Billiards and Hilbert modular surfaces

MPI Arbeitstagung 2003

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In this talk we discuss a connection between billiards in polygons and algebraic curves in the moduli space of Riemann surfaces. In genus two, we find these Teichmüller curves lie on Hilbert modular surfaces parameterizing Abelian varieties with real multiplication. Explicit examples give \( L \)-shaped billiard tables with optimal dynamical properties.

Let \( P \subset \mathbb{C} \) be a polygon whose angles are rational multiples of \( \pi \). Then a billiard ball bouncing off the sides of \( P \) moves in only finitely many directions. A typical billiard trajectory in a regular pentagon is shown in Figure 1.

Figure 1. Billiards in a regular pentagon.

By gluing together copies of \((P, dz)\) reflected through its sides, one obtains a compact Riemann surface \( X \) equipped with a holomorphic 1-form \( \omega \). Billiard trajectories in \( P \) then correspond to geodesics on the surface \((X, |\omega|)\). The metric \( |\omega| \) is flat apart from isolated singularities coming from the vertices of \( P \).

The space of all pairs \((X, \omega)\) forms a bundle \( \Omega \mathcal{M}_g \to \mathcal{M}_g \) over the moduli space of Riemann surface of genus \( g \), and admits a natural action of \( \text{SL}_2(\mathbb{R}) \). Upon projection to \( \mathcal{M}_g \), the \( \text{SL}_2(\mathbb{R}) \)-orbit of a given point \((X, \omega)\) determines a ‘complex geodesic’

\[
f : \mathbb{H} \to \mathcal{M}_g,
\]

i.e. a holomorphic and isometric immersion of the hyperbolic plane into moduli space.

Usually the image of \( f \) is dense in \( \mathcal{M}_g \). On rare occasions, however, \( f \) may cover an algebraic curve \( V \subset \mathcal{M}_g \). This happens exactly when the
stabilizer $\text{SL}(X,\omega)$ of $(X,\omega)$ is a lattice in $\text{SL}_2(\mathbb{R})$. In this case $V$ is a \textit{Teichmüller curve} and $P$ is a \textit{lattice polygon}.

Using renormalization and Teichmüller theory, Veech showed that billiards in a lattice polygon is dynamically optimal:

- every billiard trajectory is either periodic or uniformly distributed, and
- the number of classes of closed trajectories of length $\leq L$ is asymptotic to $c(P) \cdot L^2$.

It is classical that a square is a lattice polygon, with $\text{SL}(X,\omega) = \text{SL}_2(\mathbb{Z})$. Only a handful of additional examples are known: these include the regular $n$-gons \cite{V}, certain triangles \cite{Wa}, \cite{KS}, \cite{Pu}, and other polygons derived from these. Similarly, until recently only finitely many 'primitive' Teichmüller curves were known in each $\mathcal{M}_g$. (A Teichmüller curve is \textit{primitive} if it does not arise from a curve in a moduli space of lower genus via a branched covering construction.)

In \cite{Mc1} we give a synthetic construction of infinitely many primitive Teichmüller curves $V \subset \mathcal{M}_2$.

We begin by observing that if $(X,\omega) \in \Omega\mathcal{M}_2$ generates a primitive Teichmüller curve $V$, then the Jacobian of $X$ admits real multiplication by the trace field $K \cong \mathbb{Q}(\sqrt{d})$ of $\text{SL}(X,\omega)$. This observation shows $V$ lies on a certain Hilbert modular surface: we have

$$V \subset \Sigma \cong (\mathbb{H} \times \mathbb{H})/\Gamma \subset \overline{\mathcal{M}_2},$$

where $\Gamma$ is commensurable to $\text{SL}_2(\mathcal{O}_K)$, and $\Sigma$ parameterizes those $X$ admitting real multiplication by a given order in $K$.

Let us say $\omega$ is a \textit{Weierstrass form} if its zero divisor is concentrated at a single point. By imposing this additional condition, we reduce from surfaces to curves and obtain:

\textbf{Theorem 1} \textit{The locus}

$$\mathcal{W}_2 = \{X : \text{Jac}(X) \text{ admits real multiplication with}$$

$$\text{a Weierstrass eigenform}\} \subset \mathcal{M}_2$$

\textit{is a union of infinitely many primitive Teichmüller curves.}

As a concrete application, consider the $L$-shaped billiard table $P(a)$ with sides of lengths 1 and $a$ as in Figure 2. Then we find:
Theorem 2  \( P(a) \) is a lattice polygon if and only if \( a = (1 + \sqrt{d})/2 \) for some \( d \in \mathbb{Q}_+ \).

The geometry of the Teichmüller curve \( V \subset \Sigma \subset \mathcal{M}_2 \) generated by \( P(a) \) is also interesting. After passing to the universal cover, the lift

\[
\tilde{V} \subset \mathbb{H} \times \mathbb{H} = \tilde{\Sigma}
\]

becomes the graph of a holomorphic function \( F : \mathbb{H} \to \mathbb{H} \), intertwining the action of \( \Gamma = \text{SL}(X, \omega) \subset \text{SL}_2(K) \) with its Galois conjugate \( \Gamma' \). When \( d = 5 \), \( F \) is the ‘pentagon-star’ map shown in Figure 3; it can be constructed by mapping an ideal pentagon to an ideal star, and then analytically continuing to the whole of \( \mathbb{H} \) by Schwarz reflection. (In particular \( F \) is transcendental, and \( V \) is not a Shimura curve on \( \Sigma \).

Figure 3. A holomorphic map from an ideal pentagon to an ideal star.

To place these results in context, recall that Ratner has establish a powerful classification theorem for the orbits and invariant measures of unipotent flows on homogeneous spaces [Rat]. One can hope to find a similar structure for the action of \( \text{SL}_2(\mathbb{R}) \) on \( \Omega \mathcal{M}_g \). We conclude by reporting on recent progress on classifying invariant measures and orbit closures in \( \Omega \mathcal{M}_2 \).

The papers [Mc1], [Mc2] contain more details and references, and are available at [http://math.harvard.edu/~ctm/papers](http://math.harvard.edu/~ctm/papers).
References


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