Entropy on Riemann Surfaces and the Jacobians of Finite Covers

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<thead>
<tr>
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</tr>
</thead>
<tbody>
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Entropy on Riemann surfaces and the Jacobians of finite covers

Curtis T. McMullen

20 June, 2010

Abstract

This paper characterizes those pseudo-Anosov mappings whose entropy can be detected homologically by taking a limit over finite covers. The proof is via complex-analytic methods. The same methods show the natural map \( \mathcal{M}_g \rightarrow \prod A_h \), which sends a Riemann surface to the Jacobians of all of its finite covers, is a contraction in most directions.

Contents

1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . 1
2 Odd order zeros . . . . . . . . . . . . . . . . . . . . . . . . 3
3 Siegel space . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
4 Teichmüller space . . . . . . . . . . . . . . . . . . . . . . . 6
5 Contraction . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
6 Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . 8
A The hyperbolic metric via Jacobians of finite covers . . . . . . 9

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1 Introduction

Let $f : S \to S$ be a pseudo-Anosov mapping on a surface of genus $g$ with $n$ punctures. It is well-known that the topological entropy $h(f)$ is bounded below in terms of the spectral radius of $f^* : H^1(S, \mathbb{C}) \to H^1(S, \mathbb{C})$; we have

$$\log \rho(f^*) \leq h(f).$$

If we lift $f$ to a map $\tilde{f} : \tilde{S} \to \tilde{S}$ on a finite cover of $S$, then its entropy stays the same but the spectral radius of the action on homology can increase. We say the entropy of $f$ can be detected homologically if

$$h(f) = \sup \log \rho(\tilde{f} : H^1(\tilde{S}) \to H^1(\tilde{S})),$$

where the supremum is taken over all finite covers to which $f$ lifts.

In this paper we will show:

**Theorem 1.1** The entropy of a pseudo-Anosov mapping $f$ can be detected homologically if and only if the invariant foliations of $f$ have no odd-order singularities in the interior of $S$.

The proof is via complex analysis. Hodge theory provides a natural embedding $M_g \to A_g$ from the moduli space of Riemann surfaces into the moduli space of Abelian varieties, sending $X$ to its Jacobian. Any characteristic covering map from a surface of genus $h$ to a surface of genus $g$, branched over $n$ points, provides a similar map

$$M_{g,n} \to M_h \to A_h.$$ (1.1)

It is known that the hyperbolic metric on a Riemann surface $X$ can be reconstructed using the metrics induced from the Jacobians of its finite covers ([Kaz]; see the Appendix). Similarly, it is natural to ask if the Teichmüller metric on $M_{g,n}$ can be recovered from the Kobayashi metric on $A_h$, by taking the limit over all characteristic covers $C_{g,n}$. We will show such a construction is impossible.

**Theorem 1.2** The natural map $M_{g,n} \to \prod C_{g,n} A_h$ is not an isometry for the Kobayashi metric, unless $\dim M_{g,n} = 1$.

It is an open problem to determine if the Kobayashi and Carathéodory metrics on moduli space coincide when $\dim M_{g,n} > 1$ (see e.g. [FM, Prob 5.1]). An equivalent problem is to determine if Teichmüller space embeds holomorphically and isometrically into a (possibly infinite) product of bounded
symmetric domains. Theorem 1.2 provides some support for a negative answer to this question.

Here is a more precise version of Theorem 1.2, stated in terms of the lifted map

\[ T_{g,n} \to T_h \to \mathcal{H}_h \]

from Teichmüller space to Siegel space determined by a finite cover.

**Theorem 1.3** Suppose the Teichmüller mapping between a pair of distinct points \( X, Y \in T_{g,n} \) comes from a quadratic differential with an odd order zero. Then

\[
\sup d(J(\tilde{X}), J(\tilde{Y})) < d(X, Y),
\]

where the supremum is taken over all compatible finite covers of \( X \) and \( Y \).

Conversely, if the Teichmüller map from \( X \) to \( Y \) has only even order singularities, then there is a double cover such that \( d(J(\tilde{X}), J(\tilde{Y})) = d(X, Y) \) (cf. [Kra]). In particular, the complex geodesics generated by squares of holomorphic 1-forms map isometrically into \( A_g \). The only directions contracted by the map \( \mathcal{M}_g \to \prod A_h \) are those identified by Theorem 1.3.

Theorem 1.1 follows from Theorem 1.3 by taking \( X \) and \( Y \) to be points on the Teichmüller geodesic stabilized by the mapping-class \( f \). It would be interesting to find a direct topological proof of Theorem 1.1.

As a sample application, let \( \beta \in B_n \) be a pseudo-Anosov braid whose monodromy map \( f : S \to S \) (on the \( n \)-times punctured plane) has an odd order singularity. Then Theorem 1.1 implies the image of \( \beta \) under the Burau representation satisfies

\[
\log \sup_{|q|=1} \rho(B(q)) < h(f).
\]

Indeed, \( \rho(B(q)) \) at any \( d \)-th root of unit is bounded by \( \rho(\tilde{f}^*) \) on a \( \mathbb{Z}/d \) cover \( S \) [Mc2]. This improves a result in [BB]. Similar statements hold for other homological representations of the mapping-class group.

**Notes and references.** For \( C^\infty \) diffeomorphisms of a compact smooth manifold, one has \( h(f) \geq \log \sup_i \rho(f^*|H^i(X)) \) [Ym], and equality holds for holomorphic maps on Kähler manifolds [Gr]. The lower bound \( h(f) \geq \log \rho(f^*|H^1(X)) \) also holds for homeomorphisms [Mn]. For more on pseudo-Anosov mappings, see e.g. [FLP], [Bers] and [Th].

A proof that the inclusion of \( T_{g,n} \) into universal Teichmüller space is a contraction, based on related ideas, appears in [Mc1].
2 Odd order zeros

We begin with an analytic result, which describes how well a monomial $z^k$ of odd order can be approximated by the square of an analytic function.

**Theorem 2.1** Let $k \geq 1$ be odd, and let $f(z)$ be a holomorphic function on the unit disk $\Delta$ such that $\int |f(z)|^2 = 1$. Then

$$\left| \int_{\Delta} f(z)^2 \left( \frac{\overline{z}}{|z|} \right)^k \right| \leq C_k = \frac{\sqrt{k+1}(k+3)}{k+2} < 1.$$

Here the integral is taken with respect to Lebesgue measure on the unit disk.

**Proof.** Consider the orthonormal basis $e_n(z) = a_n z^n$, $n \geq 0$, $a_n = \sqrt{n+1}/\sqrt{\pi}$, for the Bergman space $L^2_\alpha(\Delta)$ of analytic functions on the disk with $\|f\|^2 = \int |f(z)|^2 < \infty$. With respect to this basis, the nonzero entries in the matrix of the symmetric bilinear form $Z(f, g) = \int f(z)g(z)\overline{z}^k/|z|^k$ are given by

$$Z(e_n, e_{k-n}) = a_na_{k-n} \int_{\Delta} |z|^k = \frac{2\sqrt{n+1}k-n+1}{k+2}.$$

In particular, $Z(e_i, e_i) = 0$ for all $i$ (since $k$ is odd), and $Z(e_i, e_j) = 0$ for all $i, j > k$.

Note that the ratio above is less than one, by the inequality between the arithmetic and geometric means, and it is maximized when $n < k/2 < n + 1$. Thus the maximum of $|Z(f, f)|/\|f\|^2$ over $L^2_\alpha(\Delta)$ is achieved when $f = e_n + e_{n+1}$, $n = (k-1)/2$, at which point it is given by $C_k$. 

3 Siegel space

In this section we describe the Siegel space of Hodge structures on a surface $S$, and its Kobayashi metric.

**Hodge structures.** Let $S$ be a closed, smooth, oriented surface of genus $g$. Then $H^1(S) = H^1(S, \mathbb{C})$ carries a natural involution $C(\alpha) = \overline{\alpha}$ fixing $H^1(S, \mathbb{R})$, and a natural Hermitian form

$$\langle \alpha, \beta \rangle = \frac{\sqrt{-1}}{2} \int_S \alpha \wedge \overline{\beta}$$
of signature \((g, g)\). A *Hodge structure* on \(H^1(S)\) is given by an orthogonal splitting

\[ H^1(S) = V^{1,0} \oplus V^{0,1} \]

such that \(V^{1,0}\) is positive-definite and \(V^{0,1} = C(V^{1,0})\). We have a natural norm on \(V^{1,0}\) given by \(\|\alpha\|^2 = \langle \alpha, \alpha \rangle\).

The set of all possible Hodge structures forms the *Siegel space* \(\mathcal{H}(S)\). To describe this complex symmetric space in more detail, fix a splitting \(H^1(S) = W^{1,0} \oplus W^{0,1}\). Then for any other Hodge structure \(V^{1,0} \oplus V^{0,1}\), there is a unique operator

\[ Z : W^{1,0} \to W^{0,1} \]

such that \(V^{1,0} = (I + Z)(W^{1,0})\). This means \(V^{1,0}\) coincides with the graph of \(Z\) in \(W^{1,0} \oplus W^{0,1}\).

The operator \(Z\) is determined uniquely by the associated bilinear form

\[ Z(\alpha, \beta) = \langle \alpha, CZ(\beta) \rangle \]

on \(W^{1,0}\), and the condition that \(V^{1,0} \oplus V^{0,1}\) is a Hodge structure translates into the conditions:

\[ Z(\alpha, \beta) = Z(\beta, \alpha) \quad \text{and} \quad |Z(\alpha, \alpha)| < 1 \text{ if } \|\alpha\| = 1. \quad (3.1) \]

Since the second inequality above is an open condition, the tangent space at the base point \(p \sim W^{1,0} \oplus W^{0,1}\) is given by

\[ T_p\mathcal{H}(S) = \{\text{symmetric bilinear maps } Z : W^{1,0} \times W^{1,0} \to \mathbb{C}\}. \]

**Comparison maps.** Any Hodge structure on \(H^1(S)\) determines an isomorphism

\[ V^{1,0} \cong H^1(S, \mathbb{R}) \quad (3.2) \]

sending \(\alpha\) to \(\text{Re}(\alpha) = (\alpha + C(\alpha))/2\). Thus \(H^1(S, \mathbb{R})\) inherits a norm and a complex structure from \(V^{1,0}\).

Put differently, \((3.2)\) gives a *marking* of \(V^{1,0}\) by \(H^1(S, \mathbb{R})\). By composing one marking with the inverse of another, we obtain the real-linear *comparison map*

\[ T = (I + Z)(I + CZ)^{-1} : W^{1,0} \to V^{1,0} \quad (3.3) \]

between any pair of Hodge structures. It is characterized by \(\text{Re}(\alpha) = \text{Re}(T(\alpha))\).
Symmetric matrices. The classical Siegel domain is given by
\[ H_g = \{ \mathbf{Z} \in M_g(\mathbb{C}) : Z_{ij} = Z_{ji} \text{ and } I - \mathbf{Z}\overline{\mathbf{Z}} \gg 0 \} \].
(cf. [Sat, Ch. II.7]). It is a convex, bounded symmetric domain in \( \mathbb{C}^N \), 
\( N = g(g + 1)/2 \). The choice of an orthonormal basis for \( W^{1,0} \) gives an isomorphism \( Z \mapsto Z(\omega_i, \omega_j) \) between \( \mathfrak{H}(S) \) and \( \mathfrak{H}_g \), sending the basepoint \( p \) to zero.

The Kobayashi metric. Let \( \Delta \subset \mathbb{C} \) denote the unit disk, equipped with the metric \( |dz|/(1-|z|^2) \) of constant curvature \(-4\). The Kobayashi metric on \( \mathfrak{H}(S) \) is the largest metric such that every holomorphic map \( f : \Delta \to \mathfrak{H}(S) \) satisfies \( \|Df(0)\| \leq 1 \). It determines both a norm on the tangent bundle and a distance function on pairs of points [Ko].

**Proposition 3.1** The Kobayashi norm on \( T_p \mathfrak{H}(S) \) is given by
\[ \|Z\|_K = \sup \{Z(\alpha, \alpha) : \|\alpha\| = 1\}, \]
and the Kobayashi distance is given in terms of the comparison map (3.3) by
\[ d(V^{1,0}, W^{1,0}) = \log \|T\|. \]

**Proof.** Choosing a suitable orthonormal basis for \( W^{1,0} \), we can assume that
\[ Z(\omega_i, \omega_j) = \lambda_i \delta_{ij} \]
with \( \lambda_1 \geq \lambda_2 \geq \cdots \lambda_g \geq 0 \). Since \( \mathfrak{H}_g \) is a convex symmetric domain, the Kobayashi norm at the origin and the Kobayashi distance satisfy
\[ \|Z\|_K = r \quad \text{and} \quad d(0, Z) = \frac{1}{2} \log \frac{1 + r}{1 - r}, \]
where \( r = \inf \{s > 0 : Z \in s \mathfrak{H}_g \} \) (see [Ku]). But clearly \( r = \lambda_1 = \sup \|Z(\alpha, \alpha)\|/\|\alpha\|^2 \), and by (3.3), we have
\[ \|T\|^2 = \|T(\sqrt{-1} \omega_1)\|^2 = \left\| \frac{\omega_1}{1 - \lambda_1} + \frac{\lambda_1 \overline{\omega_1}}{1 - \lambda_1} \right\|^2 = \frac{1 + \lambda_1}{1 - \lambda_1}, \]
which gives the expressions above. \( \blacksquare \)
4 Teichmüller space

This section gives a functorial description of the derivative of the map from Teichmüller space to Siegel space.

Markings. Let $\overline{S}$ be a compact oriented surface of genus $g$, and let $S \subset \overline{S}$ be a subsurface obtained by removing $n$ points.

Let $T\text{eich}(S) \cong \mathcal{T}_{g,n}$ denote the Teichmüller space of Riemann surfaces marked by $S$. A point in $T\text{eich}(S)$ is specified by a homeomorphism $f: S \to X$ to a Riemann surface of finite type. This means there is a compact Riemann surface $\overline{X} \supset X$ and an extension of $f$ to a homeomorphism $\overline{f}: \overline{S} \to \overline{X}$.

Metrics. Let $Q(X)$ denote the space of holomorphic quadratic differentials on $X$ such that $\|q\|_X = \int_X |q| < \infty$.

There is a natural pairing $(q, \mu) \mapsto \int_X q\mu$ between the space $Q(X)$ and the space $M(X)$ of $L^\infty$-measurable Beltrami differentials $\mu$. The tangent and cotangent spaces to Teichmüller space at $X$ are isomorphic to $M(X)/Q(X)^\perp$ and $Q(X)$ respectively.

The Teichmüller and Kobayashi metrics on $T\text{eich}(S)$ coincide [Roy1], [Hub, Ch. 6]. They are given by the norm $\|\mu\|_T = \sup \left\{ \left| \int_X q\mu \right| : \|q\|_X = 1 \right\}$ on the tangent space at $X$; the corresponding distance function

$$d(X, Y) = \inf \frac{1}{2} \log K(\phi)$$

measures the minimal dilatation $K(\phi)$ of a quasiconformal map $\phi: X \to Y$ respecting their markings.

Hodge structure. The periods of holomorphic 1-forms on $X$ serve as classical moduli for $X$. From a modern perspective, these periods give a map $J: T\text{eich}(S) \to \mathcal{H}(\overline{S}) \cong \mathcal{H}_g$, sending $X$ to the Hodge structure

$$H^1(\overline{S}) \cong H^1(\overline{X}) \cong H^{1,0}(\overline{X}) \oplus H^{0,1}(\overline{X}).$$

Here the first isomorphism is provided by the marking $\overline{f}: \overline{S} \to \overline{X}$. We also have a natural isomorphism between $H^{1,0}(\overline{X})$ and the space of holomorphic
1-forms $\Omega(X)$. The image $J(X)$ encodes the complex analytic structure of the Jacobian variety $\text{Jac}(X) = \Omega(X)^*/H_1(X,\mathbb{Z})$. (It does not depend on the location of the punctures of $X$.)

**Proposition 4.1** The derivative of the period map sends $\mu \in M(X)$ to the quadratic form $Z = DJ(\mu)$ on $\Omega(X)$ given by

$$Z(\alpha, \beta) = \int_X \alpha \beta \mu.$$ 

This is a basis-free reformulation of Ahlfors' variational formula [Ah, §5]; see also [Ra], [Roy2] and [Kra, Prop. 1]. Note that $\alpha \beta \in Q(X)$.

## 5 Contraction

This section brings finite covers into play, and establishes a uniform estimate for contraction of the mapping $T_{g,n} \to T_h \to \mathcal{F}_h$.

**Jacobian of finite covers.** A finite connected covering space $S_1 \to S_0$ determines a natural map

$$P : \text{Teich}(S_0) \to \text{Teich}(S_1)$$

sending each Riemann surface to the corresponding covering space $X_1 \to X_0$.

By taking the Jacobian of $X_1$, we obtain a map $J \circ P : \text{Teich}(S_0) \to \mathcal{F}(S_1)$. Let $q_0 \in Q(X_0)$ be a holomorphic quadratic differential with a zero of odd order $k$, say at $p \in X_0$. Let $\mu = q_0/|q_0| \in M(X_0)$, then $\|\mu\|_T = 1$. Let $\pi : X_1 \to X_0$ denote the natural covering map, and let $q_1 = \pi^*(q_0)$.

We will show that $J(X_1)$ cannot change too rapidly under the unit deformation $\mu$ of $X_0$. Indeed, if $J(X_1)$ were to move at nearly unit speed, then $\pi^*(\mu) = q_1/|q_1|$ would pair efficiently with $\alpha^2$ for some unit-norm $\alpha \in \Omega(X_1)$, which is impossible because of the many odd-order zeros of $q_1$.

To make a quantitative estimate, choose a holomorphic chart $\phi : (\Delta, 0) \to (X_0, p)$ such that $\phi^*(\mu) = z^k/|z|^k \frac{d\phi}{dz}$. Let $U = \phi(\Delta)$, and let

$$m(U) = \inf\{\|q\|_U : q \in Q(X_0), \|q\|_X = 1\}.$$ 

(Here $\|q\|_U = \int_U |q|$.) Since $Q(X_0)$ is finite-dimensional, we have $m(U) > 0$.

**Theorem 5.1** The image $Z$ of the vector $[\mu]$ under the derivative of $J \circ P$ satisfies

$$\|Z\|_K \leq \delta < 1 = \|\mu\|_T,$$

where $\delta = \max(1/2, 1-(1-C_k)m(U)/2)$ does not depend on the finite cover $S_1 \to S_0$. 


Proof. The derivative of $P$ sends $\mu$ to $\pi^*(\mu)$. By Proposition 3.1, to show $\|Z\|_K \leq \delta$ it suffices to show that

$$|Z(\alpha, \alpha)| = \left| \int_{X_1} \alpha^2 \pi^* \mu \right| \leq \delta$$

for all $\alpha \in \Omega(X_1)$ with $\|\alpha^2\|_{X_1} = 1$. Setting $q = \pi_*(\alpha^2)$, we also have

$$|Z(\alpha, \alpha)| = \left| \int_{X_0} q \mu \right| \leq \|q\|_{X_0},$$

so the proof is complete if $\|q\|_{X_0} \leq 1/2$. Thus we may assume that

$$\|\alpha^2\|_V \geq \|q\|_V \geq m(U)\|q\|_{X_0} \geq m(U)/2,$$

where $V = \pi^{-1}(U) = \bigcup_i V_i$ is a finite union of disjoint disks. Using the coordinate charts $V_i \cong U \cong \Delta$ and Theorem 2.1, we find that on each of these disks we have

$$\left| \int_{V_i} \alpha^2 \pi^*(\mu) \right| = \left| \int_{\Delta} \alpha(z)^2 \left( \frac{z}{|z|} \right)^k \right| \leq C_k\|\alpha^2\|_{V_i}.$$

Summing these bounds and using the fact that $\|\alpha^2\|_{(X_1-V)} + \|\alpha^2\|_V = 1$, we obtain

$$\left| \int_{X_1} \alpha^2 \pi^*(\mu) \right| \leq \|\alpha^2\|_{(X_1-V)} + C_k\|\alpha^2\|_V \leq 1 - \frac{(1-C_k)m(U)}{2} \leq \delta.$$

6 Conclusion

It is now straightforward to establish the results stated in the Introduction.

Proof of Theorems 1.3. Assume the Beltrami coefficient of the Teichmüller mapping between $X, Y \in T_{g,n}$ has the form $\mu = k\overline{q}/q$, where $q \in Q(X)$ has an odd order zero. Then the same is true for the tangent vectors to the Teichmüller geodesic $\gamma$ joining $X$ to $Y$. Theorem 5.1 then implies that $D(J \circ P)|_{\gamma}$ is contracting by a factor $\delta < 1$ independent of $P$, and therefore

$$d(J \circ P(X), J \circ P(Y)) = d(J(\tilde{X}), J(\tilde{Y})) < \delta \cdot d(X, Y).$$
Proof of Theorem 1.2. The contraction of \( M_{g,n} \to \prod_{Cg,n} A_h \) in some directions is immediate from the uniformity of the bound in Theorem 1.3, using the fact that the Kobayashi metric on a product is the sup of the Kobayashi metrics on each term, and that there exist \( q \in Q(X) \) with simple zeros whenever \( X \in M_{g,n} \) and \( \dim M_{g,n} > 1 \).

Proof of Theorem 1.1. Let \( f : S_0 \to S_0 \) be a pseudo-Anosov mapping. If \( f \) has only even order singularities, then its expanding foliation is locally orientable, and hence there is a double cover \( \tilde{S} \to S \) such that \( \log \rho(\tilde{f}^*) = h(f) \).

Now suppose \( f \) has an odd-order singularity. Let \( X_0 \in \text{Teich}(S_0) \) be a point on the Teichmüller geodesic stabilized by the action of \( f \) on \( \text{Teich}(S_0) \). Then \( d(f \cdot X_0, X_0) = h(f) > 0 \) (see e.g. [FLP] and [Bers]).

Let \( f : S_1 \to S_0 \) be a lift of \( f \) to a finite covering of \( S_0 \), and let \( X_1 = P(X_0) \in \text{Teich}(S_1) \). Using the marking of \( X_1 \) and the isomorphism \( H^1(X_1, \mathbb{R}) \cong H^{1,0}(X_1) \), we obtain a commutative diagram

\[
\begin{array}{ccc}
H^1(S_1, \mathbb{R}) & \overset{\tilde{f}^*}{\longrightarrow} & H^1(S_1, \mathbb{R}) \\
\downarrow & & \downarrow \\
H^{1,0}(X_1) & \overset{T}{\longrightarrow} & H^{1,0}(X_1)
\end{array}
\]

where \( T \) is the comparison map between \( J(X_1) \) and \( J(\tilde{f} \cdot X_1) \) (see equation (3.3)). Then Theorem 1.3 and Proposition 3.1 yield the bound

\[
\log \rho(\tilde{f}^*) \leq \log \|T\| = d(J(X_1), \tilde{f} \cdot J(X_1)) \leq \delta d(X_0, f \cdot X_0) = \delta h(f),
\]

where \( \delta < 1 \) does not depend on the finite covering \( S_1 \to S_0 \). Consequently, \( \sup \log \rho(\tilde{f}^*) < h(f) \). \( \blacksquare \)

A The hyperbolic metric via Jacobians of finite covers

Let \( X = \Delta/\Gamma \) be a compact Riemann surface, presented as a quotient of the unit disk by a Fuchsian group \( \Gamma \). Let \( Y_n \to X \) be an ascending sequence of finite Galois covers which converge to the universal cover, in the sense that

\[
Y_n = \Delta/\Gamma_n, \quad \Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \cdots, \quad \text{and} \quad \bigcap \Gamma_i = \{e\}. \quad (A.1)
\]
The Bergman metric on $Y_n$ (defined below) is invariant under automorphisms, so it descends to a metric $\beta_n$ on $X$. This appendix gives a short proof of:

**Theorem A.1 (Kazhdan)** The Bergman metrics inherited from the finite Galois covers $Y_n \to X$ converge to a multiple of the hyperbolic metric; more precisely, we have

$$\beta_n \to \frac{\lambda_X}{2\sqrt{\pi}}$$

uniformly on $X$.

The argument below is based on [Kaz, §3]; for another, somewhat more technical approach, see [Rh].

**Metrics.** We begin with some definitions. Let $\Omega(X)$ denote the Hilbert space of holomorphic 1-forms on a Riemann surface $X$ such that

$$\|\omega\|^2_X = \int_X |\omega|^2 < \infty.$$  

The area form of the Bergman metric on $X$ is given by

$$\beta_X^2 = \sum |\omega_i|^2,$$  

(A.2)

where $(\omega_i)$ is any orthonormal basis of $\Omega(X)$. Equivalently, the Bergman length of a tangent vector $v \in T_X$ is given by

$$\langle \beta_X, v \rangle = \sup_{\omega \neq 0} \frac{\|\omega(v)\|}{\|\omega\|_X}.$$  

(A.3)

This formula shows that inclusions are contracting: if $Y$ is a subdomain of $X$, then $\beta_Y \geq \beta_X$.

Now suppose $X$ is a compact surface of genus $g > 0$. Then (A.2) shows its Bergman area is given by

$$\int_X \beta_X^2 = \dim \Omega(X) = g.$$  

(A.4)

In this case $\beta_X$ is also the pullback, via the Abel–Jacobi map, of the natural Kähler metric on the Jacobian of $X$.

Finally suppose $X = \Delta/\Gamma$. Then the hyperbolic metric of constant curvature $-1$,

$$\lambda_\Delta = \frac{2|dz|}{1 - |z|^2},$$

10
descends to give the hyperbolic metric $\lambda_X$ on $X$. Using the fact that $\|dz\|_\Delta = \pi$, it is easy to check that $4\pi\beta_\Delta^2 = \lambda_X^2$.

**Proof of Theorem A.1.** We will regard the Bergman metric $\beta_n$ on $Y_n$ as a $\Gamma_n$-invariant metric on $\Delta$. It suffices to show that $\beta_n/\beta_\Delta \to 1$ uniformly on $\Delta$.

Let $g$ and $g_n$ denote the genus of $X$ and $Y_n$ respectively, and let $d_n$ denote the degree of $Y_n/X$; then $g_n - 1 = d_n(g - 1)$. By (A.1), the injectivity radius of $Y_n$ tends to infinity. In particular, there is a sequence $r_n \to 1$ such that $\gamma(r_n \Delta)$ injects into $Y_n$ for any $\gamma \in \Gamma$. Since inclusions are contracting, this shows

$$\beta_n \leq (1 + \epsilon_n)\beta_\Delta \quad (A.5)$$

where $\epsilon_n \to 0$.

Next, note that both $\beta_n$ and $\beta_\Delta$ are $\Gamma$-invariant, so they determine metrics on $X$. By (A.4), we have

$$\int_X \beta_n^2 = \frac{1}{d_n} \int_{Y_n} \beta_n^2 = \frac{g_n}{d_n} \to (g - 1) = \int_X \beta_\Delta^2$$

(since $\int_X \lambda_X^2 = 2\pi(2g - 2)$ by Gauss-Bonnet). Together with (A.5), this implies

$$\int_X |\beta_n - \beta_\Delta|^2 \to 0. \quad (A.6)$$

To show $\beta_n \to \beta_\Delta$ uniformly, consider any sequence $p_n \in \Delta$ and let $x \in [0,1]$ be a limit point of $(\beta_n/\beta_\Delta)(p_n)$. It suffices to show $x = 1$.

Passing to a subsequence and using compactness of $X$, we can assume that $p_n \to p \in \Delta$ and that $\beta_n(p_n) \to x\beta_\Delta(p)$. By changing coordinates on $\Delta$, we can also assume $p = 0$. By (A.6) we can find $q_n \to 0$ such that $\beta_n(q_n) \to \beta_\Delta(0)$. Then by (A.3), there exist $\Gamma_n$-invariant holomorphic 1-forms $\omega_n(z) \, dz$ on $\Delta$ such that $\int_{Y_n} |\omega_n|^2 = 1$ and

$$|\omega_n(q_n)| = \beta_n(q_n) \to \beta_\Delta(0) = \frac{|dz|}{\pi}.$$

Since $\omega_n$ is holomorphic and $\int_{r_n \Delta} |\omega_n|^2 < 1$, the equation above easily implies that $|\omega_n| \to |dz|/\pi$ uniformly on compact subsets of $\Delta$. But we also have

$$\beta_n(p_n) \geq |\omega_n(p_n)| \to \beta_\Delta(0),$$

and thus $\beta_n(p_n) \to \beta_\Delta(0)$ and hence $x = 1.$
References


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