Dynamics on the unit disk: 
Short geodesics and simple cycles

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1 Introduction

In this paper we show that rotation cycles on $S^1$ for a proper holomorphic map $f : \Delta \to \Delta$ share several of the analytic, geometric and topological features of simple closed geodesics on a compact hyperbolic surface.

**Dynamics on the unit disk.** Let $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$. For $d > 1$ let $B_d \cong \Delta^{(d-1)}$ denote the space of all proper holomorphic maps $f : \Delta \to \Delta$ of the form

$$f(z) = z \prod_{1}^{d-1} \left( \frac{z - a_i}{1 - \overline{a_i} z} \right),$$

$|a_i| < 1$. Every degree $d$ holomorphic map $g : \Delta \to \Delta$ with a fixed point in the disk can be put into the form above, by normalizing so its fixed point is $z = 0$.

The maps $f \in B_d$ have the property that $f|S^1$ is measure-preserving and $|f'| > 1$ on the circle. Moreover, there is a unique *marking* homeomorphism $\phi_f : S^1 \to S^1$ that varies continuously with $f$, conjugates $f$ to $p_d(z) = z^d$,

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and satisfies $\phi_f(z) = z$ when $f = p_d$. We define the length on $f$ of a periodic cycle $C$ for $p_d$ by
\[ L(C, f) = \log |(f^q)'(z)|, \] (1.1)
where $q = |C|$ and $\phi_f(z) \in C$.

The degree of a cycle $C$ is the least $e > 0$ such that $p_d|C$ extends to a covering map of the circle of degree $e$. We say $C$ is simple if $\deg(p_d|C) = 1$; equivalently, if $p_d|C$ preserves its cyclic ordering. A finite collection of cycles $C_i$ is binding if $\deg(\bigcup C_i) = d$ and if $\bigcup C_i$ is not renormalizable (§7).

In this paper we establish four main results.

**Theorem 1.1** Any cycle with $L(C, f) < \log 2$ is simple. All such cycles $C_i$ have the same rotation number, and $p_d|\bigcup C_i$ preserves the cyclic ordering of $\bigcup C_i$.

**Theorem 1.2** Every $f \in B_d$ has a simple cycle $C$ with $L(C, f) = O(d)$.

**Theorem 1.3** Let $(C_i)_1^n$ be a binding collection of cycles. Then for any $M > 0$, the set of $f \in B_d$ with $\sum_1^n L(C_i, f) \leq M$ has compact closure in the moduli space of all rational maps of degree $d$.

**Theorem 1.4** The closure $E \subset S^1$ of the simple cycles for a given $f \in B_d$ has Hausdorff dimension zero.

See Theorems 4.1, 5.8, 7.1 and 2.2 below.

**Hyperbolic surfaces.** The results above echo the following fundamental facts about compact hyperbolic surfaces $X$ of genus $g > 1$:

1. The closed geodesics on $X$ of length less than $\log(3 + 2\sqrt{2})$ are simple and disjoint.

2. There exists a simple closed geodesic on $X$ with length $O(\log g)$.

3. If $(\gamma_i)_1^n$ is a binding collection of closed curves, then the locus in Teichmüller space $T_g$ where $\sum L(\gamma_i, X) \leq M$ is compact for any $M > 0$.

4. The union of the simple geodesics on $X = \Delta/\Gamma$ is a closed set of Hausdorff dimension one.

See [Bus, §4, §5], [Ker, Lemma 3.1] and [BS] for proofs. Thus simple cycles behave in many ways like simple closed geodesics.

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1 A collection of closed curves is **binding** if their geodesic representatives cut $X$ into disks.
Figure 1. Tiling of $\Delta^*$ according to the slope of the shortest loop on the torus $\mathbb{C}^*/\alpha\mathbb{Z}$.

**Rotation numbers and slopes.** Next we formulate a more direct connection between short cycles and short geodesics. Suppose $f \in \mathcal{B}_d$ satisfies $\alpha = f'(0) = \exp(2\pi i \tau) \neq 0$. The action of $\langle f \rangle$ on $\Delta$ (with the orbit of $z = 0$ removed) determines a natural *quotient torus*, isomorphic to $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \tau) \cong \mathbb{C}^*/\alpha \mathbb{Z}$.

Let $L(p/q, X_\tau)$ denote the length of a closed geodesic on $X_\tau$ in the homotopy class $(-p, q)$, for the flat metric of area one. The slope $p/q \mod 1$ which minimizes $L(p/q, X_\tau)$ depends only on $f'(0) \in \Delta^*$. The regions $T(p/q) \subset \Delta^*$ where a given slope is shortest rest on the corresponding roots of unity, and form a tiling of $\Delta^*$ (see Figure 1).

In §6 we will show:

**Theorem 1.5** For any $f \in \mathcal{B}_d$ with $f'(0) \in T(p/q)$, there is a nonempty collection of compatible simple cycles $C_i$ with rotation number $p/q$ such that

$$\frac{1}{L(p/q, X_\tau)^2} \leq \sum \frac{\pi}{L(C_i, f)} \leq \frac{1}{L(p/q, X_\tau)^2} + O(d),$$

and all other cycles satisfy $L(C, f) > \epsilon_d > 0$.

(Compatibility is defined in §2.) This result implies Theorem 1.2 and gives an alternate proof of Theorem 1.1 (with log 2 replaced by $\epsilon_d$); it also yields:
Corollary 1.6 If a sequence \( f_n \in B_d \) satisfies \( L(C, f_n) \to 0 \), then \( f'_n(0) \to \exp(2\pi ip/q) \) where \( p/q \) is the rotation number of \( C \).

On the other hand, we will see in §3:

Proposition 1.7 If \( f_n \in B_d \) and \( f'_n(0) \to \exp(2\pi i\theta) \) where \( \theta \) is irrational, then \( L(C, f_n) \to \infty \) for every cycle \( C \).

Thus the cycles of moderate length guaranteed by Theorem 1.2 may be forced to have very large periods.

Petals. The proof of Theorem 1.5 is illustrated in Figure 2. Consider a map \( f \in B_2 \) with \( f'(0) = \exp(2\pi i\tau) \in T(1/3) \), \( \tau = 1/3 + i/10 \). The dark petals shown in the figure form the preimage \( \tilde{A} \subset \Delta \) of an annulus \( A \) in the homotopy class \( [3\tau - 1] \) on the quotient torus for the attracting fixed point at \( z = 0 \). Any two adjacent rectangles within a petal give a fundamental domain for the action of \( f \). The three largest petals join \( z = 0 \) to the repelling cycle on \( S^1 \) labeled by \( C = (1/7, 2/7, 4/7) \). Thus a copy of \( A \) embeds in the quotient for torus the repelling cycle as well; by the method of extremal length (§5), this gives an upper bound for \( L(C, f) \) in terms of \( L(1/3, X_\tau) \). (The lower bound comes from the holomorphic Lefschetz fixed-point theorem.)

Rational maps. Here is a related result from §5 for general rational maps \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \). Let \( L(f) = \inf \log |\beta| \), where \( \beta \) ranges over the multipliers of all repelling and indifferent periodic cycles for \( f \).
Theorem 1.8 If \( f_n \in \text{Rat}_d \) and \( L(f_n) \to \infty \), then the maps \( f_n \) have fixed points \( z_n \) with \( f'(z_n) \to 0 \).

Questions. We conclude with some natural questions suggested by the analogy with hyperbolic surfaces.

1. Let \( C \) be a simple cycle. Is the function \( L(C, f) \) free of critical points in \( B_d \)?

2. Let \((C_i)\) be a binding collection of cycles. Does \( \sum L(C_i, f) \) achieve its minimum at a unique point \( f \in B_d \)?

3. Let \( QB_d \) denote the space rational maps of the form
   \[
   f(z) = z \prod_{i=1}^{d-1} \left( \frac{z - a_i}{1 - b_i z} \right)
   \]
such that \( \prod |a_i| < 1 \), \( \prod |b_i| < 1 \), and \( J(f) \) is a Jordan curve. Each \( f \in QB_d \) can be regarded as a marked quasiblaschke product, obtained by gluing together a pair of maps \( f_1, f_2 \in B_d \) using their markings on \( S^1 \).

   Does there exist an \( \epsilon_d > 0 \) such for all \( f \in QB_d \), all cycles of length shorter than \( \epsilon_d \) are simple?

4. Suppose the cycles \((C_1, C_2)\) are binding. Does the set of \( f \in QB_d \) with \( L(C_1, f_1) + L(C_2, f_2) \leq M \) have compact closure in the moduli space of all rational maps of degree \( d \)?

The analogous questions for hyperbolic surfaces and quasifuchsian groups are known to have positive answers [Ker, §3], [Ot], [Th, Thm 4.4].

Notes and references. This paper is a sequel to [Mc4] and [Mc5], which construct a Weil-Petersson metric on \( B_d \) and an embedding of \( B_d \) into the space of invariant measures for \( p_d(z) = z^d \).

Simple cycles in degree two play a central role in the combinatorics of the Mandelbrot set [DH], [Ke], and are studied for higher degree in [Gol] and [GM]. Extremal length arguments similar to those we use in §5 are well-known both in the theory of Kleinian groups [Bers, Thm. 3], [Th, Proposition 1.3], [Mc1, §6.3], [Pet1], [Mil2] and rational maps [Pom], [Lev], [Hub], [Pet2]. The quotient Riemann surface of a general rational map is discussed in [McS]; other aspects of the dictionary between rational maps and Kleinian groups are presented in [Mc2]. See [PL] for a related discussion of spinning degenerations of the quotient torus.
2 Simple cycles

In this section we discuss the combinatorics of periodic cycles for the map $p_d(t) = d \cdot t \mod 1$, and prove the closure of the simple cycles has Hausdorff dimension zero.

Degree and rotation number. Let $S^1 = \mathbb{R}/\mathbb{Z}$. Given $a \neq b \in S^1$, let $[a,b] \subset S^1$ denote the unique subinterval that is positively oriented from $a$ to $b$. We write $a < c < b$ if $c \in [a,b]$. The length of an interval is denoted $|I|$.

Let $f : S^1 \to S^1$ be a topological covering map of degree $d > 0$, and suppose $f(X) = X$. The degree of $f|X$, denoted $\deg(f|X)$, is the least $e > 0$ such that $f|X$ extends to a topological covering $g : S^1 \to S^1$ of degree $e$.

Note that $\deg(f|X) = 1$ iff $f$ preserves the cyclic ordering of $X$, in which case $f|X$ also has a well-defined rotation number $\rho(f|X) \in S^1$. If $X$ is finite then $\rho(f|X) = p/q$ is rational and the orbits of $f|X$ have size $q$.

Example: Suppose $X = \{x_0, x_1, \ldots, x_n = x_0\}$ in increasing cyclic order, and $f|X$ is a permutation; then we have

$$\deg(f|X) = \sum_{i=0}^{n-1} |[f(x_i), f(x_{i+1})]|.$$

Indeed, an extension of $f|X$ of minimal degree is obtain by mapping $[x_i, x_{i+1}]$ homeomorphically to $[f(x_i), f(x_{i+1})]$. The degree is thus a variant of the number of descents of a permutation (see e.g. [St, §1.3]).

The model map and its modular group. Now fix $d > 1$, and let $p_d(t) = d \cdot t \mod 1$. Any expanding map $f : S^1 \to S^1$ of degree $d$ is topologically conjugate to $p_d$ [Sh].

The modular group $\text{Mod}_d \subset \text{Aut}(S^1)$ is the cyclic group of rotations generated by $t \mapsto 1/(d-1) + t \mod 1$; it coincides with the group of (degree one) topological automorphisms of $p_d$. Note that $\text{Mod}_d$ acts transitively on the fixed points of $p_d$.

Simple cycles. A finite set $C \subset S^1$ is a cycle of degree $d$ if $p_d|C$ is a transitive permutation. As in §1, we say a cycle is simple if $\deg(p_d|C) = 1$. Simple cycles $(C_1, \ldots, C_m)$ are compatible if $\deg(p_d|\bigcup C_i) = 1$.

It is elementary to see:

**Proposition 2.1** The simple cycles $(C_1, \ldots, C_m)$ are compatible iff they are pairwise compatible.
We let $C_d$ denote the set of all cycles of degree $d$, and $C_d(p/q) \subset C_d$ the simple cycles with rotation number $p/q$.

**Portraits of fixed points.** The fixed-point portrait [Gol] of a simple cycle $C \in C_d(p/q)$ is the monotone increasing function

$$\sigma : \{1, \ldots, d-2\} \to \{0, 1, \ldots, q\}$$

given by

$$\sigma(j) = |C \cap [0, j/(d-1))|.$$  

This invariant specifies how $C$ is interleaved between the fixed points of $p_d$, which are all of the form $j/(d - 1) \mod 1$.

**Basic properties.** The following results are immediate from [Gol] (see especially Lemma 2 and Theorem 7).

1. A simple cycle $C \in C_d(p/q)$ is uniquely determined by its fixed-point portrait $\sigma(j)$, and all possible monotone increasing functions $\sigma(j)$ arise.

2. The number of simple cycles of degree $d$ and rotation number $p/q$ is $\binom{d+q-2}{q}$.

3. The number of cycles of period $q$ grows like $d^q$, while the number of simple cycles is $O(q^{d-1})$; so most cycles are not simple.

4. Cycles $C_1, C_2 \in C_d(p/q)$ are compatible iff their fixed-point portraits satisfy

$$\sigma_1(j) \leq \sigma_2(j) \leq \sigma_1(j) + 1$$

for $0 \leq j \leq d-2$, or the same with $\sigma_1$ and $\sigma_2$ reversed.

5. Every maximal collection of compatible cycles has cardinality $d - 1$.

**From portraits to cycles.** A simple cycle $C \in C_d$ can be reconstructed explicitly from its rotation number $p/q$ and its fixed-point portrait $\sigma$ as follows. Let $\tau$ be the ‘transpose’ of $\sigma$, namely the monotone function $\tau : \{0, 1, \ldots, q-1\} \to \{0, 1, \ldots, d-1\}$ given by

$$\tau(i) = |\{j : \sigma(j) \leq i\}|,$$

and let

$$\tau'(i) = \tau(i) + \begin{cases} 
0 & \text{if } 0 \leq i < q - p, \text{ and} \\
1 & \text{otherwise},
\end{cases}$$

\[7\]
where \( i \) is taken mod \( q \). Then the periodic point given by \( t = 0.\tau'(0)\tau'(p)\tau'(2p)\ldots \) in base \( d \) generates \( C \); indeed, \( t \) is the ‘first point’ in the cycle \( C \).

**Examples.** To simplify notation, let \((p_1/q, \ldots, p_m/q) = (p_1, \ldots, p_m)/q\), and let \( \sigma = n_1 \ldots n_{d-1} \) denote the function with values \( \sigma(j) = n_j \).

**Degree \( d = 2 \).** In the quadratic case, \( \sigma \) is trivial and hence there is a unique simple cycle \( C(p/q) \) for each possible rotation number; e.g.

\[
\begin{align*}
C(1/2) &= (1,2)/3, \\
C(1/3) &= (1,2,4)/7, \\
C(2/5) &= (5,10,20,9,18)/31.
\end{align*}
\]

The only cycle of period \( \leq 4 \) which is not simple is \( C = (1,2,4,3)/5 \). For period 5 there are two such, namely \( C \) and \(-C\) where \( C = (3,6,12,24,17)/31 \). Any two distinct quadratic simple cycles are incompatible.

**Degree \( d = 3 \).** In the cubic case \( p_d \) has two fixed points, 0 and 1/2, and three cycles of period two, given by

\[
\begin{align*}
C(1/2,0) &= (5,7)/8, \\
C(1/2,1) &= (1,3)/4 \quad \text{and} \\
C(1/2,2) &= (1,3)/8.
\end{align*}
\]

The first and last are incompatible, while the other pairs are compatible. In general there are \( q+1 \) cubic simple cycles with rotation number \( p/q \), whose fixed-point portraits are given by \( \sigma(1) = 0,1,\ldots,q \). Only the pairs with adjacent values of \( \sigma(1) \) are compatible.

![Figure 3. Compatibility of degree 4 cycles of the form C(1/2, \sigma).](image)

**Degree \( d = 4 \).** In the quartic case there are six cycles in \( C_4(1/2) \), generated by \( t = p/15 \) with \( p = 1,2,3,6,7 \) and 11. The compatibility relation between these cycles is shown in Figure 3. The 4 visible triangles give the 4 distinct triples of compatible simple cycles with rotation number 1/2. Note that the modular group \( \text{Mod}_4 \cong \mathbb{Z}/3 \) acts by rotations on this diagram.
In general $C_d(p/q)$ can be identified with the vertices of the $q$-fold barycentric subdivision of a $(d - 2)$-simplex, with the top-dimensional cells corresponding to maximal collections of compatible cycles.

![Diagram](image.png)

Figure 4. The degree 5 simple cycle with rotation number $3/7$ and $\sigma = 013$.

**Sample computation in degree $d = 5$.** To compute $C(3/7, 013)$, we first use equation (2.1) to compute the ‘transpose’ $\tau = 1223333$ of $\sigma = 013$. Note that the graphs of $\sigma$ and $\tau$, shown in white and black in Figure 4, fit together to form a rectangle. Evaluating $\tau' = 1223444$ along the sequence $ip \mod q$, $i = 0, 1, 2, \ldots$ we obtain the base 5 expansion $t = 0.1342424_5 = 6966/19531$ for a generator of $C$.

The cycle $C$, along with the 4 fixed points of $p_5$, is drawn at the right in Figure 4. Note that $\sigma = 013$ gives the running total of the number of points of $C$ in the first three quadrants.

**Comparison with simple geodesics.** The simple cycles for $p_d|S^1$ behave in many ways like simple closed geodesics on a compact hyperbolic surface $X = \Delta/\Gamma$ of genus $g$, with compatible cycles corresponding to disjoint geodesics. For example, every maximal collection of disjoint simple closed curves on $X$ has $3g - 3$ elements, just as every maximal collection of compatible cycles for $p_d$ has $d - 1$ elements.

It is also known that the endpoints of lifts of simple geodesics lie in a closed set $\mathcal{E} \subset S^1$ of Hausdorff dimension zero [BS]. The analogous statement for simple cycles is:

**Theorem 2.2** The closure $\mathcal{E}$ of the union of all simple cycles $C \subset S^1$ of degree $d$ has Hausdorff dimension zero.

**Proof.** Let us say a finite set $P \subset S^1$ is a precycle if it is the forward orbit of preperiodic point $x \in S^1$ under $p_d$. We say $P$ is simple, with rotation number $p/q$, if $p_d|P$ extends to a continuous, monotone increasing map $f : S^1 \to S^1$ with rotation number $p/q$. Then $q \leq n$ and the periodic part $C$ of $P$ is a simple cycle.
Let $\mathcal{P}_d(n, p/q)$ denote the set of all simple precycles of length $n$ and rotation number $p/q$. The argument that shows $|\mathcal{C}_d(p/q)| = O(q^{d-2})$ can be adapted to show that $|\mathcal{P}_d(n, p/q)| = O(n^{d-2})$ as well.

Now fix $N > 0$. We claim that every $x \in E$ lies within distance $O(d^{-N})$ of a simple precycle $P$ with $|P| \leq N$. To find this precycle, simply increase $x$ continuously until two of the points among $x, f(x), \ldots, f^N(x)$ coincide. This requires moving $x$ only slightly, since $|(f^N)'(x)| = d^N$.

Thus $E$ is contained in a neighborhood of diameter $O(d^{-N})$ of the union $E_N$ of all simple precycles with $|P| \leq N$. Since $|E_N| = O(N^d)$ grows only like a polynomial in $N$, this implies $\dim(E) = 0$.

**Proof of Theorem 1.4.** The Hölder continuous conjugacy $\phi_f$ between $f$ and $p_d$ preserves sets of Hausdorff dimension zero.

**Remark: Invariant measures.** The basic properties of simple cycles can also be developed using the correspondence between invariant measures and covering relations established in [Mc5]. For example, any union $D = \bigcup C_i$ of compatible cycles in $\mathcal{C}_d(p/q)$ arises as the support of an invariant measure $\nu$ for $p_d|S^1$. Invariant measures, in turn, correspond bijectively to covering relations $(F, S)$ of degree $d$. In the case at hand, $F(t) = t + p/q \mod 1$ and $S$ is a divisor on $S^1$ of degree $d - 1$. By perturbing $S$ so its points have multiplicity one, we obtain a nearby invariant measure $\nu'$ whose support $D' \supset D$ is a maximal union of exactly $(d - 1)$ compatible cycles (property (5) above).

The compactification of the space of Blaschke products by covering relations $(F, S)$ is discussed in the following section.

**Question.** Is there a useful notion of intersection number for a pair of cycles?

### 3 Blaschke products

This section presents basic facts about marked Blaschke products, their derivatives and their images in the moduli space of all rational maps. See [Mc5] for related background material.

**Blaschke products.** Identify $S^1 = \mathbb{R}/\mathbb{Z}$ with the unit circle in the complex plane, using the coordinate $z = \exp(2\pi it)$. Let $\Delta = \{ z : |z| < 1 \}$ be the unit disk, and $\Delta^{(n)}$ its $n$-fold symmetric product.
Given $d > 1$, let $\mathcal{B}_d \cong \Delta^{(d-1)}$ denote the space of Blaschke products $f : \Delta \to \Delta$ of the form

$$f(z) = z \prod_{i=1}^{d-1} \frac{z - a_i}{1 - \overline{a}_iz}$$

with $a_i \in \Delta$. Note that $f$ extends to a rational map on the whole Riemann sphere, and $f|S^1$ is a covering map of degree $d$.

A proper holomorphic map $g : \Delta \to \Delta$ of degree $d > 1$ is conjugate to some $f \in \mathcal{B}_d$ iff $g$ has a fixed point.

**Derivatives and measure.** By logarithmic differentiation, any $f \in \mathcal{B}_d$ satisfies

$$|f'(z)| = 1 + \sum_{i=1}^{d-1} \frac{1 - |a_i|^2}{|z - a_i|^2}$$

for $z \in S^1$. In particular, $f|S^1$ is expanding.

More importantly, $f|S^1$ preserves normalized Lebesgue measure $\lambda$ on the circle; equivalently, $f_*(dz/z) = dz/z$, as can be verified by residue considerations. This means

$$\sum_{f(w) = z} |f'(w)|^{-1} = 1$$

for any $z \in S^1$.

**Markings.** All $f \in \mathcal{B}_d$ are topologically conjugate to the model mapping $p_d(z) = z^d$. A marking for $f$ the choice of one such conjugacy, i.e. the choice of a degree one homeomorphisms $\phi : S^1 \to S^1$ such that

$$f(z) = \phi^{-1} \circ p_d \circ \phi(z).$$

There is a unique marking $\phi_f$ which varies continuously in $f$ and satisfies $\phi_f(z) = z$ when $f = p_d$. Thus $\mathcal{B}_d$ can be regarded as the space of marked Blaschke products.

The modular group $\text{Mod}_d \cong \mathbb{Z}/(d-1)$ acts on $\mathcal{B}_d$ by $(a_i) \mapsto (\zeta a_i)$ where $\zeta^{d-1} = 1$. Its orbits correspond to different markings of the same map. Thus $f_1, f_2 \in \mathcal{B}_d$ are conformally conjugate on $\Delta$ iff they are in the same orbit of the modular group.

**Lengths.** The canonical marking allows one to label the cycles of $f$ by the cycles of $p_d$. We define the length on $f$ of a cycle $C \in \mathcal{C}_d$ of period $q$ by

$$L(C, f) = \log |(f^q)'(z)|$$
for any $z \in S^1$ with $\phi_f(z) \in C$.

**Limits of lower degree.** The space of Blaschke products has a natural compactification $\overline{B}_d \cong \Delta^{d-1}$, whose boundary points $(a_i)$ can be interpreted as pairs $(F, S)$ consisting of a Blaschke product

$$F(z) = z \prod_{|a_i|<1} \left( \frac{z-a_i}{1\overline{a_i}z} \right) \cdot \prod_{|a_i|=1} (-a_i)$$

and a divisor of sources

$$S = \sum_{|a_i|=1} 1 \cdot a_i \in \text{Div}(S^1),$$

satisfying $\text{deg } F + \text{deg } S = d$. It is easy to see:

**Proposition 3.1** A sequence $f_n \in B_d$ converges to $(F, S) \in \partial B_d$ iff

(i) $f_n(z) \to F(z)$ uniformly on compact subsets of $\hat{C} - \text{supp } S$, and

(ii) the zeros $Z(f_n)$ converge to $Z(F) + S$ as divisors on $\hat{C}$.

More generally, the space $\text{Rat}_d$ of degree $d$ rational maps $f : \hat{C} \to \hat{C}$ has a compactification $\overline{\text{Rat}}_d \cong \mathbb{P}^{2d+1}$, whose boundary points $(F, S)$ are pairs consisting of a rational map $F$ and an effective divisor $S \in \text{Div}(\hat{C})$ with $\text{deg}(F) + \text{deg}(S) = d$. We have $f_n \to (F, S)$ in $\overline{\text{Rat}}_d$ iff their graphs satisfy

$$\text{gr}(f_n) \to \text{gr}(F) + S \times \hat{C}$$

as divisors of degree $(1, d)$ on $\hat{C} \times \hat{C}$ (cf. [D, §1]).

**Radial bounds on $f'(z)$**. The following elementary observation is useful for studying limits as above.

**Proposition 3.2** For any proper holomorphic map $f : \Delta \to \Delta$ and $\zeta \in S^1$, we have

$$\sup_{r \in [0,1]} |f'(r\zeta)| \leq 4|f'(\zeta)|.$$  

Note that we do not require that $f(0) = 0$. This bound is sharp, as can be seen by considering $f(z) = (z + a)/(1 + az)$ as $a \to 1-$.

**Proof.** We can write

$$f(z) = e^{i\theta} \prod_{1}^{d} M_i(z),$$

(3.3)
where $M_i(z) = (z - a_i)/(1 - a_i z)$ and $a_i \in \Delta$. Composing with a rotation, we can also assume that $\zeta = 1$. For $r \in [0, 1]$ we have

$$\left| \frac{M_i'(r)}{M_i'(1)} \right| = \frac{|1 - a_i|^2}{|1 - ra_i|^2},$$

and therefore

$$|M_i'(r)| \leq 4|M_i'(1)|,$$

since the distance from 1 to $a_i$ is never more than twice the distance from 1 to $ra_i$. Differentiating the product (3.3) and using the fact that $|\prod_{j \neq i} M_j(r)| \leq 1$, we obtain:

$$|f'(r)| \leq \sum |M_i'(r)| \leq 4 \sum |M_i'(1)| = 4|f'(1)|.$$

The last equality, like equation (3.1), is verified by logarithmic differentiation.

**Corollary 3.3** If $f_n \to (F, S) \in \partial B_d$, $z_n \in S^1$, $z_n \to z$ and $|f_n'(z_n)| = O(1)$, then $\lim f_n(z_n) = F(z)$.

**Proof.** Suppose $\sup |f_n'(z_n)| = M$; then for any $r < 1$ we have

$$\limsup |f_n(z_n) - F(z)| \leq \limsup |f_n(rz_n) - F(z)| + 4M(1-r) = |F(rz) - F(z)| + 4M(1-r);$$

now let $r \to 1$.

**Irrational rotations.** As a sample application, we prove the following result stated in the Introduction:

**Corollary 3.4** If $f_n \in B_d$ satisfies $f_n'(0) \to \exp(2\pi i \theta)$ where $\theta$ is irrational, then $L(C, f_n) \to \infty$ for every cycle $C$.

**Proof.** Suppose to the contrary that $L(C, f_n)$ is bounded for some cycle $C$. Let $C_n \subset S^1$ be the corresponding periodic cycle for $f_n$. Pass to a subsequence such that $f_n \to (F, S) \in \partial B_d$ and $C_n \to D \subset S^1$ in the Hausdorff topology. Then $F(z) = \exp(2\pi i \theta)z$ and by Corollary 3.3 we have $F(D) = D$, contradicting the irrationality of $\theta$. 


Variants. Here are two useful variants of the results above:

**Proposition 3.5** For any proper holomorphic map \( f : \mathbb{H} \to \mathbb{H} \) and \( x \in \mathbb{R} \), we have
\[
\sup_y |f'(x + iy)| \leq f'(x).
\]

**Proposition 3.6** Assume \( f_n \in \text{Rat}_d \) converges to \((F, S) \in \overline{\text{Rat}}_d\), \( z_n \to z \), and \( \|Df_n(z_n)\| = O(1) \) in the spherical metric on \( \hat{\mathbb{C}} \). Then we have
\[
f_n(z_n) \to F(z)
\]
provided \( z_n \) belongs to a circle \( T_n \) with \( f_n^{-1}(T_n) = T_n \), and \( \inf_n \text{diam}(T_n) > 0 \).

**Proofs.** The first result follows directly from the representation \( f(z) = a_0 z + b_0 + \sum_{i=1}^{d-1} a_i(b_i - z) \) with \( a_i > 0 \) and \( b_i \in \mathbb{R} \), and the second follows by the same argument as Corollary 3.3.

The maps \( f_n(z) = 1/(1 + nz^2) \) satisfy \( f_n'(0) = 0 \) and \( \lim f_n(0) = 1 \neq F(0) = 0 \); thus some extra hypothesis is needed to interchange limits as in Proposition 3.6.

**Moduli space of rational maps.** Let \( \text{MRat}_d = \text{Rat}_d / \text{Aut}(\hat{\mathbb{C}}) \) denote the moduli space of holomorphic conjugacy classes of rational maps of degree \( d > 1 \). A pair of Blaschke products are conjugate iff they are related by the modular group or by \( T \mapsto 1 \) ; thus we have an inclusion
\[
\mathcal{B}_d/(\text{Mod}_d \ltimes \mathbb{Z}/2) \hookrightarrow \text{MRat}_d.
\]

The next result shows this inclusion is almost proper.

**Theorem 3.7** If \( f_n \to (F, S) \in \partial \mathcal{B}_d \) but \( [f_n] \) remains bounded in \( \text{MRat}_d \), then \( F(z) = z \) and \( \text{supp} \, S \) is a single point. In particular, we have \( f_n'(0) \to 1 \).

**Proof.** Pass to a subsequence such \( [f_n] \to [g] \in \text{MRat}_d \) and \( f_n \to (F, S) \in \partial \mathcal{B}_d \). Then there are conjugates \( h_n = A_n f_n A_n^{-1} \to g \). Since \( f_n \) diverges in \( \mathcal{B}_d \), \( A_n \to \infty \) in \( \text{Aut}(\hat{\mathbb{C}}) \). On the other hand, the measures of maximal entropy satisfy \( \mu(h_n) \to \mu(g) \) and \( \mu(f_n) \to \mu(F, S) \), by [D, Thm. 0.1] (see also [Mc5]). Since \( \mu(g) \) is nonatomic, this implies \( \mu(F, S) = \lim A_n^{*}\mu(h_n) \) is supported at a single point. But \( \text{supp} \, \mu(F, S) \) is \( F \)-invariant and includes \( \text{supp} \, S \); thus \( F(z) = z \) and \( \text{supp} \, S = \{s\} \) is itself a single point.

**Example.** The sequence \( f_n(z) = z(z + a_n)/(1 + a_n z) \), with \( a_n = 1 - 1/n \), is divergent in \( \mathcal{B}_2 \) but convergent in \( \text{MRat}_2 \). To see this, normalize so the origin is a critical point instead of a fixed point; then \( f_n(z) \) is conjugate to \( h_n(z) = (z^2 + b_n)/(1 + b_n z^2) \), and \( b_n = a_n/(2 + a_n) \to 1/3 \) as \( a_n \to 1 \).
4 The thin part of $f(z)$

Let us define the thin part of $f \in B_d$ by

$$S_{\text{thin}}^1(f) = \{z \in S^1 : |f'(z)| < 2\}.$$

In this section we will show:

**Theorem 4.1** For any $f \in B_d$, the map $f|S_{\text{thin}}^1(f)$ extends to a degree one homeomorphism of the circle.

**Corollary 4.2** All cycles of $f$ with $L(C, f) < \log 2$ are simple and compatible.

**Visual angles.** The derivative of

$$f(z) = z \prod_{1}^{d-1} \left( \frac{z - a_i}{1 - \bar{a}_i z} \right)$$

can be conveniently analyzed using the **hyperbolic visual angle**, defined for $a, z \in \overline{\Delta}$ by

$$\alpha(z, a) = 2 \text{arg}(z - a) - \text{arg}(z).$$

This is the angle at $a$ of the hyperbolic geodesic $\overline{az}$. For $z \in S^1$ we have $\text{arg}(1 - \overline{az}) = \text{arg}(z - \text{arg}(z - a)$, and thus

$$\text{arg}(f(z)) = \text{arg}(z) + \sum_{1}^{d-1} \alpha(z, a_i). \quad (4.1)$$

(Note this simplifies to $\text{arg}(f(z)) = 2 \text{arg}(z - a_1)$ when $d = 2$.) Letting $\theta = \text{arg}(z)$ and $\dot{\alpha} = d\alpha/d\theta$, we then obtain:

$$|f'(z)| = 1 + \sum_{1}^{d-1} \dot{\alpha}(z, a_i) \quad (4.2)$$

for $z \in S^1$.

**The visual density.** The **visual density** $\dot{\alpha}(z, a)$ is essentially the Poisson kernel; for $a = r \geq 0$ it is given by

$$\dot{\alpha}(z, r) = \frac{1 - r^2}{1 + r^2 - 2r \cos \theta}. \quad (4.3)$$
where $\theta = \arg z$. Geometrically, $(\dot{\alpha}(z,a)/2\pi) d\theta$ is the hitting measure on the circle for a random hyperbolic geodesic starting at $a$.

For fixed $z \in S^1$, the level sets of $\dot{\alpha}(z,a)$ are horocycles resting on $z$. Thus

$$J(a) = \{ z \in S^1 : \dot{\alpha}(z,a) < 1 \}$$

is the large arc cut off by the chord perpendicular to $0a$. This follows from the fact that the horocycle resting on one of the endpoints of $J(a)$ and passing through $0$ also passes through $a$ (see Figure 5).

![Figure 5. The arc $J(a)$ where $\dot{\alpha}(z,a) < 1$.](image)

**Proposition 4.3** The visual density $\dot{\alpha}(z,a)|_{J(a)}$ is strictly convex, and decreases as $a$ moves radially towards the circle. In other words, we have

$$\ddot{\alpha}(z,a) > 0 \text{ and } \frac{d}{ds} \dot{\alpha}(z,sa) \bigg|_{s=1} < 0$$

for all $z \in J(a)$.

**Proof.** To verify convexity, consider the case where $a = r \in [0,1)$. By (4.3), in this case we have $\dot{\alpha} = (1 - r^2)/u$ where $u = 1 + r^2 - 2r \cos \theta$. We may assume $\theta \in (0,\pi)$. Cross-multiplying and differentiating, we obtain

$$\dot{\alpha} u = 1 - r^2,$$

$$\ddot{\alpha} u + \dot{\alpha}(2r \sin \theta) = 0,$$

$$\dddot{\alpha} u + \ddot{\alpha}(4r \sin \theta) + \dot{\alpha}(2r \cos \theta) = 0.$$

Since $r, u$ and $\sin \theta$ are all positive, we have $\dot{\alpha} > 0$ and $\dddot{\alpha} < 0$. Comparing the last two equations, we find the sign of $\dddot{\alpha}$ is the same as the sign of the determinant

$$D = \det \begin{pmatrix} 2r \sin \theta & u \\ 2r \cos \theta & 4r \sin \theta \end{pmatrix} = 8r^2 \sin^2 \theta - 2ru \cos \theta.$$
We claim $D > 0$ when $z \in J(r)$, i.e. when $u = |z - r|^2 > 1 - r^2$. The claim is evident if $\cos \theta$ is negative, so assume $\theta \in (0, \pi/2)$; then

$$u = |z - r|^2 \leq |z - 1|^2 \leq 2(\text{Im} z)^2 = 2 \sin^2 \theta.$$  

We also have $\cos \theta = \text{Re}(z) < r$ for $z \in J(r)$, and thus:

$$D \geq 4r^2u - 2r^2u > 0.$$

The proof of the density decreasing property is straightforward.

Properties of the thin part of $f$. We can now show that $f|S^1_{\text{thin}}(f)$ acts like a rotation. We first observe:

**Proposition 4.4** For any $f \in B_d$,

(i) The map $f|S^1_{\text{thin}}(f)$ is injective,

(ii) We have $S^1_{\text{thin}}(f) \subset \bigcap J(a_i)$,

(iii) $S^1_{\text{thin}}(f)$ consists of at most $(d - 1)$ disjoint open intervals, and

(iv) $S^1_{\text{thin}}(f)$ increases as the zeros $a_i$ of $f$ move radially towards the circle.

**Proof.** If $f(x_1) = f(x_2)$ for two distinct points in $S^1_{\text{thin}}(f)$, then $|f'(x_1)| + |f'(x_2)| > 1/2 + 1/2 = 1$, which violates the measure-preserving property (3.2) of $f$; thus $f|S^1_{\text{thin}}(f)$ is injective. Equation (4.2) implies (ii). Since $\bigcup (S^1 - J(a_i))$ has at most $(d - 1)$ components, so does $I = \bigcap J(a_i)$. By Proposition 4.3, $|f'(z)|$ is locally convex on $I$; thus the intersection of $S^1_{\text{thin}}(f)$ with any component of $I$ is connected, and (iii) follows. The density decreasing property stated in Proposition 4.3 implies (iv).

**Proof of Theorem 4.1.** By moving the points $(a_i)$ radially to the circle, we obtain a smooth 1-parameter family of maps $f_t \in F_d$, $t \in [0,1]$, with $f_0 = f$ and $f_1 = (F, S)$. Since $\text{deg}(S) = d - 1$, we have $\text{deg}(F) = 1$. Proposition 4.4 implies that $f_t|T_t = S^1_{\text{thin}}(f_t)$ is injective, $T_s \subset T_t$ when $s < t$, and $\text{supp} S \cap T_t = \emptyset$. Thus for any three distinct points $x_i \in S^1_{\text{thin}}(f)$, the triple $(f_t(x_1), f_t(x_2), f_t(x_3))$ moves by isotopy as $t$ increases from 0 to 1, and converges to $F(x_1), F(x_2), F(x_3)$ as $t \to 1$. Since $F$ is a rotation, it preserves the cyclic ordering of the points $(x_i)$, so the same is true of $f$. Consequently $f$ extends from $S^1_{\text{thin}}(f)$ to an orientation-preserving homeomorphism of the circle.
5 Bounds on repelling cycles

In this section we show that every $f \in B_d$ has a simple cycle with $L(C, f) = O(d)$, and obtain related results for general rational maps.

Moduli and tori. We begin by summarizing some well-known facts about extremal length on tori.

Any point $\tau \in \mathbb{H}$ determines a complex torus $X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ with a flat metric inherited from the plane, and a distinguished basis $\langle 1, \tau \rangle$ for its fundamental group. Factoring the covering map $\mathbb{C} \to X_\tau$ through the map $\xi : \mathbb{C} \to \mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$ given by $\xi(z) = \exp(2\pi i z)$, we have

$$X_\tau = \mathbb{C}^*/\alpha\mathbb{Z}$$

where $\alpha = \xi(\tau)$ satisfies $0 < |\alpha| < 1$. The same construction can be made when $-\tau \in \mathbb{H}$; then $|\alpha| > 1$.

Given a slope $p/q \in \mathbb{Q} \cup \{\infty\}$, let $\gamma_{p/q} \subset X_\tau$ denote the simple closed geodesic obtained as the projection of the line $\mathbb{R} \cdot (\tau - p/q)$ from $\mathbb{C}$ to $X_\tau$. Its preimage $\tilde{\gamma}_{p/q}$ in the intermediate cover $\mathbb{C}^*$ consists of $q$ arcs joining 0 to $\infty$, cyclically permuted with rotation number $p/q$ by $z \mapsto \alpha z$.

Any annulus $A$ is conformally equivalent to a right cylinder, which is unique up to scale. The ratio $\text{mod}(A) = h/c$ between the height and circumference of this cylinder is the modulus of $A$.

The maximum modulus of an annulus $A \subset X_\tau$ homotopic to $\gamma_{p/q}$ is given by

$$\text{mod}(p/q, X_\tau) = \frac{\text{area}(X_\tau)}{L(\gamma_{p/q}, X_\tau)^2} = \frac{|\text{Im} \tau|}{|q\tau - p|^2}$$

(assuming $\gcd(p, q) = 1$). This maximum is realized by taking $A = X_\tau \setminus \gamma_{p/q}$. The set of $\tau \in \mathbb{H}$ with $\text{mod}(p/q, X_\tau) \geq m$ is a horoball of diameter $1/(mq^2)$ resting on the real axis at $p/q$. For $p/q = 1/0$ we have

$$\text{mod}(\infty, X_\tau) = |\text{Im} \tau|.$$

The intersection inequality

$$\text{mod}(p/q, X_\tau) \text{mod}(r/s, X_\tau) \leq \left(\det \begin{pmatrix} p & q \\ r & s \end{pmatrix}\right)^{-2}$$

is easily verified by considering the determinant of the lattice $\mathbb{Z}(q\tau - p) \oplus \mathbb{Z}(s\tau - r)$. This inequality implies:
There is at most one slope with \( \text{mod}(p/q, X_\tau) > 1 \).

On the other hand we have:

**Proposition 5.1** For any \( \tau \in \mathbb{H} \), there exists a slope \( p/q \in \mathbb{Q} \cup \{\infty\} \) such that
\[
\text{mod}(p/q, X_\tau) \geq \sqrt{3}/2.
\]

**Proof.** Since the statement is invariant under the action of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{H} \), it suffices to verify it when \( \tau \) lies in the fundamental domain \( |\tau| \geq 1, |\text{Re}\tau| \leq 1/2 \); and in this case, we have \( \text{mod}(\infty, X_\tau) = \text{Im}\tau \geq \sqrt{3}/2 \).

**Rational maps.** Now let \( f: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a rational map of degree \( d > 1 \). If \( z \in \hat{\mathbb{C}} \) is a point of period \( q \), its *multiplier* is given by \( \beta = (f^q)'(z) \). The *grand orbit* of \( z \) is the set \( \bigcup_{i,j>0} f^{-i} \circ f^j(z) \).

Suppose \( f \) has a fixed point at \( z = 0 \) and a periodic point \( w \neq 0 \) with period \( q \). We say \( w \) has *rotation number* \( p/q \) relative to \( z = 0 \) if there are arcs \( (\delta_i)_{q-1}^0 \subset \hat{\mathbb{C}} \) joining \( z = 0 \) to \( f^i(w) \), meeting only at \( z = 0 \), which are cyclically permuted by \( f \) with rotation number \( p/q \).

**Theorem 5.2** Let \( f \) be a rational map with an attracting fixed-point at \( z = 0 \), with multiplier
\[
\alpha = f'(0) = \exp(2\pi i \tau) \neq 0.
\]
Let \( e \) be the number of grand orbits of critical points in the immediate basin \( \Omega \) of \( z = 0 \). Then for each \( p/q \in \mathbb{Q} \), there exists a repelling or parabolic periodic point \( w \in \partial \Omega \) such that:

1. The rotation number of \( w \) relative to \( z = 0 \) is \( p/q \); and
2. Its multiplier has the form \( \beta = (f^q)'(w) = \exp(-2\pi i \sigma) \), where \( \sigma = 0 \) or
\[
\frac{\text{Im}\sigma}{|\sigma|^2} \geq \frac{\text{mod}(p/q, X_\tau)}{e}.
\]

In particular, we have
\[
|\beta| \leq \left( \exp\left( \frac{2\pi}{\text{mod}(p/q, X_\tau)} \right) \right)^e.
\]
Proof. Let $\Omega^*$ denote the immediate basin of $z = 0$ with the grand orbits of all critical points in $\Omega$ and of $z = 0$ deleted. Then $f : \Omega^* \to \Omega^*$ is a covering map. Moreover, the holomorphic linearizing map

$$\phi(z) = \lim \alpha^{-n} f^n(z)$$

is defined for all $z \in \Omega^*$, and satisfies $\phi(f(z)) = \alpha f(z)$. Consequently $\phi$ descends to an inclusion of the space of grand orbits $Y = \Omega^* / \langle f \rangle$ into the torus $X_\tau = \mathbb{C}^* / \alpha \mathbb{Z}$, making the diagram

$$\begin{array}{ccc}
\Omega & \xrightarrow{\phi} & \mathbb{C}^* \\
\downarrow & & \downarrow \\
\Omega / \langle f \rangle = Y & \hookrightarrow & X_\tau = \mathbb{C}^* / \alpha \mathbb{Z}
\end{array}$$

commute. By assumption we have $|Y - X_\tau| = e$.

For a given $p/q \in \mathbb{Q}$, the geodesics parallel to $\gamma_{p/q}$ passing through the punctures of $Y$ cut it into $\leq e$ parallel annuli, one of which satisfies

$$\text{mod}(A) \geq \text{mod}(p/q, X_\tau) / e. \tag{5.5}$$

Let $\delta \subset A$ be the core curve of $A$, and $\delta_0 \subset \Omega^*$ one of its lifts which is incident to $z = 0$. Let $\delta_i = f^i(\delta_0)$. By construction, the arc $\delta_0$ is invariant under $f^q$, and $f^q \delta_0$ is a bounded translation in the hyperbolic metric on $\Omega^*$. Consequently $\delta_0$ must join $z = 0$ to another fixed point $w$ of $f^q$ in $\partial \Omega$. By the Snail Lemma [Mil1, Lem. 16.2], $w$ is repelling or parabolic.

We have seen that the preimage of $\gamma_{p/q}$ on $\mathbb{C}^*$ consists of $q$ arcs, cyclically permuted with rotation number $p/q$ by $z \mapsto \alpha z$. Since $\phi$ is a homeomorphism near $z = 0$, the arcs $\delta_0, \ldots, \delta_{q-1}$ are also cyclically permuted with rotation number $p/q$ by $f$. In particular $w$ has rotation number $p/q$ relative to $z = 0$.

Now suppose $w$ is repelling, with multiplier $\beta$. Choose an injective branch of $f^{-q}$ defined on a punctured neighborhood $U^*$ of $w$ such that $f^{-q} : U^* \to U^*$ and

$$Z = U^* / \langle f^{-q} \rangle \cong \mathbb{C}^* / \beta \mathbb{Z} = X_\sigma,$$

where $\sigma = \log(\beta/2\pi i)$. There is a unique choice of the logarithm such that the invariant arc $\delta_0 \cap U^*$ descends to a loop isotopic to $\gamma_0$ on $X_\sigma$.

By construction, $A \subset Y$ is covered by a strip $A_0 \subset \Omega^*$ which retracts to $\delta_0$, and hence we have an inclusion

$$A \cong A_0 / \langle f^q \rangle \hookrightarrow Z \cong X_\sigma$$
in the same homotopy class as $\gamma_0$. This implies
\[
\text{mod}(0, X_\sigma) \geq \text{mod}(A),
\]
and the bound (5.3) follows from equations (5.1) and (5.5).

**Corollary 5.3** If $f \in \text{Rat}_d$ has an attracting fixed point with multiplier
satisfying
\[
|\alpha| > \exp(-\pi\sqrt{3}) = 0.0043\ldots
\]
then it also has a repelling or parabolic cycle with multiplier satisfying
\[
|\beta| \leq \exp(4\pi/\sqrt{3})^{2d-2} \leq 1416^{2d-2}.
\]

**Proof.** The lower bound on $|\alpha|$ implies $\text{Im}(\tau) = \text{mod}(\infty, X_\tau) < \sqrt{3}/2$, where $\tau = (\log \alpha)/2\pi i$. Hence $\text{mod}(p/q, X_\tau) \geq \sqrt{3}/2$ for some $p/q \in \mathbb{Q}$, by Proposition 5.1. Now apply equation (5.4) and note that $e \leq 2d - 2$.

**Corollary 5.4** If a map $f \in \text{Rat}_d$ has an attracting fixed point with mul-
tiplier $\alpha$, then it also has a repelling or parabolic cycle with multiplier satisfying
\[
|\beta| \leq \left(\exp(4\pi/\sqrt{3})/|\alpha|\right)^{2d-2}.
\]

**Proof.** Choose $\tau = (\log \alpha)/2\pi i = x + iy$ with $x \in [-1/2, 1/2]$. The previous corollary shows the desired bound holds when $y < \sqrt{3}/2$. For $y \geq \sqrt{3}/2$ we have
\[
m = \text{mod}(0, X_\tau)^{-1} \leq \frac{x^2 + y^2}{y} \leq \frac{1}{2\sqrt{3}} + y < \frac{2}{\sqrt{3}} + y,
\]
which implies $\exp(2\pi/m) \leq \exp(4\pi/\sqrt{3})/|\alpha|$; thus by (5.4) the desired bound holds in this case as well.

**The bottom of the spectrum.** Here is a qualitative consequence of the preceding corollary.

Let the spectrum $S(f) \subset \mathbb{C}$ be the set of all multipliers $\beta$ that arise from periodic points of $f \in \text{Rat}_d$, and let
\[
L(f) = \inf\{|\beta| : \beta \in S(f) \text{ and } |\beta| \geq 1\}.
\]
By the fixed-point formula for rational maps [Mil1, Thm. 12.4], the multipliers of \( f \) at its fixed points satisfy
\[
\sum \frac{1}{\mu_j - 1} = 1, \tag{5.6}
\]
provided no \( \mu_j = 1 \); in particular, \( |\mu_j| \leq d + 1 \) for some \( j \). Thus if \( f \) has no attracting fixed points, it satisfies
\[
L(f) \leq \log(d + 1).
\]
Combining this observation with Corollary 5.4, we obtain:

**Corollary 5.5** Let \( f_n \in \text{Rat}_d \) be a sequence of rational maps with \( L(f_n) \to \infty \). Then the maps \( f_n \) have fixed points with multipliers \( \alpha_n \to 0 \).

**Examples.** It is easy to see that \( f_n(z) = z^2 + n^2 \) satisfies \( L(f_n) \to \infty \) as \( n \to \infty \), since its Julia set lies close to \( \pm n \). Of course \( f_n \) has a fixed point at infinity with multiplier \( \alpha_n = 0 \).

Parabolics must be included in the definition of \( L(f) \) to obtain Corollary 5.5. In fact, if we let \( L^*(f) = \inf \{ \log |\beta| : \beta \in S(f), |\beta| > 1 \} \), then \( f_n(z) = z - 1/z + n \) satisfies \( L^*(f_n) \to \infty \) even though \( f_n \) has no attracting fixed point. (The map \( f_n(z) \) behaves like the Hecke group \( \langle z \mapsto -1/z, z \mapsto z + n \rangle \); cf. [Mc3, Thm 6.2].)

**Question.** Does Corollary 5.5 remain true if only parabolic and repelling multipliers are included in the definition of \( L(f) \)?

**Blaschke products.** We now return to the setting of a proper map \( f : \Delta \to \Delta \) fixing \( z = 0 \). In this case formula (5.6) implies:

**Proposition 5.6** The multipliers \( (\lambda_i)^{d-1} \) of \( f \in \mathcal{B}_d \) at its fixed points on the circle satisfy
\[
\sum_{i=1}^{d-1} \frac{1}{\lambda_i - 1} = \frac{1 - |\alpha|^2}{|1 - \alpha|^2},
\]
where \( \alpha = f'(0) \).

**Corollary 5.7** If \( |\alpha| < 1/2 \), then \( f \) has a repelling fixed point with multiplier satisfying \( 1 < \beta \leq 1 + (d - 1)/3 \).

**Theorem 5.8** Every \( f \in \mathcal{B}_d \) has a simple cycle with \( L(C, f) = O(d) \).

**Proof.** Combine Corollaries 5.4 and 5.7. \( \blacksquare \)
6 Short cycles and short geodesics

In this section we use the fixed-point formula for rational maps to obtain the following more detailed connection between the short cycles for \( f \) and the short geodesics on its quotient torus.

**Theorem 6.1** Given \( f \in B_d \) with \( f'(0) = \exp(2\pi i \tau) \), choose \( p/q \in \mathbb{Q} \) to maximize \( \text{mod}(p/q, X_\tau) \). Then there exist compatible simple cycles \( C_i \) with rotation number \( p/q \), such that:

1. Their lengths satisfy
   \[
   \text{mod}(p/q, X_\tau) \leq \pi \sum L(C_i, f)^{-1} \leq \text{mod}(p/q, X_\tau) + O(d); \quad (6.1)
   \]
2. All other cycles satisfy \( L(C, f) > \epsilon_d > 0 \); and
3. For any \( r > 0 \), the multipliers of \( f^r \) at its repelling fixed points satisfy
   \[
   \frac{1}{r} \sum_{j} \frac{1}{\lambda_j} - 1 = O(d),
   \]
   where the prime indicates that fixed points in \( \bigcup C_i \) are excluded.

In qualitative terms, the construction shows:

**Corollary 6.2** All cycles with \( L(C, f) < \epsilon_d \) arise from short geodesics on the quotient torus for \( f \).

**Tiling of \( \Delta^* \).** The slope \( p/q \mod 1 \) appearing in the Theorem above depends only on \( \alpha = f'(0) \in \Delta^* \). Figure 1 of the Introduction shows the regions \( T(p/q) \subset \Delta^* \) where a given slope maximizes the value of \( \text{mod}(p/q, X_\tau) = \text{mod}(p/q, \mathbb{C}/\alpha \mathbb{Z}) \).

This picture is nothing more than the image, under the covering map \( \xi: \mathbb{H} \to \Delta^* \) given by \( \xi(\tau) = \exp(2\pi i \tau) \), of the tiling of \( \mathbb{H} \) by \( \text{SL}_2(\mathbb{Z}) \) translates of the Dirichlet region

\[
F = \{ \tau \in \mathbb{H} : |\tau - n| \geq 1 \forall n \in \mathbb{Z} \}
\]

for the cusp \( \tau = \infty \). The tile \( T(\infty) = \xi(F) \) lies in a ball of radius \( \exp(-\pi \sqrt{3}) \approx 1/230 \) about the origin. In this tile the short curve is \( \gamma_\infty \subset X_\tau \), which lifts to a loop around \( z = 0 \) rather than a path connecting \( z = 0 \) to a periodic point. Thus the length of \( \gamma_\infty \) can go to zero without any cycle getting short.
Each remaining tile $T(p/q)$ contains a horocycle $H$ resting on the root of unity $\exp(2\pi ip/q) \in S^1$. Within a still smaller horocycle $H' \subset H$, $\gamma_{p/q}$ becomes very short, and hence $f$ has a very short cycle with rotation number $p/q$.

**Moduli and multipliers.** We begin the proof of Theorem 6.1 by connecting Diophantine properties of $\alpha \in \Delta^*$ to lengths of geodesics on $\mathbb{C}^*/\alpha^\mathbb{Z}$.

**Lemma 6.3** For any $\alpha = \exp(2\pi i \tau) \in \Delta^*$ and $q > 0$, we have

$$\sup_p \frac{\mod(p/q, X_\tau)}{\gcd(p,q)^2} = \frac{\pi}{q} \frac{1 - |\alpha|^2}{1 - |\alpha|^2} + O(1).$$

**Proof.** First consider the case $q = 1$, and assume $\tau$ is chosen so $|\text{Re} \tau| \leq 1/2$. Then we have $2\pi i \tau \approx 1 - \alpha$ when either side is small, and hence

$$\sup_p \mod(p, X_\tau) = \frac{\text{Im} \tau}{|\tau|^2} = \frac{1 - |\alpha|^2}{1 - |\alpha|^2} + O(1).$$

The general case follows using the fact that

$$\frac{\mod(p/q, X_\tau)}{\gcd(p,q)^2} = \frac{\mod(p, X_{q\tau})}{q}.$$

**Proof of Theorem 6.1.** Choose $p$ so that $\mod(p/q, X_\tau)$ is maximized. As in Theorem 5.2, by cutting the torus $X_\tau$ open along $e \leq d - 1$ geodesics parallel to $\gamma_{p/q}$ we obtain annuli $A_1, \ldots, A_e \subset Y$ with

$$\mod(p/q, X_\tau) = \sum \mod(A_i).$$

Each annulus $A_i$, when lifted to the unit disk, connects $z = 0$ to a simple cycle $C_i$ for $f$ with rotation number $p/q$ and multiplier $\beta_i > 1$.

The lifts of the annuli $A_i$ are disjoint, so the cycles $C_i$ are compatible. Assume for the moment they are also distinct. Since two copies of $A_i$ embed in the quotient torus $\mathbb{C}^*/\beta_i^\mathbb{Z}$ (one for the inside of the disk and one for the outside), we have

$$2 \mod(A_i) \leq \frac{2\pi}{\log \beta_i} = \frac{2\pi}{L(C_i, f)}.$$

The combination of these inequalities yields:

$$\mod(p/q, X_\tau) \leq \pi \sum L(C_i, f)^{-1}.$$
This lower bound also holds when the cycles are not distinct; then we simply have more annuli \( A_i \) embedded in a given torus \( \mathbb{C}^*/\beta^\mathbb{Z}_j \).

For the upper bound, let \( (\lambda_j) \) denote the multipliers of the repelling fixed points of \( f^q \). Note that each cycle \( C_i \) contributes \( q \) fixed points, each with multiplier \( \beta_i \). Combining Proposition 5.6 and Lemma 6.3, we obtain:

\[
\frac{1}{q} \sum \frac{1}{\lambda_j - 1} = \frac{1}{q} \sum' \frac{1}{\lambda_j - 1} + \sum \frac{1}{\beta_i - 1} = \frac{1 - |\alpha^q|^2}{q|1 - \alpha^q|^2} = \pi^{-1} \mod(p/q, X_\tau) + O(1).
\]

(Again, the prime indicates fixed points in \( \bigcup C_i \) are excluded.) Since the cycles \( C_i \) are compatible, there are no more than \( d - 1 \) of them, and hence

\[
\sum \frac{1}{\beta_i - 1} = \sum \left( \frac{1}{\log \beta_i} + O(1) \right) = \left( \sum L(C_i)^{-1} \right) + O(d).
\]

This yields the upper bound in (6.1); and it also implies

\[
\frac{1}{q} \sum' \frac{1}{\lambda_j - 1} = O(d).
\]

That is, equation (6.2) holds for \( r = q \).

To obtain (6.2) for other values of \( r \), recall that by (5.2) we have \( \mod(s/r, X_\tau) < 1 \) whenever \( s/r \neq p/q \). Thus if \( q \) does not divide \( r \), Lemma 6.3 implies

\[
\frac{1}{r} \sum' \frac{1}{\lambda_j - 1} \leq \frac{1 - |\alpha^r|^2}{r|1 - \alpha^r|^2} \leq \sup \mod(s/r, X_\tau) + O(1) = O(1);
\]

while for \( r = nq \) we obtain

\[
\frac{1}{r} \sum' \frac{1}{\lambda_j - 1} + \frac{q}{r} \sum \frac{1}{\beta_i^n - 1} = \frac{1 - |\alpha^r|^2}{r|1 - \alpha^r|^2} = \frac{\mod(p/q, X_\tau)}{\pi n^2} + O(1),
\]

which again implies (6.2), since (6.1) gives

\[
\frac{q}{r} \sum \frac{1}{\beta_i^n - 1} = \frac{1}{n} \left( \sum \frac{1}{nL(C_i, f)} + O(1) \right) = \frac{\mod(p/q, X_\tau)}{\pi n^2} + O(d).
\]

Finally note that equation (6.2) implies \( L(C, f) > \epsilon_d \approx 1/d > 0 \), since any cycle \( C \) of period \( r \) and multiplier \( \beta \), not among the \( C_i \), contributes \( 1/(\beta - 1) \) to the sum \( (1/r) \sum' (\lambda_j - 1)^{-1} \).
7 Binding and renormalization

We conclude by proving the following compactness result.

**Theorem 7.1** Let \((C_i)_{i=1}^m\) be a binding set of cycles of degree \(d\). Then for any \(M > 0\), the set of \(f \in B_d\) such that \(\sum_i L(C_i, f) \leq M\) has compact closure in \(\text{MRat}_d\).

**Corollary 7.2** The set of \(f \in B_d\) such that \(\sum_i L(C_i, f) \leq M\) and \(|f'(0) - 1| \geq 1/M\) is compact.

**Proof.** By Theorem 3.7, the only way a sequence \(f_n\) can diverge in \(B_d\) but remain bounded in \(\text{MRat}_d\) is if \(f_n'(0) \to 1\).

**Definitions.** Sets \(A, B \subset S^1\) are unlinked if they lie in disjoint connected sets; equivalently, if their convex hulls in the unit disk are disjoint. A map \(f : X \to X\) with \(X \subset S^1\) is renormalizable if there is a nontrivial partition of \(X\) into disjoint, unlinked subsets \(X_1, \ldots, X_m\), such that every \(f(X_i)\) lies in some \(X_j\).

We say a collection of degree \(d\) cycles \(C_1, \ldots, C_m\) is binding if \(\deg(p_d|\bigcup C_i) = d\) and \(p_d|\bigcup C_i\) is not renormalizable.

**Proof of Theorem 7.1.** Suppose to the contrary that we have a sequence \(f_n \in B_d\) with \(\sum_i L(C_i, f_n) \leq M\) that is divergent in moduli space. Let

\[D_n = \phi_{f_n}^{-1}\left(\bigcup C_i\right) \subset S^1\]

be the finite \(f_n\)-invariant set corresponding to the binding cycles. Since \(f_n|S^1\) is expanding, we have \(|f_n'| \leq e^M\) on \(D_n\).

Next we conjugate the entire picture by an affine transformation depending on \(n\), so that \(0 \in D_n\) and \(\text{diam}(D_n) = 1\). Then \(S^1\) goes over to a circle \(T_n \supset D_n\) invariant by \(f_n\), and we still have \(|f_n'|_{D_n} \leq e^M\).

Pass to a subsequence such that \(f_n \to (F, S) \in \text{Rat}_d\). Since \(f_n\) diverges in \(\text{MRat}_d\), we have \(\deg(F) < d\). Passing to a further subsequence, we can find a finite set \(D\) containing zero and a circle \(T \subset \hat{\mathbb{C}}\) such that \(D_n \to D\) and \(T_n \to T\) in the Hausdorff topology. Note that \(|D| > 1\) since \(\text{diam} D = 1\).

By Proposition 3.6, the map \(f_n|D_n\) converges to \(F|D\). But if \(|D| = |\bigcup C_i|\), the map \(F|D \subset T\) is combinatorially the same as \(p_d|\bigcup C_i \subset S^1\), contradicting our assumption that \(\deg(p_d|\bigcup C_i) = d\). Similarly, if \(|D| < |\bigcup C_i|\), then the collapse of \(D_n\) to \(D\) provides an invariant partition for \(\bigcup C_i\), contradicting our assumption that \(p_d|\bigcup C_i\) is not renormalizable. ■
Examples. The single cycle $C = (3, 6, 12, 24, 17)/31$ in degree 2 is already binding, as is any cycle of prime order with deg($p_d(C) = d$).

The first renormalizable cycle in degree 2 is $C = (1, 2, 4, 3)/5$. Although deg($p_2(C) = 2$, $L(C, f_n)$ remains bounded as $f_n \in B_2$ diverges along the sequence specified by $f_n'(0) = -1 + 1/n$. Indeed, $f_n^2$ can be renormalized so that $C$ converges to the cycle of period 2 for $G(z) = z - 1/z$ [Ep]; and thus $L(C, f_n) \to \log 9$. For more details, see [Mc6, §14].

References


