



# Billiards and Hilbert Modular Surfaces

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## Billiards and Hilbert modular surfaces

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In this talk we discuss a connection between billiards in polygons and algebraic curves in the moduli space of Riemann surfaces. In genus two, we find these *Teichmüller curves* lie on Hilbert modular surfaces parameterizing Abelian varieties with real multiplication. Explicit examples give  $L$ -shaped billiard tables with optimal dynamical properties.

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Let  $P \subset \mathbb{C}$  be a polygon whose angles are rational multiples of  $\pi$ . Then a billiard ball bouncing off the sides of  $P$  moves in only finitely many directions. A typical billiard trajectory in a regular pentagon is shown in Figure 1.

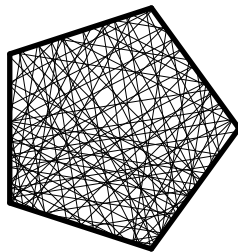


Figure 1. Billiards in a regular pentagon.

By gluing together copies of  $(P, dz)$  reflected through its sides, one obtains a compact Riemann surface  $X$  equipped with a holomorphic 1-form  $\omega$ . Billiard trajectories in  $P$  then correspond to geodesics on the surface  $(X, |\omega|)$ . The metric  $|\omega|$  is flat apart from isolated singularities coming from the vertices of  $P$ .

The space of all pairs  $(X, \omega)$  forms a bundle  $\Omega\mathcal{M}_g \rightarrow \mathcal{M}_g$  over the moduli space of Riemann surface of genus  $g$ , and admits a natural action of  $\mathrm{SL}_2(\mathbb{R})$ . Upon projection to  $\mathcal{M}_g$ , the  $\mathrm{SL}_2(\mathbb{R})$ -orbit of a given point  $(X, \omega)$  determines a ‘complex geodesic’

$$f : \mathbb{H} \rightarrow \mathcal{M}_g,$$

i.e. a holomorphic and isometric immersion of the hyperbolic plane into moduli space.

Usually the image of  $f$  is dense in  $\mathcal{M}_g$ . On rare occasions, however,  $f$  may cover an algebraic curve  $V \subset \mathcal{M}_g$ . This happens exactly when the

stabilizer  $\mathrm{SL}(X, \omega)$  of  $(X, \omega)$  is a lattice in  $\mathrm{SL}_2(\mathbb{R})$ . In this case  $V$  is a *Teichmüller curve* and  $P$  is a *lattice polygon*.

Using renormalization and Teichmüller theory, Veech showed that billiards in a lattice polygon is dynamically optimal:

- every billiard trajectory is either periodic or uniformly distributed, and
- the number of classes of closed trajectories of length  $\leq L$  is asymptotic to  $c(P) \cdot L^2$ .

It is classical that a square is a lattice polygon, with  $\mathrm{SL}(X, \omega) = \mathrm{SL}_2(\mathbb{Z})$ . Only a handful of additional examples are known: these include the regular  $n$ -gons [V], certain triangles [Wa], [KS], [Pu], and other polygons derived from these. Similarly, until recently only finitely many ‘primitive’ Teichmüller curves were known in each  $\mathcal{M}_g$ . (A Teichmüller curve is *primitive* if it does not arise from a curve in a moduli space of lower genus via a branched covering construction.)

In [Mc1] we give a synthetic construction of infinitely many primitive Teichmüller curves  $V \subset \mathcal{M}_2$ .

We begin by observing that if  $(X, \omega) \in \Omega\mathcal{M}_2$  generates a primitive Teichmüller curve  $V$ , then the Jacobian of  $X$  admits real multiplication by the trace field  $K \cong \mathbb{Q}(\sqrt{d})$  of  $\mathrm{SL}(X, \omega)$ . This observation shows  $V$  lies on a certain Hilbert modular surface: we have

$$V \subset \Sigma \cong (\mathbb{H} \times \mathbb{H})/\Gamma \subset \overline{\mathcal{M}_2},$$

where  $\Gamma$  is commensurable to  $\mathrm{SL}_2(\mathcal{O}_K)$ , and  $\Sigma$  parameterizes those  $X$  admitting real multiplication by a given order in  $K$ .

Let us say  $\omega$  is a *Weierstrass form* if its zero divisor is concentrated at a single point. By imposing this additional condition, we reduce from surfaces to curves and obtain:

**Theorem 1** *The locus*

$$\mathcal{W}_2 = \{X : \mathrm{Jac}(X) \text{ admits real multiplication with a Weierstrass eigenform}\} \subset \mathcal{M}_2$$

*is a union of infinitely many primitive Teichmüller curves.*

As a concrete application, consider the  $L$ -shaped billiard table  $P(a)$  with sides of lengths 1 and  $a$  as in Figure 2. Then we find:

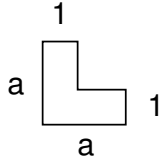


Figure 2. An  $L$ -shaped billiard table determines a Riemann surface of genus two.

**Theorem 2**  $P(a)$  is a lattice polygon if and only if  $a = (1 + \sqrt{d})/2$  for some  $d \in \mathbb{Q}_+$ .

The geometry of the Teichmüller curve  $V \subset \Sigma \subset \mathcal{M}_2$  generated by  $P(a)$  is also interesting. After passing to the universal cover, the lift

$$\tilde{V} \subset \mathbb{H} \times \mathbb{H} = \tilde{\Sigma}$$

becomes the graph of a holomorphic function  $F : \mathbb{H} \rightarrow \mathbb{H}$ , intertwining the action of  $\Gamma = \mathrm{SL}(X, \omega) \subset \mathrm{SL}_2(K)$  with its Galois conjugate  $\Gamma'$ . When  $d = 5$ ,  $F$  is the ‘pentagon-star’ map shown in Figure 3; it can be constructed by mapping an ideal pentagon to an ideal star, and then analytically continuing to the whole of  $\mathbb{H}$  by Schwarz reflection. (In particular  $F$  is transcendental, and  $V$  is *not* a Shimura curve on  $\Sigma$ .)

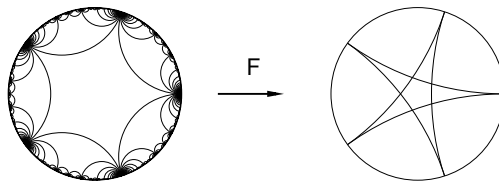


Figure 3. A holomorphic map from an ideal pentagon to an ideal star.

To place these results in context, recall that Ratner has established a powerful classification theorem for the orbits and invariant measures of unipotent flows on homogeneous spaces [Rat]. One can hope to find a similar structure for the action of  $\mathrm{SL}_2(\mathbb{R})$  on  $\Omega\mathcal{M}_g$ . We conclude by reporting on recent progress on classifying invariant measures and orbit closures in  $\Omega\mathcal{M}_2$ .

The papers [Mc1], [Mc2] contain more details and references, and are available at <http://math.harvard.edu/~ctm/papers>.

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