The Motion of the Maximal Measure of a Polynomial

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Harvard University, March 1985. Preliminary notes.

§1 Introduction

The maximal measure of a rational function is the measure with respect to which the periodic points are uniformly distributed. It also coincides with the distribution of the inverse orbit of any point. It is also the measure of maximum entropy. See Ljubic [L] for proofs of these fundamental results.

In the case of a polynomial the maximal measure takes a particularly simple form, analyzed by Brolin [B]. It is the hitting measure on the Julia set for Brownian motion initiated at infinity.

How does the maximal measure move as the polynomial varies analytically with a parameter? We find that the measures move in a strongly continuous way when paired with harmonic or analytic functions, and weakly continuously when paired with arbitrary continuous functions. This is suprising insofar as the Julia set (which is the smallest closed set supporting these measures) moves in a spectacularly discontinuous fashion. The continuity properties of the maximal measure make precise the empirical observation that computer drawings (based on a naive algorithm) have trouble detecting these bifurcations.

To show that these results are the best possible we prove a transversality result for expanding Julia sets which are quasicircles. This result implies that as ϵ varies (through small values), the maximal measures of $z^2 + \epsilon$ are mutually singular. In fact we show that the conformal shapes of the Julia sets J_ϵ are all transverse -- in the sense that the intersection of any two is geometrically scattered (has no points of density), hence has measure zero with respect to any geometrically well-distributed measure (such as the maximal measure or the appropriate Hausdorff measure).

What about families of rational maps? There is a natural proof to try to carry out, based on the same idea as Ljubic's proof of the convergence of inverse orbits to the maximal measure. We give a characterization of stability and show that to carry out this program, the family must already be stable (in which case the proof is easy). A more subtle proof requires analyzing the percentage by which the original argument breaks down; here we show that even in the case of quadratic polyno-

mials, the amount of spoilage is a fixed (nonzero) percentage (with a nice interpretation in terms of the equilibrium measure on the Mandelbrot set).

§2 Continuous Motion for Polynomials.

We consider only quadratic polynomials of the form $P_c(z) = z^2 - c$. The maximal measure $\mu = \mu_c$ (for $c \neq 0$) can be defined as follows:

$$\mu_c(f) = \lim_{n\to\infty} 2^{-n} \sum_{P_c^n(x)=0} f(x)$$

for any continuous function f on $\mathbb{P}^1$. Thus μ gives the average of f over the backward orbit of the critical point (zero). In fact, any point could play the role of zero; all backward orbits have the same asymptotic distribution.

Now suppose f(z) is itself analytic (or indeed just harmonic). Then the same is true for each of the finite sums appearing above. Since $\mu_c(f)$ is bounded by the supremum of f over the Julia set of P_c, and since the Julia set itself remains in a compact set as c varies through a small ball, it is easy to see that the finite sums are uniformly bounded analytic (harmonic) functions of c. The sums converge pointwise and a normal families argument shows that limit is itself analytic or harmonic. Then the derivate of $\mu_c(f)$ with respect to c is controlled by a bound on $\mu_c(f)$, which is (trivially) bounded by the supremum of f on the Julia set of P_c.

To sum up, let Δ_r denote the closed disk of radius r centered at 0, and let $C(\Delta_r)$ ($H(\Delta_r)$, resp. $A(\Delta_r)$) denote the spaces of continuous functions on Δ_r (which are harmonic, resp. analytic) on its interior. These are Banach spaces in the supremum norm. Let U denote an open subset of the complex plane such that the Julia set of P_c is contained in Δ_r whenever $c \in U$. Then $c \to \mu_c$ defines a map of U into the dual of any of these Banach spaces. We have skteched the proof of:

Proposition 2.1 As c varies in U, μ_c varies continuously in the strong topology on $H^*(\Delta_r)$ (and hence on $A^*(\Delta_r)$).

Concretely, this means that when c and c' are close enough together, $|\mu_c(f) - \mu_{c'}(f)| < \varepsilon$ for all harmonic f with $\|f\|_\infty \leq 1$.

It is interesting that the pairing of μ_c with polynomials can be described very concretely. Let $P(x) = P_c(x) = x^2 - c$, and let

$$S_n^k(c) = 2^{-n} \sum_{P^n(x)=0} x^k$$

Then for fixed k, $S_n^k(c) \to \mu_c(x^k)$ as $n \to \infty$.

Proposition 2.2 $S_n^k(c) = S_m^k(c)$ for all n and $m \geq \log_2 k$. The stable value $S^k(c) = \mu_c(x^k)$ satisfies

$S^0(c) = 1$;

$S^k(c) = 0$ if k is odd; and

$$S^{2k}(c) = \sum_{m=0}^{k} \binom{k}{m} S^m(c)\, c^{k-m}.$$

(These formulas give an explicit recursive description of $S^k(c)$.)

The proof is to observe that

$$S_n^{2k}(c) = 2^{-n} \sum_{P^n(x)=0} x^{2k} = 2^{-(n-1)} \sum_{P^{n-1}(y)=0} (y+c)^k$$

To get weak continuity with respect to continuous functions, it is enough to understand the pairing between μ_c and potentials (since every continuous function is uniformly near the sum of a harmonic function and the convolution of $\log|z-w|$ with a C^∞ function.) Here we use the description of μ_c as the equilibrium measure. Namely, μ_c has the property that its convolution with the harmonic kernel $\log|z-w|$ produces the Green's function G_c for the unbounded component of the complement of the Julia set. G_c is the unique harmonic function which looks like $\log|z|$ near infinity and is constant on the Julia set. Since the Julia set of a monic polynomial has capacity 1 (see [B]), the Green's function in fact vanishes on the Julia set. (For a discussion of capacity, equilibrium distribution and the Green's function, see Ahlfors [A]).

Now let f be the potential of a smooth charge distribution; i.e. f is obtained as the convolution $f(z) = \log|z-w| * \phi(w)$ where ϕ is a C^∞ function with compact support. By changing the order of integration we obtain

Proposition 2.3 The following identity holds:

$$\int f(z)\, d\mu_c(z) \quad = \quad \int G_c(w)\phi(w)\, dA(w)$$

where dA denotes two dimensional Lebesgue measure.

It is not too hard to check that G_c is a continuous fuction of c and w. This was done (independently) by Douady and Hubbard [DH]). Thus we have

Corollary 2.4 As c varies in U, μ_c varies continuously in the weak* topology on $C^*(\Delta_r)$.

Concretely, $\mu_c(f)$ is a continuous function of c for any fixed continuous function f. To prove this, approximate f by a C^∞ function with compact support, and write the smooth funciton as the sum of a harmonic function and the convolution of $\log|z-w|$ with Δf. Then it is enough to check that the pairing with the potential defines a continuous function of c, and by 2.2 this follows from continuity of G_c as a function of c.

§3. What About Rational Functions?

We now make a digression and discuss some ideas in families of rational maps. We do not prove any general continuity results for the maximal measure of a rational map, and these results will point out the stumbling blocks to one natural program of proof.

Recall that the Julia set of a rational map R is the set of z such that the iterations $\langle R^n(z)\rangle$ do not form a normal family at z. We can give a similar description of the bifurcation set of a family R_λ depending holomorphically on a parameter λ which varies in a complex manifold. Here the family is stable at a point λ if there is a bound on the period of the attracting cycles of maps appearing in a neighborhood of λ; otherwise λ is a bifurcation point.

Let us assume (for simplicity) that the critical points of R_λ are labelled by holomorphically varying functions $c_1(\lambda), \ldots, c_n(\lambda)$.

Proposotion 3.1 The bifurcation set of the family R_λ is exactly the set of parameter values λ such that the functions

$$\{c_i(\lambda),\ R_\lambda(c_i(\lambda)),\ R_\lambda{}^2(c_i(\lambda)),\ \ldots\ ;\ \text{for } i = 1,2,\ldots n\}$$

do not form a normal family.

Example The Mandelbrot set is the set of c such that $\{P_c^n(0) : n = 0,1,2,...\}$ does not form a normal family.

How might we try to prove continuity of the maximal measure in general? One way is to try to find a point x_λ, varying holomorphically in λ, such that its backward orbit under R_λ can be labelled by injective branches of the inverse iterates of R_λ, all defined on a fixed neighborhood of our original point λ_0. Then the Koebe distortion theorem gives control on the motion of these approximations to the maximal measure. This argument mimics one in Ljubic.

To make all these branches injective, we just have to assure that x_λ avoids the forward orbit of the critical points of R_λ. When can we choose a holomorphically varying point avoiding these forward orbits? It is easy to see that one such point can be used to cook up three such points, from which it follows easily (by Montel) that the forward orbits of the critical points are a normal family and the original λ was a stable point of the family. Thus it is not possible to carry out this procedure on the bifurcation set -- for example, on the boundary of the Mandelbrot set.

Corollary 3.2 Let c_0 be any point in the boundary of the Mandelbrot set. Then the forward orbits of the critical point fill up the complex plane as c varies through any neighborhood of c_0.

Perhaps, however (this is also part of Ljubic's argument), only a small percentage of the inverse branches are spoiled by critical points. Suppose, for example, we take pre-images of zero in the quadratic family. Then the nth pre-image meets the critical point again iff the critical point is periodic with period dividing n.

Proposition 3.3 Let U be an open set in the c-plane, and let p_n denote the number of values of $c \in U$ for which P_c has a periodic critical point with period n. Then $p_n/2^n \to \mu(U)$ where μ is the equilibrium measure on the boundary of the Mandelbrot set.

This says that the percentage of spoiled branches tends to zero iff U is contained entirely in the set of stable values of c. The proof uses the Douady-Hubbard theory of exterior angles on the Mandelbrot set. Thus the equilibrium measure on the Mandelbrot set gives the asymptotic distribution of c such that P_c has a superattracting cycle.

On the other hand, this shows that the attempted argument for continuity of the maximal measure breaks down already for quadratic polynomials -- even though the result is true for this family!

§4 Julia Sets as Signatures

We now use the theory of expanding conformal dynamical systems to the following:

Proposition 4.1 For $c \neq c'$ in the principal cardioid of the Mandelbrot set, the measures μ_c and $\mu_{c'}$ are mutually singular.

Here the principal cardiod (we'll denote it by C) is the set of values of c such that P_c has an attracting fixed point; and two measures are mutually singular if each one lives on a nullset of the other. Thus any two measures are distance 2 apart in the strong topology on the dual of $C(\mathbb{P}^1)$, so μ_c varies extremely discontinuously in this topology.

The result itself is perhaps less interesting than the theory of conformal shapes it motivates.

We sketch the proof. For $c \in C$, the Julia set J_c of P_c is a quasicircle and P_c is expanding on the Julia set. By the distortion lemma, any miniscule 'subsegment' of J_c can be blown up to some uniform size by a conformal map (an iterate of P_c) with bounded distortion of distances. That is, the blowing-up map is the composition of a quasi-isometry and a conformal dilation. The bound on the distortion is independent of the starting size.

Now suppose μ_c and $\mu_{c'}$ are not mutually singular. Then one of the Julia sets, say $J_{c'}$, meets J_c in a set of positive measure with respect to μ_c. We will show that this implies that the curve $J_{c'}$ follows the curve J_c very closely somewhere. We may have to look at a very small piece of the two curves to see this, but by using the expanding property described above we can apply the dynamics on the two curves to blow the picture up to uniform size. Looking more closely, continuing to blow up and passing to a limit, we conformally embed a neighborhood of a point in J_c into a neighborhood of a point in $J_{c'}$. Thus the 'conformal shapes' of some patch of these two curves agree. Now a patch is almost as good as the whole curve because of self-similarity. Using the dynamics, we show that this embedding is actually part of a conjugacy. The conjugacy, once spread out by the dynamics, becomes a conformal map on the whole sphere and hence a Möbius transformation. Then c and c' must be equal.

We now fill in some details. The maximal measure μ_c for $c \in C$ is the same as the push-forward of Lebesgue measure on S^1 via the uniformizing map carrying the unit disk Δ onto the unbounded component of the complement of the Julia set. When normalized to send 0 to ∞ and 1 to the repelling fixed point of P_c, this map is actually a conjugacy between P_c and $z \to z^2$. Since J_c is a quasicircle, the map $S^1 \to J_c$ is actually the restriction of a quasiconformal map.

To make a general statement, we define a quasiangular measure to be the pushforward of the angular measure on S^1 under a quasiconformal map. A point e in a set E contained in a metric space X is a point of geometric density if for any ϵ there is a radius r such that E comes within ϵr of every point in B(e,r), the ball of radius r about e.

Proposition 4.2 Let e be a point of geometric density of a set E in S^1. Let f be a quasiconformal map. Then f(e) is a point of geometric density of f(E) in the quasicircle $f(S^1)$.

Corollary 4.3 A set of positive quasiangular measure contains points of geometric density in the corresponding quasicircle. In particular, a set of positive measure for μ_c contains points of geometric density in J_c.

The proof of 4.2 uses the fact that a quasiconformal map cannot take evenly spaced points and unevenly spread them out (because, for instance, cross-ratios are not overly distorted.) Then the usual Lebesgue density theorem shows sets of angular positive measure have points of geometric density on S^1, so the corollary is an easy consequence.

Note, for example, that the usual Cantor set has no points of geometric density. Thus 4.2 gives another proof that the quasisymmetric image of the usual Cantor set always has linear measure 0. (There are other Cantor sets of dim < 1 which can be expanded to full measure by quasisymmetric functions. [M])

Another example of a quasiangular measures is provided by the Hausdorff measure μ_δ, where $\delta = \dim(J_c)$. This follows from the distortion lemma, which implies for $x \in J_c$, $\mu_\delta(B(x,r)) \approx r^\delta$.

Proposition 4.4 If two expanding quasicircles meet in a set of postive quasiangular measure with respect to either, then there exists a conformal map of a neighborhood of a subsegment of one to a subsegment of the other.

The proof is to look at a point of geometric density of the intersection. Then along a subsegment, the two quasicircles have to meet very frequently. Since they have bounded turning, this implies they actually follow each other quite closely. We can blow this picture up to uniform size by the dynamics on each curve. Put differently, we obtain a conformal map with domain and image of size bounded below, and bounded distortion, which nearly carries one subsegment to another. Looking at points of higher and higher density, we obtain a family of such maps, which is normal because the distortion is bounded. We pass to a subsequence such that the domain and range are converging, and the limiting map is that claimed above.

Proposition 4.5 For J_c and $J_{c'}$ expanding quasicircles (i.e. c and c' in the principal cardiod), there is a conformal map of a subsegment of one to a subsegment of another iff c=c'.

This shows, in a fairly strong sense, that the dynamics can be recovered from the Julia set -- the latter is a signature for the polynomial.

The idea of the proof it to use the conformal map to transport the self-similarities of J_c to those of $J_{c'}$. Either the similarities match up -- in which case the map establishes a conjugacy between the dynamics -- or they generate a very generous group of similarities, and we show that the only shape with that many conformal similarities is a real-analytic arc. In this case it is known that J_c is real-analytic iff J_c is a circle (in which case c=0). In fact, the existence of a map between subsegments easily implies that the Hausdorff dimensions of J_c and $J_{c'}$ agree; and it is known that the dimension of J_c is one iff c=0; so the argument so far is already enough to prove that μ_c moves discontinuously (in the strong topology) at c=0. Indeed this discussion establishes:

Proposition 4.6 If $c\neq 0$, J_c meets any real analytic arc in a set of linear measure zero and measure zero with respect to any quasiangular measure on J_c.

This is an example of the 'transversality' we are describing. Another consequence is the following curious result.

Corollary 4.7 Let $\pi : J_c \to \mathbb{R}$ be the linear projection of a Julia set onto the real axis, and let ν be the push-forward of the any quasiangular measure on J_c (such as the maximal measure or the Hausdorff measure). Then ν has a continuous density on the image of π.

This just says that the fibers of π have zero measure (so ν has no atoms). But the fiber is the intersection of J_c with a line.

Problem How smooth is ν?

It should be possible, modulo some rather forbidding combinatorics, to make a sensible statement like 4.5 in the context of general expanding conformal dynamical systems (not neccessarily even coming from a rational map.) The combinatorics disappears when the endomorphism is replaced by a group action, and we can make such a statement for expanding (i.e. convex cocompact) Kleinian groups.

non-Fuchsian

Proposition 4.8 Let Γ_0 and Γ_1 be convex cocompact Kleinian groups, and let ϕ be a conformal map of a patch of the limit set of one to a patch of the other. Then Γ_0 and Γ_1 are commensurable, ϕ is the restriction of a Möbius transformation M and M establishes a conjugacy between two appropriate subgroups of finite index.

To prove 4.6 we use some very special properties of P_c : the uniformizing map on the outside of J_c and the primality of 2, for instance. The idea is to use the uniformizing maps and lift the conformal map between subsegments to a real-analytic map between subsegments of S^1. Then it is not hard to see that this analytic map is affine ($\theta \to a\theta + b$). We transport the dynamics of $\theta \to 2\theta$ via this map, and study how it interacts with the original dynamics. These maps on S^1 all descend to conformal similarities between patches of the Julia sets and in fact the circle is used mostly to clarify the combinatorics.

As mentioned above, the similarities cannot be too rich unless the Julia set is a circle, and this places strong enough restrictions on a and b to prove that the maps are conjugate.

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