Knockdown Factors for Buckling of Cylindrical and Spherical Shells Subject to Reduced Biaxial Membrane Stress

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Knockdown Factors for Buckling of Cylindrical and Spherical Shells Subject to Reduced Biaxial Membrane Stress

John W. Hutchinson
School of Engineering and Applied Sciences
Harvard University
Cambridge, MA 02138

Abstract

Cylindrical shells under uniaxial compression and spherical shells under equi-biaxial compression display the most extreme buckling sensitivity to imperfections. In engineering practice, the reduction of load carrying capacity due to imperfections is usually addressed by use of a knockdown factor to lower the critical buckling stress estimated or computed without accounting for imperfections. For thin elastic cylindrical shells under uniaxial compression and spherical shells under equi-biaxial compression, the knockdown factor is typically as small as 0.2. This paper explores the alleviation of imperfection-sensitivity for loadings with a reduced circumferential (transverse) membrane stress component. The analysis of W.T. Koiter (1963) on the effect of an axisymmetric imperfection on the elastic buckling of a cylindrical shell under uniaxial compression is extended to both cylinders and spheres for loadings that produce general combinations of biaxial membrane stresses. Increases in the knockdown factor due to a reduction of the transverse membrane component are remarkably similar for cylindrical and spherical shells.

1. Introduction

Design of structures comprising thin cylindrical and spherical shells subject to compressive membrane stresses makes use of a knockdown factor, α, to account for the fact that imperfections can reduce the compressive stress at buckling to a small fraction of the critical stress at which the perfect shell buckles. For all the cases considered in this paper, the relevant compressive stress at buckling of the imperfect shell is given by

$$\sigma_1 = \alpha\,\sigma_C \tag{1}$$

where σ_C is the critical compressive stress component in the 1-direction of the perfect shell. Cylindrical shells under uniaxial compression and spherical shells under equi-

biaxial compression are the most imperfection-sensitive of all shell structure/loading combinations. As established by Koiter (1945, 2009), their extreme sensitivity is due to the nonlinear post-buckling interaction among the many simultaneous buckling modes associated with buckling of the perfect structure. By far, the most information is available for knockdown factors for cylindrical shells under uniaxial compression based on experimental data collected by NASA (1965). Data assembled by Seide, Weingarten and Morgan (1960) is plotted in Fig. 1 as α versus the shell radius to thickness ratio, R/t. Very thin elastic cylindrical shells under uniaxial compression buckle at stresses as low as $\alpha \cong 0.2$, and possibly even lower for $R/t > 1500$. The body of experimental data for spherical shells under external pressure (equi-biaxial compression) is much smaller. However, due to similarity with the cylindrical shell under axial compression noted above and the data that does exist, it has been common practice to adopt the knockdown factor for cylindrical shells plotted in Fig. 1 for the spherical shells under equi-biaxial compression.

The issue addressed in this paper is the extent to which imperfection-sensitivity of elastic buckling is alleviated by loadings which alter the circumferential (transverse) membrane stress component. Throughout this paper, σ_1 and σ_2 denote the membrane stresses in the axial and circumferential directions (Fig. 2), *taken positive in compression*. For a perfect cylinder, σ_1 is proportional to the axial load and σ_2 is related to an external pressure, p, by $\sigma_2 = pR/t$. For perfect spherical shell segments, the membrane stresses depend on the combination of axial load (and possibly transverse load) and external pressure, p, such that $\sigma_1 + \sigma_2 = pR/t$ for all combinations. A spherical shell loaded solely by external pressure has equi-biaxial compression (i.e., $\sigma_1 = \sigma_2 = pR/2t$). A spherical equatorial segment (a belt-line segment) loaded only by a compressive axial load, P, has axial compression and circumferential tension with $\sigma_1 = -\sigma_2 = P/(2\pi Rt)$. For imperfect cylindrical and spherical shells, there will be short wavelength spatial variations in the membrane stresses, but their average values are the same as for the perfect shells. Consequently, throughout this paper, σ_1 and σ_2 can be regarded as the values associated with loads on the perfect shell whether the shell is perfect or not.

In all cases considered here, σ_1 is the largest compressive component such that it drives buckling. Attention is limited to loadings that produce circumferential tension in the cylindrical shell ($\sigma_2 \leq 0$) and that reduce circumferential compression in the spherical shell ($\sigma_2 \leq \sigma_1$). Commonly encountered loadings produce these reductions, as will be illustrated later. Experiment (Seide, et al., 1960; Limam, et al, 1991; Mathon and Limam, 2006) and theory (Hutchinson, 1965; Rotter and Zhang, 1990; Teng and Rotter, 1992) have revealed that the knockdown is less severe for axial buckling of cylindrical shells under internal pressure. One set of tests on spherical shell segments by Yao (1963) discussed later also reveals reduced imperfection-sensitivity when the loading produces reduced transverse stress. Efforts are underway by both ASME and NASA to refine the knockdown prescriptions as a function of the circumferential membrane stress to take advantage of the reduced imperfection-sensitivity. For the ranges of σ_2 considered here, for both the cylindrical shell and the spherical shell, σ_2 has no effect on the buckling stress of the perfect shell:

$$\sigma_1 = \sigma_C \equiv \frac{E}{\sqrt{3(1-v^2)}} \frac{t}{R} \tag{2}$$

with E as Young's modulus and v as Poisson's ratio (c.f. analysis in *Section3*).

For the ranges of σ_2 considered in this paper, the deflection normal to the shell associated with the buckling mode of the perfect shell has the form (away from the ends of the cylinder or away from the boundaries of a spherical segment)

$$W = \xi t \cos\left(q \frac{x_1}{R}\right) \quad \text{with} \quad q = \left[12(1-v^2)\right]^{1/4}\left(\frac{R}{t}\right)^{1/2} \tag{3}$$

with ξ as the buckling amplitude. For thin shells, the wavelength of this mode,

$$\ell = 2\pi R / q = \left(3.38/(1-v^2)^{1/4}\right)\sqrt{Rt}, \tag{4}$$

is short compared to the shell radius. For a full cylindrical shell, the mode in (3) is axisymmetric, as in the analysis by Koiter (1963) for uniaxial compression. It also applies to a "large" circumferential segment of a cylindrical shell away from the boundaries as long as the boundary edges are well supported. Large in this context means that the circumferential width and axial height of the segment are both large

compared to ℓ. Similarly, (3) is an axisymmetric mode for a full equatorial segment of a spherical shell (Fig. 2b) whose height, H, is large compared to ℓ but such that the shell is still shallow in the axial direction, i.e., $H/(2R)$ not larger than about ½. The mode (3) also represents the buckling deflection away from adequately supported edges for any "large" shallow segment of a sphere whose dimensions are large compared to ℓ, as discussed by Hutchinson (1967).

2. Buckling strength reduction due to an axisymmetric imperfection

An imperfection in the form of a normal displacement, $\overline{W}$, of the shell middle surface is considered. Following Koiter (1963) for the cylinder and Hutchinson (1967) for the sphere, an imperfection in the shape of the buckling mode (3) is assumed:

$$\overline{W} = \overline{\xi} t \cos\left(q \frac{x_1}{R} \right) \tag{5}$$

with $\overline{\xi}$ as the normalized imperfection amplitude. As just discussed, (5) can be viewed as an axisymmetric imperfection for full circumferential shell segments or as a local imperfection varying in only the x_1-direction in a sufficiently large shell segment (that is also shallow in the case of the sphere). For simplicity, the imperfection (5) will be referred to as being axisymmetric and the associated pre-buckling deflections of the loaded shell will also be referred to as axisymmetic in the sequel. The shells, or shell segments, are assumed to meet the conditions described in the previous section with regard to their dimensions. It is also assumed that the support conditions are sufficiently strong such that buckling is dominated by the imperfection and not by a weakly supported edge.

With due regard for readers not interested in the details of the analysis, the results for the knockdown factor will be presented in this section followed by presentation of details in *Section 3*. Nevertheless, a brief outline of the analysis, following the steps laid out by Koiter (1963), is described because it highlights the validity of the predictions.

(i) Given the imperfection (5), an exact axisymmetric solution to the Donnell-Mushtari-Vlassov (DMV) nonlinear shell equations is obtained.

(ii) Bifurcation from the axisymmetric state is determined in the form of a non-axisymmetric mode,

$$w = At \sin\left(\frac{1}{2}q\frac{x_1}{R}\right)\sin\left(\gamma q\frac{x_2}{R}\right), \tag{6}$$

where γ is a minimization parameter in the analysis that sets the wavelength in the circumferential direction.

(iii) While the bifurcation mode (6) is not exact, the analysis carried out ensures that the prediction for the knockdown factor, α, is both accurate and an upper bound for the prescribed imperfection.

Insight into the selection of the bifurcation mode (6) was provided by Koiter (1963, 2009) and repeated here in Fig. 2c. In the axisymmetric state the loaded imperfect shell experiences alternating compressive (+) and tensile (-) enhancements of the circumferential membrane stress, as noted in Fig. 2c, depending on whether the deflection is inward or outward. Compressive enhancement favors circumferential variations in non-axisymmetric bifurcation mode while tensile enhancement discourages such variations. Thus, the axial wavelength of the bifurcation mode is exactly twice that of the axisymmetric deformation with the associated deflection phased such that the circumferential nodal lines coincide with the peaks of enhanced tension. The largest circumferential variations align with regions of enhanced compression. In many cases, but depending on the way the loads are applied and the imperfection amplitude, the post-bifurcation interaction of the axisymmetric and non-axisymmetric deflections gives rise to dynamic snap buckling at bifurcation (Budiansky and Hutchinson, 1972).

The knockdown factors for the cylindrical shell under uniaxial compression and the spherical under equi-biaxial compression based on the results of *Section 3* are plotted as a function of the normalized imperfection amplitude in Fig. 3. Plotted in this manner, the curves are independent of ν. The curve for cylindrical shells was originally given by Koiter (1963) and the curve for spherical shells was given by Hutchinson (1967). Based on asymptotic methods for small imperfections, these authors also established that the axisymmetric imperfection (5) gives rise to knockdowns that are the largest for the cylinder, and almost the largest for the sphere, among all competing imperfections with the same amplitude. The extreme sensitivity to the geometric imperfection of the middle surface of these two shells under their respective loadings is evident. The fact that the sensitivity of the cylindrical and spherical shells is nearly the same for the imperfection

(5) provides some justification for the common practice of applying knockdown factors based on experimental data for cylindrical shell under uniaxial compression to the spherical shell under equi-biaxial compression.

The effect of a circumferential membrane stress component that departs from uniaxial compression for the cylinder or from equi-biaxial compression for the sphere is shown in Fig. 4. This figure plots four sets of curves with one for the cylinder and the other for the sphere in each set. For each set of paired curves, the normalized imperfection amplitude, $\sqrt{(1-v^2)}\bar{\xi}$, is chosen such that the knockdown factor, α, for the cylinder under uniaxial compression ($\sigma_2 = 0$) and that for the sphere under equi-biaxial compression ($\sigma_2 = \sigma_1$) coincide. The abscissa in Fig. 4 is taken as the normalized decrease in the circumferential membrane component from the uniaxial state for the cylinder, measured by σ_2/σ_1, and from the equi-biaxial state for the sphere, measured by $\sigma_2/\sigma_1 - 1$.

It is particularly striking in Fig. 4 that *the effect on the knockdown factor of the decrease in the circumferential membrane stress is almost the same for the two shell structures.* This observation provides further justification and guidance for using experimental data obtained from cylindrical shell compression tests to assign knockdown factors to spherical shells, especially since data for cylindrical shells is more readily acquired that the corresponding data for spherical shells.

It is also evident from Fig. 4 that a reduction in the compressive circumferential membrane stress lowers the imperfection-sensitivity, the primary concern of this paper. If one assumes that the results in Fig. 4 are accurate for the prescribed axisymmetric imperfection, then it can be argued that they should provide a conservative prediction for the *increase in knockdown factor* above the values for uniaxial compression for the cylinder and equi-biaxial compression for the sphere.

The argument for the conservatism follows from the fact that, while σ_2 has no effect on pre-buckling behavior for axisymmetric imprefections, σ_2 diminishes pre-buckling deflections for non-axisymmetric imperfections—in effect, smoothing out the circumferential variations (assuming circumferential tension for the cylinder and reduced circumferential compression, or tension, for the sphere). Thus, the knockdown factor

based on non-axisymmetric imperfections increases more rapidly with reduction in σ_2 than for axisymmetric imperfections. This was established for cylindrical shells by Hutchinson (1965) using asymptotic methods for small imperfections. More recently, this point was emphasized by Teng and Rotter (1992) who also studied the effect of pressure for a variety of axisymmetric imperfection shapes using finite element simulations. It is not possible to carry out an analysis such as that in *Section 3* for the σ_2-dependence of α for non-axisymmetric imperfections without resorting to nonlinear finite element computations. Such an analysis would almost certainly produce a more rapid increase of α with decreases in σ_2 than those in Fig. 4. Nevertheless, it remains true that any shell dominated by axisymmetric imperfections would be expected to approximately follow the trends in Fig. 4 as is also evident in the results of Teng and Rotter (1992).

An illustration of a loading case that produces reduced circumferential membrane stress is provided by the tests on spherical shell segments conducted by Yao (1963). Yao subjected clamped fully circumferential spherical shell segments to an overall axial tensile load, P, with no normal pressure applied to the shell. These shell segments experienced a tensile axial membrane stress $P/(2\pi Rt)$ and an equal and opposite compressive circumferential stress. Thus, in the present notation with the axial and circumferential directions interchanged, Yao's tests have $\sigma_2/\sigma_1 = -1$. The range of the experimentally measured knockdown in Yao's tests is plotted in Fig. 5. The seven shells tested by Yao had R/t ranging from 455 to 1600 with the ratio of the experimental to the theoretical buckling loads ranging from 0.38 to 0.67. Stress estimates indicated that the shells all buckled in the elastic range. Two shells buckled at $\alpha = 0.38$, including one shell with $R/t = 476$. The imperfection associated with the theoretical curve in Fig. 5 has been set at $\sqrt{(1-\nu^2)}\overline{\xi} = 0.843$ such that the spherical shell under equi-biaxial compression buckles at $\alpha = 0.2$, consistent with the factor commonly used for the range of R/t for Yao's shells (c.f., Fig.1).

A large body of experimental data for spherical shells under loadings other than equi-biaxial compression does not exist, and even that is scarcer than cylindrical shell data. It is significant that two of Yao's seven shells buckled just above the theoretical

prediction in Fig. 5. This limited data set suggests that the present theoretical prediction for α as a function of σ_2/σ_1 may not be overly conservative. In other words, axisymmetric imperfections can be expected to dominate buckling in some shells. A similar conclusion was drawn for cylindrical shells subject to combined axial compression and internal pressure (Hutchinson, 1965).

The present study has relevance to cylindrical shells subject to combined bending and internal pressure where experimental data provides clear evidence of reduced imperfection-sensitivity as the pressure is increased (Seide, et al., 1960; Limam, et al., 1991; Mathon and Limam, 2006). For thin shells subject to bending, it is common practice to estimate the buckling stress based on the local stress in the most highly compressed region. Imperfection-sensitivity is less severe than for shells under uniaxial compression due to fact that a far smaller region of the imperfect shell is sampled by the highest compression and the mode is more localized. In principle, however, a local "axisymmetric" imperfection in the most highly stressed region should be almost as deleterious as for the uniaxial loading. The trends seen in the experimental data presented by Limam, et al. (1991) indicate a clear reduction in imperfection-sensitivity with increased internal pressure and, further, that the present results are conservative when applied in the manner just stated.

Lastly, it should be emphasized again that the present results focus on bifurcation from the axisymmetric state. It is well known from experiments and theory (e.g., Koiter, 2009) that bifurcation of cylindrical shells under uniaxial compression is unstable giving rise to dynamic collapse even under prescribed overall end shortening, except for highly imperfect shells (Budiansky and Hutchinson, 1972). It is also known from experimental observation that sufficiently high internal pressure can lead to stable post-bifurcation behavior in which the buckled shell supports an imposed axial load. This paper has not addressed the issue of post-bifurcation stability. It is expected that over much of the range of σ_2/σ_1 plotted in Figs. 4 and 5, the bifurcation will be unstable, but that has not been established.

3. Knockdown factor α for buckling of cylindrical and spherical shell segments with an axisymmetric imperfection

Essential details of the buckling analysis are presented in this section. The shells are elastic and isotropic with uniform thickness t. Middle surface coordinates, x_1 and x_2, are aligned with the axial (meridional) direction and the circumferential direction, respectively. An axisymmetric imperfection, $\bar{W}(x_1)$, specifies the normal deflection of the middle surface of the unloaded shell. The nonlinear Donnell-Mushtari-Vlasov (DMV) shell equations are

$$D\nabla^4 W = -p - \frac{1}{R}\left(F_{,11} + \beta F_{,22}\right) + F_{,11}W_{,22} + F_{,22}W_{,11} - 2F_{,12}W_{,12} + F_{,22}\bar{W}_{,11}$$

$$\frac{1}{Et}\nabla^4 F = -\frac{1}{R}\left(W_{,11} + \beta W_{,22}\right) + W_{,12}{}^2 - W_{,11}W_{,22} - W_{,22}\bar{W}_{,11}$$

(7)

with $D = E/[12(1-v^2)]$, $W(x_1, x_2)$ as the additional normal deflection of the shell, and $F(x_1, x_2)$ as the stress function giving resultant membrane stresses

$$N_{11} = F_{,22}, \quad N_{22} = F_{,11}, \quad N_{12} = -F_{,12}$$

For the cylindrical shell, R is the radius and $\beta = 0$; for the spherical shell, R is the radius of curvature and $\beta = 1$. The equations provide an accurate description of shell behavior for modes of deformation that are shallow, characteristic of the shortwave length buckling modes pertinent to the present study. The equations are also limited to middle surface rotations that do not exceed about 15° which is well below the rotations relevant to the results obtained below. In addition, the dimensions of the shells, or shell segments, must be large compared to ℓ in (4) and, for spheres, the segments must be shallow, as enunciated earlier. The edge support is assumed to be sufficiently robust such that the dominant buckling deflections are away from the boundaries. The analysis given below focuses on the behavior in the interior area of the shell and ignores a layer of width ℓ at the boundaries, as in the approach of Hutchinson (1967).

The analysis which follows applies to both the cylinder and the sphere depending on whether β is 0 or 1. The pre-buckling state of stress of the perfect shell ($\bar{W} = 0$) is uniform with

$$N_{11} = -\sigma_1 t, \quad N_{22} = -\sigma_2 t, \quad N_{12} = 0, \quad \text{with } \left(\beta\sigma_1 + \sigma_2 = pR/t\right)$$

(8)

The classical buckling equations for the perfect shell are obtained by linearizing (7) about the prebuckling state:

$$D\nabla^4\tilde{W} = \frac{1}{R}\left(\tilde{F}_{,11} + \beta\tilde{F}_{,22}\right) - \sigma_1 t\tilde{W}_{,11} - \sigma_2 t\tilde{W}_{,22}$$

$$\frac{1}{Et}\nabla^4\tilde{F} = -\frac{1}{R}\left(\tilde{W}_{,11} + \beta\tilde{W}_{,22}\right)$$

(9)

with $(\tilde{W}, \tilde{F})$ representing the linearized mode. Eqs. (9) admit eigenmodes of the form

$$\tilde{W} = \cos\left(\lambda_1 qx_1/R\right)\cos\left(\lambda_2 qx_2/R\right), \ \tilde{F} = F_0\cos\left(\lambda_1 qx_1/R\right)\cos\left(\lambda_2 qx_2/R\right) \quad (10)$$

where q is defined in (3). For the cylinder with $\sigma_2 = 0$, the critical (lowest) eigenvalue is given by (2) for any combination of λ_1 and λ_2 satisfying $\lambda_1^2 - \lambda_1 + \lambda_2^2 = 0$ (Koiter, 1945, 2009). For the sphere with $\sigma_2 = \sigma_1$, the critical eigenvalue is also given by (2) for any combination satisfying $\lambda_1^2 + \lambda_2^2 = 1$ (Hutchinson, 1967). Note that the axisymmetric mode (3) with $\lambda_1 = 1$ is one of the possible modes for both cases. When the loading is such that $\sigma_2 < 0$ for the cylinder, or $\sigma_2 < \sigma_1$ for the sphere, the critical eigenvalue (2) with the axisymmetric mode (3) is still valid, but now the non-axisymmetric modes with $\lambda_2 \neq 0$ are associated with higher eigenvalues. Thus, when the circumferential membrane stress is reduced below uniaxial compression for the cylinder, or below equi-biaxial compression for the sphere, the axisymmetric mode (3) becomes unique.

An axisymmetric imperfection (5) in the shape of the buckling mode is assumed. The nonlinear DMV equations for the imperfect shell admit an exact axisymmetric solution which written in terms of the normal displacement and the Airy stress function is

$$W_A = -\frac{R}{E}(\sigma_2 - \nu\sigma_1) + \bar{\xi}t\frac{\bar{\sigma}_1}{1-\bar{\sigma}_1}\cos(\frac{qx_1}{R})$$

$$F_A = -\frac{t}{2}\left(\sigma_1 x_2^2 + \sigma_2 x_1^2\right) - \bar{\xi}t\frac{ERt}{q^2}\frac{\bar{\sigma}_1}{1-\bar{\sigma}_1}\cos(\frac{qx_1}{R})$$

(11)

with $\bar{\sigma}_1 \equiv \sigma_1/\sigma_C$.

Following Koiter's (1963) approach for cylindrical shells under uniaxial compression, the bifurcation problem for buckling from the axisymmetric state into a non-axisymmetic mode is analyzed. An important feature of this approach is that it is carried out in such a way that the result is not only accurate but also provides an upper-

bound to the bifurcation stress for the specific imperfection, with a caveat mentioned below.

The solution in the buckled state is written as

$$W = W_A + w(x_1, x_2) \quad \& \quad F = F_A + f(x_1, x_2) \tag{12}$$

A non-axisymmetric deflection, w, of the form of (6) is assumed where γ sets the wavelength in the circumferential direction and is determined in the solution process. The motivation underlying the choice of this mode was given in connection with Fig. 2c. The next step is to substitute (12) into the nonlinear DMV compatibility equation and to note that the resulting equation can be solved *exactly* for f in terms of w:

$$f = \frac{AEt^3}{c}\left[b_1 \sin\left(\frac{1}{2}\frac{qx_1}{R}\right) + b_2 \sin\left(\frac{3}{2}\frac{qx_1}{R}\right)\right]\sin\left(\gamma \frac{qx_2}{R}\right) \tag{13}$$

with $c = \sqrt{3(1-v^2)}$ and

$$b_1 = \frac{1}{2}\left(\frac{1}{4}+\gamma^2\right)^{-2}\left[-\left(\frac{1}{4}+\beta\gamma^2\right)^{-1} + \frac{c\bar{\xi}\gamma^2}{1-\bar{\sigma}_1}\right], \quad b_2 = -\frac{1}{2}\left(\frac{9}{4}+\gamma^2\right)^{-2}\frac{c\bar{\xi}\gamma^2}{1-\bar{\sigma}_1}$$

Because only terms linear in A will be required, the quadratic terms in A are not shown.

The final step in the analysis is to evaluate the potential energy difference of the shell in buckled state from that in the axisymmetric state. The eigenvalue problem for the bifurcation problem only requires the quadratic terms in w and f in the potential energy change. Koiter's notation for this term is $P_2(w, f)$, and for DMV theory:

$$P_2(w, f) = \frac{1}{2}\int_S\left\{ D\left[(1-v)w_{,\alpha\beta}w_{,\alpha\beta} + vw_{,\gamma\gamma}^2\right] + \frac{1}{Et}\left[(1+v)f_{,\alpha\beta}f_{,\alpha\beta} - vf_{,\gamma\gamma}^2\right]\right.$$
$$\left. - \sigma_1 t\, w_{,1}^2 + \left(-\sigma_2 t + \frac{\bar{\xi}Et^2}{R}\frac{\bar{\sigma}_1}{1-\bar{\sigma}_1}\cos\left(\frac{qx_1}{R}\right)\right)w_{,2}^2 \right\}dS \tag{14}$$

The potential energy change can be evaluated in closed form:

$$P_2 = \frac{Et^3SA^2}{4R^2}\left\{\left(\frac{1}{4}+\gamma^2\right)^2 + 4\left(b_1\left(\frac{1}{4}+\gamma^2\right)\right)^2 + 4\left(b_2\left(\frac{9}{4}+\gamma^2\right)\right)^2 - \left(\frac{1}{2}+2\gamma^2 r\right)\bar{\sigma}_1 - c\bar{\xi}\gamma^2\frac{\bar{\sigma}_1}{1-\bar{\sigma}_1}\right\}(15)$$

where S is the area of the spherical segment and $r = \sigma_2/\sigma_1$. The only approximation in the above calculation occurs in the final step where contributions in the boundary layer of width ℓ are ignored. The neglected terms are of order ℓ/L relative to those retained

where L is the minimum in-plane dimension of the shell segment. In Koiter's (1963) analysis of infinitely long cylindrical shells, a similar approximation is made in ignoring the requirement that there must be an integral number of waves around the circumference.

For prescribed $\bar{\xi}$ and r with specified, γ, the eigenvalue for bifurcation from the axisymmetric state, $\bar{\sigma}_1$, is given by $P_2 = 0$. For prescribed $\bar{\xi}$ and r, the lowest buckling stress is obtained by minimizing this eigenvalue with respect to γ. Note that the normalized lowest buckling stress, $\bar{\sigma}_1 = \sigma_1 / \sigma_C$, is precisely the desired knockdown factor, α. The fact that the result so obtained is an upper-bound to the factor follows because the field used to evaluate P_2 is kinematically admissible due to fact that f is obtained exactly in terms of w. For the infinitely long cylinder constrained to have an integral number of waves around the circumference, the upper-bound is rigorous. For the finite cylinder or for spherical shell segments, the upper-bound is only rigorous as $R/t \rightarrow \infty$ due to the neglect of terms of order $1/q$.

Inspection of P_2 shows that $c\bar{\xi}$ appears in combination with no other dependence on ν and, thus, the curves in Figs. 3 and 4 do not depend on ν.

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J.-G. Teng and J.M. Rotter, Buckling of pressurized axisymmetrically imperfect cylinders under axial loads. J. Engineering Mechanics V. 118/2, 229-247, 1992.

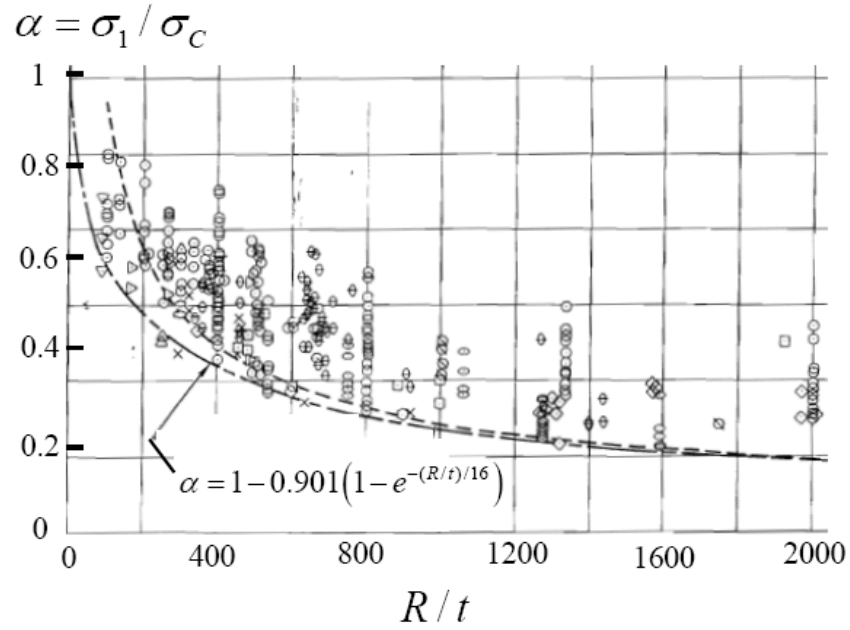J. C. Yao, Buckling of a truncated hemispherical under axial tension. AIAA J. 1, 2316-2319 (1963).

Fig. 1 Experimental data on the knockdown factor plotted against the radius to thickness ratio, R/t, for cylindrical shells under uniaxial compression. Reproduced from the report of Seide, Weingarten and Morgan (1960) who assembled the data for shells from various sources. Included in the figure is the empirical NASA (1965) recommendation for the knockdown factor for cylinders under uniaxial compression:
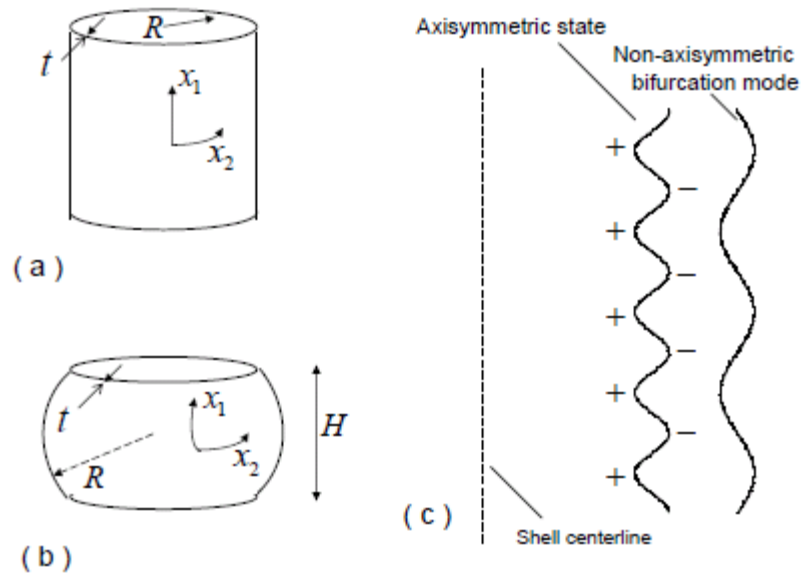
$$\alpha = 1 - 0.901\left(1 - e^{-(R/t)/16}\right).$$

Fig. 2 (a) Cylindrical and (b) spherical shell segments. (c) Interaction between the axisymmetric deformation associated with the initial imperfection and the non-axisymmetric bifurcation mode.
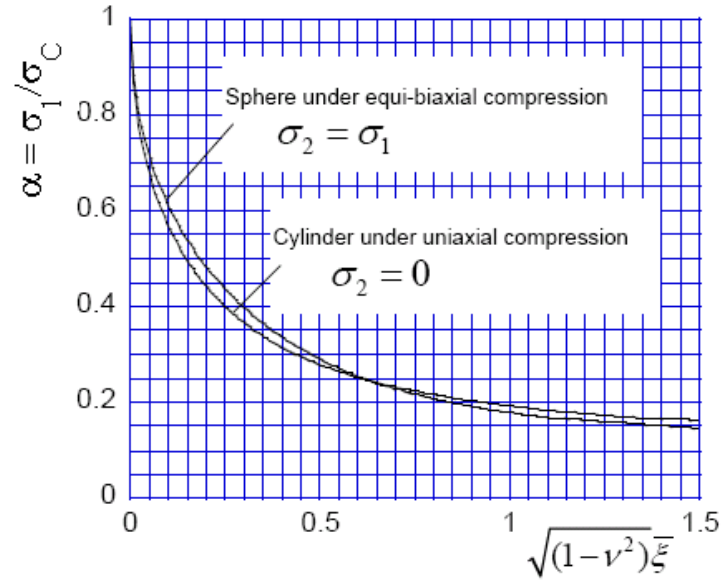
Fig. 3 Imperfection-sensitivity of cylindrical shells under uniaxial compression and spherical shells under external pressure (equi-biaxial compression) based on axisymmetric imperfections (5).
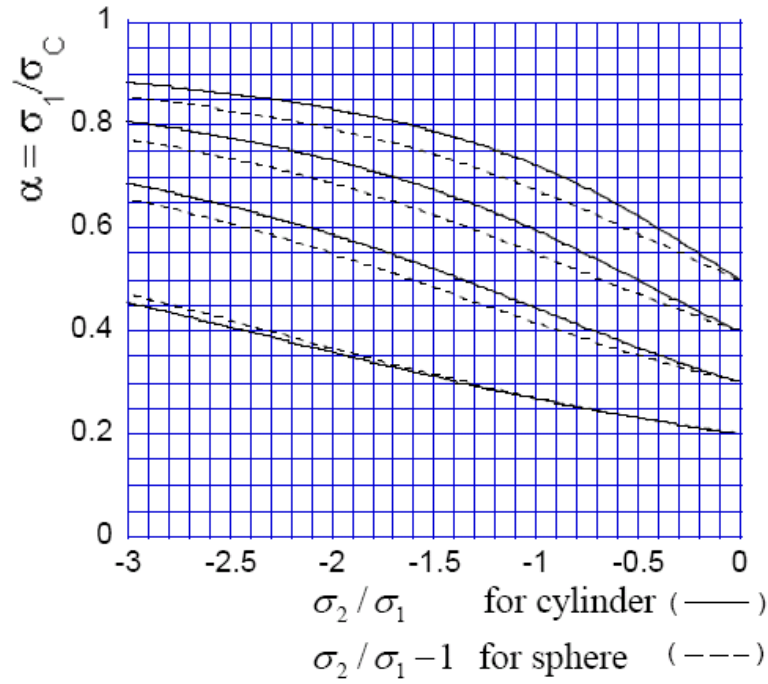
Fig. 4 The effect of a reduction in circumferential (transverse) membrane stress on the knockdown factor for cylindrical and spherical shells with an axisymmetric imperfection (5). The membrane stresses, σ_1 and σ_2, are taken positive in compression. For each of the four pairs of curves, the normalized imperfection amplitude, $\sqrt{(1-v^2)}\bar{\xi}$, is chosen separately for the cylinder and the sphere such that α the respective values coincide at the ordinate on the right. For the cylinder curves, from bottom up: $\sqrt{(1-v^2)}\bar{\xi} = 0.924$, 0.329, 0.250 and 0.147. The corresponding values for the sphere are: 0.843, 0.475, 0.296 and 0.183. The curves are independent of v.
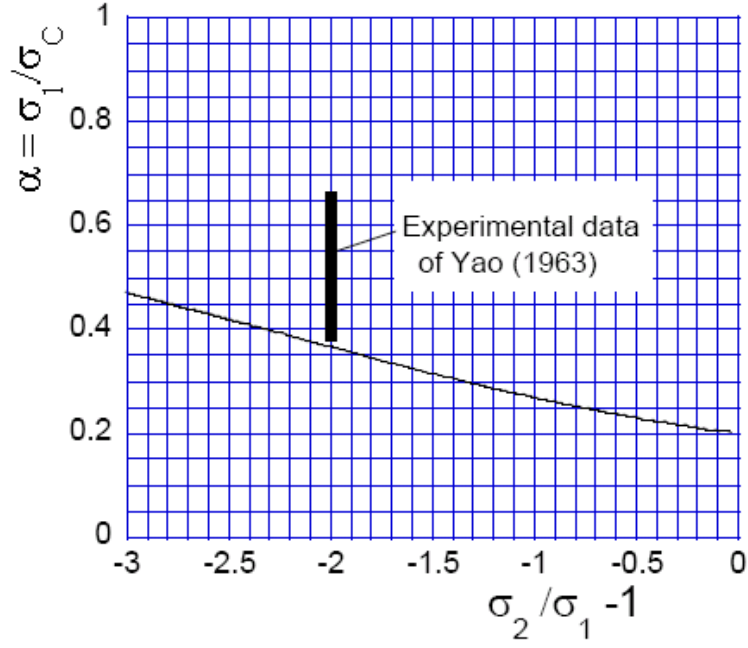
Fig. 5 Range of the buckling loads of seven thin spherical shell segments tested by Yao (1963) for a loading that generates equal and opposite membrane stresses. The theoretical curve is that for the sphere from Fig. 4 with the imperfection amplitude set such that $\alpha = 0.2$ for equi-biaxial compression ($\sqrt{(1-v^2)}\bar{\xi} = 0.843$).