Trees, Berkovich Spaces and the Barycentric Extension in Complex Dynamics

Trees, Berkovich spaces and the barycentric extension in complex dynamics

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Abstract

A metric space T is called an $\mathbb{R}$-tree if any two points $x, y \in T$ can be connected by a unique topological arc $[x, y] \subset T$ which is isometric to an interval in $\mathbb{R}$. $\mathbb{R}$-trees are natural generalizations to finite trees and simplicial trees, and have many applications in mapping class groups, Teichmüller theory, hyperbolic 3-manifold and Kleinian groups and etc.

In this work, we will give a new construction of $\mathbb{R}$-trees in complex dynamics using barycentric extensions. We will establish the relation between the barycentric construction and the Berkovich construction via the *complexified Robison's field*. As an application, we will also use our construction to classify all hyperbolic components that admits degenerating sequences with bounded multipliers.

Contents

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Introduction

Let $\mathrm{Rat}_d(\mathbb{C})$ denote the space of rational maps of degree d, i.e., the set of maps

$$f(z:w) = (P(z,w) : Q(z,w))$$

where P, Q are homogeneous polynomials of degree d with no common factors. A sequence $f_n \in \mathrm{Rat}_d(\mathbb{C})$ is called *a degenerating sequence of rational maps* if f_n escapes any compact set of $\mathrm{Rat}_d(\mathbb{C})$. The group of Möbius transformations $\mathrm{PSL}_2(\mathbb{C})$ naturally acts on $\mathrm{Rat}_d(\mathbb{C})$ by conjugation. The quotient space $M_d = \mathrm{Rat}_d(\mathbb{C})/\mathrm{PSL}_2(\mathbb{C})$ is called the *moduli space of rational maps*. We call a sequence $[f_n] \in M_d$ *degenerating as conjugacy classes* if $[f_n]$ escapes any compact set of M_d.

In this work, we shall study the limiting dynamics of $f_n : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ via investigating its barycentric extension $\mathscr{E} f_n : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ (see Section 1.1). A metric space T is called an $\mathbb{R}$-tree if any two points $x, y \in T$ can be connected by a unique topological arc $[x, y] \subset T$ which is isometric to an interval in $\mathbb{R}$. We will construct a limiting branched covering map on an $\mathbb{R}$-tree for the sequence of barycentric extensions of a degenerating sequence of rational maps. The construction is in the same spirit of the isometric group actions on an $\mathbb{R}$-tree studied in [MS84] [Bes88] [Pau88] for Kleinian groups, and is a generalization of the Ribbon $\mathbb{R}$-tree for degenerating sequence of Blaschke products introduced by McMullen in [McM09b] (see Section 1.3).

$\mathbb{R}$-trees also appear in Berkovich projective spaces of a non-Archimedean field, which have many applications in the study of degenerating families of rational maps (see [Kiw15]). One of our main result is the establishment of the relation between the barycentric construction and the Berkovich construction via the *complexified Robison's field* (see Section 1.2).

As an application, we will use our constructions to study hyperbolic components that admits degenerating sequences with bounded multipliers. We will show that these hyperbolic components are exactly those with 'nested Julia sets' (see Section 1.4).

We now turn to detailed statement of results.

1.1 Barycentric extension and limiting dynamics

Let $(\mathbb{H}^n, d_{\mathbb{H}^n}) := (\{\boldsymbol{x} = (x_1, .., x_n) \in \mathbb{R}^n : |\boldsymbol{x}| < 1\}, \frac{2|d\boldsymbol{x}|}{1-|\boldsymbol{x}|^2})$ be the Hyperbolic n-space in the standard ball model. Denote $S^{n-1} := \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| = 1\}$ as the *conformal boundary* of $\mathbb{H}^n$.

Given a continuous map $f : S^{n-1} \longrightarrow S^{n-1}$, the barycentric extension / Douady-Earle extension, which was first introduced in dimension 2 in [DE86], is a continuous extension to $\mathscr{E} f : \mathbb{H}^n \longrightarrow \mathbb{H}^n$. Indeed, given a point $p \in \mathbb{H}^n$, one can associate a visual measure v_p on S^{n-1}. The barycentric extension $\mathscr{E} f$ sends p to the barycenter of the measure $f_*(v_p)$.

In dimension $n = 1$, if $f : S^1 \longrightarrow S^1$ is the restriction of a finite Blaschke product, or more generally, an *inner function*, the barycentric extension $\mathscr{E}f : \mathbb{H}^2 \cong \Delta \longrightarrow \Delta \cong \mathbb{H}^2$ satisfies $\mathscr{E}f = f$. Hence, the Schwarz lemma implies that, with respect to the hyperbolic metric, the extension $\mathscr{E}f$ is uniformly 1-Lipschitz. Our first result is a generalization of the Schwarz lemma to higher dimensions*:

Theorem 1.1. *Let $f : \mathbb{P}^1_{\mathbb{C}} \cong S^2 \longrightarrow S^2 \cong \mathbb{P}^1_{\mathbb{C}}$ be a rational map of degree d, then the barycentric extension of f*

$$\mathscr{E}f : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$$

is uniformly Cd-Lipschitz, for some universal constant C.

Branched covering on the asymptotic cone of $\mathbb{H}^3$

Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$ be a sequence of rational maps, and

$$r_n := \max_{y \in \mathscr{E}f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0}),$$

where $\mathbf{0}$ is the origin of the ball model of $\mathbb{H}^3$. Then f_n is degenerating as rational maps if and only if $r_n \to \infty$. To capture the large scale dynamics for the sequence of extensions $\mathscr{E}f_n$, we rescale the metrics on $\mathbb{H}^3$ by r_n. For technical reasons, we will fix a non-principal ultrafilter ω on $\mathbb{N}$, which is a finitely additive measure on $\mathbb{N}$ taking values in $\{0, 1\}$. We will consider the *ultra-limit* $({}^r\mathbb{H}^3, x^0, d)$ of the pointed metric spaces $(\mathbb{H}^3, \mathbf{0}, d^3_{\mathbb{H}}/r_n)$. This metric space $({}^r\mathbb{H}^3, x^0, d)$ is also known as the *asymptotic cone* of $\mathbb{H}^3$. It is well known that the asymptotic cone of $\mathbb{H}^3$ (or any Gromov hyperbolic metric space) is an $\mathbb{R}$-tree (see [Roe03]).

Theorem 1.1 allows us to construct a limiting map $\mathscr{E}_{bc}(f_n)$ on ${}^r\mathbb{H}^3$. We will prove the following in Section 3.8.

*See Chapter 2 for more general result

Theorem 1.2. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$ be a degenerating sequence,*

$$r_n := \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$$

and ${}^r\mathbb{H}^3$ be the asymptotic cone of $\mathbb{H}^3$ with rescaling r_n. Then the limiting map

$$\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$$

is a branched covering† *of degree d.*

1.2 Connections to Berkovich dynamics

Let K be a non-Archimedean field, then the *Berkovich affine line* $\mathbb{A}^1_{Berk}(K)$ of K is the space of multiplicative semi-norms on the polynomial ring $K[T]$ extending the absolute value on K. The elements of K embed into $\mathbb{A}^1_{Berk}(K)$ by evaluation, and the one point compactification is the *Berkovich projective line* $\mathbb{P}^1_{Berk}(K)$. The space $\mathbb{P}^1_{Berk}(K) - \mathbb{P}^1(K)$ is called the *Berkovich hyperbolic space*, and is denoted by $\mathbb{H}_{Berk}(K)$. There is a natural metric one can put on $\mathbb{H}_{Berk}(K)$ so that it is an $\mathbb{R}$-tree. A Rational map $\mathbf{f} : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ naturally extends to a map on $\mathbb{H}_{Berk}(K)$, which we denote it by $\mathscr{E}_{Berk}(\mathbf{f}) : \mathbb{H}_{Berk}(K) \longrightarrow \mathbb{H}_{Berk}(K)$.

Recall for a non-principal ultrafilter ω on $\mathbb{N}$, we defined $({}^r\mathbb{H}^3, x^0, d)$ as the ultralimit of the sequence of pointed metric spaces $(\mathbb{H}^3, \mathbf{0}, d_{\mathbb{H}^3}/r_n)$. The sequence $\rho_n := e^{-r_n}$ also naturally gives a spherically complete non-Archimedean field ${}^\rho\mathbb{C}$ which is a complexified version of the Robinson's field [LR75]. As a set, ${}^\rho\mathbb{C}$ is the quotient M_0/M_1, where

$$M_0 = \{(z_n) : \text{There exists some } N \in \mathbb{N} \text{ such that } |z_n| < \rho_n^{-N} \ \omega\text{-almost surely}\}$$

†The definition of branched coverings on an $\mathbb{R}$-tree can be found in Section 3.1.

and

$$M_1 = \{(z_n) : \text{For all } N \in \mathbb{N}, |z_n| < \rho_n^N \ \omega\text{-almost surely}\}$$

By choosing representatives of the coefficients, we can associate a sequence of rational maps $f_n \in \mathrm{Rat}_d(\mathbb{C})$ to a rational map $\mathbf{f} \in \mathrm{Rat}_d({}^\rho\mathbb{C})$. Conversely, if $f_n \in \mathrm{Rat}_d(\mathbb{C})$, then by representing coefficients, we can associate a rational map $\mathbf{f}$ potentially of lower degrees.

In Section 4.3, we will prove that

Theorem 1.3. *Let $r_n \to \infty$ be a positive sequence and $\rho_n = e^{-r_n}$, there is a canonical isometry*

$$\Phi : \mathbb{H}_{Berk}({}^\rho\mathbb{C}) \longrightarrow {}^r\mathbb{H}^3$$

If $\mathbf{f} \in \mathrm{Rat}_d({}^\rho\mathbb{C})$, the limit of the associated sequence of barycentric extensions satisfies

$$\Phi \circ \mathscr{E}_{Berk}(\mathbf{f}) = \mathscr{E}_{bc}(f_n) \circ \Phi$$

Conversely, if $f_n \in \mathrm{Rat}_d(\mathbb{C})$ with $r_n = \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$, then the associated rational map $\mathbf{f} \in \mathrm{Rat}_d({}^\rho\mathbb{C})$ and

$$\Phi \circ \mathscr{E}_{Berk}(\mathbf{f}) = \mathscr{E}_{bc}(f_n) \circ \Phi$$

Let $T \subset {}^r\mathbb{H}^3$ be the minimal tree containing x^0, J_{Berk} be the Berkovich Julia set in $\mathbb{P}^1_{Berk}({}^\rho\mathbb{C})$, then $J_{Berk} \subset \Phi^{-1}(T \cup \epsilon(T))$.

A version for holomorphic families

Let f_t is a holomorphic family of rational maps of degree $d > 1$ defined over the punctured unit disk $\Delta^* = \{t \in \mathbb{C} : 0 < |t| < 1\}$. We also assume that all the coefficients of f_t extend to meromorphic functions on the unit disk Δ. We may also view $\mathbf{f} = f_t$ as a rational map with coefficients in the field of formal Puiseux series $\mathbb{L}$. The field $\mathbb{L}$ embeds into ${}^{\rho}\mathbb{C}$, and this allows us to show

Theorem 1.4. *Let $\rho_n \to 0$ and $r_n = |\log|\rho_n||$, there is an isometric embedding*

$$\Phi : \mathbb{H}_{Berk}(\mathbb{L}) \hookrightarrow {}^{r}\mathbb{H}^3$$

Moreover, if $\mathbf{f} = f_t$ is a holomorphic family of rational maps of degree $d > 1$ defined over Δ^*, the limit of the associated sequence of barycentric extensions $\mathscr{E}_{bc}(f_{\rho_n}) : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ satisfies

$$\Phi \circ \mathscr{E}_{Berk}(\mathbf{f}) = \mathscr{E}_{bc}(f_{\rho_n}) \circ \Phi$$

1.3 The minimal trees and Blaschke products

It is natural to consider the *minimal tree* $T \subset {}^{r}\mathbb{H}^3$ containing x^0, that is, the smallest closed tree which is invariant under $\mathscr{E}_{bc}(f_n)$ and $\mathscr{E}_{bc}(f_n)^{-1}$ that contains x^0. We say the sequence $f_n : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ converges *geometrically* to $F : T \longrightarrow T$ if there exists $h_n : (T, x^0) \longrightarrow (\mathbb{H}^3, \mathbf{0})$ such that

1. *Rescaling:* We have
$$d(x, y) = \lim d_{\mathbb{H}^3}(h_n(x), h_n(y))/r_n$$
for all $x, y \in T$.

2. *Conjugacy:* We have for all $x \in T$,
$$d_{\mathbb{H}^3}(h_n(F(x)), f_n(h_n(x)))/r_n \to 0$$
as $n \to \infty$.

In Section 3.10, we will show that

Theorem 1.5. *Let f_n be a degenerating sequence rational maps of degree d, $r_n := \max_{y \in \mathscr{E}f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$ and ${}^{r}\mathbb{H}^3$ be the asymptotic cone of $\mathbb{H}^3$ with rescaling r_n. Let T be the minimal tree containing x^0, then* *possibly passing to a subsequence, the minimal tree T is the union of Gromov-Hausdorff limit of convex hulls of*

$$\bigcup_{i=-k}^{k} \mathscr{E} f_n^i(\{\mathbf{0}\})$$

rescaled by r_n and $\mathscr{E} f_n$ converges geometrically to $\mathscr{E}_{bc}(f_n)$ on T.

In [McM09b], McMullen constructed a branched covering on a Ribbon $\mathbb{R}$-tree as geometric limits of divergent sequence Blaschke products. The tree (T_{Ribbon}, p) is the union of Gromov-Hausdorff limit of a sequence of hyperbolic polygons in $\Delta \cong \mathbb{H}^2$, namely the convex hulls of finite sets of the form

$$\bigcup_{i=-k}^{k} f_n^i(\{0\}).$$

The hyperbolic plane $\mathbb{H}^2$ bounded by the equator in the ball model of $\mathbb{H}^3$ is totally invariant under the barycentric extension $\mathscr{E} f$ of a Blaschke product f. In Section 3.11, we will prove

Theorem 1.6. *Let $f_n(z) = z \prod_{i=1}^{d-1} \frac{z-a_{i,n}}{1-\overline{a}_{i,n} z}$ be a degenerating sequence of Blaschke products which converges to a branched covering $f_{Rib} : (T_{Rib}, p) \longrightarrow (T_{Rib}, p)$ in the sense as in [McM09b]. Let $r_n := \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$ and ${}^r\mathbb{H}^3$ be the asymptotic cone of $\mathbb{H}^3$ with rescaling r_n. Let T be the minimal tree containing x^0, then*

$$\mathscr{E}_{bc}(f_n) : (T, x^0) \longrightarrow (T, x^0)$$

is isometrically conjugate to

$$f_{Rib} : (T_{Rib}, p) \longrightarrow (T_{Rib}, p).$$

The ends of a tree and translation lengths

A ray α in the $\mathbb{R}$-tree T is a subtree isometric to $[0, \infty) \subset \mathbb{R}$. Two rays are *equivalent* if $\alpha_1 \cap \alpha_2$ is still a ray. The collection $\epsilon(T)$ of all equivalence classes of rays forms the set of *ends* of T. We will let α denote both a ray and the end it represents. We also say a sequence of points x_i converges to an end $\alpha \subset T$, denoted by $x_i \to \alpha$, if $d(x_1, x_i) \to \infty$ and $x_i \in \alpha$ for all sufficiently large i.

For the asymptotic cone ${}^r\mathbb{H}^3$ of $\mathbb{H}^3$, an end $\alpha \in \epsilon({}^r\mathbb{H}^3)$ can be represented by a sequence of points in the conformal boundary $x_n \in \mathbb{P}^1_{\mathbb{C}} \cong S^2$. The limiting map $\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$ determines a degree d map on the the set of ends $\epsilon({}^r\mathbb{H}^3)$. We define the *translation length* of an end α by

$$L(\alpha, \mathscr{E}_{bc}(f_n)) = \lim_{x_i \to \alpha} d(x_i, x^0) - d(\mathscr{E}_{bc}(f_n)(x_i), x^0)$$

We will show the translation length is well-defined and $< +\infty$ (possibly equals to $-\infty$). If $C = \{\alpha_1, ..., \alpha_q\}$ is a periodic cycle of ends, we define the *translation length* of the periodic cycle C by

$$L(C, \mathscr{E}_{bc}(f_n)) = \sum_{i=1}^{q} L(\alpha_i, \mathscr{E}_{bc}(f_n))$$

Theorem 1.7. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$ be a degenerating sequence,*

$$r_n := \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$$

and ${}^r\mathbb{H}^3$ be the asymptotic cone of $\mathbb{H}^3$ with rescaling r_n.

If $C_n = \{z_{1,n}, ..., z_{q,n}\} \subset \mathbb{P}^1_{\mathbb{C}}$ is a sequence of periodic cycles of f_n, then there exists a periodic cycle of ends $C = \{\alpha_1, ..., \alpha_q\} \subset \epsilon({}^r\mathbb{H}^3)$ so that after passing to a subsequence

$$\lim_{n \to \infty} \frac{\log |(f_n^q)'(z_{1,n})|}{r_n} = L(C, \mathscr{E}_{bc}(f_n))$$

1.4 Applications

Let $[f] \in M_d$, we define

$$r([f]) := \min_{x \in \mathbb{H}^3} \max_{y \in \mathscr{E} f_n^{-1}(x)} d_{\mathbb{H}^3}(y, x)$$

where f is a representative rational map of $[f]$. We will show that a sequence $[f_n]$ is degenerating as conjugacy classes if and only if $r([f_n]) \to \infty$. We can choose representative f_n for each each conjugacy class $[f_n]$ so that

$$r([f_n]) = \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0}).$$

This allows us to construct a limiting dynamics $\mathscr{E}_{bc}(f_n)$ on the $\mathbb{R}$-tree for the sequence of conjugacy classes $[f_n]$. In the following, we will use the limiting dynamics to study hyperbolic components.

Marking on the Julia set and Length Spectra

A conjugacy class of rational map $[f]$ is called *hyperbolic* if the orbit of every critical point converges to some attracting periodic cycle. The space of hyperbolic rational maps is open in M_d, and a connected component of it is called a *hyperbolic component*. For each hyperbolic component H, there is a topological dynamical system

$$\sigma : J \longrightarrow J$$

such that for any $[f] \in H$, there is a homeomorphism

$$\phi(f) : J \longrightarrow J(f)$$

which conjugates σ and f. A particular choice of such $\phi(f)$ will be called a *marking* of the Julia set.

Let $[f] \in H \subset M_d$ be a hyperbolic rational map with a marking $\phi : J \longrightarrow J(f)$. We let S be the space of periodic cycles of the topological model $\sigma : J \longrightarrow J$. We define the length on $[f]$ of a periodic cycle

$C \in S$ by

$$L(C, [f]) = \log |(f^q)'(z)|,$$

where $q = |C|$ and $z \in \phi(C)$. The collection $(L(C, [f]) : C \in S) \in \mathbb{R}_+^S$ will be called the *marked length spectrum* of $[f]$. As $[f]$ varies over the hyperbolic component, we are interested in understanding how the length spectrum changes. In particular, we will investigate the behavior of the length spectrum for a degenerating sequence $[f_n]$.

Bounded escape and nested Julia sets

Let $[f_n] \in H$ be a sequence of conjugacy classes of marked rational maps in H. One way for $[f_n]$ to be degenerating is that the length $L(C, [f_n])$ is going to infinity for some periodic cycle C. In fact, for Kleinian groups, this is the only way to get degeneracy: a sequence of representation of a finitely generated group in $\mathrm{PSL}_2(\mathbb{C})$ is degenerating if and only if the lengths of some closed geodesic are going to infinity. As we shall see, in the rational map setting, it is possible that $L(C, [f_n])$ stays bounded for every $C \in S$. Hence, we define

Definition 1.8. *Let H be a hyperbolic component, we say H admits* bounded escape *if there exists a sequence $[f_n] \in H$ (with a marking ϕ_n) so that*

1. *$[f_n]$ is degenerating;*
2. *For any periodic cycle $C \in S$ of the topological model $\sigma : J \longrightarrow J$, the sequence of lengths $L(C, [f_n])$ is bounded.*

Since there are only finitely many periodic points of a fixed period, we can formulate the definition without using the markings and replace the second condition by

2’ For any $p \in \mathbb{N}$ and any sequence of periodic points x_n of f_n with period p, the multipliers of f_n at x_n stay bounded.

The sequence $f_n(z) = z^2 + \frac{1}{nz^3}$ provides such an example (or more precisely, the subsequence of conjugacy classes of this sequence with $n \geq N$ for some large N). The Julia set J for this hyperbolic component is homeomorphic to a Cantor set of circles [McM88]. In particular, any Julia component separates the two points $0, \infty$, and the Julia set is disconnected. We will show these two characteristics actually classify all examples of hyperbolic components admitting bounded escape:

Definition 1.9. *Let $f \in \mathrm{Rat}_d(\mathbb{C})$ be a hyperbolic rational map, we say $J(f)$ is* nested *if*

1. *There are two points $p_1, p_2 \in \mathbb{P}^1_{\mathbb{C}}$ such that any component of $J(f)$ separates p_1 and p_2;*
2. *$J(f)$ contains more than one component.*

A hyperbolic component H is said to have nested Julia set if the Julia set of any rational map in H is nested.

Theorem 1.10. *Let H be a hyperbolic component. H admits bounded escape if and only if H has nested Julia set.*

In Chapter 5, we will see that any example of rational maps of nested Julia set can be essentially built from 2 hyperbolic polynomials via a ‘nested mating’ procedure. Using this, one can prove the direction nested Julia set implies bounded escape.

To prove the other direction, we get a degenerating sequence of rational maps with bounded multipliers from the bounded escape condition. This gives a limiting dynamics on an $\mathbb{R}$-tree with no repelling periodic ends. We will classify these dynamics in Chapter 5, and use our classification to conclude the topological properties of the Julia set. In the course of the proof, we will also show

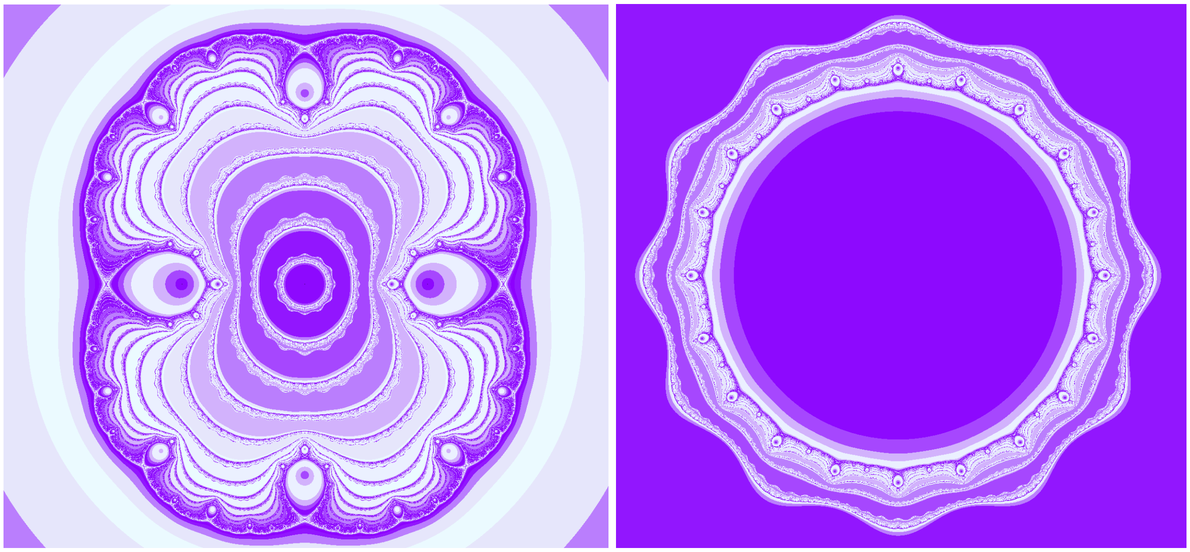

Figure 1.1: The Julia set of $z^2/(1-z^2) + p/z^{10}$ with $p = 10^{-7}$ on the left, and a zoom of the Julia set near 0 on the right. The Julia set is a Cantor set of closed curves. Any 'buried' closed curve is a circle. Any boundary component of the 'gaps' is a covering of the Julia set of z^2-1 (which is conjugate via $z \mapsto 1/z$ to $z^2/(1-z^2)$).

Theorem 1.11. *Let H be a hyperbolic component which does not have nested Julia set, and $[f_n] \in H$ be a degenerating sequence with markings. Let $r([f_n]) := \min_{\boldsymbol{x}\in\mathbb{H}^3} \max_{y\in\mathscr{E} f_n^{-1}(\boldsymbol{x})} d_{\mathbb{H}^3}(y, x)$, then after passing to a subsequence, we have*

$$L(C, \mathscr{E}_{bc}(f_n)) = \lim_{n\to\infty} L(C, [f_n])/r([f_n])$$

and there exists some $C \in S$ with $L(C, \mathscr{E}_{bc}(f_n)) \neq 0$.

In other words, if H does not have nested Julia set and $[f_n]$ is degenerating in H, then any periodic cycles escape to infinity at most comparable to $r([f_n])$, and there exist some periodic cycles escaping to infinity comparable to $r([f_n])$.

We end our discussion with the following open question:

Question. *Is the length spectrum bounded throughout a hyperbolic component with nested Julia set?*

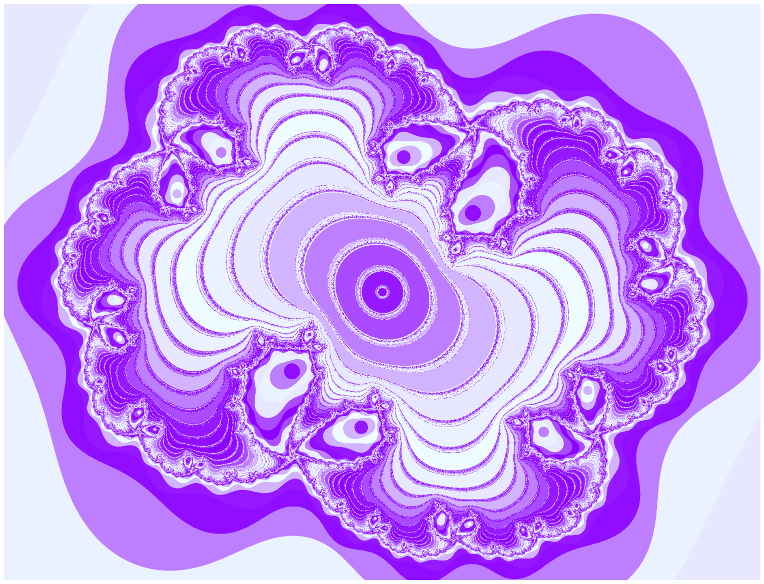

Figure 1.2: The Julia set of $z^2/(1+cz^2)+p/z^{10}$ with $p=10^{-7}$ and c in the 'rabbit' component of the Mandelbrot set. Each Julia component is either a circle or a covering of the Julia set of the quadratic polynomial z^2+c.

This question is related to the conjecture the hyperbolic components with Sierpinski carpet Julia set is bounded. See Chapter 5 for more discussions on this.

1.5 Notes and references

The use of Berkovich space of formal Puiseux series and 'rescalings' to understand asymptotic behaviors for a degenerating sequence of rational maps was introduced and made precise in [Kiw15]. Similar ideas have been also explored in [Sti93], [Eps00], [DeM07] and [Arf17]. Other application of trees in complex dynamics can be found in [Shi89] and [DM08].

Barycentric extension for rational maps has also been studied in [Pet11]. The construction of limiting branched coverings on an $\mathbb{R}$-tree suggests that, at least in the asymptotic sense, the barycentric extension of a rational map plays similar role as a hyperbolic 3-manifold to a Kleinian group. For other 3-

dimensional objects associated to a rational map, see [LM97]. Other applications of barycentric extensions in negatively curved Riemannian manifold can be found in [**?**] [BCG96].

R-trees also arise naturally in the study of degenerating sequences of representations of $\mathrm{Isom}(\mathbb{H}^n)$: in [MS84], based on the study of valuations on the function field of the character variety of representations of a finitely generated group G into $\mathrm{SL}_2(\mathbb{C})$, Morgan and Shalen showed how to assign an isometric action of a surface group on a $\mathbb{R}$-tree to such a sequence of representations that 'degenerate'. Bestvina and Paulin gave a new, and more geometric point of view of this theory in [Bes88] and [Pau88]. The equivalence of various constructions in Kleinian groups is analogous to the connections between the barycentric construction and Berkovich constructions for rational maps explained in Theorem 1.3. The use of asymptotic cone and the connection of $\mathbb{R}$-trees with the nonstandard analysis are developed and explained in [KL95] and [Chi91].

Rational maps with disconnected Julia sets are studied in details in [PT00], and some examples of rational maps with nested Julia sets also appear there. Many examples of degenerating families of rational maps with bounded multipliers are studied in [FRL10] using Berkovich dynamics. The geometric limit as a branched covering on an $\mathbb{R}$-tree for a sequence of degenerating Blaschke products is first introduced [McM09b], where the boundary of B_2 under this compactification is also studied there. We also refer to [McM08] [McM09a] [McM10] for more comparisons between the space of Blaschke products and the Teichmüller spaces.

2

Barycentric extensions

2.1 The definition of the barycentric extension

The theory of Barycentric extension was extensively studied for circle homeomorphisms in [DE86]. The construction can be easily generalized to any continuous maps on sphere S^{n-1}, (see [McM96][Pet11]).

Given a probability measure μ on S^{n-1} with no atoms of mass $\geq 1/2$, then there is a unique point $\beta(\mu) \in \mathbb{H}^n$ called the *barycenter* of μ for which the measure is *balanced* (see [DE86], [Hub06] or [Pet11] for a proof). A measure is said to be *balanced* at a point x if one moves x to the origin in the ball model of hyperbolic space using isometry, the push forward of the measure has Euclidean barycenter at the origin.

We fix a ball model of the hyperbolic space $\mathbb{H}^n$, in other words, we choose a base point, which we call it $\mathbf{0}$ and a base frame at $\mathbf{0}$. Let $\mu_{S^{n-1}}$ be the probability measure coming from the spherical metric on S^{n-1}, and we say a map is *admissible* if $f_*\mu_{S^{n-1}}$ has no atoms.

Let $f : S^{n-1} \longrightarrow S^{n-1}$ be an admissible continuous map, then *the barycentric extension* $\mathscr{E}f$ is a map from $\mathbb{H}^n \longrightarrow \mathbb{H}^n$ which sends the point $\boldsymbol{x} \in \mathbb{H}^n$ to the barycenter of the measure $f_*(M_{\boldsymbol{x}})_*(\mu_{S^{n-1}})$, where $M_{\boldsymbol{x}}$ is any isometry sending the origin $\mathbf{0}$ of the ball model to $\boldsymbol{x}$.

The extension is conformally natural in the sense if $M_1, M_2 \in \operatorname{Isom}\mathbb{H}^n$, then they are conformal maps of the conformal boundary S^{n-1}, and the extension satisfies

$$M_1 \circ \mathscr{E}(f) \circ M_2 = \mathscr{E}(M_1 \circ f \circ M_2)$$

Computing the derivatives

Given a point $\boldsymbol{x} \in \mathbb{H}^n \cong B(0,1) \in \mathbb{R}^n$, the map

$$M_{\boldsymbol{x}}(\boldsymbol{y}) = \frac{\boldsymbol{y}(1-|\boldsymbol{x}|^2) + \boldsymbol{x}(1+|\boldsymbol{y}|^2 + 2 < \boldsymbol{x}, \boldsymbol{y} >)}{1 + |\boldsymbol{x}|^2|\boldsymbol{y}|^2 + 2 < \boldsymbol{x}, \boldsymbol{y} >}$$

is an isometry sending the origin $\mathbf{0}$ to $\boldsymbol{x}$. In fact, this is the unique isometry preserving the frame along the geodesic connecting $\mathbf{0}$ to $\boldsymbol{x}$ with respect to the Levi-Civita connection. Note that $M_{\boldsymbol{x}}^{-1} = M_{-\boldsymbol{x}}$. Restricting $M_{\boldsymbol{x}}$ to S^{n-1}, an easy computation shows that the Jacobian

$$\text{Jac} M_{\boldsymbol{x}}(\boldsymbol{\zeta}) = (\frac{1-|\boldsymbol{x}|^2}{|\boldsymbol{\zeta}+\boldsymbol{x}|^2})^{n-1}$$

with $\boldsymbol{\zeta} \in S^{n-1}$.

Let $F(\boldsymbol{x}, \boldsymbol{y})$ denote the function

$$\begin{aligned} F(\boldsymbol{x}, \boldsymbol{y}) &= \int_{S^{n-1}} M_{\boldsymbol{y}}^{-1}(f(M_{\boldsymbol{x}}(\boldsymbol{\zeta})))d\mu_{S^{n-1}}(\boldsymbol{\zeta}) \\ &= \int_{S^{n-1}} M_{-\boldsymbol{y}}(f(\boldsymbol{\zeta}))(M_{\boldsymbol{x}})_* d\mu_{S^{n-1}}(\boldsymbol{\zeta}) \\ &= \int_{S^{n-1}} M_{-\boldsymbol{y}}(f(\boldsymbol{\zeta}))(\frac{1-|\boldsymbol{x}|^2}{|\boldsymbol{\zeta}-\boldsymbol{x}|^2})^{n-1} d\mu_{S^{n-1}}(\boldsymbol{\zeta}) \end{aligned}$$

With the notations as above, the barycentric extension of f is the unique solution of the implicit formula: $F(\boldsymbol{x}, \mathscr{E} f(\boldsymbol{x})) = \vec{0}$.

With this formula, implicit differentiation allows us to compute the derivative of the extension:

$$\mathrm{D}\,\mathscr{E}(f)(\boldsymbol{x}) = -F_{\boldsymbol{y}}^{-1}(\boldsymbol{x}, \mathscr{E}(f)(\boldsymbol{x}))F_{\boldsymbol{x}}(\boldsymbol{x}, \mathscr{E}(f)(\boldsymbol{x}))$$

We can compute this derivative very explicitly. Since $\mathrm{Isom}^+(\mathbb{H}^n) \times \mathrm{Isom}^+(\mathbb{H}^n)$ acts transitively on pairs of points in $\mathbb{H}^n$, we can assume that $\mathscr{E}(f)(\mathbf{0}) = \mathbf{0}$, i.e.,

$$\int_{S^{n-1}} f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta}) = \vec{0} \tag{2.1.1}$$

and compute the derivative at the origin $\mathbf{0}$.

Proposition 2.1. *Assume that* $\int_{S^{n-1}} f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta}) = \vec{0}$*, then we have*

$$F_{\boldsymbol{y}}(\vec{0}, \vec{0})(\vec{v}) = -2\vec{v} + 2\int_{S^{n-1}} < \vec{v}, f(\boldsymbol{\zeta}) > f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta})$$

Similarly,

$$F_{\boldsymbol{x}}(\vec{0}, \vec{0})(\vec{v}) = 2(n-1)\int_{S^{n-1}} < \vec{v}, \boldsymbol{\zeta} > f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta})$$

Proof. For the first equality, we have

$$
\begin{aligned}
F_{\boldsymbol{y}}(\vec{0},\vec{0})(\vec{v}) &= \lim_{t\to 0}\frac{F(\vec{0},t\vec{v})}{t}\\
&= \lim_{t\to 0}\frac{\int_{S^{n-1}} M_{-t\vec{v}}(f(\boldsymbol{\zeta}))d\mu_{S^{n-1}}(\boldsymbol{\zeta})}{t}\\
&= \lim_{t\to 0}\int_{S^{n-1}}\frac{1}{t}\cdot\frac{f(\boldsymbol{\zeta})(1-t^2|\vec{v}|^2)-t\vec{v}(1+|f(\boldsymbol{\zeta})|^2-2<t\vec{v},f(\boldsymbol{\zeta})>)}{1+t^2|\vec{v}|^2|f(\boldsymbol{\zeta})|^2-2t<\vec{v},f(\boldsymbol{\zeta})>}d\mu_{S^{n-1}}(\boldsymbol{\zeta})\\
&= \lim_{t\to 0}\int_{S^{n-1}}\frac{1}{t}\cdot(f(\boldsymbol{\zeta})-2t\vec{v}+O(t^2))(1+2t<\vec{v},f(\boldsymbol{\zeta})>+O(t^2))d\mu_{S^{n-1}}(\boldsymbol{\zeta})\\
&= -2\vec{v}+2\int_{S^{n-1}}<\vec{v},f(\boldsymbol{\zeta})>f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta})
\end{aligned}
$$

Similarly, for the second equality, we have

$$
\begin{aligned}
F_{\boldsymbol{x}}(\vec{0},\vec{0})(\vec{v}) &= \lim_{t\to 0}\frac{F(t\vec{v},\vec{0})}{t}\\
&= \lim_{t\to 0}\frac{\int_{S^{n-1}} f(\boldsymbol{\zeta})(\frac{1-|t\vec{v}|^2}{|\boldsymbol{\zeta}-t\vec{v}|^2})^{n-1}d\mu_{S^{n-1}}(\boldsymbol{\zeta})}{t}\\
&= \lim_{t\to 0}\int_{S^{n-1}}\frac{1}{t}\cdot f(\boldsymbol{\zeta})(1+t^2|\vec{v}|^2)^{n-1}(1-2t<\vec{v},\boldsymbol{\zeta}>+t^2|\vec{v}|^2)^{1-n}d\mu_{S^{n-1}}(\boldsymbol{\zeta})\\
&= \lim_{t\to 0}\frac{1}{t}\cdot f(\boldsymbol{\zeta})(1+O(t^2))(1+2(n-1)t<\vec{v},\boldsymbol{\zeta}>+O(t^2))d\mu_{S^{n-1}}(\boldsymbol{\zeta})\\
&= 2(n-1)\int_{S^{n-1}}<\vec{v},\boldsymbol{\zeta}>f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta})
\end{aligned}
$$

□

In order to bound the norm of the derivative, it is now sufficient to bound $F_{\boldsymbol{x}}$ from above and $F_{\boldsymbol{y}}$ from below. Since $f(\boldsymbol{\zeta}),\boldsymbol{\zeta}$ has norm 1, and $\mu_{S^{n-1}}$ is a probability measure, it is easy to see that $||F_{\boldsymbol{x}}||\leq 2(n-1)$.

Note that $F_{\boldsymbol{y}}$ is a self-adjoint operator, to bound the norm $F_{\boldsymbol{y}}^{-1}$, it is sufficient to bound the eigenvalues of $F_{\boldsymbol{y}}$ from below, or equivalently, bound the eigenvalues of $\int_{S^{n-1}}<\vec{v},f(\boldsymbol{\zeta})>f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta})$ away from 1.

Note that the quantity $\int_{S^{n-1}} < \vec{v}, f(\zeta) > f(\zeta) d\mu_{S^{n-1}}(\zeta)$ can be thought of the second moment of the function $f(\zeta)$. Hence, the bound we are going to get can be interpreted as bounding the second moment of $f(\zeta)$ under the condition that f is *balanced*, i.e., $\int_{S^{n-1}} f(\zeta) d\mu_{S^{n-1}}(\zeta) = \vec{0}$.

2.2 Quasiregular maps

We shall follow [Ric93] as our main reference for quasiregular maps. Roughly speaking, quasiregular maps are natural extensions of the concept of quasiconformal maps, similar to holomorphic maps to conformal maps, i.e., away from critical locus it is locally quasiconformal.

Let U be a domain in $\mathbb{R}^n$, and $W_n^1(U)$ be the Soblev space, consisting of all real valued functions $u \in L^n(U)$ with weak first order partial derivatives which are themselves in $L^n(U)$. By $W^1_{n,loc}(U)$ we denote functions which locally belong to W_n^1. By considering component functions, we can extend these definitions to $\mathbb{R}^m$-valued mappings without separate notation.

Definition 2.2. *A mapping $f : U \longrightarrow \mathbb{R}^n$ of a domain $U \subset \mathbb{R}^n$ is* quasiregular *if*

1. $f \in C^0(U) \cap W^1_{n,loc}(U)$
2. *there exists K, $1 \leq K < \infty$ such that*

$$|f'(x)|^n \leq K \operatorname{Jac} f(x) \textit{ a.e.}$$

The smallest K is called the *outer dilatation* $K_O(f)$ of f. If f is quasiregular, then it is also true that

$$J_f(x) \leq K' l(f'(x))^n \text{ a.e.}$$

for some $K' \geq 1$ where $l(f'(x)) := \inf_{|h|=1} |f'(x)h|$. The smallest K' is called the *inner dilatation* $K_I(f)$ of f. The maximal of K_O and K_I is called the *dilatation* of f, and denoted by $K(f)$. A quasiregular map is called K-quasiregular if $K(f) \leq K$. In dimension 2, the inner dilatation coincides with the outer dilatation.

A quasiregular map shares many nice properties with holomorphic functions. A non-constant quasiregular map is discrete and open, i.e., the preimage of a point is discrete, and the image of open set is open. This allows us to define the *local degree* $i(f, x)$ at a point x by considering the induced action on the homology. A non-constant quasiregular map always has positive local degrees. A domain $D \subset U$ is called *normal* if $f(\partial D) = \partial f(D)$.

Similar to the study of quasiconformal and conformal maps in dimension 2, the moduli of curve systems play an essential role in controlling the geometry of the image of quasiregular maps. For our purpose, we will be considering an equivalent formulation of *capacity* of a *condenser*.

Definition 2.3. *A* condenser *in $\mathbb{R}^n$ is a pair $E = (A, C)$ where A is open in $\mathbb{R}^n$ and $C \neq \emptyset$ is a compact subset of A.*

The (conformal) capacity *of E is defined by*

$$\operatorname{cap} E := \inf_u \int_A |\nabla u|^n dm$$

where the infimum is taken over all nonnegative functions $u \in C^0(A) \cap W^1_{n,loc}(A)$ with compact support and $u|_C \geq 1$.

It is not hard to see that we can restrict to $u \in C_c^\infty(A)$ or u as piecewise linear maps in the definition without changing the value (see Section 5 and 7 in [Geh61]).

Let $E = (A, C)$ be a condenser with A bounded. We say a closed set σ is *separating* if $\sigma \subset A - C$ and if there are two open sets U_1 and U_2 of $\overline{A}$ with $C \subset U_1$, $\partial A \subset U_2$ and $\overline{A} - \Sigma = U_1 \cup U_2$.

Definition 2.4. *Let Σ be the set of all separating sets of a bounded condenser $E = (A, C)$ in $\mathbb{R}^n$, the* modulus of separating sets *is defined by*

$$M(\Sigma) := \inf_{f \wedge \Sigma} \int_{\mathbb{R}^n} f^{\frac{n}{n-1}} dm$$

where $f \wedge \Sigma$ means f is a nonnegative Borel function on $\mathbb{R}^n$ such that

$$\int_{\sigma} f d\mathcal{H}^{n-1} \geq 1 \textit{ for all } \sigma \in \Sigma$$

where $d\mathcal{H}^{n-1}$ is the $n-1$-dimensional Hausdorff measure.

It can be seen that if we denote Σ' to be the set of separating with infinite $n-1$-dimensional Hausdorff measure, then $M(\Sigma') = 0$ (See Section 3.1 in[Zie67]).

We may also assume that the separating sets are piecewise linear without changing the value (See Section 3.3 in [Zie67] and Section 7 in [Geh61]).

We have the following theorem proved in [Zie67] [Geh62]

Theorem 2.5. *Let $E = (A, C)$ be a bounded condenser in $\mathbb{R}^n$ and Σ be the set of all separating sets, then $M(\Sigma) = \operatorname{cap}(E)^{-\frac{1}{n-1}}$.*

This theorem is more well-known in dimension 2. Assuming that A is a topological disk, and C is a closed disk in A, then $A - C$ is an open annulus. Let Γ_1 be the family of curves connecting two components of the boundary of $A - C$, and Γ_2 be the family of curves homotopic to the core of the annulus $A - C$. Then the capacity $\operatorname{cap}(E) = 1/\mathscr{L}(\Gamma_1)$, where $\mathscr{L}(\Gamma_1)$ is the extremal length of family Γ_1. And the modulus of separating sets $M(\Sigma) = \mathscr{L}(\Gamma_2)$. The theorem says that $1/\mathscr{L}(\Gamma_1) = \mathscr{L}(\Gamma_2)$.

Let $f : U \longrightarrow \mathbb{R}^n$ be a quasiregular map, and $E = (A, C)$ be a condenser. Since the map f is open, $f(E) = (f(A), f(C))$ is another condenser. The following two theorems allows us to control the geometry of the image of the condensers. The proofs can be found in [Ric93].

Recall that A is normal domain if $f(\partial A) = \partial f(A)$. Let

$$N(f, A) := \sup_y |f^{-1}(y) \cap A|$$

be the maximal number of preimages in A.

Theorem 2.6. *Let $f : U \longrightarrow \mathbb{R}^n$ be a non-constant quasiregular map and $E = (A, C)$ be a condenser in U with A normal and $N(f, A) < \infty$. Then*

$$\operatorname{cap}(E) \leq K_O(f) N(f, A) \operatorname{cap}(f(E))$$

Recall that $i(f, x)$ is the local degree at x. Let

$$M(f, C) := \inf_{y \in C} \sum_{x \in f^{-1}(y) \cap C} i(x, f)$$

be the minimal multiplicity of f on C.

Theorem 2.7. *Let $f : U \longrightarrow \mathbb{R}^n$ be a non-constant quasiregular map and $E = (A, C)$ be a condenser in U. Then*

$$\operatorname{cap}(f(E)) \leq \frac{K_I(f)}{M(f, C)} \operatorname{cap}(E)$$

2.3 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 by proving the following more general result which is interesting in its own right:

Theorem 2.8. *Let $f : S^{n-1} \longrightarrow S^{n-1}$ be a proper K-quasiregular map of degree d, then the barycentric* *extension of f*

$$\mathscr{E}f : \mathbb{H}^n \longrightarrow \mathbb{H}^n$$

is uniformly $C(n) \cdot (Kd)^{\frac{1}{n-2}}$-Lipschitz, where the constant $C(n)$ only depends only on n.

Note that Theorem 1.1 follows from Theorem 2.8 by setting $K = 1$ and $n = 3$.

Recall that

$$F(\boldsymbol{x}, \boldsymbol{y}) = \int_{S^{n-1}} M_{\boldsymbol{y}}^{-1}(f(M_{\boldsymbol{x}}(\boldsymbol{\zeta})))d\mu_{S^{n-1}}(\boldsymbol{\zeta})$$

We will first use the theory developed in the previous sections to prove the following proposition which controls the norm of $||F_{\boldsymbol{y}}(\vec{0}, \vec{0})^{-1}||$ under the balanced condition 2.1.1.

Proposition 2.9. *Let $f : S^{n-1} \longrightarrow S^{n-1}$ be a proper K-quasiregular map of degree d such that*

$$\int_{S^{n-1}} f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta}) = \vec{0}.$$

then

$$||F_{\boldsymbol{y}}(\vec{0}, \vec{0})^{-1}|| \leq C(Kd)^{\frac{1}{n-2}} \tag{2.3.1}$$

for some constant $C = C(n)$ only depends on the dimension n.

Proof. Recall from the computation in Section 2.1, we have

$$F_{\boldsymbol{y}}(\vec{0}, \vec{0})(\vec{v}) = -2\vec{v} + 2\int_{S^{n-1}} < \vec{v}, f(\boldsymbol{\zeta}) > f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta})$$

Hence, we need to bound the norm of the linear operator

$$T : \vec{v} \to \int_{S^{n-1}} < \vec{v}, f(\boldsymbol{\zeta}) > f(\boldsymbol{\zeta})d\mu_{S^{n-1}}(\boldsymbol{\zeta})$$

away from 1 subject to the balanced condition.

Since the operator T is self-adjoint, after change of variable, we may assume the largest eigenvalue is associated to $\boldsymbol{e}_0 \in \mathbb{R}^n$.

Let $A' = B(0, \sqrt{3}) \in \mathbb{R}^{n-1}$ and $C' = \overline{B(0, \frac{1}{\sqrt{3}})} \subset A'$. Equivalently, $A' - C'$ is the image under stereographic projection of 'belt' $-\frac{1}{2} < e_0 < \frac{1}{2}$. Let $E = (A, C) = (f^{-1}(A'), f^{-1}(C'))$. We may assume that the preimage of A does not contain ∞ so E is a condenser in $\mathbb{R}^n$.

Recall that $\mu_{S^{n-1}}$ is the probability measure induced by the spherical metric on S^{n-1}. Let $V := \mu_{S^{n-1}}(A - C)$, $V_1 := \mu_{S^{n-1}}(C)$ and $V_2 := \mu_{S^{n-1}}(S^{n-1} - A)$ be the (normalized) spherical volumne of the $A - C$, C and $S^{n-1} - A$ (under the stereographic projection) respectively. Then there are two cases to consider:

1. $V \geq \frac{1}{3}$
2. $V < \frac{1}{3}$

Note that we have

$$\begin{aligned}
< \boldsymbol{e}_0, T(\boldsymbol{e}_0) > &=< \boldsymbol{e}_0, \int_{S^{n-1}} < \boldsymbol{e}_0, f(\boldsymbol{\zeta}) > f(\boldsymbol{\zeta}) d\mu_{S^{n-1}}(\boldsymbol{\zeta}) > \\
&=< \boldsymbol{e}_0, \int_{S^{n-1}} < \boldsymbol{e}_0, \boldsymbol{\zeta} > \boldsymbol{\zeta} f_* d\mu_{S^{n-1}}(\boldsymbol{\zeta}) > \\
&=< \boldsymbol{e}_0, \int_{A'-C'} < \boldsymbol{e}_0, \boldsymbol{\zeta} > \boldsymbol{\zeta} f_* d\mu_{S^{n-1}}(\boldsymbol{\zeta}) > + \\
&\quad + < \boldsymbol{e}_0, \int_{(S^{n-1}-A') \cup C'} < \boldsymbol{e}_0, \boldsymbol{\zeta} > \boldsymbol{\zeta} f_* d\mu_{S^{n-1}}(\boldsymbol{\zeta}) > \\
&\leq (\frac{1}{2})^2 \cdot V + 1 \cdot (1 - V) = 1 - \frac{3}{4} V \qquad (2.3.2)
\end{aligned}$$

Hence in the first case, we have

$$< \boldsymbol{e}_0, T(\boldsymbol{e}_0) > \leq \frac{3}{4}$$

In the second case, by the balanced condition, we have

$$\begin{aligned}
0 &=< \boldsymbol{e}_0, \int_{S^{n-1}} f(\boldsymbol{\zeta}) d\mu_{S^{n-1}}(\boldsymbol{\zeta}) > \\
&=< \boldsymbol{e}_0, \int_{S^{n-1}} \boldsymbol{\zeta} f_* d\mu_{S^{n-1}}(\boldsymbol{\zeta}) > \\
&=< \boldsymbol{e}_0, \int_{S^{n-1}-A'} \boldsymbol{\zeta} f_* d\mu_{S^{n-1}}(\boldsymbol{\zeta}) + \int_{A'-C'} \boldsymbol{\zeta} f_* d\mu_{S^{n-1}}(\boldsymbol{\zeta}) + \int_{C'} \boldsymbol{\zeta} f_* d\mu_{S^{n-1}}(\boldsymbol{\zeta}) > \\
&\leq 1 \cdot V_2 + \frac{1}{2} \cdot V + (-\frac{1}{2}) \cdot V_1 = V_2 + \frac{1}{2} V - \frac{1}{2}(1 - V_2 - V) \\
&= \frac{3}{2} V_2 + V - \frac{1}{2} \leq \frac{3}{2} V_2 - \frac{1}{6}
\end{aligned}$$

Hence $V_2 \geq \frac{1}{9}$, similarly, $V_1 \geq \frac{1}{9}$. By exchange the role of the set associated to V_1 and V_2, we may assume that $V_1 < 1/2$.

Note that $f(E)$ is the condenser $E' = (A', C')$, which has capacity

$$\operatorname{cap}(f(E)) = \frac{\omega_{n-2}}{(\log 3)^{n-2}}$$

where ω_{n-2} is the $n-2$-measure of the unit sphere S^{n-2}. Since f has degree d, we know $N(f, A) \leq d$. Hence by Theorem 2.6, we have

$$\operatorname{cap}(E) \leq \frac{\omega_{n-2} K d}{(\log 3)^{n-2}} \tag{2.3.3}$$

Let Σ be the set of separating sets associated to E, and $\Sigma' \subset \Sigma$ be the set of separating sets which is piecewise linear. Let $\phi^{n-1} dm$ be the push forward of the spherical measure under the stereographic projection. The isoperimetric inequality for sphere (Troisième Partie, Chapitre I in [Lév22] and [Sch39]) says that if $E \subset S^{n-1}$ is a measurable set, then

$$\mathbf{P}(E) \geq \mathbf{P}(B_\theta)$$

where B_θ is a geodesic ball of the same volume as E and $\mathbf{P}(E)$ is the perimeter of E (See [Mag12]). If the boundary of E is smooth enough, in particular, if ∂E is piecewise linear, then $\mathbf{P}(E) = \mathcal{H}^{n-2}_{S^{n-1}}(\partial E)$. Here $\mathcal{H}^{n-2}_{S^{n-1}}$ is the $n-2$ dimensional Hausdorff measure on the sphere.

Let $\sigma \in \Sigma'$ and let E denote the set bounded by σ containing C. Then $|E| \in (\frac{1}{9}\omega_{n-1}, \frac{8}{9}\omega_{n-1})$. Therefore, we have that

$$\int_\sigma \phi^{n-2} d\mathcal{H}^{n-2}_{\mathbb{R}^{n-1}} = \mathcal{H}^{n-2}_{S^{n-1}}(\sigma) \geq \Omega_{n-2}(\frac{1}{9}\omega_{n-1}).$$

Here $\Omega_{n-2}(V)$ denotes the $n-2$-measure of the boundary of a ball of volume V. Hence if we consider the function

$$g(x) := \frac{\phi^{n-2}}{\Omega_{n-2}(\frac{1}{9}\omega_{n-1})} \text{ for } x \in A - C$$

and $g(x) = 0$ elsewhere, then

$$\int_\sigma g d\mathcal{H}^{n-2}_{\mathbb{R}^{n-1}} \geq 1 \text{ for all } \sigma \in \Sigma'.$$

Hence we have

$$\begin{aligned}
M(\Sigma) &= \inf_{h \wedge \Sigma} \int_{\mathbb{R}^{n-1}} h^{\frac{n-1}{n-2}} dm \\
&= \inf_{h \wedge \Sigma'} \int_{\mathbb{R}^{n-1}} h^{\frac{n-1}{n-2}} dm \\
&\leq \int_{\mathbb{R}^{n-1}} g^{\frac{n-1}{n-2}} dm \\
&= (\frac{1}{\Omega_{n-2}(\frac{1}{9}\omega_{n-1})})^{\frac{n-1}{n-2}} \int_{A-C} \phi^{n-1} dm \\
&= (\frac{1}{\Omega_{n-2}(\frac{1}{9}\omega_{n-1})})^{\frac{n-1}{n-2}} V \cdot \omega_{n-1}
\end{aligned}$$

By Theorem 2.5, we have

$$M(\Sigma) = \text{cap}(E)^{-\frac{1}{n-2}}$$

so we have

$$\operatorname{cap}(E) = M(\Sigma)^{-(n-2)} \geq \frac{(\Omega_{n-2}(\frac{1}{9}\omega_{n-1}))^{n-1}}{V^{n-2}\omega_{n-1}^{n-2}} \tag{2.3.4}$$

Now combining inequality 2.3.3 and 2.3.4, we get

$$\frac{\omega_{n-2}Kd}{(\log 3)^{n-2}} \geq \frac{\Omega_{n-2}(\frac{1}{9}\omega_{n-1}))^{n-1}}{V^{n-2}\omega_{n-1}^{n-2}}$$

which gives

$$V \geq \frac{\log 3 \cdot (\Omega_{n-2}(\frac{1}{9}\omega_{n-1}))^{\frac{n-1}{n-2}}}{(Kd)^{\frac{1}{n-2}}\omega_{n-1}}$$

Now plug this in inequality 2.3.2, we get

$$< \boldsymbol{e}_0, T(\boldsymbol{e}_0) > \leq 1 - \frac{3 \cdot \log 3 \cdot (\Omega_{n-2}(\frac{1}{9}\omega_{n-1}))^{\frac{n-1}{n-2}}}{4 \cdot (Kd)^{\frac{1}{n-2}}\omega_{n-1}}$$

so we have

$$||T|| \leq \max(\frac{3}{4}, 1 - \frac{3 \cdot \log 3 \cdot (\Omega_{n-2}(\frac{1}{9}\omega_{n-1}))^{\frac{n-1}{n-2}}}{4 \cdot (Kd)^{\frac{1}{n-2}}\omega_{n-1}})$$

Since $F_{\boldsymbol{y}}(\vec{0},\vec{0}) = 2I - 2T$, we have

$$||F_{\boldsymbol{y}}(\vec{0},\vec{0})^{-1}|| \leq \max(2, \frac{2 \cdot (Kd)^{\frac{1}{n-2}}\omega_{n-1}}{3 \cdot \log 3 \cdot (\Omega_{n-2}(\frac{1}{9}\omega_{n-1}))^{\frac{n-1}{n-2}}}) \tag{2.3.5}$$

which proves the result. □

Proof of Theorem 2.8. We assume that $\mathscr{E}f(\mathbf{0}) = \mathbf{0}$ first.

Note that under this condition, we have the bound on

$$||F_{\boldsymbol{x}}(\vec{0},\vec{0})|| \leq 2(n-1),$$

combining this with the bound 2.3.1, we get

$$||\,\mathrm{D}\,\mathscr{E}f(\mathbf{0})|| \leq ||F_{\boldsymbol{x}}(\vec{0},\vec{0})||||F_{\boldsymbol{y}}(\vec{0},\vec{0})^{-1}|| \leq C(Kd)^{\frac{1}{n-2}}$$

Since $\mathrm{Isom}^+\,\mathbb{H}^n \times \mathrm{Isom}^+\,\mathbb{H}^n$ acts transitively on pairs of points in $\mathbb{H}^n$, and the extension is natural, we get the result. □

2.4 Asymptotic translation length

Let $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a rational map of degree d and $\boldsymbol{\zeta} \in S^2$. We define the *asymptotic translation length* of f at point $\boldsymbol{\zeta}$ to be

$$L(\boldsymbol{\zeta}) := \lim_{t\to\infty} \mathrm{d}_{\mathbb{H}^3}(\boldsymbol{x}_t, \mathbf{0}) - \mathrm{d}_{\mathbb{H}^3}(\mathscr{E}f(\boldsymbol{x}_t), \mathbf{0})$$

where **0** is the origin in the ball model of $\mathbb{H}^3$, and $\boldsymbol{x}_t$ is the point distance t away from **0** on the geodesic along the direction of $\boldsymbol{\zeta}$.

We have the following basic but important proposition:

Proposition 2.10. *Let $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a rational map of degree d and $\boldsymbol{\zeta} \in S^2$. The asymptotic translation length*

$$L(\boldsymbol{\zeta}) = \log|f'(\boldsymbol{\zeta})|_{S^2}$$

where $|f'(\boldsymbol{\zeta})|_{S^2}$ is the spherical derivative of f on the sphere S^2.

Proof. First we assume that $\boldsymbol{\zeta}$ is not a critical point of f. Under a change of variable by the action of $SO(3)$, we may assume that $\boldsymbol{\zeta} \in S^2$ is mapped to $0 \in \mathbb{C}$ under stereographic projection, and $f(0) = 0$. Then the isometry $M_t(z) = \frac{z}{e^t}$ sends **0** to $\boldsymbol{x}_t$. We denote $N_t^{-1}(z) = \frac{e^t z}{a}$ where a is the derivative f at 0, and consider the family

$$F_t(z) = N_t^{-1} \circ f \circ M_t$$

Note that F_t converges uniformly on compact subset of $\mathbb{C}$ to the identity map. Let U_t be the isometry such that

$$\mathscr{E}(F_t)(\mathbf{0}) = U_t(\mathbf{0})$$

Hence, we have the balanced condition:

$$\int_{S^2} U_t^{-1} \circ F_t(\boldsymbol{\xi}) d\mu(\boldsymbol{\xi}) = \vec{0}$$

If U_t converges to a constant map, then $(U_t^{-1} \circ F_t)_* d\mu$ converges weakly to a delta measure supported at the constant value. This will be a contradiction to the balanced condition.

Therefore, U_t converges to an isometry, which is necessarily the identity map by the balanced condition. Hence, by naturality of the extension we have

$$\mathscr{E} f(\boldsymbol{x}_t) = N_t \circ U_t(\mathbf{0})$$

Since $U_t \to id$, we have

$$\begin{aligned}
\mathrm{d}_{\mathbb{H}^3}(\mathscr{E} f(\boldsymbol{x}_t), \mathbf{0}) &= \mathrm{d}_{\mathbb{H}^3}(N_t \circ U_t(\mathbf{0}), \mathbf{0}) \\
&= \mathrm{d}_{\mathbb{H}^3}(U_t(\mathbf{0}), N_t^{-1}(\mathbf{0})) \\
&= \mathrm{d}_{\mathbb{H}^3}(\mathbf{0}, N_t^{-1}(\mathbf{0})) + o(1) \\
&= t - \log|a| + o(1)
\end{aligned}$$

This implies the asymptotic translation length is $\log|a| = \log|f'(\boldsymbol{\zeta})|_{S^2}$.

If $\boldsymbol{\zeta}$ is a critical point, similar argument shows that

$$\lim_{t\to\infty} \mathrm{d}_{\mathbb{H}^3}(\boldsymbol{x}_{t^n}, \boldsymbol{x}_0) - \mathrm{d}_{\mathbb{H}^3}(\mathscr{E}(f)(\boldsymbol{x}_t), \boldsymbol{x}_0)$$

exists and finite where n is the order of the critical point. So the asymptotic translation length is $-\infty$, which proves the proposition.

$\square$

2.5 Examples of degree 2 rational maps

In this section, we will consider some examples of the barycentric extensions of a rational map on S^2. We have not tried to state our result in the most general form, and for simplicity of the presentation, we will now restrict ourselves to the case of degree 2. In this case, $\operatorname{Isom}\mathbb{H}^3 \times \operatorname{Isom}\mathbb{H}^3$ acts transitively on $\mathrm{Rat}_2(\mathbb{C})$, so by naturality, there is only one map $f(z) = z^2$ to study.

Let (r, θ, h) be the cylindrical coordinate system for the hyperbolic 3-space $\mathbb{H}^3$. In the ball model, r and h represents the hyperbolic distance to z-axis and xy-plane, and θ is the angle to the positive x-axis projected to the xy-plane. We identify $\mathbb{P}^1(\mathbb{C})$ with the unit sphere in $\mathbb{R}^3$ via standard stereographic projection sending $0, 1, \infty$ to $(0,0,1), (1,0,0), (0,0,-1)$ respectively. In the coordinate (r, θ, h), the extension of $f(z) = z^2$ has the following form:

Proposition 2.11. *Let $f(z) = z^2$, then in the cylindrical coordinate system, the extension has the form*

$$\begin{aligned} \mathscr{E}f : \mathbb{H}^3 &\longrightarrow \mathbb{H}^3 \\ (r, \theta, h) &\mapsto (\log(\cosh(r)) - \delta(r), 2\theta, 2h) \end{aligned}$$

where $\delta(r) > 0$ when $r > 0$, and $\delta(r) \to 0$ as $r \to \infty$.

Note that in (r, θ) coordinate, z^2 has the form

$$(r, \theta) \mapsto (\log(\cosh(r)), 2\theta)$$

Since $\delta(r) > 0$, $\mathscr{E}f|_{\mathbb{H}^2_0}$ is not isometrically conjugate to $z^2|_{\mathbb{H}^2}$. Since $\operatorname{Isom}\mathbb{H}^3 \times \operatorname{Isom}\mathbb{H}^3$ acts transitively, we have the following immediate corollary, which answers the question asked by Petersen in [Pet11]:

Corollary 2.12. *Let $f(z)$ be a degree 2 Blaschke product, and $\mathbb{H}^2_0$ be the hyperbolic plane bounded by the invariant unit circle, then the barycentric extension preserves this hyperbolic plane $\mathscr{E}f : \mathbb{H}^2_0 \longrightarrow \mathbb{H}^2_0$, but $\mathscr{E}f|_{\mathbb{H}^2_0}$ is not isometrically conjugate to f.*

Proof. Since the extension is natural, and $e^{2t}f(z/e^t) = f(z)$, so $\mathscr{E}f$ sends the hyperbolic plane of $h = t$ to $h = 2t$. Hence, $\mathscr{E}f$ preserves the hyperbolic plane $\mathbb{H}^2_0$ of $h = 0$. Also by naturality, the restriction of $\mathscr{E}f$ on each hyperbolic plane $h = t$ to $h = 2t$ is the same as $\mathscr{E}f|_{\mathbb{H}^2_0}$. Hence, in order to prove the proposition, we only need to show

$$\begin{aligned} \mathscr{E}f|_{\mathbb{H}^2_0} : \mathbb{H}^2_0 &\longrightarrow \mathbb{H}^2_0 \\ (r,\theta) &\mapsto (\log(\cosh(r)) - \delta(r), 2\theta) \end{aligned}$$

Since $f(z) = e^{2\pi 2\theta i} f(z/e^{2\pi\theta i})$, by naturality, $\mathscr{E}f|_{\mathbb{H}^2_0}(r,\theta) = (g(r), 2\theta)$ for some function g. Hence, we only need to figure out the $\mathscr{E}f|_{\mathbb{H}^2_0}(r,0)$. To compute this, we need the following lemma:

Lemma 2.13. *Let $f_t(z) = z\frac{z-t}{1-tz}$ for $t \in (0,1)$, then there exists a positive function $\delta : (0,1) \longrightarrow \mathbb{R}_+$ such that $\mathscr{E}f(\mathbf{0}) = (-\delta(t), 0, 0)$ for all $t \in (0,1)$.*

Proof. Let

$$\begin{aligned} P : \mathbb{C} &\longrightarrow \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3 \\ z &\mapsto (\frac{2z}{1+|z|^2}, \frac{|z|^2-1}{|z|^2+1}) \end{aligned}$$

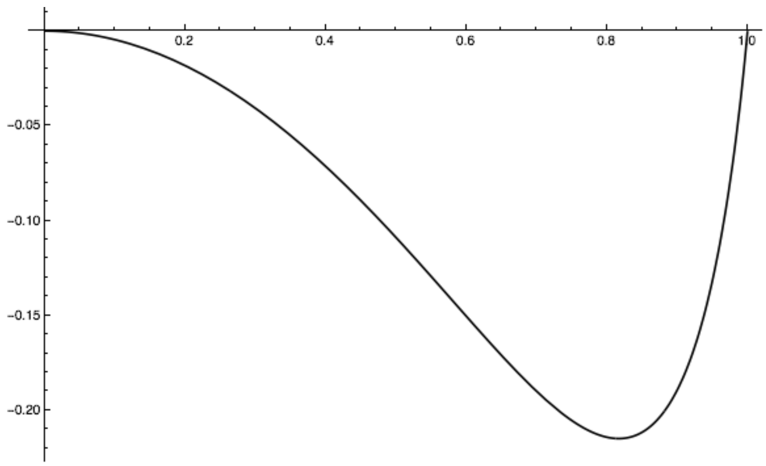

Figure 2.1: The graph of the function I_t

be the stereographic projection By definition, $\mathscr{E} f_t(\mathbf{0}) = \mathbf{0}$ if and only if

$$\int_{S^2} P(f_t(P^{-1}(\vec{x})))d\mu_{S^2}(\vec{x}) = \vec{0}$$

We will compute this integral and it is of the form $(-v, 0, 0)$. Note that by symmetry, the second component of the integral is always 0. Changing the variables to z, we get the first component of the integral equals to

$$I_t := \int_{\mathbb{C}} \frac{2z\frac{z-t}{1-tz}}{1+|z\frac{z-t}{1-tz}|^2} \frac{4}{(1+|z|^2)^2} \frac{i}{2}|dz|^2$$

Using a simple change of variables in polar coordinate, we may express the integral as

$$\begin{aligned} I_t &= \int_0^\infty \int_0^{2\pi} \frac{2z\frac{z-t}{1-tz}}{1+|z\frac{z-t}{1-tz}|^2} \frac{4}{(1+|z|^2)^2} r d\theta dr \\ &= \int_0^\infty (\int_{\partial B_r} \frac{2z\frac{z-t}{1-tz}}{1+|z\frac{z-t}{1-tz}|^2} \frac{4}{(1+|z|^2)^2} \frac{|z|dz}{iz}) dr \end{aligned}$$

where B_r is the disk centered at 0 of radius r. Let $J_{t,r}$ be the inner integral, we will show that $J_{t,r}$ is negative for all $r \neq 1$. Note that on ∂B_r, $\bar{z} = r^2/z$, so we have

$$
\begin{aligned}
J_{t,r} &= \int_{\partial B_r} \frac{2z\frac{z-t}{1-tz}}{1+|z\frac{z-t}{1-tz}|^2} \frac{4}{(1+|z|^2)^2} \frac{|z|dz}{iz} \\
&= \int_{\partial B_r} \frac{2z\frac{z-t}{1-tz}}{1+r^2\frac{(z-t)(\bar{z}-t)}{(1-tz)(1-t\bar{z})}} \frac{4}{(1+r^2)^2} \frac{rdz}{iz} \\
&= \frac{8r}{i(1+r^2)^2} \int_{\partial B_r} \frac{\frac{z-t}{1-tz}}{1+r^2\frac{(z-t)(r^2/z-t)}{(1-tz)(1-tr^2/z)}} dz \\
&= \frac{8r}{i(1+r^2)^2} \int_{\partial B_r} \frac{(z-t)(z-tr^2)}{(1-tz)(z-tr^2)+r^2(z-t)(r^2-tz)} dz \\
&= \frac{16r\pi}{(1+r^2)^2} \mathrm{Res}_{z\in B_r} \frac{(z-t)(z-tr^2)}{(1-tz)(z-tr^2)+r^2(z-t)(r^2-tz)}
\end{aligned}
$$

Let $F(z) = (z-t)(z-tr^2)$ and $G(z) = (1-tz)(z-tr^2)+r^2(z-t)(r^2-tz)$, note that $G(r) = r(1-tr)^2+r^3(r-t)^2 > 0$ as $t \in (0,1)$ and $r > 0$. Since the coefficient of z^2 in $G(z)$ is negative, there are two real roots x_1, x_2 with $x_1 < r < x_2$. Note that $G(t) = t(1-t^2)(1-r^2)$ and $G(tr^2) = tr^4(r^2-1)(1-t^2)$ have different sign when $r \neq 1$, and $G(\max\{t, tr^2\}) > 0$, so the smaller root x_1 is in between t and tr^2. Hence, $F(x_1) < 0$. Therefore, when $r \neq 1$

$$
\mathrm{Res}_{z\in B_r} \frac{(z-t)(z-tr^2)}{(1-tz)(z-tr^2)+r^2(z-t)(r^2-tz)} = \frac{F(x_1)}{-t(1+r^2)(x_1-x_2)} < 0
$$

Hence, $J_{t,r} < 0$ for all $r \neq 1$, so $I_t < 0$. So

$$
\int_{S^2} P(f_t(P^{-1}(\vec{x})))d\mu_{S^2}(\vec{x}) = (-v(t), 0, 0)
$$

for some positive function $v(t) > 0$. This means that the barycenter of $(f_t)_* d\mu_{S^2}$ is in the negative x-axis, so

$$
\mathscr{E} f(\mathbf{0}) = (-\delta(t), 0, 0)
$$

for some positive function $\delta(t) > 0$ which proves the lemma. $\square$

Let $M_t(z) = \frac{z-t}{1-tz}$, then $M_{t^2} \circ f \circ M_{-t}^{-1}(z) = f_{\frac{2t}{1+t^2}}(z)$. Note that the the distance $\mathrm{d}_{\mathbb{H}^3}(\mathbf{0}, M_t(\mathbf{0})) = \log \frac{1+t}{1-t}$. Hence, by the previous lemma and naturality and the fact that $\mathscr{E}f|_{\mathbb{H}^2_0}(r,0) = \mathscr{E}f|_{\mathbb{H}^2_0}(r,\pi)$, we have

$$\mathscr{E}f|_{\mathbb{H}^2_0}(r,0) = ((\log(\cosh(r)) - \delta(r), 0)$$

Note that the spherical derivative of $1 \in \mathbb{P}^1(\mathbb{C}) \cong S^2$ is 2, so by Proposition 2.10

$$\begin{aligned}
\log \frac{1}{2} &= \lim_{r\to\infty} \mathrm{d}_{\mathbb{H}^3}(\mathscr{E}f|_{\mathbb{H}^2_0}(r,0), \mathbf{0}) - \mathrm{d}_{\mathbb{H}^3}((r,0), \mathbf{0}) \\
&= \lim_{r\to\infty} (\log(\cosh(r)) - \delta(r) - r)
\end{aligned}$$

Note that

$$\lim_{r\to\infty} \log(\cosh(r)) - r = \log \frac{1}{2}$$

Therefore $\delta(r) \to 0$ as $r \to \infty$.

$\square$

3

Limiting branched coverings on $\mathbb{R}$-trees

3.1 $\mathbb{R}$-trees and branched coverings

In this section, we give various definitions and notations related to branched coverings between $\mathbb{R}$-trees.

$\mathbb{R}$-trees

An $\mathbb{R}$-tree is a nonempty metric space (T, d) such that any two points $x, y \in T$ are connected by a unique topological arc $[x, y] \subset T$, and every arc of T is isometric to an interval in $\mathbb{R}$.

We say x is an endpoint of T if $T - \{x\}$ is connected; otherwise x is an interior point. If $T - \{x\}$ has three or more components, we say x is a branch point. The set of branch points will be denoted $B(T)$.

We say T is a *finite tree* if $B(T)$ is finite. Note that we allow a finite tree to have an infinite end, so a finite tree may not be compact. We will write $[x, y)$ and (x, y) for $[x, y]$ with one or both of its endpoints removed.

A ray α in the $\mathbb{R}$-tree T is a subtree isometric to $[0, \infty) \subset \mathbb{R}$. Two rays are *equivalent* if $\alpha_1 \cap \alpha_2$ is still a ray. The collection $\epsilon(T)$ of all equivalence classes of rays forms the set of *ends* of T. We will let α denote both a ray and the end it represents. We also say a sequence of points x_i converges to an end $\alpha \subset T$, denoted by $x_i \to \alpha$, if $d(x_1, x_i) \to \infty$ and $x_i \in \alpha$ for all sufficiently large i.

Let $x \in T$, two segments $[x, y_1]$ and $[x, y_2]$ are said to represent the same tangent vector at x if $[x, y_1] \cap [x, y_2]$ is another non-degenerate segment. The set of equivalence classes of tangent vectors at x is called the *tangent space* at x, and denoted by $\mathrm{T}_x T$. Equivalently, the tangent space $\mathrm{T}_x T$ can be identified with the set of components of $T - \{x\}$. Let $\vec{v} \in \mathrm{T}_x T$, we will use $U_{\vec{v}}$ to denote the component of $T - \{x\}$ corresponding to $\vec{v}$. More generally, we will use $U^{a_1,...,a_n}$ to denote a connected open set with finitely many boundary points $a_1, ..., a_n \in T$.

Convexity and subtrees

A subset S of T is called *convex* if $x, y \in S \implies [x, y] \subset S$. The smallest convex set containing $E \subset T$ is called the *convex hull* of E, and is denoted by $\mathrm{hull}(E)$. More generally, we can easily extend the definition of convex, convex hull to subset $S \subset T \cup \epsilon(T)$. Note that subset $S \subset T$ is convex if and only if S is connected if and only if S is a subtree. Moreover, S is a *finite subtree* of T if and only if S is the convex hull of a finite set $E \subset T \cup \epsilon$.

Branched coverings between $\mathbb{R}$-trees

We now give the definition of a branched covering between $\mathbb{R}$-trees:

Definition 3.1. *Let $f : T_1 \longrightarrow T_2$ be a continuous map between two $\mathbb{R}$-trees, we say f is a degree d*

branched covering if there is a finite subtree $S \subset T_1$ such that

1. *S is nowhere dense in T_1, and $f(S)$ is nowhere dense in T_2.*

2. *For every $y \in T_2 - f(S)$, there are exactly d preimages in T_1.*

3. *For every $x \in T_1 - S$, f is a local isometry.*

4. *For every $x \in S$, and any sufficiently small neighborhood U of $f(x)$, $f : V - f^{-1}(f(V \cap S)) \longrightarrow U - f(V \cap S)$ is an isometric covering, where V is the component of $f^{-1}(U)$ containing x.*

Local degree and critical sets

Let $f : T_1 \longrightarrow T_2$ be a degree d branched covering, and $x \in T_1$, we define the *local degree* at x, denoted as $\deg_x(f)$ as the degree of the isometric covering of $f(x)$, $f : V - f^{-1}(f(V \cap S)) \longrightarrow U - f(V \cap S)$ for sufficiently small neighborhood U of $f(x)$. We define $C(f) = \{x \in T_1 : \deg_x(f) \geq 2\}$. Note that $C(f) \subset S$. The image of $C(f)$ are called the *critical values* of f and denoted by $CV(f)$

Here are some properties which can be easily verified using definitions:

Proposition 3.2. *Let $f : T_1 \longrightarrow T_2$ be a degree d branched covering, then*

1. *$C(f)$ is a closed set.*

2. *For $y \in T_2$, $\sum_{f(x)=y} \deg_x(f) = d$.*

3. *If $\deg_x(f) = 1$, then f is a local isometry near x.*

4. *For every $x \in T_1$, and any sufficiently small neighborhood U of $f(x)$, $f : V - f^{-1}(f(V \cap C(f))) \longrightarrow U - f(V \cap C(f))$ is an isometric covering where V is the component of $f^{-1}(U)$ containing x.*

5. *If U is a subtree disjoint from $C(f)$, then f maps U isometrically to $f(U)$.*

Tangent maps

Let $f : T_1 \longrightarrow T_2$ be a degree d branched covering, note that if $[x, y]$ is sufficiently small, $f([x, y])$ is contained in a connected component of $T - \{x\}$. Hence, we define the tangent map for f at x

$$D_x f : \mathrm{T}_x T_1 \longrightarrow \mathrm{T}_{f(x)} T_2$$
$$\vec{v} \mapsto \vec{w}$$

if $f([x, y]) \subset U_{\vec{w}}$ where $[x, y]$ is any segment representing $\vec{v}$. Given $\vec{v}$, we define the local degree of $x \in T$ at direction $\vec{v}$, denoted as $\deg_{\vec{v}}(D_x f)$, as the degree of the local isometric covering

$$f : U_{\vec{v}} \cap (V - f^{-1}(f(V \cap C(f)))) \longrightarrow U_{D_x f(\vec{v})} \cap (U - f(V \cap C(f)))$$

for sufficiently small neighborhood U of $f(x)$.

The following can be easily verified from the definitions:

Proposition 3.3. *Let $f : T_1 \longrightarrow T_2$ be a degree d branched covering, and $D_x f : \mathrm{T}_x T_1 \longrightarrow \mathrm{T}_{f(x)} T_2$ be the tangent map at x, and $\vec{w} \in \mathrm{T}_{f(x)} T_2$, then*

$$\sum_{D_x f(\vec{v}) = \vec{w}} \deg_{\vec{v}}(D_x f) = \deg_x(f)$$

In particular, $D_x f$ is surjective.

3.2 Algebraic limit of rational maps

The space $\mathrm{Rat}_d(\mathbb{C})$ of rational maps of degree d is an open variety of $\mathbb{P}^{2d+1}_{\mathbb{C}}$. More concretely, fixing a coordinate system of $\mathbb{P}^1_{\mathbb{C}}$, then a rational can be expressed as a ratio of homogeneous polynomials $f(z : w) =$

$(P(z,w) : Q(z,w))$, where P and Q have degree d with no common divisors. Using the coefficients of P and Q as parameters, we have

$$\mathrm{Rat}_d(\mathbb{C}) = \mathbb{P}_{\mathbb{C}}^{2d+1} - V(\mathrm{Res})$$

where Res is the resultant of the two polynomials P and Q.

One natural compactification of $\mathrm{Rat}_d(\mathbb{C})$ is $\overline{\mathrm{Rat}_d(\mathbb{C})} = \mathbb{P}_{\mathbb{C}}^{2d+1}$. We will call this compactification the *algebraic compactification*. Every map in $f \in \overline{\mathrm{Rat}_d(\mathbb{C})}$ determines the coefficients of a pair of homogeneous polynomials, and we write

$$f = (P : Q) = (Hp : Hq) = H\varphi_f$$

where $H = \gcd(P, Q)$ and $\varphi_f = (p : q)$ is a rational map of degree at most d. A zero of H is called a *hole* of f and the set of zeros of H is denoted by $\mathcal{H}(f)$. We will also define the degree of $f \in \overline{\mathrm{Rat}_d(\mathbb{C})}$ as the degree of φ_f.

The following lemma is standard (see Lemma 4.2 in [DeM05]):

Lemma 3.4. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$ be a sequence converges algebraically to $f = H\varphi_f \in \overline{\mathrm{Rat}_d(\mathbb{C})}$, then f_n converges compactly to φ_f on $\mathbb{P}_{\mathbb{C}}^1 - \mathcal{H}(f)$.*

The following lemma is also standard (Cf. Lemma 4.5 and Lemma 4.6 in [DeM05]):

Lemma 3.5. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$ be a sequence converges algebraically to $f = H\varphi_f \in \overline{\mathrm{Rat}_d(\mathbb{C})}$, and h is a hole of f.*

1. *If $\varphi_f \equiv c$, then for any neighborhood U of h, any point $a \in \mathbb{P}_{\mathbb{C}}^1 - \{c\}$, there exists a N such that for all $n \geq N$, $f_n^{-1}(a) \cap U \neq \emptyset$.*
2. *If $\deg(\varphi_f) \geq 1$, then for any neighborhood U of h, there exists a N such that for all $n \geq N$, $f_n(U) = \mathbb{P}_{\mathbb{C}}^1$, and U contains some critical points of f_n.*

We also have a similar result for barycentric extensions:

Lemma 3.6. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$ be a sequence converges algebraically to $f = H\varphi_f \in \overline{\mathrm{Rat}_d(\mathbb{C})}$, and assume $\deg(\varphi_f) \geq 1$, then $\mathscr{E} f_n$ converges compactly to $\mathscr{E}\varphi_f$.*

Proof. We first claim that $\mathscr{E} f_n$ coverges to $\mathscr{E}\varphi_f$ pointwise. By naturality, it suffices to show that $\mathscr{E} f_n(\mathbf{0})$ converges to $\mathbf{0}$ under the assumption that $\mathscr{E}\varphi_f(\mathbf{0}) = \mathbf{0}$.

Let $M_n(\mathbf{0}) = \mathscr{E} f_n(\mathbf{0})$. First note that M_n is bounded in $\mathrm{PSL}_2(\mathbb{C})$, as otherwise, there is a subsequence so that $(M_n^{-1} \circ f_n)_*\mu_{S^2}$ converges weakly to a delta measure, which is a contradiction as $(M_n^{-1} \circ f_n)_*\mu_{S^2}$ is balanced (see 2.1.1) for all n. We claim $M_n(\mathbf{0})$ converges to $\mathbf{0}$, as otherwise, there is a subsequence so that M_n^{-1} converges to $M^{-1} \in \mathrm{PSL}_2(\mathbb{C})$ with $M(\mathbf{0}) \neq \mathbf{0}$. This means that $(M^{-1} \circ \varphi_f)_*\mu_{S^2}$ is balanced (see 2.1.1), so $\mathscr{E}\varphi_f(\mathbf{0}) = M(\mathbf{0}) \neq \mathbf{0}$ which is a contradiction. Hence $\mathscr{E} f_n$ coverges to $\mathscr{E}\varphi_f$ pointwise.

Now by Theorem 1.1, the sequence $\mathscr{E} f_n$ is Cd-Lipschitz, so the pointwise convergence can be promoted to uniform convergence on any compact set. Therefore, $\mathscr{E} f_n$ converges compactly to $\mathscr{E}\varphi_f$. $\square$

We define a notion of convergence of annulus as follows (Cf. Carathéodory topology in Section 5 of [McM94]).

Definition 3.7. *Let U_n be topological disks of $\mathbb{C}$, and K_n be a compact and connected subset of U_n respectively. Let $A_n = U_n - K_n$ and $u_n \in K_n$. We say (A_n, u_n) converges to the annulus (A, u) if $A = U - K$ where U is a topological disk and $K \subset U$ is compact and connected such that*

1. *u_n converges to u.*
2. *K_n converges in the Hausdorff topology on compact subset of $\mathbb{P}^1_{\mathbb{C}}$ to K.*
3. *$\mathbb{P}^1_{\mathbb{C}} - U_n$ converges in the Hausdorff topology on compact subset of the sphere to D, and U is a component of $\mathbb{P}^1_{\mathbb{C}} - D$ containing u.*

Equivalently, the annuli $A_n = U_n - K_n$ and $u_n \in K_n$ converges to $A = U - K$ and $u \in K$ if (U_n, u_n) converges to (U, u) in Carathéodory topology and K_n converges to K in Hausdorff topology.

The proof of the following lemma can be found in the proof of Theorem 5.8 in [McM94].

Lemma 3.8. *The space of pairs (A, u) with $m(A) \geq m$ is compact up to affine conjugacy.*

More precisely, any sequence (A_n, u_n) with $m(A_n) \geq m$, normalized so that $u_n = 0$ and the diameter of the bounded component of $\mathbb{C} - A_n$ is 1, has a convergent subsequence.

The following lemma will be used to give criterions to bound the degree of the algebraic limit.

Lemma 3.9. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$, and $A \subset \mathbb{C} - \{0\}$ be an annulus of modulus $\geq m$. Let A_n be an annulus of $\mathbb{C} - \{0\}$ such that the diameter of the bounded component of $\mathbb{C} - A_n$ is 1 and $f_n : A_n \longrightarrow A$ is a degree e_n covering map. Then after passing to a subsequence and perturbing A if necessary, we have $f = \lim_{n\to\infty} f_n = H\varphi_f$, $e_n = e$ for all n, $(A_n, 0)$ converges to an annulus $(A_\infty, 0)$, and $\varphi_f : A_\infty \longrightarrow A$ is a degree e covering map. In particular $\deg(\lim_{n\to\infty} f_n) \geq e$.*

Proof. After passing to a subsequence, we may assume $f = \lim_{n\to\infty} f_n = H\varphi_f$, $e = e_n$ for all n. Since $m(A_n) \geq m/d$, by Lemma 3.8, we may also assume that $(A_n, 0)$ converges to an annulus $(A_\infty, 0)$.

Perturb A if necessary, we may assume there are no holes of f on ∂A_∞. We will now show that $\varphi_f : A_\infty \longrightarrow \varphi_f(A_\infty)$ is a degree e covering map. First we claim $A_\infty \cap \mathcal{H}(f) = \emptyset$. Otherwise, there is an open set $U \subset A_\infty$ and a point $a \in \mathbb{P}^1_{\mathbb{C}} - A$ so that $a \in f_n(U)$ for all sufficiently large n by Lemma 3.5. This is a contradiction to $f_n(U) \subset A$. Hence f_n converges uniformly on a neighborhood of A_∞ to φ_f by Lemma 3.4, so φ_f is proper on A_∞. We note that A_∞ contains no critical point of φ_f as otherwise there is a critical point of f_n near c for all sufficiently large n which is not possible. Hence φ_f is a covering map. The degree is e as f_n converges uniformly on a neighborhood of A_∞ to φ_f. $\square$

3.3 Ultrafilters and ultralimits

In this section, we review the theory of ultrafilter on a set. Refer to [KL95, LR75, Roe03] for more details. Given a countable set I, a subset $\omega \subset \wp(I)$ of the powerset of I is called an ultrafilter if

1. If $A, B \in \omega$, then $A \cap B \in \omega$;
2. If $A \in \omega$ and $A \subset B$, then $B \in \omega$;
3. $\emptyset \notin \omega$;
4. If $A \subset I$, then either $A \in \omega$ or $I - A \in \omega$

By the virtue of the 4 properties of an ultrafilter, one can think of a ultrafilter ω as defining a *finitely additive* $\{0,1\}$*-valued probability measure* on I: the sets of measure 1 are precisely those belonging to the filter ω. We will call such sets as ω-*big* or simply *big*. Its complement is called ω-*small* or simply *small*. If a certain property is satisfied by a ω-big set, then we will also say this property holds ω-almost surely.

Example. *Let $a \in I$, we define*

$$\omega_a := \{A \subset \wp(I) : a \in A\}.$$

It can be easily verified that ω_a is an ultrafilter on I.

An ultrafilter of the above type will be called a *principal ultrafilter*. It can be shown that an ultrafilter is principal if and only if it contains a finite set. An ultrafilter which is not principal is called a *non-principal ultrafilter*. The existence of a non-principal ultrafilter is guaranteed by Zorn's lemma.

Limit of a sequence of points with respect to an ultrafilter

Let ω be a non-principal ultrafilter on $\mathbb{N}$. If x_n be a sequence in a metric space (X,d) and $x \in X$, we say x is the ω-*limit* of x_n, denoted by

$$\lim_{\omega} x_n = x$$

if for every $\epsilon > 0$, the set $\{n : d(x_n, x) < \epsilon\}$ is big.

It can be easily verified (see [KL95]) that

1. If the ω-limit exists, then it is unique.
2. If x_n is contained in a compact set, then the ω-limit exists.
3. If $x = \lim_{n\to\infty} x_n$ in the standard sense, then $x = \lim_\omega x_n$.
4. If $x = \lim_\omega x_n$, then there exists a subsequence n_k such that $x = \lim_{k\to\infty} x_{n_k}$ in the standard sense.

From these properties, one should intuitively think (as one of the benefits) of the non-principal ultrafilter ω as performing all the subsequence-selection in advance, and all sequences in compact spaces will automatically converge without the need to pass to any further subsequences.

From now on and throughout the rest of the thesis, we will fix a non-principal ultrafilter ω on $\mathbb{N}$.

3.4 Barycentric extensions of a sequence on ${}^*\mathbb{H}^3$

We define two sequences $(x_n), (y_n)$ of $\mathbb{H}^3$ to be equivalent if

$$\lim_\omega d_{\mathbb{H}^3}(x_n, y_n) < \infty$$

The space of equivalence classes is denoted as ${}^*\mathbb{H}^3$. We will abuse the notation a little bit and use a sequence (x_n) to denote a point in ${}^*\mathbb{H}^3$. We will also use x^0 to denote the point $(\mathbf{0}) \in {}^*\mathbb{H}^3$.

Given a sequence $f_n \in \mathrm{Rat}_d(\mathbb{C})$, Theorem 1.1 immediately implies the map

$$\begin{aligned} \mathscr{E}(f_n) : {}^*\mathbb{H}^3 &\longrightarrow {}^*\mathbb{H}^3 \\ (x_n) &\mapsto (\mathscr{E} f_n(x_n)) \end{aligned}$$

is well defined.

We first note that

Lemma 3.10. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$, $x, y \in {}^*\mathbb{H}^3$, and $M_n, L_n \in \mathrm{PSL}_2(\mathbb{C})$ be such that $x = (M_n(\mathbf{0}))$ and $y = (L_n(\mathbf{0}))$. Then $\mathscr{E}(f_n)(x) = y$ if and only if*

$$\deg(\lim_{\omega} L_n^{-1} \circ f_n \circ M_n) \geq 1$$

Moreover, $\deg(\lim_{\omega} L_n^{-1} \circ f_n \circ M_n)$ *is well-defined, and for any (y_n) representing $y \in {}^*\mathbb{H}^3$, there is a sequence (x_n) representing $x \in {}^*\mathbb{H}^3$ such that $\mathscr{E} f_n(x_n) = y_n$ ω-almost surely.*

Proof. By naturality of the barycentric extension, it suffices to prove that $\mathscr{E}(f_n)(x^0) = x^0$ if and only if $\deg(\lim_{\omega} f_n) \geq 1$. Note that since the algebraic compactification $\overline{\mathrm{Rat}_d(\mathbb{C})} = \mathbb{P}^{2d+1}_{\mathbb{C}}$ is compact, so $\lim_{\omega} f_n$ always exists.

If $\mathscr{E}(f_n)(x^0) = x^0$, let $y_n = \mathscr{E} f_n(\mathbf{0})$, and $L_n(\mathbf{0}) = y_n$, Since (y_n) represents $x^0 \in {}^*\mathbb{H}^3$, we know

$$\lim_{\omega} L_n \in \mathrm{PSL}_2(\mathbb{C}) \tag{3.4.1}$$

By naturality of the extension, we have $\mathscr{E}(L_n^{-1} \circ f_n)(\mathbf{0}) = \mathbf{0}$. This means that the push forward of the spherical measure $(L_n^{-1} \circ f_n)_* \mu_{S^2}$ is balanced. Hence the degree of $\lim_{\omega} L_n^{-1} \circ f_n$ cannot be 0. Therefore, combining with equation (3.4.1), we have

$$\deg(\lim_{\omega} f_n) \geq 1.$$

Conversely, if $\mathscr{E}(f_n)(x^0) \neq x^0$, let $L_n \in \mathrm{PSL}_2(\mathbb{C})$ be such that $L_n(\mathbf{0}) = \mathscr{E}(f_n)(\mathbf{0})$. Then $\deg(\lim_{\omega} L_n) = 0$. Hence we have

$$\deg(\lim_{\omega} L_n^{-1} \circ f_n) \geq 1.$$

Suppose for contradiction that $\lim_\omega f_n = H\varphi_f$ with $\deg(\varphi_f) \geq 1$, then by Lemma 3.4, after passing to a subsequence, f_n converges compact to φ away from finitely many points. So $L_n^{-1} \circ f_n$ converges compactly to a constant map away from finitely many points, which is a contradiction.

Since changing representatives x and y only changes M_n and L_n by elements of a compact subset of $\mathrm{PSL}_2(\mathbb{C})$, the degree is well-defined.

To prove the last statement, by naturality, it suffices to prove if $\mathscr{E}(f_n)(x^0) = x^0$, then there is a sequence (x_n) representing $x^0 \in {}^*\mathbb{H}^3$ and $\mathscr{E}f_n(x_n) = \mathbf{0}$. Let $f := \lim_\omega f_n = H\varphi_f$, by Lemma 3.6, $\mathscr{E}f_n$ converges compactly (as ultralimit) to $\mathscr{E}\varphi_f$. Recall that we identify $\mathbb{H}^3$ with the unit ball in $\mathbb{R}^3$ and we choose a Euclidean ball $B(\mathbf{0}, r) \subset \mathbb{R}^3$ with $r < 1$ such that $(\mathscr{E}\varphi_f)^{-1}(\mathbf{0}) \cap (B(\mathbf{0}, 1) - B(\mathbf{0}, r)) = \emptyset$. This is possible as $\mathscr{E}\varphi_f$ is proper.

Now we claim that $(\mathscr{E}f_n)^{-1}(\mathbf{0}) \cap \overline{B(\mathbf{0}, r)} \neq \emptyset$. Indeed, suppose for contradiction that $(\mathscr{E}f_n)^{-1}(\mathbf{0}) \cap \overline{B(\mathbf{0}, r)} = \emptyset$, we can define a sequence of new maps

$$
\begin{aligned}
F_n : \overline{B(\mathbf{0}, r)} &\longrightarrow S^2 \\
x &\mapsto \mathscr{E}f_n(x)/|\mathscr{E}f_n(x)|_{\mathbb{R}^3}
\end{aligned}
$$

and

$$
\begin{aligned}
F : \partial B(\mathbf{0}, r) &\longrightarrow S^2 \\
x &\mapsto \mathscr{E}\varphi_f(x)/|\mathscr{E}\varphi_f(x)|_{\mathbb{R}^3}
\end{aligned}
$$

Since $\mathscr{E}f_n$ converges uniformly (as ultralimit) to $\mathscr{E}\varphi_f$ on $\partial B(\mathbf{0}, r)$, $F_n|_{\partial B(\mathbf{0},r)}$ is homotopic F ω-almost surely. But F is homotopic to φ_f, and since φ_f has degree ≥ 1, $F_n|_{\partial B(\mathbf{0},r)}$ has degree ≥ 1. So F_n cannot extend to a continuous map from $\overline{B(\mathbf{0}, r)}$ to S^2, which is a contradiction.

We now choose $x_n \in \overline{B(\mathbf{0}, r)}$ such that $\mathscr{E}f_n(x_n) = \mathbf{0}$, then (x_n) gives the sequence in the last statement. $\square$

Definition 3.11. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$, $x, y \in {}^*\mathbb{H}^3$, with $\mathscr{E}(f_n)(x) = y$. Let $M_n, L_n \in \mathrm{PSL}_2(\mathbb{C})$ be such that $x = (M_n(\mathbf{0}))$ and $y = (L_n(\mathbf{0}))$. We define the* multiplicity *of $\mathscr{E}(f_n)$ at x as* $\deg(\lim_\omega L_n^{-1} \circ f_n \circ M_n)$.

We will now prove the following:

Proposition 3.12. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$, then every point $y \in {}^*\mathbb{H}^3$ has exactly d preimages under $\mathscr{E}(f_n)$ counted multiplicities.*

Proof. By naturality, we only need to consider the preimages of the point $x^0 \in {}^*\mathbb{H}^3$.

Let

$$CV = \{\lim_\omega z_n : z_n \text{ is a critical value of } f_n\}$$

Note that $|CV| \leq 2d-2$. Let $A \subset \mathbb{P}^1_{\mathbb{C}} - CV$ be an annulus of modulus m. We choose A so that A contains no critical values of f_n ω-almost surely. By Riemann-Hurwitz formula, each component of $f_n^{-1}(A)$ is an annulus ω-almost surely, and there are at most d components.

In the following, to avoid cumbersome notations, the definitions should be understood as for an ω-big subset of $\mathbb{N}$. We label the components $f_n^{-1}(A)$ as $A_{1,n}, A_{2,n}, ..., A_{k,n}$. Each component $A_{i,n}$ is a covering of A, and we denote $d_{i,n}$ as the degree of the cover. Note that $\sum_{i=1}^k d_{i,n} = d$.

We first show that there are at most d preimages of x^0 counted multiplicities. Let $x_n = M_n(\mathbf{0})$ and assume that $\mathscr{E} f_n(x_n) = \mathbf{0}$. Let $f := \lim_\omega f_n \circ M_n = H\varphi_f$, then by Lemma 3.10, we have $q := \deg(\varphi_f) \geq 1$. Let $A_1, ..., A_p$ denote the component of $\varphi_f^{-1}(A)$. Note that each A_i is an annulus as A contains no critical values of φ_f.

Perturbing A if necessary, we may assume that ∂A_i contains no holes of f. We claim that $A_i \cap \mathcal{H}(f) = \emptyset$. Indeed, if not, then there is an open ball $U \subset A_i$ such that $f_n(U) = \mathbb{P}^1_{\mathbb{C}}$ ω-almost surely by Lemma 3.5. By Lemma 3.4, there is an index j, such that $\lim_\omega M_n^{-1}(\partial A_{j,n}) = \partial A_i$ in the sense of Hausdorff distance. Then $U \subset M_n^{-1}(A_{j,n})$ ω-almost surely, which is a contradiction to $f_n(A_{j,n}) = A$.

Therefore, we can associate each component A_i an index j_i such that $\lim_\omega M_n^{-1}(A_{j_i,n}) = A_i$ with $\sum_{i=1}^p d_{j_i,n} = q$ ω-almost surely. So there are at most d preimages of $(\mathbf{0})$ counted multiplicities.

To show there are at least d preimages of x^0 counted multiplicities, for each i, we choose sequence $M_{i,n} \in \mathrm{PSL}_2(\mathbb{C})$ so that $M_{i,n}^{-1}(A_{i,n})$ is an annulus in $\mathbb{C}$, 0 is in the bounded component $K_{i,n}$ of $\mathbb{C} - M_{i,n}^{-1}(A_{i,n})$ and $\mathrm{diam}(K_{i,n}) = 1$. Let $x = (M_{i,n}(\mathbf{0}))$, then by Lemma 3.9, $\mathscr{E}(f_n)(x) = x^0$ and has multiplicity at least d_i. Note that if $(M_{i,n}(\mathbf{0}))$ and $(M_{j,n}(\mathbf{0}))$ represent the same point in ${}^*\mathbb{H}^3$, then the multiplicity of that point is at least $d_i + d_j$. So there are at least d preimages counted multiplicities.

Therefore, there are exactly d preimages counted multiplicities. $\square$

Given two points $x, y \in \mathbb{H}^3$, we can associate an annulus, denoted by $A(x, y)$, defined as follows. Let H_1 and H_2 be two geodesics planes perpendicular to the geodesic segment $[x, y]$ and passing though x and y respectively. The boundary of H_1 and H_2 in $\mathbb{P}^1_{\mathbb{C}} \cong \partial\mathbb{H}^3$ are two circles C_1 and C_2. The annulus $A(x, y)$ is defined as the region bounded by C_1 and C_2. Note that the modulus $m(A(x, y)) = d_{\mathbb{H}^3}(x, y)/2\pi$. We say a sequence of annuli A_n *approximates* A'_n if there is a sequence of annuli $C_n \subset A_n \cap A'_n$ such that $\lim_\omega m(C_n)/m(A_n)$ and $\lim_\omega m(C_n)/m(A'_n)$ exist and are not zero.

The following lemma will be used later:

Lemma 3.13. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$, $x_n, y_n \in \mathbb{H}^3$ such that $(x_n), (y_n)$ represents different points $x \neq y \in {}^*\mathbb{H}^3$ and $A(x_n, y_n)$ contains no critical values of f_n ω-almost surely. Let $a_1 = (a_{1,n}), ..., a_k = (a_{k,n})$ and $b_1 = (b_{1,n}), ..., b_l = (b_{l,n})$ be the preimages of x and y with multiplicity $e_1, ..., e_k$ and $e'_1, ..., e'_l$ respectively. Let $I = \{1, ..., k\}$ and $J = \{1, ..., l\}$. Then there exists a local degree function $d : I \times J \longrightarrow \mathbb{N}$ such that*

1. *If $d(i, j) \neq 0$, then f_n is a degree $d(i, j)$ covering on the component $f_n^{-1}(A(x_n, y_n))$ approximating $A(a_{i,n}, b_{j,n})$ (defined for ω-big set).*
2. *$e_i = \sum_{j=1}^l d(i, j)$ and $e'_j = \sum_{i=1}^k d(i, j)$*
3. *If $d(i, j) \neq 0$, then $\lim_\omega d_{\mathbb{H}^3}(a_{i,n}, b_{j,n})/d_{\mathbb{H}^3}(x_n, y_n) = 1/d(i, j)$.*

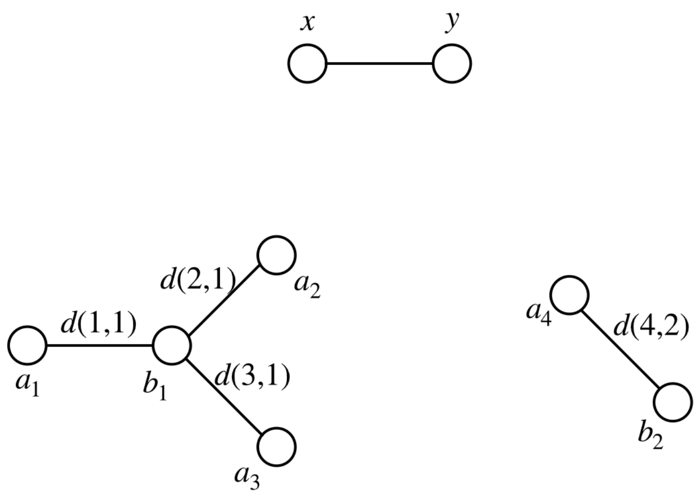

Figure 3.1: A schematic picture for the local degree function. a_i, b_j is mapped to $x, y \in {}^*\mathbb{H}^3$ respectively. Two circles are a_i, b_j are connected if $d(i,j) \neq 0$. The edge (a_i, b_j) of the graph should be thought of as an annulus which is mapped to the annulus (x, y) by a degree $d(i,j)$ covering.

Proof. We will construct the local degree function for $i = 1$. Other values of i can be treated in a similar fashion. By naturality of the extension, we may assume $x_n = \mathbf{0}$ and $a_{1,n} = \mathbf{0}$. Applying a sequence of rotations, we can also assume that one of the boundary of $A_n := A(x_n, y_n)$ is the unit circle S^1, and the other boundary is denoted by C_n. Since $(y_n) \neq (x_n) \in {}^*\mathbb{H}^3$, the modulus $m(A_n) = d_{\mathbb{H}^3}(\mathbf{0}, y_n)/2\pi$ is unbounded.

In the following, to avoid cumbersome notations, the definitions should be understood as for an ω-big subset of $\mathbb{N}$. Since A_n contains no critical values of f_n, by Riemann-Hurwitz formula, each component of $f_n^{-1}(A_n)$ is an annulus. Denote these annuli by $A_{1,n}, ..., A_{k,n}$.

Let $f = \lim_\omega f_n = H\varphi_f$. Note that $e_1 = \deg(\varphi_f) \geq 1$ by Lemma 3.10. Perturb S^1 if necessary, we may assume there is no holes of f on $\varphi_f^{-1}(S^1)$. Relabeling the indices if necessary, and we assume that $A_{1,n}, ..., A_{k_1,n}$ correspond to the sequences of components so that one of the boundaries converges to a component of $\varphi_f^{-1}(S^1)$.

For $u = 1, ..., k_1$, we denote $C_{u,n}$ be the boundary component of $A_{u,n}$ which is mapped to C_n and $C'_{u,n}$ be the boundary component which is mapped to S^1. By applying a bounded sequence in $\mathrm{PSL}_2(\mathbb{C})$, we may assume that 0 and ∞ are in the component of $\mathbb{P}^1_{\mathbb{C}} - A_{u,n}$ bounded by $C_{u,n}$ and $C'_{u,n}$ respectively. We let $e = e(u)$ denote the degree of the covering $f_n : A_{u,n} \longrightarrow A_n$. We let $D_{u,n} = \max_{z \in C_{u,n}} |z|$. Since each $C'_{u,n}$ converges to a component of $\varphi_f^{-1}(S^1)$ and there are no holes on $\varphi_f^{-1}(S^1)$, so $\min_{z \in C'_{u,n}} |z| \geq c > 0$ is bounded below. Hence, $m(A_{u,n}) = \frac{\log(1/D_{u,n})}{2\pi} + O(1)$ (see Theorem 2.1 in [McM94]). Let $M_{u,n} \in \mathrm{PSL}_2(\mathbb{C})$ be hyperbolic transformation fixing 0 and ∞ such that $\max_{z \in M^{-1}_{u,n}(C_{u,n})} |z| = 1$. Let $b = (M_{u,n}(\mathbf{0}))$, by Lemma 3.9 and naturality of the extension, b is mapped to y. So there is an index $j \in J$ so that $b = (b_{j,n}) \in {}^*\mathbb{H}^3$. We define $d(1, j) = e$.

Note that

$$\begin{aligned}
\lim_{\omega} d_{\mathbb{H}^3}(\mathbf{0}, b_{j,n})/d_{\mathbb{H}^3}(\mathbf{0}, y_n) &= \lim_{\omega}(\log(1/D_{u,n}) + O(1))/d_{\mathbb{H}^3}(\mathbf{0}, y_n) \\
&= \lim_{\omega}(2\pi m(A_{u,n}) + O(1))/2\pi m(A_n) \\
&= \lim_{\omega}(2\pi m(A_n)/e + O(1))/2\pi m(A_n) = 1/e
\end{aligned}$$

The last equality holds as $\lim_\omega 2\pi m(A_n) = \infty$.

We set $d(1, j) = 0$ if the index j does not appear after all k_1 constructions as above. Since e_1 equals to the sum of the local degree of coverings of φ_f on $\varphi_f^{-1}(S^1)$, we immediately get $e_1 = \sum_{j=1}^{l} d(1, j)$. We can define the local degree functions for other values of i's. $e_i = \sum_{j=1}^{l} d(i, j)$ and $e'_j = \sum_{i=1}^{k} d(i, j)$ can be checked in a similar fashion. $\square$

Using similar proof, we also have

Lemma 3.14. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$, $x_n, y_n \in \mathbb{H}^3$ such that $(x_n), (y_n)$ represents different points $x \neq y \in {}^*\mathbb{H}^3$ and $A(x_n, y_n)$ contains no critical points of f_n ω-almost surely. Then there are (a_n) and (b_n) repre-*

senting $\mathscr{E}(f_n)(x), \mathscr{E}(f_n)(y) \in {}^*\mathbb{H}^3$ so that f_n is a degree e covering on the component of $f_n^{-1}(A(a_n, b_n))$ approximating $A(x_n, y_n)$ (defined for ω-big set) and

$$\lim_{\omega} d_{\mathbb{H}^3}(\mathscr{E} f_n(x_n), \mathscr{E} f_n(y_n))/d_{\mathbb{H}^3}(x_n, y_n) = e$$

3.5 Criterions for degenerating sequences

Recall that a sequence $f_n \in \mathrm{Rat}_d(\mathbb{C})$ is called degenerating if f_n escapes every compact subset of $\mathrm{Rat}_d(\mathbb{C})$. We first give a criterion for a sequence of rational maps to be degenerating

Lemma 3.15. *Let $f_n \in \mathrm{Rat}_d(\mathbb{C})$, then f_n is degenerating if and only if*

$$r_n := \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0}) \to \infty$$

Proof. If f_n is degenerating, and suppose for contradiction that $r_n \not\to \infty$. Then after passing to a subsequence, we can assume that r_n is bounded. After passing to a further subsequence, we can assume $\lim_{n\to\infty} f_n = f \in \overline{\mathrm{Rat}_d(\mathbb{C})}$. Note that $\deg(f) < d$ as f_n is degenerating. Hence $x^0 \in {}^*\mathbb{H}^3$ has multiplicity $\deg f < d$ under $\mathscr{E}(f_n) : {}^*\mathbb{H}^3 \longrightarrow {}^*\mathbb{H}^3$. But by Lemma 3.10 and Proposition 3.12, we know there exists a sequence (x_n) such that $\mathscr{E} f_n(x_n) = \mathbf{0}$ and $(x_n) \neq x^0 \in {}^*\mathbb{H}^3$. Therefore, $\lim_\omega d_{\mathbb{H}^3}(x_n, \mathbf{0}) = \infty$, which is a contradiction to r_n is bounded.

Conversely, if $r_n \to \infty$, and suppose for contradiction that f_n is not degenerating. Then $f = \lim_\omega f_n \in \mathrm{Rat}_d(\mathbb{C})$, so $x^0 \in {}^*\mathbb{H}^3$ has multiplicity d. Let x_n be such that $\mathscr{E} f_n(x_n) = \mathbf{0}$ and $d_{\mathbb{H}^3}(x_n, \mathbf{0}) \to \infty$. Note that $(x_n) \neq x^0 \in {}^*\mathbb{H}^3$. By Proposition 3.12, x^0 has exactly d preimages in ${}^*\mathbb{H}^3$ counted multiplicities, so this is a contradiction. $\square$

Recall that the group of Möbius transformations $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathrm{Rat}_d(\mathbb{C})$ by conjugation and the moduli space of rational maps is its quotient $M_d(\mathbb{C}) = \mathrm{Rat}_d(\mathbb{C})/\mathrm{PSL}_2(\mathbb{C})$. Given a sequence $[f_n] \in M_d(\mathbb{C})$, it is degenerating as conjugacy classes if for every sequence g_n of representatives for $[f_n]$, g_n is degenerating as rational maps. Equivalently, $[f_n]$ escapes every compact subset of $M_d(\mathbb{C})$ equipped with the quotient topology. We define

$$r([f]) := \min_{x\in\mathbb{H}^3} \max_{y\in\mathscr{E} f_n^{-1}(x)} d_{\mathbb{H}^3}(y, x)$$

and note that the definition does not depend on the choice of representatives, and the minimum is achieved as $\max_{y\in\mathscr{E} f^{-1}(x)} d_{\mathbb{H}^3}(y, x) \to \infty$ as $x \to S^2$. The following is an easy corollary of Lemma 3.15 and the definitions:

Lemma 3.16. *Let $[f_n] \in M_d(\mathbb{C})$ be a sequence of conjugacy classes of rational maps, then $[f_n]$ is degenerating (as conjugacy classes) if and only if*

$$r([f_n]) \to \infty$$

3.6 Ultralimit and asymptotic cone of metric spaces

For most of the applications, ${}^*\mathbb{H}^3$ is too big to work with. In this section, we will introduce a quotient of a subspace of ${}^*\mathbb{H}$, which can be naturally equipped with a metric.

Let (X_n, p_n, d_n) be a sequence of pointed metric spaces with basepoints p_n. Let $\mathcal{X}$ denote the set of sequences $(x_n), x_n \in X_n$ such that $d_n(x_n, p_n)$ is a bounded function of n. We also define an equivalence relation $\sim$ by

$$(x_n) \sim (y_n) \Leftrightarrow \lim_{\omega} d_n(x_n, y_n) = 0$$

Let $X_\omega = \mathcal{X}/\sim$, and we define

$$d_\omega((x_n), (y_n)) = \lim_{\omega} d_n(x_n, y_n)$$

We shall abuse notation a little bit and use the sequence (x_n) to denote a point in X_ω, even though strictly speaking one should deal with equivalence classes of sequences. The distance function d_ω makes $(X_\omega, (p_n), d_\omega)$ a pointed metric space, and is called the *ultralimit* of (X_n, p_n, d_n), and is written as

$$\lim_\omega (X_n, p_n, d_n)$$

or simply $\lim_\omega X_n$ for short.

The ultralimit of X_n has many of the desired properties (see Section 7.5 in [Roe03] and [KL95] for associated definitions and proofs):

1. The ultralimit X_ω is always a complete metric space.
2. The ultralimit of a *length space* is a length space.
3. The ultralimit of a *geodesic space* is a geodesic space.
4. If X_n are *proper* metric spaces, with $(X_n, p_n) \to (Y, y)$ in the sense of Gromov-Hausdorff, then

$$(Y, y) \cong \lim_\omega (X_n, p_n).$$

Now let (X, p) be a fixed pointed metric space. Given a positive sequence r_n with $\lim_\omega r_n = \infty$, which will be called a *rescaling*, the *asymptotic cone* of X with respect to the rescaling r_n and the base point p is the ultralimit of the sequence $(X, p, \frac{1}{r_n} d)$, and is denoted by $({}^r X, (p), d_\omega)$ or simply ${}^r X$ for short.

Asymptotic cone of $\mathbb{H}^3$

Let $r_n \to \infty$ be a rescaling, we let ${}^r\mathbb{H}^3$ to be the asymptotic cone of $\mathbb{H}^3$ with rescaling r_n. It is well known that ${}^r\mathbb{H}^3$ is an $\mathbb{R}$-tree (see [KL95] and [Roe03]).

Let $z \in \mathbb{P}^1_{\mathbb{C}} \cong S^2$, we denote $\gamma(t, z) \in \mathbb{H}^3$ as the point at distance t away from $\mathbf{0}$ in the direction corresponding to z. Then given any sequence $z_n \in \mathbb{P}^1_{\mathbb{C}}$, the ray

$$s(t) = (\gamma(t \cdot r_n, z_n))$$

is a geodesic ray parameterized by arc length in ${}^r\mathbb{H}^3$. So we can associate a sequence (z_n) to an end in $\epsilon({}^r\mathbb{H}^3)$. Conversely, if $s(t)$ is a geodesic ray starting from $(\mathbf{0})$. Let $(\gamma(k \cdot r_n, z_{k,n}))$ represent the point $s(k)$, then the geodesic ray $s'(t) = (\gamma(t \cdot r_n, z_{n,n}))$ represents the same end as $s(t)$. It is not hard to work out in detail when two sequences (z_n) and (w_n) represent the same end. As we shall see in Chapter 4, the set of ends corresponds to the projective space of a non-Archimedean field ${}^\rho\mathbb{C}$.

Given $x \in {}^r\mathbb{H}^3$, the tangent space $T_x{}^r\mathbb{H}^3$ is uncountable. For example, the geodesic rays $s(t) = (\gamma(t \cdot r_n, z))$ represents different tangent vectors at x^0 for different $z \in \mathbb{P}^1_{\mathbb{C}}$, so $T_{x^0}{}^r\mathbb{H}^3$ contains $\mathbb{P}^1_{\mathbb{C}}$. In Chapter 4, we shall see that the tangent space corresponds to the projective space of the residual field of ${}^\rho\mathbb{C}$.

Let $\vec{v} \in T_x{}^r\mathbb{H}^3$, recall that $U_{\vec{v}}$ is the component of ${}^r\mathbb{H}^3 - \{x\}$ corresponding to the direction $\vec{v}$. We can associate a sequence of hyperbolic half space to $U_{\vec{v}}$ as follows. Let $y \in U_{\vec{v}}$, and $(x_n), (y_n)$ representing $x, y \in {}^r\mathbb{H}^3$. Let $H_n^{\vec{v}} \subset \mathbb{H}^3$ be the hyperbolic half space bounded by the hyperbolic plane passing through x_n and perpendicular to the geodesic $[x_n, y_n]$ that contains y_n. Let $S_n^{\vec{v}}, D_n^{\vec{v}} \subset \mathbb{P}^1_{\mathbb{C}} \cong S^2$ denote the boundary circle and disk in S^2 associated to $H_n^{\vec{v}}$. Note that the definitions require a lot of choices, but to avoid cumbersome notations, we will drop the dependence of choices in the notations.

If $U^{a_1,\ldots,a_k}$ is an open connected subset of ${}^r\mathbb{H}^3$ with finitely many boundary points $a_1, ..., a_k$. We let

$$H_n^{a_1,\ldots,a_k} = \bigcap_{j=1}^{k} H_n^{\vec{v}_j}$$

where $\vec{v}_j$ is the tangent vector at a_j associated to U. We define similarly for $S_n^{a_1,\ldots,a_k}$ and $D_n^{a_1,\ldots,a_k}$. Using the definition, we have

Lemma 3.17. *Let $U^{a_1,\ldots,a_k}$ be an open connected subset of ${}^r\mathbb{H}^3$ with boundary points $a_1, ..., a_k$, then $x \in$*

U if and only if for any representative $x = (x_n)$ and any choice $H_n^{a_1,\ldots,a_k}$, $x_n \in H_n^{a_1,\ldots,a_k}$ ω-almost surely.

The following definition will be useful to study the limiting maps on ${}^r\mathbb{H}^3$.

Definition 3.18. *Let $U^{a_1,\ldots,a_k}$ be an open connected subset of ${}^r\mathbb{H}^3$ with boundary points $a_1, \ldots, a_k$, we say $D_n \subset \mathbb{P}^1_{\mathbb{C}} \cong S^2$ is a domain approximating U if there exist two choices ${}^1H_n^{a_1,\ldots,a_k}, {}^2H_n^{a_1,\ldots,a_k}$ so that*

$${}^1H_n^{a_1,\ldots,a_k} \subset \operatorname{hull} D_n \subset {}^2H_n^{a_1,\ldots,a_k}$$

ω-almost surely. Or equivalently,

$${}^1D_n^{a_1,\ldots,a_k} \subset D_n \subset {}^2D_n^{a_1,\ldots,a_k}$$

ω-almost surely.

3.7 Barycentric extensions of a sequence on ${}^r\mathbb{H}^3$

In this section, we will construct a map on ${}^r\mathbb{H}^3$. Let $f_n \in \operatorname{Rat}_d(\mathbb{C})$ be a degenerating sequence of rational maps which will be fixed throughout this section. Lemma 3.15 gives us a natural choice of rescalings $r_n := \max_{y \in \mathscr{E}f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$. Let ${}^r\mathbb{H}^3$ be the asymptotic cone of $(\mathbb{H}^3, \mathbf{0})$ with respect to the rescaling r_n.

We define the map

$$\begin{aligned} \mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 &\longrightarrow {}^r\mathbb{H}^3 \\ (x_n) &\mapsto (\mathscr{E}f_n(x_n)) \end{aligned}$$

We will first show that it is well defined:

Lemma 3.19. *$\mathscr{E}_{bc}(f_n)$ is well defined and Lipschitz.*

Proof. Let $x_n \in (\mathscr{E} f_n)^{-1}(\mathbf{0})$, then (x_n) represents a point in ${}^r\mathbb{H}^3$ as $d_{\mathbb{H}^3}(\mathbf{0}, x_n)/r_n$ is bounded. If $(y_n) \in {}^r\mathbb{H}^3$, then $\lim_\omega d_{\mathbb{H}^3}(x_n, y_n)/r_n < \infty$. By Theorem 1.1, $\mathscr{E} f_n$ is Cd-Lipschitz, so their image

$$\lim_\omega d_{\mathbb{H}^3}(\mathbf{0}, \mathscr{E} f_n(y_n))/r_n = \lim_\omega d_{\mathbb{H}^3}(\mathscr{E} f_n(x_n), \mathscr{E} f_n(y_n))/r_n \leq Cd \lim_\omega d_{\mathbb{H}^3}(x_n, y_n)/r_n < \infty$$

Therefore $(\mathscr{E} f_n(y_n))$ represents a point in ${}^r\mathbb{H}^3$.

If two sequences $(a_n), (b_n)$ represents the same point in ${}^r\mathbb{H}^3$, then $\lim_\omega d_{\mathbb{H}^3}(a_n, b_n)/r_n = 0$. Hence their image

$$\lim_\omega d_{\mathbb{H}^3}(\mathscr{E} f_n(a_n), \mathscr{E} f_n(b_n))/r_n \leq Cd \lim_\omega d_{\mathbb{H}^3}(a_n, b_n)/r_n = 0$$

Therefore $\mathscr{E}_{bc}(f_n)$ is well defined.

The fact that $\mathscr{E}_{bc}(f_n)$ is Lipschitz follows from the fact that $\mathscr{E} f_n$ is Cd-Lipschitz for a universal C. □

Local degrees

We will now give a definition of multiplicity of the map $\mathscr{E}_{bc}(f_n)$ using the multiplicities of $\mathscr{E}(f_n) : {}^*\mathbb{H}^3 \longrightarrow {}^*\mathbb{H}^3$. Let $x = (x_n), y = (y_n) \in {}^r\mathbb{H}^3$ be such that $\mathscr{E}_{bc}(f_n)(x) = y$. Note (y_n) represents a point in $\tilde{y} \in {}^*\mathbb{H}^3$, and we let $x_1, ..., x_k \in {}^*\mathbb{H}^3$ be its preimages with multiplicities $e_1, ..., e_k$. We define the multiplicity of $\mathscr{E}_{bc}(f_n)$ at x, denoted by $\deg_x \mathscr{E}_{bc}(f_n)$ as

$$\deg_x \mathscr{E}_{bc}(f_n) = \sum_{x_i \in {}^*\mathbb{H}^3 \text{ represents } x \in {}^r\mathbb{H}^3} e_i$$

Proposition 3.20. $\deg_x \mathscr{E}_{bc}(f_n)$ *is well defined, and every* $y \in {}^r\mathbb{H}^3$ *has exactly* d *preimages counted multiplicities.*

Proof. We will show that the definition of the multiplicity does not depend on the choice of representatives. Let $(y_n), (w_n)$ represents the same point in $y \in {}^r\mathbb{H}^3$ but different points $\tilde{y} \neq \tilde{w} \in {}^*\mathbb{H}^3$, and

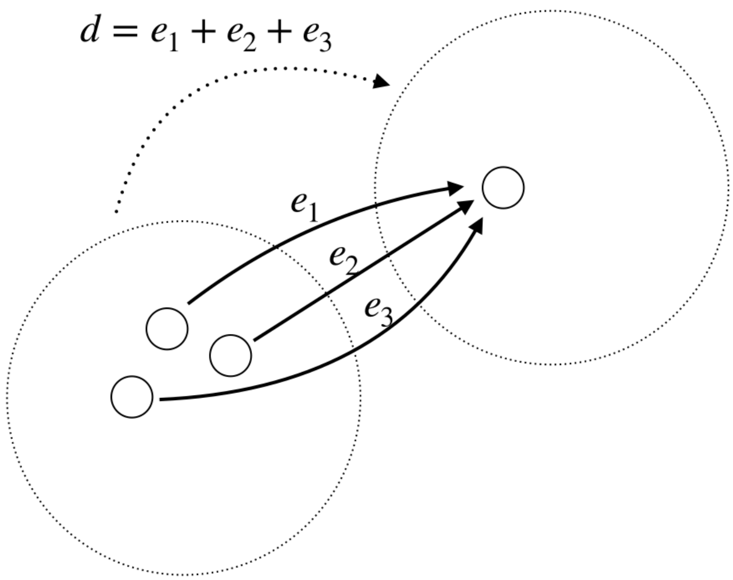

Figure 3.2: A schematic picture of the definition of local degrees. The small bulb represents a point in ${}^*\mathbb{H}^3$ and the large dotted bulb represents a point in ${}^r\mathbb{H}^3$. A point in ${}^r\mathbb{H}^3$ is an equivalence class of points in ${}^*\mathbb{H}^3$. If $\mathscr{E}_{bc}(f_n)$ sends x to $y \in {}^r\mathbb{H}^3$, and let $\tilde{y} \in {}^*\mathbb{H}^3$ representing y, then the degree is defined as the sum over all the local degrees of preimages of $\tilde{y}$ representing x in ${}^*\mathbb{H}^3$.

$(a_{1,n}), ..., (a_{k,n})$ and $(b_{1,n}), ..., (b_{l,n})$ be the preimages of $\tilde{y}$ and $\tilde{w}$ in ${}^*\mathbb{H}^3$ repectively, with index set $I = \{1, ..., k\}$ and $J = \{1, ..., l\}$. We also assume that first u points $(a_{1,n}), ..., (a_{u,n})$ represent the same point as x in ${}^r\mathbb{H}^3$.

Assume first that the annulus $A(y_n, w_n)$ contains no critical values of f_n ω-almost surely, then by Lemma 3.13, there exists a local degree function $d : I \times J \longrightarrow \mathbb{N}$. We let $S \subset J$ be the set of index j such that $d(i, j) \neq 0$ for some $i = 1, ..., u$. By property (2) of the local degree function in Lemma 3.13, for $j \in S$, $(b_{j,n})$ represents the same point as x in ${}^r\mathbb{H}^3$. Note that for $j \in S$ and $i > u$ $d(i, j) = 0$, as otherwise $(a_{i,n})$ represents x in ${}^r\mathbb{H}^3$ by property (2) in Lemma 3.13. Therefore, we have $\sum_{i=1}^{u} e_i = \sum_{j \in S} e'_j$

by property (1) in Lemma 3.13. Therefore, the definition of the degree agrees for (y_n) and (w_n).

More generally, we can choose a finite points $p_{1,n} := y_n < q_{1,n} < p_{2,n} < ... < p_{k,n} < q_{k,n} := w_n$ on the geodesic segment $[y_n, w_n]$ such that

1. $d_{\mathbb{H}^3}(p_{i,n}, q_{i,n}) \leq 1$
2. The annulus $A(q_{i,n}, p_{i+1,n})$ contains no critical values of f_n

Then the definition of multiplicity at x agrees for $(q_{i,n})$ and $(p_{i+1,n})$ by the previous argument. Note that $(p_{i,n})$ and $(q_{i,n})$ represent the same point in ${}^*\mathbb{H}^3$, so the definition of multiplicity also agrees. Therefore, $\deg_x \mathscr{E}_{bc}(f_n)$ is well defined.

Let $y = (y_n) \in {}^r\mathbb{H}^3$, if $A(\mathbf{0}, y_n)$ contains no critical values ω-almost surely, then Lemma 3.13 implies that for any preimage $(x_n) \in {}^*\mathbb{H}^3$ of (y_n), $\lim_\omega d_{\mathbb{H}^3}(\mathbf{0}, x_n)/r_n < \infty$. Hence (x_n) represents a point in ${}^r\mathbb{H}^3$. More generally, we can cut the geodesic segment $[\mathbf{0}, y_n]$ into finite pieces and argue as above. Therefore, every point y has exactly d preimages counted multiplicities in ${}^r\mathbb{H}^3$ by Proposition 3.12. $\square$

Critical subtree

Let $c_{1,n}, ..., c_{2d-2,n}$ be the $2d-2$ critical points of f_n. Then $\kappa_1 = (c_{1,n}), ..., \kappa_{2d-2} = (c_{2d-2,n})$ represent $2d-2$ ends (which may be the same) of the tree ${}^r\mathbb{H}^3$. Let $S = \mathrm{hull}(\kappa_1, ..., \kappa_{2d-2})$, then S is a finite subtree of ${}^r\mathbb{H}^3$. We will call S the *critical subtree*.

The following lemma gives criterions for points in the convex hull.

Lemma 3.21. *Let $x \in {}^r\mathbb{H}^3$ and assume $x = (M_n(\mathbf{0}))$ with $M_n \in \mathrm{PSL}_2(\mathbb{C})$. Let $E = \{(a_{1,n}), (a_{2,n})\} \subset \epsilon({}^r\mathbb{H}^3)$, and $a_i = \lim_\omega M_n^{-1}(a_{i,n}) \in \mathbb{P}^1_{\mathbb{C}}$. If $a_1 \neq a_2$, then $x \in \mathrm{hull}(E)$.*

Proof. Note that it suffices to prove the case for $x_n = \mathbf{0}$. Consider the geodesic rays $s_1(t) = (\gamma(t \cdot r_n, a_{1,n}))$ and $s_2(t) = (\gamma(t \cdot r_n, a_{2,n}))$. Since $\lim_\omega a_{1,n} \neq \lim_\omega a_{2,n}$, $s_1((0,\infty))$ is disjoint from $s_2((0,\infty))$. Hence $x^0 = s_1(0) = s_2(0)$ is in the convex hull of E. $\square$

3.8 Proof of Theorem 1.2

We will now prove Theorem 1.2 by proving three lemmas.

Lemma 3.22. *Let U be a subtree of ${}^r\mathbb{H}^3 - S$, then $\mathscr{E}_{bc}(f_n) : U \longrightarrow \mathscr{E}_{bc}(f_n)(U)$ is an isometry.*

Proof. Let $x, y \in U$ with $d(x, y) = l \neq 0$, we will show that $d(\mathscr{E}_{bc}(f_n)(x), \mathscr{E}_{bc}(f_n)(y)) = l$.

If the projection of S onto the geodesic segment $[x, y] \subset {}^r\mathbb{H}^3$ is one of the two end points, we can choose representatives of (x_n) and (y_n) of x, y so that only one component of $\mathbb{P}^1_{\mathbb{C}} - A(x_n, y_n)$ contains critical points ω-almost surely. Therefore, f_n is an isomorphism on an annulus approximating $A(x_n, y_n)$, so by Lemma 3.14, $d(\mathscr{E}_{bc}(f_n)(x), \mathscr{E}_{bc}(f_n)(y)) = l$

If the projection of S onto the geodesic segment $[x, y] \subset {}^r\mathbb{H}^3$ is an interior point a, then by naturality, we may assume that $a = x^0$ and $\mathscr{E} f_n(\mathbf{0}) = \mathbf{0}$. We claim that $\deg(\lim_\omega f_n) = 1$. Indeed, by Lemma 3.10, $\deg(\lim_\omega f_n) \geq 1$. If $\deg(\lim_\omega f_n) > 1$, then there are at least two distinct limits of critical points of f_n in $\mathbb{P}^1_{\mathbb{C}}$, contradicting Lemma 3.21. Therefore, the images of $[x, a)$ and $(a, y]$ are disjoint. Now apply the previous argument to the geodesic segment $[x, a]$ and $[a, y]$ separately, we get the result.

Therefore, $\mathscr{E}_{bc}(f_n)$ is an isometry on U. $\square$

Lemma 3.23. *Let $y \in {}^r\mathbb{H}^3 - \mathscr{E}_{bc}(f_n)(S)$, then it has exactly d preimages.*

Proof. Let $x \in \mathscr{E}_{bc}(f_n)^{-1}(y)$, by Lemma 3.13, if the the local degree at x is strictly greater than 1, then $\mathscr{E}_{bc}(f_n)$ is not a local isometry near x. Hence by Lemma 3.22, we conclude that the local degree at x is 1. Therefore, there are exactly d preimages by Proposition 3.20. $\square$

Lemma 3.24. *Let $x \in {}^r\mathbb{H}^3$, then for sufficiently small neighborhood U of $\mathscr{E}_{bc}(f_n)(x)$,*

$$\mathscr{E}_{bc}(f_n) : V - \mathscr{E}_{bc}(f_n)^{-1}(\mathscr{E}_{bc}(f_n)(V \cap S)) \longrightarrow U - \mathscr{E}_{bc}(f_n)(V \cap S)$$

is an isometric covering of degree $\deg_x \mathscr{E}_{bc}(f_n)$ where V is the component of $\mathscr{E}_{bc}(f_n)^{-1}(U)$ containing x.

Proof. Since $\mathscr{E}_{bc}(f_n)(x)$ has exactly d preimages counted multiplicities, in particular, there are at most d preimages. We choose a neighborhood U of $\mathscr{E}_{bc}(f_n)(x)$ so small so that the component V of $\mathscr{E}_{bc}(f_n)^{-1}(U)$ containing x contains no other preimages of $\mathscr{E}_{bc}(f_n)(x)$.

Let $y \in U - \mathscr{E}_{bc}(f_n)(S)$, we will show that y has exactly $\deg_{(x_n)} \mathscr{E}_{bc}(f_n)$ preimages in U (each with multiplicity 1 by Lemma 3.23).

Let $x = (x_n)$ and $y = (y_n)$, if $A(\mathscr{E} f_n(x_n), y_n)$ contains no critical values of f_n ω-almost surely, since V contains no other preimages of $\mathscr{E}_{bc}(f_n)(x)$, so by Lemma 3.13, there are exactly $\deg_x \mathscr{E}_{bc}(f_n)$ preimages in V counted multiplicities.

More generally, we choose $p_{1,n} := \mathscr{E} f_n(x_n) < q_{1,n} < p_{2,n} < \ldots < p_{k,n} < q_{k,n} := y_n$ on the geodesic segment $[\mathscr{E} f_n(x_n), y_n]$ such that

1. $d_{\mathbb{H}^3}(p_{i,n}, q_{i,n}) \leq 1$
2. The annulus $A(q_{i,n}, p_{i+1,n})$ contains no critical values of f_n

Inductively, there are exactly $\deg_x \mathscr{E}_{bc}(f_n)$ preimages in V for $(p_{i,n}) = (q_{i,n})$. Hence there are exactly $\deg_x \mathscr{E}_{bc}(f_n)$ preimages of y in V and each has multiplicity 1 by Lemma 3.23.

The covering map is isometric follows from Lemma 3.22. ☐

Combining the three Lemmas above, we proved Theorem 1.2:

Proof of Theorem 1.2. It is easy to check that S and $\mathscr{E}_{bc}(f_n)(S)$ are both no where dense in ${}^r\mathbb{H}^3$. So by Lemma 3.23 and Lemma 3.24, $\mathscr{E}_{bc}(f_n)$ is a branched covering of degree d. Note that by Lemma 3.24, the multiplicity of $\mathscr{E}_{bc}(f_n)$ agrees with the definition of local degree of a branched covering on an $\mathbb{R}$-tree. ☐

3.9 Expanding property and the tangent map $\mathscr{E}_{bc}(f_n)$

In this section, let f_n be a degenerating sequence of rational maps of degree d with

$$r_n := \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$$

and ${}^r\mathbb{H}^3$ be the asymptotic cone of $\mathbb{H}^3$ with rescaling r_n. We will prove the expanding property and study the tangent map of $\mathscr{E}_{bc}(f_n)$ on ${}^r\mathbb{H}^3$.

Let $U \subset {}^r\mathbb{H}^3$, we say a critical end κ is persistent if κ is an end in U. Similarly, we define persistent end for critical values. If $U = U^{a_1,...,a_k}$ is an open connected subset of ${}^r\mathbb{H}^3$ with boundary points $a_1, ..., a_k$, we can always choose $D_n^{a_1,...,a_k}$ so that it only contains persistent critical points or critical values.

Proposition 3.25. *Let $U = U^{a_1,...,a_k}$ be an open connected subset of ${}^r\mathbb{H}^3$ with boundary points $a_1, ..., a_k$, and V be a component of $\mathscr{E}_{bc}(f_n)^{-1}(U)$. If $D_n^{a_1,...,a_k}$ associated to U contains only persistent critical values, then there is a component D_n of $f_n^{-1}(D_n^{a_1,...,a_k})$ which is a domain approximating V.*

Moreover, the degree of the branched coverings of f_n on D_n agrees with the degree of $\mathscr{E}_{bc}(f_n)$ on V.

Proof. We may assume $k = 1$, the general case can be proved by taking intersections.

By naturality of the extension, we may assume $x^0 \in V$ is mapped to $x^0 \in U$. We may further assume that $\mathscr{E} f_n(\mathbf{0}) = \mathbf{0}$. Let S_n denote the boundary of $D_n^{a_1}$, we may assume there is an annulus neighborhood A_n so that each component of $A_n - S_n$ has modulus K sufficiently large, and A_n contains no critical values of f_n ω-almost surely. Choose $b \in U$ close to a_1 so that no end for critical values projects to the interior of $[a_1, b]$. Since $D_n^{a_1}$ contains only persistent critical values, we let $D_n^{a_1,b} \subset D_n^{a_1}$ contains no critical values ω-almost surely. Let $x_1, ..., x_j$ be the preimages of a_1 that are also the boundary points of V, and $y_1, ..., y_j$ be the preimages of b in V, then $[x_i, y_i]$ is mapped to $[a_1, b]$. Let $D_{k,n}$ denote the preimages of $D_n^{a_1,b}$, where $D_{i,n}$ are associated to $[x_i, y_i]$ for $i = 1, ..., j$. Since $x^0 \in V$, the diameters of annuli $D_{k,n}$

all converge to 0. We claim that the outermost annuli are exactly those associated to $[x_i, y_i]$ for $i = 1, ..., j$. Indeed, in order to access the geodesic segment associated to the inner annuli from $x^0 \in {}^r\mathbb{H}^3$, one has to go through the segment $[x_i, y_i]$ associated to the outer ones. Conversely, if x is a preimage of a_1 such that $[x_i, y_i] \subset [x^0, x]$, then the annulus associated to x is nested inside of the annulus D_i.

Now let D_n be the component $f_n^{-1}(D_n^{a_1})$ containing these outermost annuli. Let $S_{i,n}$ denote the preimage of S_n associated to these outermost annulus. Note that the preimages $A_{i,n}$ of A_n satisfy that each component $A_{i,n} - S_{i,n}$ has large modulus, so $S_{i,n}$ is contained in some annulus with circular boundaries and with modulus bounded above by $2K$. Replace each boundary of D_n by these round circles, we conclude that D_n is a domain approximating V.

Let e be the degree of the branched covering f_n on D_n . Since there are exactly e preimages of $x \in U - F(S)$ in V, the two degrees agree. $\square$

Theorem 3.26. *Let $E \subset {}^r\mathbb{H}^3$ be a segment such that the projection of critical ends are not in the interior. Then $\mathscr{E}_{bc}(f_n)$ is linear with derivative e on E, where $e = \deg_x \mathscr{E}_{bc}(f_n)$ for any (and all) interior point of E.*

Proof. Since the projection of a critical end $\kappa_i = (c_{i,n}) \in \epsilon({}^r\mathbb{H}^3)$ to E is not an interior point, so if $x = (x_n), y = (y_n)$ are two distinct interior points of S, the annulus $A(x_n, y_n)$ contains no critical points of f_n ω-almost surely. By Lemma 3.14, there exists an $e \in \mathbb{N}$ such that $d(\mathscr{E}_{bc}(f_n)(x), \mathscr{E}_{bc}(f_n)(y)) = ed(x, y)$. This is true for any pairs of interior points, so $\mathscr{E}_{bc}(f_n)$ has to be linear with derivative e.

We will now show that $e = \deg_x \mathscr{E}_{bc}(f_n)$ for an interior point of E. Let $y \in E$ close to x so that no ends of critical values projects to $(a = \mathscr{E}_{bc}(f_n)(x), b = \mathscr{E}_{bc}(f_n)(y))$. Hence, we can choose representatives (a_n) and (b_n) for a, b so that $A(a_n, b_n)$ contains no critical values of f_n ω-almost surely. Let A_n be the component of $f_n^{-1}(A(a_n, b_n))$ approximating $D^{x,y}$. Then any other component of $f_n^{-1}(A(a_n, b_n))$ does not intersect $A(u_n, v_n)$ ω-almost surely. Hence by Lemma 3.13, $e = \deg_x \mathscr{E}_{bc}(f_n)$. $\square$

If E is an edge of the critical tree S, then the projection of critical ends are not in the interior of E, so we get

Corollary 3.27. *If E is an edge of the critical tree S, then $\mathscr{E}_{bc}(f_n)$ is linear with derivative e on S, where $e = \deg_x \mathscr{E}_{bc}(f_n)$ for any interior point of E.*

We will now prove some properties for the tangent maps. Recall the definition of tangent maps for a branched covering on an $\mathbb{R}$-tree is given in Section 3.1.

Lemma 3.28. *Let $a \in {}^r\mathbb{H}^3$, and E be a segment with an end point a. Let $\vec{v} \in T_a{}^r\mathbb{H}^3$ be the tangent vector associated to E, then*

$$\deg_{\vec{v}}(D_a\mathscr{E}_{bc}(f_n)) = \deg_x(\mathscr{E}_{bc}(f_n))$$

for $x \in E$ sufficiently close to a.

Proof. Let $x \in E$ be sufficiently close to a so that we can choose a neighborhood U containing $\mathscr{E}_{bc}(f_n)(a)$ and $\mathscr{E}_{bc}(f_n)(x)$ and V of $\mathscr{E}_{bc}(f_n)^{-1}(U)$ containing a such that $V \cap U_{\vec{v}}$ contains no other preimages $\mathscr{E}_{bc}(f_n)(x)$ than x. Then $\deg_x(\mathscr{E}_{bc}(f_n))$ and $\deg_{\vec{v}}(D_a\mathscr{E}_{bc}(f_n))$ both equal to the number of preimages of $y \in U \cap U_{\vec{v}} - \mathscr{E}_{bc}(f_n)(V \cap U_{\vec{v}} \cap S)$ in $V \cap U_{\vec{v}} - \mathscr{E}_{bc}(f_n)^{-1}(\mathscr{E}_{bc}(f_n)(V \cap U_{\vec{v}} \cap S))$, so they are equal. □

Lemma 3.29. *Let $a \in {}^r\mathbb{H}^3$ with $e = \deg_a(\mathscr{E}_{bc}(f_n))$, then*

$$\sum_{\deg_{\vec{v}}(D_a\mathscr{E}_{bc}(f_n)) \geq 2} \deg_{\vec{v}}(D_a\mathscr{E}_{bc}(f_n)) - 1 = 2e - 2$$

Proof. Let $\vec{v}_i \in T_a{}^r\mathbb{H}^3$ be those vectors associated to the critical tree, $i = 1, ..., k$. Note that $\vec{v}_i$ contains all vectors with degree ≥ 2. Choose $x_i \in U_{\vec{v}_i}$ close to a so that $\deg_{\vec{v}_i}(D_a\mathscr{E}_{bc}(f_n)) = \deg_{x_i}(\mathscr{E}_{bc}(f_n))$. Denote $y_i = \mathscr{E}_{bc}(f_n)(x_i)$, and let U be the component of ${}^r\mathbb{H}^3 - \{y_1, .., y_k\}$ containing $\mathscr{E}_{bc}(f_n)(a)$, and let V be the component of $\mathscr{E}_{bc}(f_n)^{-1}(U)$ containing a. Note that the critical ends project to the boundary of V, so we can choose $D_n^{x_1,...,x_k}$ such that the component D_n of $f_n^{-1}(D_n^{x_1,...,x_k})$ approximating V contains no critical points. Note the boundary of D_n associated to x_i is mapped as degree $\deg_{\vec{v}_i}(D_a\mathscr{E}_{bc}(f_n))$ covering to the boundary of $D_n^{x_1,\dots,x_k}$ associated to y_i. Therefore, by Riemann-Hurwitz formula for the degree e coverings on D_n we have

$$e(k-2) = (k \cdot e - \sum(\deg_{\vec{v}_i}(D_a \mathscr{E}_{bc}(f_n)) - 1)) - 2$$

so rearrange, we get

$$\sum_{\deg_{\vec{v}}(D_a \mathscr{E}_{bc}(f_n)) \geq 2} \deg_{\vec{v}}(D_a \mathscr{E}_{bc}(f_n)) - 1 = 2e - 2.$$

$\square$

Recall the *critical locus* $C := \{x \in {}^r\mathbb{H}^3 : \deg_x \mathscr{E}_{bc}(f_n) \geq 2\} \subset S$. We will call an edge E of S a *critical edge* if E is in C. We will show that

Theorem 3.30. *Let C be the critical locus for the map $\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$, then there is a partition $\{\kappa_1, ..., \kappa_{i_1}\} \cup ... \cup \{\kappa_{i_{l-1}+1}, ...\kappa_{2d-2}\}$ of the critical ends so that each set has at least 2 elements and*

$$C = \bigcup_{j=1}^{l} \mathrm{hull}(\kappa_{i_{j-1}+1}, ..., \kappa_{i_j})$$

with the convention $i_0 = 0$ and $i_l = 2d-2$.

Proof. Let κ be a critical end, and let E be the (infinite) edge of S associated to κ. We will show that the local degree is ≥ 2 at any interior point of E. Suppose not, then any interior point has degree 1. Let $x \in E$, and $\vec{v} \in T_x{}^r\mathbb{H}^3$ is the tangent vector associated to the end κ. Then $U_{\vec{v}}$ is mapped isometrically to $U_{\vec{w}}$ for some $\vec{w} \in T_y{}^r\mathbb{H}^3$ and $y = \mathscr{E}_{bc}(f_n)(x)$ as every point in $U_{\vec{v}}$ has degree 1. This is a contradiction to Proposition 3.25 as any $D_n^{\vec{v}}$ contains critical points associated to κ ω-almost surely.

Let E be an edge where the local degree is ≥ 2 at any interior point of E, and assume $a \in {}^r\mathbb{H}^3$ be its end point. By Lemma 3.28 and Lemma 3.29, there is another edge E' of S with a is an end point and the local degree is ≥ 2 on any interior point of E'.

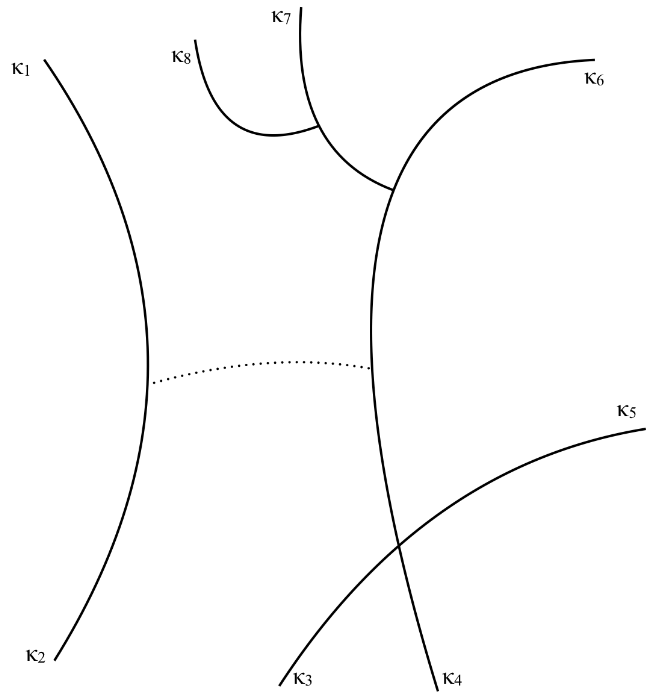

Figure 3.3: A schematic picture for the critical locus C. The ends are denoted by κ_i. The union of solid lines represents the critical locus C, and the union of solid and dotted lines represents the critical tree S. The map $\mathscr{E}_{bc}(f_n)$ has degree 1 at any point in the interior of the dotted edge, so it is an isometry on it. The map $\mathscr{E}_{bc}(f_n)$ has degree $e \geq 2$ at any point in the interior of the solid edge and is linearly expanding with factor e on it.

Now one can start from any critical end, and follow critical edges of S. We will not stop at an interior point by the argument in the previous paragraph. So we can define the partition by requiring critical ends that can be connected via critical edges to form a single set. The convex hull of this partition gives the critical locus. □

3.10 The minimal tree

Let f_n be a degenerating sequence of rational maps of degree d, with $r_n := \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$. Let $x^0 = (\mathbf{0}) \in {}^r\mathbb{H}^3$, and let $T \subset {}^r\mathbb{H}^3$ be the minimal tree containing x^0, i.e., smallest closed tree containing

x^0 that is invariant under $\mathscr{E}_{bc}(f_n)$ and $\mathscr{E}_{bc}(f_n)^{-1}$. We will show that the minimal tree T can be realized as union of Gromov-Hausdorff limit of the rescaled convex hulls of

$$\bigcup_{i=-k}^{k} \mathscr{E} f_n^i(\{\mathbf{0}\})$$

Recall that the *Hausdorff distance* between a pair of sets A, B in a metric space is defined by

$$d_H(A, B) = \max(\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)).$$

The *Gromov-Hausdorff distance* $d_{GH}(A, B)$ between a pair of abstract metric spaces is defined as the infimum of $d_H(A, B)$ over all metrics on $A \cup B$ extending the given metrics on A and B. The following compactness criterion is due to Gromov (See [Gro81] and Proposition 3.2 in [Bes88]):

Proposition 3.31. *Let A_n be a sequence of connected compact metric spaces. Suppose for any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that each A_n can be covered by $N(\epsilon)$ ϵ-balls. Then there is a subsequence of A_n that converges to a compact metric space in Gromov-Hausdorff metric.*

The following proposition is well known (See Proposition 11.2 in [McM09b] and Theorem 3.4 in [Bes88]):

Proposition 3.32. *Let $K_n \subset \mathbb{H}^3$ is a sequence of hyperbolic polyhedrons with k vertices. Let $r_n \to \infty$, so that the metric spaces $(K_n, d_{\mathbb{H}^3}/r_n)$ has bounded diameter, then $(K_n, d_{\mathbb{H}^3}/r_n)$ converges in Gromov-Hausdorff metric to a finite tree T.*

We remark that a finite tree T is uniquely determined by the distances between its endpoints. Thus a sequence of rescaled hyperbolic k-polyhedrons converges, in the Gromov-Hausdorff metric, as soon as the k^2 distances between its pairs of vertices converge.

Let $W_n^k = \{(\mathscr{E} f_n)^i(\mathbf{0}) : i = -k, ..., k\}$, and let $F_n^k := \mathrm{hull}(W_n^k) \subset \mathbb{H}^3$. We will also denote (X_n^k, d_n) the abstract compact metric space that is set-wise equal to F_n^k but $d_n(a, b) = d_{\mathbb{H}^3}(a, b)/r_n$ for $a, b \in X_n^k = F_n^k$.

Lemma 3.33. *Fix $k > 0$, there is a subsequence of X_n^k converges to a compact tree*

$$T^k = \mathrm{hull}(\bigcup_{i=-k}^{k} \mathscr{E}_{bc}(f_n)^i(x^0)) \subset {}^r\mathbb{H}^3$$

in Gromov-Hausdorff metric.

Proof. Since T^k is compact, we can choose a finite set of points $x_1, ..., x_k \in {}^r\mathbb{H}^3$ so that T^k is contained in the interior of the tree of $S^k := \mathrm{hull}(x_1, ..., x_k)$. Let $x_i = (x_{i,n})$, and we claim W_n^k is contained in the convex hull of $x_{1,n}, ..., x_{k,n}$ ω-almost surely. Suppose not, then there exists a sequence $x_n \in W_n^k$ such that (x_n) represents a point in $\bigcup_{i=-k}^{k} \mathscr{E}(f_n)^i(x^0) \subset {}^*\mathbb{H}^3$ but outside of T^k, which is a contradiction.

Since $\lim_\omega d_{\mathbb{H}^3}(x_{i,n}, x_{j,n})/r_n = d(x_i, x_j)$, passing to a subsequence, we may assume for all i, j,

$$\lim_{n\to\infty} d_{\mathbb{H}^3}(x_{i,n}, x_{j,n})/r_n = d(x_i, x_j).$$

Therefore $(\mathrm{hull}(x_{1,n}, ..., x_{k,n}), d_{\mathbb{H}^3}/r_n)$ converges in Gromov-Hausdorff metric to the tree S^k by Proposition 3.32 and the remark after it. We can assume X_n^k also converges in Gromov-Hausdorff metric to a subspace of S^k by passing to a further subsequence as X_n^k satisfies the criterion of Proposition 3.31 (as it is contained in the rescaled hyperbolic k-polyhedrons with bounded diameters).

Since for any point $x \in \bigcup_{i=-k}^{k} \mathscr{E}_{bc}(f_n)^i(x^0) \subset {}^r\mathbb{H}^3$, there exist $x_n \in W_n^k$ such that $(x_n) = x \in {}^r\mathbb{H}^3$ by Lemma 3.10, so the Gromov-Hausdorff limit of X_n^k contains T^k. It cannot contain more points by similar argument in the first paragraph. Therefore, X_n^k converges to T^k in Gromov-Hausdorff metric. $\square$

Hence, by a diagonal argument, we can pass to a subsequence, (which we still use index n to denote) so that X_n^k converges to T^k in the Gromov-Hausdorff metric for $k = 1, 2, ...$ Since $T = \bigcup_{k=1}^{\infty} T^k$, we have

Proposition 3.34. *The minimal tree T is the union of Gromov-Hausdorff limit of convex hull of*

$$\bigcup_{i=-k}^{k} \mathscr{E} f_n^i(\{\mathbf{0}\}).$$

Geometric limit

Recall that we say a sequence $f_n : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ converges *geometrically* to $F : T \longrightarrow T$ if there exists $h_n : (T, x^0) \longrightarrow (\mathbb{H}^3, \mathbf{0})$ such that

1. *Rescaling:* We have
$$d(x,y) = \lim d_{\mathbb{H}^3}(h_n(x), h_n(y))/r_n$$
for all $x, y \in T$.
2. *Conjugacy:* We have for all $x \in T$,
$$d_{\mathbb{H}^3}(h_n(F(x)), f_n(h_n(x)))/r_n \to 0$$
as $n \to \infty$.

We will now construct the approximating maps h_n. Recall that $T^k = \text{hull}(\bigcup_{i=-k}^{k} \mathscr{E}_{bc}(f_n)^i((\mathbf{0}))) \subset {}^r\mathbb{H}^3$. Set $T^0 = \{x^0\}$, and define

$$h_n^0 : T^0 \longrightarrow \mathbb{H}^3$$
$$x^0 \mapsto \mathbf{0}$$

Assume we have defined the map $h_n^k : T^k \longrightarrow \mathbb{H}^3$, such that for any $x \in T^k$, the sequence $(h_n(x))$ represents $x \in T^k \subset {}^r\mathbb{H}^3$. We will construct $h_n^{k+1} T^{k+1} \longrightarrow \mathbb{H}^3$ extending h_n^k. Since $T^k \subset T^{k+1}$ and T^{k+1} is a compact tree, we can construct the sequence $T^k = T_0^k \subset T_1^k \subset ... \subset T_m^k = T^{k+1}$ so that T_{i+1}^k is constructed from T_i^k by attaching a new edge. Let $[a, b]$ be the attaching edge to T_i^k, and say a is the attaching point, we can represent $a = (h_n(a))$. Choose a representative $b = (b_n)$ Let t denotes the point on $[a, b]$ of distance t to a, and we define h_n^{k+1} on $[a, b]$ so that $h_n^{k+1}(t)$ is the point on the geodesic $[h_n(a), b_n]$ of distance $t \cdot r_n$ away from a_n. Note that the sequence $(h_n^{k+1}(t))$ represents the point $t \in {}^r\mathbb{H}^3$. So after finitely many steps, we construct the map $h_n^{k+1} : T^{k+1} \longrightarrow \mathbb{H}^3$ such that for any $x \in T^{k+1}$, the sequence $(h_n(x))$ represents $x \in T^{k+1} \subset {}^r\mathbb{H}^3$.

Let $h_n : T \longrightarrow \mathbb{H}^3$, we will now prove Theorem 1.5. The idea of the proof is that we can find a countable dense subset W of T, and use a diagonal argument to construct a subsequence so that the limit of the rescaled distance between any two points in W exists and equals to the ultralimit. The rest is to check the rescaled distance actually converges to the ultralimit for any two points in T.

Proof of Theorem 1.5. The first statement follows from Proposition 3.34. To prove that $\mathscr{E} f_n$ converges geometrically to $\mathscr{E}_{bc}(f_n)$, we will show that after passing to a subsequence, h_n satisfies the rescaling and conjugacy conditions.

Let W be a countable dense subset of T, for any $x, y \in W$,

$$\lim_{\omega} d_{\mathbb{H}^3}(h_n(x), h_n(y))/r_n = d(x, y)$$

and for any $x \in W$,

$$\lim_{\omega} d_{\mathbb{H}^3}(h_n(\mathscr{E}_{bc}(f_n)(x)), \mathscr{E} f_n(h_n(x)))/r_n = 0$$

Hence, after passing to a subsequence and using a diagonal argument, we can assume for any points $x, y \in W$,

$$\lim_{n\to\infty} d_{\mathbb{H}^3}(h_n(x), h_n(y))/r_n = d(x, y)$$

and for any $x \in W$,

$$\lim_{n\to\infty} d_{\mathbb{H}^3}(h_n(\mathscr{E}_{bc}(f_n)(x)), \mathscr{E} f_n(h_n(x)))/r_n = 0$$

Now if $x, y \in T$, and suppose for contradiction that

$$\lim_{n\to\infty} d_{\mathbb{H}^3}(h_n(x), h_n(y))/r_n \neq d(x, y),$$

then there exists an $\epsilon > 0$ and a subsequence (which we still use index n to denote) so that $|d_{\mathbb{H}^3}(h_n(x), h_n(y))/r_n - d(x, y)| > \epsilon$. We may now choose $x', y' \in W$ such that

1. $d(x, x') < \epsilon/8$ and $d(y, y') < \epsilon/8$.
2. x, x' (and y, y') are in an edge of T^k for some k.

Then by the definition of h_n, we have $d_{\mathbb{H}^3}(h_n(x), h_n(x')) = d(x, x') \cdot r_n$ and similarly for y and y'. Therefore, for all sufficiently large n,

$$\begin{aligned}
|d_{\mathbb{H}^3}(h_n(x), h_n(y))/r_n - d(x, y)| &\leq |d_{\mathbb{H}^3}(h_n(x), h_n(y))/r_n - d_{\mathbb{H}^3}(h_n(x'), h_n(y'))/r_n| \\
&+ |d_{\mathbb{H}^3}(h_n(x'), h_n(y'))/r_n - d(x, y)| \\
&\leq d_{\mathbb{H}^3}(h_n(x), h_n(x'))/r_n + d_{\mathbb{H}^3}(h_n(y), h_n(y'))/r_n \\
&+ |d_{\mathbb{H}^3}(h_n(x'), h_n(y'))/r_n - d(x, y)| \\
&< \epsilon
\end{aligned}$$

where the last inequality holds as $d_{\mathbb{H}^3}(h_n(x'), h_n(y'))/r_n \to d(x', y')$. This is a contradiction, so

$$\lim_{n\to\infty} d_{\mathbb{H}^3}(h_n(x), h_n(y))/r_n = d(x, y).$$

Let $x \in T$, and suppose for contradiction that

$$\lim_{n\to\infty} d_{\mathbb{H}^3}(h_n(\mathscr{E}_{bc}(f_n)(x)), \mathscr{E} f_n(h_n(x)))/r_n \neq 0$$

then there exists an $\epsilon > 0$ and a subsequence (which we still use index n to denote) so that

$$d_{\mathbb{H}^3}(h_n(\mathscr{E}_{bc}(f_n)(x)), \mathscr{E} f_n(h_n(x)))/r_n > \epsilon.$$

We choose $x' \in W$ such that

1. $d(x, x') < \epsilon/3Cd$ where C is the constant in Theorem 1.1.
2. x, x' are in an edge of T^k for some k.

Then by the definition of h_n, we have $d_{\mathbb{H}^3}(h_n(x), h_n(x')) = d(x, x') \cdot r_n$. Therefore, for all sufficiently large n,

$$\begin{aligned} d_{\mathbb{H}^3}(h_n(\mathscr{E}_{bc}(f_n)(x)), \mathscr{E} f_n(h_n(x)))/r_n &\leq d_{\mathbb{H}^3}(h_n(\mathscr{E}_{bc}(f_n)(x)), h_n(\mathscr{E}_{bc}(f_n)(x')))/r_n \\ &+ d_{\mathbb{H}^3}(h_n(\mathscr{E}_{bc}(f_n)(x')), \mathscr{E} f_n(h_n(x')))/r_n \\ &+ d_{\mathbb{H}^3}(\mathscr{E} f_n(h_n(x')), \mathscr{E} f_n(h_n(x)))/r_n \\ &< \epsilon \end{aligned}$$

This is a contradiction, so

$$\lim_{n\to\infty} d_{\mathbb{H}^3}(h_n(\mathscr{E}_{bc}(f_n)(x)), \mathscr{E} f_n(h_n(x)))/r_n = 0$$

Therefore, $\mathscr{E} f_n : \mathbb{H}^3 \longrightarrow \mathbb{H}^3$ converges geometrically to $\mathscr{E}_{bc}(f_n) : T \longrightarrow T$.

□

3.11 Degenerating sequences of Blaschke products

Let $f_n(z) = z \prod_{i=1}^{d-1} \frac{z - a_{i,n}}{1 - \overline{a}_{i,n} z}$ with $|a_{i,n}| < 1$. Note that $f_n(z)$ commutes with $z \to \overline{z}$, so by naturality of the barycentric extension, $\mathscr{E} f_n$ preserves the hyperbolic plane bounded by the equator, which we denote it as

$\mathbb{H}^2$.

In [McM09b], McMullen constructed a geometric limit of a branched covering on a Ribbon $\mathbb{R}$-tree for a degenerating sequence f_n. The tree $(T_{Rib}, p) = \bigcup_{k=1}^{\infty}(T_{Rib}^k, p)$ where (T_{Rib}^k, p) is the Gromov-Hausdorff limit of the hyperbolic convex hull $(W_n^k, 0)$ of $\bigcup_{i=-k}^{k} f_n^i(0) \subset \mathbb{H}^2$ rescaled by $r_{n,Rib} = d_{\mathbb{H}^2}(0, f_n^{-1}(0))$. The limiting map $f_{Rib} : (T_{Rib}, p) \longrightarrow (T_{Rib}, p)$ is constructed by the limit of maps $f_n : (W_n^k, 0) \longrightarrow (W_n^{k+1}, 0)$. In [McM09b], it is proven that $f_{Rib} : (T_{Rib}, p) \longrightarrow (T_{Rib}, p)$ is minimal.

We will show that this geometric limit as branched covering on a Ribbon $\mathbb{R}$-tree coincide with the dynamics $\mathscr{E}_{bc}(f_n)$ on the minimal tree $T \subset {}^r\mathbb{H}^3$. First, we prove the following

Lemma 3.35. *Let $f_n(z) = z\prod_{i=1}^{d-1} \frac{z-a_{i,n}}{1-\bar{a}_{i,n}z}$ with $|a_{i,n}| < 1$ be a degenerating sequence of rational maps. If $f_n(x_n) = y_n$ and let $x_n = M_n(0)$, $y_n = L_n(0)$ where $M_n, L_n \in \mathrm{PSL}_2(\mathbb{C})$ preserving the unit circle, then $\deg(\lim_\omega L_n^{-1} \circ f_n \circ M_n) \geq 1$.*

Proof. Note that $L_n^{-1}\circ f_n \circ M_n$ is still of the form $z\prod_{i=1}^{d-1} \frac{z-b_{i,n}}{1-\bar{b}_{i,n}z}$ for some $|a_{i,n}| < 1$ as $L_n^{-1}\circ f_n\circ M_n(0) = 0$. Hence the degree of the ultralimit is greater or equal to 1 follows from direct computation. $\square$

As a corollary, we have

Corollary 3.36. *Let $f_n(z) = z\prod_{i=1}^{d-1} \frac{z-a_{i,n}}{1-\bar{a}_{i,n}z}$ with $|a_{i,n}| < 1$ be a degenerating sequence of rational maps. Let $r_{n,Rib} = d_{\mathbb{H}^2}(0, f_n^{-1}(0))$ and $r_n = d_{\mathbb{H}^3}(\mathbf{0}, (\mathscr{E} f_n)^{-1}(\mathbf{0}))$, then*

$$\lim_{\omega} r_{n,Rib}/r_n = 1$$

Proof. Let $z_n \in f_n^{-1}(0)$ which achieves $r_{n,Rib}$, and $M_n \in \mathrm{PSL}_2(\mathbb{C})$ preserving the unit circle and $z_n = M_n(0)$, then $\deg(\lim_\omega f_n \circ M_n) \geq 1$ by Lemma 3.35. So $\mathscr{E}(f_n)((M_n(\mathbf{0}))) = x^0 \in^* \mathbb{H}^3$ by Lemma 3.10 and hence $\lim_\omega r_{n,Rib}/r_n \leq 1$.

Similarly, let $z_n \in \mathscr{E} f_n^{-1}(\mathbf{0})$ which achieves r_n, and $M_n \in \mathrm{PSL}_2(\mathbb{C})$ preserving the unit circle and $z_n = M_n(\mathbf{0})$. Then $\mathscr{E}(f_n)((M_n(\mathbf{0}))) = x^0 \in^* \mathbb{H}^3$, so $\deg(\lim_\omega f_n \circ M_n) \geq 1$ by Lemma 3.10. Hence we can choose a bounded sequence L_n preserving the unit circle so that $f_n \circ M_n \circ L_n(0) = 0$. Hence $r_n \leq r_{n,Rib} + C$ for some constant C ω-almost surely. Since $\lim_\omega r_n = \infty$, so $\lim_\omega r_{n,Rib}/r_n \geq 1$.

Hence $\lim_\omega r_{n,Rib}/r_n = 1$ □

We will now prove Theorem 1.6:

Proof of Theorem 1.6. We will identify $\mathbb{H}^2$ with the hyperbolic plane bounded by the equator. We will now construct the conjugacy map $\Psi : (T_{Rib}, p) \longrightarrow (T, x^0)$ as follows. Let $x \in T_{Rib}$, then $x \in T^k_{Rib}$ for some k, and we let $(x_n) \in W^k_n \subset \mathbb{H}^2$ such that the Gromov-Hausdorff limit of x_n is x. We define $\Psi(x) = (x_n) \in {}^r\mathbb{H}^3$. We first note that this is well defined as if x'_n is another sequence, then $\lim_{n\to\infty} d_{\mathbb{H}^2}(x_n, x'_n)/r_{n,Rib} = 0$ so $\lim_\omega d_{\mathbb{H}^3}(x_n, x'_n)/r_n = \lim_\omega d_{\mathbb{H}^2}(x_n, x'_n)/r_n = 0$ by Corollary 3.36. Ψ is an isometry and a conjugacy can be checked similarly.

Note that each point has exactly d preimages counted for multiplicities for the map $f_{Rib} : (T_{Rib}, p) \longrightarrow (T_{Rib}, p)$, the image is invariant under $\mathscr{E}_{bc}(f_n)$ and $\mathscr{E}_{bc}(f_n)^{-1}$, so the image contains (T, x^0). If the image is not (T, x^0), then $\Psi^{-1}((T, x^0))$ will be invariant under f_{Rib} and f^{-1}_{Rib}. This is a contradiction to (T_{Rib}, p) is minimal. Hence the image is the minimal tree (T, x^0). □

We remark that the minimal tree (T, x^0) has a natural Ribbon structure as for any point $x \in T$, we can choose representatives (x_n) so that $x_n \in \mathbb{H}^2$. The isometry Ψ preserves the Ribbon structure.

3.12 Periodic ends and translation lengths

Let $\alpha : [0, \infty) \longrightarrow {}^r\mathbb{H}^3$ represents an end. If α is not a critical end, then $\mathscr{E}_{bc}(f_n)$ is isometry on $\alpha([K, \infty))$ for a sufficiently large K. This is because the end is eventually disjoint from the critical locus C. Hence $d(\alpha(t), x^0) - d(\mathscr{E}_{bc}(f_n)(\alpha(t)), x^0)$ is constant for $t \geq K$, so

$$L(\alpha, \mathscr{E}_{bc}(f_n)) = \lim_{x_i \to \alpha} d(x_i, x^0) - d(\mathscr{E}_{bc}(f_n)(x_i), x^0)$$

is well defined.

If α is a critical end, then by Corollary 3.27 and Theorem 3.30, $\mathscr{E}_{bc}(f_n)$ is expanding with derivative $e \in \mathbb{N}_{\geq 2}$ on $\alpha([K, \infty))$ for a sufficiently large K. Hence,

$$(d(\alpha(t), x^0) - d(\mathscr{E}_{bc}(f_n)(\alpha(t)), x^0)) - (d(\alpha(K), x^0) - d(\mathscr{E}_{bc}(f_n)(\alpha(K)), x^0)) = (1-e)(t-K)$$

for all $t \geq K$, so

$$L(\alpha, \mathscr{E}_{bc}(f_n)) = \lim_{x_i \to \alpha} d(x_i, x^0) - d(\mathscr{E}_{bc}(f_n)(x_i), x^0) = -\infty$$

We will now give a description of the action of $\mathscr{E}_{bc}(f_n)$ on the ends $\epsilon({}^r\mathbb{H}^3)$.

Proposition 3.37. *Let f_n be a degenerating sequence rational maps of degree d,*

$$r_n := \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0}),$$

and ${}^r\mathbb{H}^3$ be the asymptotic cone of $\mathbb{H}^3$ with rescaling r_n.

Let (z_n) represents an end $\alpha \in \epsilon({}^r\mathbb{H}^3)$, then $(f_n(z_n))$ represents the end $\mathscr{E}_{bc}(f_n)(\alpha) \in \epsilon({}^r\mathbb{H}^3)$.

Proof. Let $\alpha : [0, \infty) \longrightarrow {}^r\mathbb{H}^3$ be the end associated to (z_n). Let V_t denote the component of ${}^r\mathbb{H}^3 - \{\alpha(t)\}$ associated to the end α. We choose t large enough so that there is no preimage of x^0 in V_t. Let U_t be the component of ${}^r\mathbb{H}^3 - \{\mathscr{E}_{bc}(f_n)(\alpha(t))\}$ associated to the end $\mathscr{E}_{bc}(f_n)(\alpha)$. We claim that V_t is a component of $\mathscr{E}_{bc}(f_n)^{-1}(U_t)$. Indeed, if there exists $a \in V_t$ mapping to $\mathscr{E}_{bc}(f_n)(\alpha(t))$, then by choosing the component ${}^r\mathbb{H}^3 - \{\mathscr{E}_{bc}(f_n)(\alpha(t))\}$ containing x^0 and applying Proposition 3.25, we can find a preimage of x^0 in V_t which is a contradiction.

Let $V_{t,n}$ and $U_{t,n}$ be the domains approximating V_t and U_t, and $f_n : V_{t,n} \longrightarrow U_{t,n}$ is a covering. Hence $f_n(z_n) \in U_{t,n}$ ω-almost surely. Since this is true for all t, $f_n(z_n)$ represents the end $\mathscr{E}_{bc}(f_n)(\alpha)$. □

In particular, we have

Corollary 3.38. *If $z_{1,n}, ..., z_{q,n}$ is a periodic cycles of f_n of period q, then $(z_{i,n})$ represents a periodic cycle of ends with period q.*

We now prove Theorem 1.7:

Proof of Theorem 1.7. We may prove the case when $q = 1$. The general case can be proved by considering iterations.

By changing coordinates, we let 0 be the fixed point of f_n, and $\alpha : [0, \infty) \longrightarrow {}^r\mathbb{H}^3$ be the fixed end associated to (0) and we may assume that $\alpha(0) = x^0$. We first consider the case when α is not a critical end. Let $M_{t,n}(z) = e^{t \cdot r_n} z$, then the sequence $M_{t,n}$ represents $\alpha(t)$. Let $L_{t,n}(\mathbf{0}) = \mathscr{E} f_n \circ M_{t,n}(\mathbf{0})$ such that $L_{t,n}(0) = 0$. Let $f_t = \lim_\omega L_{t,n}^{-1} \circ f_n \circ M_{t,n} = H_t \varphi_{f_t}$ with $\deg(\varphi_{f_t}) = 1$ for sufficiently large t. Similar to the proof of Propostion 3.37, for sufficiently large t, a neighborhood of 0 is mapped to a neighborhood of 0 by $L_{t,n}^{-1} \circ f_n \circ M_{t,n}$, so 0 is not a hole of f. Therefore, we have

$$\lim_{\omega} (L_{t,n}^{-1} \circ f_n \circ M_{t,n})'(0) = \varphi'_{f_t}(0) \neq 0$$

Note that $\lim_\omega \log |M'_{t,n}(0)|/r_n = d(\alpha(t), x^0)$ and $\lim_\omega \log |L'_{t,n}(0)|/r_n = d(\mathscr{E}_{bc}(f_n)(\alpha(t)), x^0)$, so we get the result as $\log |\varphi'_{f_t}(0)| = O(1)$. Hence by passing to a subsequence, we get $\lim_{n\to\infty} \frac{\log |(f_n^q)'(z_{1,n})|}{r_n} = L(C, \mathscr{E}_{bc}(f_n))$.

If α is a critical end, then we can show $\lim_{n\to\infty} \frac{\log |(f_n^q)'(z_{1,n})|}{r_n} = -\infty$ in a similar way. $\square$

4

Connections with Berkovich dynamics

4.1 Rational maps on Berkovich projective space $\mathbb{P}^1_{Berk}$

In this section, we will give a brief introduction of the Berkovich projective space $\mathbb{P}^1_{Berk}$ for a complete, algebraically closed non-Archimedean field K, and the dynamics of rational maps on it. We are going to summarize some of the properties, and refer the readers to [BR10] for more detailed exposition of this grand theory.

Let K be a complete, algebraically closed non-Archimedean field, we will use notations $B(a,r) := \{z \in K : |z-a| \le r\}$ and $B(a,r)^- := \{z \in K : |z-a| < r\}$ to denote the *closed ball* and *open ball* centered at a with radius r respectively. Recall that in an non-Archimedean field, any point $z \in B(a,r)$

(or $z \in B(a,r)^-$) is the center of the ball. If two balls intersect, then one is contained in the other.

The valuation ring of K will be denoted as $\mathfrak{O}_K = B(0,1)$, and its maximal ideal is $\mathfrak{M}_K = B(0,1)^-$. The residual field is $\tilde{K} = \mathfrak{O}_K/\mathfrak{M}_K$.

Let a rational map $f \in \mathrm{Rat}_d(K)$ with coefficients in K, after multiplying the denominator and numerator by a common factor, we may assume that the maximum norm of the coefficients is 1. The reduction map $\tilde{f}$ is given by taking the reduction on its coefficients.

The Berkovich affine space and the Berkovich projective space

As a topological space, $\mathbb{A}^1_{Berk}$ can be defined as follows. The underlying point set is the collection of all the multiplicative seminorms $[\,]_x$ on the polynomial ring $K[T]$ which extend the absolute value on K. The topology on $\mathbb{A}^1_{Berk}$ is the weakest one for which $x \to [f]_x$ is continuous for all $f \in K[T]$. The field K can be thought of as a subspace of $\mathbb{A}^1_{Berk}$, via the evaluation map. That is, we can associate to a point $x \in K$ the seminorm

$$[f]_x = |f(x)|$$

Those seminorms of this form will be called *classical points*.

The Berkovich projective space $\mathbb{P}^1_{Berk}$ is the one point compactification of $\mathbb{A}^1_{Berk}$. The extra point, which is denoted, as usual, by ∞, can be regarded as the point $\infty \in \mathbb{P}^1_K$ embedded in $\mathbb{P}^1_{Berk}$.

This definition, however, does not make clear why a rational map $f \in \mathrm{Rat}_d(K)$ induces a map on $\mathbb{P}^1_{Berk}$. We will now give many different but equivalent ways of viewing this space.

Before proceeding further, we note that given closed ball $B(a,r)$, one can construct the *supremum norm*

$$[f]|_{B(a,r)} = \sup_{z \in B(a,r)} |f(z)|.$$

One of the miracles of the non-Archimedean universe is that this norm is *multiplicative*. More generally, given any decreasing sequence of closed balls $x = \{B(a_i, r_i)\}$, we can consider the limit seminorm

$$[f]_x = \lim_{i \to \infty} [f]_{B(a_i, r_i)}.$$

Berkovich's classification asserts that every point $x \in \mathbb{A}^1_{Berk}$ arises in this way, and we can classify them into 4 types:

1. Type I: Points in $\mathbb{A}^1_K$, which we will also call the *classical points*;
2. Type II: Points corresponding to a closed ball $B(a, r)$ with $r \in |K^\times|$;
3. Type III: Points corresponding to a closed ball $B(a, r)$ with $r \notin |K^\times|$;
4. Type IV: Points corresponding to a nested sequence $\{B(a_i, r_i)\}$ with empty intersection.

Type I, II and III can be thought of a special case of Type IV: the classical points correspond to a nested sequence $\{B(a_i, r_i)\}$ with $\lim r_i = 0$; the Type II points correspond to a nested sequence $\{B(a_i, r_i)\}$ with nonempty intersection and $r = \lim r_i > 0$ belongs to the value group $|K^\times|$; the Type III points correspond to a nested sequence $\{B(a_i, r_i)\}$ with nonempty intersection but $r = \lim r_i > 0$ does not belong to the value group $|K^\times|$. We will call the point corresponding to $B(0, 1)$ the *gauss point* and is denoted by x_g.

Another way of viewing the Berkovich projective space $\mathbb{P}^1_{Berk}$ is to use the 'Proj' construction. This point of view allows us to construct a natural action of $f \in \mathrm{Rat}_d(K)$.

We consider S as the set of multiplicative seminorms on the two-variable polynomial ring $K[X, Y]$ which extend the absolute value on K, and which are not identically zero on the maximal ideal (X, Y) of $K[X, Y]$. We will use $\llbracket\,\rrbracket$ to emphasize that these are seminorms on the two-variable ring. We put an equivalence relation on S by declaring that $\llbracket\,\rrbracket_1 \sim \llbracket\,\rrbracket_2$ if and only if there exists a constant $C > 0$ such that for all $d \in \mathbb{N}$ and all homogeneous polynomials $G \in K[X, Y]$ of degree d, $\llbracket G \rrbracket_1 = C^d \llbracket G \rrbracket_2$.

As a set, $\mathbb{P}^1_{Berk}$ is the collection of equivalence classes of S. One can choose a representative $\llbracket\,\rrbracket^*$ so that $\max(\llbracket X \rrbracket^*, \llbracket Y \rrbracket^*) = 1$, which we will call it *normalized*. Note that by the equivalence relation, it is easy to check that for any normalized seminorm $[\![\ ]\!]^*$ in an equivalence class, it gives the same values on homogeneous polynomials. We put the topology on $\mathbb{P}^1_{Berk}$ to be the weakest so that $x \to [\![G]\!]^*_x$ is continuous for any homogeneous $G \in K[X,Y]$.

$\mathbb{P}^1_K$ naturally embeds into $\mathbb{P}^1_{Berk}$ via $[a:b] \to [\![G]\!]_{[a:b]} = |G(a,b)|$. It can be checked that $\mathbb{P}^1_{Berk} - \{\infty\}$ and $\mathbb{P}^1_{Berk} - \{0\}$ are both homeomorphic to $\mathbb{A}^1_{Berk}$, and one can construct $\mathbb{P}^1_{Berk}$ by gluing two copies of $\mathbb{A}^1_{Berk}$ on their common intersection $\mathbb{A}^1_{Berk} - \{0\}$.

Rational maps on $\mathbb{P}^1_{Berk}$

Let $f \in \mathrm{Rat}_d(K)$, we can write f is the ratio of two homogeneous $f(T) = F_1(X,Y)/F_2(X,Y)$ where $T = X/Y$. Let $x \in \mathbb{P}^1_{Berk}$, we can define $[\![\ ]\!]_{f(x)}$ by

$$[\![G]\!]_{f(x)} := [\![G(F_1(X,Y), F_2(X,Y))]\!]_x$$

for $G \in K[X,Y]$. It can be checked that $[\![\ ]\!]_{f(x)}$ is a multiplicative seminorm on $K[X,Y]$ which extend the absolute value on K, and which are not identically zero on the maximal ideal (X,Y) of $K[X,Y]$. This gives the natural action of f on $\mathbb{P}^1_{Berk}$. Note this defines the usual action on $\mathbb{P}^1_K$, hence we can regard this natural action of f as an extension to $\mathbb{P}^1_{Berk}$. It can also be shown that this action preserves the types of the points (see Proposition 2.15 in [BR10]).

If M is a rational map of degree 1, i.e., $M \in \mathrm{PSL}_2(K)$, then this action can be viewed via the action on the balls: if $x \in \mathbb{P}^1_{Berk}$ corresponds to a nested sequence of balls $\{B(a_i, r_i)\}$, then $M(x)$ corresponds to the nested sequence of balls $\{M(B(a_i, r_i))\}$. Given any Type II point x, there exists $M \in \mathrm{PSL}_2(K)$ such that $M(x_g) = x$. We will regard M as 'change of coordinates'. We now give another point of view of the natural action of the rational map via change of coordinates. The following proposition can be proved using Lemma 2.17 in [BR10]:

Proposition 4.1. *Let $f \in \mathrm{Rat}_d(K)$, $x, y \in \mathbb{P}^1_{Berk}$ are two type II points. Assume that $x = M(x_g)$ and*

$y = L(x_g)$ with $M, L \in \mathrm{PSL}_2(K)$, then $f(x) = y$ if and only if $L^{-1} \circ f \circ M$ has non constant reduction.

The tree structure on $\mathbb{H}_{Berk}$ and $\mathbb{P}^1_{Berk}$

The *Berkovich hyperbolic space* $\mathbb{H}_{Berk}$ is defined by

$$\mathbb{H}_{Berk} = \mathbb{P}^1_{Berk} - \mathbb{P}^1_K = \mathbb{A}^1_{Berk} - \mathbb{A}^1_K$$

Note that $\mathbb{H}_{Berk}$ is also the space of Type II, III and IV points.

Given two Type II or III points x, y corresponding to the balls $B(a, r)$ and $B(b, s)$ respectively, we let $B(a, R)$ be the smallest ball containing both $B(a, r)$ and $B(b, s)$. Note that $R = \max(r, s, |a - b|)$. We define the distance function

$$d(x, y) = 2 \log R - \log r - \log s$$

Note that if $B(a, r)$ is contained in $B(b, s)$, then $d(x, y) = \log s - \log r = \log s/r$, which should be interpreted roughly as the modulus of the open 'annulus' $B(b, s)^- - B(a, r)$. In general, the distance is the sum of modulus of $B(a, R)^- - B(a, r)$ and $B(b, R)^- - B(b, s)$.

One can extend this distance formula continuously to arbitrary points $x, y \in \mathbb{H}_{Berk}$. The metric space $(\mathbb{H}_{Berk}, d)$ can be shown to be a complete $\mathbb{R}$-tree (see Proposition 2.29 in [BR10]). Moreover, the finite ends of the $\mathbb{R}$-tree correspond to the Type IV points, while the infinite ends of the $\mathbb{R}$-tree correspond to the classical (Type I) points.

We should remark that the topology generated by the metric d is strictly finer than the subspace topology of the Berkovich topology on $\mathbb{H}_{Berk}$.

4.2 Robinson's field

In this section, we will introduce a complete, algebraically closed, non-Archimedean field ${}^{\rho}\mathbb{C}$. This field is introduced by Robinson in the real case in the study of non-standard analysis. It should be thought of as a complexified version of Robinson's field.

Recall that we have fixed a non-principal ultrafilter ω of $\mathbb{N}$. Consider $\mathbb{C}^{\mathbb{N}}$ consisting of all sequences in $\mathbb{C}$. We say two sequence (z_n) and (w_n) are equivalent if

$$z_n = w_n \quad \omega - \text{alomost surely}$$

The set of equivalence classes will be denoted by ${}^*\mathbb{C}$.

We define addition and multiplication as follows: let $x, y \in {}^*\mathbb{C}$ be represented by (x_n) and (y_n), then we define $x+y$ and $x \cdot y$ as the class represented by (x_n+y_n) and $(x_n \cdot y_n)$. It can be checked that these are indeed well defined, and make ${}^*\mathbb{C}$ a field. This field is usually referred to as the *ultrapower construction* for $\mathbb{C}$ (Cf. Chapter 2 in [LR75]).

To simplify the notations, we will sometimes use a single roman letter to represent a number in ${}^*\mathbb{C}$. Given two numbers $x, y \in {}^*\mathbb{C}$ represented by (x_n) and (y_n), we write $|x| \leq |y|$ or $|x| < |y|$ if $|x_n| \leq |y_n|$ or $|x_n| < |y_n|$ ω-almost surely.

The field ${}^*\mathbb{C}$ is usually too big to work with in our applications, and is not equipped with a norm. We will construct a more useful field ${}^{\rho}\mathbb{C}$ as the quotient of a subspace of ${}^*\mathbb{C}$, similar as ${}^{r}\mathbb{H}^3$ to ${}^*\mathbb{H}^3$ in Chapter 3.

Given a positive sequence $\rho_n \to 0$, which we can regard as $\rho \in {}^*\mathbb{C}$. With the notations above, we construct

$$M_0 = \{t \in {}^*\mathbb{C} : \text{There exists some } N \in \mathbb{N} \text{ such that } |t| < \rho^{-N}\}$$

and

$$M_1 = \{t \in {}^*\mathbb{C} : \text{For all } N \in \mathbb{N}, |t| < \rho^N\}$$

It is easy to show that both M_0 and M_1 form rings with respect to the addition and multiplication of ${}^*\mathbb{C}$. It can also be shown that M_1 is a maximal ideal of ring M_0 (Cf. Chapter 3.3 in [LR75]). We define ${}^\rho\mathbb{C} = M_0/M_1$ as the quotient field. Note that $\mathbb{C}$ embeds into ${}^\rho\mathbb{C}$ via constant sequences.

Intuitively, the field ${}^\rho\mathbb{C}$ lies in between $\mathbb{C}$ and ${}^*\mathbb{C}$ consisting of those large infinitesimals and small infinite numbers. We shall regard each member of $t \in M_1$ as a small infinitesimal, and its multiplicative inverse (provided that $t \neq 0$) a large infinite number. Using the terminologies in [LR75], each number in M_1 will be called an *iota* and the multiplicative inverse of a non zero number in M_1 will be called a *mega*.

We can define an equivalence relation on ${}^*\mathbb{C}$ by declaring $x \sim y$ if $x - y$ is an iota. Note that in particular, if $y \in M_0$, then $x \sim y$ if and only if $x \in [y]$ as a member of ${}^\rho\mathbb{C}$.

Non-Archimedean norm on ${}^\rho\mathbb{C}$

One of the many desired properties of ${}^\rho\mathbb{C}$ is that we can put a norm on it. Let $x \in M_0 - M_1$ and $i \in M_1$ represented by (x_n) and (i_n) respectively. Note that there exist n, m such that $\rho^n \leq |x| < \rho^{-m}$, hence the ultralimit

$$\log_\rho |x| := \lim_\omega \log |x_n| / \log \rho_n$$

is finite. Since $i \in M_1$, so $|i_n| < \rho_n^n$ for any $n \in \mathbb{N}$. Note

$$\log_\rho |x+i| - \log_\rho |x| = \lim_\omega \log_{\rho_n} |\frac{x_n + i_n}{x_n}| = \lim_\omega \frac{\log |1 + i_n/x_n|}{\log \rho_n}$$

Since $x_n \notin M_1$, $\lim_\omega i_n/x_n = 0$, but $\lim_\omega |\log \rho_n| = \infty$. Hence $\log_\rho |x+i| - \log_\rho |x| = 0$.

We now define a valuation of an element $[x] \in {}^{\rho}\mathbb{C}$ by

$$\nu([x]) = \log_{\rho} |x|$$

where $x \in {}^{\rho}\mathbb{C}$ is a representative of $[x]$.

To illustrate the definition, notice that

$$\nu([\rho]) = 1$$

or more generally,

$$\nu([\rho^n]) = n \text{ for } n \in \mathbb{R}$$

To simplify the notations, from now on, we will use a single roman letter to represent a number in ${}^{\rho}\mathbb{C}$, and drop the square bracket.

It can be easily checked that for $x, y \in {}^{\rho}\mathbb{C}$ (Cf. Chapter 3 Lemma 3.1 and 3.2 in [LR75]), we have

$$\nu(x \cdot y) = \nu(x) + \nu(y)$$

$$\nu(x + y) \geq \min(\nu(x), \nu(y))$$

Hence, ν defines a non-Archimedean valuation on ${}^{\rho}\mathbb{C}$, and this valuation naturally gives rise to a non-Archimedean norm via

$$|x|_{\nu} = e^{-\nu(x)}$$

The distance function is given by

$$d(x, y) = |x - y|_{\nu}$$

$({}^{\rho}\mathbb{C}, d)$ is complete and spherically complete

Recall that a metric space X is said to be *spherically complete* if for any nested sequence of (closed) balls $B_0 \supset B_1 \supset$ ..., their intersection $\bigcap_j B_j$ is non-empty. In this subsection, we will show that ${}^{\rho}\mathbb{C}$ is spherically complete:

Theorem 4.2. *The field* $({}^{\rho}\mathbb{C}, d)$ *is spherically complete.*

Proof. Let $B_0' \supset B_1' \supset$ ... be a decreasing sequence of closed balls. We consider a decreasing sequence of open balls B_n so that $B_n' \supset B_n \supset B_{n+1}'$, and assume that B_i has radius r_i, and denote $q_i = -\log r_i$. Pick $\alpha_i \in B_i$, and assume that α_i is represented by $(a_{i,n})$. Since $B_j \subset B_i$ for all $j \geq i$, we know

$$|\alpha_i - \alpha_j| < r_i$$

Equivalently,

$$\nu(\alpha_i - \alpha_j) = \lim_{\omega} \log |a_{i,n} - a_{j,n}| / \log \rho_n > q_i$$

We can construct inductively a decreasing sequence $\mathbb{N} = N_0 \supset N_1 \supset$ ... such that

1. N_k is ω-big;
2. $\bigcap_{k=1}^{\infty} N_k = \emptyset$;
3. For any $i \leq j \leq k$ and $l \in N_k$, we have

$$\nu_l(a_{i,l} - a_{j,l}) := \log |a_{i,l} - a_{j,l}| / \log \rho_l > q_i.$$

Indeed, we can set $N_0 = \mathbb{N}$ as the base case. Assume that N_k is constructed, to construct N_{k+1}, we note that for any $i \leq k+1$,

$$\nu(\alpha_i - \alpha_{k+1}) = \lim_{\omega} \log |a_{i,n} - a_{k+1,n}| / \log \rho_n > q_i.$$

Hence, there exists an ω-big set N so that for all $i \leq k+1$ and $l \in N$,

$$\nu_l(a_{i,l} - a_{k+1,l}) > q_i.$$

We define $N_{k+1} = N \cap N_k \cap \{n : n \geq k+1\}$, then $N_{k+1} \subset N_k$ is still ω-big. Property (3) is satisfied by induction hypothesis and by the definition of N. Property (2) holds as $N_k \subset \{n : n \geq k\}$ by construction.

We now define the sequence $a_n := a_{k,j}$ for $j \in N_k - N_{k-1}$, and let $\alpha = (a_n)$. Note that for any $l \in N_i$, by Property (2), $l \in N_k - N_{k-1}$ for some $k \geq i$. Hence for any $i \in \mathbb{N}$ and $l \in N_i$,

$$\nu_l(a_{i,l} - a_l) = \nu_l(a_{i,l} - a_{k,l}) > q_i.$$

Therefore, $\nu(\alpha_i - \alpha) > q_i$. This means that $|\alpha_i - \alpha| < r_i$, so $\alpha \in B_i$.

Since this holds for any i, we conclude that $\alpha \in \bigcap_i B_i$, so $\bigcap_i B_i \neq \emptyset$. Therefore, $\bigcap_i B_i' \neq \emptyset$ as well. $\square$

As an immediate corollary, we have (cf. Chapter 3 Theorem 4.1 in [LR75]):

Corollary 4.3. *The field $({}^\rho\mathbb{C}, d)$ is complete.*

${}^\rho\mathbb{C}$ is algebraically closed

Theorem 4.4. *${}^\rho\mathbb{C}$ is algebraically closed.*

Proof. Let $z^d + a_{d-1}z^{d-1} + ... + a_0$ be a monic polynomial with coefficients $a_n = (a_{n,k}) \in {}^\rho\mathbb{C}$. We assume that $M < \min(0, \nu(a_0), ..., \nu(a_{d-1}))$. Hence there is a ω-big set $N \subset \mathbb{N}$ so that for all $k \in N$ and $n = 0, ..., d-1$,

$$|a_{n,k}| < \rho_k^M$$

Now let $f_k(z) = a_{d-1,k}z^{d-1} + ... + a_{0,k}$ and $g(z) = z^d$. Note that for any $k \in N$, we have on the circle centered at 0 of radius $d \cdot \rho_k^M$ (note that $\rho_k^M > 1$ as $M < 0$) that

$$\begin{aligned} |f_k(z)| &\leq |a_{d-1,k}| \cdot (d \cdot \rho_k^M)^{d-1} + ... + |a_{0,k}| \\ &< \rho_k^M \cdot d \cdot (d \cdot \rho_k^M)^{d-1} = |g(z)| \end{aligned}$$

By Roché's theorem, there are d solutions of $g + f_k(z) = 0$ in the ball $B(0, d \cdot \rho_k^M)$. Let x_k be such a root. Note that x_k is defined on an ω-big set N. So $x = (x_k)$ represents a point in ${}^\rho\mathbb{C}$ as $|x_k| < \rho_k^{M+1}$ for all $k \in N$. Moreover, x satisfies the equation $z^d + a_{d-1}z^{d-1} + ... + a_0 = 0$. Therefore, ${}^\rho\mathbb{C}$ is algebraically closed. □

The residue field of ${}^\rho\mathbb{C}$ and a cascade of Robinson's fields

Recall that the residual field of a non-Archimedean field K is the quotient $\tilde{K} = \mathfrak{O}_K/\mathfrak{M}_K$ where $\mathfrak{O}_K = B(0,1)$ and $\mathfrak{M}_K = B(0,1)^-$. For the field ${}^\rho\mathbb{C}$, one can represent a non-zero element in the residual field by a sequence (z_n) with $\lim_\omega \log|z_n|/\log\rho_n = 0$. Two sequences (z_n) and (w_n) are said to be equivalent if $\lim_\omega \log|z_n - w_n|/\log\rho_n > 0$.

Let σ_n be a positive sequence with $\sigma \to 0$, which we can regard as $\sigma \in {}^*\mathbb{C}$. We also assume that $\lim_\omega \log\sigma_n/\log\rho_n = 0$, in other words, ρ_n goes to 0 super-polynomially compared to ω_n.

We consider the following subset of $\widetilde{{}^\rho\mathbb{C}}$

$$M_0^\sigma = \{[t] \in \widetilde{{}^\rho\mathbb{C}} : t \in {}^*\mathbb{C}, |t| < \sigma^{-N} \text{for some } N \in \mathbb{N}\}$$

Note that M_0^σ is well defined. Indeed, if $t' \in {}^*\mathbb{C}$ is another representation of $[t]$, then $|t - t'| < \rho^\alpha$ for some $\alpha > 0$. Therefore

$$|t'| < |t| + |t - t'| < \sigma^{-N} + \rho^\alpha < \sigma^{-N-1}$$

Similarly, the set

$$M_1^\sigma = \{[t] \in \widetilde{{}^\rho\mathbb{C}} : t \in {}^*\mathbb{C}, |t| < \sigma^N \text{for any } N \in \mathbb{N}\}$$

is well defined. It follows directly from the definition, the field M_0^σ / M_1^σ is isomorphic to ${}^\sigma\mathbb{C}$. Inductively, we can construct another Robinson's field as a quotient of the subset of the Residual field ${}^\sigma\mathbb{C}$, we summarize as follows.

Given a sequence of positive sequences $\rho_{n,k}$ with $\lim_k \rho_{n,k} = 0$, such that for any n,

$$\lim_\omega \log \rho_{n+1,k} / \log \rho_{n,k} = 0$$

We can construct a sequence of Robinson's field $\rho_n\mathbb{C}$. Each one $\rho_n\mathbb{C}$ can constructed as a quotient of the subset of the Residual field of the previous one $\rho_{n-1}\mathbb{C}$. We will call such a configuration a *cascade of Robinson's field*.

Embedding of the field of Puiseux series $\mathbb{L}$

In this subsection, we will show how to embed the *field of formal Puiseux series* $\mathbb{L}$ into the Robinson's field ${}^\rho\mathbb{C}$ (Cf. Chapter 3 Section 6 in [LR75]).

The field $\mathbb{L}$ is the algebraic closure of the completion of the *field of formal Laurent series* $\mathbb{C}((t))$. An element in $\mathbf{a} \in \mathbb{L}$ can be represented by a formal series

$$\mathbf{a} = \sum_{j \geq 0} a_j t^{\lambda_j}$$

where $a_j \in \mathbb{C}$, $\lambda_j \in \mathbb{Q}$ if a_j does not vanish for sufficiently large j, then $\lambda_j \to \infty$ as $j \to \infty$. The absolute value is given by

$$|\mathbf{a}| = \exp(-\min\{\lambda_j : a_j \neq 0\})$$

provided $\mathbf{a} \neq \mathbf{0}$.

To show we have an embedding we first prove the following lemma about convergence of series in ${}^{\rho}\mathbb{C}$.

Lemma 4.5. *Let $a_j \in \mathbb{C}$, and λ_j be an unbounded increasing sequence of $\mathbb{R}$. Then the series*

$$\sum_{j=0}^{\infty} a_j \rho^{\lambda_j}$$

converges in ${}^{\rho}\mathbb{C}$.

Moreover, $|\sum_{j=0}^{\infty} a_j \rho^{\lambda_j}| = \exp(-\min\{\lambda_j : a_j \neq 0\})$.

Proof. Let $\alpha_j = a_j \rho^{\lambda_j}$. If $a_j = 0$, then $\nu(\alpha_j) = \infty$. Otherwise, $\nu(\alpha_j) = \nu(\rho^{\lambda_j}) = \lambda_j$. Since $\lim \lambda_j = \infty$, so $\lim_{\nu}(\alpha_j) = \infty$. Hence, the series $\sum \alpha_j$ converges in ${}^{\rho}\mathbb{C}$ by the convergence criterion in non-Archimedean field.

For the moreover part, let $\sigma_n = \sum_{j=0}^{n} a_j \rho^{\lambda_j}$ be the associated partial sums. Without loss of generality, we assume that $a_0 \neq 0$, then $\nu(\sigma_n) = \lambda_0$ for all n by the strong triangle inequality. Therefore $|\sum_{j=0}^{\infty} a_j \rho^{\lambda_j}| = \exp(-\min\{\lambda_j : a_j \neq 0\})$. □

We will now define $\Psi : \mathbb{L} \longrightarrow {}^{\rho}\mathbb{C}$ as follows. Let $\mathbf{a} = \sum_{j \geq 0} a_j t^{\lambda_j} \in \mathbb{L}$, we define

$$\Psi(\mathbf{a}) = \sum_{j \geq 0} a_j \rho^{\lambda_j} \in {}^{\rho}\mathbb{C}$$

Note that the series converges by Lemma 4.5. One can easily verify that $\Psi(\mathbf{a} + \mathbf{b}) = \Psi(\mathbf{a}) + \Psi(\mathbf{b})$ and $\Psi(\mathbf{a} \cdot \mathbf{b}) = \Psi(\mathbf{a}) \cdot \Psi(\mathbf{b})$. Hence we have

Proposition 4.6. *The map*

$$\Psi : \mathbb{L} \longrightarrow {}^{\rho}\mathbb{C}$$

$$\mathbf{a} = \sum_{j \geq 0} a_j t^{\lambda_j} \mapsto \sum_{j \geq 0} a_j \rho^{\lambda_j}$$

is an embedding of fields and preserves the non-Archimedean norms.

4.3 Proof of Theorem 1.3 and 1.4

In this section, it is better to use the upper space model H of the hyperbolic 3-space $\mathbb{H}^3$. We can identify $H = \mathbb{C} \times \mathbb{R}_{>0}$, and a linear map $M(z) = Az + B$ extends to an isometry on H given by

$$M(z, h) = (Az + B, |A|h). \tag{4.3.1}$$

The distance between two points (z_1, h_1) and (z_2, h_2) is given by the formula

$$d((z_1, h_1), (z_2, h_2)) = 2 \log \frac{\sqrt{|z_1 - z_2|^2 + (h_1 - h_2)^2} + \sqrt{|z_1 - z_2|^2 + (h_1 + h_2)^2}}{2\sqrt{h_1 h_2}} \tag{4.3.2}$$

Since the field ${}^{\rho}\mathbb{C}$ is spherically complete (see Theorem 4.2) and has valuation group $|{}^{\rho}\mathbb{C}^{\times}| = \mathbb{R}$, we know $\mathbb{H}_{Berk}$ consists of only of Type II points. Hence, by Berkovich's classification, every point $x \in \mathbb{H}_{Berk}$ can be represented by a closed ball $B(p, R)$. We consider a linear polynomial of the form

$$M(z) = az + b \in \mathrm{PSL}_2({}^{\rho}\mathbb{C}),$$

with $M(B(0, 1)) = B(p, R)$. Representing a and b by the sequences (a_n) and (b_n), we get a sequence of Möbius transformations

$$M_n(z) = a_n z + b_n.$$

Let ${}^{r}\mathbb{H}^3$ be the asymptotic cone of $\mathbb{H}^3$ with respect to rescaling $r_n = -\log \rho_n$. We define $\Phi : \mathbb{H}_{Berk} \longrightarrow {}^{r}\mathbb{H}^3$

$$\Phi(x) = (M_n(\mathbf{0})) \in {}^{r}\mathbb{H}^3.$$

We will now check that this definition is well defined. If we have a different representation $L_n(z) = a'_n z + b'_n$, where $(a'_n), (b'_n)$ represent a' and b', then $|a| = |a'| = R$ and $|b - b'| \leq R$. Without loss of generality, we may assume $|a_n| \geq |a'_n|$ ω-almost surely. Hence we have given any $\epsilon > 0$,

1. $\log |b_n - b'_n| / \log \rho_n > -\log R - \epsilon$ ω-almost surely;
2. $\frac{|a_n|}{|a'_n|} < \rho_n^{-\epsilon}$ ω-almost surely.

Rearranging the first inequality and using the fact that $|a'| = R$, we conclude that for any $\epsilon > 0$,

1. $|b_n - b'_n| / |a'_n| < \rho_n^{-\epsilon}$ ω-almost surely;
2. $\frac{|a_n|}{|a'_n|} < \rho_n^{-\epsilon}$ ω-almost surely.

Consider $L_n^{-1} \circ M_n(z) = \frac{a_n}{a'_n} z + \frac{(b_n - b'_n)}{a'_n}$, then using equations 4.3.1 and 4.3.2, we conclude that for any $\epsilon > 0$, on an ω-big set,

$$
\begin{aligned}
d(L_n(\mathbf{0}), M_n(\mathbf{0})) &= d(\mathbf{0}, L_n^{-1} \circ M_n(\mathbf{0})) \\
&= d((0,1), (|\frac{(b_n - b'_n)}{a'_n}|, |\frac{a_n}{a'_n}|)) \\
&< 2 \log \frac{\sqrt{\rho^{-\epsilon} + (\rho^{-\epsilon} - 1)^2} + \sqrt{\rho^{-\epsilon} + (\rho^{-\epsilon} + 1)^2}}{2} \\
&< 2 \log (2\rho^{-\epsilon}) \\
&= 2 \log 2 + \epsilon r_n = O(\epsilon \cdot r_n).
\end{aligned}
$$

Since ϵ is arbitrary, we conclude $(L_n(\mathbf{0}))$ and $(M_n(\mathbf{0}))$ represent the same point in ${}^r\mathbb{H}^3$. Therefore, Φ is a well-defined map.

We will now show that Φ is bijective. To show this, we will construct the inverse map $\Xi : {}^r\mathbb{H}^3 \longrightarrow \mathbb{H}_{Berk}$. Given a point $x \in {}^r\mathbb{H}^3$, we can represent it as $x = (M_n(\mathbf{0}))$, where $M_n(z) = a_n z + b_n$. Using equations 4.3.1 and 4.3.2, we conclude that $|a_n| < \rho_n^{-N}$ and $|b_n| < \rho_n^{-N}$ for some $N \in \mathbb{N}$ ω-almost surely.

Hence $(a_n), (b_n)$ represent $a, b \in {}^\rho\mathbb{C}$, with $a \neq 0$. Denote $M(z) = az + b \in \mathrm{PSL}_2({}^\rho\mathbb{C})$, and we define

$$\Xi(x) = M(B(0,1)) \in \mathbb{H}_{Berk}$$

In a similar fashion, we can easily check that Ξ is well defined, and $\Phi \circ \Xi$, $\Xi \circ \Phi$ are identity maps. Therefore Φ is bijective.

We will now show that Φ is an isometry. Note that given $a, b \in {}^\rho\mathbb{C}$ represented by (a_n) and (b_n), $M(z) = az + b \in \mathrm{PSL}_2({}^\rho\mathbb{C})$ and $(x_n) \mapsto (M_n(x_n)) \in {}^r\mathbb{H}^3$ where $M_n(z) = a_n z + b_n$ are isometries of $\mathbb{H}_{Berk}$ and ${}^r\mathbb{H}^3$ respectively. Hence to show $d(x, y) = d(\Phi(x), \Phi(y))$, it suffices to show $d(x_g, M(x_g)) = d((\mathbf{0}), (M_n(\mathbf{0})))$.

If $M(x_g)$ is represented by a closed ball contained or containing $B(0,1)$, then we can choose $M(z) = az$, and $d(x_g, M(x_g)) = |\log|a||$. A direct computation using equation 4.3.2, we have $d(\mathbf{0}, M_n(\mathbf{0})) = |\log|a_n||$, so $d((\mathbf{0}), (M_n(\mathbf{0}))) = \lim_\omega -|\log|a_n||/\log\rho_n = |\log|a||$, where the last equality holds by the definition of norm on ${}^\rho\mathbb{C}$.

More generally, if $M(x_g)$ is represented by a closed ball $B(p, R)$ disjoint from $B(0,1)$, one can construct a geodesic by connecting $B(0,1)$ to $B(0,|p|)$ and then connecting $B(0,|p|)$ to $B(p,R)$. By the above argument, one can show that Φ is an isometry on either geodesic segment. Since Φ is a bijection, and ${}^r\mathbb{H}^3$ is a tree, this means $d(x_g, M(x_g)) = d((\mathbf{0}), (M_n(\mathbf{0})))$. Therefore, Φ is an isometry.

Before proving Theorem 1.3 and Theorem 1.4, we need the following lemma.

Lemma 4.7. *Let* $\mathbf{f} \in \mathrm{Rat}_d({}^\rho\mathbb{C})$*, if the reduction of* $\mathbf{f}$ *has degree* ≥ 1*, then* $\mathscr{E}_{bc} f_n(x^0) = x^0$*, where* f_n *is the sequence associated to* $\mathbf{f}$.

Proof. Since the reduction of $\mathbf{f}$ has degree ≥ 1, we can represent

$$\mathbf{f}(z) = \frac{a_d z^d + ... + a_0}{b_d z^d + ... + b_0}$$

with $\max\{|a_d|, ..., |a_0|\} = 1$ and $\max\{|b_d|, ..., |b_0|\} = 1$. We denote

$$f_n(z) = \frac{a_{d,n}z^d + ... + a_{0,n}}{b_{d,n}z^d + ... + b_{0,n}}.$$

Note that $(a_{k,n})$ and $(b_{k,n})$ represent a_k and b_k in ${}^{\rho}\mathbb{C}$. Let i_{top} and i_{bot} be the largest index so that $|a_{i_{top},n}| \geq |a_{j,n}|$ ω-almost surely and $|b_{i_{bot},n}| \geq b_{j,n}$ ω-almost surely respectively.

If $i_{top} \neq i_{bot}$, we let $L_n(z) = b_{i_{bot},n}/a_{i_{top},n}z$, then

$$\lim_{\omega} L_n \circ f_n$$

has degree ≥ 1, so $\lim_\omega d_{\mathbb{H}^3}(\mathbf{0}, L_n \circ \mathscr{E}_{bc} f_n(\mathbf{0})) = 0$. But since $|b_{i_{bot}}| = |a_{i_{top}}| = 1$, so $d_{\mathbb{H}^3}(\mathbf{0}, L_n(\mathbf{0})) < \epsilon r_n$ for any $\epsilon > 0$. Hence we have $\mathscr{E}_{bc} f_n(x^0) = x^0$.

If $i_{top} = i_{bot}$, we let $L_n(z) = z - a_{i_{top},n}/b_{i_{bot},n}$ and consider $g_n = L_n \circ f_n$. Since $|b_{i_{bot}}| = |a_{i_{top}}| = 1$, $d_{\mathbb{H}^3}(\mathbf{0}, L_n(\mathbf{0})) < \epsilon r_n$ for any $\epsilon > 0$. Moreover, note that for $\mathbf{g}$ represented by g_n has non trivial reduction, and the indices i_{top} and i_{bot} for g_n will be different. Hence apply the previous argument for g_n, we conclude that $\mathscr{E}_{bc} f_n(x^0) = x^0$. □

Proof of Theorem 1.3. Let $\Phi : \mathbb{H}_{Berk} \longrightarrow {}^{r}\mathbb{H}^3$ be the map defined as above, then Φ is an isometry. Let $\mathbf{f} \in \mathrm{Rat}_d({}^{\rho}\mathbb{C})$, then by representing the coefficients with sequences, we may associate a sequence $f_n \in \mathrm{Rat}_d(\mathbb{C})$ to $\mathbf{f}$.

Given $x \in \mathbb{H}_{Berk}$ represented by $x = M(x_g)$ where $M(z) \in \mathrm{PSL}_2({}^{\rho}\mathbb{C})$. Assume $y = \mathscr{E}_{Berk}(\mathbf{f})$ is represented by $y = L(x_g)$ where $L(z) \in \mathrm{PSL}_2({}^{\rho}\mathbb{C})$. Then the reduction

$$L^{-1} \circ \mathbf{f} \circ M$$

has degree ≥ 1. By Lemma 4.7 and the definition of Φ, $\mathscr{E}_{bc}(f_n)(\Phi(x)) = \Phi(y)$. Hence Φ is a conjugacy between $\mathscr{E}_{Berk}(\mathbf{f})$ and $\mathscr{E}_{bc}(f_n)$.

Conversely, if $f_n \in \mathrm{Rat}_d(\mathbb{C})$ with $r_n = \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$, and let $\mathbf{f}$ be the associated rational map with coefficients in ${}^{\rho}\mathbb{C}$. If the degree $\mathbf{f}$ is strictly smaller than d, then $\mathscr{E}_{bc}(f_n)$ has degree strictly smaller than d as Φ is a conjugacy. Hence, $\mathbf{f} \in \mathrm{Rat}_d({}^{\rho}\mathbb{C})$. The fact that Φ is a conjugacy follows immediately.

The fact that the Berkovich Julia set is contained in the $\Phi^{-1}(T \cup \epsilon(T))$ follows immediately from the fact that Julia set is the limit set of preimages of x^0. $\square$

Proof of Theorem 1.4. By Proposition 4.6, the field $\mathbb{L}$ naturally embeds into ${}^{\rho}\mathbb{C}$. Such an embedding also gives an embedding of $\mathbb{H}_{Berk}(\mathbb{L})$ into $\mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$. A rational map $\mathbf{f}$ with coefficients in $\mathbb{L}$ can be naturally thought of as a rational map with coefficients in ${}^{\rho}\mathbb{C}$ via the embedding. Its action on $\mathbb{H}_{Berk}(\mathbb{L})$ naturally extends to $\mathbb{H}_{Berk}({}^{\rho}\mathbb{C})$. The theorem now follows from Theorem 1.3. $\square$

As an immediate corollary, we have

Corollary 4.8. *Let $\rho_n \to 0$ and $r_n = -\log \rho_n$, then we have*

1. *If $\mathbf{f} \in \mathrm{Rat}_d({}^{\rho}\mathbb{C})$, and f_n represents $\mathbf{f}$. Let $s_n := \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$, then $\lim_\omega s_n / r_n$ is bounded. Moreover, if we assume the reduction of $\mathbf{f}$ has degree $< d$, then s_n is comparable to r_n.*
2. *If f_n is a degenerating sequence with $\max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0}) = r_n$, then f_n represents a rational map $\mathbf{f} \in \mathrm{Rat}_d({}^{\rho}\mathbb{C})$. Moreover, the reduction of $\mathbf{f}$ has degree $< d$.*

5

Applications on hyperbolic components

5.1 Markings and Length Spectra

Recall that for a conjugacy class $[f] \in M_d(\mathbb{C})$, we define

$$r([f]) := \min_{x \in \mathbb{H}^3} \max_{y \in \mathscr{E} f_n^{-1}(x)} d_{\mathbb{H}^3}(y, x)$$

where f is a representative rational map of $[f]$. In Lemma 3.16, we proved that a sequence $[f_n]$ is degenerating as conjugacy classes if and only if $r([f_n]) \to \infty$. We may choose a sequence of representatives f_n of $[f_n]$ so that

$$r([f_n]) = \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0}).$$

We construct a limiting dynamics associated to the degenerating sequence of conjugacy classes $[f_n]$:

$$\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3,$$

where the rescalings $r_n = r([f_n])$.

The limiting dynamics can tell us a lot about the asymptotic behaviors of the sequence f_n, e.g, the growth rate of the multipliers (see Theorem 1.7). To talk about the multipliers for a sequence $[f_n]$, it is more natural to consider the case where all the rational maps $[f_n]$ come from a single hyperbolic component, and we have a marking on their Julia sets.

Recall that a conjugacy class of rational map $[f]$ is called *hyperbolic* if any of the following equivalent definition holds (see Theorem 3.13. in [McM94]):

1. The postcritical set $P(f)$ is disjoint from the Julia set $J(f)$.
2. There are no critical points or parabolic cycles in the Julia set.
3. Every critical point of f tends to an attracting cycle under forward iteration.
4. There is a smooth conformal metric ρ defined on a neighborhood of the Julia set such that $|f'(z)|_\rho > C > 1$ for all $z \in J(f)$.
5. There is an integer $n > 0$ such that f^n strictly expands the spherical metric on the Julia set.

The space of hyperbolic rational maps is open in $M_d(\mathbb{C})$, and a connected component of it is called a *hyperbolic component*. For each hyperbolic component H, there is a topological dynamical system

$$\sigma : J \longrightarrow J$$

such that for any $[f] \in H$, there is a homeomorphism

$$\phi(f) : J \longrightarrow J(f)$$

which conjugates σ and f. A particular choice of such $\phi(f)$ will be called a *marking* of the Julia set.

Let $[f] \in H \subset M_d(\mathbb{C})$ be a hyperbolic rational map with a marking $\phi : J \longrightarrow J(f)$. We let S be the space of periodic cycles of the topological model $\sigma : J \longrightarrow J$. We define the length on $[f]$ of a periodic cycle $C \in S$ by

$$L(C, [f]) = \log |(f^q)'(z)|,$$

where $q = |C|$ and $z \in \phi(C)$. The collection $(L(C, [f]) : C \in S) \in \mathbb{R}_+^S$ will be called the *marked length spectrum* of $[f]$.

Let $[f_n] \in H \subset M_d(\mathbb{C})$ be a degenerating sequence with markings ϕ_n. Note that the ends $\epsilon({}^r\mathbb{H}^3)$ can be represented by a sequence of points in the conformal boundary $\mathbb{P}^1_{\mathbb{C}}$, so the sequence of markings ϕ_n also provides a marking on the end of the tree via

$$\begin{aligned} \phi_\infty : J &\longrightarrow \epsilon({}^r\mathbb{H}^3) \\ t &\mapsto [(\phi_n(t))] \end{aligned}$$

Hence a periodic cycle $C \in S$ is identified with a periodic cycle of ends for ${}^r\mathbb{H}^3$. Theorem 1.7 then implies that after passing to a subsequence, we have for all $C \in S$,

$$L(C, \mathscr{E}_{bc}(f_n)) = \lim_{n\to\infty} \frac{L(C, [f_n])}{r([f_n])}.$$

Note that it is possible to have a degenerating situation where $L(C, [f_n])$ stay bounded for all $C \in S$. If this is the case, then $L(C, \mathscr{E}_{bc}(f_n)) = 0$ for all $C \in S$. In the upcoming sections, we are going to classify this degenerating case.

5.2 Hyperbolic component with nested Julia set

In this section, we will study a special type of hyperbolic component. We begin with the following definition.

Definition 5.1. *Let $f \in \mathrm{Rat}_d(\mathbb{C})$ be a hyperbolic rational map, we say $J(f)$ is* nested *if*

1. *There are two points $p_1, p_2 \in \mathbb{P}^1_{\mathbb{C}}$ such that any component of $J(f)$ separates p_1 and p_2;*
2. *$J(f)$ contains more than one component.*

A hyperbolic component H is said to have nested Julia set if the Julia set of any rational map in H is nested.

A typical example of the Julia set may look like Figure 1.1. We remark that as soon as $J(f)$ has more than 1 component, it must have uncountably many components by taking preimages and their accumulation points. We shall see in a moment that there is a continuous map $\pi : J(f) \longrightarrow C$, where C is a Cantor set, such that $\pi^{-1}(x)$ is a continuum.

The first example of hyperbolic rational map with nested Julia set was introduced by McMullen in [McM88], where the Julia set is homeomorphic to a Cantor set times a circle. In their study of rational maps with disconnected Julia set (see Section 8 in[PT00]), Pilgrim and Tan constructed an example where the Julia set is nested, but not homeomorphic to a Cantor set times a circle. In this section, we shall classify these hyperbolic rational maps with nested Julia sets. We begin by introducing some terminologies and deduce some topological properties of the Julia sets.

Let J be a nested Julia set of a hyperbolic rational map f. Since each component of $J(f)$ separates 2 points, it can be easily verified that any Fatou component of f is either simply connected, or isomorphic to an annulus. We will call an annulus Fatou component a *gap*. Note that gaps are nested, and is backward invariant. We will call a component K of J an *extremal* if there is no other Julia component separating K

and p_1 or p_2. We say K is *buried* if K does not intersect the boundary of any gap. We say K is *unburied* otherwise.

Since f is hyperbolic, f is expanding in the hyperbolic metric on $\mathbb{P}^1_{\mathbb{C}} - P(f)$ where $P(f)$ is the postcritical set. If K is a buried component which is not extremal, then K is the accumulation set of Julia component from *both* sides. A standard argument using the expanding property and a converse of Jordan curve theorem, one can show that any buried Julia component is a Jordan curve (see Section 5 of [PT00] and Chapter 11.8 of [Bea91] for detailed arguements).

Since f is hyperbolic, a gap is eventually mapped to a simply connected Fatou component. We will call those gaps which is mapped to simply connected Fatou component the *critical gaps*. By Riemann-Hurwitz formula, those critical gaps are exactly those gaps containing critical points of f. Let U be a critical gap, and K_1, K_2 be two component of the Julia set J containing ∂U. Then K_1 and K_2 are both mapped to K, a component of J. This component K must be an extremal Julia component. Indeed, K cannot be unburied, as otherwise, K contains a boundary component of a gap and the preimage of this gap must also be a gap, which has to be U and we get a contradiction as U is mapped to a simply connected Fatou component. If K is buried but not extremal, then K is a Jordan curve, so it cannot contain a boundary of a Fatou component. Therefore, any unburied component is eventually mapped to an extremal Julia component by a degree e covering for some e. A similar argument also shows that the extremal Julia components are mapped to extremal ones.

We summarize these topological properties in the following lemma.

Lemma 5.2. *Let f be a hyperbolic rational map with nested Julia set J, then*

1. *A Fatou component is either simply connected, or isomorphic to an annulus, which will be called a* gap.
2. *The gaps are nested, and backward invariant.*
3. A critical gap *is a gap which contains critical points, and a critical gap is mapped to a simply connected Fatou component whose boundary is contained in an extremal Julia component.*

4. *The extremal Julia components are mapped to extremal ones.*
5. *Any unburied Julia component K is eventually mapped to an extremal Julia component by a degree $e = e(K)$ covering.*
6. *Any buried Julia component K except for the extremal ones is a Jordan curve.*

Shishikura tree for nested Julia set

We shall see that the dynamics on the gaps for a hyperbolic rational map f with nested Julia set can be well explained using the Shishikura tree. The Shishikura tree was first introduced by Shishikura in [Shi89] in the study of rational maps with Herman rings. We will give a brief introduction of the special case that we are interested, and refer the readers to [Shi89] for details and more general theories.

Let A be an annulus of $\mathbb{C}$ with modulus M, then there is a conformal map unique up to post composing with ratation $\phi_A : A \longrightarrow \{z : 1 < |z| < e^{2\pi M}\}$ sending the inner boundary to the inner one, and outer to the outer one. We define

$$A[z] := \phi_A^{-1}(\{\zeta : |\zeta| = |\phi_A(z)|\})$$

$$A(x, y) = \{z \in A : A[z] \text{ separates } x \text{ and } y\}$$

Let f be a hyperbolic rational map with nested Julia set J of degree d, and let $\mathscr{A}$ be the collection of gaps. We note that by sub-additivity of moduli, $\sum_{A\in\mathscr{A}} m(A) < \infty$. We define a pseudo metric on $\mathbb{P}^1_{\mathbb{C}}$ by

$$d(x, y) = \sum_{A\in\mathscr{A}} A(x, y)$$

In the usual fashion, we identify two points $x \sim y$ if $d(x, y) = 0$. It can be easily verified that $\mathbb{P}^1_{\mathbb{C}}/\sim$ is isometric to a closed interval I as the gaps are nested, and we denote

$$\pi : \mathbb{P}^1_{\mathbb{C}} \longrightarrow I$$

as the projection map.

The dynamics of f on $\mathbb{P}^1_{\mathbb{C}}$ determines an associated map on I via

$$\begin{aligned} f_* : I &\longrightarrow I \\ x &\mapsto \pi \circ f(\partial \pi^{-1}(x)) \end{aligned}$$

where $\partial \pi^{-1}(x)$ is the boundary of $\pi^{-1}(x) \subset \mathbb{P}^1_{\mathbb{C}}$. It can be verified that f_* is well defined and continuous. We will now prove some properties of the map f_*.

Lemma 5.3. *Let $I = [a, b]$, then there exists $a = a_1 < b_1 < a_2 < b_2 < ... < a_k < b_k = b$ such that*

1. *$f_* : [a_i, b_i] \longrightarrow I$ is a linear isometry with derivative $\pm d_i$ and $d_i \in \mathbb{Z}_{\geq 2}$ and the $\pm$ sign alternating;*
2. *U is a critical gap if and only if $U = \pi^{-1}((b_i, a_{i+1}))$ and $f_*([b_i, a_{i+1}]) \subset \{a, b\}$;*
3. *$d = \sum_{i=1}^{k} d_i$, where $d = \deg(f)$, and $\sum_{i=1}^{k} 1/d_i < 1$.*

Proof. Consider the annulus $A = \pi^{-1}((a, b))$, the boundary ∂A equals to the two extremal Julia components. Since the Julia set is nested, each component A_i of $f^{-1}(A)$ is an annulus. Let a_i, b_i be the projection of the boundary $\pi(\partial A_i)$, and we order them so that $a_1 < b_1 \leq a_2 < b_2 \leq ... \leq a_k < b_k$. We note $k \geq 2$, as otherwise, we have an invariant annulus, which is not possible as the map f is hyperbolic. Since the extremal Julia components are mapped to extremal ones, we know $a = a_1$ and $b_k = b$. Suppose for contradiction that $b_k = a_{k+1}$. Without loss of generality, we assume that $f_*(b_k) = a$, and let $C = \pi^{-1}(b_k)$ Note that C does not intersect gap, and π is constant on a simply connected Fatou component, so ∂C is

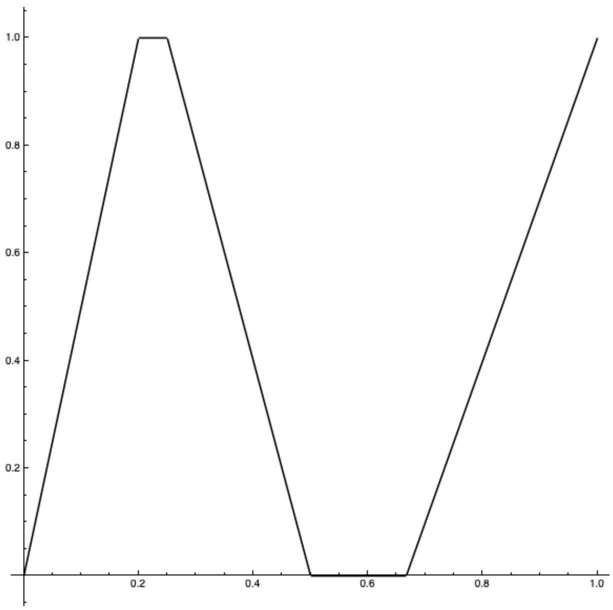

Figure 5.1: An example of the graph of the map $f_* I \longrightarrow I$. It has invariants $k = 3$, $d_1 = 5, d_2 = 4, d_3 = 3$.

in the Julia set. ∂C is connected, as otherwise, C must contain a gap. Since there is a sequence of Julia components accumulating to the extremal Julia component associated to a, taking the preimages of the sequence, we conclude that there are two different sequences of Julia components accumulating to ∂C from two different sides. Therefore ∂C is buried, so it is a Jordan curve by Lemma 5.2, which is a contradiction as $f(C)$ has interior (as $f(C)$ contains the interior of $\pi^{-1}(a)$ by assumption). Hence, we have $a = a_1 < b_1 < a_2 < b_2 < ... < a_k < b_k = b$.

Let $d_k = \deg(f : A_i \longrightarrow A)$, since $A_i \subset A$, each $d_i \geq 2$. Now restricting to $U \subset A_i$ and $U \in \mathscr{A}$, then $f(U) \in \mathscr{A}$. The map $f : U \longrightarrow f(U)$ is a degree d_i covering, so f_* restricting to $\pi(U)$ is linear with derivative $\pm d_i$. This is tree for any such U, so f_* is a linear on $[a_i, b_i]$ with derivative $\pm d_i$.

Since the critical gap is mapped to a simply connected Fatou component with boundary contained in the extremal Julia component, and all the other gaps are mapped to other gaps, the property (2) follows immediately.

For the last property, we note that the gaps are backward invariant. Hence we pick an arbitrary gap, there are exactly d preimages counted multiplicities. Hence $d = \sum_{i=1}^{k} d_i$. $\sum_{i=1}^{k} 1/d_i < 1$ follows as $|b_i - a_i| = |b - a|/d_i$. $\square$

Let $C = \bigcap_{i=1}^{\infty} \overline{f^{-n}((a,b))} \subset I$, and P be the set of periodic points in I. By Proposition 5.3, C is a Cantor set and $P \subset C$. We have the following corollary.

Corollary 5.4. *The restriction π on J gives a surjective continuous map $\pi : J \longrightarrow C$ with connected fiber. Moreover, the repelling periodic points of f are contained in $\pi^{-1}(P)$.*

Proof. Since π is a semi conjugacy, it is easy to verify that $I - C = \pi(\cup_{A \in \mathscr{A}} A)$. Hence $\pi : J \longrightarrow C$ is surjective and continuous. $\pi^{-1}(t) \cap J$ is connected as otherwise, $\pi^{-1}(t)$ contains a gap which is a contradiction. The moreover part follows directly from the fact π is a semi conjugacy. $\square$

Let $f_* : I = [a, b] \longrightarrow I$. Switching a and b, and take the second iteration of f if necessary, we may assume that $f_*(a) = a$. Note if k is even, then $f_*(b) = a$, and if k is odd, then $f_*(b) = b$.

Recall that $g : U \longrightarrow V$ is called a *polynomial like map* if g is a proper holomorphic map, and $\overline{U} \subset V$. The degree of a polynomial like map is defined as the degree of the proper map. The filled Julia set of the polynomial like map is defined as $K = \cap_{k=1}^{\infty} g^{-1}(V)$. If K is connected, then g is quasiconformally conjugate (in fact, hybrid conjugate) to a polynomial P of the same degree which is unique up to affine conjugation.

For sufficiently small $\epsilon > 0$, we have

$$f_*([a, a + \epsilon)) = [a, a + d_1\epsilon).$$

Let $U = \pi^{-1}([a, a + \epsilon))$ and $V = \pi^{-1}([a, a + d_1\epsilon))$, then U, V are open sets with $\overline{U} \subset V$ and $f : U \longrightarrow V$ is proper of degree d_1. Hence, $f : U \longrightarrow V$ is a polynomial like map, with connected filled Julia set $K = \pi^{-1}(a)$. Let P_a be the polynomial for which f is quasiconformally conjugate to, then P_a is hyperbolic with connected Julia set as f is hyperbolic. Similarly, if $f_*(b) = b$, then we may associate a hyperbolic polynomial with connected Julia set P_b to the end b. Note that if f varies in the hyperbolic component, P_a (and P_b) also varies in the corresponding hyperbolic component of polynomials. Hence, combining Lemma 5.3, we summarize the invariants that we can associate to a hyperbolic component with nested Julia set in the following proposition.

Proposition 5.5. *Let H be a hyperbolic component with nested Julia set in $M_d(\mathbb{C})$, and $f \in H$. Taking the second iteration f^2 if necessary, we assume that f fixes one of the two extremal Julia component. We can associate the following set of invariants to H:*

1. *A natural number $k \geq 2$, and a sequence $d_1, ..., d_k$ such that $\sum d_i = d$, and $\sum 1/d_i < 1$, where these numbers are associated to H as in Lemma 5.3.*
2. *If k is even, a hyperbolic component H_a in $\mathrm{Poly}_{d_1}(\mathbb{C})$ with connected Julia set;*
3. *If k is odd, a hyperbolic component H_a in $\mathrm{Poly}_{d_1}(\mathbb{C})$ with connected Julia set, and a hyperbolic component H_b in $\mathrm{Poly}_{d_k}(\mathbb{C})$ with connected Julia set.*

We shall see next that given any set of data as above, one can construct a hyperbolic component with nested Julia set with that set of data as invariants. This set of invariants, however, is not complete. For example, the critical gap may map to different Fatou component with boundary contained in the extremal Julia component. One can indeed introduce the itinerary of the critical points not included in the polynomial like maps, and try to construct a full set of invariants. The combinatorics becomes harder to maneuver, and we shall not pursue it here.

Construction of hyperbolic component with nested Julia set

In this subsection, we will prove the following using quasiconformal surgery.

Proposition 5.6. *Given the set of data as in Proposition 5.5, there is a hyperbolic rational map f with nested Julia set having the set of data as its invariants.*

Proof. Let $k \geq 2$, and a sequence $d_1, ..., d_k$ with $\sum 1/d_i < 1$. We may assume k is an even number, the case k is an odd number can be treated in a similar way. Let P be a monic hyperbolic polynomial with connected Julia set, we arrange so that

1. $B(0,1)$ is a contained in a bounded Fatou component of P.
2. Let R be large enough so that $\mathbb{C} - B(0,R)$ is contained in the unbounded Fatou component of P, and $P^{-1}(\partial B(0,R)) \subset B(0,(1+\epsilon)R^{1/d_1})$, where $\epsilon > 0$ is a sufficiently small constant determined in (3).
3. For $i = 2, ..., k$ let A_i be a round annulus centered at 0 with modulus $\frac{\log R}{2d_i\pi}$ contained in $B(0,R) - B(0,(1+\epsilon)R^{1/d_1})$. We assume A_is are arranged so that A_i is contained in the bounded component of $\mathbb{C} - A_{i+1}$ with disjoint closure, and $A_k = B(0,R) - \overline{B(0,R^{1-1/d^k})}$.

Note for any $\epsilon > 0$, (2) is possible to achieve, by choosing R large enough, as P is a monic polynomial with degree d_i. Given a sufficiently small $\epsilon > 0$, (3) is possible to achieve as $\sum 1/d_i < 1$.

We now define $F = P$ on $P^{-1}(B(0,R))$. For $i = 2, ..., k-1$, we define $F(z) = C_i z^{(-1)^{i+1}d_i}$ on A_i where $C_i > 0$ is chosen so that F sends the boundary of A_i to the $\partial B(0,1) \cup \partial B(0,R)$. Note this is possible as A_i is a round annulus centered at 0 with modulus $\frac{\log R}{2d_i\pi}$. Finally, we define $F(z) = C_k z^{-d_k}$ on $\mathbb{P}^1_{\mathbb{C}} - \overline{B(0,R^{1-1/d^k})}$, where $C_k > 0$ is chosen so that F sends $\partial B(0,R^{1-1/d^k})$ to $\partial B(0,R)$. Note by our construction, F sends $\mathbb{P}^1_{\mathbb{C}} - \overline{B(0,R)}$ to $B(0,1)$.

Let

$$U := P^{-1}(B(0,R)) \cup (\bigcup_{i=2}^{k-1} A_i) \cup (\mathbb{P}^1_{\mathbb{C}} - \overline{B(0,R^{1-1/d^k})}),$$

then F extends continuously to $\overline{U}$.

Let U_i be a component of $\mathbb{C} - U$, then each U_i is an annulus, and F maps ∂U_i to either $\partial B(0,1)$ or $\partial B(0,R)$. One can extend F to a quasiregular map on U_i which sends U_i to either $B(0,1)$ or $\mathbb{P}^1_{\mathbb{C}} - \overline{B(0,R)}$ depending on where F sends the boundary ∂U_i. Therefore, we get a quasiregular map defined on $\mathbb{P}^1_{\mathbb{C}}$.

Note that each U_i is mapped to $B(0,1)$ under either first iteration or the second iteration. Since we assume $B(0,1)$ is contained in a bounded Fatou component of P, each U_i is eventually mapped to a periodic

Fatou component of P. Therefore, using Shishikura's principle on quasiconformal surgery (see Proposition 5.2 in [BF14]), we conclude F is quasiconformally conjugate to a rational map f. Note that f is hyperbolic. Indeed, the critical points of F are contained either in the bounded Fatou component of P or one of U_i's, and each U_i is mapped to a bounded Fatou component of P under second iteration.

Now it can be easily verified that the Julia set f is nested, and it has the invariants the set of invariants $k, d_1, ..., d_k$ and P. □

We remark that in [QYY15], Qin, Yang and Yin have a similar result for rational maps with Cantor set of circles.

5.3 Hyperbolic component with nested Julia set admits bounded escape

Recall that we have the following definition of *bounded escape*:

Definition 5.7. *Let H be a hyperbolic component, we say H admits* bounded escape *if there exists a sequence $[f_n] \in H$ with a marking ϕ_n so that*

1. *$[f_n]$ is degenerating;*
2. *For any periodic cycle $C \in S$ of the topological model $\sigma : J \longrightarrow J$, the sequence of lengths $L(C, [f_n])$ is bounded.*

In this section, we will show the following

Theorem 5.8. *Let H be a hyperbolic component with nested Julia set, then it admits bounded escape.*

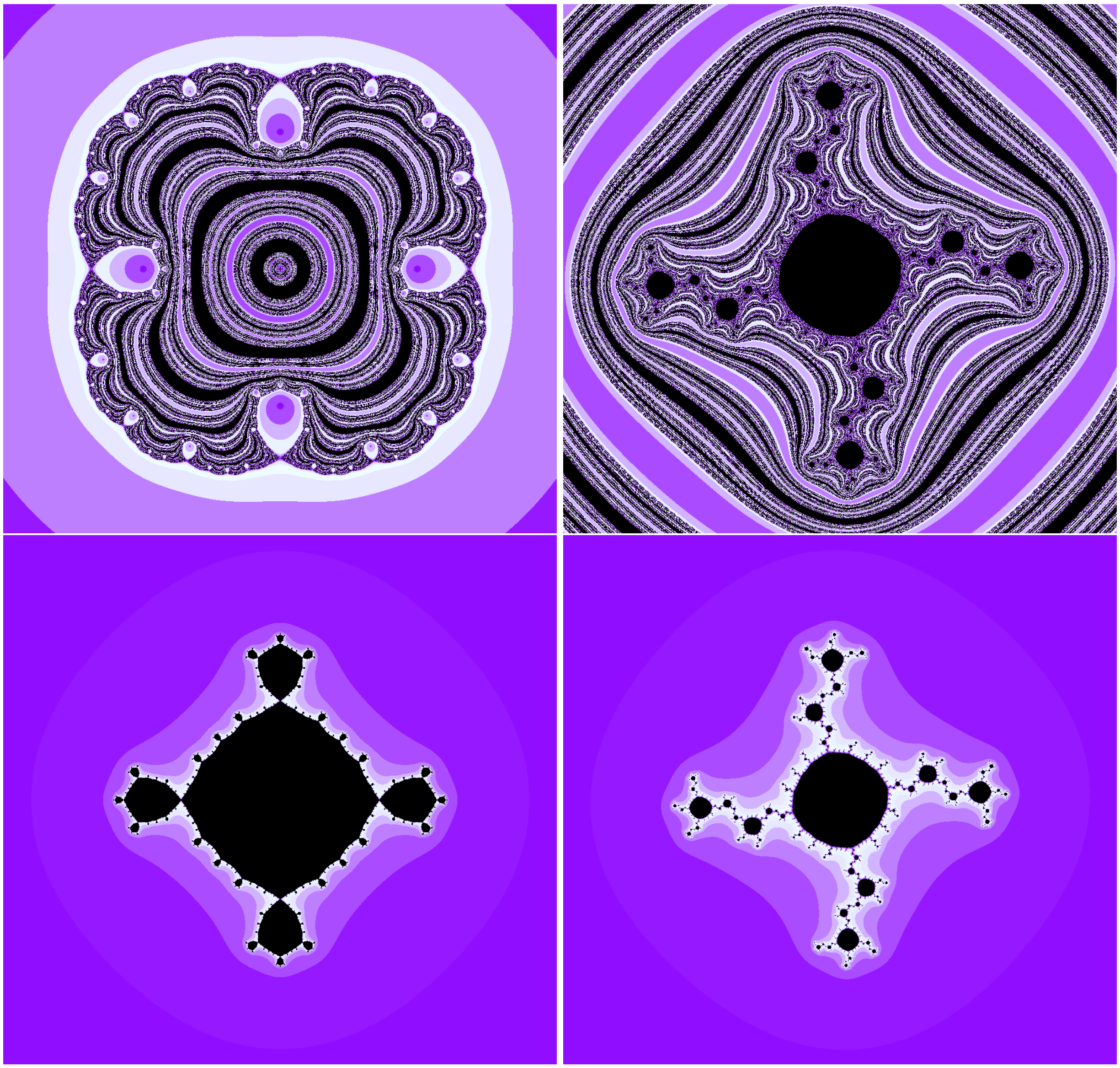

Figure 5.2: An example of the Julia set of 'nested mating' of $z^4 - 1$ (see lower left figure) and $z^4 - 1.10658 - 0.24848i$ (see lower right figure). The outermost Julia component (see upper left figure) is an inverted copy the Julia set of $z^4 - 1$, while the innermost Julia component (see upper right figure) is a copy of the Julia set of $z^4 - 1.10658 - 0.24848i$. The two component is separated by a Cantor set of nontrivial continuum, with countably many being coverings of either the Julia set of $z^4 - 1$ or $z^4 - 1.10658 - 0.24848i$.

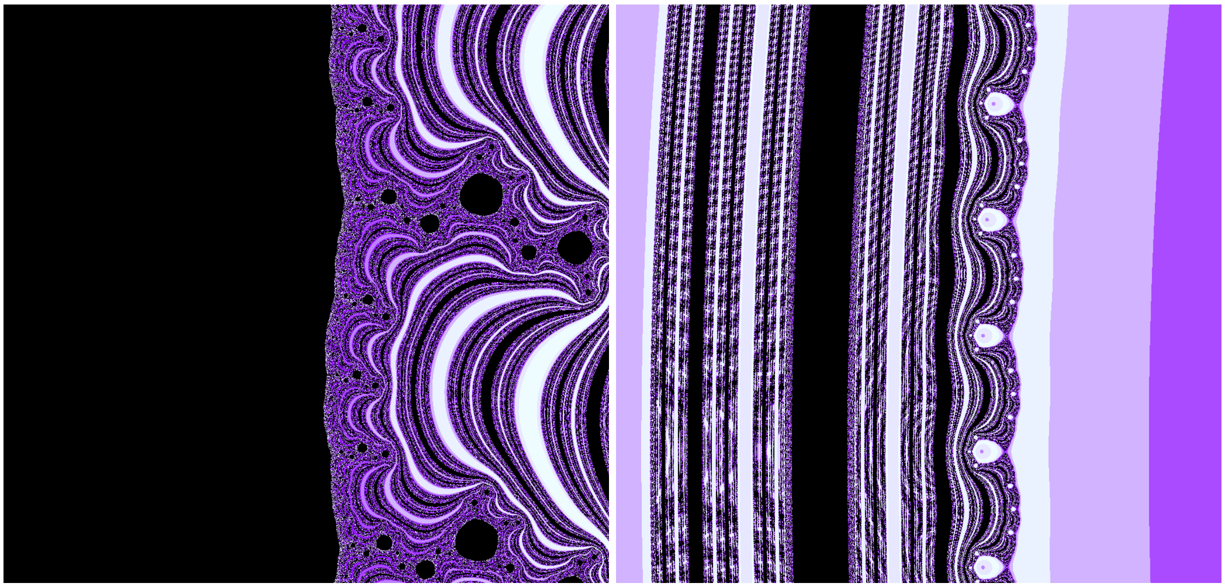

Figure 5.3: A zoom near the boundary of the black region (see left figure) shows that it is a covering of the Julia set of $z^4 - 1.10658 - 0.24848i$, while a zoom near the boundary of the pink region (see right figure) shows that it is a covering of the Julia set of $z^4 - 1$.

Proof. Let $f \in H$ be such a hyperbolic rational map. We first construct a sequence f_n which is degenerating.

Let A be a critical gap, and A is mapped to a simply connected Fatou component D. We define a quasiregular map F_n as follows. On a small neighborhood U of $\mathbb{P}^1_{\mathbb{C}} - A$, we set $F_n = f$. On A, we can construct $F_n : A \longrightarrow D$ using interpolation so that

1. f_n is quasiregular on A.
2. $f_n = f$ on $A \cap U$.
3. If we pull back the standard complex structure on D to A, the modulus of $m(A)$ with respect to the new complex structure satisfies $m(A) \geq n$.

Using Shishikura's principle on quasiconformal surgery (see Proposition 5.2 in [BF14]), F_n is quasiconformally conjugate via ϕ_n to a rational map f_n. By construction, $f_n \in H$, and the modulus of the gap of $\phi_n(A)$ goes to infinity. This implies that f_n is degenerating.

Note that the quasiconformal conjugacy also provides a marking $\phi_n : J(f) \longrightarrow J(f_n)$. Let x be a periodic point of f of period p. First assume that x is on the extremal Julia component K. Then we can find U containing K such that $f^p : U \longrightarrow V = f^p(U)$ is a polynomial like map. We may choose U small enough so that $U, f(U), ..., f^{p-1}(U)$ does not intersect the critical gap A, so $f^p|U$ is conjugate to $f_n^p|_{\phi_n(U)}$ via ϕ_n. We let $U_n = \phi_n(U)$ and $V_n = \phi(V)$, then $f_n^p : U_n \longrightarrow V_n$ is a polynomial like map. Note that by construction, we have $m(V_n - U_n) \to \infty$. Since polynomial like map of degree e $f : U \longrightarrow V$ with $m(V - U)$ bounded below is compact up to affine conjugacy (see Theorem 5.8 in [McM94]), so f_n^p converges compactly to $f_\infty : U_\infty \longrightarrow V_\infty$ of the same degree, so the multipliers of $\phi_n(x)$ stay bounded.

If x is on a buried Julia component K, then we can find an annulus neighborhood U of K with boundary contained in two Julia components so that $f^p : U \longrightarrow U'$ is a degree e covering map. We may also assume that $U, f(U), ..., f^{p-1}(U)$ do not intersect the gap A. Let $U_n = \phi_n(U)$ and $U_n' = \phi_n(U')$, then the modulus of the two annulus $U_n - \phi_n(K)$ both go to infinity and f_n^p on U_n is conjugate via ϕ_n to f^p on U. We normalize so that U_n' separates two fixed points $0, \infty$ of f_n^p, and $\phi_n(x) = 1$. Since the modulus of the two annulus $U_n - \phi_n(K)$ both go to infinity, the two boundary components of U_n go to 0 and ∞ respectively (and similarly for U_n'). Hence, both U_n and U_n' converges to $\mathbb{C} - \{0\}$. Using a similar argument as in Lemma 3.9, one can show that after passing to a subsequence, f_n^p converges compactly on $\mathbb{C} - \{0\}$ to a non-constant rational map. This shows that the multiplier at $\phi_n(x)$ is bounded in the subsequence.

Since the set of periodic cycles is countable, a diagonal argument allows us to construct a subsequence f_{n_k} so that for any periodic cycle C of $f : J(f) \longrightarrow J(f)$, the multipliers of $\phi_{n_k}(C)$ of f_{n_k} stay bounded. So H admits bounded escape. $\square$

5.4 Hyperbolic component which admits bounded escape has nested Julia set

In this section, we shall prove the converse of Theorem 5.8. Let $H \subset M_d(\mathbb{C})$ be a hyperbolic component which admits bounded escape. Let $\sigma : J \longrightarrow J$ be the topological model for the actions on the Julia set for H, and S be the set of periodic cycles of σ. Then we have a sequence $f_n \in H$ with markings ϕ_n which is degenerating as conjugacy classes, and for any $C \in S$, the multipliers of $\phi_n(C)$ stay bounded. Since f_n is degenerating as conjugacy classes, $r([f_n]) \to \infty$ by Lemma 3.16. We construct a limiting dynamics

$$\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3,$$

where the rescalings $r_n = r([f_n])$ as in Section 5.1. Recall that the markings provides a marking on the end of the tree (see Section 5.1), so each periodic cycle $C \in S$ represents a periodic cycle of ends for $\mathscr{E}_{bc}(f_n)$ on ${}^r\mathbb{H}^3$. We say a periodic end $\alpha \in \epsilon({}^r\mathbb{H}^3)$ a *repelling periodic end* if $L(\alpha, \mathscr{E}_{bc}(f_n)) > 0$. If $\mathscr{E}_{bc}(f_n)$ has a repelling periodic end of period p, then there is a sequence of periodic points (x_n) of period p which has unbounded multipliers. Hence, $\mathscr{E}_{bc}(f_n)$ has no repelling periodic ends. We shall now classify those limiting dynamics with no repelling periodic ends.

We will assume f_n is degenerating as conjugacy classes, $r_n = r([f_n]) \to \infty$. We normalize each f_n so that $r([f_n]) = \max_{y \in \mathscr{E} f_n^{-1}(\mathbf{0})} d_{\mathbb{H}^3}(y, \mathbf{0})$ for all n. Let $\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$ be the associated limiting map on the $\mathbb{R}$-tree.

Lemma 5.9. *Let $\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$, and $x_0, x_1 \in {}^r\mathbb{H}^3$. Let $\vec{v_0} \in T_{x_0}{}^r\mathbb{H}^3$ associated to x_1, and $\vec{v_1} \in T_{x_1}{}^r\mathbb{H}^3$ such that*

1. $\mathscr{E}_{bc}(f_n)(x_1) = x_0$;
2. $D_{x_1}\mathscr{E}_{bc}(f_n)(\vec{v_1}) = \vec{v_0}$;
3. *The component $U_{\vec{v_1}}$ does not intersect the critical tree nor x_1.*

Then $\mathscr{E}_{bc}(f_n)$ has a repelling fixed end.

Proof. Since $U_{\vec{v_1}}$ does not intersect the critical tree, $\mathscr{E}_{bc}(f_n)$ is an isometry from $U_{\vec{v_1}}$ to its image $U_{\vec{v_0}}$. Since $U_{\vec{v_1}}$ does not intersect x_0, so $U_{\vec{v_1}} \subset U_{\vec{v_0}}$.

We let x_2 be the preimage of x_1 in $U_{\vec{v_1}}$, then $d(x_0, x_1) = d(x_1, x_2)$, and $x_2 \in U_{\vec{v_0}}$. Hence, we can define x_n inductively by taking the preimage of x_{n-1} in $U_{\vec{v_1}}$. The union of the geodesic segments $\alpha := \cup_{k=0}^{\infty}[x_k, x_{k+1}]$ is an end which is fixed by $\mathscr{E}_{bc}(f_n)$. It is repelling as $L(\alpha, \mathscr{E}_{bc}(f_n)) = d(x_0, x_1) > 0$. $\square$

The following lemma follows from our Theorem 1.3 and Theorem 10.83 in [BR10] (see also Proposition 9.3 in [RL05] and Lemma 6.2 in [RL03]). For completeness, we produce a proof here as well.

Lemma 5.10. *Assume the degree d branched covering $\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$ has no repelling periodic ends, then it has a fixed point $x \in {}^r\mathbb{H}^3$ which has multiplicity ≥ 2.*

Proof. We say x is *strongly involutive* if either

1. x is fixed by $\mathscr{E}_{bc}(f_n)$, or
2. $x \neq y := \mathscr{E}_{bc}(f_n)(x)$ and if the tangent vectors $\vec{v}$ at x associated to y and $\vec{w}$ at y associated to x satisfies $D_x\mathscr{E}_{bc}(f_n)(\vec{v}) = \vec{w}$ and $\vec{v}$ is the only vector in $T_x{}^r\mathbb{H}^3$ that is mapped to $\vec{w}$.

We will consider two cases. If every point is strongly involutive, then by Theorem 3.30, the critical locus is non empty. Let x be a point in the critical locus. If x is a fixed point, then we are done. Otherwise, consider the geodesic segment $[x, y := \mathscr{E}_{bc}(f_n)(x)]$. Since x is strongly involutive, and the multiplier at x is $e \geq 2$, $\mathscr{E}_{bc}(f_n)$ has derivative e near x on $[x, y]$. The isometry from $[x, y]$ to $[0, d(x, y)]$ gives a natural ordering on $[x, y]$, and we let $t := \sup\{s \in [x, y] : \mathscr{E}_{bc}(f_n)(s) \in [x, y] \text{ and } s \geq \mathscr{E}_{bc}(f_n)(s)\}$. Note that this set is non empty, and by continuity, $\mathscr{E}_{bc}(f_n)(t) \in [x, y]$ and $t \geq \mathscr{E}_{bc}(f_n)(t)$. Since t is strongly involutive, the maximal property guarantees that t is a fixed point. Hence $\mathscr{E}_{bc}(f_n)$ maps $[x, t]$ homeomorphically to $[y, t]$. Since every point on $s \in [x, t]$ is strongly involutive, the local multiplicity of $D_s\mathscr{E}_{bc}(f_n)$ at $\vec{v_s}$ associated to t is e. Hence $\mathscr{E}_{bc}(f_n)$ is linear and has derivative e on $[x,t]$. Hence t has multiplicity at least $e \geq 2$. This proves the first case.

Now assume that there is a point $x_1 \in {}^r\mathbb{H}^3$ which is not strongly involutive, let $x_0 := \mathscr{E}_{bc}(f_n)(x_1)$. Since x_1 is not strongly involutive, by Lemma 3.29, there is $\vec{v}$ at x_1 maps to $\vec{w}$ at x_0 associated to x_1 such that $x_0 \notin U_{\vec{v}}$. Hence by Proposition 3.25, we can construct a $x_2 \in U_{\vec{v}}$ such that $\mathscr{E}_{bc}(f_n)$ maps $[x_1, x_2]$ homeomorphically to $[x_0, x_1]$. Note that the vector $\vec{v_2}$ at x_2 associated to x_1 is mapped to $\vec{v_1}$ at x_1 associated to x_0, so x_2 is not strongly involutive. Therefore, inductively, we construct x_n so that $[x_{n-1}, x_n]$ is mapped homeomorphically to $[x_{n-2}, x_{n-1}]$. Consider the union of the geodesic segment $l := \cup_{k=0}^{\infty}[x_k, x_{k+1}]$, then l has finite length, as otherwise, we will have a repelling fixed end. Let x denote the end of l (other than x_0), then x is a fixed point in ${}^r\mathbb{H}^3$. It has multiplicity ≥ 2 as $\mathscr{E}_{bc}(f_n)$ is locally expanding in the direction associated to l. This proves the second case. □

Proposition 5.11. *Assume the degree d branched covering $\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$ has no repelling periodic ends, let $x \in {}^r\mathbb{H}^3$ be a fixed point of multiplicity ≥ 2 (which exists by Lemma 5.10), then the set*

$$P = \bigcup_{i=0}^{\infty} \mathscr{E}_{bc}(f_n)^{-i}(x)$$

is contained in a geodesic segment.

Proof. Since we define $r_n = r([f_n])$, the preimage of x contains more than 1 point. Note it suffices to show P is contained in a line. If we prove this, and P escapes to one end, then replace $\mathscr{E}_{bc}(f_n)$ by its second iterate if necessary, we get a repelling fixed end which is a contradiction.

We will now argue by contradiction to prove P is contained in a line. Suppose not, then there are two points y, y' which are eventually mapped to x and the convex hull x, y, y' is a 'tripod'. Replace $\mathscr{E}_{bc}(f_n)$ by some iterates, we may assume that

$$\mathscr{E}_{bc}(f_n)(y) = \mathscr{E}_{bc}(f_n)(y') = x.$$

Let $\vec{v}$ be the tangent vector at x associated to y (or equivalently, y'). There are two cases to consider:

Case (1): The preimage of $\vec{v}$ under $D_x\mathscr{E}_{bc}(f_n)$ in $T_x{}^r\mathbb{H}^3$ is infinite. Then we can construct a 'fan' as follows (see Figure 5.4). Let $z_0 = y, z_1, ..., z_n, ...$ constructed inductively so that $\mathscr{E}_{bc}(f_n)$ sends $[x, z_{n+1}]$ homeomorphically to $[x, z_n]$. Let $\vec{w}_k$ denote the tangent vector at x associated to z_k. We let $\vec{u}_{0,k}$ be tangent vectors at $z_0 = 0$ which is mapped to $\vec{w}_k$ (there might be many such vectors, if that's the case, we just choose one). Inductively, we let $\vec{u}_{n,k}$ be vectors at z_n which is mapped to $\vec{u}_{n-1,k}$. Note that the vectors $\vec{u}_{n,k}$ are all different. Since the critical tree for $\mathscr{E}_{bc}(f_n)$ is a finite tree, there is a K, such that for all $k \geq K$ and all n, the component $U_{\vec{u}_{n,k}}$ does not intersect the critical set. Since $\vec{u}_{n,K}$ is mapped to $\vec{u}_{n-1,K}$, $\mathscr{E}_{bc}(f_n)^{K+1}$ is an isometry from $U_{\vec{u}_{K,K}}$ to its image $U_{\vec{w}_K}$. Since the critical tree intersect $[x, y]$, so $x \notin U_{\vec{u}_{K,K}}$. Now by Lemma 5.9, we conclude that there exists a repelling periodic end of period K, which is a contradiction.

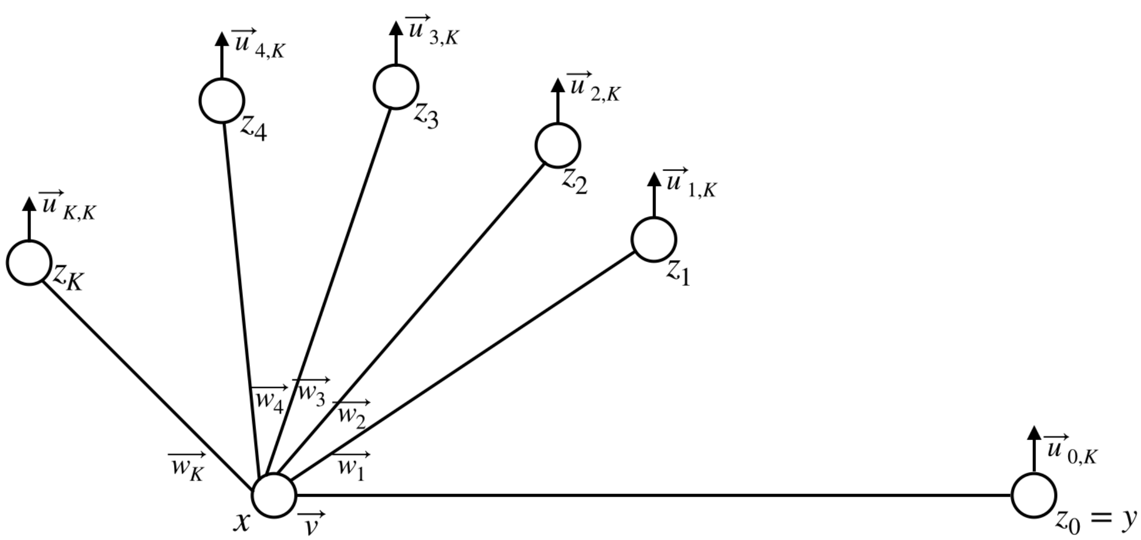

Figure 5.4: The 'fan' for Case (1).

Case (2): The preimage of $\vec{v}$ under $D_x\mathscr{E}_{bc}(f_n)$ in $T_x{}^r\mathbb{H}^3$ is infinite. By Lemma 3.29, replace $\mathscr{E}_{bc}(f_n)$ by its second iterate if necessary, we may assume $\vec{v}$ is totally invariant under $D_x\mathscr{E}_{bc}(f_n)$. Hence $\mathscr{E}_{bc}(f_n)$ is (locally) expanding in the direction $\vec{v}$ by Lemma 3.27. We choose $[x, z_1]$ and $[x, z_1']$ which are mapped homeomorphically to $[x, y]$ and $[x, y']$ respectively. Let p be the middle point of the tripod $\mathrm{hull}(x, y, y')$, we may assume $[x, q] := [x, z_1] \cap [x, z_1']$ is mapped homeomorphically to $[x, p]$. We note that either $[x, z_1]$ is not contained in $[x, y]$ or $[x, z_2]$ is not contained in $[x, y']$. Indeed, since $\mathscr{E}_{bc}(f_n)$ is (locally) expanding in the direction $\vec{v}$, $d(x, q) < d(x, p)$, so at least one of $[x, z_1] \cap [x, z_1']$ must branch off $[x, p]$ (note that it may even happen before q). We assume $[x, z_1]$ is not contained in $[x, y]$, and denote $z_0 = y$. Then we can construct a generalized 'fan' as follows (see Figure 5.4). Let $z_0 = y, z_1, ..., z_n, ...$ constructed inductively so that $\mathscr{E}_{bc}(f_n)$ sends $[x, z_{n+1}]$ homeomorphically to $[x, z_n]$, let q_k denote the middle point of the tripod $\mathrm{hull}(x, z_0, z_n)$. Note that by construction, $\mathscr{E}_{bc}(f_n)(q_n) = \mathscr{E}_{bc}(f_n)(q_{n-1})$. Since $\mathscr{E}_{bc}(f_n)$ is (locally) expanding in the direction $\vec{v}$, $d(x, q_n) < d(x, q_{n-1})$ and $d(x, q_n) \to 0$. Let $\vec{w}_k$ denote the tangent vector at q_k associated to z_k. We define $q_{0,k}$ so that $[z_0, q_{o,k}]$ is mapped homeomorphically to $[x, q_k]$, and let $\vec{u}_{0,k}$ be a tangent vector at $q_{0,k}$ which is mapped $\vec{w}_k$. Inductively, we define $q_{n,k}$ so that $[z_n, q_{n,k}]$ is mapped homeomorphically to $[z_{n-1}, q_{n-1,k}]$, and let $\vec{u}_{n,k}$ be a tangent vector at $q_{n,k}$ which is mapped $\vec{u}_{n-1,k}$. For sufficiently large k, we may assume $q_{1,k} \in U_{\vec{w}_1}$. Therefore, inductively, we can assume that $q_{n,k} \in U_{\vec{w}_n}$ for all n. Now the argument is similar to the Case (1). Note that the vectors $\vec{u}_{n,k}$ are all different. Since the critical tree for $\mathscr{E}_{bc}(f_n)$ is a finite tree, there is a K, such that for all $k \geq K$ and all n, the component $U_{\vec{u}_{n,k}}$ does not intersect the critical set. Since $\vec{u}_{n,K}$ is mapped to $\vec{u}_{n-1,K}$, $\mathscr{E}_{bc}(f_n)^{K+1}$ is an isometry from $U_{\vec{u}_{K,K}}$ to its image $U_{\vec{w}_K}$. Since the critical tree intersect $[x, y]$, and $q_{K,K} \in U_{\vec{w}_K}$, so $q_K \notin U_{\vec{u}_{K,K}}$. Now by Lemma 5.9, we conclude that there exists a repelling periodic end of period K, which is a contradiction.

$\square$

Let $I = [a, b]$ be the smallest geodesic segment containing P, then $\mathscr{E}_{bc}(f_n)$ sends the boundary $\{a, b\}$ to the boundary $\{a, b\}$. As otherwise, we can find a point in P with preimage outside of $[a, b]$, which is a

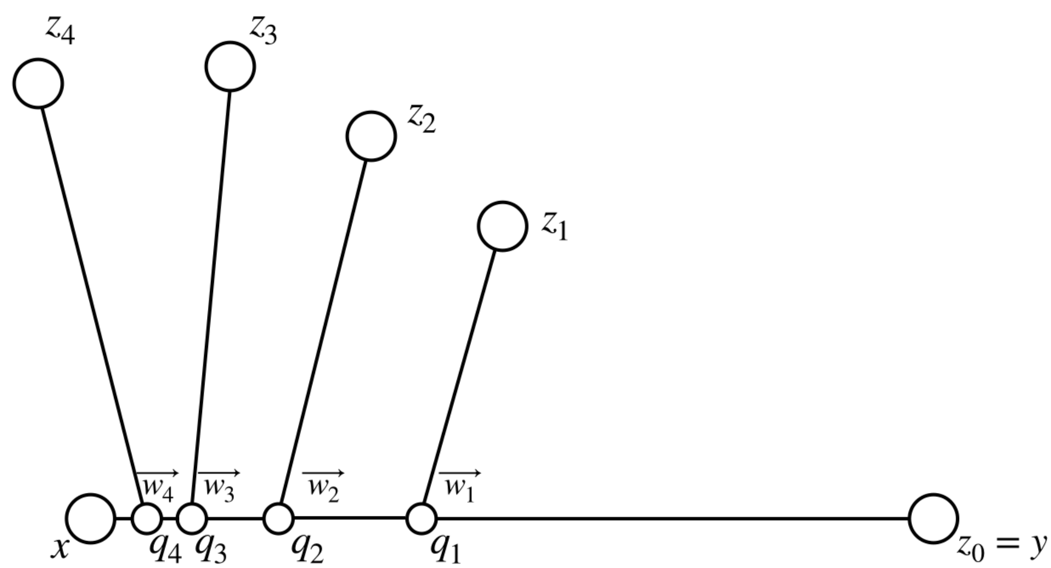

Figure 5.5: The generalized 'fan' for Case (2).

contradiction.

Let $J \subset I$ be a component of $\mathscr{E}_{bc}(f_n)^{-1}(I)$ intersecting I, then $\mathscr{E}_{bc}(f_n)$ maps J homeomorphically to I. Indeed, if the map is not injective, then there is a point $t \in J$ with tangent vectors $\vec{v}_1, \vec{v}_2$ at t associated to a and b respectively so that $D_t\mathscr{E}_{bc}(f_n)(\vec{v}_1) = D_t\mathscr{E}_{bc}(f_n)(\vec{v}_2)$. But $D_t\mathscr{E}_{bc}(f_n)$ is surjective by Lemma 3.29, so there is a tangent vector $\vec{v}$ which is mapped to the tangent vector at $\mathscr{E}_{bc}(f_n)(t)$ associated to either a or b. This means that P intersect non-trivially with $U_{\vec{v}}$, which is a contradiction. The map is surjective by a similar argument: if the map is not surjective and let $I' \subset I$ be the image, then by Lemma 3.29, the preimage of $I - I'$ is not contained in I, which is a contradiction as P intersect $I - I'$ non-trivially.

We also note that $\mathscr{E}_{bc}(f_n)$ has constant derivative on J. Indeed, if not, then we can find a point $t \in J$ with tangent vectors $\vec{v}_1, \vec{v}_2$ at t associated to a and b respectively so that the local degrees at $\vec{v}_1$ and $\vec{v}_2$ are different. Then applying Lemma 3.29, one of $D_t\mathscr{E}_{bc}(f_n)(\vec{v}_i)$ has a preimage $\vec{v}$ in $T_t{}^r\mathbb{H}^3$ other than $\vec{v}_i$.

Then there is a point of P in $U_{\vec{v}}$, which is a contradiction.

By looking at the local degrees at the preimages of the point x, we conclude the sum of the derivatives on different components J equals to d.

To summarize, we have the following Proposition which describes the limiting dynamics with no repelling periodic ends.

Proposition 5.12. *Assume the degree* d *branched covering* $\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$ *has no repelling periodic ends. Let* $x \in {}^r\mathbb{H}^3$ *be a fixed point with multiplicity* ≥ 2 *and* $P = \cup_{i=0}^{\infty}\mathscr{E}_{bc}(f_n)^{-i}(x)$*. Let* $I = [a, b]$ *be the smallest geodesic segment that contains* P*, then there exists* $a = a_1 < b_1 \leq a_2 < b_2 <\leq .. \leq a_k < b_k = b$ *such that*

1. $\mathscr{E}_{bc}(f_n) : [a_i, b_i] \longrightarrow I$ *is a linear isometry with derivative* $\pm d_i$ *and* $d_i \in \mathbb{Z}_{\geq 2}$ *and the* $\pm$ *sign alternating;*
2. $d = \sum_{i=1}^{k} d_i$.

An immediate corollary of the above Proposition is the following:

Corollary 5.13. *Let* $t \in I = [a, b]$ *which is mapped into* (a, b)*, then* $U_{\vec{v}}$ *contains no critical ends for all* $\vec{v}$ *at* t *not associated to* a *or* b.

Remark 5.14. *We remark that we did not use the fact that* f_n *all come from a single hyperbolic component yet. In our communication with Favre, this classification also appears in an unpublished manuscript by Charles Favre and Juan Rivera-Letelier, though the author never had a chance to read the manuscript. We would refer to [FRL10] where many such examples are studied.*

We also remark the similarities and the distinctions of the classification with the induced map on the Shishikura's tree (see Lemma 5.3). In Proposition 5.12, it is possible for $b_k = a_{k+1}$*, and* $\sum_{i=1}^{k} 1/d_i = 1$*. Neither equality can occur in Lemma 5.3.*

We will now further assume that the sequence f_n is an example of bounded escape in the hyperbolic component H. Then $\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$ has no repelling periodic ends. Let $I = [a, b]$ be the geodesic segment as in Proposition 5.12, and x be a fixed point in (a, b). Then there exists an open set $U^{x-t,x+t} \subset {}^r\mathbb{H}^3$ with boundary points $x - t$ and $x + t$ which is mapped to $V^{x-et,x+et}$ for some integer $e \geq 2$. Note that $U^{x-t,x+t}$ does not contain critical ends. Let V_n be the sequence of annulus associated to $V^{x-et,x+et}$, then for all large n, there is a component U_n approximating $U^{x-t,x+t}$ so that $\overline{U}_n \subset V_n$, and is mapped under f_n to V_n as a degree e covering (see Proposition 3.25). Fix such a large N, we let $U_{0,N} = V_N$ and $U_{k,N}$ be the component of $f^{-1}(U_{k-1,N})$ contained in $U_{k-1,N}$. Let $K = \cap_{k=0}^{\infty} \overline{U}_{k,N}$. Note that $K \subset J_N = J(f_N)$, and f_N is hyperbolic, so by an argument as in Section 5 of [PT00] or Chapter 11.8 of [Bea91], K is a Jordan curve. Since all f_n comes from a single hyperbolic component, we will abuse notations and regard K as in the topological model $\sigma : J \longrightarrow J$ of the Julia set. The realization of K in $J_n = J(f_n)$ will be denoted by $\phi_n(K)$, where ϕ_n is the marking.

Let $\mathcal{K}_n := \cup_{i=0}^{n} \sigma^{-i}(K)$ and $\mathcal{K} := \cup_{n=0}^{\infty} \mathcal{K}_n$.

Lemma 5.15. *$\mathcal{K}$ is a nested set of circles.*

Proof. Indeed, this can be done by induction: we assume that $\mathcal{K}_n$ is a nested set of circles. Let $\mathcal{P}_n := \cup_{i=0}^{n} \mathscr{E}_{bc}(f_n)^{-1}(x)$, if $y \in \mathcal{P}_{n-1} - \mathcal{P}_n$ with $\mathscr{E}_{bc}(f_n) = w \in \mathcal{P}_n$, then there is a open set $U^{y-t,y+t} \subset {}^r\mathbb{H}^3$ with boundary points $y - t$ and $y + t$ which is mapped to $V^{w-et,w+et}$ for some integer $e \geq 2$. We may choose t small enough so that $U^{y-t,y+t} \cap \mathcal{P}_n = \emptyset$, and note that $U^{y-t,y+t}$ contains no critical ends. Similar as before, let V_n be the sequence associated to $V^{w-et,w+et}$, then for a sufficiently large N, there exists U_N approximating $U^{y-t,y+t}$ which maps to V_N by f_N as a degree e covering. Since $U^{y-t,y+t} \cap \mathcal{P}_n = \emptyset$, $U_N \cap \mathcal{K}_n = \emptyset$. If we denote $\phi_N(C)$ be the component of $\phi_N^{-(n+1)}(K)$ in U_N, then $\phi_N(C) \cup \phi_N(\mathcal{K}_n)$ is still a nested set of circles, so $C \cup \mathcal{K}_n$ is a nested set of circles. We can now add more $n+1$-th preimage of K into $C \cup \mathcal{K}_n$ in a similar way. Therefore, $\mathcal{K}$ is a nested set of circles. □

From the construction above, we also have

Proposition 5.16. *The natural ordering on the nested set $\mathcal{K}$ is compatible with the linear ordering on $\mathcal{P} = \cup_{n=0}^{\infty}\mathcal{P}_n$, and the map π sending a component C of $\mathcal{K}$ to the associated point in $\mathcal{P}$ is a semi-conjugacy.*

We are now ready to prove the main theorem of this section:

Theorem 5.17. *Let H be a hyperbolic component, and $[f_n] \in H$ be degenerating such that*

$$\mathscr{E}_{bc}(f_n) : {}^r\mathbb{H}^3 \longrightarrow {}^r\mathbb{H}^3$$

has no repelling periodic ends, then H has nested Julia set.

Proof. Let f_n be an example of bounded escape in the hyperbolic component H. We shall use the notations in this section.

First, we will show the Julia set is disconnected. To show this, we will show $\sum_{i=1}^{k} 1/d_i < 1$, where d_i is defined as in Proposition 5.12. Replace $\mathscr{E}_{bc}(f_n)$ by its second iterates and switch the role of a and b if necessary, we may assume a is fixed by $\mathscr{E}_{bc}(f_n)$. Let $p_n, q_n \in \mathcal{P}$ with $p_n \to a$, $q_n \to b$, and $C_n = \pi^{-1}(p_n)$, $D_n = \pi^{-1}(q_n)$. We also define A_n be the annulus bounded by C_n and D_n, and $A = \cup_{n=1}^{\infty} A_n$ which is again an annulus. Let $p_{i,n}$ and $q_{i,n}$ be the $i-th$ in the linear ordering on $[a, b]$ of the preimages of p_n and q_n, and $C_{i,n} = \pi^{-1}(p_{i,n})$ and $D_{i,n} = \pi^{-1}(q_{i,n})$ respectively. Let $A_{i,n}$ be the annulus bounded by $C_{i,n}$ and $D_{i,n}$, and $A_i = \cup_{n=1}^{\infty} A_{i,n}$. Then each $A_i \subset A$ and the inclusion map is an isomorphism on fundamental group. Also note that A_i is mapped to A as a degree d_i covering, so $m(A_i) = m(A)/d_i$. If $\sum_{i=1}^{k} 1/d_i = 1$, then by the equality case of the subadditivity of moduli, A_i and A_{i+1} shares a Jordan curve boundary. This forces f_n to have a critical point on this boundary, which is a contradiction as this boundary is in the Julia set, and f_n is hyperbolic.

Since $\sum_{i=1}^{k} 1/d_i < 1$, we conclude that the $\mathcal{P}$ is not dense in I. This means that $J = \overline{\mathcal{K}}$ is not connected.

We will now prove the every component separates two points. Since f_n is hyperbolic, $\mathcal{K}$ separates two Fatou components. Since $J = \overline{\mathcal{K}}$, every component of J separates these two components. $\square$

Remark 5.18. *If we have a degenerating sequence of flexible Lattès maps of degree* d^2*, the limiting map* $\mathscr{E}_{bc}(f_n)$ *also provides an example with no repelling periodic ends. In this case, we have* $k = d$ *and each* $d_i = d$ *(so* $\sum_{i=1}^{k} 1/d_i = 1$*). Furthermore, the nested set of circle* $\mathcal{K}$ *is dense in* $\mathbb{P}^1_{\mathbb{C}}$ *giving* $J = \mathbb{P}^1_{\mathbb{C}}$.

5.5 Proof of Theorem 1.10 and 1.11

Proof of Theorem 1.10. Combining Theorem 5.8 and Theorem 5.17, we get Theorem 1.10. ☐

Proof of Theorem 1.11. Combining Theorem 1.7 and Theorem 5.17, we get Theorem 1.11 ☐

It is interesting to know

Question 5.19. *Is the length spectrum bounded* throughout *a hyperbolic component with nested Julia set?*

Let $A_{1,n}, ..., A_{k,n}$ denote the critical gaps of f_n. If there is an $i \in \{1, ..., k\}$ such that the moduli $m(A_{i,n})$ are bounded from below, then a similar argument as in the proof of Theorem 5.8 can be used to show that the lengths of a periodic cycle C stay bounded. Therefore, in order to get unbounded length spectrum, all moduli of the critical gaps tend to 0. Conversely, if $[f_n]$ is degenerating and has bounded length spectrum, the proof of Theorem 5.17 implies that some moduli of critical gaps have to tend to ∞.

Therefore, the question above is equivalent to the following:

Question 5.20. *Does there exist a degenerating sequence* $[f_n] \in H$ *with all moduli of the critical gaps tend to* 0*?*

If the answer to Question 5.20 is 'Yes', then such a sequence will provide an example where the length spectrum is unbounded, and so the answer to Question 5.19 is 'No'. Otherwise, the answer to Question 5.19 is 'Yes'.

We conjecture the answer to Question 5.20 is 'No'.

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