K3 Surfaces, Entropy and Glue

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Accessibility
K3 surfaces, entropy and glue

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13 September, 2009

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1 Introduction

In this paper we use the gluing theory of lattices to construct K3 surface automorphisms with small entropy.

Algebraic integers. A Salem number $\lambda > 1$ is an algebraic integer which is conjugate to $1/\lambda$, and whose remaining conjugates lie on $S^1$. There is a unique minimum Salem number $\lambda_d$ of degree $d$ for each even $d$. The smallest known Salem number is Lehmer’s number, $\lambda_{10}$. These numbers and their minimal polynomials $P_d(x)$, for $d \leq 14$, are shown in Table 1.

<table>
<thead>
<tr>
<th>$\lambda_d$</th>
<th>$P_d(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_2$</td>
<td>$2.61803398$</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>$1.72208380$</td>
</tr>
<tr>
<td>$\lambda_6$</td>
<td>$1.40126836$</td>
</tr>
<tr>
<td>$\lambda_8$</td>
<td>$1.28063815$</td>
</tr>
<tr>
<td>$\lambda_{10}$</td>
<td>$1.17628081$</td>
</tr>
<tr>
<td>$\lambda_{12}$</td>
<td>$1.24072642$</td>
</tr>
<tr>
<td>$\lambda_{14}$</td>
<td>$1.20002652$</td>
</tr>
</tbody>
</table>

Table 1. The smallest Salem numbers by degree, and their minimal polynomials.

Surface dynamics. Now let $F : X \to X$ be an automorphism of a compact complex surface. It is known that the topological entropy $h(F)$ is determined by the spectral radius of $F^*$ acting on $H^*(X)$. More precisely, we have

$$h(F) = \log \rho(F^*|H^2(X)),$$

and if $h(F) > 0$, then a minimal model for $X$ is either a K3 surface, an Enriques surface, a complex torus or a rational surface [Ca]. The lower bound

$$h(F) \geq \log \lambda_{10}$$

holds for all surface automorphisms of positive entropy, by [Mc3].

In this paper, we will show that the lower bound (1.2) can be achieved on a K3 surface.

Theorem 1.1 There exists an automorphism of a K3 surface with entropy $h(F) = \log \lambda_{10}$.
Although the entropy in Theorem 1.1 is the minimum possible, the associated K3 surface is not projective. For projective surfaces, we will show:

**Theorem 1.2** There exists an automorphism of a projective K3 surface with entropy $h(F) = \log \lambda_6$.

As a complement, we note:

**Theorem 1.3** There exists an automorphism of a complex torus $\mathbb{C}^2/\Lambda$ with $h(F) = \log \lambda_6$, and an automorphism of an Abelian surface with $h(F) = \log \lambda_4$. In each case, no smaller positive entropy is possible.

In particular, the automorphisms provided by Theorems 1.1 and 1.2 have lower entropy than any example that can be obtained from a complex torus automorphism by passing to the associated Kummer surface (cf. [Mc2, §4]).

**Proof of Theorem 1.3.** For the first example, let $A \in \text{SL}_4(\mathbb{Z})$ be a matrix with $\det(xI - A) = x^4 + x + 1$. Then $A$ gives an automorphism $F$ of $X = \mathbb{R}^4/\mathbb{Z}^4$ preserving a complex structure, since the roots of $P$ occur in conjugate pairs; and the characteristic polynomial of $\Lambda^2 A$ is $P_6(x)$, so $h(F) = \log \lambda_6$ (compare [Mc2, §5]). No smaller entropy can arise, since $\exp h(F)$ must be a Salem number of degree at most $\dim H^2(X) = 6$.

For the second example, let $\zeta_d = \exp(2\pi i/d)$, let $E = \mathbb{C}/\mathbb{Z}[\zeta_d]$, let $X = E \times E$, and let $A \in M_2(\mathbb{Z}[\zeta_d])$ be any matrix with $(\text{tr} A, \det A) = (1, \zeta_6)$. Then the largest eigenvalue of $A$ satisfies $|\lambda|^2 = \lambda_4$. It follows that the induced automorphism $F : X \rightarrow X$ has entropy $h(F) = \log \lambda_4$. No smaller entropy is possible because, in the projective case, the entropy is given by the log of the leading eigenvalue of $F^*$ acting on the Néron-Severi group $\text{NS}(X) \subset H^2(X, \mathbb{Z})$, and the rank of $\text{NS}(X)$ is at most four.

It is known that the lower bound (1.2) can be realized on a rational surface [BK, Appendix], [Mc3], but not on an Enriques surface [Og, Thm 1.2]. At present there is no known automorphism $F$ of a projective K3 surface with $0 < h(F) < \log \lambda_6$.

**Glue groups.** To explain how the examples underlying Theorems 1.1 and 1.2 were found, suppose $F : X \rightarrow X$ is a K3 surface automorphism of positive entropy, and let $f = F^*$ acting on the even unimodular lattice $L = H^2(X, \mathbb{Z})$ of signature $(3, 19)$. Then we can write $S(x) = \det(xI - f) = S_1(x)S_2(x)$, where $S_1(x)$ is a Salem polynomial and $S_2(x)$ is a product of cyclotomic polynomials $C_n(x)$. There is a corresponding splitting $f = f_1 \oplus f_2$, leaving
invariant a sum of lattices $L_1 \oplus L_2$ with finite index in $L$. Passing to the
\textit{glue groups} $G(L_i) = L_i^\vee / L_i$, we obtain an isomorphism
\[
\phi : G(L_1) \to G(L_2)
\]
intertwining the quotient actions of $f_1$ and $f_2$. If these glue groups happen
to be nontrivial vector spaces over $\mathbb{F}_p$, then $S_1(x)$ and $S_2(x)$ must have a
common factor when reduced mod $p$. (Compare [Og, §4]).

In these terms, Theorems 1.1 and 1.2 were suggested by the fact that,
when reduced modulo $p = 3$, the Salem polynomial $P_{10}(x)$ is divisible by
$C_3(x) = x^2 + x + 1$, and $P_6(x)$ divides $C_{13}(x) = (x^{13} - 1)/(x - 1)$.

To actually construct examples, in §2–§4 we develop the general theory
of equivariant gluing, Coxeter groups and twists. These results provide
tools for producing a model $f : L \to L$ of the desired lattice automorphism
$F^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$. Care must be taken to construct a candidate
for the Kähler cone of $X$ (§5). Then the strong Torelli theorem and surjectivity of the period map (reviewed in §6) show one can realize $f : L \to L$
by a holomorphic automorphism $F : X \to X$ of a K3 surface. Detailed
constructions adapted to the Salem numbers $\lambda_{10}$ and $\lambda_{6}$ are given in §7 and
§8.

Many variations on these constructions, adapted to other Salem numbers
and to other properties of the resulting K3 surface, remain to be explored.

\textbf{Notes and references.} This paper is a sequel to [Mc2] and [GM], and
was inspired by Oguiso’s recent example of a K3 surface automorphism
with entropy $\log \lambda_{14}$ [Og]. I would like to thank B. Gross for many useful
discussions, and for pointing out the positive automorphism of $A_2 \oplus A_2$ used
in §7.

\section{Lattices and glue}

We begin by reviewing the construction of lattices and their automorphisms
using glue groups. This technique goes back to Witt and Kneser [Kn]; for
more details see e.g. [CoS].

\textbf{Lattices.} A \textit{lattice} $L$ of rank $r$ is a free abelian group $L \cong \mathbb{Z}^r$, equipped
with a nondegenerate inner product $\langle x, y \rangle$ taking values in $\mathbb{Z}$. The inner
product determines natural inclusions
\[
L \subset L^\vee \subset L \otimes \mathbb{Q}
\] (2.1)
where
\[
L^\vee = \text{Hom}(L, \mathbb{Z}) \cong \{x \in L \otimes \mathbb{Q} : \langle x, L \rangle \subset \mathbb{Z}\}.
\]
We say \( L \) has signature \((p, q)\) if the associated quadratic form
\[
x^2 = \langle x, x \rangle
\]
on \( L \otimes \mathbb{R} \) is equivalent to
\[
x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2.
\]
(2.2)

**The glue group.** The finite abelian group \( G(L) = L^\vee / L \) is the glue group of \( L \). It comes equipped with a nondegenerate fractional form \( \langle \langle x, y \rangle \rangle \) taking values in \( \mathbb{Q}/\mathbb{Z} \), characterized by
\[
\langle \langle x, y \rangle \rangle = \langle \tilde{x}, \tilde{y} \rangle \mod 1
\]
for any \( \tilde{x}, \tilde{y} \in L^\vee \) representing \( x, y \in G(L) \).

Concretely, if \((e_i)\) is an integral basis for \( L \) with Gram matrix \( B_{ij} = \langle e_i, e_j \rangle \), and \( d_i \in G(L) \) are the classes represented by a dual basis for \( L^\vee \), then the glue group has order
\[
|G(L)| = \det(L) = |\det B_{ij}|,
\]
and its fractional form is given by
\[
\langle \langle d_i, d_j \rangle \rangle = (B^{-1})_{ij} \mod 1.
\]

**Primary decomposition** The glue group can be written canonically as an orthogonal direct sum of \( p \)-groups,
\[
G(L) = \bigoplus_{p} G(L)_p,
\]
where \( p \) ranges over the primes dividing \( \det(L) \). The fractional form on \( G(L)_p \) takes values in \( \mathbb{Z}[1/p^e]/\mathbb{Z} \) for some \( e \).

In the special case where every element of \( G(L)_p \) has order \( p \), we can regard \( G(L)_p \) as a vector space over \( \mathbb{F}_p \), and consider the fractional form as an inner product with values in \( \mathbb{Z}[1/p]/\mathbb{Z} \cong \mathbb{F}_p \); see §3.

**Extensions of \( L \).** The glue group provides a useful description of all the lattices \( M \supset L \) such that \( M/L \) is finite. Indeed, since \( M \) pairs integrally with \( L \), any such extension can be regarded as a subgroup of \( L^\vee \); and the condition that the inner product on \( M \) is integral is equivalent to the condition that
\[
\overline{M} = M/L \subset G(L)
\]
is isotropic, i.e. $\langle x, y \rangle = 0$ for all $x, y \in \overline{M}$. Thus we have a bijective correspondence:

\[ \left\{ \text{Lattices } M \text{ with } L \subset M \subset L^\vee \right\} \leftrightarrow \left\{ \text{Isotropic subgroups } \overline{M} \text{ with } 0 \subset \overline{M} \subset G(L) \right\}. \]

Note that $[M : L] = |M|$, $\det(M) = \det(L)/[M : L]^2$, and the glue group of the extension is given by

\[ G(M) \cong \overline{M}^\perp / \overline{M}. \]

**Gluing a pair of lattices.** Now suppose $L = L_1 \oplus L_2$. A gluing map is an isomorphism $\phi : H_1 \to H_2$ between a pair of subgroups $H_i \subset G(L_i)$, $i = 1, 2$, satisfying

\[ \langle \langle x, y \rangle \rangle = -\langle \langle \phi(x), \phi(y) \rangle \rangle. \tag{2.3} \]

This condition guarantees that $\overline{M} = \{(x, \phi(x)) : x \in H_1 \} \subset G(L_1) \oplus G(L_2) = G(L)$ is isotropic, and hence $\phi$ determines a lattice

\[ M = L_1 \oplus_\phi L_2 \]

obtained by gluing $L_1$ and $L_2$ along $H_1 \cong H_2$. The extension $L_1 \oplus L_2 \subset M$ is primitive in the sense that $L_i = M \cap (L_i \otimes \mathbb{Q})$, or equivalently $M/L_i$ is torsion-free. Any primitive extension arises in this way, and hence we also have a natural correspondence:

\[ \left\{ \text{Primitive extensions } L_1 \oplus L_2 \subset M \right\} \leftrightarrow \left\{ \text{Gluing maps } \phi : H_1 \to H_2 \text{ between subgroups of } G(L_1) \text{ and } G(L_2) \right\}. \]

**Even lattices.** A lattice $L$ is even if $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$. In this case we have a natural quadratic form $q : G(L) \to \mathbb{Q}/\mathbb{Z}$ defined by

\[ q(x) = (1/2)\langle \bar{x}, \bar{x} \rangle \mod 1. \tag{2.4} \]

An extension $L \subset M$ is even iff $q|\overline{M} = 0$; similarly, a gluing $M = L_1 \oplus_\phi L_2$ of even lattices is even iff $q(x) + q(\phi(x)) = 0$ for all $x \in H_1$.

Note that $M \supset L$ is even whenever $L$ is even and $d = [M : L]$ is odd, for in this case we have $(dx)^2 = x^2 \mod 2$. 

5
Since \( q(x+y) = q(x) + q(y) + \langle x, y \rangle \), the fractional form determines \( q|G(L)_p \) for all odd primes \( p \) (but not for \( p = 2 \)).

**Extending isometries.** A bijective map from one lattice to another is an *isometry* if it preserves the inner product and group structure.

The *orthogonal group* \( O(L) \) consists of the isometries \( f : L \to L \). For simplicity, we also use \( f \) to denote its linear extensions to \( L^\vee, L \otimes \mathbb{R}, L \otimes \mathbb{C} \), etc. We let \( \overline{f} \) denote the induced isometry of \( G(L) \).

An isometry \( f \in O(L) \) extends to \( M \supset L \) iff \( \overline{f}(M) = M \). Similarly, equivariant gluing maps allow one to glue together isometries; we have a natural correspondence:

\[
\left\{ \begin{array}{l}
\text{Extensions } f \in O(M) \text{ of } f_1 \oplus f_2 \in O(L_1 \oplus L_2) \\
\text{satisfying } \phi \circ f_1 = f_2 \circ \phi
\end{array} \right\} \leftrightarrow \left\{ \text{Gluing maps } \phi : H_1 \to H_2 \right\}.
\]

**Roots and the Weyl group.** A vector \( e \in L \) is a *root* if \( \langle e, e \rangle = \pm 1 \) or \( \pm 2 \). Any root determines an isometric reflection \( s \in O(L) \) by the formula

\[
s(x) = x - \frac{2\langle x, e \rangle}{\langle e, e \rangle} e.
\]

The subgroup generated by all such reflections is the *Weyl group* \( W(L) \subset O(L) \). Note that \( s(x) - x \) is an integral multiple of \( e \) for all \( x \in L^\vee \). This shows:

*The Weyl group acts trivially on the glue group.*

**Root lattices.** We say \( L \) is a *root lattice* if it has an integral basis of roots. We conclude with some examples of root lattices that will be useful later. For more details, see [CoS], [Hum].

**Odd unimodular lattices.** Let \( \mathbb{Z}^{p,q} \) denote \( \mathbb{Z}^n \) with the inner product associated to the quadratic form (2.2). This is an odd unimodular root lattice, so it has trivial glue group.

**Coxeter diagrams.** Let \( \Gamma \) be a graph with vertices labeled \( 1, 2, \ldots n \). Then \( \Gamma \) determines a symmetric form with matrix

\[
B_{ij} = \langle e_i, e_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \text{ and } j \text{ are joined by an edge, and} \\ 0 & \text{otherwise.} \end{cases}
\]
Provided det $B_{ij} \neq 0$, this form makes $L = \oplus \mathbb{Z}e_i$ into an even root lattice of rank $n$. The product of the basic reflections $s_i$ determined by $e_i$ yields the Coxeter element

$$f = s_1s_2 \cdots s_n \in W(L) \subset O(L).$$

If $\Gamma$ is a tree, then the conjugacy class of $f$ is independent of the ordering of the vertices of $\Gamma$. Since $f$ lies in the Weyl group, $f$ acts trivially on $G(L)$.

**A$_n$ and E$_n$.** The diagrams for the lattices $A_n$ and $E_n$ are shown in Figure 2. The $A_n$ lattice can be regarded as the sublattice of $\mathbb{Z}^{n+1}$ defined by the equation $\sum x_i = 0$. Equivalently, $A_n$ is the orthogonal complement of $v_n = (1, 1, \ldots, 1)$. Since $\mathbb{Z}^{n+1}$ is unimodular, this shows

$$\mathbb{Z}^{n+1} = A_n \oplus \phi(\mathbb{Z}v_n)$$

where $\phi : G(A_n) \to G(\mathbb{Z}v_n)$ is an isomorphism. Since $\langle v_n, v_n \rangle = n + 1$, this implies

$$G(A_n) \cong G(\mathbb{Z}v_n) \cong \mathbb{Z}/(n + 1).$$

Similarly, $E_n$ can be regarded as the sublattice of $\mathbb{Z}^{n+1}$ perpendicular to $k_n = (1, 1, 1, \ldots, 1, -3)$.

(This vector represents the canonical class on the blowup of $\mathbb{P}^2$ at $n$ points; cf. [Mc3, §3].) Note that $\langle k_n, k_n \rangle = n - 9$. Excluding the case $n = 9$ (since the bilinear form on $E_9$ is degenerate), we find that

$$G(E_n) \cong \mathbb{Z}/|9 - n|.$$

The signature of $E_n$ is $(n, 0)$ for $n \leq 8$ and $(n - 1, 1)$ for $n \geq 10$.

**Even unimodular lattices.** The inner product $\langle e_i, e_j \rangle = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ on $\mathbb{Z}^2$ gives the unique even unimodular lattice $H$ of signature $(1, 1)$. More generally, for any $p, q \geq 1$ with $p \equiv q \mod 8$, there is a unique even unimodular lattice $\Pi_{p,q}$ of signature $(p, q)$ [MH], [Ser, §5].

We have just seen that $E_8$ and $E_{10}$ are unimodular, so we have $E_{10} \cong E_8 \oplus H \cong \Pi_{9,1}$; and in general $\Pi_{p,q} \cong aE_8 \oplus bH$ for suitable integers $a, b$. 

\[ \begin{array}{c c c c c c c c c}
A_n & & & & & & & & \\
1 & 2 & 3 & \ldots & n \\
& \bullet & \bullet & \bullet & & & & \bullet & \\
E_n & & & & & & & & \\
1 & 2 & 3 & 5 & \ldots & n \\
& \bullet & \bullet & \bullet & \bullet & & \bullet & \\
\end{array} \]
3 Isometries over finite fields

In this section we give a criterion for certain lattice automorphisms to automatically glue together.

**Theorem 3.1** Let $f_i \in O(L_i)$, $i = 1, 2$ be a pair of lattice isometries, and let $p$ be a prime. Suppose

1. Each glue group $G(L_i)_p$ is a vector space over $\mathbb{F}_p$;
2. The maps $f_i$ on $G(L_i)_p$ have the same characteristic polynomial $S(x)$; and
3. $S(x) \in \mathbb{F}_p[x]$ is a separable polynomial, with $S(1)S(-1) \neq 0$.

Then there is a gluing map $\phi : G(L_1)_p \to G(L_2)_p$ such that $f_1 \oplus f_2$ extends to $L_1 \oplus_{\phi} L_2$.

The proof is based on general properties of isometries over finite fields.

**Inner products spaces.** Let $k$ be a field. An inner product space over $k$ is a finite-dimensional vector space $V$ equipped with nondegenerate, symmetric bilinear form $\langle x, y \rangle : V \times V \to k$. With respect to a basis, the form is given by a symmetric matrix $B_{ij} = \langle e_i, e_j \rangle$; and the class

$$\det(V) = [\det B_{ij}] \in k^*/(k^*)^2$$

is an invariant of $V$.

**Example.** The sum $W = V \oplus V^\vee$ of a vector space with its dual carries a natural split inner product with $\det(W) = (-1)^{\dim V}$. Its matrix with respect to a pair of dual bases is given by $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

**Polynomials.** Given a degree $d$ monic polynomial $S \in k[x]$ with $S(0) \neq 0$, let

$$S^*(x) = x^d S(x^{-1})/S(0).$$

This is again a monic polynomial, whose roots are the inverses of the roots of $S$. If $S = S^*$ we say $S$ is a reciprocal polynomial. In this case $S(0) = \pm 1$. If $S(1)S(-1) \neq 0$, then the degree $d = 2e$ of $S$ is even, and there is a unique trace polynomial $R$ (of degree $e$) such that

$$S(x) = x^e R(x + x^{-1}).$$

**Isometries.** Let $f : V \to V$ be an isometry. Then $f^\vee = f^{-1}$, and hence the characteristic polynomial $S(x) = \det(xI - f)$ is reciprocal. Similarly

$$(\ker P(f))^\perp = \text{Im}(P(f)^\vee) = \text{Im} P^*(f) \quad (3.1)$$
for any $P \in k[x]$.

**Finite fields.** Now let $f : V \to V$ be an isometry of an inner product space over a finite field $k$.

We first note that $V$ is almost determined, up to isometry, by its dimension. In fact:

1. If $\text{char } k$ is odd, then $V$ is uniquely determined by $\dim(V)$ and by $\det(V) \in k^*/(k^*)^2 \cong \mathbb{Z}/2$; while

2. If $\text{char } k$ is 2, then $V$ is uniquely determined by $\dim(V)$ and the parity of $V$ (which is even if $\langle x, x \rangle = 0$ for all $x \in V$, and otherwise odd).

Even forms exists only in even dimensions.

See e.g. [MH, App. 2], [Ger, §2.8].

We now turn to the problem of classifying the pair $(V, f)$ up to isometry.

**Proposition 3.2** If $S(x) = \det(xI - f)$ is irreducible and $\dim V > 1$, then $(V, f)$ is determined up to isometry by $S$.

**Proof.** We claim $(V, f)$ is isometric to $(K, g)$, where $K = k[t]/S(t)$, $g(x) = tx$ and the inner product on $K$ is given by $\langle x, y \rangle_K = \text{Tr}_K^k(xy')$. Here $x \mapsto x'$ is the Galois involution on $K$ sending $t$ to $t^{-1}$, whose existence is guaranteed by the fact that $S(t)$ is a reciprocal polynomial.

To make this identification, first observe that $t \mapsto f$ gives an isomorphism $K \cong k[t] \subset \text{End}_K(V)$ sending the Galois involution to the adjoint involution (since $f^\vee = f^{-1}$). Upon choosing a nonzero vector $v \in V$, we obtain an isomorphism $V \cong Kv \cong K$ sending $f$ to $g$. By nondegeneracy of the trace form, there is then a unique $k$-linear map $\xi : K \to K$ such that

$$\langle x, y \rangle = \text{Tr}_K^k(\xi(x)y').$$

Using the fact that $\langle f(x), f(y) \rangle = \langle x, y \rangle = \langle y, x \rangle$, we find that $\xi(x) = bx$ where $b = b' \in K$. Since $\deg(S) > 1$, the Galois involution is nontrivial, and hence $b = aa'$ for some $a \in K$ (as a counting argument shows). But then we can simply replace $v$ by $av$ to obtain a new identification $V \cong Kv \cong K$ such that $\langle x, y \rangle = \text{Tr}_K^k(xy')$.

**Proposition 3.3** If $\det(xI - f) = Q(x)Q^*(x)$, where $Q(x)$ and $Q^*(x)$ are distinct irreducible monic polynomials, then $(V, f)$ is determined up to isometry by $Q(x)$.
Proof. In this case $V = \text{Ker} \, Q(f) \oplus \text{Ker} \, Q^*(f) = W \oplus W^\vee$, where the inner product identifies the second summand with the dual of the first. Since the linear map $f|W$ is determined by $Q(x)$, the pair $(V, f)$ is determined up to isometry by the same information.

**Proposition 3.4** If $S(x) = \det(xI - f)$ is separable and $S(1)S(-1) \neq 0$, then $(V, f)$ is determined up to isometry by $S$.

Proof. Since $S$ is a separable, reciprocal polynomial, it factors as a product of distinct irreducible polynomials

$$S(x) = S_1(x) \cdots S_r(x)Q_1(x)Q_1^*(x) \cdots Q_s(x)Q_s^*(x)$$

where $S_i = S_i^*$. Thus $V$ splits as an $f$-invariant orthogonal direct sum

$$V = \left( \bigoplus_{i=1}^r \text{Ker} \, S_i(f) \right) \oplus \left( \bigoplus_{i=1}^s \text{Ker} \, Q_i(f)Q_i^*(f) \right).$$

(Orthogonality follows from (3.1).) The assumption $S(1)S(-1) \neq 0$ insures $\dim \text{Ker} \, S_i(f) > 1$ for each $i$. Thus the preceding two propositions can be applied, term to term, to show that $(V, f)$ is determined up to isometry by $S$.

**Proof of Theorem 3.1.** The fractional form makes $G(L_i)_p$ into an inner product space over $\mathbb{F}_p \cong \mathbb{Z}[1/p]/\mathbb{Z}$. Since $\overline{f}_i$ acts isometrically, we may applying the preceding result (after reversing the sign of one of the forms) to obtain the desired gluing map $\phi$.

**The glue group of $A_{p-1}$.** In the absence of an automorphism, the isometry type of a glue group may need to be determined directly. For later use, we record a particular case:

**Proposition 3.5** The fractional form makes $V = G(A_{p-1})$ into an inner product space over $k = \mathbb{F}_p$ with $\det(V) = [-1] / (k^*/(k^*)^2)$.

Proof. The vector $x = (1, 1, \ldots, 1, 1 - p)/p \in A_{p-1}^\vee \subset \mathbb{R}^p$ satisfies $\langle x, x \rangle = (p - 1)/p = -1/p \mod 1$.

**Notes and references.** The results above can be regarded as special cases of the fact that a Hermitian space over a finite field is determined up to isomorphism by its dimension; see [MH, App. 2].
4 Twists

In this section we discuss the twists of a lattice $L$ by a self-adjoint endomorphism $a : L \to L$. Twisting allows one to adjust the signature and glue group of $L$ while respecting the action of a given isometry.

Twisting lattices. Let $L$ be a lattice of rank $r$. Suppose $a \in \text{End}(L)$ satisfies $a = a^\vee$ and $\det(a) \neq 0$. Then

$$\langle x, y \rangle_a = \langle ax, y \rangle$$

defines a new inner product on $L$, giving us a new lattice $L(a)$ called the twist of $L$ by $a$.

It is easy to see that $G(L(a)) = L^\vee/aL$ and $\det(L(a)) = \det(L)|\det(a)|$. More precisely, we have an exact sequence

$$0 \to L/aL \to G(L(a)) \to G(L) \to 0,$$

(4.1)

which splits if $\det(a)$ and $\det(L)$ are relatively prime.

Twisting isometries. Now suppose $L$ is equipped with an isometry $f : L \to L$. Let $\mathbb{Z}[f] \subset \text{End}(L)$ be the ring generated by $f$, and suppose $a \in \mathbb{Z}[f + f^{-1}]$ and $\det(a) \neq 0$. Then $a = a^\vee$ and $af = fa$, so $f \in O(L(a))$ as well. Thus we can regard $L, L(a)$ and their glue groups as modules over $\mathbb{Z}[f]$. With this understanding, (4.1) is an exact sequence of $\mathbb{Z}[f]$-modules.

Proposition 4.1 If $a \in \mathbb{Z}[f + f^{-1}]$ and $L$ is even, then so is $L(a)$.

Proof. Write $a = \sum_{i=0}^{n} a_i (f^i + f^{-i})$ with $a_i \in \mathbb{Z}$, and observe that $\langle f^{-1}x, x \rangle = \langle fx, x \rangle$; thus for all $x \in L$, we have

$$\langle ax, x \rangle = a_0 \langle x, x \rangle + \sum_{i=1}^{n} 2\langle f^i x, x \rangle \in 2\mathbb{Z}.$$ 

Primes and divisors. For more detailed results, we fix a prime $p$ not dividing $\det(L)$, and let $P \mapsto \overline{P}$ denote the natural map $\mathbb{Z}[x] \to \mathbb{F}_p[x]$. Then the twist $M = L(p)$ of a lattice of rank $r$ satisfies

$$G(M)_p \cong \mathbb{F}_p^r \quad \text{and} \quad \det(xI - f|G(M)_p) = \overline{S}(x),$$

where $S(x) = \det(xI - f)$.

By twisting with a divisor of $p$ in the ring $\mathbb{Z}[f + f^{-1}]$, we can sometimes arrange that the characteristic polynomial of $\overline{f|G(M)_p}$ is a given divisor of $\overline{S}(x)$. To state a result in this direction, assume that:
1. The polynomial $S(x) \in \mathbb{F}_p[x]$ is separable; and

2. We have $pL \subset aL$, where $a \in \mathbb{Z}[f + f^{-1}]$.

Then the result of twisting by $a$ can be described as follows.

**Theorem 4.2** The lattice $M = L(a)$ has glue group

$$G(M) \cong G(M)_p \oplus G(L)$$

(4.2)
as a $\mathbb{Z}[f]$-module. Moreover $G(M)_p$ is a vector space over $\mathbb{F}_p$, and the characteristic polynomial of $f | G(M)_p$ is given by

$$Q(x) = \gcd(A(x), S(x)) \in \mathbb{F}_p[x],$$

where $a = A(f) \in \mathbb{Z}[f]$.

**Proof.** Since $pL \subset aL$, $\det(a)$ is a power of $p$; and since $p$ does not divide $\det(L)$, the exact sequence (4.1) splits, which gives (4.2). The assumption $pL \subset aL$ also implies that $G(M)_p \cong L/aL$ is a quotient of $V = L/pL \cong \mathbb{F}_p^r$, so it is a vector space over $\mathbb{F}_p$. By separability, we have $V \cong \mathbb{F}_p[x]/(S)$, and hence

$$G(M)_p \cong V/aV \cong \mathbb{F}_p[x]/(\overline{A}, \overline{S}) \cong \mathbb{F}_p[x]/(\overline{Q})$$
as modules over $\mathbb{Z}[f]$.

**Dedekind domains.** The existence of a desired twist is guaranteed in certain situations by the following result.

**Theorem 4.3** Suppose $\mathcal{O} = \mathbb{Z}[f + f^{-1}]$ is a Dedekind domain of class number one,

$$\overline{S}(x) = \det(xI - f) \mod p$$
is separable, $\overline{S}(1)\overline{S}(-1) \neq 0$ and $\gcd(p, \det L) = 1$. Let $\overline{S}_1(x)$ be a reciprocal factor of $\overline{S}(x)$. Then there exists a twist $M = L(a)$, with $a \in \mathbb{Z}[f + f^{-1}]$ dividing $p$, such that

$$\overline{S}_1(x) = \det(xI - \overline{f}G(M)_p).$$

(4.3)

**Proof.** Let $R(y)$ be the trace polynomial associated to $S(x)$, so $S(x) = x^eR(x + x^{-1})$. Let $\overline{R} = \overline{R}_1\overline{R}_2$ be the factorization of $\overline{R}$ corresponding to
the given factorization $S = S_1S_2$. Then $O \cong \mathbb{Z}[y]/(R)$, so by basic number theory (see e.g. [La, I, §8]), there is a factorization $p = a_1a_2$ in $O$ such that

$$O/(a_iO) \cong \mathbb{F}_p[y]/(\overline{R}_i)$$

for $i = 1, 2$. Equivalently, if $a_i = A_i(f + f^{-1})$ with $A_i \in \mathbb{Z}[y]$, then $(\overline{A}_i) = (\overline{R}_i)$ as ideals in $\mathbb{F}_p[y]/(\overline{R})$.

Now we can also write $a_1 = A(f)$, since $\mathbb{Z}[f^{-1}] = \mathbb{Z}[f]$. Then $A(x) = A_1(x + x^{-1})$ in the ring $\mathbb{Z}[x]/(S)$. Consequently

$$\overline{(A)(x)} = (\overline{A_1}(x + x^{-1})) = (\overline{R_1}(x + x^{-1})) = (\overline{S_1}(x))$$

as ideals in $\mathbb{F}_p[x]/(\overline{S})$, and hence $\overline{S_1} = \gcd(\overline{A}, \overline{S})$, which gives (4.3) for $M = L(a_1)$.

**Signature.** To conclude, we relate the signatures of $L$ and $L(a)$.

Let $f : L \to L$ be an isometry of a lattice of signature $(p, q)$ such that $S(x) = \det(xI - f)$ is separable and $S(1)S(-1) \neq 0$.

Since $S(x)$ is reciprocal, it has $2s$ roots outside $S^1$ and $2t$ roots on $S^1$. The map $\lambda \mapsto \lambda + \lambda^{-1}$ sends the roots on $S^1$ to a set $T \subset (-2, 2)$ with $|T| = t$. As in [GM], we define the sign invariant $\epsilon_L : T \to \langle \pm 1 \rangle$ by

$$\epsilon_L(\tau) = \begin{cases} +1 & \text{if } E_\tau \text{ has signature } (2, 0), \\ -1 & \text{if } E_\tau \text{ has signature } (0, 2), \end{cases}$$

where

$$E_\tau = \text{Ker}(f + f^{-1} - \tau I) \subset L \otimes \mathbb{R} \cong \mathbb{R}^{p,q}.$$ 

Then the signature of $L$ is given by

$$(p, q) = (s, s) + \sum_T \begin{cases} (2, 0) & \text{if } \epsilon_L(\tau) = +1, \\ (0, 2) & \text{if } \epsilon_L(\tau) = -1. \end{cases} \quad (4.4)$$

Now for any twisting parameter $a = A(f + f^{-1}) \in \mathbb{Z}[f + f^{-1}]$, define $\epsilon_a : T \to \langle \pm 1 \rangle$ so that $\epsilon_a(\tau)A(\tau) > 0$. We then have

$$\epsilon_{L(a)}(\tau) = \epsilon_L(\tau)\epsilon_a(\tau), \quad (4.5)$$

and by (4.4) this determines the signature of $L(a)$.
5 Positivity

In this section we discuss the notion of a positive automorphism of a Euclidean or Lorentzian lattice. (The Lorentzian case is not needed until §8).

The Euclidean case. To begin with, assume $L$ is an even, positive-definite lattice. Let

$$\Phi = \{ y \in L : y^2 = 2 \}$$

be the finite set of roots in $L$. We say $\Phi^+ \subset \Phi$ is a system of positive roots if there is an $x \in L$ such that

$$\Phi = \Phi^+ \cup (-\Phi^+) \quad \text{and} \quad \langle x, y \rangle > 0 \ \forall y \in \Phi^+.$$  \hspace{1cm} (5.2)

Such an $x$ exists iff the convex hull of $\Phi^+$ does not contain the origin.

We say an isometry $f \in \text{O}(L)$ is positive if it preserves a system of positive roots.

![Figure 3. Reflection through the $x$-axis preserves a set of positive roots in the $A_2$ lattice.](image)

Example. The hexagonal root lattice $A_2$ admits a positive involution $f$ which interchanges the basic roots $e_1$ and $e_2$; see Figure 3. This map is not in the Weyl group $W(L)$; it comes from a symmetry of the $A_2$ diagram. We have $f(x) = -x$ on the glue group $G(A_2) \cong \mathbb{Z}/3$, and in fact $f$ generates $\text{O}(L)/W(L) \cong \mathbb{Z}/2$.

Basic properties. If $L = L_1 \oplus L_2$, and $f_i \in \text{O}(L_i)$ are positive, then so is $f = f_1 \oplus f_2$. This is because every root of $L$ lies in $L_1$ or $L_2$.

So long as $L$ has at least one root, any positive map $f \in \text{O}(L)$ has 1 as an eigenvalue, since $f$ must fix $\sum \{ y : y \in \Phi^+ \}$. By splitting along this eigenspace, we can present $(L, f)$ as a gluing $(L_1 \oplus L_2, f_1 \oplus f_2)$ where $f_1$ is the identity map and $L_2$ has no roots. Conversely, any map of this form is positive.

The Lorentzian case. We now turn to the case where $L$ is an even lattice of signature $(n,1)$. 

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In this case we say $\Phi^+$ is a positive root system if it satisfies (5.2) for some $x \in L$ with $x^2 < 0$. Geometrically, the roots $y \in \Phi$ define a locally finite system of hyperplanes $y^\perp$ in the hyperbolic space $\mathbb{H}^n \subset \mathbb{P}(L \otimes \mathbb{R})$, cutting it into open chambers. The choice of a positive root system (up to sign) is the same as the choice of one of those chambers; and a chamber can be specified by giving a representative point $[x] \in \mathbb{H}^n$ satisfying $\langle x, \Phi^+ \rangle > 0$.

Note that $\Phi$ excludes any roots of $L$ with $y^2 = -2$.

**Example.** Let $L$ be the Lorentz lattice $\mathbb{Z}^2$ with $(a, b)^2 = 2(a^2 + ab - b^2)$. Its roots $\Phi$ include the Fibonacci pairs $(1, 1), (2, 3), (5, 8), \ldots$. Let $e_1 = (1, 0), e_2 = (0, 1)$, and let $\Phi_1 = \{y \in \Phi : \langle e_i, y \rangle > 0\}$. Then $\Phi_2$ is a positive root system but $\Phi_1$ is not, essentially because $e_2^2 < 0$ but $e_1^2 > 0$.

**The light cone condition.** In the Lorentzian case, we say $f : L \to L$ is positive if it preserves a positive root system and it stabilizes each component of the light cone defined by $x^2 < 0$ in $L \otimes \mathbb{R} \cong \mathbb{R}^{n,1}$. (Thus $f(x) = -x$ is never positive).

**Gluing.** Let $L = L_1 \oplus \varphi L_2$ be an even lattice obtained by gluing a Lorentzian lattice $L_1$ to a Euclidean lattice $L_2$. We wish to investigate when an automorphism $f = f_1 \oplus f_2$ of $L$ is positive.

We remark that the timelike vectors in any Lorentzian lattice satisfy a reverse Cauchy-Schwarz inequality:

$$x^2, y^2 \leq 0 \implies |x^2 y^2| \leq |\langle x, y \rangle|^2,$$

as is easily verified by reducing to the case $x = e_{n+1}$ in $\mathbb{R}^{n,1}$.

**Theorem 5.1** Suppose $x^2 \in 2a_i \mathbb{Z}$ for all $x \in L_i$, $a_i > 1$, $bL \subset L_1 \oplus L_2$, and $b^2 \notin \mathbb{Z} + a_1 + \mathbb{Z} + a_2$. Then $f_1 \oplus f_2$ is a positive automorphism of $L$ provided $f_1$ has an eigenvalue $\lambda > 1$.

Here $\mathbb{Z}_+ = \{1, 2, 3 \ldots\}$ denotes the set of positive integers.

**Proof.** Let $y \in \Phi$ be a root of $L$. Then $y = (y_1, y_2) \in L_1^\perp \oplus L_2^\perp$ satisfies $y^2 = y_1^2 + y_2^2 = 2$, and $by \in L_1 \oplus L_2$. Therefore

$$(by)^2 = 2b^2 = (by_1)^2 + (by_2)^2 = 2(a_1 n_1 + a_2 n_2)$$

with $n_1, n_2 \in \mathbb{Z}$ and $n_2 \geq 0$. By assumption, this equation has no solutions with $n_1, n_2 > 0$. Our assumptions also imply that $L_1$ and $L_2$ have no roots, so we cannot have $n_1 = 0$ or $n_2 = 0$. Thus we must have $n_1 < 0$ and hence $y_1^2 < 0$; more precisely, we have $y_1^2 \leq -a_1/b^2$. 

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Pick any \( x_1 \in L_1 \) with \( x_1^2 < 0 \). Then by (5.3), we have

\[
|\langle x_1, y \rangle|^2 = |\langle x_1, y_1 \rangle|^2 \geq |x_1^2 y_1^2| \geq a_1 |x_1^2|/b_2 > 0 \quad \forall y \in \Phi.
\]

Thus \( \Phi^+ = \{ y \in \Phi : \langle x_1, y \rangle > 0 \} \) is a positive root system for \( L \). It consists exactly of the roots which project into the component of the lightcone of \( L_1 \otimes \mathbb{R} \) as \( x_1 \). Since \( f_1 \) has an eigenvalue \( \lambda > 1 \), it preserves this component, and hence \( f \) preserves \( \Phi^+ \) and is positive. \( \blacksquare \)

Geometrically, the proof shows that the inclusion \( L_1 \subset L \) determines an \( f \)-invariant hyperbolic subspace \( \mathbb{H}^{n_1} \subset \mathbb{H}^n \) lying completely inside one of the chambers defined by \( \Phi \).

**Theorem 5.2** Suppose \( f_1 \) and \( f_2 \) are positive, and every root \( (y_1, y_2) \in L \) which is not in \( L_1 \oplus L_2 \) satisfies \( y_2^2 \geq 2 \). Then \( f_1 \oplus f_2 \) is a positive automorphism of \( L \).

**Proof.** Let \( \Phi \) denote the roots of \( L \), let \( \Phi_i = \Phi \cap L_i \) for \( i = 1, 2 \), and let

\[
\Phi_3 = \{(y_1, y_2) \in \Phi : y_1^2 \leq 0, y_1 \neq 0 \}.
\]

Consider any root \( y = (y_1, y_2) \) of \( L \) that is not in \( \Phi_1 \cup \Phi_2 \). Since \( L_1 \) and \( L_2 \) are primitive, neither \( y_1 \) nor \( y_2 \) is zero. If \( (y_1, y_2) \in L_1 \oplus L_2 \), this implies \( y_1^2 = 2 - y_2^2 \leq 0 \); and the same is true, by assumption, if \( (y_1, y_2) \not\in L_1 \oplus L_2 \). Therefore

\[
\Phi = \Phi_1 \cup \Phi_2 \cup \Phi_3.
\]

Let \( \Phi_i^+ \subset \Phi_i \) be an \( f_i \)-invariant system of positive roots for \( i = 1, 2 \), and choose \( x_i \in L_i \) such that \( \langle x_i, y \rangle > 0 \) for all \( y \in \Phi_i^+ \). We may assume \( x_i^2 < 0 \).

Note that any \( (y_1, y_2) \in \Phi_3 \) satisfies \( \langle x_1, y_1 \rangle \neq 0 \), since \( y_1^2 \leq 0 \) and \( y_1 \neq 0 \). Let

\[
\Phi_3^+ = \{(y_1, y_2) \in \Phi_3 : \langle x_1, y_1 \rangle > 0 \}.
\]

This subset just consists of the elements of \( \Phi_3 \) that project to the component of the lightcone in \( L_1 \otimes \mathbb{R} \) that contains \(-x_1 \). Since \( f_1 \) preserves this component, we have \( f(\Phi_3^+) = \Phi_3^+ \).

We claim that \( \Phi_+ = \Phi_1^+ \cup \Phi_2^+ \cup \Phi_3^+ \) is a positive root system for \( L \). Since \( f(\Phi_+) = \Phi_+ \), this will complete the proof of positivity of \( f = f_1 \oplus f_2 \).

Evidently \( \Phi^+ \cup (-\Phi^+) = \Phi \). Let \( x = (Mx_1, x_2) \) for some integer \( M > 0 \). For all \( M \) large enough, we have \( x^2 < 0 \). It remains only to show that for all \( M \) large enough, we also have \( \langle x, y \rangle > 0 \) for all \( y \in \Phi^+ \).
The desired inequality is immediate for all \( y \in \Phi_1^+ \cup \Phi_2^+ \). Now suppose \( y = (y_1, y_2) \in \Phi_3^+ \). Choose \( d > 0 \) such that \( dL \subset L_1 \oplus L_2 \). Then \( dy_1 \in L_1 \), and hence \( \langle x_1, y_1 \rangle \geq 1/d \). Similarly, we have \( |y_2^2| \geq 1/d^2 \) provided \( y_1^2 \neq 0 \).

We claim that
\[
y_2^2 \leq (2d^2 + 1)|x_1, y_1|^2
\]
for all \( (y_1, y_2) \in \Phi_3^+ \). Indeed, if \( y_1^2 \neq 0 \) then \( |y_2^2| \geq 1/d^2 \) and hence
\[
y_2^2 = 2 + |y_1^2| \leq (2d^2 + 1)|y_1^2|.
\]
The reverse Cauchy-Schwarz inequality together with the fact that \( |x_1^2| \geq 1 \) then gives
\[
|y_1^2| \leq |x_1^2 y_1^2| \leq \langle x_1, y_1 \rangle^2,
\]
which yields (5.4). For the case \( y_1^2 = 0 \) we just observe that \( \langle x_1, y_1 \rangle^2 \geq 1/d^2 \), and hence
\[
y_2^2 = 2 \leq 2d^2 \langle x_1, y_1 \rangle^2,
\]
so (5.4) holds in this case as well.

Now the usual Cauchy-Schwarz inequality implies
\[
\langle x_2, y_2 \rangle^2 \leq x_2^2 y_2^2 \leq x_2^2 (2d^2 + 1) \langle x_1, y_1 \rangle^2.
\]
So for any \( M \) large enough that \( M^2 > (2d^2 + 1)x_2^2 \), we have \( \langle Mx_1, y_1 \rangle > |\langle x_2, y_2 \rangle| \), and hence \( \langle x, y \rangle = \langle Mx_1, y_1 \rangle + \langle x_2, y_2 \rangle > 0 \) for all \( y = (y_1, y_2) \in \Phi_3^+ \). Thus \( \Phi^+ \) is a positive, \( f \)-invariant root system, and \( f \) is a positive automorphism of \( L \).

6 Automorphisms of K3 surfaces

In this section we relate automorphisms of lattices to automorphisms of K3 surfaces.

**K3 structures.** Fix an even, unimodular lattice \( L \) of signature \((3, 19)\). A **K3 structure** on \( L \) consists of following data:

1. A *Hodge decomposition*
\[
L \otimes \mathbb{C} = L^{2,0} \oplus L^{1,1} \oplus L^{0,2}
\]
such that \( L^{i,j} = \overline{L^{j,i}} \), and the Hermitian spaces \( L^{1,1} \) and \( L^{2,0} \oplus L^{0,2} \) have signatures \((1, 19)\) and \((2, 0)\) respectively;
2. A positive cone $C \subset L^{1,1} \cap (L \otimes \mathbb{R})$, forming one of the two components of the locus $x^2 > 0$; and

3. A set of effective roots

$$\Psi^+ \subset \Psi = \{x \in L \cap L^{1,1} : x^2 = -2\},$$

satisfying $\Psi = \Psi^+ \cup (-\Psi^+)$. We require that the Kähler cone

$$C^+(L) = \{x \in C : \langle x, y \rangle > 0 \ \forall y \in \Psi^+\}$$

defined by this data is nonempty.

**Realizability.** A K3 structure on $L$ is realized by a K3 surface $X$ if there exists an isomorphism

$$\iota : L \to H^2(X, \mathbb{Z})$$

sending $L_{i,j}$ to $H^{i,j}(X)$ and sending $C^+(L)$ to the Kähler cone in $H^{1,1}(X, \mathbb{R})$. Similarly, an isometry $f : L \to L$ is realized an automorphism $F : X \to X$ if $\iota$ can be chosen so that the diagram

$$
\begin{array}{ccc}
L & \xrightarrow{\iota} & H^2(X, \mathbb{Z}) \\
\downarrow f & & \downarrow F^* \\
L & \xrightarrow{\iota} & H^2(X, \mathbb{Z})
\end{array}
$$

commutes.

The following fundamental theorem [BPV, VIII] encapsulates the strong Torelli theorem for K3 surfaces as well as surjectivity of the period map:

**Theorem 6.1** Any K3 structure on $L$ is realized by a unique K3 surface $X$, and any $f \in O(L)$ preserving a given K3 surface structure is realized by a unique automorphism $F : X \to X$.

**Remarks.** The Hodge structure on $L$ determines $X$ up to isomorphism, while the Kähler cone $C^+(L)$ pins down the isomorphism $\iota$. The Néron-Severi group $\text{NS}(X) \cong \text{Pic}(X) \subset H^2(X, \mathbb{Z})$ is given by

$$\text{NS}(X) = \iota(L \cap L^{1,1}).$$

To conclude, we address the problem of finding an $f$-invariant K3 structure.
Theorem 6.2 Let \( f \in \text{O}(L) \) be an isometry with spectral radius \( \rho(f) > 1 \). Suppose \( \rho(f) \) is an eigenvalue of \( f \), and there is a unique \( \tau \in (-2, 2) \) such that
\[
E_\tau = \text{Ker}(f + f^{-1} - \tau I) \subset L \otimes \mathbb{R}
\]
has signature \((2, 0)\). Then \( f \) is realizable by a K3 surface automorphism iff \( f|_{M(-1)} \) is positive, where \( M = L \cap E_\tau^\perp \).

Proof. Our assumptions imply there is an \( f \)-invariant Hodge structure with \( L^{2,0} \oplus L^{0,2} \cong E_\tau \otimes \mathbb{C} \). Moreover, this is the unique \( f \)-invariant Hodge structure of K3 type, up to interchanging the roles of \( L^{2,0} \) and \( L^{0,2} \). In particular, \( L^{1,1} \) is uniquely determined, as is the candidate Néron-Severi group
\[
M = L \cap L^{1,1} = L \cap E_\tau^\perp
\]
and the set of roots \( \Psi = \{ x \in M : x^2 = -2 \} \).

Suppose \( f|_{M(-1)} \) is positive. Let \( \Psi^+ \subset \Psi \) be a system of positive roots preserved by \( f \). If \( M \) has signature \((1, n)\), then (by the definitions in §5, taking into account the reversal of signs) there is an \( x \in M \) with \( x^2 > 0 \) such that \( \langle x, y \rangle > 0 \) for all \( y \in \Psi^+ \). Choose \( \mathcal{C} \) so \( x \in \mathcal{C}^+ \); then \( \mathcal{C}^+ \) is nonempty, and \( f(\mathcal{C}) = \mathcal{C} \) because the leading eigenvalue of \( f \) is positive. Thus \( f \) is realizable. If \( M \) has signature \((0, n)\), then the same argument applies, except \( x^2 < 0 \). But we can then simply replace \( x \) with \( x' = x + z \), where \( z \in L^{1,1} \cap M^\perp \) and \( z^2 + x^2 > 0 \); then \( (x')^2 > 0 \), so \( \mathcal{C}^+ \neq \emptyset \), and hence \( f \) is realizable in this case as well.

Conversely, if \( f \) is realizable by \( F : X \to X \), then \( F \) preserves the Kähler cone in \( H^{1,1}(X, \mathbb{R}) \), and hence \( f \) preserves the dual system of positive roots \( \Psi^+ \subset \text{NS}(X)(-1) \cong M(-1) \), so it is positive.

The proof shows the pair \((X, F)\) realizing \( f \) is unique up to complex conjugation, and that \( M \cong \text{NS}(X) \).

7 Minimum entropy

In this section we will show:

Theorem 7.1 There exists an automorphism \( F : X \to X \) of a non-algebraic K3 surface with entropy \( h(F) = \log \lambda_{10} \approx \log 1.17628 \).
Building blocks. To exhibit $F$, we will construct a lattice automorphism $f : L \to L$ with characteristic polynomial

$$\det(xI - f) = P_{10}(x)(x - 1)^9(x + 1)(x^2 + 1)$$

satisfying the realizability criterion stated in Theorem 6.2. The pair $(L, f)$ will in turn be obtained as a gluing of $(L_1, f_1)$ and $(L_2, f_2)$. We begin by describing these two constituents of $f$.

The Salem factor. Recall from §2 that $E_{10}$ is an even, unimodular lattice of signature $(9, 1)$. The Salem polynomial $P_{10}(x)$ arises naturally as the characteristic polynomial $S_1(x) = P_{10}(x) = \det(xI - f_1)$ of the Coxeter automorphism $f_1 : E_{10} \to E_{10}$ (see e.g. [Mc1]). Reducing mod $p$ for $p = 3$, we find

$$S_1(x) = (x^2 + 1)(x^8 + x^7 + 2x^6 + x^5 + 3x^2 + x + 1)$$

in $\mathbb{F}_3[x]$. This factorization suggests that with suitable twisting, we may be able to arrange that $\overline{f}_1$ acts with characteristic polynomial $(x^2 + 1)$ on a glue group isomorphic to $\mathbb{F}_3^2$. And indeed this is the case: if we let

$$a = 2(f_1 + f_1^{-1}) + 3 \in \text{End}(E_{10}),$$

then $|\det(a)| = 9$ and $3E_{10} \subset aE_{10}$, as can be checked by a matrix computation (e.g. $3a^{-1}$ is integral). It then follows from Theorem 4.2 that for $L_1 = E_{10}(a)$, we have

$$G(L_1) \cong \mathbb{F}_3^2 \quad \text{and} \quad \overline{S}_1(x) = \det(xI - \overline{f}_1) = (x^2 + 1) \mod 3.$$

To determine the signature of $L_1$, let $R_1(y) = y^5 + y^4 - 5y^3 - 5y^2 + 4y + 3$ be the trace polynomial of $S_1(x)$, and let

$$T = \{\tau_1, \tau_2, \tau_3, \tau_4\} \approx \{-1.886, -1.468, -0.584, 0.913\}$$

denote the roots of $R_1(y)$ which lie in $(-2, 2)$. The associated eigenspaces of $f + f^{-1}$ all have signature $(2, 0)$, since $E_{10}$ has signature $(9, 1)$ (see equation 4.4). On the other hand, the polynomial $P(y) = 2y + 3$ is negative for $y = \tau_1$ but positive for $y = \tau_2, \tau_3, \tau_4$; since $a = P(f + f^{-1})$, this implies $L_1 = E_{10}(a)$ has signature $(7, 3)$ (by equation 4.5). Finally $L_1$ is an even lattice, by Proposition 4.1.
The cyclotomic factor. Now recall from §5 that there is a positive automorphism \(g : A_2 \to A_2\) such that \(g\) acts with order two on \(G(A_2) \cong \mathbb{F}_3\). Let \(L_2 = E_8 \oplus A_2 \oplus A_2\). Note that \(L_2\) has signature \((12,0)\), and every root of \(L_2\) lies in one of its summands. It follows that the order four map \(f_2 : L_2 \to L_2\) given by

\[ f_2(x, y, z) = (x, g(z), y) \]

is also positive. Its characteristic polynomial is given by

\[ S_2(x) = (x - 1)^9(x + 1)(x^2 + 1), \]

while its action on \(G(L_2) \cong \mathbb{F}_3\) has characteristic polynomial

\[ Q_2(x) = \det(xI - f_2) = x^2 + 1. \]

Proof of Theorem 7.1. Since \(Q_1(x) = Q_2(x) = x^2 + 1 \in \mathbb{F}_3[x]\) is a separable polynomial, nonvanishing at \(x = \pm 1\), there is a gluing map \(\phi : G(L_1) \to G(L_2)\) conjugating \(f_1\) to \(f_2\) by Theorem 3.1.

Let \(L = (L_1 \oplus \phi L_2)(-1)\). This is a lattice of signature \((3,7) + (0,12) = (3,19)\). Since we are gluing at the odd prime \(p = 3\), \(L\) is still even. By construction, \(f_1 \oplus f_2\) extends to an isometry \(f : L \to L\), with characteristic polynomial \(S(x) = S_1(x)S_2(x)\).

Since \(S_1(x)\) is a Salem polynomial and \(S_2(x)\) is a product of cyclotomic polynomials, the spectral radius

\[ \rho(f) = \max\{|\lambda| : S(\lambda) = 0\} = \lambda_{10} \]

is an eigenvalue of \(f\). Moreover, the twist by \(a\) provides us with a unique eigenspace

\[ E_\tau = \text{Ker}(f + f^{-1} - \tau I) \subset L_1 \otimes \mathbb{R} \cong \mathbb{R}^{3,19} \]

with signature \((2,0)\), coming from \(\tau = \tau_1\).

Since \(S_1(x)\) is irreducible, no element of \(L_1\) lies in \(E_\tau^\perp\); thus \(M = L \cap E_\tau^\perp = L_2(-1)\). By construction, \(f_2|L_2 \cong f|M(-1)\) is positive. Thus by Theorem 6.2 there is a K3 surface automorphism \(F : X \to X\) realizing \(f\); and \(h(F) = \rho(f) = \lambda_{10}\) by equation (1.1).

Remarks.

1. We have \(\text{NS}(X)(-1) \cong E_8 \oplus A_2 \oplus A_2\). Thus by Grauert’s criterion [BPV, III.2], the exceptional curves on \(X\) can be blown down to yield a singular complex manifold \(Y\) with no curves at all. The map \(F\) descends to an automorphism of \(Y\) which exchanges its two singular points of type \(A_2\), and fixes its unique singular point of type \(E_8\).
2. The ring $\mathbb{Z}[f_1 + f_1^{-1}]$ is in fact a Dedekind domain of class number one, so by Theorem 4.3, the existence of the desired twist of $E_{10}$ is automatic once one has the factorization (7.1).

3. Oguiso gives an example of a K3 surface automorphism with entropy $\log \lambda_{14} \approx \log 1.20002$ [Og]. This example is analogous to the one above, but with $L_1 = \Pi_{11,3}$ and $L_2 = E_8$. Here both lattices are unimodular, so no glue is necessary, and one can take $f_2(x) = x$. The existence of an $f_1 \in \mathcal{O}(L_1)$ with characteristic polynomial $P_{14}(x)$ is guaranteed by [GM].

4. Many examples of K3 surface automorphisms based on Salem numbers of degree 22 are given in [Mc2]. In these examples no gluing takes place (there is no cyclotomic factor), and positivity is automatic, because the Néron-Severi group is trivial.

5. We remark that gluing theory is also useful for describing the Kummer surface $X$ associated to a complex torus $A$: indeed, $H^2(X, \mathbb{Z})$ is obtained by gluing $H^2(A, \mathbb{Z})(2)$ along $\mathbb{F}_2^6$ to a fixed lattice of rank 16 (see [BPV, VIII.5]).

8. A projective example

In this section we will show:

**Theorem 8.1** There exists an automorphism $F : X \to X$ of a projective K3 surface with entropy $h(F) = \log \lambda_6 \approx \log 1.40126$.

For the proof, we will construct a model for $F|H^2(X, \mathbb{Z})$ by gluing together four lattice automorphisms $f_i : L_i \to L_i$, $i = 1, \ldots, 4$ (see Figure 4), and twisting by $-1$. The result will be an automorphism $f : L \to L$ of an even, unimodular lattice of signature $(3, 19)$ with characteristic polynomial

$$S(x) = P_6(x)C_{13}(x)(x^2 + x + 1)(x - 1)^2,$$

where $P_6(x)$ is the Salem polynomial for $\lambda_6$ (see Table 1) and $C_{13}(x) = (x^{13} - 1)/(x - 1)$. Theorem 6.2 will imply that $f$ is realizable.

We now turn to the construction of the building blocks $f_i : L_i \to L_i$. For each $i$ we will determine the signature of $L_i$, the glue group $G(L_i)$, and the characteristic polynomial

$$\overline{Q}_{i,p}(x) = \det(xI - f_i|G(L_i)_p) \in \mathbb{F}_p[x]$$
of $\bar{f}_i$ for each prime $p$ where it is nontrivial.

1. The Salem factor. Let $W = \wedge^2 \mathbb{Z}^4$, with the natural inner product

$$\langle \alpha, \beta \rangle = \alpha \wedge \beta \in \wedge^4 \mathbb{Z}^4 \cong \mathbb{Z}.$$ 

Then $W \cong \Pi_{3,3}$ is an even, unimodular lattice of signature (3, 3), and any $g \in \text{SL}_4(\mathbb{Z})$ gives rise to an isometry $f = \wedge^2 g \in \text{O}(W)$. In particular, if we take $g = g_1$ to be the companion matrix for $x^4 + x + 1$, then we obtain a map $f_1 \in \text{O}(W)$ with characteristic polynomial

$$S_1(x) = \det(I - f_1) = P_6(x).$$

(This is related to the fact that $\log \lambda_6$ can be realized as the entropy of an automorphism of a complex 2-torus [Mc2, §5].) Reducing mod $p = 2$, we find

$$S_1(x) = (1 + x + x^2)(1 + x + x^2 + x^3 + x^4)$$

in $\mathbb{F}_2[x]$. This factorization suggests we can find a twist $W(a)$ such that

$$G(W(a)) = \mathbb{F}_2^2 \quad \text{and} \quad \det(xI - \bar{f}_1 | W(a)) = (1 + x + x^2).$$

Indeed, this is the case for $a = -(1 + f + f^{-1})$, as can be verified with the help of Theorem 4.2. Similarly, if we set $L_1 = W(3a)$, then we find

$$G(L_1) = G(L_1)_2 \oplus G(L_1)_3 = \mathbb{F}_2^2 \oplus \mathbb{F}_3^6,$$

$$\bar{Q}_{1,2}(x) = (1 + x + x^2) \quad \text{and}$$

$$\bar{Q}_{1,3}(x) = (2 + x + x^2 + x^3)(2 + 2x + 2x^2 + x^3).$$
Note that $Q_{1,3}(x)$ is a reciprocal polynomial, even though its irreducible factors are not. Since $W$ is even, and we have twisted by $3a$, we have $\langle x, x \rangle \in \mathbb{Z}$ for all $x \in L_1$. In particular, $L_1$ is an even lattice with no roots. It signature is $(5, 1)$, as can be checked using equation (4.5).

2. The order 13 factor. Next consider the Coxeter automorphism $f_2 : A_{12} \to A_{12}$. This map has order 13, so its characteristic polynomial is given by $S_2(x) = C_{13}(x)$, which factors modulo $p = 3$ as

$$S_2(x) = (2 + x + x^2 + x^3)(2 + 2x + 2x^2 + x^3)(2 + 2x + x^3)(2 + x^2 + x^3).$$

The map $\overline{f}_2|G(A_{12}) \cong \mathbb{Z}/13$ is the identity, since $f_2$ belongs to the Weyl group of $A_{12}$ (see §2). If we set $L_2 = A_{12}(a)$ where $a = 1 + (f_2 + f_2^{-1}) - (f_2 + f_2^{-1})^3$, then $aA_{12} \subset 3A_{12}$ and we find

$$G(L_2) = G(L_2)_3 \oplus G(L_1)_1 = \mathbb{F}_3^2 \oplus \mathbb{F}_{13},$$

$$\overline{Q}_{2,3}(x) = (2 + x + x^2 + x^3)(2 + 2x + 2x^2 + x^3),$$

and

$$\overline{Q}_{2,13}(x) = (x - 1).$$

(Note that we have chosen $a$ so that $\overline{Q}_{1,3} = \overline{Q}_{2,3}$.) In this case $L_2$ is an even lattice of signature $(10, 2)$.

The action of $\overline{f}_2$ determines the fractional form on $G(L_2)_3$, but not on $G(L_2)_1$ (where it acts by the identity). For later use we note:

The determinant of $G(L_2)_1$ is not a square.

That is, the nonzero values of the form $13\langle x, x \rangle$ consist of the non-square numbers $\{2, 4, 6, 7, 8, 11\} \mod 13$. This can be verified by a direct matrix computation, or by noting that $\det G(A_{12}) = -1$ (as computed in Proposition 3.5) is a square mod 13 but 8 is not, and $ax = 8x$ for $x \in G(A_{12})$, since $\overline{f}_2(x) = x$.

3. The order 3 factor. Let $f_3 : A_2 \to A_2$ be the Coxeter automorphism, and let $L_3 = A_2(2)$. Then the characteristic polynomial of $f_3$ is given by $S_3(x) = x^2 + x + 1$, and $\overline{f}_3|G(A_2)$ is the identity. It follows that

$$G(L_3) = G(L_3)_2 \oplus G(L_3)_3 = \mathbb{F}_2^2 \oplus \mathbb{F}_3,$$

$$\overline{Q}_{2,2}(x) = x^2 + x + 1,$$

and

$$\overline{Q}_{2,3}(x) = (x - 1).$$

The lattice $L_2$ has signature $(2, 0)$, and $\langle x, x \rangle \in 4\mathbb{Z}$ for all $x \in L_2$. Since $-1$ is not a square mod 3, and neither is the twisting parameter 2, using Proposition 3.5 again we find:
The determinant of $G(L_3)$ is a square.

4. The identity factor. Finally let $L_4 \cong \mathbb{Z}^2$ be the even lattice of signature $(2, 0)$ and determinant 39 with inner product matrix $B = \left( \begin{smallmatrix} 2 & 1 \\ 1 & 20 \end{smallmatrix} \right)$. Then

$$G(L_4) = G(L_4)_3 \oplus G(L_4)_{13} = \mathbb{F}_3 \oplus \mathbb{F}_{13}.$$ 

To control later gluings, the following two properties are important.

If $x \in L_4^\vee$ projects to a nontrivial element of $\mathbb{F}_3 \subset G(L_4)$, then $x^2 \geq 2$.

In fact, $(1, 1)/3$ and $(-1, -1)/3$ are the minimal norm vectors in $L_4^\vee$ representing the nonzero elements of $\mathbb{F}_3$, and each satisfies $x^2 = 8/3$.

Neither $\det G(L_4)_3$ nor $\det G(L_4)_{13}$ is a square.

To see this, note that $2/39$ appears on the diagonal of $B^{-1}$. Thus $\langle x, x \rangle = 2/39 \mod 1$ for some $x$ in $G(L_4)$; this implies $\det G(L_4)_3 = |3\langle 13x, 13x \rangle| = [2 \mod 3]$ and $\det G(L_4)_{13} = |13\langle 3x, 3x \rangle| = [6 \mod 13]$, and neither class is a square.

As an automorphism of $L_4$, we simply take $f_4(x) = x$.

Assembly. By construction, we have gluing isometries

$$\begin{align*}
\phi_{12} & : G(L_1)_3 \to G(L_2)_3 \cong \mathbb{F}_3^6, \\
\phi_{13} & : G(L_1)_2 \to G(L_3)_2 \cong \mathbb{F}_2^2, \\
\phi_{24} & : G(L_2)_{13} \to G(L_4)_{13} \cong \mathbb{F}_{13}, \text{ and} \\
\phi_{34} & : G(L_3)_3 \to G(L_4)_3 \cong \mathbb{F}_3,
\end{align*}$$

satisfying $\phi_{ij} f_i = f_j \phi_{ij}$. The first glue map $\phi_{12}$ exists by Theorem 3.1, since $Q_{1,3} = Q_{2,3}$. Similar reasoning applies to the second. The third map $\phi_{24}$ exists because both the domain and range have non-square determinant, and because $(-1)$ is a square mod 13. (Recall that a gluing map must reverse the sign of the bilinear form.) The last map exists because its domain has square determinant, while its range does not, and because $-1$ is not a square mod 3.

Let $\bigoplus \phi_i L_i$ denote the unimodular lattice of signature $(19, 3)$ obtained by gluing together all four lattices, as shown Figure 4. Let $L = (\bigoplus \phi_i L_i)(-1)$, and let $f : L \to L$ denote the linear extension of $\bigoplus f_i$. Then the characteristic polynomial of $f$ is given by $S(x) = \prod_i S_i(x)$, which agrees with equation (8.1).
Evenness. We claim $L$ is even. This is almost automatic, since its constituents $L_i$ are even and since almost all the gluings take place over groups of odd order. The one exception comes from $\phi_{13}$. To verify evenness, we must check that the $\mathbb{Q}/\mathbb{Z}$-valued quadratic forms $q_i(x)$ on $G(L_i)_2$, $i = 1, 3$, defined by equation (2.4), satisfy $q_1(x) + q_3(\phi_{13}(x)) = 0 \mod 1$. But $q_i$ is invariant under $T_i$, which cyclically permutes the three nonzero vectors in $G(L_i)_2 \cong \mathbb{F}_2^2$. This easily implies that $q_i(x) = 1/2$ for all $x \neq 0$, so the sum vanishes and hence $L$ is even.

Proof of Theorem 8.1. Clearly $\rho(f) = \lambda_6 > 1$, since all other roots of $S(x)$ are inside or on the unit circle. By construction, $f + f^{-1}$ has a unique eigenspace of signature $(2, 0)$, namely

$$E_\tau = \text{Ker}(f + f^{-1} - \tau I) \subset L_2 \otimes \mathbb{R}$$

where $\tau = 2 \cos(2\pi/13)$. Since the action of $f$ on $L_2$ is irreducible over $\mathbb{Q}$, the candidate Néron-Severi group $M = L \cap E_\tau$ satisfies

$$M(-1) = L_1 \oplus_{\phi_{13}} L_3 \oplus_{\phi_{34}} L_4.$$

We claim $f_{13} = f_1 \oplus f_3$ is positive on the Lorentzian lattice

$$L_{13} = L_1 \oplus_{\phi_{13}} L_3.$$

Indeed, we have $x^2 \in 2a_i \mathbb{Z}$ for all $x \in L_i$, where $a_1 = 3$ and $a_3 = 2$; and $bL \subset L_1 \oplus L_3$ for $b = 2$, since we are gluing along $\mathbb{F}_2$. Since $b^2 = 4 \not\in \mathbb{Z} + a_1 + \mathbb{Z} + a_3$, positivity follows from Theorem 5.1.

Note that, since $f_4$ is the identity, it gives a positive automorphism of the Euclidean lattice $L_4$.

We now claim that the sum of positive automorphisms, $f_{134} = f_{13} \oplus f_4$, is positive on $L_{134} = L_{13} \oplus_{\phi_{34}} L_4 = M(-1)$. Here the gluing takes place over $\mathbb{F}_3$. In fact, the desired positivity follows from Theorem 5.2. For if $(x, y) \in L_{134}$ is a root but $(x, y) \not\in L_{13} \oplus L_4$, then $y \in L_4^\vee$ represents a nonzero element $[x_4] \in \mathbb{F}_3 \subset G(L_4)$, and hence (as we have seen above) $y^2 \geq 8/3 > 2$.

Thus $f|M(-1)$ is positive. By Theorem 6.2, there is a K3 surface automorphism $F : X \to X$ realizing $f : L \to L$; by construction, $h(F) = \log \rho(f) = \log \lambda_6$; and since $M \cong \text{NS}(X)$ has signature $(1, 9)$, $X$ is projective.
Remarks.

1. Since $P_6(x)$ is an unramified Salem polynomial, the fact that it can be realized by an isometry $f_1 \in O(11,3)$ also follows from general results [GM].

2. The existence of the twists producing $L_1$ and $L_2$ can, as in §7, be explained by the fact that $\mathbb{Z}[f_1 + f_1^{-1}]$ and $\mathbb{Z}[f_2 + f_2^{-1}]$ are Dedekind domains of class number one.

3. We note that $\text{NS}(X)$ has signature $(1,9)$ and determinant $3^6 \cdot 13 = 9477$, and $F^*|H^{2,0}(X)$ has order 13.

4. The Coxeter polynomial for $Eh_8$ (the ‘hyperbolic extension of $E_6$’, with diagram $Y_{3,3,4}$) is the same as the characteristic polynomial for $f|L_1 \oplus L_3$, namely $P_6(x)(1 + x + x^2)$ (see [Mc1, Table 5]). In fact the Coxeter automorphism of $Eh_8$ could have been taken as the starting point for the construction of $f$, just as the Coxeter automorphism of $E_{10}$ (the hyperbolic extension of $E_8$) was the starting point for the construction in §7.

References


