 3.96 for $J \geq 2$. Then we consider equations 3.97 and show that EFT is the unique solution for one Regge trajectory $N = 1$. 199
3.5.4 **Solution of Smoothness Conditions for** $J \geq 2$.

Let us introduce the notation\textsuperscript{13}

\[ z \equiv \Omega_J^2, \quad d_i(z) \equiv d_{J,i}, \quad \epsilon_i(z) \equiv \epsilon_{J,i}^2 , \]  

(3.98)

where analyticity in spin for $\text{Re} \ J \geq 2$ implies that the functions $d_i(z), \epsilon_i(z)$ are analytic for $\text{Re} \ z \geq \sqrt{3}$. Moreover, for real $z \geq \sqrt{3}$ the functions $d_i(z), \epsilon_i(z)$ are real and positive and $\epsilon_i(z) \neq \epsilon_j(z)$ for $i \neq j$. Then the equations\textsuperscript{3.96} take the form

\[ \sum_{i=1}^{N} d_i(z) \epsilon_i(z)^n = W_n(z), \quad n = 0, 1, 2, \ldots . \]  

(3.99)

The solutions of these equations are derived in appendix C. The result is as follows. The scaling dimensions $\epsilon_i(z)$ are given by the $N$ distinct solutions of the

\textsuperscript{13}This $z$ parameterizes dependence on spin and has nothing to do with $z$ in previous sections, which was the position of an operator insertion.
$N$-th order algebraic equation

$$
\prod_{i=1}^{N}(x - \epsilon_i(z)) = x^N - \mathcal{P}_1(z)x^{N-1} + \mathcal{P}_2(z)x^{N-2} + \cdots + (-1)^N\mathcal{P}_N(z) = 0 ,
$$

(3.100)

where $\mathcal{P}_n(z)$ is an $n$-th order polynomial in $z$.\footnote{These solutions can be traced back to the 17th century works of Albert Girard and Isaac Newton.} The three-point functions $d_i(z)$ are given by

$$
d_k(z) = \prod_{i<j}[\epsilon_j(z) - \epsilon_i(z)]^{-1} \det \begin{pmatrix}
1 & \ldots & W_0 & \ldots & 1 \\
\epsilon_1(z) & \ldots & W_1(z) & \ldots & \epsilon_N(z) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\epsilon_1(z)^{N-1} & \ldots & W_{N-1}(z) & \ldots & \epsilon_N(z)^{N-1}
\end{pmatrix},
$$

(3.101)

where the $k$-th column is made of polynomials $W_n(z)$. The equations $3.100$, $3.101$ give a complete solution of the smoothness conditions $3.99$ for spins $J \geq 2$. Finally, one needs to impose constraints on $W_n(z), \mathcal{P}_i(z)$ to ensure that $\epsilon_i, d_i(z)$ are real and positive for real $z \geq \sqrt{3}$. One simple consequence of this is that $\mathcal{P}_i(z)$ is real and positive for real $z \geq \sqrt{3}$.\footnote{These solutions can be traced back to the 17th century works of Albert Girard and Isaac Newton.}
The solution 3.100, 3.101 is parameterized by \( N(N + 2) \) real constants contained in polynomials \( W_0, \ldots, W_{N-1}, P_1, \ldots, P_N \).

Now, let us explain why 3.100, 3.101 is a solution. The equation 3.101 is simply the solution of the first \( N \) equations in 3.99 considered as a linear system for \( d_i(z) \) due to Cramer’s rule. The solution for \( \epsilon_i(z) \) comes from the rest of the equations in 3.99. Generically, the scaling dimensions \( \epsilon_i(z) \), being the solutions of the equation 3.100, will contain branch cuts and will define an analytic function on an \( N \)-sheeted Riemann surface. Then the LHS of 3.99 sums over all sheets of this Riemann surface and cancels the branch cuts to reproduce the polynomial on the RHS.

Further, \( d_k(z) \underline{3.101} \) has poles at \( \epsilon_k = \epsilon_j, j \neq k \). This pole is cancelled on the LHS of 3.99 between \( d_k \) and \( d_j \), which is guaranteed by 3.101. Essentially, this follows from the anti-symmetry of the determinant in 3.101 under exchange of two columns.

Finally, the choice of powers of polynomials \( P_n(z) \) ensures correct behavior \( d_i(z) \sim 1, \epsilon_i(z) \sim z \) when \( z \to \infty \) to match the behavior of the RHS of 3.99.

The proof of the uniqueness of the solution 3.100, 3.101 and more details about its derivation can be found in appendix C.
3.5.5 One Regge Trajectory

Now, we can consider the smoothness conditions for \( J = 0, 1 \). Unlike the situation for \( J \geq 2 \), these equations do not have a nice analytic structure. Therefore, we are forced to deal with them separately for different numbers of Regge trajectories. We start with \( N = 1 \).

In this case, \( 3.100, 3.101 \) are reduced to

\[
\epsilon_j^2 = \mathcal{P}_1(\Omega_j^2) = c^2\Omega_j^2 + m^2, \quad d_J = W_0, \quad J \geq 2 ,
\]

where \( c, m, W_0 \) are arbitrary parameters. The polynomials \( W_n \) are given by

\[
W_n(\Omega_j^2) = W_0(c^2\Omega_j^2 + m^2)^n, \quad n = 0, 1, 2, \ldots
\]

Then, we would like to solve the equations for \( J = 0, 1 \). It is convenient to consider two cases separately:

1) \( N_0 = 0 \). The equation for \( J = 0 \) gives

\[
W_0 = 1, \quad m = 0
\]
For spin $J = 1$ we have

\begin{align*}
1 + \sum_{i=1}^{N_1} D_{1,i} &= 1 , \\
1 + \sum_{i=1}^{N_1} D_{1,i} E_{1,i}^2 &= c^2 , \\
1 + \sum_{i=1}^{N_1} D_{1,i} E_{1,i}^4 &= c^4 .
\end{align*}

(3.105)

Since $D_{1,i} > 0$, the only solution is $N_1 = 0, c = 1$. All free parameters are fixed in this case and we recover the EFT solution

\[ f = -|\tau| + 3xe^{-|\tau|} + \sum_{J=2}^{\infty} \frac{2J + 1}{\Omega_J} e^{-\Omega_J|\tau|} P_J(x) . \]  

(3.106)

2) $N_0 > 0$. In this case \textcolor{red}{3.97} for $J = 0$ gives

\begin{align*}
1 + \sum_{i=1}^{N_0} D_{0,i} &= W_0 , \\
\sum_{i=1}^{N_0} D_{0,i} E_{0,i}^{2n} &= W_0 m^{2n} , \quad n = 1, 2, \ldots .
\end{align*}

(3.107)

These do not have nontrivial unitary solutions. Indeed, without loss of generality we assume that $E_{0,1}^2 > \cdots > E_{0,N_0}^2$. Taking the large $n$ limit of the
second equation in the 3.107 we find $D_{0,1}E_{0,1}^{2n} = W_0m^{2n}, n \gg 1$. This implies $D_{0,1} = W_0, E_{0,1} = m, N_0 = 1$, which contradicts the first equation in 3.107.

We, thus, conclude that the EFT of [21, 22] is the unique solution to the crossing equations if we assume that only one Regge trajectory appears in the OPE.

### 3.5.6 Two Regge Trajectories

For two Regge trajectories the equations 3.100, 3.101 reduce to

\[
x^2 - \mathcal{P}_1(\Omega_J^2)x + \mathcal{P}_2(\Omega_J^2) = 0,
\]

\[
d_1 = \frac{W_0\epsilon_2 - W_1}{\epsilon_2 - \epsilon_1}, \quad d_2 = \frac{W_1 - W_0\epsilon_1}{\epsilon_2 - \epsilon_1}, \quad J \geq 2, \quad (3.108)
\]

where the scaling dimensions $\epsilon_i$ are the solutions of the first equation in 3.108 and the dependence on $\Omega_J^2$ is suppressed for brevity. Thus, we have

\[
\epsilon_1(\Omega_J^2) = \frac{1}{2} \left( \mathcal{P}_1 + \sqrt{\mathcal{P}_1^2 - 4\mathcal{P}_2} \right), \quad \epsilon_2(\Omega_J^2) = \frac{1}{2} \left( \mathcal{P}_1 - \sqrt{\mathcal{P}_1^2 - 4\mathcal{P}_2} \right),
\]

\[
d_1(\Omega_J^2) = \frac{W_0}{2} + \frac{W_1 - \frac{1}{2}W_0\mathcal{P}_1}{\sqrt{\mathcal{P}_1^2 - 4\mathcal{P}_2}}, \quad d_2(\Omega_J^2) = \frac{W_0}{2} - \frac{W_1 - \frac{1}{2}W_0\mathcal{P}_1}{\sqrt{\mathcal{P}_1^2 - 4\mathcal{P}_2}}, \quad J \geq 2 (3.109)
\]

Now we need to solve the smoothness conditions 3.97 for $J = 0, 1$. Instead of writing down the most general solution, we will consider two simple examples,
which will demonstrate what kind of solution one might have. The first solution will have one Regge trajectory with a dispersion relation of a free massive particle and one Regge trajectory of the Goldstone mode. The other solution will have neither a Goldstone mode nor any other interpretation in terms of quasiparticles and might be considered as some strongly interacting CFT.

As the first example let us consider the following solution

\[
f(\tau, \theta) = D(\tau, x) + (W_0 - 1) D_{c,m}(\tau, x) = \\
= -|\tau| + \sum_{J=1}^{\infty} \frac{2J + 1}{\Omega_J} e^{-\Omega_J |\tau|} P_J(x) + (W_0 - 1) \sum_{J=0}^{\infty} \frac{2J + 1}{\sqrt{c^2 \Omega_J^2 m^2}} e^{-\sqrt{c^2 \Omega_J^2 m^2} |\tau|} P_J(x)
\]

where \( D(\tau, x) \) is the propagator of the Goldstone boson \( N \), and \( D_{c,m}(\tau, x) \) stands for a propagator of a free particle of mass \( m \) moving at the speed \( \frac{c}{\sqrt{2}} \), satisfying the equation

\[
(\partial_\tau^2 + \frac{c^2}{2} \Delta_{S^2} - m^2) D_{c,m}(\tau, x) = -4\delta(\tau)\delta(x-1) .
\] (3.111)

One can check that \( 3.110 \) is indeed a particular case of \( 3.109 \) and that it solves the \( J = 0, 1 \) constraints \( 3.97 \) as well.

As the second example, let us consider \( 3.109 \) and set \( W_1 = \frac{1}{2} W_0 P_1 \) to cancel the second term in \( d_i \). To further simplify our lives, let us take \( N_0 = N_1 = 0 \)
in \[3.97\]. Now it is easy to solve \[3.97\] and find that \(W_0 = 1, W_1(z) = z, P_1(z) = 2z, P_2(z) = (1 - a^2)z^2 + a^2z\) and \(d_1 = d_2 = \frac{1}{2}, \epsilon_{1,2} = z \pm az\sqrt{z - 1}\) with an arbitrary constant \(a\). Thus we have

\[
f(\tau, \theta) = -|\tau| + 3e^{-|\tau|}x + \frac{1}{2} \sum_{j=2}^{\infty} \frac{2J + 1}{\sqrt{\Omega_j^2 + a\Omega_j\sqrt{\Omega_j^2 - 1}}} e^{-|\tau|\sqrt{\Omega_j^2 + a\Omega_j\sqrt{\Omega_j^2 - 1}}} P_J(x) + e^{-|\tau|\sqrt{\Omega_j^2 - a\Omega_j\sqrt{\Omega_j^2 - 1}}} P_J(x)
\]

For \(a = 0\) we recover EFT, but for \(a \neq 0\) this solution does not have a quasiparticle interpretation.

### 3.6 Extensions

In this section we sketch several extensions of the previous analysis. First, we consider external operators that carry spin. Second, we comment on correlation functions at the next order in \(\frac{1}{Q}\). Third, we consider the correlation function in the limit \(w \gg \bar{w} \gg 1\) and discuss matching to the \(t\)-channel OPE expansion.
3.6.1 External Operators With Spin

In the sections above the external operator was a heavy charged scalar. It corresponds to the Goldstone vacuum. On the other hand, if we act on the vacuum with the Goldstone creation operators $a_{j,m}^\dagger |Q\rangle$ we end up with the state which corresponds to a primary operator $O_{\mu_1,\ldots,\mu_J}^{\Delta_Q+\Omega_J,J}$ of spin $J$ and dimension $\Delta_Q + \Omega_J$.

Taking this operator as an external state, we can repeat the computation of both the three- and four-point functions. We limit ourselves only to the leading nontrivial corrections in this case.

The three-point function takes the form

$$\langle Q-q, J, m' | O_{-q}(\tau, n) | Q, J, m \rangle_{cyl} =$$

$$= c_q \mu^{\Delta_q} e^{-\mu q \tau} \left( \langle J, m' | J, m \rangle_{cyl} - \frac{1}{8} q^2 \alpha \sqrt{Q} \langle J, m' | \pi^2 (\tau, n) | J, m \rangle_{cyl} \right)$$

$$= c_q \alpha^{\Delta_q} Q^{\frac{\Delta_q}{2}} e^{-\alpha \sqrt{Q} q \tau} \left( \delta_{m,m'} \left( 1 - \frac{\beta q}{2 \sqrt{Q} \tau} \right) - \frac{1}{4} q^2 \alpha \frac{4 \pi}{Q} \Omega_J Y_{jm}(n) Y^*_{jm'}(n) \right)$$

where $|Q, J, m\rangle = a_{j,m}^\dagger |Q\rangle$ and we used 3.25 to compute the second line. One can relate this result to a more familiar basis of structures in flat space considered in [87]. For example, $\delta_{m,m'}$ corresponds to $H_{13}^J$, whereas $Y_{jm}(n) Y^*_{jm'}(n)$ involves $V^J_1 V^J_3$. 

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Next, let us compute the four-point function. The result takes the form

\[
\langle Q, J, m'|O_q(\tau, n_1)O_{-q}(0, n_2)|Q, J, m \rangle = c_q e^{-\alpha \sqrt{Q} \tau} Q^{q\Delta} e^{-\alpha \sqrt{Q} \tau} \\
\left( \delta_{m, m'} \left( 1 - \frac{\beta}{2} \frac{q}{\sqrt{Q}} \tau + \frac{\alpha}{4} \frac{q^2}{\sqrt{Q}} D(\tau, x) \right) - \right. \\
- \frac{1}{4} \frac{\alpha q^2}{\sqrt{Q} \Omega} \left[ Y_{J,m}(n_1) - Y_{J,m}(n_2) \right] \left[ Y_{J,m'}(n_1) - Y_{J,m'}(n_2) \right],
\]

(3.114)

where the only difference with the scalar case is the possibility of contracting the Goldstone field with the external state. This is the source of the spherical harmonics \(Y_{J,m}(n)\) in the third line of 3.114. Note that the second and the third lines are crossing symmetric by themselves. Moreover, the second line, that is the \(\delta_{m, m'}\) term, is identical to the scalar correlator. Decomposing this expression into conformal blocks we find the tower of operators with dimensions \(\Delta_H + \Omega_J + \Omega_{J'}\).

Again, in principle, we could have used the technology of [88] to decompose this expression into conformal blocks. This is, however, completely unnecessary, since by inserting a complete set of states in 3.114 we explicitly get a sum over exchanges of states \(|Q-q, J, \tilde{m}\rangle\) as well as \(|Q-q, J, m_1; J', m_2\rangle\) with positive coefficients. The only nontrivial terms which are not of this type are \(-\frac{\beta}{2} \frac{q}{\sqrt{Q}} \tau, \frac{\alpha q^2}{4 \sqrt{Q}} \tau\) coming from the correction to the dimension of \(\Delta_Q\), and \(J = 1\) term in the ex-
pansion of the propagator $D(\tau, x)$ which corresponds to the contribution of the descendant.

Let us check that the descendant comes with the correct coefficient. The first level descendants contribute as follows

$$
\sum_{\mu} \sum_{m'} \frac{\langle J, m | O_{q}(\tau, n_1) P_{\mu} | J, m' \rangle \langle J, m' | K_{\mu} O_{-q}(0, n_2) | J, m \rangle}{\langle J, m' | K_{\mu} P_{\mu} | J, m' \rangle} = - \sum_{\mu} \sum_{m'} \frac{\langle J, m | [P_{\mu}, O_{q}(\tau, n_1)] | J, m' \rangle \langle J, m' | [K_{\mu}, O_{-q}(0, n_2)] | J, m \rangle}{\langle J, m' | [K_{\mu}, P_{\mu}] | J, m' \rangle} = \alpha^2 q^2 Q^{n_1, n_2} c_q c_{-q} \alpha^{2\Delta_q} Q^{\Delta_q} e^{-\alpha \sqrt{Q} q \tau} e^{\tau}
$$

$$
= c_q c_{-q} \alpha^{2\Delta_q} Q^{\Delta_q} e^{-\alpha \sqrt{Q} q \tau} \left( \frac{3}{4} \frac{\alpha q^2}{\sqrt{Q} e^{\tau}} \right),
$$

(3.115)

which is exactly what we have in 3.114. In evaluating 3.115 we used that $[K_{\mu}, P_{\nu}] = 2D_{\mu\nu} - 2M_{\mu\nu}$, as well as the action of $P_{\mu}$ and $K_{\mu}$ on the primaries which can be found, for example, in [23] (see formula (3.11) in that paper).

In principle, at this point we can repeat the bootstrap analysis. The only complication is the fact that external operators in this case carry spin and multiple tensor structures have to be taken into account. We leave the detailed analysis of this case for the future, but it is clear that the same type of structure appears as with the external scalar operator. Namely, requiring smoothness of the correlator would lead to a similar set of equations. Assuming that only one
Regge trajectory is present, the solution is given by the Goldstone propagator.

This time, however, we have operators of the type $\Delta_H + \Omega_J + \Omega_{J'}$. Then we can make this “two-particle” state an external operator and repeat the argument. In this way, we see that the spectrum of operators exhibits an additive structure, as observed in the EFT.

3.6.2 Comment on Bootstrap at Order $\frac{1}{Q}$

In our analysis of the correlator it was crucial that the number of Regge trajectories that appear at leading order is finite. One can wonder if this structure persists at higher orders in $\frac{1}{\sqrt{Q}}$.

Consider the correction to the correlator of the type

$$\delta G(z, \bar{z}) \sim \frac{1}{Q} \hat{f}(\tau, \theta) .$$

From the existence of the macroscopic limit we learn that $\lambda^2 \hat{f}(\lambda \tau, \lambda \theta)$ is finite in the limit of small $\lambda$. This means that when we compute the contribution of the spin $J$ operators at order $\frac{1}{Q}$, by an argument identical to the one in section
5, we potentially get a $\frac{1}{\epsilon}$ singularity

$$- \frac{1}{2J + 1} \lim_{\epsilon \to 0} \sum_i c_{J,i} \epsilon_{J,i} e^{-\epsilon_{J,i} \epsilon} =$$

$$= \lim_{\epsilon \to 0} \int_0^\pi d\theta \sin \theta \left[ \partial_r \hat{f}(\epsilon, \theta) - \partial_r \hat{f}(-\epsilon, \theta) \right] P_J(\cos \theta) \sim \frac{1}{\epsilon}.$$  \hspace{1cm} (3.117)

If the coefficient in front of $\frac{1}{\epsilon}$ happens to be zero the situation is identical to the one encountered at leading order. Otherwise, 3.117 is consistent if the number of operators with spin $J$ on the LHS of 3.117 is infinite. This situation is identical to the discussion of Tauberian theorem in \[75\]. This is also precisely what happens in the case of EFT.

At this point a careful reader could be puzzled by how little mileage we get from the constraints in this case compared to the leading correction, where it was possible to bootstrap the answer completely. The crucial point is that in this case we have an infinite number of Regge trajectories on the LHS of 3.117. A remarkable efficiency of this simple matching at the order $\frac{1}{\sqrt{Q}}$ was due to a finite number of Regge trajectories. Here we see that at the order $\frac{1}{Q}$ the equation

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requires that

\[ \frac{1}{2J+1} \sum_i c_{J,i} e^{J,i} \rho^{-J,i} \epsilon = \frac{2}{\epsilon} \int_0^1 dy \left( 2\hat{F}(y) + y\hat{F}'(y) - \frac{1}{y}\hat{F}'(y) \right), \quad \epsilon \to 0, \]

(3.118)

where we used \( \hat{f}(\tau, \theta) = \frac{1}{\tau^2 + \theta^2} \hat{F}(\frac{\tau}{\sqrt{\tau^2 + \theta^2}}) \). We have not explored if \( 3.118 \) being zero is consistent with the leading order solutions found in the previous sections.

3.6.3 “Light-cone” Bootstrap in the Macroscopic limit: \( w \gg \bar{w} \gg 1 \)

Let us recall the structure of the macroscopic limit in the \( t \)-channel. The macroscopic limit of the \( t \)-channel conformal block is given by

\[ Q^{-\frac{\Delta}{2}} (w\bar{w})^{\frac{\Delta}{2}} P_J \left( \frac{w+\bar{w}}{2\sqrt{w\bar{w}}} \right), \]

where descendants are again further suppressed. We do not know the convergent properties of the \( t \)-channel OPE after we take the macroscopic limit. It should be definitely a reliable expansion for \( w\bar{w} \ll 1 \), but in this section we assume that it converges for \( w\bar{w} \gg 1 \) as well.

We can try to match the macroscopic limit of the EFT result \( 3.65 \) to the \( t \)-channel. The EFT description is only valid for \( w, w \gg 1 \), therefore, we get the
following relation

\[ c_0 e^{\frac{2}{2} q (w + \bar{w})} \left( 1 + O \left( \frac{1}{w}, \frac{1}{\bar{w}} \right) \right) = \sum_{\Delta, J} Q^{-\frac{\Delta}{2}} c_{\Delta, J}(w \bar{w})^{\frac{\Delta - 2 \Delta q}{2}} P_J \left( \frac{w + \bar{w}}{2 \sqrt{\bar{w}} \sqrt{w}} \right), \quad w, \bar{w} \gg 1, \]

\[ c_{\Delta, J} = \lambda_{Q, -Q, O_{\Delta, J}} \lambda_{q, -q, O_{\Delta, J}}, \]

(3.119)

where for simplicity we kept only the leading order answer on the LHS. It is clear from (3.119) that only operators that saturate the bound on the three-point function (3.59) \[ \lambda_{Q, -Q, O_{\Delta, J}} \sim Q^{\frac{\Delta}{2}} \] contribute in the macroscopic limit. We can integrate over \( x = \frac{w + \bar{w}}{2 \sqrt{w \bar{w}}} \) to project both sides on the sector with given spin and derive an asymptotic density of states in each spin sector that is dictated by the LHS in (3.119).

In the limit \( w \gg \bar{w} \gg 1 \) the mapping can be made more explicit. In this case the argument of the Legendre polynomial is \( x \gg 1 \) and the leading asymptotic of the block simply becomes

\[ c_0 e^{\frac{2}{2} q w} \left( 1 + O \left( \frac{\bar{w}}{w} \right) \right) = \sum_{\Delta, J} \frac{4^{-J} \Gamma(1 + 2J)}{\Gamma(1 + J)^2} Q^{-\frac{\Delta}{2}} c_{\Delta, J}(w \bar{w})^{\frac{\Delta - 2 \Delta q}{2}} w^J, \quad w \gg \bar{w} \gg 1. \]

(3.120)

There are natural candidate operators on the RHS to reproduce the LHS. These
are the usual double-twist operators $O_q \partial^J O_{-q}$ which have an asymptotic twist
$\Delta - J = 2 \Delta_q$. Note, however, the difference with the usual light-cone bootstrap.

Here, the spin $J$ is not the largest parameter in the problem and we first take
the large $Q$ limit. Remembering that $c_{\Delta, J} = \lambda_{q, -q, O_q \partial^J O_{-q}}\lambda_{Q, -Q, O_q \partial^J O_{-q}}$ and
using the usual light-cone bootstrap result \cite{73, 74} for $\lambda_{q, -q, O_q \partial^J O_{-q}}$

$$\lambda_{q, -q, O_q \partial^J O_{-q}} = \frac{\pi^{\frac{5}{2}} J^{\Delta_q - \frac{3}{2}}}{2J + \Delta_q - 1 \Gamma(\Delta_q)} + ...$$

(3.121)

we can derive a formula for $\lambda_{Q, -Q, O_q \partial^J O_{-q}}$. In this way we get

$$c_0 e^{\frac{q}{2} w} = \sum J \frac{1}{J! 2^{\frac{J}{2}} \pi^{\frac{J}{2}}} \frac{\pi^{\frac{5}{2}} J^{\Delta_q - \frac{3}{2}}}{2J + \Delta_q - 1 \Gamma(\Delta_q)} Q^{-\frac{2\Delta_q + J}{2}} \lambda_{Q, -Q, O_q \partial^J O_{-q}} w^J, \quad w \gg 1 .$$

(3.122)

From \ref{3.122} we see that the LHS can be reproduced if the three-point couplings
$\lambda_{Q, -Q, O_q \partial^J O_{-q}}$ take the following value at large spin

$$\lim_{J \gg 1} \lim_{Q \gg 1} \lambda_{Q, -Q, O_q \partial^J O_{-q}} = c_0 2^{\Delta_q - 1} Q^{\Delta_q} \left(\frac{\alpha \sqrt{Q}}{J}\right)^J J^{\frac{5}{2} - \Delta_q} \Gamma(\Delta_q) \frac{1}{\pi^{\frac{1}{2}}} .$$

(3.123)

This is in accord with expectations from the EFT. Indeed, the operators $O_q \partial^J O_{-q}$
could be represented in EFT schematically as $|\partial^J e^{i q \chi} \partial^J | \partial^J e^{-i q \chi}$, see \ref{3.29}.
The leading contribution at large $Q$ comes when all $J$ derivatives act on the exponential factor. This brings a factor of $q\mu \simeq \alpha \sqrt{Q} q$ in agreement with 3.123. Let us emphasize again that 3.123 is different from the usual light-cone bootstrap, where spin $J$ and not the charge $Q$ is the largest parameter in the problem. It would be interesting to understand the interpolation between these two regimes.
Appendices to chapter 1

A.1 Laplace Transform

In this appendix we prove a complex tauberian theorem for Laplace transform that we used in section 2. The proof is basically a review of results of [17] where many extra details can be found.

We will write $O(x)$ to estimate the magnitude of different quantities. Let us
remind the reader that

\[ f(x) = O(g(x)),\quad x \to \infty \quad (x \to a) \quad \text{(A.1)} \]

iff there exist numbers \( M, x_0 (M, \delta) \) s.t.

\[ |f(x)| < M|g(x)|, \quad \forall \ x > x_0 \quad (\forall |x - a| < \delta) \quad \text{(A.2)} \]

We start with the following useful lemma \[25, 17\].

**Lemma 1:** Let \( 0 < \sigma < \Lambda \). Then for arbitrary real \( \nu \) we have an estimate

\[ \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dt \frac{(\Lambda^2 - t^2)^2}{\sigma + it} e^{(\sigma + it)\nu} = (\Lambda^2 + \sigma^2)^2 \theta(\nu) + O(e^{\sigma \nu} \min[\Lambda^4, \Lambda^2 \nu^{-2}]). \quad \text{(A.3)} \]

where \( \theta(\nu) \) is the Heaviside function.

Consider say \( \nu \geq 0 \). We set \( z = \eta + it \) and consider a closed contour \( C \) in the \( z \)-plane that consists of vertical segment \([-\Lambda, \Lambda]\) at \( \eta = 0 \) and a part of the
circle $K$ centered at $z = -\sigma$, with the radius $R = |\sigma + i\Lambda|$. In this way we get

$$\frac{1}{2\pi i} \oint_K \frac{(\Lambda^2 + z^2)^2}{\sigma + z} e^{(\sigma + z)\nu} = \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dt \frac{(\Lambda^2 - t^2)^2}{\sigma + it} e^{(\sigma + it)\nu} + \frac{1}{2\pi i} \int_K \frac{(\Lambda^2 + z^2)^2}{\sigma + z} e^{(\sigma + z)\nu}$$

$$= (\Lambda^2 + \sigma^2)^2,$$

(A.4)

where in the last line we evaluated the integral by taking the residue at $z = -\sigma$.

We can estimate the integral over $K$ in two different ways. First, we have

$$\frac{1}{2\pi i} \int_K \frac{(\Lambda^2 + z^2)^2}{\sigma + z} e^{(\sigma + z)\nu} = O\left(\frac{\Lambda^4}{R} e^{\sigma\nu} \int_K |dz| \right) = O(e^{\sigma\nu} \Lambda^4),$$

(A.5)

where we used that $|z| \leq R + \sigma \leq 3\Lambda$, $|\sigma + z| = R$, $|e^{z\nu}| \leq 1$ along $K$ for $\nu > 0$.

Another estimate comes from writing $e^{z\nu} = \frac{1}{\nu^2} \frac{d^2}{dz^2} e^{z\nu}$ and integrating by parts.

For $\nu < 0$ we construct the contour $C$ by attaching to the vertical segment a small part of the same circle.

The integral in (A.3) could be of course computed exactly. The point of (A.3) is that it provides a convenient estimate of the correction to $\theta(\nu)$ for arbitrary $\nu$ which will be very useful in proving tauberian theorems.

To see utility of the lemma above recall that we are studying the Laplace
transform (which is just the OPE expansion in case of CFTs)

\[ \mathcal{L}_b(s) \equiv \int_0^\infty e^{-su}db(u) \quad (A.6) \]

and we would like to derive some statements about the integrated spectral density

\[ F_b(x) \equiv \int_0^x db(u) \quad . \quad (A.7) \]

Using the lemma above we establish the following useful statement.

**Lemma 2:** Assume that \( \mathcal{L}_b(s) \) converges absolutely for \( \text{Re}[s] > 0 \) and let \( 0 < \sigma < \Lambda \). Then

\[
F_b(x) = \frac{1}{2\pi(\Lambda^2 + \sigma^2)^2} \int_{\Lambda}^{-\Lambda} dt \frac{(\Lambda^2 - t^2)^2}{\sigma + it} \mathcal{L}_b(\sigma + it)e^{(\sigma + it)x} \\
+ O \left( \int_0^\infty e^{\sigma(x-u)} \min[1, \Lambda^{-2}(x-u)^{-2}] |db(u)| \right) . \quad (A.8)
\]

This lemma expresses the integrated spectral density \( F_b(x) \) in terms of the Laplace integral \( \mathcal{L}_b(s) \) plus a correction. This lemma follows from applying \( A.3 \) to the integral in the RHS of the first line in \( A.8 \) and exchanging two integrations.

Having this two lemmas we are ready to prove an almost relevant theorem (a
simplified version of theorem 2.3.1 in [17]).

**Theorem I:** Let the functions \( \phi(u) \) and \( \psi(u) \) be defined for \( u \geq 0 \). We assume that they are non-decreasing and positive-definite. Moreover, we assume that \( \phi(u) \) locally does not grow faster than a power, namely there exist a positive constant \( b \) such that

\[
  u\phi'(u) < b\phi(u), \quad u \geq u_0.
\]

(A.9)

We also assume that Laplace transforms \( \mathcal{L}_\phi(s) \) and \( \mathcal{L}_\psi(s) \) satisfy

\[
  \mathcal{L}_\psi(s) = \mathcal{L}_\phi(s) + O(|s|^\alpha), \quad s = \sigma + it
\]

(A.10)

in the region

\[
  |t| \leq c\sigma^\omega, \quad 0 \leq \omega \leq 1.
\]

(A.11)

The strength of the result will depend on the value of \( \omega \). The larger is the complex domain (smaller \( \omega \)'s) in which the estimate A.10 holds, the better is the bound. The case relevant for CFTs is \( \omega = 0 \).
Let us also assume that

\[ \psi(0) - \phi(0) = 0. \tag{A.12} \]

This is just a technical assumption that does not play any important role.

Then for every \( m \geq 0 \) we have

\[
\int_0^x (x - u)^m d\psi(u) = \int_0^x (x - u)^m d\phi(u) + O\left( \frac{\phi(x)}{x} x^{\omega(m+1)} \right) + O(\max[x^{m-\alpha}, (x^{-\omega})^{\alpha-m}]), \quad m \neq \alpha. \tag{A.13}
\]

If \( m = \alpha \) the estimate in the second line becomes \( \ln x \). Let us go through the proof of A.13. For further details see [17].

A.1.1 Estimate for \( \psi(u) \)

It is convenient to integrate A.9 to get

\[
\frac{\phi(v)}{\phi(u)} < \left( \frac{v}{u} \right)^b. \tag{A.14}
\]

We will use this estimate extensively below.
Let us first prove that

\[ \psi(u) = O(\phi(u)). \quad (A.15) \]

To show this it is crucial that \( \psi(u) \) is non-decreasing and positive. We set \( \sigma = \frac{1}{u} \) and do the following estimate

\[
\psi(u) = O(\sigma \psi(u) \int_u^\infty e^{-\sigma v} dv) = O(\sigma \int_u^\infty \psi(v)e^{-\sigma v} dv) = O(\sigma \int_0^\infty \psi(v)e^{-\sigma v} dv) \\
= O(\sigma \int_0^\infty \phi(v)e^{-\sigma v} dv) + O(\sigma^{1+\alpha}), \quad (A.16)
\]

where we used \( A.10 \) to switch from the Laplace transform of \( \psi \) to the one of \( \phi \). Then we can use the power-like bound on \( \phi \) \( A.14 \) to estimate

\[
O(\sigma \int_0^\infty \phi(v)e^{-\sigma v} dv) = O(\phi(u)) + O(\sigma \int_0^\infty \phi(v)e^{-\sigma v} dv) \\
= O(\phi(u)) + O(\frac{\sigma \phi(u)}{u^b} \int_u^\infty e^{-\sigma v} v^b dv) = O(\phi(u)), \quad (A.17)
\]

where we used that \( \sigma = \frac{1}{u} \).

As a last step note that \( A.10 \) is only meaningful if the second term in the RHS is small compared to the first. Therefore, we can drop \( O(\sigma^{1+\alpha}) \) in the last
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Next we use lemma 2 from above to write

$$
\psi(x) - \phi(x) = O \left( \int_0^\infty |d\psi(u) - d\phi(u)| e^{\sigma(x-u)} \min[1, \Lambda^{-2}(u-x)^{-2}] \right) 
+ \frac{1}{2\pi(\Lambda^2 + \sigma^2)^2} \int_{-\Lambda}^\Lambda dt \frac{(\Lambda^2 - t^2)^2}{\sigma + it} e^{(\sigma + it)x} \left[ \mathcal{L}_\psi(\sigma + it) - \mathcal{L}_\phi(\sigma + (\Lambda)^2) \right].
$$

(A.18)

In bounding these terms we set $\sigma = \frac{1}{x}$ and $\Lambda = c\sigma^\omega$. We also think of $x$ as being large.

Below we will use a series of estimates to show that (see equation 2.3.9 in [17])

$$
O \left( \int_0^\infty |d\psi(u) - d\phi(u)| e^{\sigma(x-u)} \min[1, \Lambda^{-2}(u-x)^{-2}] \right) = O \left( \frac{\phi(x)}{\Lambda x} \right) 
+ O \left( \frac{1}{\Lambda} \max_{\frac{x}{2} \leq y \leq 2x} \left| \int_{-\Lambda}^\Lambda dt (1 - \frac{|t|}{\Lambda}) e^{itv}(\mathcal{L}_\psi(\frac{1}{v} + it) - \mathcal{L}_\phi(\frac{1}{v} + it)) \right| \right). \quad (A.19)
$$

The argument goes as follows. The idea is to split the $u$ integral as $\int_0^{x-y} + \int_{x-y}^{x+y} + \int_{x+y}^\infty$, where

$$
y = \left( \frac{x}{\Lambda} \right)^{1/2} \ll x. \quad (A.20)
$$

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The point of this splitting is that for $0 < u < x - y$ and $u > x + y$ we have $\min[1, \Lambda^{-2}(u - x)^{-2}] = \Lambda^{-2}(u - x)^{-2}$, whereas for $x - y < u < x + y$ it is not necessarily true.

Let us first bound the $d\phi(u)$ terms. We can use integration by parts to estimate

$$\frac{1}{\Lambda^2} \int_0^{x-y} \frac{d\phi(u)}{(x-u)^2} = O \left( \frac{\phi(x)}{\Lambda^2 y^2} \right) = O \left( \frac{\phi(x)}{\Lambda x} \right). \quad (A.21)$$

And we use power-like bound to show that

$$\frac{1}{\Lambda^2} \int_{x+y}^\infty \frac{e^{\sigma(x-u)} d\phi(u)}{(x-u)^2} = O \left( \frac{1}{\Lambda^2} \int_{x+y}^\infty \frac{e^{-\sigma u} d\phi(u)}{(x-u)^2} \right)$$

$$= O \left( \frac{\phi(x)}{\Lambda^2 y^2} \right) + O \left( \frac{\sigma}{\Lambda^2 y^2} \int_{x+y}^\infty du \, e^{-\sigma u} \phi(u) \right)$$

$$= O \left( \frac{\phi(x)}{\Lambda^2 y^2} \right) + O \left( \frac{\sigma \phi(x+y)}{\Lambda^2 y^2} \left( x+y \right)^b \int_{x+y}^\infty du \, e^{-\sigma u} u^b \right)$$

$$= O \left( \frac{\phi(x)}{\Lambda x} \right), \quad (A.22)$$

where we again used A.14.

Finally we want to show that

$$\int_{x-y}^{x+y} e^{-\sigma u} \min[1, \Lambda^{-2}(u - x)^{-2}] d\phi(u) = O \left( \frac{\phi(x)}{\Lambda x} \right). \quad (A.23)$$
The basic observation is that for $\frac{x}{2} < \bar{x} < 2x$ we can write

$$\phi\left(\bar{x} + \frac{1}{\Lambda}\right) - \phi(\bar{x}) < \phi(\bar{x}) \left(\left(\frac{\bar{x} + \frac{1}{\Lambda}}{\bar{x}}\right)^b - 1\right) = O\left(\frac{\phi(\bar{x})}{\Lambda\bar{x}}\right).$$  \hspace{1cm} (A.24)$$

This bound is not very surprising and is another way of saying that $\phi(x)$ grows locally at most like a power for purposes of estimates.

Now we split $\int_{x-y}^{x+y} du$ into many intervals of the size $\Lambda$ and to each of them we will apply [A.24]. We get

$$\int_{x-y}^{x+y} e^{-\sigma u \min[1, \Lambda^{-2}(u-x)^{-2}]} d\phi(u) = O\left(\frac{\phi(x)}{\Lambda x}\right) = i_1 + i_2 + i_3,$$

$$i_1 = \frac{1}{\Lambda^2} \sum_{k=2}^{y\Lambda} \int_{x-k\Lambda}^{x-k\Lambda-1} \frac{d\phi(u)}{(x-u)^2},$$

$$= O\left(\sum_{k=2}^{y\Lambda} \frac{1}{k^2} \int_{x-k\Lambda}^{x-k\Lambda-1} d\phi(u)\right) = O\left(\frac{\phi(x)}{\Lambda x} \sum_{k=2}^{y\Lambda} \frac{1}{k^2}\right) = O\left(\frac{\phi(x)}{\Lambda x}\right),$$

$$i_2 = \int_{x-k\Lambda}^{x-k\Lambda+1} d\phi(u) = O\left(\frac{\phi(x)}{\Lambda x}\right),$$

$$i_3 = \frac{1}{\Lambda^2} \sum_{k=2}^{y\Lambda} \int_{x+k\Lambda}^{x+k\Lambda-1} \frac{d\phi(u)}{(x-u)^2} = O\left(\frac{\phi(x)}{\Lambda x}\right),$$ \hspace{1cm} (A.25)

where we used that $y\Lambda = (x\Lambda)^{1/2} \gg 1$ since $\omega < 1$.

For $d\psi(u)$ terms we also split the integral as $\int_0^{x-y} + \int_{x-y}^{x+y} + \int_{x+y}^{\infty}$ and then bound separately each of the terms.
For integrals $\int_0^{x-y}$ and $\int_{x+y}^{\infty}$ we integrate by parts, use $\text{A.15}$ and estimates for $\phi(u)$ from above. Therefore we are left with the estimate

$$\int_{x-y}^{x+y} e^{-\sigma u} \min[1, \Lambda^{-2}(u - x)^{-2}] d\psi(u) = i_1 + i_2 + i_3,$$  

(A.26)

where we split the integral as we did above for $\phi$. In this integral we cannot integrate by parts and simply use $\text{A.15}$ because of $1$ inside $\min[1, \Lambda^{-2}(u - x)^{-2}]$, which leads to boundary terms in the integration by parts $O(\phi(x))$.

To circumvent this difficulty we need another auxiliary lemma (lemma 2.1.6 in [17]). It states that for $x/2 < \tilde{x} < 2x$ (this interval could be changed) and functions that satisfy the conditions that we used above we have

$$\psi \left( \tilde{x} + \frac{1}{\Lambda} \right) - \psi \left( \tilde{x} - \frac{1}{\Lambda} \right) = O \left( \frac{\phi(x)}{\Lambda x} \right)$$

$$+ O \left( \left| \frac{1}{\Lambda} \int_{-\Lambda}^{\Lambda} dt \left( 1 - \frac{|t|}{\Lambda} \right) e^{i\tilde{x}t} \left( L_\psi(\tilde{\sigma} + it) - L_\phi(\tilde{\sigma} + it) \right) \right| \right)$$  

(A.27)

where $\tilde{\sigma} = \frac{1}{\tilde{x}}$. In this way to estimate $\text{A.26}$ we just repeat the analysis for $\phi$ while keeping the difference of Laplace transform terms.
The way this lemma works is as follows. Start with the following relation

\[
\frac{1}{\Lambda} \int_{-\Lambda}^{\Lambda} \left(1 - \frac{|t|}{\Lambda}\right)e^{i\nu t} = \left(\frac{\sin \frac{\Lambda \nu}{2}}{\Lambda \nu / 2}\right)^2. \tag{A.28}
\]

We can use it to write

\[
\psi \left(\tilde{x} + \frac{1}{\Lambda}\right) - \psi \left(\tilde{x} - \frac{1}{\Lambda}\right) = \int_{\tilde{x} - \frac{1}{\Lambda}}^{\tilde{x} + \frac{1}{\Lambda}} d\psi(u) = O \left(\int_{0}^{\infty} e^{-\tilde{\sigma}u} \left(\frac{\sin \frac{\Lambda(\tilde{x} - u)}{2}}{\Lambda(\tilde{x} - u)/2}\right)^2 d\psi(u)\right) . \tag{A.29}
\]

To estimate the RHS we use A.28

\[
\frac{1}{\Lambda} \int_{-\Lambda}^{\Lambda} dt \left(1 - \frac{|t|}{\Lambda}\right)e^{i\tilde{\sigma}t} \left(\mathcal{L}_\psi(\tilde{\sigma} + it) - \mathcal{L}_\phi(\tilde{\sigma} + it)\right)
= \int_{0}^{\infty} e^{-\tilde{\sigma}u} \left(\frac{\sin \frac{\Lambda(\tilde{x} - u)}{2}}{\Lambda(\tilde{x} - u)/2}\right)^2 \left(d\psi(u) - d\phi(u)\right) . \tag{A.30}
\]

Estimating all the terms that involve \(\phi\) by methods identical to above we arrive at A.28.
At this point let us reiterate an important intermediate result

\[
\psi(x) - \phi(x) = O \left( \frac{\phi(x)}{Ax} \right) + O \left( \frac{1}{\Lambda} \max_{\frac{1}{2} \leq v \leq 2x} \left| \int_{-\Lambda}^{\Lambda} dt(1 - \left| \frac{t}{\Lambda} \right|) e^{itv} (L_\psi(\frac{1}{v} + it) - L_\phi(\frac{1}{v} + it)) \right| \right)
\]

\[
+ \frac{1}{2\pi(\Lambda^2 + \sigma^2)^2} \int_{-\Lambda}^{\Lambda} dt \frac{(\Lambda^2 - t^2)^2}{\sigma + it} e^{(\sigma + it)x} |L_\psi(\sigma + it) - L_\phi(\sigma + it)|
\]

To prove the desired statement for \( m = 1 \) we simply need to estimate the integrals that involve difference of Laplace transforms in \[A.31\].

**A.1.3 Estimate For the Difference \( L_\psi - L_\phi \)**

Next, we estimate the rest of the terms that involve the difference of Laplace transforms. We get

\[
\frac{1}{2\pi(\Lambda^2 + \sigma^2)^2} \int_{-\Lambda}^{\Lambda} dt \frac{(\Lambda^2 - t^2)^2}{\sigma + it} e^{(\sigma + it)x} |L_\psi(\sigma + it) - L_\phi(\sigma + it)|
\]

\[
= O \left( \int_{-\Lambda}^{\Lambda} dt \frac{|L_\psi(\sigma + it) - L_\phi(\sigma + it)|}{|\sigma + it|} \right) = O \left( \int_{0}^{\Lambda} |\sigma + it|^{\alpha-1} dt \right)
\]

\[
= O(\max[\sigma^\alpha, \Lambda^\alpha]) = O(\max[x^{-\alpha}, \Lambda^\alpha]).
\]

(A.32)
and similarly

\[
O \left( \frac{1}{\Lambda} \max_{\frac{1}{2} \leq v \leq 2x} \left| \int_{-\Lambda}^{\Lambda} dt (1 - \frac{|t|}{\Lambda}) e^{itv} (L_\psi (\frac{1}{v} + it) - L_\phi (\frac{1}{v} + it)) \right| \right)
\]

\[
= O \left( \frac{1}{\Lambda} \max_{\frac{1}{2} \leq v \leq 2x} \int_{-\Lambda}^{\Lambda} dt |L_\psi (\frac{1}{v} + it) - L_\phi (\frac{1}{v} + it)| \right) = O(\max[x^{-\alpha}, \Lambda^\alpha])\quad (A.33)
\]

Therefore, we showed that

\[
\psi(x) - \phi(x) = O \left( \frac{\phi(x)}{\Lambda x} \right) + O(\max[x^{-\alpha}, \Lambda^\alpha]). \quad (A.34)
\]

Recall that \( \sigma = \frac{1}{x} \) and \( \Lambda = c \sigma^\omega \), from which \( m = 0 \) claim of the theorem follows.

### A.1.4 Higher Cauchy Moments

For higher \( m \)'s the theorem is proved by induction. Imagine it holds for \( m \)'th moment and let us try to prove it for \( (m+1) \)'th moment. Consider \( m \)'th Cauchy moment

\[
\Phi_m(x) = \frac{1}{m!} \int_0^x (x-u)^m d[\psi(u) - \phi(u)]. \quad (A.35)
\]
Differentiating \( m \) times by parts we get (here we use the condition \( \phi(0) - \psi(0) = 0 \))

\[
L_\psi(s) - L_\phi(s) = s^{m+1} \int_0^\infty du \, e^{-su} \Phi_m(s) \tag{A.36}
\]

from which an estimate

\[
H(s) = \int_0^\infty du \, e^{-su} \Phi_m(s) = O(|s|^{\alpha-m-1}) . \tag{A.37}
\]

immediately follows. We then apply lemma 2 to get

\[
(\Lambda^2 + \sigma^2)^2 \int_0^x du \, \Phi_m(u) = O \left( \int_0^\infty |\Phi_m(u)| e^{(x-u)\sigma} \min[\Lambda^4, \Lambda^2(x-u)^{-2}] \right)
+ \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} dt \frac{(\Lambda^2 - t^2)^2}{\sigma + it} H(\sigma + it)e^{(\sigma + it)x} \tag{A.38}
\]

Then we apply the \( m \)'th step estimate to the first line of \( \text{A.38} \) and \( \text{A.37} \) to estimate the second line in \( \text{A.38} \). From this theorem for \( (m + 1) \)-th moment follows.

**A.1.5 Important Ingredients**

Sign-definiteness of \( d\psi \) and \( d\phi \) is clearly very important for the proof. Also the power-like bound on local behavior of \( \phi(u) \) is extensively used. Other assump-
tions, e.g. the ones stated in theorem 2.3.1 [17], seem to be technical details that could be easily relaxed.

A.1.6 Case of CFTs

In the case of CFTs as we discussed in the main text we have the $t$-channel OPE expansion for the Laplace transform

$$L_\psi(\beta) = \frac{1}{\beta^{2\Delta_x}} \sum_{\Delta_i} c_{\Delta_i} \beta^{\Delta_i} + \ldots,$$  \hfill (A.39)

for any $|\beta| \ll 1$. Therefore we can set $\omega = 0$ in the previous section and take $\Lambda$ to be constant. Let us review the $m = 0$ part of the theorem. We take $d\phi(u)$ to be simply a set of powers $u^\alpha$ with proper coefficients so that in the difference $L_\psi(\beta) - L_\phi(\beta)$ all the singular terms cancel.\footnote{A slightly better prescription is to choose $d\phi(u) = \sum_k c_k u^k \theta(0 < u < 1) + \sum_\alpha d_\alpha u^\alpha \theta(u \geq 1)$. In this way $\alpha$ could be arbitrarily small.} Therefore we get the OPE expansion

$$L_\psi(\beta) - L_\phi(\beta) = c_\alpha \beta^\alpha + \ldots, \quad \alpha > 0,$$  \hfill (A.40)

where by ellipses we denoted higher order terms in the OPE. Usually, the $t$-channel OPE is formulated in terms of $(1 - z)^h(1 - \bar{z})^{\bar{h}}$. While for us $z = \bar{z} = e^{-\beta}$.
and we expand each term \((1 - z)^h(1 - \bar{z})^{\bar{h}}\) for small \(\beta\) and then swap the small \(\beta\) expansion with the sum over operators. This is possible due to the absolute convergence of the series. Since we have a convergent OPE expansion around \(s = 0\) we can make a better estimate of the integrals \(A.32\) and \(A.33\) using the OPE for small but constant \(\Lambda\). We get the following estimate of the relevant integrals:\(^2\)

\[
\frac{1}{2\pi(\Lambda^2 + \sigma^2)^2} \int_{-\Lambda}^{\Lambda} dt \frac{(\Lambda^2 - t^2)^2}{\sigma + it} e^{(\sigma + it)x} (\sigma + it)^\alpha = \frac{\sin \pi \alpha \Gamma(\alpha)}{\pi x^\alpha} \left(1 + O\left(\frac{1}{x^2}\right)\right) + \frac{8\epsilon \Lambda^{\alpha-3}}{\pi x^3} \cos\left(\frac{\pi \alpha}{2} + x\Lambda\right) + ...,
\]

\(\text{(A.41)}\)

The leading asymptotic is different for integer \(\alpha\) in which case the first term in the RHS \(\text{A.41}\) vanishes. The leading asymptotic is then captured by the second line in \(\text{A.41}\). Let us now estimate the second integral

\[
\frac{1}{\Lambda} \max_{\frac{1}{2} \leq v \leq 2x} | \int_{-\Lambda}^{\Lambda} dt (1 - \frac{|t|}{\Lambda}) e^{iv\left(\frac{1}{v} + it\right)^\alpha} | = O\left(\frac{1}{x^{1+\alpha}}\right) + O\left(\frac{1}{x^2}\right) = O\left(\frac{1}{x}\right).
\]

\(\text{(A.42)}\)

\(^2\)A very similar statement is theorem 2.3.2 in \cite{17}. Instead of regularity used in theorem 2.3.2 we used here the fact that we have a convergent OPE expansion for \(L_\psi(s) - L_\phi(s)\).
Using this better estimates we have

\[
\psi(x) - \phi(x) = O \left( \frac{\phi(x)}{x} \right). 
\]  
\[
(A.43)
\]

We then set up an induction. As we consider higher Cauchy moments more and more terms in the OPE become effectively singular. We simply add those extra terms to the naive spectral density and repeat the argument. The endpoint of this reasoning is \[1.17\]. Powers of \(E\) in the RHS of \[1.17\] is simply the contribution of \(\phi\) in the Cauchy moments that we discussed in this section.

### A.2 Stieltjes Transform

In this section we prove a complex tauberian theorem for the Stieltjes transform used in section 5. We extensively use methods of \[26\], where similar theorems had been proved.

Let us define curves \(\gamma_{\pm}\) by

\[
\gamma_{\pm} = \{ z = x + iy; \ |y| = \Lambda(x), \pm x \geq 0 \} 
\]
\[
(A.44)
\]

where \(\Lambda(x)\) is a positive-definite function of \(x\) (in particular it could be constant) s.t. curves \(\gamma_{\pm}\) are smooth (including at \(x = 0\)), see \[A.1\]. By \(G\) we will
Figure A.1: $\gamma_+$ (solid line) and $\gamma_-$ (dashed line) integration contour in the $z$-plane.

denote a complex region outside of $\gamma_{\pm}$

$$G = \{ z = x + iy; \ |y| \geq \Lambda(x) \}.$$  \hspace{1cm} (A.45)

Then the following theorem is true.

**Theorem II:** Suppose we are given two functions $\rho(\nu), \phi(\nu)$ s.t.

$$\rho(\nu), \phi(\nu) \geq 0.$$  \hspace{1cm} (A.46)

For $|\nu| > a$ the function $\phi(\nu)$ is smooth, $|\phi(\nu)|$ is monotonically decreasing with
$|\nu|$ and locally polynomially bounded

\[
\alpha|\phi(\nu)| < -|\nu \phi'(\nu)| < \beta|\phi(\nu)|, \quad |\nu| > a, \quad \alpha < \beta < 0. \quad (A.47)
\]

Furthermore, the following condition holds

\[
\int_{-\infty}^{\infty} d\nu \frac{\rho(\nu)}{\nu - z} - \int_{-\infty}^{\infty} d\nu \frac{\phi(\nu)}{\nu - z} = R(z), \quad z \in G, \quad (A.48)
\]

where the function $R(z)$ is analytic in the complex region $G$ and

\[
R(z) = O(|z|^{-\omega}), \quad |z| \to \infty, \quad z \in G, \quad \omega > m \quad (A.49)
\]

Then Cauchy moments of $\rho$ and $\phi$ are related by $(m = 1, 2, \ldots)$

\[
F_m(x) = \Phi_m(x) + \sum_{k=1}^{m} b_k \frac{x^{m-k}}{(m-k)!} + O(y^m \phi(x)) + O(y^m x^{m-\omega})
\]

\[
+ O \left( \frac{y^{(m+1)}|\phi(-x)|}{x} \right) \quad (A.50)
\]
where \( y \equiv \Lambda(x) \) and we defined

\[
F_m(\nu) = \frac{1}{(m-1)!} \int_0^\nu d\nu' (\nu - \nu')^{m-1} \rho(\nu') ,
\]

\[
\Phi_m(\nu) = \frac{1}{(m-1)!} \int_0^\nu d\nu' (\nu - \nu')^{m-1} \phi(\nu') , \tag{A.51}
\]

and the constants \( b_k \) are determined by \( R(z) \).

Before giving a proof of the theorem, let us make two comments in the context of dispersion relations. The spectral density \( \rho(\nu) \) is in general not a smooth function and e.g. may contain delta-function components. While the function \( \phi(\nu) \) can be thought of as naive spectral density given by a sum of powers. In particular, the condition \( A.47 \) says that it behaves like a power of \( \nu \).

Both \( \phi(\nu) \) and \( R(z) \) are defined by the \( t \)-channel expansion. The splitting of the \( t \)-channel OPE into \( \phi(\nu) \) and \( R(z) \) is completely arbitrary. Every term in the large \( z \) expansion of \( R(z) \) can be rewritten as a power of \( \nu \) term in \( \phi(\nu) \).

Thus, we can make \( \omega \) as large as we like in \( A.49 \).

A.2.1 Lemma

The following lemma will be useful in the proof of the theorem \( [25, 26] \). Consider a finite part of the contour \( \gamma_+ \) defined as \( \Gamma_x = \{ z' = x' + iy' \in \gamma_+ ; \ x' < x \} \).
Then we have

\[
\frac{1}{2\pi i} \int_{\Gamma_x} \frac{dz'}{\nu - z'} = \theta(0 < \nu < x) + \frac{y}{\pi} \text{Re} \frac{1}{\nu - z} + O\left(\frac{y^2}{(\nu - x)^2 + y^2}\right) \tag{A.52}
\]

where \( z = x + iy \) and \( y = \Lambda(x) \). Indeed, we have

\[
\frac{1}{2\pi i} \int_{\Gamma_x} \frac{dz'}{\nu - z'} = \theta(0 < \nu < x) + \frac{1}{2\pi i} \int_K \frac{dz'}{\nu - z'} = \theta(0 < \nu < x) + \frac{\nu - x}{\pi} \int_0^y \frac{dy'}{(\nu - x)^2 + y'^2}, \tag{A.53}
\]

where in the first line we added and subtracted an integral over a vertical segment \( K = \{ z = x + iy'; |y'| < y = \Lambda(x) \} \) to close the contour \( \Gamma_x \). The integral on the RHS is, of course, easy to do explicitly. However, it will prove useful to estimate it as follows instead

\[
(\nu - x) \int_0^y \frac{dy'}{(\nu - x)^2 + y'^2} = \frac{y(\nu - x)}{(\nu - x)^2 + y^2} + 2(\nu - x) \int_0^y \frac{dy'}{\left[(\nu - x)^2 + y'^2\right]^2} = \frac{y(\nu - x)}{(\nu - x)^2 + y^2} + 2 \int_0^{y/(\nu - x)} \frac{dy'}{1 + y^2} \frac{y'^2}{1 + y'^2} = \frac{y(\nu - x)}{(\nu - x)^2 + y^2} + O\left(\frac{y^2}{(\nu - x)^2 + y^2} \int_0^{y/(\nu - x)} \frac{dy'}{1 + y'^2}\right) \tag{A.54}
\]
where in the first equality we integrated by parts and in the last equality we
substituted a monotonically increasing function $\frac{y'^2}{1+y'^2}$ by its value at the upper
limit. The remaining integral in the third line of A.54 is a bounded function.
Therefore, we get A.52.

The virtues of the formula A.52 are twofold. First, it relates the Stieltjes ker-
nel $\frac{1}{\nu-z}$ to the indicator function $\theta(0 < \nu < x)$, needed to obtain Cauchy mo-
ments A.51. Second, the remainder terms on the RHS of A.52 are again given
by the Stieltjes kernel. This will allow us to estimate the remainder terms us-
ing the original condition A.48.

A.2.2 First Cauchy Moment

We start with the proof of A.50 for $m = 1$. Integrating A.48 over $\Gamma_x$ and using
A.52 we obtain

$$F_1(x) - \Phi_1(x) + \frac{y}{\pi} \Re R(z) + O \left( y^2 \int_{-\infty}^{\infty} d\nu \frac{\left| \rho(\nu) - \phi(\nu) \right|}{(\nu - x)^2 + y^2} \right) = \frac{1}{2\pi i} \int_{\Gamma_x} dz' R(z')$$

(A.55)

The last term in A.52 is the imaginary part of the Stieltjes kernel.
The $R(z)$ term on the LHS is $O(x^{-\omega})$. The RHS is $b_1 + O(|z|^{1-\omega})$. Indeed, for instance the integral over the arc in the upper-half plane is

$$
\int_0^z dz' R(z') = \int_0^\Lambda dz' R(z') + \int_\Lambda^z dz' R(z') = 
\int_0^\Lambda dz' R(z') + \int_\Lambda^z dz' \left( \frac{r_1}{z^{\omega}} + \frac{r_2}{z^{\omega+1}} + \ldots \right) = \text{const} + O(|z|^{1-\omega}) \quad (A.56)
$$

Therefore, (A.5.5) becomes

$$
F_1(x) - \Phi_1(x) - b_1 = O(x^{1-\omega}) + 
+ O\left(y^2 \int_{-\infty}^{0} d\nu \frac{\rho(\nu)}{(\nu - x)^2 + y^2}\right) + O\left(y^2 \int_{-\infty}^{0} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2}\right) \quad (A.57)
$$

First, we estimate the integral with $\phi$

$$
O\left(y^2 \int_{-\infty}^{\infty} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2}\right) = 
O\left(y^2 \left[ \int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{x/2} + \int_{x/2}^{x-y} + \int_{x-y}^{x+y} + \int_{x+y}^{\infty} \right] d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2}\right) = 
= i_1 + \cdots + i_7 \quad (A.58)
$$
Due to monotonicity of $|\phi(\nu)|$ we have

\begin{align*}
    i_1 &= O \left( y^2 \int_{-\infty}^{-x} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2} \right) = O \left( y^2 \int_{x}^{\infty} d\nu \frac{|\phi(-\nu)|}{(\nu + x)^2} \right) = O \left( \frac{y^2 |\phi(-x)|}{x} \right), \\
    i_5 &= O \left( y^2 \int_{x/2}^{x-y} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2} \right) = O \left( y^2 |\phi(x/2)| \int_{x/2}^{x-y} \frac{d\nu}{(\nu - x)^2 + y^2} \right) = O(y\phi(x)) \\
    i_6 &= O \left( y^2 \int_{x-y}^{x+y} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2} \right) = O \left( \int_{x-y}^{x+y} d\nu |\phi(\nu)| \right) = O(y\phi(x)), \\
    i_7 &= O \left( y^2 \int_{x+y}^{\infty} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2} \right) = O \left( y^2 \phi(x) \int_{x+y}^{\infty} \frac{d\nu}{(\nu - x)^2} \right) = O(y\phi(x))
\end{align*}

(W.59)

We also have

\[ i_3 = O \left( y^2 \int_{-a}^{a} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2} \right) = O(y^2/x^2) \quad \text{(A.60)} \]

Further, to estimate $i_2, i_4$ we use that \[ A.47 \] implies

\[ \left( \frac{\mu}{\nu} \right)^{\alpha} < \frac{\phi(\mu)}{\phi(\nu)} < \left( \frac{\mu}{\nu} \right)^{\beta}, \quad |\mu| > |\nu| > a \quad \text{(A.61)} \]
and therefore

\[ i_2 = O \left( y^2 \int_{-x}^{x} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2} \right) = O \left( y^2 \int_{a}^{x} d\nu \frac{|\phi(-\nu)|}{(\nu + x)^2} \right) = \\
= O \left( y^2 \frac{|\phi(-x)|}{x^{2+\alpha}} \int_{a}^{x} d\nu \nu^{\alpha} \right) = O \left( \frac{y^2 |\phi(-x)|}{x} \right), \\
i_4 = O \left( y^2 \int_{a}^{x/\sqrt{2}} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2} \right) = O \left( \frac{y^2 \phi(x)}{x^{\alpha+2}} \int_{a}^{x/\sqrt{2}} d\nu \nu^{\alpha} \right) = O \left( \frac{y^2 \phi(x)}{x^\alpha} \right) \] (A.62)

Collecting A.59 - A.62 we obtain

\[ O \left( y^2 \int_{-\infty}^{\infty} d\nu \frac{|\phi(\nu)|}{(\nu - x)^2 + y^2} \right) = O(y\phi(x)) + O \left( \frac{y^2 |\phi(-x)|}{x} \right) + O(y^2/x^2). \] (A.63)

Finally, we need to estimate the \( \rho \) integral in A.57. Since \( \rho(\nu) \geq 0 \) for all \( \nu \) we have by taking the imaginary part of A.48 and using A.63

\[ O \left( y^2 \int_{-\infty}^{\infty} d\nu \frac{|\rho(\nu)|}{(\nu - x)^2 + y^2} \right) = O \left( y^2 \int_{-\infty}^{\infty} d\nu \frac{\phi(\nu)}{(\nu - x)^2 + y^2} \right) + O(y|R(z)|) = \\
= O(y\phi(x)) + O \left( \frac{y^2 |\phi(-x)|}{x} \right) + O(y^2/x^2) + O(yx^{1-\omega}) \] (A.64)
Therefore, A.63 and A.64 imply for A.57

\[ F_1(x) - \Phi(x) - b_1 = O(y\phi(x)) + O\left(\frac{y^2|\phi(-x)|}{x}\right) + O(x^{1-\omega}). \quad (A.65) \]

This finishes the proof of \( m = 1 \) case of A.50.

A.2.3 Higher Cauchy Moments

Integrating A.48 by parts we have

\[ \int_{-\infty}^{\infty} d\nu \frac{F_1(\nu) - \Phi_1(\nu) - b_1}{(\nu - z)^2} = R(z) \quad (A.66) \]

where we also added 0 as \( b_1 \) term in the integral. Integrating this from \( z \) to \( \infty \) along \( \gamma_+ \) we get

\[ \int_{-\infty}^{\infty} d\nu \frac{F_1(\nu) - \Phi_1(\nu) - b_1}{\nu - z} = \int_{z}^{\infty} dz' R(z') = O(|z|^{1-\omega}) \quad (A.67) \]
Notice that \( F_1(\nu) - \Phi_1(\nu) - b_1 \to 0 \) as \( \nu \to \infty \) due to \( \text{A.80} \), so that the integral on the LHS of \( \text{A.67} \) converges. Integrating \( \text{A.67} \) over \( \Gamma_x \) we obtain

\[
F_2(x) - b_1 x - \frac{y}{\pi} \text{Re} \int_{z}^{\infty} dz'R(z') + O \left( y^2 \int_{-\infty}^{\infty} d\nu \frac{|F_1(\nu) - \Phi_1(\nu) - c_1|}{(\nu - x)^2 + y^2} \right) =
\]

\[
= -\frac{1}{2\pi i} \int_{\Gamma_x} dz' \int_{z'}^{\infty} dz'' R(z'')
\]

(A.68)

Using \( \text{A.80} \) we estimate similarly to \( \text{A.63} \)

\[
O \left( y^2 \int_{-\infty}^{\infty} d\nu \frac{|F_1(\nu) - \Phi_1(\nu) - b_1|}{(\nu - x)^2 + y^2} \right) =
\]

\[
= O \left( y^2 \int d\nu \frac{|L(\nu)\phi(\nu)| + |L(\nu)\nu^{-1}\phi(-\nu)| + |\nu|^{1-\omega}}{(\nu - x)^2 + y^2} \right) =
\]

\[
= O(y^2\phi(x)) + O \left( \frac{y^3|\phi(-x)|}{x} \right) + O(yx^{1-\omega}).
\]

(A.69)

Therefore \( \text{A.68} \) gives

\[
F_2(x) - \Phi_2(x) - b_1 x - b_2 = O(y^2\phi(x)) + O \left( \frac{y^3|\phi(-x)|}{x} \right) + O(x^{2-\omega}), \quad (A.70)
\]

where the constant \( b_2 \) comes from the finite \( u \) part of the integral on the RHS of \( \text{A.68} \), similarly to \( \text{A.56} \). This proves \( \text{A.50} \) for \( m = 2 \). Iterating this argument we obtain the tauberian theorem \( \text{A.50} \) for all \( m = 1, 2, \ldots \).
A.2.4 Odd Densities

The theorem above is not quite what we need in bootstrap applications. Instead we would like to consider parity odd densities that satisfy

$$\rho(-\nu) = -\rho(\nu), \quad \phi(-\nu) = -\phi(\nu). \quad (A.71)$$

Most of the proof goes intact apart from application of the tauberian condition [A.48] in [A.64]. Indeed in this case we have

$$y^2 \int_{-\infty}^{\infty} d\nu \frac{|\rho(\nu)|}{(\nu-x)^2 + y^2} = y^2 \int_{-\infty}^{\infty} d\nu \frac{\rho(\nu)}{(\nu-x)^2 + y^2} + 2y^2 \int_{-\infty}^{0} d\nu \frac{-\rho(\nu)}{(\nu-x)^2 + y^2}.$$  

(A.72)

For the first term we could use the estimates above but the second term should be estimated separately. We get

$$y^2 \int_{-\infty}^{0} d\nu \frac{-\rho(\nu)}{(\nu-x)^2 + y^2} = O(y^2/x^2) + O \left( y^2 \int_{0}^{\infty} d\nu \frac{\nu \rho(\nu)}{(\nu+x)^2 + y^2} \right)$$

$$= O \left( y^2 \int_{0}^{\infty} d\nu \frac{\nu \rho(\nu)}{\nu^2 + x^2} \right) + O(y^2/x^2), \quad (A.73)$$
where we used the fact that \( \nu, x > 0 \) and \( \text{(A.71)} \). Note that

\[
\int_{-\infty}^{\infty} d\nu \frac{\rho(\nu)}{\nu - ix} = \int_{0}^{\infty} d\nu \rho(\nu) \left( \frac{1}{\nu - ix} + \frac{1}{\nu + ix} \right) = 2 \int_{0}^{\infty} d\nu \frac{\rho(\nu) \nu}{\nu^2 + x^2}, \tag{A.74}
\]

where we again used \( \text{(A.71)} \).

Therefore we have

\[
y^2 \int_{-\infty}^{0} d\nu \frac{-\rho(\nu)}{(\nu - x)^2 + y^2} = O \left( y^2 \int_{-\infty}^{\infty} d\nu \frac{\rho(\nu)}{\nu - ix} \right). \tag{A.75}
\]

To estimate this we can use the tauberian condition for \( z = ix \). Therefore we get

\[
y^2 \int_{-\infty}^{0} d\nu \frac{-\rho(\nu)}{(\nu - x)^2 + y^2} = O \left( y^2 |R(ix)| \right) + O \left( y^2 \int_{-\infty}^{\infty} d\nu \frac{\phi(\nu)}{\nu - ix} \right). \tag{A.76}
\]

Now we can estimate the last integral

\[
O \left( y^2 \int_{-\infty}^{\infty} d\nu \frac{\phi(\nu)}{\nu - ix} \right) = O \left( y^2 \int_{0}^{\infty} d\nu \frac{\phi(\nu) \nu}{\nu^2 + x^2} \right) = i_1 + i_2 + i_3, \tag{A.77}
\]

where we split the integral into \( \int_{0}^{\infty} + \int_{a}^{x} + \int_{x}^{\infty} \). Let us estimate each integral us-
ing the usual techniques

\[ i_1 = O \left( y^2 \int_0^a d\nu \frac{\phi(\nu)\nu}{\nu^2 + x^2} \right) = O(y^2/x^2), \]
\[ i_2 = O \left( y^2 \int_a^x d\nu \frac{\phi(\nu)\nu}{\nu^2 + x^2} \right) = O \left( \frac{y^2\phi(x)}{x^{2+\alpha}} \int_a^x d\nu \nu^{1+\alpha} \right) = O(y^2\phi(x)), \]
\[ i_3 = O \left( y^2 \int_x^\infty d\nu \frac{\phi(\nu)\nu}{\nu^2 + x^2} \right) = O \left( \frac{y^2\phi(x)}{x^{\beta}} \int_x^\infty d\nu \frac{\nu^{1+\beta}}{\nu^2 + x^2} \right) = O(y^2\phi(x)) \tag{A.78} \]

Thus, we get the following estimate

\[ y^2 \int_{-\infty}^0 d\nu \frac{-\rho(\nu)}{\nu - x + y^2} = O(y^2x^{-\omega}) + O(y^2/x^2) + O(y^2\phi(x)) . \tag{A.79} \]

The conclusion is that the estimate in this case takes the form

\[ F_1(x) - \Phi(x) - b_1 = O(y^2\phi(x)) + O(x^{1-\omega}) , \tag{A.80} \]

where as usual \( y \equiv \Lambda(x) \). For higher \( m \) the argument is identical the one discussed in appendix B.3. This theorem is what we leads to the statement 1.83.
A.2.5 Construction of $\Phi_m$  

Let us understand better how to construct $\Phi_m$. Consider the following ansatz for the subtraction density

$$
\rho_J^{\text{naive}}(\nu) = \theta(0 < \nu < 1) \sum_i \tilde{\alpha}_i \nu^i + \theta(\nu > 1) \sum_i \alpha_i \nu^{-\delta_i - 1} \cos \frac{\pi \delta_i}{2}. \tag{A.81}
$$

It has the following large $\nu$ expansion

$$
\int_0^\infty d\nu' \rho_J^{\text{naive}}(\nu) \frac{2\nu'\nu}{\nu'^2 + \nu^2} = \sum_i \alpha_i \left( \nu^{-\delta_i} + \frac{2 \cos \frac{\pi \delta_i}{2}}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(\delta_i - 1 - 2k)\nu^{1+2k}} \right)
+ \sum_i \tilde{\alpha}_i \sum_{k=0}^{\infty} \frac{(-1)^k}{(i + 2 + 2k)\nu^{1+2k}}. \tag{A.82}
$$

This takes care of all non-integer powers that appear in the OPE. In addition, it contributes to the non-universal terms in the dispersion relations, namely to $\frac{1}{\nu^{1+2k}}$ in [1.82]. To cancel those we can add terms $\theta(0 < \nu < 1)\tilde{\alpha}_i \nu^i$ to the naive density and fix the coefficients so that they cancel the RHS of [1.82] to any required order. We can then apply the theorem and compute $\Phi_m(\nu)$. Note that non-integer powers of $\nu$ that we are interested at only come from the term $\nu^{-\delta_i}$.

One can use the explicit form of $\phi(\nu)$ [A.81] to check estimates from the proof as
well as to analyze the contribution of operators that produce $\log \nu$ terms.

A careful reader might have noticed that $\phi(\nu) = \rho^{naive}_{\nu}(\nu)$ above is not necessarily positive for $0 < u < 1$, whereas in the assumptions of the theorems we assumed that it is. It is trivial to check that the behavior of $\phi(\nu)$ on a finite interval is completely immaterial for the proof apart from cluttering it a bit, see again [17, 26].

A.2.6 Case With Subtractions

Similarly, we need a version of the theorem for the case with subtractions 1.54. It is useful to consider the following identity [26]

$$
\int_{\nu}^{z} d\tilde{\nu} \frac{(\nu - \tilde{\nu})^{n-1}}{(\nu - \tilde{\nu})^{n+1}} = \frac{1}{n} \left( \frac{z}{\nu} \right)^n \frac{1}{\nu - z}. \quad (A.83)
$$

Let us rewrite the dispersion relation 1.54 as follows

$$
\int_{-\infty}^{\infty} d\nu \frac{\rho(\nu)}{(\nu - z)^{N+1}} = \frac{1}{\Gamma(N + 1)} \frac{\partial^{N}_{z} c_{J}(z)}{\nu - z}, \quad (A.84)
$$

where we defined the full density at negative $\nu$ through $\rho(-\nu) = -\rho(\nu)$. Applying $A.83$ to $A.84$ we get

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\[
\int_{-\infty}^{\infty} d\nu \frac{\rho(\nu)}{\nu^N(\nu - z)} = \frac{1}{z^N} c_J(z) - \sum_{k=0}^{N-1} \frac{1}{z^{N-k}(k)!} \partial_z^k c_J(0) \tag{A.85}
\]

where only even \( k \) contribute since \( c_J(z) \) is an even function of \( z \). Therefore, for even \( N \) we get the following structure of dispersion relations at large \( z \)

\[
\int_{-\infty}^{\infty} d\nu \frac{\rho^{\text{OPE}}(\nu)}{\nu^N(\nu - z)} = \sum_i \alpha_i z^{-N-\delta_i} + \sum_{k=1}^{\infty} d_k z^{-2k}, \tag{A.86}
\]

where again \( \alpha_i \) are computable in terms of the OPE and \( \rho^{\text{OPE}}(-\nu) = -\rho^{\text{OPE}}(\nu) \).

The naive spectral density that will reproduce the RHS is of course exactly the same as before.

In principle, one can try to derive a separate tauberian theorem for the kernel \[ \text{A.86}, \text{ see e.g. } [26]. \text{ For us however it suffices to consider } \tilde{\rho}(\nu) = \frac{\rho^{\text{OPE}}(\nu)}{\nu^N} \text{ and apply the theorem II. Note also that we have the following identity between the moments of } \rho \text{ and } \tilde{\rho} \]

\[
F_m(\nu) = (-1)^N \frac{(N + m - 1)!}{(m - 1)!} \tilde{G}_{N+m,N}(\nu), \tag{A.87}
\]

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where recall that $G_{m,k}$ was defined in [1.90]. This completes the consideration of the case with subtractions.

The symmetry of $\tilde{\rho}$ is different for odd and even $N$'s $\tilde{\rho}(\nu) = (-1)^{N+1} \tilde{\rho}(\nu)$. For even $N$ we can directly use the formulas from our analysis of the unsubtracted dispersion relations. Consider a term $\alpha_i z^{-N-\delta_i}$ in A.86. Using 1.91 and A.87 (and being careful about factors of $i$ in A.86, 1.54) we recover the result 1.83. For odd $N$ the conclusion is the same. It had to be the case by self-consistency of the whole construction, namely for a given external scaling dimension we could have considered dispersion relations with different numbers of subtractions, but this should not affect the result for the leading asymptotic. This is indeed the case.

A.3 Tauberian Optimality Example

Here we would like to understand properties of the Laplace transform in the complex $\beta$ plane of the spectral density [1.18]

$$f(E) = (1 + \sin[(\log E)^2]) \theta(E - 1). \quad \text{(A.88)}$$
The relevant integral to study is the following

\[ \mathcal{L}(\beta) = \int_0^\infty dE e^{-\beta E} f(E) = \frac{e^{-\beta}}{\beta} + \int_1^\infty dE \ e^{-\beta E} \sin[(\log E)^2]. \] (A.89)

To analyze the second integral it is convenient to use the standard Mellin representation for \( e^{-\beta E} \)

\[ e^{-x} = \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{d\delta}{2\pi i} \Gamma(-\delta)x^\delta, \quad \epsilon > 0. \] (A.90)

Convergence of this integral depends on the argument of \( x \). Denoting \( \delta = \delta_R + i\delta_I \) we get the asymptotic behavior

\[ \Gamma(-\delta)x^\delta \sim |\delta|^{\delta_R-1/2} e^{\delta_I \arg[x] - \frac{\pi}{2}|\delta_I|}. \] (A.91)

In particular, the integral converges only for \( |\arg[x]| < \frac{\pi}{2} \).
We then have for the second term in (A.89) \((x = \log E)\)

\[
\int_1^\infty dE \ e^{-\beta E} \sin[(\log E)^2] = \int_0^\infty dx \ e^x \sin(x^2) e^{-\beta e^x} =
\]

\[
= \int_{-\epsilon+i\infty}^{-\epsilon-i\infty} \frac{d\delta}{2\pi i} \Gamma(-\delta) \beta^\delta \int_0^\infty dx \ e^{(\delta+1)x} \sin(x^2) =
\]

\[
= \frac{1}{2} \sqrt{\frac{\pi}{2}} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{d\delta}{2\pi i} \Gamma(-\delta) \beta^\delta \times
\]

\[
\times \left[ \cos \left( \frac{(\delta + 1)^2}{4} \right) \left( 1 + 2C \left( \frac{\delta + 1}{\sqrt{2\pi}} \right) \right) + \sin \left( \frac{(\delta + 1)^2}{4} \right) \left( 1 + 2S \left( \frac{\delta + 1}{\sqrt{2\pi}} \right) \right) \right],
\]

where \(C(x), S(x)\) denote the corresponding Fresnel integrals

\[
C(z) \equiv \int_0^z dt \cos \frac{\pi t^2}{2}, \quad S(z) \equiv \int_0^z dt \sin \frac{\pi t^2}{2}.
\]  

(A.93)

In doing the \(x\) integral we assumed \(\epsilon > 1\) for convergence. The result however is regular at \(\epsilon = -1\) and we can shift the \(\delta\) contour to estimate the asymptotic of the integral (A.89).

Let us analyze the convergence properties of the integral (A.92) in the complex \(\beta\) plane. We set \(\beta = |\beta| e^{i\phi}\). The danger is that now we have an extra \(e^{i\phi}\delta\) factor which blows up in the lower half-plane. Evaluating the asymptotic of the
integrand we get

\[ e^{-\phi \delta} e^{-\frac{\pi}{2} |\delta_l| e^{\frac{(1+i\delta_R)\delta_l}{2}}} |\beta|^{|\delta_R+i\delta_l|}. \]  

(A.94)

The integral converges for

\[ \phi + \frac{1 + \delta_R}{2} < \frac{\pi}{2}. \]  

(A.95)

Therefore we cannot evaluate the integral by simply shifting the contour to the right, namely increasing \( \delta_R \).

It does however allow us to evaluate the leading asymptotic for \( \phi = 0 \). From the first three poles at \( \delta = 0, 1, 2 \) we get

\[
\int_1^\infty dE \ e^{-\beta E} \sin[(\log E)^2] = c_0 + c_1 \beta + c_2 \beta^2 + \ldots,
\]

\[
c_0 = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left[ \cos \left( \frac{1}{4} \right) \left( 1 + 2C \left( \frac{1}{\sqrt{2\pi}} \right) \right) + \sin \left( \frac{1}{4} \right) \left( 1 + 2S \left( \frac{1}{\sqrt{2\pi}} \right) \right) \right],
\]

(A.96)

and similar expressions for \( c_1 \) and \( c_2 \). Pushing contour to further poles is not possible because it would violate \( \delta_{\text{max}}^R = \pi - 1 \). As we increase \( \phi \) the range of maximal allowed \( \delta_R \) decreases. Therefore we cannot use the estimate above.
anymore. By studying numerically the integral we observed that the next term in the expansion is of the type \( \beta^{\pi-1} \cos(\log \beta)^2 L(\beta) \) where \( L(\beta) \) is slowly varying function. We also observed that this term captures the asymptotic behavior correctly for complex \( \beta \) as well. As we increase the argument of \( \beta \) the term \( \beta^{\pi-1} \cos(\log \beta)^2 L(\beta) \) becomes dominant and as \( \beta \) becomes imaginary it behaves as \( \frac{1}{\beta} L(i\beta) \) which is again consistent with our numerical observations.
B.1 Estimate of $G(\nu)$

In this appendix we derive the estimate $2.50$

$$|G_{\pm}(\nu)| \leq 2e^{-\beta\nu} \min[1, (\Lambda\nu)^{-2}] .$$  \hspace{1cm} (B.1)
First, let’s consider $G_+$

$$G_+(\nu) = \frac{1}{2\pi i} \int_{\beta-i\Lambda}^{\beta+i\Lambda} \frac{dz}{z} \frac{\Lambda^2 + (z - \beta)^2}{\Lambda^2 + \beta^2} e^{-\nu z}. \quad (B.2)$$

We will estimate $B.2$ in two different ways. First way is to deform the contour to the right to $\gamma_1, \gamma_2$, see $B.1$. Let’s call the integrals over these contours $\Gamma_{1,2}$. Then we estimate as follows

$$|\Gamma_1| = \left| \frac{1}{2\pi} \int_0^{\infty} \frac{dx}{\beta - i\Lambda + x} \frac{x(x - 2i\Lambda)}{\Lambda^2 + \beta^2} e^{-\nu(\beta - i\Lambda + x)} \right|$$

$$\leq e^{-\beta \nu} \int_0^{\infty} e^{-y} \sqrt{\frac{y^2/\nu^2 + 4\Lambda^2}{(\beta + y/\nu)^2 + \Lambda^2}}. \quad (B.3)$$
Assuming $\nu > 0$ the square root in the last integral is bounded by 2. Therefore we have

$$|\Gamma_1| \leq \frac{1}{\pi} \frac{e^{-\beta \nu}}{(\Lambda^2 + \beta^2)\nu^2} \leq \frac{1}{2} e^{-\beta \nu}(\Lambda \nu)^{-2}. \quad (B.4)$$

The contribution $\Gamma_2$ is bounded in the same way. In total we get

$$|G_+ (\nu)| \leq e^{-\beta \nu}(\Lambda \nu)^{-2}. \quad (B.5)$$

On the other hand we can estimate $B.1$ as follows. First, we change variables in $z = \beta + \Lambda t$. After some algebra we get

$$G_+ (\nu) = \frac{\Lambda^3 e^{-\beta \nu}}{\pi (\Lambda^2 + \beta^2)} \int_0^1 dt \frac{(1 - t^2)}{\beta^2 + \Lambda^2 t^2} [\beta \cos (t \Lambda \nu) - \Lambda t \sin (t \Lambda \nu)] . \quad (B.6)$$

Suppose $\Lambda |\nu| < 1$. Then we estimate

$$|\beta \cos (t \Lambda \nu) - \Lambda t \sin (t \Lambda \nu)| \leq \beta + \Lambda^2 t^2 |\nu| \leq \beta + \Lambda t^2 , \quad (B.7)$$
so that (B.6) becomes

\[ |G_+(\nu)| \leq \frac{\Lambda^3 e^{-\beta \nu}}{\pi (\Lambda^2 + \beta^2)} \int_0^1 dt \frac{\beta + t^2 \Lambda^2}{\beta^2 + \Lambda^2 t^2} \]

\[ = \frac{\Lambda^2 e^{-\beta \nu}}{\pi (\Lambda^2 + \beta^2)} \left[ 1 + \left( 1 - \frac{\beta}{\Lambda} \right) \arctan \left( \frac{\Lambda}{\beta} \right) \right] \]

\[ \leq \frac{\Lambda^2 e^{-\beta \nu}}{\pi (\Lambda^2 + \beta^2)} \left( 1 + \frac{\pi}{2} \right) \leq e^{-\beta \nu} . \quad (B.8) \]

Combining (B.5), (B.8) we get (B.1) for \( G_+ \).

Now we estimate \( G_- \) for \( \nu < 0 \). First, note that (B.8) is valid for \( \nu < 0 \) as well.

Then we can use it to write

\[ |G_-(\nu)| = |G_+(\nu) - 1| \leq |G_+(\nu)| + 1 \leq 1 + e^{-\beta \nu} \leq 2e^{-\beta \nu}, \quad \nu < 0 . \quad (B.9) \]

On the other hand we can deform the contour to the left and again get two

contributions \( \Gamma_{3,4} \), see (B.2). Then we estimate similarly to (B.3)

\[ |\Gamma_3| \leq \frac{e^{-\beta \nu}}{2\pi (\Lambda^2 + \beta^2) \nu^2} \int_0^\infty dy \ e^{-y} \sqrt{\frac{y^2/\nu^2 + 4\Lambda^2}{(\beta + y/\nu)^2 + \Lambda^2}} , \quad (B.10) \]

but now with \( \nu < 0 \). Then we estimate for \( \Lambda |\nu| > 1 \)

\[ \sqrt{\frac{y^2/\nu^2 + 4\Lambda^2}{(\beta + y/\nu)^2 + \Lambda^2}} \leq 2 \sqrt{1 + \frac{y^2}{4\Lambda^2 \nu^2}} \leq \sqrt{1 + \frac{y^2}{4}} \quad (B.11) \]
so that $B.10$ becomes

$$\left| \Gamma_3 \right| \leq \frac{e^{-\beta \nu}}{2\pi(\Lambda^2 + \beta^2)\nu^2} \int_{0}^{\infty} dy \, e^{-y} y \sqrt{1 + \frac{y^2}{4}} \leq \frac{3}{2\pi} \frac{e^{-\beta \nu}}{(\Lambda^2 + \beta^2)^{\frac{3}{2}}} \leq e^{-\beta \nu} (\Lambda \nu)^{-2} \tag{B.12}$$

Making the same estimate for $\Gamma_4$ we finally get

$$\left| G_-(\nu) \right| \leq 2e^{-\beta \nu} (\Lambda \nu)^{-2} \tag{B.13}$$

And combining this with $B.9$ gives $B.1$ for $G_-(\nu)$. 

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**Figure B.2:** Contour deformation for $G_-$. 

\[ \text{Figure B.2: Contour deformation for } G_- \]
B.2 Power Corrections

We can consider multiple integrals of the density of states \[ F^m_\rho(\Delta) = \frac{1}{(m-1)!} \int_0^\Delta d\Delta' (\Delta - \Delta')^{m-1} \rho(\Delta') = \]

\[ = \int_0^\Delta d\Delta_m \int_0^{\Delta_m} d\Delta_{m-1} \cdots \int_0^{\Delta_2} d\Delta_1 \rho(\Delta_1). \] (B.14)

The tauberian theorem for \[ B.14 \] takes the form

\[ F^m_\rho(\Delta) = \frac{1}{(m-1)!} \int_0^\Delta d\Delta' (\Delta - \Delta')^{m-1} \rho_0(\Delta') + O\left(\Delta^{-3/4} e^{2\pi\sqrt{3/\Delta}}\right) = \]

\[ = \frac{1}{2\pi} \left(\frac{3}{c\Delta}\right)^{1/4} e^{2\pi\sqrt{3/\Delta}} \left[ \sum_{i=1}^m c_i \Delta^{i-1} + O(\Delta^{-1/2}) \right], \quad \Delta \to \infty. \] (B.15)

The coefficients \( c_i \) can be computed explicitly using the crossing kernel \[ 2.15 \].

Again, all the spectral density moments \( F^m_\rho(\Delta) \) are controlled by the unit operator in the dual channel. The intuition behind \[ B.14 \] is that each integration enhances smooth power-like terms while keeping intact oscillating non-universal terms.

The derivation of \[ B.15 \] is analogous to the one in section 5. We consider \[ 2.46 \].
with a higher order pole

\[
\frac{1}{2\pi i} \int_{\beta-i\Lambda}^{\beta+i\Lambda} \frac{dz}{z^{m+1}} \left[ \Lambda^2 + (z - \beta)^2 \right] e^{\beta \Delta} \delta \mathcal{L}(z) .
\] (B.16)

When deforming the contour to the left, the pole contribution will produce the desired kernel \((\Delta - \Delta')^m\) from the expansion of \(e^{\beta(\Delta - \Delta')}\) near \(z = 0\).\(^1\) The rest of the argument is identical to the \(m = 0\) case.

### B.3 Local bound at large \(c\)

Let’s estimate the sum

\[
A = e^{\beta \Delta} \sum_{\Delta_L \leq \Delta_H} \int_0^\infty d\Delta' \rho_{\Delta_L}(\Delta') e^{-\beta \Delta'} \phi_+(\Delta') ,
\] (B.17)

where the crossing kernel of operator \(\Delta_L\) is

\[
\rho_{\Delta_L}(\Delta) = 2\pi \sqrt{\frac{\xi}{\Delta - \frac{c}{12}} I_1 \left( 4\pi \sqrt{\left( \frac{c}{12} - \Delta_L \right) \left( \Delta - \frac{c}{12} \right)} \right)} \theta(\Delta - c/12) + \delta(\Delta - c/12) .
\] (B.18)

\(^1\)Plus lower orders of \(\Delta - \Delta'\) due to the expansion of the polynomial in (B.16).
It reproduces the contribution of the operator $\Delta_L$ in the dual channel

$$\int_{0}^{\infty} d\Delta \, \rho_{\Delta_L}(\Delta) e^{-\beta \Delta} = e^{-\frac{4\pi^2}{\beta} (\Delta_L - c/12)}. \quad (B.19)$$

Computing each integral by a saddle approximation we get

$$A = O \left( c^{-1/2} e^{2\pi c \sqrt{\frac{3}{\epsilon}}} \sum_{\Delta_L \leq \Delta_H} e^{-4\pi \sqrt{3\epsilon} \Delta_L} \right) = O \left( c^{-1/2} e^{2\pi c \sqrt{\frac{3}{\epsilon}}} \right)$$

$$= O(\rho_0(\Delta)), \quad \epsilon > \frac{1}{12}, \quad (B.20)$$

where we used the sparseness condition $2.75$. 263
Appendices to chapter 3

In this section we will derive all solutions of \(3.99\). It is convenient to separate the equations in \(3.99\) into groups of \(N\) equations, namely the equations 1 to \(N\), 2 to \(N + 1\), 3 to \(N + 2\) and so on. Each of these systems determines \(d_i(z)\). To write this more concisely, let us define \(N \times N\) matrices \(V_n\) through recursion relations
\[ V_{n+1} = V_n \mathcal{E}, \quad n = 0, 1, 2, \ldots, \quad (C.1) \]

where \( V_0 \) is Vandermonde matrix and \( \mathcal{E} \) is diagonal

\[
V_0(z) = \begin{pmatrix}
1 & \ldots & 1 \\
\epsilon_1(z) & \ldots & \epsilon_N(z) \\
\ldots & \ldots & \ldots \\
\epsilon_1(z)^{N-1} & \ldots & \epsilon_N(z)^{N-1}
\end{pmatrix}, \quad \mathcal{E}(z) = \begin{pmatrix}
\epsilon_1(z) & 0 \\
& \ddots \\
0 & \epsilon_N(z)
\end{pmatrix}. \quad (C.2)
\]

Also, define columns of \( d_i(z) \) and \( W_n(z) \)

\[
A(z) = \begin{pmatrix}
d_1(z) \\
\ldots \\
d_N(z)
\end{pmatrix}, \quad A_n(z) = \begin{pmatrix}
W_n(z) \\
W_{n+1}(z) \\
\ldots \\
W_{n+N-1}(z)
\end{pmatrix}. \quad (C.3)
\]

Then the equations \(3.99\) become

\[
V_0 \Lambda = A_0, \quad V_1 \Lambda = A_1, \quad V_2 \Lambda = A_2, \quad \ldots. \quad (C.4)
\]
Each of these matrix equations is a linear system of \( N \) equations for \( N \) unknowns \( \Lambda \). The determinants of the matrices \( V_n \) are not zero in the physical regime of real \( z > \sqrt{3} \)

\[
\det V_n = \left( \prod_{i=1}^{N} \epsilon_i \right)^n \prod_{i<j} (\epsilon_j - \epsilon_i) \neq 0 \tag{C.5}
\]

since we assumed that \( \epsilon_i \neq 0 \) and \( \epsilon_i \neq \epsilon_j \) for \( i \neq j \). Thus, we can invert the matrices \( V_n \) and solve \( C.4 \) to find \( \Lambda \)

\[
\Lambda = V_0^{-1} A_0 = V_1^{-1} A_1 = V_2^{-1} A_2 = \ldots \ . \tag{C.6}
\]

Using the relations \( C.1 \), we can write this as

\[
\Lambda = V_0^{-1} A_0 ,
\]

\( (V_0 \mathcal{E} V_0^{-1}) A_0 = A_1 , \)

\( (V_0 \mathcal{E} V_0^{-1}) A_1 = A_2 , \)

\( (V_0 \mathcal{E} V_0^{-1}) A_2 = A_3 , \)

\( \ldots \ . \tag{C.7} \)
Here, the first equation gives the solution for \( d_i \) in terms of \( \epsilon_i, W_n \) and the rest of the equations give an infinite number of constraints on \( \epsilon_i, W_n \).

The matrix \( V_0 \mathcal{E} V_0^{-1} \) can be directly computed and is given by

\[
V_0 \mathcal{E} V_0^{-1} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
(-1)^{N-1} e_N(z) & (-1)^{N-2} e_{N-1}(z) & (-1)^{N-3} e_{N-2} & \ldots & e_1(z)
\end{pmatrix},
\]

where \( e_k(z) \) are symmetric polynomials in \( \epsilon_i(z) \)

\[
e_1(z) = \sum_i \epsilon_i(z), \quad e_2(z) = \sum_{i<j} \epsilon_i(z)\epsilon_j(z), \quad \ldots, \quad e_N(z) = \prod_i \epsilon_i(z). \tag{C.9}
\]

Only the last row of \( \text{C.8} \) gives nontrivial equations in \( \epsilon_i(z) \). These are

\[
\begin{pmatrix}
W_0 & \ldots & W_{N-1} \\
\vdots & \ddots & \vdots \\
W_{N-1} & \ldots & W_{2(N-1)} \\
\vdots & \ddots & \vdots 
\end{pmatrix}
\begin{pmatrix}
(-1)^{N-1} e_N \\
\vdots \\
e_1 \\
\vdots 
\end{pmatrix}
= \begin{pmatrix}
W_N \\
\vdots \\
W_{2N-1} \\
\vdots 
\end{pmatrix}, \tag{C.10}
\]
where we omitted the dependence on $z$ for the sake of brevity. A simple solution of the overdetermined linear system (C.10) is for $e_k(z)$ to be a polynomial of order $k$

$$e_k(z) = \mathcal{P}_k(z) = \sum_{n=0}^{k} a_n^{(k)} z^n, \quad k = 1, \ldots, N.$$ (C.11)

Indeed, taking independent parameters to be the coefficients of polynomials $W_0, \ldots, W_{N-1}, \mathcal{P}_1, \ldots, \mathcal{P}_N$, the first equation in (C.10) defines $W_N$, the second equation in (C.10) defines $W_{N+1}$, etc.

Solving the first $N$ equations in (C.10) we find that $e_k(z)$ is a rational function of $z$. As we prove below, to satisfy the rest of the equations it cannot have poles, so that (C.11) is the only solution of (C.10).

Since $e_k(z)$ are symmetric polynomials (C.9), by Vieta's formula the scaling dimensions $\epsilon_i(z)$ are given by $N$ different solutions of the $N$th order algebraic equation

$$\prod_{i=1}^{N} (x - \epsilon_i(z)) = x^N - \mathcal{P}_1(z)x^{N-1} + \mathcal{P}_2(z)x^{N-2} + \cdots + (-1)^N \mathcal{P}_N(z) = 0.$$ (C.12)

Finally, the three-point functions $d_i(z)$ are given by the first equation in (C.7).
Using Cramer’s rule, we find

\[
d_k(z) = \prod_{i<j} (\epsilon_j(z) - \epsilon_i(z))^{-1} \det \begin{pmatrix}
1 & \cdots & W_0 & \cdots & 1 \\
\epsilon_1(z) & \cdots & W_1(z) & \cdots & \epsilon_N(z) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\epsilon_1(z)^{N-1} & \cdots & W_{N-1}(z) & \cdots & \epsilon_N(z)^{N-1}
\end{pmatrix},
\]

(C.13)

where the \( k \)-th column is made out of polynomials \( W_n(z) \). The equations C.12, C.13 give a complete solution of the smoothness conditions 3.99 for spins \( J \geq 2 \).

Now let us show that C.12, C.13 is the only solution. We need to show that \( e_k(z) \) cannot have poles. Equivalently, we need to show that

\[
\text{Tr} \mathcal{E}^n = \sum_{i=1}^{N} \epsilon_i(z)^n
\]

(C.14)

does not have poles in \( z \) for any positive integer \( n \). Consider the equations C.7. They can be combined into matrix equations

\[
(V_0 \mathcal{E} V_0^{-1}) M_n = M_{n+1}, \quad n = 0, 1, 2, \ldots,
\]

\[
M_n = (A_n, \ldots, A_{n+N-1}),
\]

(C.15)
and are solved by

\[ V_0^* V_0^{-1} = M_1 M_0^{-1} = M_2 M_1^{-1} = \ldots . \]  \hfill (C.16)

Taking the trace of the \( n \)-th power of these equations we have

\[ \text{Tr} \mathcal{E}^n = \text{Tr}(M_1 M_0^{-1})^n . \]  \hfill (C.17)

On the other hand, the equations (C.16) require that \( M_n M_0^{-1} = (M_1 M_0^{-1})^n \) and

\[ \text{Tr} \mathcal{E}^n = \text{Tr} M_n M_0^{-1} . \]  \hfill (C.18)

Suppose that one of the \( \epsilon_i(z) \) develops a singularity at \( z_0 \). This singularity must be consistent with (C.18). Let us expand (C.18) around \( z_0 \). The crucial point is that a potential singularity on the RHS of (C.18) could only come from \( \det M_0 = 0 \) and its maximal order does not depend on \( n \). The LHS of (C.18), which is equal to (C.14), has a singularity \( \epsilon_i(z_0)^n \), whose strength is unbounded in contrast with the RHS of (C.18) (it becomes more and more singular as \( n \) grows). To reconcile these two facts, other \( \epsilon_j(z) \) should soften \( \epsilon_i(z_0)^n \) for large enough \( n \).

\(^1\text{The actual behavior depends on the behavior of } \text{Tr} M_n M_0^{-1}, \text{but it cannot be more singular than } \frac{1}{\det M_0}.\)
Imposing this cancelation it is trivial to see that for a finite number of Regge trajectories \( N \) it is not possible for every \( n \). Without loss of generality assume that close to \( z_0 \) we have \( \epsilon_i(z) = c_i(z - z_0)^{-\alpha} + \ldots \). The conditions for cancelation of the singularity become

\[
\sum_{i=1}^{N} c_i^n = 0, \quad n = n_0, n_0 + 1, \ldots , \tag{C.19}
\]

where \( c_i \) are complex number (analogs of residues), which control the behavior of \( \epsilon_i(z) \) near the singularity. The only solution to \( \text{(C.19)} \) is

\[
c_i = 0 . \tag{C.20}
\]

Indeed, consider \( \text{(C.19)} \) as a linear system of a Vandermonde matrix of \( c_i \)'s acting on \((1, \ldots, 1)^T\). If \( c_i \neq c_j \) and \( c_i \neq 0 \) the determinant of Vandermonde is non-zero and the system is inconsistent. If \( c_i = c_j \) for some \( i, j \), the system \( \text{(C.19)} \) can be reduced to a similar one with smaller \( N \). Thus, we conclude that the only solution is \( \text{(C.20)} \). Therefore, the assumed singularity was absent in the first place.

In the derivation above we tacitly assumed that \( \det M_n(z) \) is not identically zero for all \( n \) and \( z \). Suppose one of them is identically zero. Without loss of
generality assume \( \det M_0 \equiv 0 \). This implies

\[
\det M_0 = \det(A_0, \ldots, A_{N-1}) = \det(A_0, V_0 \mathcal{E} V_0^{-1} A_0, \ldots, (V_0 \mathcal{E} V_0^{-1})^{N-1} A_0) \equiv 0 .
\]

(C.21)

Therefore, there must exist non-zero \( \lambda_k(z) \) such that

\[
\sum_{k=0}^{N-1} \lambda_k(V_0 \mathcal{E} V_0^{-1})^k A_0 \equiv 0 .
\]

(C.22)

Using the first equation in C.7 and C.3, we can write C.22 as

\[
\sum_{k=0}^{N-1} \lambda_k(z) \epsilon_i(z)^k d_i(z) \equiv 0, \quad i = 1, \ldots, N .
\]

(C.23)

Consider this equation in the physical regime \( z \geq \sqrt{3} \). We can divide by \( d_i(z) > 0 \) since these are nonzero in the physical region. Further, since \( \epsilon_i(z) \neq \epsilon_j(z) \) for \( i \neq j \), the determinant of the system C.23 is non-zero. Indeed, it is given by the Vandermonde determinant made of \( \epsilon_i(z) \)'s. Consequently, the system C.23 has only trivial solution \( \lambda_k(z) \equiv 0 \). Thus, physical constraints rule out the possibility that \( \det M_n(z) \equiv 0 \).
References


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