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Equivariant higher algebra

Natalie Stewart

A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

in the subject of Mathematics

Harvard University
Cambridge, MA
April 2026

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Equivariant higher algebra

Abstract. In this thesis, we set out foundations for a homotopy-coherently parameterized algebra suited towards equivariant homotopy theory and algebraic topology. In particular, the first chapter after the introduction introduces parameterized $(\infty, 2)$-category theory and a theory of parameterized algebraic patterns lifting the theory of Chu-Haugseng; intuivitvely, for base of parameterization the orbit ∞-category $\mathcal{O}_G$, a G-$(\infty, 2)$-category is an ∞-category enriched in ∞-categories enriched in G-spaces (i.e. enriched in G-∞-categories) and $\mathcal{O}_G$-parameterized algebraic patterns are G-∞-categorical objects which functorially present G-$(\infty, 2)$-categories of generalized G-operads and G-symmetric monoidal ∞-categories via the fibrational perspective. We present a Morita theory for such parameterized patterns lifting the approximation theory of Lurie (and Barkan, Barkan-Haugseng-Steinebrunner) and apply it to identify many models for parameterized operads, compatibly with the corresponding theories of parameterized symmetric monoidal categories (considered as normed algebras in ∞-categories).

In the second chapter, we apply this to develop some rudiments of equivariant higher algebra. In particular, we center the roles of *(co)cartesian G-symmetric monoidal structures*, *parameterized Segal-object models for algebras, Boardman-Vogt tensor products/homotopy-coherent interchange*, and *parameterized distributivity*. At the end of the day, we use these to understand a parameterized version of Lurie, Gepner-Groth-Nikolaus, Harpaz, and Carmeli-Schlank-Yanovski's theory of *modes*, giving existence and uniqueness of a presentable G-symmetric monoidal structure on G-spectra when G is a compact Lie group (with unit $\Sigma^\infty S^0$) incorporating finite-index Hill-Hopkins-Ravanel norms.

In the final chapter, we use these developments to perform two important arguments regarding interchange: we prove a parameterized version of Schlank-Yanovski's *∞-categorical Eckmann-Hilton arguments* (realizing a mild generalization of Blumberg-Hill's $\mathcal{N}_\infty$-operads precisely as the G-operads arising from infinitely iterated Eckmann-Hilton arguments), and we lift Dunn-Lurie additivity to little disk G-operads for finite G, as well as versions with tangential structure specified by an arbitrary map of G-spaces $B \to B_G O(n)$.

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Acknowledgements

I'd first like to thank my advisor, Mike Hopkins, for so much mathematical and personal insight over the years. Of course, it takes a village to raise a mathematician; I must also thank Jeremy Hahn and Andy Senger for playing a crucial part in my mathematical development, as role-models and as mentors.

In addition to them, I have a laudry list of people who I must thank for discussions or collaborations surrounding this material. First and foremost, large chunks of Section I.3.3 were written in collaboration with Kaif Hilman and Branko Juran, who I have to thank for many helpful conversations about other material in this thesis. Likewise I have been blessed to have informative conversations on these topics with Clark Barwick, Andrew Blumberg, Mike Hill, Dhilan Lahoti, Piotr Pstrągowski, Maxime Ramzi, and Jan Steinebrunner, among others.

Outside of explicit conversations, this work stands on the shoulders of giants, logically and intuitively; it would be impossible without crucial insights present in the work of Rune Haugsengeng and Hongyi Chu, Shaul Barkan and Jan Steinebrunner, Kaif Hilman, and the collaborative works of Clark Barwick, Emanuele Dotto, Saul Glasman, Denis Nardin, and Jay Shah.

None of this would be possible without the support and friendship of many within the community of young mathematicians within Cambridge, MA. Within that group, I would especially like to thank Amanda, August, Dhilan, Dylan, Elliot, Eunice, Grant, Gwyn, Howard, Isabel, Katia, Keeley, Kush, Matt, Niven, Ollie, Merrick, Rafael, Rushil, Thibault, Tyler, and Wyatt. I can't imagine the last 5 years without yall.

Outside of this, I've been blessed to be supported throughout grad school by my community, friends, and loved ones. Online, I'd like to thank James, Luna, Irony, Snakey, Andrea, Aden, Alex, and Nully for all of the conversations and support over the years. I'm also grateful to my parents for all of the support for the last 27 years, and for raising me to be the person that I am today.

To Madison, Tara, and Tali, it'd be hard to overstate the positive impact each of you have had on me, and I'm endlessly grateful. To Willow, thanks for always being there and for all of the years of hanging out. To Liz and Baz, June and Kepler, Nabeel, Eve, Ingrid: you all have been my bedrock, and I owe you so much, even outside of splitwise.

Chapter 0

Introduction

A motivating conjecture from chromatic homotopy theory

Let MU denote the spectrum representing *complex bordism*; that is MU represents a homology theory such that, for X a smooth manifold, $\mathrm{MU}_k(X) \simeq \pi_k(\mathrm{MU} \otimes \Sigma^\infty_+ X)$ is naturally equivalent to the Abelian group of smooth maps $M \to X$ with complex structure on their stable normal bundle modulo structure-preserving bordism. Unsurprisingly, the conjugation C_2-action on stable normal bundles lifts to a C_2-action on MU itself; we refer to $\mathrm{MU}_{\mathbb{R}}$ as the associated C_2-spectrum, representing the associated C_2-equivariant homology theory. Each object is multiplicative in a strict way, and yields a strict commutative algebra object in suitably chosen model categories of spectra and C_2-spectra.

Through the advent of chromatic homotopy theory, the "rings" MU and $\mathrm{MU}_{\mathbb{R}}$ have become a crucial object within stable homotopy theory; roughly, the observation is that the associated graded diagram $\mathrm{MU}_*\mathrm{MU} \rightrightarrows \mathrm{MU}_*$ presents the moduli stack of (1-dimensional, commutative) formal groups, providing a lift of $\mathrm{MU}_*(-)$ to a cohomology theory valued in quasicoherent sheaves over $\mathcal{M}_{f.g.}$. It turns out that the associated *descent spectral sequence* over this presentation of $\mathcal{M}_{f.g.}$ (the Adams-Novikov spectral sequence) ends up converging to stable homotopy (see [Hop99]), so knowledge of $\mathrm{MU}_*(-)$ may be lifted to knowledge of $\pi^{\mathrm{st}}_*(-)$. There is a corresponding theory of *Real orientations* and *Real Adams-Novikov spectral sequences* for $\mathrm{MU}_{\mathbb{R}}$ [Ara79; HK01].

Now, through the usual *arithmetic fracture square*, we may in principle recover all of this from what occurs after localization to each prime. Here, a key observation is that, after choosing parameters, every p-local formal group law admits a *p-typical* parameterization. On the spectral side, this corresponds with a splitting of $\mathrm{MU}_{(p)}$ to various suspensions of a p-local (and $\oplus$-indecomposable) spectrum BP; in particular, there is a split monomorphism $BP \to \mathrm{MU}_{(p)}$ which on homotopy represents p-typicalization of p-local formal group *laws*.

Chromatic homotopy theory then arises from the observation that the *height* stratification of p-typical formal group laws may be lifted to a sequence of spectra

$$BP \to \cdots \to E(n+1) \to E(n) \to \cdots,$$

with the corresponding Bousfield localizations presenting $\mathrm{Sp}_{(p)}$ as a sequence of *recollement situations* with quotients having extraordinarily pleasant properties.[1] These spectra $E(n)$ arise by taking localizations of a sequence of quotients

$$BP = BP\langle\infty\rangle \to \cdots \to BP\langle n+1\rangle \to BP\langle n\rangle \to \cdots,$$

so one recovers a version of the Adams-Novikov spectral sequence by studying modules over $BP\langle n\rangle_* BP\langle n\rangle$-comodules.

A question naturally arises: *may the modules-and-sheaves program be lifted to $BP\langle n\rangle$-modules?* Now, for this, $BP\langle n\rangle$ must have either a strict commutative algebra structure or its homotopy-coherent analog, an $\mathbb{E}_k$-algebra structure for some $k \in [2,\infty]$. The former constructs the latter for $k = \infty$, so we may focus on constructing an $\mathbb{E}_k$-algebra structure; unfortunately, for $n \gg 0$, $BP\langle n\rangle$ is known to not even admit an $\mathbb{E}_{2(p^2+2)}$-structure [Law18; Sen22].

Luckily, it does attain *some structure*; Basterra-Mandell construct an essentially unique $\mathbb{E}_4$-structure on BP [BM13], and Hahn-Wilson an $\mathbb{E}_3$-BP-algebra form of $BP\langle n\rangle$ [HW22], which is where the nonequivariant story lands. The latter has been useful e.g. in the recent disproof of the telescope conjecture [BHLS23].

Now¸ while extending the universal property of MU in terms of *complex orientations* to one of $\mathrm{MU}_{\mathbb{R}}$ for *Real orientations*, Araki constructed a C_2-equivariant lift of the Quillen idempotent [Ara79], the resulting summand $BP_{\mathbb{R}} \subset \mathrm{MU}_{\mathbb{R}}$ playing an analogous role to $BP_{\mathbb{R}}$ in a "Real-oriented" chromatic homotopy theory of C_2-spectra [HK01].

Moving forwards, $BP_{\mathbb{R}}$ itself admits a series of truncations

$$BP_{\mathbb{R}} = BP_{\mathbb{R}}\langle\infty\rangle \to \cdots \to BP_{\mathbb{R}}\langle n+1\rangle \to BP_{\mathbb{R}}\langle n\rangle \to \cdots,$$

lifting the nonequivariant truncations, which are invaluable in chromatic homotopy theory. Unfortunately, their multiplicative nature is quite mysterious; for instance, $BP_{\mathbb{R}}\langle n\rangle^{C_2}$ is not even known to have a *homotopy associative* multiplication, though it's conjectured to admit an $\mathbb{E}_1$-structure [HW22, Rmk. 1.0.14]. Indeed, $BP_{\mathbb{R}}\langle n\rangle$ is obstructed from admitting any construction of any C_2-equivariant multiplicative structure yielding an $\mathbb{E}_{2(p^2+2)}$-structure on underlying spectra, so it certainly can not be made strictly commutative, or $\mathbb{E}_\infty$. We arrive at a question.

[1] These quotients are the $K(n)$-*local categories*; they are pleasant e.g. in that they admit only vanishing tate cohomology for finite groups, and furthermore are ∞-*semiadditive*.

Question. *What multiplicative structure does $BP_{\mathbb{R}}$ attain? What about $BP_{\mathbb{R}}\langle n\rangle$?*

Experts share conjectural answers to this question, with a genuine-equivariant theory of little disk algebras. That is, work of Hauschild [Hau80] and Rourke-Sanderson [RS00], repackaged by Guillou-May [GM17], and modernized and generalized by Juran [Jur25a] identify preimages of a pointed G-space under $\Omega^V(-) \simeq \mathrm{map}_*(S^V -)$ with suitably grouplike algebra structures over a so-called *G-operad* $\mathbb{E}_V$, where V is an orthogonal G-representation and S^V its one-point compactification; $\mathrm{MU}_{\mathbb{R}}$ is the Thom C_2-spectrum of a $\Omega^{\rho\infty}$-map, so in a sensible theory, it would attain an $\mathbb{E}_{\rho\infty}$-algebra structure with respect to a canonical "C_2-symmetric monoidal" structure on C_2-spectra. From here, the conjecture often shared by experts is the following.

Conjecture. *$BP_{\mathbb{R}}$ admits an essentially unique $\mathbb{E}_{2\rho}$-$\mathrm{MU}_{\mathbb{R}}$-algebra structure, and there exists an $\mathbb{E}_{2\rho-1}$-$BP_{\mathbb{R}}$-algebra form of $BP_{\mathbb{R}}\langle n\rangle$.*

Work on this thesis began with the author attempting to prove this conjecture; this work is still in progress, now jointly with Christian Carrick, Mike Hill, and Ryan Quinn.

Now, the lion's share of work on $\mathbb{E}_k$-algebras, including the construction of the $\mathbb{E}_k$-structures on BP and $BP\langle n\rangle$, eventually compiles down to two fundamental observations:

(i) There exists a natural equivalence $\mathrm{Alg}_{\mathbb{E}_{k+1}}(-) \simeq \mathrm{Alg}_{\mathbb{E}_1}\mathrm{Alg}_{\mathbb{E}_k}(-)$, and

(ii) $\mathbb{E}_1$ has n-ary structure space which is a torsor over Σ_n; equivalently, its free algebras are tensor algebras, generalizing the 1-categorical associative situation.

This is miraculous; it can be said that $\mathbb{E}_k$ is complicated because it admits interesting power operations (i.e. its n-ary structure space has homotopically interesting homotopy Σ_n-orbits), but these together reduce study of $\mathbb{E}_k$-algebras to a case with no interesting power operations. This is by far the hardest part of establishing the higher Hochschild (co)homology formula for $\mathbb{E}_k$-centers and enveloping algebras of [HA], which is the fundamental technical aspect of constructing the $\mathbb{E}_3$-structure on $BP\langle n\rangle$ in [HW22]; moreover, it may be used to reduce the construction of the *iterated bar construction* presentation of $\mathbb{E}_k$-derived indecomposables/cotangent complexes which is fundamental to [BM13] (as well as the Galatius–Kupers–Randall-Williams $\mathbb{E}_k$-cellularity regime for homological stability [GKR23]).

Thus, to approach this conjecture, it may help to first answer the following question.

Question. *Does there exist a theory of G-equivariant higher algebra wherein*

(i) Sp_G admits a canonical "G-symmetric monoidal structure" such that Thom G-spectra of V-fold loop spaces attain functorial $\mathbb{E}_V$-algebras structures, and

(ii) there exists a natural equivalence $\mathrm{Alg}_{\mathbb{E}_V}\mathrm{Alg}_{\mathbb{E}_W}(-) \simeq \mathrm{Alg}_{\mathbb{E}_{V\oplus W}}(-)$, all terms suitably interpreted?

In this thesis, we will answer Statement (i) in the affirmative when G is a compact Lie group, and Statement (ii) in the affirmative in the case that G is a finite group; this suffices to study $BP_{\mathbb{R}}\langle n\rangle$. The author intends to approach Statement (ii) in the more general case of compact Lie groups in future work.

Now, an $\mathbb{E}_V$-space X is given, for all closed subgroups $H \subset G$ and finite H-sets $S \in \mathsf{Fin}_H$, actions

$$\mu_S \colon \mathrm{Conf}_S^H(V) \simeq \mathrm{Emb}^{H,V-\mathrm{Fr}}\left(\coprod_{[H/K]\subset S} H \times_K D(V), D(V)\right) \longrightarrow \mathrm{Map}^H\left(\prod_{[H/K]\subset S} \mathrm{CoInd}_K^H X, X\right),$$

compatible with restriction, action of the H-equivariant automorphism group of S, and "composition" of equivariant little disk embeddings; here, the leftmost space is the space of H-equivariant embeddings of S into V under the usual compact-open topology, the middle the space of pairs of an H-equivariant embedding (under the C^∞-topology) and homotopies making the distinguished trivializations of their H-equivariant vector bundles compatible, and on the righthand side $\mathbf{CoInd}_K^H X$ is the right adjoint to restriction.

For instance, by taking fixed points, an $\mathbb{E}_V$-space X receives a family of transfers

$$\left(\mathrm{Res}_H^G V\right)_{(K)} \longrightarrow \mathrm{Map}\left(X^K, X^H\right)$$

parameterized by the K-isotropy points in the restriction of V to H, for all finite-index inclusions $K \subset H$ between closed subgroups of G. This space $\left(\mathrm{Res}_H^G V\right)_{(K)}$ is an interesting complement of central real hyperspaces; the (co)homology of fixed points of $\mathbb{E}_V$-spaces correspondingly acquires an action of an interesting Orlik-Solomon algebra, i.e. an interesting collection of equivariant Dyer-Lashof operations. Moreover, each X^H receives an $\mathbb{E}_{\dim V^H}$-space structure, making this structure even richer.

The reader would be justified at worrying about Statements (i) and (ii) at this point. Among other things, it involves the notions of *restriction* and *coinduction*, so it may only be applied to inputs sophisticated enough to parameterize those. In G-spaces, at least, coinduction is right adjoint to restriction; however, the functors $\Sigma_+^\infty \colon \mathcal{S}_H \to \mathrm{Sp}_H$ *do not intertwine the right adjoints to restriction.* Instead, it takes coinduction to the *Hill-Hopkins-Ravanel norm* of [HHR16], which is neither a left adjoint nor a right adjoint, and is essentially only defined by this property, sifted continuity, and symmetric monoidality.

As a consequence, we will need heavy machinery; we will need a notion of *families of categories* (one for each conjugacy class of subgroups), connected contravariantly along morphisms subgroup inclusions to describe objects; we will need symmetric monoidal structures on the value of each orbit, with compatible *covariant* norms along some class of "admissible" subgroup inclusions to describe algebras. Following pioneering work of Hill-Hopkins [HH16], succeeded by Barwick-Dotto-Glasman-Nardin-Shah [BDGNS16a] and many others, we will accomplish this by working in a carefully constructed form of *parameterized higher category theory* and *parameterized higher algebra.*

Parameterized category theory

To understand our parameterization, recall Elmendorf's theorem [Elm83]; a modern variant of it proves that the restricted Yoneda embedding

$$\mathcal{S}_G \hookrightarrow \mathrm{Fun}\left(\mathcal{S}_G^{\mathrm{op}}, \mathcal{S}\right) \xrightarrow{\iota^*} \mathrm{Fun}\left(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}\right)$$

is an equivalence of ∞-categories, where $\mathcal{S}_G$ denotes G-spaces (i.e. topological spaces with G-action, after inverting equivariant homotopy), and $\mathcal{O}_G \subset \mathcal{S}_G$ is the full subcategory spanned by transitive G-spaces, called the *orbit category*. Noting the equivalence $\mathrm{Map}^G([G/H], X) \simeq X^H$ with *point-set* fixed points, this can be said to recover the fixed points of X, connected by restriction maps and conjugation actions, and no more.

It is easy to show that this yields an equivalence

$$\mathrm{CSeg}(\mathcal{S}_G) \simeq \mathrm{Fun}\left(\mathcal{O}_G^{\mathrm{op}}, \mathrm{CSeg}(\mathcal{S})\right) \simeq \mathrm{Fun}\left(\mathcal{O}_G^{\mathrm{op}}, \mathrm{Cat}\right) \simeq \mathrm{Cat}^{\mathrm{cocart}}_{/\mathcal{O}_G^{\mathrm{op}}}$$

between complete Segal objects in G-spaces and presheaves of ∞-categories on the orbit category, or equivalently, cocartesian fibrations to the opposite orbit category. The value on or fiber over the orbit $[G/H]$ may be understood as the "category of H-equivariant objects," connected by conjugation actions and restriction functors; we will henceforth abusively refer to these conjugation actions themselves as "restrictions," thinking of restrictions as going along *subconjugacy relations*.

A canonical example of this is the restricted self-indexing

$$\underline{\mathcal{S}}_G \colon \mathcal{O}_G^{\mathrm{op}} \hookrightarrow \mathcal{S}_G^{\mathrm{op}} \xrightarrow{\mathcal{S}_{G,/(-)}} \mathrm{Cat}.$$

Taking fibers over the identity coset underlies an equivalence $\mathcal{S}_{G,[G/H]} \xrightarrow{\sim} \mathcal{S}_H$ (inverse to the topological induction $X \mapsto G \times_H X$), under which the pullback along $[G/H] \to [G/K]$ indeed implements restriction from H-spaces to K-spaces.

We eventually want to describe algebraic operations specified by *finite H-sets*, so we point out the "full G-subcategory" (i.e. restriction-stable family of full subcategories) $\underline{\mathrm{Fin}}_G \subset \underline{\mathcal{S}}_G$ specified by the value

$$\underline{\mathrm{Fin}}_G^H := \mathrm{Fin}_H \subset \mathcal{S}_H \simeq \underline{\mathcal{S}}_G^H.$$

This admits an interpretation from the perspective of self-indexing: the unstraightening $\mathrm{Tot}\,\underline{\mathcal{S}}_G \subset \mathrm{Ar}(\mathcal{S}_G)$ is the full subcategory with transitive domain, and $\mathrm{Tot}\,\underline{\mathrm{Fin}}_G \subset \mathrm{Ar}(\mathcal{S}_G)$ the further subcategory whose maps are finite disjoint unions of orbit maps with finite fibers.

For the purpose of genuineness over families, proper equivariance, global equivariance, or any number of other parameterized areas of topology admitting an analogy to the equivariant setting, we will boldly replace $\mathcal{S}_G$ with a sufficiently similar ∞-topos:

Definition. If $\mathcal{T}$ is an ∞-category, then $\mathcal{R} \subset \mathcal{T}$ is an *orbital subcategory* if the collection of disjoint unions of $\mathcal{R}$-maps $\coprod_i U_i \to V$ in $\mathcal{S}_\mathcal{T} \coloneqq \mathsf{Psh}(\mathcal{T})$ is closed under base-change along morphisms of representable preseheaves. We call $\mathcal{R}$ *atomic* if it consists of 0-truncated maps when interpreted as a subcategory of in $\mathcal{S}_\mathcal{T}$ ◁

We will generally refer to representable presheaves as *orbits*, and we will define $\underline{\mathcal{S}}_\mathcal{T}$, $\mathsf{Cat}_\mathcal{T}$, restriction, etc. similarly. We will write $\underline{\mathbb{F}}_\mathcal{R}$ for the analog of $\underline{\mathsf{Fin}}_\mathcal{T}$.

Atomicity may seem mysterious, but it is quite a basic feature in the equivariant setting: it is equivalent to the diagonal $V \to V \times_U V$ being a monomorphism (equivalently, summand inclusion, as both are finite disjoint unions of representables). This plays the role of the *identity double coset*, ensuring that it has appropriate isotropy. It turns out that every ∞-category $\mathcal{T}$ admits a maximal atomic orbital subcategory $\mathfrak{o}(\mathcal{T})$, so as long as we set out to *internalize* the notion $\underline{\mathbb{F}}_\mathcal{R}$ and other notions that come along the way in the maximal case, we will usually safely push to the case $\mathcal{R} = \mathfrak{o}(\mathcal{T})$. $\mathfrak{o}(\mathcal{O}_G)$ turns out to correspond with orbit maps with finite fibers, so we safely write $\underline{\mathsf{Fin}}_\mathcal{T} \coloneqq \underline{\mathbb{F}}_{\mathfrak{o}(\mathcal{T})}$.

Now, we automatically acquire an internal notion of induction and coinduction: they are the left and right adjoint to restriction. More generally, we define a notion of *parameterized colimits and limits* in terms of left and right adjoints between parameterized functor categories. For instance, under the inclusion $\mathcal{S}_\mathcal{T} \hookrightarrow \mathsf{Cat}_\mathcal{T}$ induced by $\mathcal{S} \hookrightarrow \mathsf{Cat}$, this causes all cocomplete $\mathcal{T}$-∞-categories to essentially be "tensored" over $\mathcal{S}_\mathcal{T}$, by taking constant $\mathcal{T}$-colimits.

This parameterized (co)limit notion allows us to define myriad internal versions of pointedness, (semi)additivity, stability, etc. In particular, we acquire a $\mathcal{T}$-pointing $\underline{\mathsf{Fin}}_{\mathcal{T},*}$ of $\underline{\mathsf{Fin}}_\mathcal{T}$, whose V-value turns out to just be the pointed objects $\mathsf{Fin}_{\mathcal{T}_{/V},*}$ for all $V \in \mathcal{T}$, functorially under restriction. Most interestingly, we acquire a G-∞-category $\underline{\mathsf{Sp}}_G$ of G-spectra by "formally inverting" finite-dimensional representation spheres in the formal pointing $\underline{\mathcal{S}}_{G,*}$, i.e. in the G-∞-category of G-spaces.

This yields an internal analog to Segal's category Γ, gotten by taking the fiberwise-opposite of $\underline{\mathsf{Fin}}_{\mathcal{T},*}$. We can correspondingly define internal notions of *Segal objects*. Following Chu-Haugseng [CH21], we do so via a parameterized version of their theory of *algebraic patterns*.

Parameterized algebraic patterns, parameterized operads

We define a $\mathcal{T}$*-algebraic pattern* to be a $\mathcal{T}$-∞-category $\mathfrak{B}$, together with a $\mathcal{T}$-factorization system $\left(\mathfrak{B}^{\mathrm{int}}, \mathfrak{B}^{\mathrm{act}}\right)$ of *parameterized inert and active maps*, and a full $\mathcal{T}$-subcategory $\mathfrak{B}^{\mathrm{el}} \subset \mathfrak{B}^{\mathrm{int}}$ of *elementary objects*. We call a $\mathcal{T}$-functor $\mathfrak{B} \to \mathcal{C}$ a *Segal* $\mathfrak{B}$*-object* if its restriction to $\mathfrak{B}^{\mathrm{int}}$ is $\mathcal{T}$-right Kan extended from $\mathfrak{B}^{\mathrm{el}}$.

Of course, $\underline{\mathsf{Fin}}_{\mathcal{T},*}$ admits a $\mathcal{T}$-algebraic pattern structure analogous to Fin_*; viewing its V-fiber as a span category on $\mathsf{Fin}_{\mathcal{T}_{/V}}$ whose backwards maps are summand inclusions and forward morphisms arbitrary, the backwards V-morphisms are inert and forwards active, and the contractible V-object $*_V$ is elementary. It turns out that the Segal objects for this structure are precisely the $\mathfrak{o}(\mathcal{T})$*-semiadditive* $\mathcal{T}$*-functors* $\underline{\mathsf{Fin}}_{\mathcal{T}} \to \mathcal{C}$ in the sense of [CLL23], in direct analogy to the non-parameterized case. In the case $\mathcal{T} = \mathcal{O}_G$ for G finite, these agree with the construction shown to be equivalent to *Mackey functors* first in [Nar16], so we have defined an interesting and useful parameterized algebraic structure. But a question arises once more:

Question. *Can we describe* $\mathbb{E}_V$*-algebras in* G*-spectra using* G*-algebraic patterns?*

The short answer is that Segal objects are not rich enough to describe this theory; indeed, we have not even constructed Hill-Hopkins-Ravanel norms yet. The long answer is more interesting.

We may take inspiration from the nonparameterized picture, from Lurie (who was clearly influenced greatly by May-Thomason). Symmetric monoidal ∞-categories, as commutative monoids in Cat, are Segal ∞-categories over Fin_*, so they unstraighten to cocartesian fibrations $\mathcal{C}^{\otimes} \to \mathsf{Fin}_*$ satisfying a *Segal condition*. Operads (possibly with multiple colors) also admit a mirror here, as fibrations to Fin_* admitting cocartesian lifts for *inert morphisms* and a pair of custom-fit Segal conditions which themselves are described in terms of inert morphisms and elementary objects. To wit, $\mathcal{O}$-algebras in $\mathcal{C}^{\otimes}$ are identified with colored operad maps to $\mathcal{C}^{\otimes}$, i.e. functors over Fin_* preserving inert-cocartesian lifts.

Following this, Chu-Haugseng defined a notion of *weak Segal fibration* to an algebraic pattern encapsulating this [CH21]; this was later riffed upon under the name *fibrous patterns* [BHS22] and recently under the name *algebrads* [BLR26].[2] We define a parameterized version of this, creatively dubbed *fibrous patterns*. As a formal corollary to the work of [BHS22], we find that these admit an *envelope adjunction* extending [HA], giving a translation of these ideas into maps between Segal ∞-categories at the end of the day.

In the special case of $\underline{\mathsf{Fin}}_{\mathcal{T},*}$, we refer to Segal $\underline{\mathsf{Fin}}_{\mathcal{T},*}$-$\infty$-categories as $\mathcal{T}$*-symmetric monoidal* ∞*-categories* and fibrous $\underline{\mathsf{Fin}}_{\mathcal{T},*}$-patterns as $\mathcal{T}$*-operads*. This matches [NS22] when applicable. In particular, we are led to a definition of G*-operads* and G*-symmetric monoidal* ∞*-categories*. We will find that G-symmetric monoidal ∞-categories have exactly our desired functoriality, with norms along finite-index subconjugacy relations.

[2] The author likes the name *algebrad* quite a bit, but the aformentioned paper was released too late for her to adopt its conventions in this thesis.

We will eventually exhibit the space of suitably well-behaved G-symmetric monoidal ∞-categories of G-spectra to be contractible; in particular, finite-index Hill-Hopkins-Ravanel norms and coherences making them compatible will come essentially uniquely. We first study the primordial ooze of equivariant topology.

Equivariant manifolds and little disks

When G is a compact Lie group, we have a preferred setting for G-equivariant topology: *smooth G-manifolds*. We'll henceforth assume all G-manifolds smooth, i.e. we'll consider smooth manifolds with a distinguished action of a compact Lie group G through diffeomorphisms, and equivariant maps between them.

Following in the examples of factorization homology and embedding calculus, one expects *C^∞-spaces of embeddings between G-manifolds* to play an outsized role in algebraic topology. We define a G-symmetric monoidal ∞-category $\underline{\mathrm{Mfld}}_G$ whose H-objects are H-manifolds, whose H-equivariant mapping spaces are C^∞-embedding spaces (under the strong C^∞-topology), whose tensor products are disjoint unions, and whose norms take $M \mapsto H \times_K M$. This admits full G-symmetric monoidal subcategories

$$\underline{\mathrm{Disk}}^n_G \subset \underline{\mathrm{Mfld}}^n_G \subset \underline{\mathrm{Mfld}}_G$$

spanned by n-dimensional representation disks and n-dimensional manifolds, respectively.

More generally, given a G-space map $\Theta\colon B_\Theta \to B_G O(n)$, we define G-symmetric monoidal ∞-categories $\underline{\mathrm{Disk}}^\Theta_G \subset \underline{\mathrm{Mfld}}^\Theta_G$ over $\underline{\mathrm{Mfld}}^n_G$ whose H-objects consist of H-manifolds M together with factorizations of their maps $M \to B_H O(n)$ classifying their H-equivariant tangent bundles through $\mathrm{Res}^G_H \Theta$. We show that $\underline{\mathrm{Disk}}^\Theta_G$ arises as the G-symmetric monoidal envelope of an (essentially unique) G-operad $\mathbb{E}_\Theta$. In particular, in the case that B_Θ is contractible and $\Theta\colon * \to B_G O(n)$ classifies the representation V as an equivariant vector bundle over the point, we acquire a G-symmetric monoidal ∞-category of V-framed G-manifolds and a G-operad $\mathbb{E}_V$; its "structure spaces" are V-framed little V-disk embedding spaces, which turn out to be equivalent to configuration spaces in V.

Taking one-point compactifications of the underlying spaces (with their contravariant functoriality along embeddings) yields a *cocartesian $\mathbb{E}_V$-coalgebra* structure on $S^V \in \underline{\mathcal{S}}_{G,*}$; we find a corresponding cartesian $\mathbb{E}_V$ structure on $\Omega^V X \simeq \mathrm{Map}_*\left(S^V, X\right)$. One correspondingly expects these structures to be abundant.

In the case that G is finite, this is retreading known ground; as a consequence, we already know there is a recognition principle for $\mathbb{E}_V$-spaces whose H-fixed points are grouplike whenever $\dim V^H > 0$ [GM17; Hau80; Jur25a; RS00]. We know in that case that we absolutely have the right structure.

From here, we may turn towards additivity: we define a closed *Boardman-Vogt* symmetric monoidal structure on $\mathcal{T}$-operads (satisfying a similar uniqueness result to [BS24a]), which allows us to prove the following, as the final theorem of this thesis.

Theorem. *If G is a finite group, then there exists a natural equivalence $\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V\oplus W}$, i.e.*

$$\mathrm{Alg}_{\mathbb{E}_V}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_W}(-) \simeq \mathrm{Alg}_{\mathbb{E}_{V\oplus W}}(-),$$

where $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(-)$ is the Boardman-Vogt internal hom G-operad structure, i.e. the pointwise structure.

We more generally construct equivalences $\mathbb{E}_\Theta \otimes \mathbb{E}_\Psi$ and $\underline{\mathrm{Disk}}^{\Theta}_G \circledast \underline{\mathrm{Disk}}^{\Psi}_G \simeq \underline{\mathrm{Disk}}^{\Theta\oplus\Psi}_G$, where $\circledast$ is the tensor product of G-symmetric monoidal ∞-categories; we will identify G-symmetric monoidal ∞-categories with Mackey functors valued in ∞-categories in the case that G is finite, in which case this is the box product structure. This is done by a *disintegration and assembly procedure* extending [HA], which recognizes $\mathbb{E}_\Theta$ and $\underline{\mathrm{Disk}}^{\Theta}_G$ as B_Θ-indexed G-colimits of the fully-framed objects (with framing at a point given by the tangent representation).

Having accomplished Statement (ii), we're left with the structure on G-spectra and Thom spectra of V-fold loop spaces. Unlike the finite group case, this had not yet been done before this thesis, so we have the opportunity to take such a construction seriously. Following [HA], we define it via a version of *mode theory*.

Parameterized modes

[Hil24; MW22] yield parameterized versions of *presentable ∞-categories*: a $\mathcal{T}$-∞-category is presentable if it is $\mathcal{T}$-cocomplete and there exists a regular cardinal κ such that it factors as

$$\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Pr}^L_\kappa \longrightarrow \widehat{\mathrm{Cat}}.$$

We refer to the $\mathcal{T}$-∞-category of presentable $\mathcal{T}$-∞-categories as $\underline{\mathrm{Pr}}^L_{\mathcal{T}}$.

Here is where we, as Nardin did [Nar17], find ourselves at a subtlety: following the nonequivariant case, we would like to define a $\mathcal{T}$-symmetric monoidal structure on $\underline{\mathrm{Pr}}^L_{\mathcal{T}}$ whose $\mathcal{O}$-algebras can fairly be called "presentably $\mathcal{O}$-monoidal ∞-categories." Nonequivariantly, these are the $\mathcal{O}$-monoidal ∞-categories $\mathcal{C}$ whose underlying ∞-category is presentable and such that $\mathcal{O}$ acts $\mathcal{C}$ through multi-functors which preserve colimits separately in each variable. Equivariantly, the latter condition can't be so simple: Hill-Hopkins-Ravanel norms do not commute with all colimits, only sifted colimits.

Following Nardin [Nar17], we spend significant time developing such a structure, through what we call *distributive $\mathcal{T}$-multifunctors*. We show that its algebras agree with the distributive normed ∞-categories described in [EH23; LLP25] in the parameterized case, so long as they have presentable underlying $\mathcal{T}$-∞-categories. Moreover, we lift Lurie's identification of smashing localizations and idempotent algebras to *$\mathcal{R}$-symmetric monoidal smashing localizations* and *idempotent $\mathcal{R}$-commutative algebras* when $\mathcal{R}$ is an atomic orbital subcategory (i.e. transfer system), exhibiting either emptiness or contractibility of such structures

extending a given idempotent $\mathbb{E}_0$-algebra. As a consequence, we attain a theory of $\mathcal{R}$-*modes*, as those idempotent $\mathcal{R}$-commutative algebras in $\underline{\mathrm{Pr}}^L_{\mathcal{T}}$.

We show that the parameterized formal inversion procedure of [Cno23; Rob15] is as parameterized-multiplicative as those objects you invert; in particular, we find that $\underline{\mathsf{Sp}}_G$ is a G-mode.

We also independently develop parameterized Day convolution on this level of generality (noting that [NS22] does not cover compact Lie groups). We use this to construct the structure on Thom spectra of $\mathbb{E}_V$-maps, hence of V-fold loop maps, finally fulfilling Statement (i).

Other things

The reader should be informed that this introduction has omitted many technical aspects that will be sticking points throughout this thesis; many of them are explained in the introductions of the papers which the author wrote and generalized into this thesis including [Ste24; Ste25a; Ste25b; Ste25c; Ste26] and some so-far unpublished work. In addition to this, we have not mentioned it, but much of the work in parameterized higher algebra will be realized on the level of a functor from $\mathcal{T}$-algebraic patterns to right-adjoint functors between $\mathcal{T}$-$(\infty,2)$-categories; a theme throughout the text is that we lift Barkan and Barkan-Haugseng-Steinebrunner's lift [Bar23a] of Lurie's approximation theory to identify morphisms of $\mathcal{T}$-algebraic patterns which become equivalences of pairs of $\mathcal{T}$-∞-categories, called *strong Morita equivalences*, in terms of the category theory of "active arrow $\mathcal{T}$-∞-categories;" this and universal properties of (co)cartesian structures comprise the main category theoretic difficulties of the proof of equivariant Dunn additivity.

Notation and conventions

Throughout this thesis, we assert an $\omega+2$-indexed collection of Grothendieck universes

$$\kappa_0 \subsetneq \kappa_1 \subsetneq \cdots \subsetneq \kappa_\omega \subsetneq \kappa_{\omega+1}.$$

We refer to κ_0-small objects (sets, spaces, categories, etc.) as *small*, κ_1-small objects as *large*, κ_2-small objects as *extremely large*, κ_n-small objects for finite n as *non-massive*, κ_ω-small objects as *massive*, and $\kappa_{\omega+1}$-small objects as *extremely massive*. We moreover set the following size-related notation:

- Cat is the large ∞-category of small ∞-categories,
- $\widehat{\mathsf{Cat}}$ is the extremely large ∞-category of large ∞-categories, and
- $\widetilde{\mathsf{Cat}}$ is the massive ∞-category of non-massive ∞-categories, and
- $\widehat{\widetilde{\mathsf{Cat}}}$ is the extremely massive ∞-category of massive ∞-categories.

We will use similar notation for set theoretically small, large, etc. objects in other contexts.

All subcategories will be assumed replete, including wide subcategories. More generally, we will attempt to speak the language of ∞-categories *model-independently*, assuming an ambient well-behaved (extremely massive) ∞-category of κ_i-small ∞-categories. For concreteness, this is incarnated in [HTT] and [kerodon].

Even more generally, we will assume a reasonably well behaved theory of $(\infty,2)$-categories, satisfying e.g. the axioms of [BS20]; the unicity and nonzero number of examples exhibited in [BS20] ensure that we do not lose anything by refusing to pick a model. We will generally refer to particular $(\infty,2)$-categories with all-capital roman font (such as CAT), and arbitrary ones with blackboard-bold (such as $\mathbb{C}$); particular ∞-categories will be the usual roman font and arbitrary ones caligraphic (such as Cat and $\mathcal{C}$), and particular topological categories bold-face (such as **Top**).

For parameterized objects, when it helps avoid confusion, we will underline them; for instance, $\underline{\mathsf{Cat}}_{\mathcal{T}}$ is the $\mathcal{T}$-∞-category of small $\mathcal{T}$-∞-categories, and $\underline{\mathrm{CAT}}_{\mathcal{T}}$ the $\mathcal{T}$-$(\infty,2)$-category of them.

We will generally write algebraic patterns and $\mathcal{T}$-algebraic patterns with fraktur font (such as $\mathfrak{C}$), and we will decorate $\mathcal{T}$-operads and $\mathcal{T}$-symmetric monoidal ∞-categories with a tensor superscript (such as $\mathcal{C}^{\otimes}$).

Chapter I

Parameterized category theory and Segal conditions

In this section, we exposit and expand upon a version of ∞-category theory which has recently gained traction within equivariant homotopy theory. To motivate this, consider the following.

In nonequivariant higher category theory, there is a colocalizing (and localizing) full subcategory inclusion $\mathcal{S} \subset \mathsf{Cat}$ of *spaces* into ∞-*categories*; its essential image consists of those ∞-categories whose morphisms are all equivalences, i.e. our theory satisfies Grothendieck's homotopy hypothesis. Given an ∞-category $\mathcal{C}$ and space X, we then acquire an equivalence of spaces

$$\mathsf{Fun}(X,\mathcal{C})^{\simeq} \simeq \mathsf{Map}(X,\mathcal{C}^{\simeq});$$

in particular, *local systems of $\mathcal{C}$-objects* are represented by the moduli space $\mathcal{C}^{\simeq}$.

Many moduli spaces of interest in algebraic topology appear in this way; for instance, if $\mathsf{Vect}^{f.d.}$ denotes the ∞-category of finite dimensional real vector spaces and linear isometric embeddings (say, arising from the usual topological category, with the spaces underlying the usual manifold structure on hom spaces), choosing bases yields an equivalence

$$\mathsf{Vect}^{f.d.,\simeq} \xrightarrow{\sim} \coprod_{i\in\mathcal{R}} B\mathrm{GL}_n(\mathbb{R}).$$

Now, given G a compact Lie group, the *equivariant homeomorphism classes* of genuine G-equivariant real vector bundles occur as path components in the mapping space $\mathsf{Map}_G(Y, \coprod_{i\in\mathcal{R}} B_G\mathrm{GL}_n(\mathbb{R}))$. Working the above idea in reverse, we may posit the following.

Hypothesis. *There exists a localizing and colocalizing full subcategory inclusion from $\mathcal{S}_G$ into an ∞-category of "G-∞-categories," so that $B_G\mathrm{GL}_n(\mathbb{R})$ is the core of finite-dimensional real representations, and moreover the various objects appearing in equivariant algebraic topology have G-categorical universal properties.*

Now, Elmendorf's theorem gives us a hint: as G-spaces are equivalent to presheaves of spaces on the orbit category, perhaps G-∞-categories must be equivalent to presheaves of ∞-*categories* on the orbit category.

From this perspective, we may probe our example a bit further: the G-space $B_G\mathrm{GL}_n(\mathbb{R})$ is assembled from its H-fixed points for all closed subgroups $H \subset G$, together with restriction maps associated with each

subconjugacy relation. In fact, we may interpret this via the equivalence

$$\left(\coprod_{n\in\mathcal{R}} B_G \mathrm{GL}_n(\mathbb{R})\right)^H \simeq \mathrm{Map}^H\left(*_H, \coprod_{n\in\mathcal{R}} B_H \mathrm{GL}_n(\mathbb{R})\right)$$
$$\simeq \mathbf{Rep}^{H,f.d.}(\mathbb{R})^{\simeq};$$

the analog of H-fixed points of a G-∞-category are its *H-equivariant objects*. Indeed, a G-∞-category will consist of an H-equivariant ∞-category for all closed $H \subset G$, connected by restriction functors, and a G-functor will simply be an H-equivariant functor for all H which coherently intertwines restrictions.

This perspective is greatly influenced by pioneering work of Hill-Hopkins [HH16] and ensuing work of Nardin-Shah along with Barwick-Dotto-Glasman [BDGNS16a; BDGNS16b; Nar16; Nar17; NS22; Sha22; Sha23], which aimed to create a homotopical theory encapsulating the *multiplicative* aspects of equivariant stable homotopy theory that Hill-Hopkins had developed with Ravanel in their solution to the Kervaire Invariant One problem away from dimension 126.[1]

In this endeavor, additional data is needed to encapsulate the tensor product and Hill-Hopkins-Ravanel norm, which were together to form a *G-symmetric monodial ∞-category*; following Hill-Hopkins once more, Nardin-Shah developed such a theory for finite groups, showing it to be equivalent to Hill-Hopkins' hopeful model of *(semi)-Mackey functors of ∞-categories* [NS22]. None of this turned out to depend so strongly on the group; rather, it only required that the orbit category $\mathcal{O}_G$ could be modified into the effective Burnside category $\mathsf{Span}(\mathbb{F}_G)$, so Barwick-Dotto-Glasman-Nardin-Shah elected to work directly with such categories, called *orbital categories*.

An obvious snag appears: *the category of finite disjoint unions of G-orbits does not admit a span category if G is positive-dimensional, as it does not admit pullbacks.* The best one may say is that arbitrary morphisms in $\mathcal{O}_G$ admit well-behaved pullbacks along *finite index closed subcategory inclusions*. A similar snag makes the global homotopy theory of [Sch18] only *almost* accessible from the orbital setting.

Seeing this, Cnossen-Lenz-Linskens developed a framework generalizing the inclusion $\mathcal{O}_G^{\mathrm{f.i.}} \subset \mathcal{O}_G$, called *atomic orbital subcategories*, which can handle a higher algebra of G-equivariant homotopy admitting HHR norms along finite-index closed subgroup inclusions [CLL24].

In another direction, Nardin-Shah furthered the study of equivariant higher algebra by developing *$\mathcal{T}$-operads* whenever $\mathcal{T}$ is an atomic orbital ∞-category [NS22], though their framework notably compiled down often to *equivariant* ∞-category theory. In an attempt to simplify this, Barkan-Haugseng-Steinebrunner

[1] There were some historical antecedents to this, for instance as far back as tom Dieck's construction of the *discrete fundamental group category* [Die87], which is easy to see as the result of applying the Grothendieck construction to the G-category whose H-equivariant category is the fundamental group of the H-fixed point space.

cast these objects into a much more lightweight framework of *algebraic patterns* in [BHS22], enabling more modern higher-categorical approaches to be applied.

In this section, we summarize the aspects of these developments which we need, as well as develop some key new work. Indeed, in Section I.1, we review the parameterized category theory, as well as develop some rudimentary equivariant $(\infty,2)$-category theory, as many of our G-categories of interest will appear as functor G-categories therein, and many of our boilerplate equivalences of G-categories boil down to adjunctions of G-$(\infty,2)$-categories. Additionally, we develop terminology roughly equivalent to Cnossen-Haugseng-Lenz-Linsken's *weakly extensive categories* [CHLL24b] and generalizing Blumberg-Hill's *indexing systems* [BH15], as well as some necessary rudiments of the corresponding theory of (incomplete) G-commutative monoids.

Moving on, in Section I.2, we review the theory of algebraic patterns, relate it to Lurie's theory of categorical patterns, and express the variants of Chu-Barkan-Haugseng-Steinebrunner's and Lurie's results which we will need throughout this text.

Additionally, in Section I.3, we define a theory of *parameterized algebraic patterns* over an atomic orbital subcategory, as it is needed in order to develop a necessary form of Barkan's pattern Morita theory and Lurie's operad approximation theory; we develop a fair bit of this Morita and approximation theory for parameterized patterns, as well as rudiments of several of the constructions of [HA, § 2] on this level of generality.

Finally, in Section I.4, we define and characterize the algebraic patterns which we will find foundationally useful to equivariant higher algebra, mostly as variations on Segal's Γ-category and the effective Burnside category.

I.1 Parameterized and equivariant category theory

We begin by working on parameterized $(\infty,1)$- and $(\infty,2)$-category theory, with *no assumptions* on the base, in Sections I.1.1 and I.1.2. Following this, in Section I.1.3 we define our categorical model for the finite-index and general orbit categories, called *atomic orbital subcategories*. Generalizing this somewhat, in Section I.1.4 we introduce *weak indexing systems*.

I.1.1 Recollections of parameterized ∞-category theory

We start with the basics.

I.1.1.1 The $\mathcal{T}$-∞-category of small $\mathcal{T}$-∞-categories

We make the following definition.

Definition I.1.1. The *∞-category of small $\mathcal{T}$-∞-categories* is

$$\mathsf{Cat}_{\mathcal{T}} \coloneqq \mathsf{Fun}(\mathcal{T}^{\mathrm{op}}, \mathsf{Cat}),$$

where Cat is the ∞-category of small ∞-categories. ◁

Notation I.1.2. Fix $\mathcal{C} \in \mathrm{Cat}_{\mathcal{T}}$. We refer to the value of $\mathcal{C}$ at $V \in \mathcal{T}^{\mathrm{op}}$ as the *V-value category of $\mathcal{C}$*, written as $\mathcal{C}^V$; given $f\colon V \to W$, we refer to the associated functor as *restriction*

$$\mathrm{Res}_V^W\colon \mathcal{C}^W \to \mathcal{C}^V. \qquad \triangleleft$$

Remark I.1.3. The Grothendieck construction, imported to ∞-category theory as the straightening unstraightening equivalence in [HTT, Thm. 3.2.0.1], produces an equivalence

$$\mathrm{Cat}_{\mathcal{T}} \simeq \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}}^{\mathrm{cocart}},$$

the latter denoting the (non-full) subcategory of $\mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}}$ whose objects are cocartesian fibrations and whose morphisms are functors over $\mathcal{T}^{\mathrm{op}}$ which preserve cocartesian arrows. Under this identification, the fiber of $\mathrm{Tot}\,\mathcal{C} \to \mathcal{T}^{\mathrm{op}}$ over V is identified with the V-value $\mathcal{C}^V$ and the restriction functors are identified with cocartesian transport, where Tot denotes the total ∞-category of the unstraightening. $\triangleleft$

Given $\mathcal{C}, \mathcal{D}$ a pair of $\mathcal{T}$-∞-categories, we may define the *$\mathcal{T}$-functor category* to be the full subcategory

$$\mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}) \coloneqq \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}^{\mathrm{cocart}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}_{/\mathcal{T}^{\mathrm{op}}}(\mathcal{C}, \mathcal{D})$$

consisting of functors over $\mathcal{T}^{\mathrm{op}}$ which preserve cocartesian lifts of the structure maps.

Example I.1.4. For any object $V \in \mathcal{T}$, the forgetful functor $(\mathcal{T}_{/V})^{\mathrm{op}} \to \mathcal{T}^{\mathrm{op}}$ is a cocartesian fibration classified by the representable presheaf $\mathrm{Map}_{\mathcal{T}}(-, V)$. We refer to the associated $\mathcal{T}$-∞-category as $\underline{V}$. This is covariantly functorial in V, since postcomposition yields functors $f_!\colon \mathcal{T}_{/V} \to \mathcal{T}_{/W}$ for all maps $f\colon V \to W$. $\triangleleft$

These play an important role in equivariant higher category theory.

Notation I.1.5. Given $\mathcal{C}$ a $\mathcal{T}$-∞-category, we define the *restricted* $\mathcal{T}_{/V}$-category by

$$\mathrm{Res}_V^{\mathcal{T}} \coloneqq \mathcal{C}^{\underline{V}} \coloneqq \mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} (\mathcal{T}_{/V})^{\mathrm{op}}. \qquad \triangleleft$$

Proposition I.1.6 ([BDGNS16b, Thm 9.7])**.** $\mathrm{Cat}_{\mathcal{T}}$ *has exponential objects* $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$ *classified by the functor*

$$V \mapsto \mathrm{Fun}_{\mathcal{T}_{/V}}\left(\mathcal{C}_{\underline{V}}, \mathcal{D}_{\underline{V}}\right).$$

We refer to monomorphisms[2] in $\mathrm{Cat}_{\mathcal{T}}$ as *$\mathcal{T}$-subcategories*, and $\mathcal{T}$-functors which are fiberwise-fully faithful as *full $\mathcal{T}$-subcategories*, or *$\mathcal{T}$-fully faithful functors*.

[2] Following [HTT, § 5.5.6], we refer to a morphism $X \to Y$ in $\mathcal{C}$ as a *monomorphism* if the canonical map $X \to X \times_Y X$ is an equivalence, or equivalently, if the pullback functor $f^*\colon \mathcal{C}_{/Y} \to \mathcal{C}_{/Y}$ is fully faithful.

Observation I.1.7. By the fiberwise expression for limits in functor categories (c.f. [HTT, Cor. 5.1.2.3]), a $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-subcategory inclusion if and only if $F^V\colon \mathcal{C}^V \to \mathcal{D}^V$ is a subcategory inclusion for all $V \in \mathcal{T}$. ◁

Definition I.1.8. Let $\mathcal{T}$ be an ∞-category. Then, a full subcategory $\mathcal{F} \subset \mathcal{T}$ is a *$\mathcal{T}$-family* if whenever $V \in \mathcal{F}$ and $W \to V$ is a map, we have $W \in \mathcal{F}$.[3] The poset of $\mathcal{T}$-families under inclusion is denoted $\mathsf{Fam}_{\mathcal{T}}$. ◁

Example I.1.9. The terminal $\mathcal{T}$-∞-category $\underline{*}_{\mathcal{T}}$ is classified by the constant functor $V \mapsto *$. The poset of *subterminal objects in* $\mathsf{Cat}_{\mathcal{T}}$ (i.e. monomorphisms with codomain $\underline{*}_{\mathcal{T}}$) is isomorphic to $\mathsf{Fam}_{\mathcal{T}}$; the $\mathcal{T}$-∞-category $\underline{*}_{\mathcal{F}}$ associated with $\mathcal{F}$ is determined by the values

$$\underline{*}_{\mathcal{F},V} \simeq \begin{cases} * & V \in \mathcal{F}; \\ \varnothing & \text{otherwise.} \end{cases}$$

In fact, the "∞-groupoid" inclusion $\mathcal{S} \hookrightarrow \mathsf{Cat}$ induces an inclusion $\mathcal{S}_{\mathcal{T}} \hookrightarrow \mathsf{Cat}_{\mathcal{T}}$ sending the universal space $E\mathcal{F}$ to $\underline{*}_{\mathcal{F}}$. ◁

The ∞-category $\mathsf{Cat}_{\mathcal{T}}$ participates in an adjunction

$$\mathsf{Tot}\colon \mathsf{Cat}_{\mathcal{T}} \rightleftarrows \mathsf{Cat}\colon \underline{\mathsf{Coeff}}^{\mathcal{T}}$$

whose left adjoint Tot is the total category of cocartesian fibrations, and whose right adjoint has V-value

$$(\underline{\mathsf{Coeff}}^{\mathcal{T}}\mathcal{C})^V \simeq \mathsf{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{C}\right)$$

where the functoriality on f is given by $(f_!)^*$ [BDGNS16b, Thm 7.8]. We refer to $\underline{\mathsf{Coeff}}^{\mathcal{T}}$ as the *$\mathcal{T}$-∞-category of coefficient systems in* $\mathcal{C}$.[4]

Example I.1.10. There is an equivalence $\underline{*}_{\mathcal{T}} = \underline{\mathsf{Coeff}}^{\mathcal{T}} * \in \mathsf{Cat}_{\mathcal{T}}$ since right adjoints preserve terminal objects. ◁

We may additionally construct the *associated ∞-category*

$$\Gamma^{\mathcal{T}}\mathcal{C} \coloneqq \mathsf{Fun}_{\mathcal{T}}(\underline{*}, \mathcal{C}),$$

whose objects consist of cocartesian sections of the structure functor $\mathcal{C} \to \mathcal{T}^{\mathrm{op}}$. We refer to this as the *∞-category of $\mathcal{T}$-objects in* $\mathcal{C}$. For instance, if $\mathcal{T}$ has a terminal object V, [BDGNS16b, Lemma 2.12] shows

[3] These are named *families* after subconjugacy closed families of subgroups, which frequently occur in equivariant homotopy; these are referred to as *sieves* in [BH15; NS22] and *upwards-closed subcategories* in [Gla17].

[4] These are referred to as the *cofree* parameterization $\underline{\mathsf{CoFree}}(\mathcal{C})$ in [Hil24] and as the *$\mathcal{T}$-∞-category of $\mathcal{T}$-objects* $\underline{\mathcal{C}}_{\mathcal{T}}$ in [Nar17]. We avoid the former for clarity (as we do not view Tot as a forgetful functor), and we avoid the latter as it conflicts with the $\mathcal{T}$-∞-category of $\mathcal{T}$-spectra $\underline{\mathrm{Sp}}_{\mathcal{T}}$; instead, our name is chosen to evoke the *coefficient systems* used in equivariant cohomology.

that we have an equivalence

$$\Gamma^{\mathcal{T}}\mathcal{C} \simeq \mathcal{C}^{V};$$

more generally, this implies that $\Gamma^{\mathcal{T}}\mathcal{C} \simeq \lim_{V\in\mathcal{T}^{\mathrm{op}}} \mathcal{C}^{V}$, i.e. it is the $\mathcal{T}$*-fixed points* (or the limit of $\mathcal{C}$ viewed as a $\mathcal{T}^{\mathrm{op}}$ functor). Defining the $\mathcal{T}$*-inflation* to have V-values

$$\left(\mathrm{Infl}_{e}^{\mathcal{T}}\mathcal{D}\right)^{V} \coloneqq \mathcal{D}$$

for any $\mathcal{D}\in\mathrm{Cat}$ and $V\in\mathcal{T}$, the adjunction between limits and diagonals immediately yields the following.

Proposition I.1.11. *The functor* $\mathrm{Infl}_{e}^{\mathcal{T}} : \mathrm{Cat} \to \mathrm{Cat}_{\mathcal{T}}$ *is left adjoint to* $\Gamma^{\mathcal{T}} : \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}$.

Using this adjunction, given $\mathcal{C}\in\mathrm{Cat}$, we define the ∞-category

$$\mathrm{Coeff}^{\mathcal{T}}\mathcal{C} \coloneqq \Gamma^{\mathcal{T}}\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C} \simeq \mathrm{Fun}(\mathcal{T}^{\mathrm{op}},\mathcal{C});$$

then, we have $\mathrm{Cat}_{\mathcal{T}} = \mathrm{Coeff}^{\mathcal{T}}\mathrm{Cat}$, and Elmendorf's theorem states that $\mathcal{S}_{G} \simeq \mathrm{Coeff}^{\mathcal{O}_{G}}\mathcal{S}$, motivating the following.

Definition I.1.12. The $\mathcal{T}$*-*∞*-category of small* $\mathcal{T}$*-*∞*-categories* is $\underline{\mathrm{Cat}}_{\mathcal{T}} \coloneqq \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathrm{Cat})$; the $\mathcal{T}$-∞-category of $\mathcal{T}$-spaces is $\underline{\mathcal{S}}_{\mathcal{T}} \coloneqq \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{T})$, and the ∞-category of $\mathcal{T}$-spaces is $\mathcal{S}_{\mathcal{T}} \coloneqq \mathrm{Coeff}^{\mathcal{T}}(\mathcal{S}) \simeq \Gamma^{\mathcal{T}}\underline{\mathcal{S}}_{\mathcal{T}}$. ◁

Observation I.1.13. The V*-value of* $\underline{\mathrm{Cat}}_{\mathcal{T}}$ is $\left(\underline{\mathrm{Cat}}_{\mathcal{T}}\right)_{V} = \mathrm{Cat}_{\mathcal{T}_{/V}}$; we henceforth refer to this as Cat_{V}. The restriction functor $\mathrm{Res}_{V}^{W} : \mathrm{Cat}_{W} \to \mathrm{Cat}_{V}$ is presented from the perspective of cocartesian fibrations by the pullback

$$\begin{array}{ccc} \mathrm{Res}_{W}^{V}\mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow \\ (\mathcal{T}_{/V})^{\mathrm{op}} & \longrightarrow & (\mathcal{T}_{/W})^{\mathrm{op}} \end{array}$$

In particular, given a map $U \to V \to W$, abusively referring to $(U \to V)\in\mathcal{T}_{/V}$ as U, this is characterized by the formula

$$\left(\mathrm{Res}_{W}^{V}\mathcal{C}\right)^{U} \simeq \mathcal{C}^{U}.$$ ◁

I.1.1.2 Join, slice, and (co)limits We now summarize some elements of [Sha22; Sha23].

Definition I.1.14 ([Sha23, Def 4.1])**.** Let $\iota\colon \mathcal{T}^{\mathrm{op}}\times\partial\Delta^{1} \hookrightarrow \mathcal{T}^{\mathrm{op}}\times\Delta^{1}$ be the evident inclusion. Then, the $\mathcal{T}$*-join* is the top horizontal functor

$$\begin{array}{ccccc} \mathrm{Cat}_{\mathcal{T}}^{2} & & \xrightarrow{\quad -\star_{\mathcal{T}}- \quad} & & \mathrm{Cat}_{\mathcal{T}} \\ \downarrow & & & & \downarrow \\ \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}\times\partial\Delta^{1}} & \xrightarrow{\iota^{*}} & \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}\times I} & \xrightarrow{\pi_{!}} & \mathrm{Cat}_{/\mathcal{T}^{\mathrm{op}}} \end{array}$$

which exists by [Sha22, Prop 4.3]. We write

$$K^{\triangleright} := K \star_{\mathcal{T}} \underline{*}_{\mathcal{T}};$$

$$K^{\triangleleft} := \underline{*}_{\mathcal{T}} \star_{\mathcal{T}} K. \qquad \triangleleft$$

Definition I.1.15. If $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{\mathcal{T},\mathcal{E}/}$ are $\mathcal{T}$-∞-categories under $\mathcal{E}$, the $\mathcal{T}$-∞-*category of* $\mathcal{T}$-*functors under* $\mathcal{E}$ is defined by the pullback of $\mathcal{T}$-∞-categories

$$\begin{array}{ccc} \underline{\mathrm{Fun}}_{\mathcal{T},\mathcal{E}/}(\mathcal{C},\mathcal{D}) & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\pi_{\mathcal{C}})^*} \\ \underline{*}_{\mathcal{T}} & \xrightarrow[\{\pi_{\mathcal{D}}\}]{} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{E},\mathcal{D}) \end{array}$$

The $\mathcal{T}$-∞-category of $\mathcal{T}$-functors over $\mathcal{E}$ is defined similarly. If $p\colon K \to \mathcal{C}$ is a $\mathcal{T}$-functor, then the $\mathcal{T}$-*undercategory and* $\mathcal{T}$-*overcategory* are the functor $\mathcal{T}$-∞-categories

$$\mathcal{C}^{p/} := \underline{\mathrm{Fun}}_{\mathcal{T},K/}(K^{\triangleright}, \mathcal{C});$$

$$\mathcal{C}^{/p} := \underline{\mathrm{Fun}}_{\mathcal{T},K/}(K^{\triangleleft}, \mathcal{C}) \qquad \triangleleft$$

In the case $p\colon \underline{*}_{\mathcal{T}} \to \mathcal{C}$ corresponds with the $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$, we simply write $\mathcal{C}^{X/} := \mathcal{C}^{p/}$ and similar for overcategories. In general, the categories $\mathcal{C}^{p/}$ take part in a functor out of $\mathrm{Cat}_{\mathcal{T},K/}$. Of fundamental importance is the adjoint relationship between these functors:

Theorem I.1.16 ([Sha23, Cor 4.27])**.** *The* $\mathcal{T}$*-join forms the left adjoint in a pair of adjunctions*

$$K \star_{\mathcal{T}} -\colon \mathrm{Cat}_{\mathcal{T}} \rightleftarrows \mathrm{Cat}_{\mathcal{T},K/} :(-)^{(-,\mathcal{T})/},$$

$$- \star_{\mathcal{T}} K\colon \mathrm{Cat}_{\mathcal{T}} \rightleftarrows \mathrm{Cat}_{\mathcal{T},K/} :(-)^{/(-,\mathcal{T})}.$$

We say a $\mathcal{T}$-functor $\underline{p}\colon K^{\triangleleft} \to \mathcal{C}$ extends $p\colon K \to \mathcal{C}$ if the composite $K \to K^{\triangleleft} \to \mathcal{C}$ is homotopic to p.

Definition I.1.17. Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category. A $\mathcal{T}$-object $X \in \Gamma^{\mathcal{T}}\mathcal{C}$ is *final* if for all $V \in \mathcal{T}$, the object $X_V \in \mathcal{C}^V$ is final; a $\mathcal{T}$-functor $\underline{p}\colon K^{\triangleleft} \to \mathcal{C}$ extending $p\colon K \to \mathcal{C}$ is a *limit diagram for* p if the corresponding cocartesian section $\sigma_{\underline{p}}\colon *_{\mathcal{T}} \to \mathcal{C}^{/(p,\mathcal{T})}$ is a final $\mathcal{T}$-object. $\triangleleft$

The *fiberwise opposite* (or vertical opposite) functor $\mathsf{op}\colon \mathrm{Cat}_{\mathcal{T}} \to \mathrm{Cat}_{\mathcal{T}}$ is the $\mathcal{T}$ functor induced under $\mathrm{Coeff}^{\mathcal{T}}$ by the *opposite category* functor $\mathsf{op}\colon \mathrm{Cat} \to \mathrm{Cat}$; the notions of initial $\mathcal{T}$-objects and $\mathcal{T}$-colimits are defined dually as final $\mathcal{T}$-objects and $\mathcal{T}$-limits in the fiberwise opposite.

In many cases, these are familiar; for instance, *trivially indexed* (co)limits are non-equivariant.

Proposition I.1.18 ([Sha22, Thm 8.6]). *A diagram $\underline{p}\colon \left(\mathrm{Infl}_e^{\mathcal{T}} K\right)^{\triangleleft} \to \mathcal{C}$ is a limit diagram for $p\colon \mathrm{Infl}_e^{\mathcal{T}} K \to \mathcal{C}$ if and only if for all V, the associated diagram $\underline{p}_V\colon K^{\triangleleft} \to \mathcal{C}_V$ is a limit diagram for p_V.*

Similarly, indexed (co)limits in coefficient systems may be converted into non-equivariant colimits.

Proposition I.1.19 ([Sha23, Prop 5.6-7]). *Let $\mathcal{T}$ be an atomic orbital ∞-category and $F\colon \mathcal{C} \to \underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D})$ a $\mathcal{T}$-functor. Then, the indexed limit and colimit of F have values computed by ordinary limits and colimits:*

$$(\underline{\mathrm{colim}} F)^V \simeq \mathrm{colim}\left(\mathcal{C}^V \to \mathrm{Tot}^V \mathcal{C} \to \mathrm{Coeff}^V(\mathcal{D}) \xrightarrow{(-)^V} \mathcal{D}\right);$$

$$(\underline{\lim} F)^V \simeq \lim\left(\mathrm{Tot}^V \mathcal{C} \to \mathrm{Coeff}^V(\mathcal{D}) \xrightarrow{(-)^V} \mathcal{D}\right).$$

Now, mirroring the nonequivariant story, cofinality is measured in $\mathcal{T}$-spaces, which is a category of coefficient systems. Thus, cofinality of a $\mathcal{T}$-functor may be converted into a nonequivariant statement using Proposition I.1.19, which turns out to be particularly simply.

Proposition I.1.20 ([Sha23, Thm. 6.7]). *Let $\mathcal{T}$ be an atomic orbital ∞-category and $F\colon \mathcal{C} \to \mathcal{D}$ a $\mathcal{T}$-functor. The following conditions are equivalent:*

(1) For all $\mathcal{T}$-colimit diagrams $\overline{p}\colon \mathcal{D}^{\triangleright} \to \mathcal{E}$, the composite functor $\overline{p} \circ F^{\triangleright}\colon \mathcal{C}^{\triangleright} \to \mathcal{E}$ is a $\mathcal{T}$-colimit diagram.

(2) For all $V \in \mathcal{T}$, the associated functor $F_V\colon \mathcal{C}_V \to \mathcal{D}_V$ is final.

For instance, the following corollary is immediate.

Corollary I.1.21. *If $\mathcal{K}$ has a terminal $\mathcal{T}$-object $*_\bullet$, then every diagram $\mathcal{K} \to \mathcal{C}$ has a $\mathcal{T}$-colimit, and a diagram $\overline{p}\colon \mathcal{K} \star_{\mathcal{T}} \{c_\bullet\} \to \mathcal{C}$ is a $\mathcal{T}$-colimit diagram if and only if, for all $V \in \mathcal{K}$, the V-map $\overline{p}(*_V) \to \overline{p}(c_V)$ is an equivalence.*

We also record a useful special interaction of $\mathcal{T}$-colimits and *initial* $\mathcal{T}$-objects.

Proposition I.1.22. *Suppose a diagram $p\colon \mathcal{K} \to \mathcal{C}$ admits an initial full $\mathcal{T}$-subcategory $\mathcal{L} \subset \mathcal{K}$ such $p|_{\mathcal{L}}$ is constant with value on initial V-objects. If $\mathcal{K}' \subset \mathcal{K}$ is a complementary full $\mathcal{T}$-subcategory to $\mathcal{L}$, then Then, an extension $\overline{p}\colon \mathcal{K} \to \mathcal{C}$ is a $\mathcal{T}$-colimit diagram if and only if its restriction to $\mathcal{K}'^{\triangleright}$ is a $\mathcal{T}$-colimit diagram.*

Proof. It suffices to note that the restriction $\mathcal{T}$-functor $\mathcal{C}^{p/} \to \mathcal{C}^{p|_{\mathcal{K}'}/}$ is an equivalence. □

Definition I.1.23. Let $\mathcal{C}$ be a $\mathcal{T}$-∞-category and let $\underline{\mathcal{K}}_{\mathcal{T}} = (\mathcal{K}_V)_{V \in \mathcal{T}} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ be a restriction-stable collection of V-categories. We say that $\mathcal{C}$ *strongly admits $\mathcal{K}$-shaped limits* if for each $V \in \mathcal{T}$, each V-category $K \in \mathcal{K}_V$ and each V-functor $p\colon K \to \mathcal{C}_{\underline{V}}$, there exists a limit diagram for p. We say $\mathcal{C}$ is *$\mathcal{T}$-complete* if it strongly admits $\underline{\mathrm{Cat}}_{\mathcal{T}}$-shaped limits.

If $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{T}$-∞-categories which strongly admit all $\mathcal{K}$-shaped limits and $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$, functor, we say F *strongly preserves $\mathcal{K}$-shaped limits* if for all $V \in \mathcal{T}$ and all $K \in \mathcal{K}_V$, postcomposition with the V-functor $F_{\underline{V}}\colon \mathcal{C}_{\underline{V}} \to \mathcal{D}_{\underline{V}}$ sends K-shaped limits diagrams to limits diagrams.

If $\mathcal{C} \subset \mathcal{D}$ is a full $\mathcal{T}$-subcategory whose inclusion strongly preserves $\mathcal{K}$-shaped limits, we say that $\mathcal{C}$ is *strongly closed under $\mathcal{K}$-shaped limits*. ◁

An important class of examples is *indexed (co)products*.

Definition I.1.24. Consider $S \in \mathbb{F}_V$, considered as a V-category under the inclusion $\mathsf{Set}_V \hookrightarrow \mathsf{Cat}_V$ extending the *representable V-category* functor $\mathcal{T}_{/V} \to \mathsf{Cat}_V$ via coproducts. Then, we refer to S-shaped V-limits as *S-indexed products* and S-shaped V-colimits as *S-indexed coproducts*.

If $\mathcal{C} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory, we refer to $\mathcal{T}$-colimits of the corresponding class as *$\mathcal{C}$-indexed coproducts*; similarly, following [Ste24], if $I \subset \mathsf{Set}_{\mathcal{T}}$ is a pullback-stable and core-full subcategory, we define the full $\mathcal{T}$-subcategory $\underline{\mathsf{Set}}_I \subset \underline{\mathsf{Set}}_{\mathcal{T}}$ of *I-admissible $\mathcal{T}$-sets* by

$$\left(\underline{\mathsf{Set}}_I\right)_V \coloneqq \mathsf{Set}_{I,V} \coloneqq \left\{S \;\middle|\; \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I\right\} \subset \mathsf{Set}_V.$$

We refer to the class of $\underline{\mathsf{Set}}_I$-indexed coproducts as *$I$-indexed coproducts*, and use the dual language for I-indexed products. If $\mathcal{D}$ strongly admits $\underline{\mathsf{Set}}_I$-shaped limits, we simply say *$\mathcal{D}$ admits I-indexed coproducts*; we use the following language.

- $\mathsf{Set}_{\mathcal{T}}$-indexed coproducts are *small indexed coproducts*;
- $\mathbb{F}_{\mathcal{T}}$-indexed coproducts are *finite indexed coproducts*;
- $\{\nabla\colon n \cdot S \to S\}$-indexed coproducts are *trivially indexed coproducts* (or *ordinary coproducts*). ◁

Notation I.1.25. Given $\mathcal{C}$ a $\mathcal{T}$-category and $S \in \mathsf{Set}_{\mathcal{T}}$, we write

$$\begin{aligned}\mathcal{C}_S &\coloneqq \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U \\ &\simeq \mathsf{Fun}_{\mathcal{T}}(S, \mathcal{C});\end{aligned}$$

more generally, given $S \in \mathsf{Set}_V$, we write $\mathcal{C}_S$ for $\mathcal{C}_{\mathrm{Ind}_V^{\mathcal{T}} S}$. where $\mathrm{Orb}(S)$ is the set of *orbits* expressing S as a disjoint union of elements of $\mathcal{T}$. Given $S \in \mathsf{Set}_{I,V}$, and $(X_U) \in \mathcal{C}_S$, we write the S-indexed products and coproducts as

$$\begin{array}{ccc} \mathcal{C}_S & \xrightarrow{\prod^S} & \mathcal{C}_V \\ \cup & & \cup \\ (X_U)_{U \in \mathrm{Orb}(S)} & \longmapsto & \prod_U^S X_U \end{array} \qquad \begin{array}{ccc} \mathcal{C}_S & \xrightarrow{\coprod^S} & \mathcal{C}_V \\ \cup & & \cup \\ (X_U)_{U \in \mathrm{Orb}(S)} & \longmapsto & \coprod_U^S X_U \end{array}$$

In particular, in the case that S has one orbit U, we write $\mathsf{Ind}_U^V(-)$ and $\mathsf{CoInd}_U^V(-)$ for S-indexed coproducts and products, respectively. ◁

Given $\mathcal{K} \subset \underline{\mathsf{Cat}}_{\mathcal{T}}$ a restriction-stable collection of V-categories and $W \in \mathcal{T}$, we let $\mathcal{K}_{\underline{W}} \subset \underline{\mathsf{Cat}}_W$ be the corresponding restriction-stable collection V-categories, where V ranges over $\mathcal{T}_{/W}$. We will use the following notation for strongly (co)limit-presereving functors.

Notation I.1.26. Let $I \subset \mathbb{F}_{\mathcal{T}}$ be a pullback-stable subcategory. Following and slightly extending [Sha22, Notn 1.15], we use the following notation for the described distinguished full $\mathcal{T}$-subcategories of $\underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$:

(1) $\underline{\mathsf{Fun}}_{\mathcal{T}}^{\mathcal{K}\text{-}L}(\mathcal{C},\mathcal{D})$: the V-functors which strongly preserve $\mathcal{K}_{\underline{V}}$-indexed colimits;
(2) $\underline{\mathsf{Fun}}_{\mathcal{T}}^{\mathcal{K}\text{-}R}(\mathcal{C},\mathcal{D})$: the V-functors which strongly preserve $\underline{\mathcal{K}}_{\underline{V}}$-indexed limits;
(3) $\underline{\mathsf{Fun}}_{\mathcal{T}}^{L}(\mathcal{C},\mathcal{D})$: the V-functors which strongly preserve small V-colimits;
(4) $\underline{\mathsf{Fun}}_{\mathcal{T}}^{R}(\mathcal{C},\mathcal{D})$: the V-functors which strongly preserve small V-limits;
(5) $\underline{\mathsf{Fun}}_{\mathcal{T}}^{I\text{-}\sqcup}(\mathcal{C},\mathcal{D})$: the V-functors which (strongly) preserve I-indexed coproducts;
(6) $\underline{\mathsf{Fun}}_{\mathcal{T}}^{I\text{-}\times}(\mathcal{C},\mathcal{D})$: the V-functors which (strongly) preserve I-indexed products.
(7) $\underline{\mathsf{Fun}}_{\mathcal{T}}^{\sqcup}(\mathcal{C},\mathcal{D})$: the V-functors which (strongly) preserve finite ordinary coproducts;
(8) $\underline{\mathsf{Fun}}_{\mathcal{T}}^{\times}(\mathcal{C},\mathcal{D})$: the V-functors which (strongly) preserve finite ordinary products. ◁

I.1.1.3 Parameterized adjunctions Related to indexed colimits, there is a theory of *parameterized adjunctions*.

Definition I.1.27. A $\mathcal{T}$-functor $L\colon \mathcal{C} \to \mathcal{D}$ is *left adjoint* to $R\colon \mathcal{D} \to \mathcal{C}$ if the associated functors $L_V\colon \mathcal{C}_V \to \mathcal{D}_V$ are left adjoint to $R_V\colon \mathcal{D}_V \to \mathcal{C}_V$ for all $V \in \mathcal{T}$. ◁

These have a pointwise description:

Proposition I.1.28 ([Hil24, Cor. 2.27])**.** *A $\mathcal{T}$-functor $L\colon \mathcal{C} \to \mathcal{D}$ admits a $\mathcal{T}$-right adjoint if and only if the following conditions hold:*

(1) for each $V \in \mathcal{T}$, $L^V\colon \mathcal{C}^V \to \mathcal{D}^V$ admits a right adjoint R^V, and
(2) for all maps $U \to V$, the Beck-Chevalley transformation $\mathrm{Res}_U^V R^V \implies R^U \mathrm{Res}_U^V R^U$ is an equivalence.

Remark I.1.29. By [Sha22, Rmk 5.4], $\mathcal{T}$-limits form a (partially defined) right $\mathcal{T}$-adjoint $\underline{\lim}\colon \underline{\mathsf{Fun}}_{\mathcal{T}}(K,\mathcal{C}) \to \mathcal{C}$ to the "diagonal" $\mathcal{T}$-functor $\Delta^K\colon \mathcal{C} \to \underline{\mathsf{Fun}}_{\mathcal{T}}(K,\mathcal{C})$, which itself may be computed as precomposition along the canonical $\mathcal{T}$-functor $K \to *_{\mathcal{T}}$. ◁

As observed in [Ste24], diagonals are functorial, so composing right adjoints to the diagonal of the "orbit set" factorization $\mathrm{Ind}_V^{\mathcal{T}} S \to \coprod_{U\in \mathrm{Orb}(S)} V \to V$ thus yields natural equivalences

$$\coprod_U^S X_U \simeq \coprod_{U\in\mathrm{Orb}(S)} \mathrm{Ind}_U^V X_U; \qquad \prod_U^S X_U \simeq \prod_{U\in\mathrm{Orb}(S)} \mathrm{CoInd}_U^V X_U. \tag{1}$$

We may construct many more $\mathcal{T}$-adjunctions using $\underline{\mathrm{Coeff}}^{\mathcal{T}}$:

Lemma I.1.30. *Suppose $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ is an adjunction of ∞-categories. Then,*

$$\underline{\mathrm{Coeff}}^{\mathcal{T}} L\colon \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{C} \rightleftarrows \underline{\mathrm{Coeff}}^{\mathcal{T}} \mathcal{D}\colon \underline{\mathrm{Coeff}}^{\mathcal{T}} R$$

is an adjunction of $\mathcal{T}$-∞-categories.

Proof. This follows from the fiberwise description of $\underline{\mathrm{Coeff}}^{\mathcal{T}}(-)$; indeed, the V-values

$$L_* : \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{C}\right) \rightleftarrows \mathrm{Fun}\left((\mathcal{T}_{/V})^{\mathrm{op}}, \mathcal{D}\right) : R_*$$

are adjoint. □

Example I.1.31. We may use Lemma I.1.30 to realize the full $\mathcal{T}$-subcategory of $\mathcal{T}$-spaces whose fixed points are d-connected or d-truncated as (co)localizing $\mathcal{T}$-subcategories

$$\underline{\mathcal{S}}_{\mathcal{T},\geq d} \overset{\perp}{\rightleftarrows} \underline{\mathcal{S}}_{\mathcal{T}} \overset{\perp}{\rightleftarrows} \underline{\mathcal{S}}_{\mathcal{T},\leq d}$$

We will use this line of thought to understand *truncatedness and connectedness of $\mathcal{T}$-operads and $\mathcal{T}$-symmetric monoidal categories.* ◁

Example I.1.32. By Lemma I.1.30, the *classifying space and core* double adjunction $(-)_{\simeq} \dashv \iota \dashv (-)^{\simeq}$ yields a double $\mathcal{T}$-adjunction

$$\underline{\mathrm{Cat}}_{\mathcal{T}} \underset{\overset{\perp}{(-)^{\simeq}}}{\overset{(-)_{\simeq}}{\overset{\perp}{\leftarrow}}} \underline{\mathcal{S}}_{\mathcal{T}}$$

◁

We may construct even more adjunctions by *slicing* existing adjunctions; the following follows immediately from the case $\mathcal{T} = *$ (which is well-known), applied fiberwise.

Proposition I.1.33. *Given $L\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon R$ an adjoint pair of $\mathcal{T}$-functors and $C \in \mathcal{C}$ an object, the corresponding lift*

$$L_{/C}\colon \mathcal{C}_{/C} \longrightarrow \mathcal{D}_{/LC}$$

is $\mathcal{T}$-left adjoint to the composition

$$\mathcal{D}_{/LC} \xrightarrow{R_{/LC}} \mathcal{C}^V_{/RLC} \xrightarrow{\eta^*} \mathcal{C}^V_{/C},$$

where $\eta\colon C \to RLC$ is the unit of the adjunction.

Additionally, we can make genuine adjunction *non-genuine* using [HA, Prop 7.3.2.1].

Proposition I.1.34. *If $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ are adjoint $\mathcal{T}$-functors, then $\operatorname{Tot} L\colon \operatorname{Tot}\mathcal{C} \rightleftarrows \operatorname{Tot}\mathcal{D}\colon \operatorname{Tot} R$ and $\Gamma L\colon \Gamma\mathcal{C} \rightleftarrows \Gamma\mathcal{D}\colon \Gamma R$ are adjoint pairs.*

Proof. The adjunction on Tot is [HA, Prop 7.3.2.1], and it induces an adjunction

$$\operatorname{Tot} L_*\colon \operatorname{Fun}_{/\mathcal{T}}(\mathcal{T}, \operatorname{Tot}\mathcal{C}) \rightleftarrows \operatorname{Fun}_{/\mathcal{T}}(\mathcal{T}, \operatorname{Tot}\mathcal{D})\colon \operatorname{Tot} R_*,$$

which restricts to the full subcategories of cocartesian sections, and hence yields an adjunction

$$\Gamma^{\mathcal{T}} L\colon \Gamma^{\mathcal{T}}\mathcal{C} \rightleftarrows \Gamma^{\mathcal{T}}\mathcal{D}\colon \Gamma^{\mathcal{T}}. \qquad \square.$$

We will need the following lemma and proposition later.

Lemma I.1.35. *Suppose a $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ has $F_V : \mathcal{C}_V \to \mathcal{D}_V$ conservative for all $V \in \mathcal{T}$; then, $\Gamma^{\mathcal{T}} F$ is conservative.*

Proof. Suppose $f_\bullet : X_\bullet \to Y_\bullet$ is a map of $\mathcal{T}$-objects in $\mathcal{C}$, i.e. a natural transformation of cocartesian sections of $\operatorname{Tot}\mathcal{C} \to \mathcal{T}^{\mathrm{op}}$. Then, $f_\bullet$ is an equivalence if and only if f_V is an equivalence for each V; by conservativity of F_V, this is true if and only if $F_v f_v$ is an equivalence for each V, i.e. if and only if $F f_\bullet$ is an equivalence, so $\Gamma^{\mathcal{T}} F$ is conservative. $\square$

Proposition I.1.36. *Suppose $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ is a $\mathcal{T}$-adjunction such that R_V is monadic for all $V \in \mathcal{T}$; Then, $\Gamma^{\mathcal{T}} R\colon \Gamma^{\mathcal{T}}\mathcal{D} \to \Gamma^{\mathcal{T}}\mathcal{C}$ is monadic.*

Proof. We verify that $\Gamma^{\mathcal{T}} R$ satisfies the conditions of the ∞-categorical Barr-Beck theorem [HA, Thm 4.7.3.5(c)]. First, by Proposition I.1.34 and Lemma I.1.35, $\Gamma^{\mathcal{T}} R$ is a conservative right adjoint. Second, note that a simplicial object $Z_\bullet(-)$ in $\Gamma^{\mathcal{T}}\mathcal{D}$ corresponds to a family of simplicial objects $Z_V(-)$ in $\mathcal{D}_V$, and a $\Gamma^{\mathcal{T}} R$-splitting of $Z_\bullet(-)$ corresponds with a restriction-stable family of R_V-splittings of $Z_V(-)$. Thus R_V creates a colimit of Z_V for all V, and the resulting cocartesian section creates a colimit for $Z_\bullet$, i.e. $\Gamma^{\mathcal{T}} R$ createss $\Gamma^{\mathcal{T}} R$-split simplicial colimits, so $\Gamma^{\mathcal{T}} R$ is monadic by [HA, Thm 4.7.3.5(c)]. $\square$

I.1.1.4 Parameterized Kan extensions Fix a (not necessarily commuting) triangle of $\mathcal{T}$-functors

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ {\scriptstyle\varphi}\downarrow & \nearrow_{G} & \\ \mathcal{D} & & \end{array}$$

and $x \in \mathcal{D}_V$ a V-object. Assume $\mathcal{D}$ has a final V-object. We define the composite V-functor

$$G^x \colon \left(\mathcal{C}^{/x}_{\underline{V}}\right)^{\triangleright} \xrightarrow{\varphi} \left(\mathcal{D}^{/x}_{\underline{V}}\right)^{\triangleright} \xrightarrow{(H',\pi)} \mathcal{D}^{/x}_{\underline{V}} \times \Delta^1 \xrightarrow{H} \mathcal{D}_{\underline{V}} \xrightarrow{G} \mathcal{E}_{\underline{V}}$$

where H' takes the cone point to a $\underline{V}$-final object and H is adjunct to the evident map $\mathcal{D}^{/x}_{\underline{V}} \to \mathrm{Ar}(\mathcal{D}_{\underline{V}})$.

Theorem I.1.37 ([Sha22, Thm 2.13])**.** *The pullback $\mathcal{T}$-functor*

$$\varphi^* \colon \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D},\mathcal{E}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{E})$$

*has a partially defined left adjoint $\varphi_!$ whose values are uniquely characterized by the property that $(\varphi_! F)^x$ is a V-colimit diagram for all $V \in \mathcal{T}$ and $x \in \mathcal{D}_V$. We call this the $\mathcal{T}$-*left Kan extension of F along φ.

For instance, $\mathcal{T}$-left Kan extensions along $\mathcal{C} \to *_{\mathcal{T}}$ are precisely $\mathcal{T}$-colimits. More generally, we will view the above colimit formula via the shorthand

$$\phi_! F(y) = \underline{\mathrm{colim}}_{\varphi(x)\to y} F(x).$$

Of course, $\mathcal{T}$-right Kan extensions are defined dually, and denoted φ_*.

Lastly, we may describe of (co)limits in *relative* functor $\mathcal{T}$-categories in the cocartesian case.

Proposition I.1.38 ([Sha22, Thm. 4.16])**.** *Fix $\pi \colon \mathcal{X} \to \mathcal{B}$ a cocartesian $\mathcal{T}$-fibration, $\mathcal{K}$ a $\mathcal{T}$-∞-category, and $r \colon \mathcal{C} \to \mathcal{B}$ a $\mathcal{T}$-functor.*

(1) A V-diagram

$$\overline{p} \colon K^{\triangleleft} \to \underline{\mathrm{Fun}}_{V,/\mathcal{B}_{\underline{V}}}\left(\mathcal{C}_{\underline{V}}, \mathcal{X}_{\underline{V}}\right)$$

is a V-limit if and only if, for all $U \in \mathcal{T}^{/V}$, $B \in \mathcal{B}^U$, and $C \in \mathcal{C}_B$, the composite $\mathcal{T}$-functor

$$K^{\triangleleft}_{\underline{U}} \xrightarrow{\mathrm{Res}^{\underline{V}}_{\underline{U}}\overline{p}} \underline{\mathrm{Fun}}_{\mathcal{T},/\mathcal{B}_{\underline{V}}}\left(\mathcal{C}_{\underline{V}}, \mathcal{X}_{\underline{V}}\right) \xrightarrow{\mathrm{ev}_C} \mathcal{X}_B$$

is a U-limit diagram; in particular, if $\mathcal{X}$ is fiberwise $\mathcal{T}$-complete, then $\underline{\mathrm{Fun}}_{\mathcal{T},/\mathcal{B}}(\mathcal{C},\mathcal{X})$ is $\mathcal{T}$-complete.

(2) A V-diagram

$$\overline{p} \colon K^{\triangleright} \to \underline{\mathrm{Fun}}_{V,/\mathcal{B}_{\underline{V}}}\left(\mathcal{C}_{\underline{V}}, \mathcal{X}_{\underline{V}}\right)$$

is a V-colimit diagram if and only if, for all $U \in \mathcal{T}^{/V}$, $B \in \mathcal{B}^U$, $C \in \mathcal{C}_B$, and $\mathcal{C}$-arrows $g\colon B \to B'$, the composite $\mathcal{T}$-functor

$$K^{\triangleright}_{\underline{U}} \xrightarrow{\mathrm{Res}^V_U \overline{p}} \underline{\mathrm{Fun}}_{\mathcal{T}/\mathcal{B}_{\underline{V}}}\left(\mathcal{C}_{\underline{V}}, \mathcal{X}_{\underline{V}}\right) \xrightarrow{\mathrm{ev}_C} \mathcal{X}_B \xrightarrow{g_!} \mathcal{X}_{B'}$$

is a U-limit diagram; moreover, if $\mathcal{X}$ is fiberwise $\mathcal{T}$-cocomplete, then $\underline{\mathrm{Fun}}_{\mathcal{T}/\mathcal{B}}(\mathcal{C},\mathcal{X})$ is $\mathcal{T}$-cocomplete.

In particular, we immediately acquire colimits in $\mathcal{T}$-categories of *cocartesian* sections (c.f. [NS22, Thm. 5.1.3] and [NS22, Thm. 5.1.4]).

Corollary I.1.39. *Suppose $\mathcal{B}^{\mathrm{int}} \subset \mathcal{B}$ is a wide $\mathcal{T}$-subcategory, $\mathcal{X} \to \mathcal{B}$ is a cocartesian $\mathcal{T}$-fibration, $\mathcal{C} \to \mathcal{B}$ a $\mathcal{T}$-functor with cocartesian lifts for arrows lying in $\mathcal{B}^{\mathrm{int}}$, and $\mathcal{K} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ a collection of small $\mathcal{T}$-∞ categories such that cocartesian transport in $\mathcal{X}$ preserves $\mathcal{K}$-colimits. Denote by $\underline{\mathrm{Fun}}^{\mathrm{int-cocart}}_{\mathcal{B}}(\mathcal{C},\mathcal{X}) \subset \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C},\mathcal{X})$ the full $\mathcal{T}$-subcategory of V-functors compatible with cocartesian arrows over $\mathcal{B}^{\mathrm{int}}$.*

(1) $\underline{\mathrm{Fun}}^{\mathrm{int-cocart}}_{\mathcal{T}/\mathcal{B}}(\mathcal{C},\mathcal{X}) \subset \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C},\mathcal{X})$ is closed under $\mathcal{T}$-limits.

(2) $\underline{\mathrm{Fun}}^{\mathrm{int-cocart}}_{\mathcal{T}/\mathcal{B}}(\mathcal{C},\mathcal{X}) \subset \underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{C},\mathcal{X})$ is closed under $\mathcal{K}$-colimits.

Lastly, we note that this is yields to parameterized Kan extensions.

Lemma I.1.40. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-functor and $L\colon \mathcal{C} \to \mathcal{E}$ is $\mathcal{T}$-left adjoint to $R\colon \mathcal{E} \to \mathcal{C}$. Then, FR is the $\mathcal{T}$-left Kan extension of F along L.*

Proof. Using [Sha23, Thm 10.5] we simply repeat the nonequivariant proof: Yoneda's lemma yields

$$\begin{aligned}\mathrm{Nat}_{\mathcal{T}}(L_!F, G) &\simeq \mathrm{Nat}_{\mathcal{T}}(F, GL)\\ &\simeq \mathrm{Nat}_{\mathcal{T}}(FR, G),\end{aligned}$$

so another application of Yoneda's lemma constructs a natural equivalence $L_!F \sim FR$. □

I.1.1.5 The parameterized Yoneda embedding [BDGNS16b, Def. 10.2] defined a *Yoneda embedding*:

$$\text{よ}_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathrm{Psh}_{\mathcal{T}}(\mathcal{C}) \coloneqq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}}\right).$$

We will need some properties of this:

Proposition I.1.41 ([BDGNS16b, Prop. 10.3], [Sha23, Thm. 11.5], [Sha23, Cor. 11.10])**.** *Fix $\mathcal{C} \in \widehat{\underline{\mathrm{Cat}}}_{\mathcal{T}}$.*

(1) For all V-presheaves $F\colon \mathcal{C}_{\underline{V}} \to \underline{\mathcal{S}}_{\underline{V}}$, there exists a natural equivalence

$$F(-) \simeq \underline{\mathrm{Map}}_{\mathrm{Psh}_{\underline{V}}(\mathcal{C}_{\underline{V}})}\left(\text{よ}_{\mathcal{C}}(-), F\right);$$

in particular, $\text{よ}_{\mathcal{C}}$ *is fully faithful.*

(2) $\text{よ}_{\mathcal{C}}$ *factors through an equivalence* $\text{よ}_{\mathcal{C}}\colon \mathcal{C} \xrightarrow{\sim} \underline{\mathrm{Fun}}^{R}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{C})$.

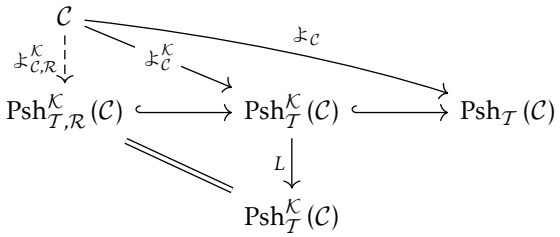
(3) $\text{よ}_{\mathcal{C}}$ *preserves and detects* $\mathcal{T}$*-limits.*

(4) $\mathrm{Psh}_{\mathcal{T}}(\mathcal{C})$ *is cocomplete; moreover, if* $\mathcal{D}$ *is a cocomplete* $\mathcal{T}$*-*∞*-category, we acquire an equivalence*

$$\text{よ}^{*}_{\mathcal{C}}\colon \underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\mathrm{Psh}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}).$$

We may view $\text{よ}_{\mathcal{C}}$ as exhibiting $\mathrm{Psh}_{\mathcal{T}}(\mathcal{C})$ as the $\mathcal{T}$-cocompletion of $\mathcal{C}$. Hilman subsequently developed a vast generalization of this, which we will eventually need within our discussions of distributivity and presentability.

Proposition I.1.42 ([Hil24, Thm. 6.4.2])**.** *Suppose* $\mathcal{C}$ *is a* $\mathcal{T}$*-*∞*-category,* $\mathcal{K} \subset \mathrm{Cat}_{\mathcal{T}}$ *a full* $\mathcal{T}$*-subcategory, and* $\mathcal{R}$ *a collection of cocone diagrams lying in* $\mathcal{C}$ *with domains of the form* $K^{\triangleright}$ *with* $K \in \mathcal{K}$*. Define the solid diagram*



where $\mathrm{Psh}^{\mathcal{K}}_{\mathcal{T}}(\mathcal{C}) \subset \mathrm{Psh}_{\mathcal{T}}(\mathcal{C})$ *is the full* $\mathcal{T}$*-subcategory generated under* $\mathcal{K}$*-colimits by* $\text{よ}_{\mathcal{C}}(\mathcal{C})$ *and* L *is the localization which forces the image of* $\mathcal{R}$ *to consist of colimit cocones. Then, the composite functor*

$$\text{よ}^{\mathcal{K}}_{\mathcal{C},\mathcal{R}}\colon \mathcal{C} \xrightarrow{\text{よ}^{\mathcal{K}}_{\mathcal{C}}} \mathrm{Psh}^{\mathcal{K}}_{\mathcal{T}}(\mathcal{C}) \xrightarrow{L} \mathrm{Psh}^{\mathcal{K}}_{\mathcal{T},\mathcal{R}}(\mathcal{C})$$

satisfies the following universal property:

$$\underline{\mathrm{Fun}}^{L\text{-}\mathcal{K}}_{\mathcal{T}}\left(\mathrm{Psh}^{\mathcal{K}}_{\mathcal{T},\mathcal{R}}(\mathcal{C}), \mathcal{D}\right) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{L\text{-}\mathcal{R}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}),$$

whose righthand side consists of those $\mathcal{T}$*-functors* $\mathcal{C} \to \mathcal{D}$ *taking elements of* $\mathcal{R}$ *to* V*-colimit cocones. Moreover, a factorization exists as in the dashed arrow if and only if* $\mathcal{R}$ *consisted of cocone diagrams in the first place; in particular, in this case,* $\text{よ}^{\mathcal{K}}_{\mathcal{C},\mathcal{R}}$ *is fully faithful.*

I.1.1.6 Parameterized localizations We use the following terminology.

Definition I.1.43. A $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is a *$\mathcal{T}$-localization* if it admits a fully faithful $\mathcal{T}$-right adjoint. A full $\mathcal{T}$-subcategory $\mathcal{D} \subset \mathcal{C}$ is *localizing* if its inclusion is right $\mathcal{T}$-adjoint to a $\mathcal{T}$-localization. ◁

Now, many of these are *Dwyer-kan localizations* in the following sense.

Proposition I.1.44 ([Hil24, Thm. 6.3.7]). *Given a small $\mathcal{T}$-set of morphisms $\mathcal{B} \subset \mathcal{C}$, there exists an essentially unique accessible $\mathcal{T}$-localization $\mathcal{C} \to L_{\mathcal{B}}\mathcal{C}$ satisfying the universal property*

$$\underline{\mathrm{Fun}}_{\mathcal{T}}(L_{\mathcal{B}}\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}_{\mathcal{T},\mathcal{B}^{-1}}(\mathcal{C}, \mathcal{D}),$$

where the latter denotes $\mathcal{B}$-inverting $\mathcal{T}$-functors.

These are described by a closure property on $\mathcal{T}$-classes of morphisms.

Definition I.1.45 ([Hil24, Def. 6.3.5]). A $\mathcal{T}$-class of morphisms $\overline{\mathcal{B}} \subset \mathcal{B}$ is *strongly saturated* if

(1) $\overline{\mathcal{B}}$ is closed under cobase change,

(2) $\overline{\mathcal{B}} \subset \mathrm{Ar}(\mathcal{C})$ is closed under $\mathcal{T}$-colimits, and

(3) $\overline{\mathcal{B}}^V$ satisfies two-out-of-three for all $V \in \mathcal{T}$.

We say that $\overline{\mathcal{B}}$ is the *strong saturation of* $\mathcal{B} \subset \overline{\mathcal{B}}$ if it is the smallest strongly saturated class containing $\mathcal{B}$. ◁

Note that whenever $\mathcal{C}$ admits $\mathcal{T}$-colimits, this is equivalent to a strongly saturated class in $\mathcal{C}^V$ for each V, stable under restriction and induction. These are relevant for the following reason.

Proposition I.1.46 ([Hil24, Prop. 6.3.6]). *Given a $\mathcal{T}$-localization $L\colon \mathcal{C} \to \mathcal{D}$, the collection of arrows inverted by L is a strongly saturated class; moreover, there exists an equivalence $L_{\mathcal{B}}\mathcal{C} \simeq L_{\overline{\mathcal{B}}}\mathcal{C}$ under $\mathcal{C}$.*

I.1.1.7 Parameterized presentability $\mathcal{T}$-left adjoints strongly preserve small $\mathcal{T}$-colimits and $\mathcal{T}$-right adjoints strongly preserve small $\mathcal{T}$-limits [Hil24, Thm 3.1.10], and they satisfy a parameterized version of the adjoint functor theorem [Hil24, Thm 6.2.1]. To describe this fully, we make a few definitions.

Definition I.1.47. We say that a $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is *$\mathcal{T}$-accessible* if it preserves fiberwise κ-filtered colimits for some regular cardinal κ. ◁

We call a $\mathcal{T}$-∞-category *presentable* if it satisfies the following equivalent conditions:

Proposition I.1.48 ([Hil24, Thm. 6.1.2]). *The following conditions on a $\mathcal{T}$-∞-category $\mathcal{C}$ are equivalent:*

(a) There exists a small idempotent complete $\mathcal{T}$-∞-category $\mathcal{D}$ and accessible $\mathcal{T}$-localization $\mathrm{Psh}_{\mathcal{T}}(\mathcal{D}) \to \mathcal{C}$.

(b) $\mathcal{C}$ is cocomplete and there exists a regular cardinal κ and factorization

$$\mathcal{C}\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Pr}^{L,\kappa} \to \widehat{\mathrm{Cat}}.$$

Now, for a convenient statement of Hilman's parameterized adjoint functor theorem, we define the following full $\mathcal{T}$-subcategories:

(1) $\underline{L\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}) \subset \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$: the $\mathcal{T}$-left adjoints.

(2) $\underline{\mathrm{Fun}}^{\mathrm{acc}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D}) \subset \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathcal{D})$: the accessible $\mathcal{T}$-functors.

Proposition I.1.49 ([Hil24, Prop. 3.3.11] and [Hil24, Thm. 6.2.1])**.** *Suppose $\mathcal{C}$ and $\mathcal{D}$ are presentable $\mathcal{T}$-∞-categories. Then, taking adjoints and inclusions yield equivalences*

$$\underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\mathcal{C},\mathcal{D}) \xrightarrow{\sim} \underline{L\,\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D}) \simeq \underline{R\,\mathrm{Fun}}_{\mathcal{T}}(\mathcal{D},\mathcal{C})^{\mathrm{op}} \xleftarrow{\sim} \underline{\mathrm{Fun}}^{R,\mathrm{acc}}_{\mathcal{T}}(\mathcal{D},\mathcal{C})^{\mathrm{op}}$$

Warning I.1.50. Our notion of accessibility from skips an assumption of *accessibility of the domain* stipulated in [Hil24, Def. 5.2.6], so as to avoid discussion of κ-Ind-completion. This will not cause issues, as we will only ever discuss accessible $\mathcal{T}$-functors with *presentable* domain, and presentable $\mathcal{T}$-∞-categories are accessible by [Hil24, Thm. 6.1.2]. ◁

Example I.1.51. The following $\mathcal{T}$-∞-categories are presentable.

(1) Accessible $\mathcal{T}$-localizations of presentable $\mathcal{T}$-∞-categories; that is, $\mathcal{T}$-localizations $\mathcal{C} \to L\mathcal{C}$ with $\mathcal{C} \in \mathrm{Pr}^{L}_{\mathcal{T}}$ whose codomain and $\mathcal{T}$-right adjoint are accessible.[5]
(2) Indexed products of presentable $\mathcal{T}$-categories [Hil24, Lem. 6.6.1].
(3) $\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$, whenever $\mathcal{C}$ is small and $\mathcal{D}$ is presentable [Hil24, Lem. 6.7.1].
(4) $\underline{\mathrm{Fun}}^{L}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ whenever $\mathcal{C}$ and $\mathcal{D}$ are presentable [Hil24, Lem. 6.7.4].
(5) $\mathcal{C}_{/X}$, whenever $X \in \Gamma^{\mathcal{T}}\mathcal{C}$ is a $\mathcal{T}$-object of a presentable $\mathcal{T}$-category [MW22, Cor. 2.4.2.8]. ◁

I.1.1.8 The parameterized pairing construction Now, one of the crucial and subtle constructions of [Sha22; Sha23] is the *parameterized pairing construction*, based off of the following.

Proposition I.1.52 (Exponentiability is measured fiberwise)**.** *Given $\mathcal{T}$-functor $f\colon \mathcal{C} \to \mathcal{D}$, the following conditions are equivalent.*

(a) The pullback $\mathcal{T}$-functor $f^{*}\colon \underline{\mathrm{Cat}}_{\mathcal{T},/\mathcal{D}} \to \underline{\mathrm{Cat}}_{\mathcal{T},/\mathcal{C}}$ admits a $\mathcal{T}$-right adjoint f_{*}; and
(b) For all $V \in \mathcal{T}$, the pullback functor $f^{V,*}\colon \mathrm{Cat}_{/\mathcal{D}^{V}} \to \mathrm{Cat}_{/\mathcal{C}^{V}}$ admits a right adjoint f_{*}^{V}.

In this case, there is a natural equivalence $f_{*}^{V} \simeq (f_{*})^{V}$.

We call such $\mathcal{T}$-functors *exponentiable*; this is equivalent to flatness by [kerodon, Tag 05N8].

Proof. Note that the inclusion $F(-) \coloneqq \mathrm{Infl}^{V}_{e}(-) \times_{F(\mathcal{D}^{V})} \mathcal{D} \to \mathcal{D}$ admits natural equivalences

$$\mathrm{Fun}_{\mathcal{T},/\mathcal{D}_{\underline{V}}}(F(\mathcal{B}),\mathcal{E}) \simeq \mathrm{Fun}_{/\mathcal{D}^{V}}\left(\mathcal{B},\mathcal{E}^{V}\right);$$

$$f^{*}F(\mathcal{B}) \simeq F(f^{V,*}\mathcal{B}).$$

[5] This follows by Proposition I.1.48 and transitivity of accessibility.

In particular, in the case (b), we acquire a natural equivalence

$$\begin{aligned}\mathsf{Fun}_{/\mathcal{C}^V}\left(f^{V,*}\mathcal{B},\mathcal{E}^V\right) &\simeq \mathsf{Fun}_{\mathcal{T}_{/\mathcal{C}_{\underline{V}}}}\left(F(f^{V,*}B),\mathcal{E}\right)\\ &\simeq \mathsf{Fun}_{\mathcal{T}_{/\mathcal{C}_{\underline{V}}}}(f^*F(B),\mathcal{E})\\ &\simeq \mathsf{Fun}_{\mathcal{T}_{/\mathcal{D}_{\underline{V}}}}(F(B),f_*\mathcal{E})\\ &\simeq \mathsf{Fun}_{/\mathcal{D}^V}\left(B,(f_*\mathcal{E})^V\right),\end{aligned}$$

so taking cores, $\mathsf{Fun}_{/\mathcal{C}^V}(f^*(-),\mathcal{E})^{\simeq}$ is representable by $(f_*\mathcal{E})^V$, i.e. that computes a right adjoint to $f^{V,*}$ for all V, implying (a).

Now, assume condition (b) holds. In [Sha22, Thm. 4.2] it is shown how to modify [Sha23, Thm. 9.3.] to prove that the span marked simplicial sets

$$\mathsf{Tot}(\mathcal{D}) \leftarrow \mathrm{Ar}^{\mathrm{cocart}}(\mathsf{Tot}(\mathcal{D})) \times_{\mathsf{Tot}(\mathcal{D})} \mathsf{Tot}(\mathcal{C}) \to \mathsf{Tot}(\mathcal{C}),$$

for cocartesian markings on the left and righthand side and pullback marking on middle, satisfies the assumptions of [HA, Thm. B.4.2] in case (a) when given the categorical pattern structure when $\mathcal{T}$ everything and no Segal conditions. The evident variation of [HA, Ex. B.0.21] shows that the corresponding fibrous model structure of [HA, Thm. B.0.20] for the left and righthand side agrees with the *partially* cocartesian model structure described in [Sha23, Thm. 2.13]; in particular, it induces a Quillen adjunction

$$\pi_!\,\mathrm{ev}_0^*\colon \mathsf{sSet}^+_{/\mathsf{Tot}(\mathcal{D})} \rightleftarrows \mathsf{sSet}^+_{/\mathsf{Tot}(\mathcal{C})}\colon \mathrm{ev}_{0*}\,\pi^*$$

with respect to these cocartesian model structures. The important observation is that there exists a natural transformation of functors of relative marked simplicial sets

$$\pi_!\,\mathrm{ev}_0^*\,\mathcal{E} \implies \mathsf{Tot}(f)^*\mathcal{E},$$

which is an equivalence on fibrant-cofibrant objects; in particular, the left derived functor of $\pi_!\,\mathrm{ev}_0^*$ and the right derived functor of $\mathsf{Tot}(f)^*\mathcal{E}$ agree. As a consequence, the associated right derived functor of $\mathrm{ev}_{0*}\,\pi^*$ supplies a right adjoint f_* to the *functor* $f^*\colon \mathsf{Cat}_{\mathcal{T}_{/\mathcal{D}}} \to \mathsf{Cat}_{\mathcal{T}_{/\mathcal{C}}}$. Restriction-stability of everything then shows that the $\mathcal{T}$-functor f^* fiberwise admits right adjoints.

In sight of Proposition I.1.28, we are left with verifying that the Beck-Chevalley transformation $\mathrm{Res}^V_U f_* \implies f_*\mathrm{Res}^V_U$ is an equivalence; but this is a natural isomorphism on the level of marked simplicial sets, so we are done. □

Along the way, the following was verified:

Corollary I.1.53. *If $f : \mathcal{C} \to \mathcal{D}$ is exponentiable, then* $\mathrm{Tot}\, f : \mathrm{Tot}\, \mathcal{C} \to \mathrm{Tot}\, \mathcal{D}$ *is exponentiable.*

By combining [kerodon, Tag 05MM] and [kerodon, Tag 05NJ], we immediately acquire the following.

Corollary I.1.54. *A $\mathcal{T}$-functor $f : \mathcal{C} \to \mathcal{D}$ is exponentiable if and only if, for all composable sequences of V-arrows $X \to Y \to Z$ in $\mathcal{D}$ and $\mathcal{C}$-lifts of their composite $\overline{X} \to \overline{Z}$, the ∞-category $\mathcal{C}^V_{\overline{X}//\overline{Z}} \times_{\mathcal{D}^V_{X//Z}} \{Y\}$ is weakly contractible.*

Now, an example of importance is the following.

Construction I.1.55 ([Sha22, Thm 4.2])**.** Given $p : \mathcal{X} \to \mathcal{B}$ and $q : \mathcal{Y} \to \mathcal{B}$ a pair of $\mathcal{T}$-functors, we define the pairing construction $\widetilde{\underline{\mathrm{Fun}}}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y}) := p_* p^* \mathcal{Y} \in \mathrm{Cat}_{\mathcal{T},/\mathcal{B}}$. ◁

The following result is quite important.

Proposition I.1.56 ([Sha22, Prop. 4.5])**.** *Given a functor $\varphi : \mathcal{A} \to \mathcal{B}$, the Beck-Chevalley natural transformation*

$$\widetilde{\underline{\mathrm{Fun}}}_{\mathcal{A}}(\varphi^* -, \varphi^* -) \longrightarrow \varphi^* \widetilde{\underline{\mathrm{Fun}}}_{\mathcal{B}}(-, -)$$

is an equivalence; in particular, given a V-object $B \in \mathcal{B}$, we acquire a natural equivalence

$$\underline{\mathrm{Fun}}_V\left((-)_{\underline{B}}, (-)_{\underline{B}}\right) \simeq \widetilde{\underline{\mathrm{Fun}}}_{\mathcal{B}}(-, -)_{\underline{B}},$$

where we write $(-)_{\underline{B}} : \mathrm{Cat}_{\mathcal{T},/\mathcal{B}} \to \mathrm{Cat}_V$ for the fiber over X.

We will want to use the following desirable interaction of this with (co)cartesian $\mathcal{T}$-fibrations.

Proposition I.1.57 ([Sha22, Thm 4.9] [Sha22, Rec. 4.3])**.** *There exists a equivalence, natural in $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{C}$,*

$$\underline{\mathrm{Fun}}_{\mathcal{T},/\mathcal{B}}\left(\mathcal{C}, \widetilde{\underline{\mathrm{Fun}}}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})\right) \simeq \underline{\mathrm{Fun}}_{\mathcal{T},/\mathcal{B}}(\mathcal{C} \times_{\mathcal{B}} \mathcal{X}, \mathcal{Y}).$$

Moreover, if $\mathcal{X}$ is cartesian over $\mathcal{B}$ and $\mathcal{Y}$ cocartesian, then $\widetilde{\underline{\mathrm{Fun}}}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})$ is cocartesian. Under the equivalence

$$\mathrm{Fun}_{V,/\mathcal{B}}\left(\underline{\Delta}^1, \widetilde{\underline{\mathrm{Fun}}}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})\right) \simeq \underline{\mathrm{Fun}}_{\mathcal{T},/\mathcal{B}}\left(\underline{\Delta}^1 \times_{\mathcal{B}} \mathcal{X}, \mathcal{Y}\right),$$

a V-arrow of $\underline{\mathrm{Fun}}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})$ is cocartesian if and only if the corresponding functor $\Delta^1 \times_{\mathcal{B}} \mathcal{X} \to \mathcal{Y}$ takes cartesian V-arrows to cocartesian V-arrows; in particular, the cocartesian arrows are preserved under $\mathcal{T}$-functors of cartesian $\mathcal{T}$-fibrations in $\mathcal{X}$ and cocartesian $\mathcal{T}$-fibrations in $\mathcal{Y}$.

I.1.1.9 Language in the case $\mathcal{T} = \mathcal{O}_G$ When G is a finite group, the category $\mathcal{O}_G$ has objects the homogeneous G-sets $[G/H]$ and morphisms the G-equivariant maps $[G/K] \to [G/H]$; tracking the image of

the identity, the hom set from $[G/K]$ to $[G/H]$ may alternatively be presented as

$$\mathrm{Hom}([G/K],[G/H]) \simeq \frac{\left\{a \in G \mid aKa^{-1} \subset H\right\}}{a \sim b \quad \text{when} \quad ab^{-1} \in K}$$

(see e.g. [Die09, Prop 1.3.1] for details). In particular, the endomorphism monoid of $[G/K]$ is the Weyl group $W_G H = N_G(H)/H$. Using this, one may see that when G is a finite group, the map $\mathrm{Ind}_H^G \colon \mathcal{O}_H \to \mathcal{O}_{G,/(G/H)}$ is an equivalence of categories. Thus we may set the following notation without creating clashes.

Notation I.1.58. In the setting that $\mathcal{T} = \mathcal{O}_G$, we use the following notation:

(1) we refer to $\underline{[G/H]}$ as $\underline{H}$;

(2) we refer to $\mathcal{O}_G$-∞-categories as G-∞-categories and $\underline{\mathrm{Cat}}_{\mathcal{O}_G}$ as $\underline{\mathrm{Cat}}_G$; we refer to $\mathcal{O}_G$-spaces as G-spaces and $\underline{\mathcal{S}}_{\mathcal{O}_G}$ as $\underline{\mathcal{S}}_G$;

(3) we refer to $\mathcal{C}_{[G/H]}$ as $\mathcal{C}_H$ and $\mathrm{Res}^{[G/H]}_{[G/K]}$ as Res^H_K; the superscripts and subscripts of Ind, CoInd, Γ, $\underline{\mathsf{Coeff}}$, $\star$, $(-)^{(-,\mathcal{T})/}$, and $*$ are determined similarly.

(4) we refer to $\coprod^S_{[H/K]} X_K$ as $\coprod^S_K X_K$, and similar for $\prod^S$. ◁

I.1.2 Rudiments of parameterized $(\infty,2)$-category theory We now assume familiarity with the basics of $(\infty,2)$-category theory.

Remark I.1.59. As much as possible, we will only reference objects in the $(\infty,2)$-category of (large) $(\infty,2)$-categories, without specific details about the presentation for this $(\infty,2)$-category. Nevertheless the literature almost always chooses a concrete presentation; as such, we will mean either a Cat-enriched ∞-category in the sense of [GH15, Rmk. 5.7.13], a Cat-module in $\widehat{\mathsf{Cat}}$, or a doubly complete Segal space in the sense of [BB25, Def. A.1.8] (i.e. a double ∞-category which happens to be an $(\infty,2)$-category). The first two models are equivalent by [Hei23, Thm. 1.8], and the third to the first two by either by work of Haugseng [Hau15, Cor. 7.21], ultimately relying on an extremely strong unicity result of Bariwck–Schommer-Preis [BS20, Thm. 1.1]. This latter unicity result includes self-enrichment, and in particular, canonically lifts to identifications of $(\infty,3)$-*categories*¸ yielding the claim on $(-)^{\leq 2}$. ◁

Construction I.1.60. Let CAT and $\mathrm{CAT}_{(\infty,2)}$ denote the $(\infty,2)$-categories of ∞-categories and $(\infty,2)$-categories; these admit an adjunction of $(\infty,2)$-categories

$$\mathrm{CAT} \underset{(-)^{\leq 1}}{\overset{}{\rightleftarrows}} \mathrm{CAT}_{(\infty,2)} \qquad (\perp)$$

we write $\mathsf{Cat}_{(\infty,2)} \coloneqq \mathrm{CAT}^{\leq 1}_{(\infty,2)}$. We define the *$(\infty,2)$-categories of $\mathcal{T}$-∞-categories and $\mathcal{T}$-$(\infty,2)$-categories* as

$$\mathrm{CAT}_{\mathcal{T}} \coloneqq \mathrm{FUN}(\mathcal{T}^{\mathrm{op}}, \mathrm{CAT}); \qquad \mathrm{CAT}_{\mathcal{T},(\infty,2)} \coloneqq \mathrm{FUN}\left(\mathcal{T}^{\mathrm{op}}, \mathrm{CAT}_{(\infty,2)}\right).$$

We similarly write $\mathrm{Cat}_{\mathcal{T},(\infty,2)} \coloneqq \mathrm{CAT}^{\leq 1}_{\mathcal{T},(\infty,2)}$. ◁

I.1.2.1 The yoneda embedding We use naturality of Yoneda due to Ben-Moshe.

Proposition I.1.61. *There exists a functor*

$$\mathrm{Fun}_{(-)} \colon \mathrm{Cat}_{(\infty,2)} \longrightarrow \mathrm{Fun}\left(\Delta^1, \widehat{\mathrm{Cat}}_{(\infty,2)}\right)^{\mathrm{lax}} \times_{\mathrm{ev}_1, \widehat{\mathrm{Cat}}_{(\infty,2)}} \{\mathrm{const}_{\mathrm{CAT}}\}$$

whose composite functor $\mathrm{Cat}_{(\infty,2)} \to \widehat{\mathrm{Cat}}_{(\infty,2)}$ *is* $\iota(-)^{\mathrm{op}} \times \iota(-)$*, and whose value on an* $(\infty,2)$*-category is the mate* $\mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{CAT}$ *to the Yoneda embedding.*

Proof. Now, in sight of [Hei23, Thm. 1.8], [Ben24, Thm. A] specializes to construction of a partial left adjoint of the inclusion $\mathrm{LMod}_{\mathrm{Cat}}(\mathrm{Pr}^L) \hookrightarrow \widehat{\mathrm{Cat}}_{(\infty,2)}$ defined on small $(\infty,2)$-categories whose unit is the $(\infty,2)$-categorical Yoneda embedding. This natural transformation

$$\text{よ} \colon \iota(\mathbb{C}) \Longrightarrow \underline{\mathrm{Fun}}(\iota(\mathbb{C})^{\mathrm{op}}, \mathrm{CAT})$$

is mate to a lax natural transformation

$$F \colon \iota(\mathbb{C}) \times \iota(\mathbb{C})^{\mathrm{op}} \Longrightarrow \mathrm{CAT}$$

i.e. a functor

$$\mathrm{FUN} \colon \mathrm{Cat}_{(\infty,2)} \longrightarrow \mathrm{Fun}\left(\Delta^1, \widehat{\mathrm{Cat}}_{(\infty,2)}\right)^{\mathrm{lax}}$$

The desired properties follow straightforwardly by construction □

Lemma I.1.62. *There exists a natural equivalence*

$$\mathrm{FUN}\left(\mathbb{C}, \mathrm{Infl}^{\mathcal{T}}_{e}\, \mathbb{D}\right)^{\mathrm{lax}} \simeq \mathrm{FUN}\left(\mathbb{C}, \underline{\mathrm{COEFF}}^{\mathcal{T}}\, \mathbb{D}\right).$$

Construction I.1.63. Currying this aggressively, we acquire a lax functor

$$\Delta^1 \to \{\mathbb{C}\} \times_{\widehat{\mathrm{Cat}}_{\mathcal{T},(\infty,2)}} \mathrm{FUN}\left(\mathcal{T}^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{(\infty,2)}\right) \times_{\mathrm{ev}_1, \widehat{\mathrm{Cat}}_{\mathcal{T},(\infty,2)}} \left\{\mathrm{Infl}^{\mathcal{T}}_{e}\, \mathrm{CAT}\right\};$$

that is, we acquire a lax natural transformation $\mathbb{C} \times \mathbb{C}^{\mathrm{op}} \Longrightarrow \Delta_{\mathcal{T}^{\mathrm{op}}} \mathrm{CAT}$. Now, we additionally have a natural transformation $\Delta_{\mathcal{T}^{\mathrm{op}}} \Gamma^{\mathcal{T}} (\mathbb{C} \times \mathbb{C}^{\mathrm{op}}) \Longrightarrow \mathbb{C} \times \mathbb{C}^{\mathrm{op}}$, so we acquire a composite lax natural transformation

$$\Delta_{\mathcal{T}^{\mathrm{op}}} \Gamma^{\mathcal{T}} (\mathbb{C} \times \mathbb{C}^{\mathrm{op}}) \to \Delta_{\mathcal{T}^{\mathrm{op}}} \mathrm{CAT},$$

which corresponds under the universal property of lax limits with a functor

$$\underline{\mathrm{FUN}}_{\mathbb{C}}(-,-) \colon \Gamma^{\mathcal{T}} \mathbb{C} \times \Gamma^{\mathcal{T}} \mathbb{C}^{\mathrm{op}} \to \mathrm{CAT}_{\mathcal{T}};$$

now, this entire procedure is natural in $\mathcal{T}$, so in particular it lifts to a functor of $\mathcal{T}$-$(\infty,2)$-categories

$$\underline{\mathrm{FUN}}_{\mathbb{C}}(-,-) : \mathbb{C} \times \mathbb{C}^{\mathrm{op}} \to \underline{\mathrm{CAT}}_{\mathcal{T}},$$

which is in fact lax-natural in $\mathbb{C}$; unwinding definitions, we find given $V \in \mathcal{T}$ and $B, C \in \mathbb{C}_V$ that

$$\underline{\mathrm{FUN}}_{\mathbb{C}}(B,C)^V \simeq \mathrm{FUN}_{\mathbb{C}_V}(B,C).$$

Finally, we may curry this procedure, yielding a Yoneda functor of $\mathcal{T}$-$(\infty,2)$-categories

$$\yoneda : \mathbb{C} \to \underline{\mathrm{FUN}}_{\mathcal{T}}\left(\mathbb{C}^{\mathrm{op}}, \underline{\mathrm{CAT}}_{\mathcal{T}}\right)$$

◁

I.1.2.2 Parameterized $(\infty,2)$-categorical adjunctions We will often access the associated functor $\mathcal{T}$-categories using *adjunctions*; importantly, these work as expected.

Proposition I.1.64. *Fix $F : \mathbb{C} \to \mathbb{D}$ a $\mathcal{T}$-$(\infty,2)$-functor. The following conditions are equivalent.*

(a) F admits a right adjoint in $\mathrm{CAT}_{\mathcal{T},(\infty,2)}$.

(b) For all $V \in \mathcal{T}$, the $(\infty,2)$-functor $F_V : \mathbb{C}_V \to \mathbb{D}_V$ admits a right adjoint R_V and, for all maps $U \to V$, the Beck-Chevalley transformation

$$\beta : R_U \mathrm{Res}^V_U(-) \implies \mathrm{Res}^V_U R_V(-)$$

is an equivalence.

(c) The unstraightening $\mathrm{Tot}\, F : \mathrm{Tot}\, \mathcal{C} \to \mathrm{Tot}\, \mathcal{D}$ admits an $(\infty,2)$-right adjoint $\mathrm{Tot}\, R$ over $\mathcal{T}$ whose unit $x \to \mathrm{Tot}\, R\, \mathrm{Tot}\, F x$ lies over an equivalence in $\mathcal{T}$ and which is unstraightened from a $\mathcal{T}$-$(\infty,2)$-functor.

By *unstraightening* we mean Nuiten–Abellán-Stern's $(\infty,2)$-categorical straightening equivalence

$$\mathrm{FUN}\left(\mathbb{C}, \mathrm{CAT}_{(\infty,2)}\right) \simeq \mathrm{CAT}^{2-\mathrm{cocart}}_{(\infty,2),\mathbb{C}};$$

indeed this follows from the $d = 3$ case of [Nui23, Thm. C] in the special case that $\mathbb{C}$ is a 2-category or from [AS23, Cor. 3.90]

Proof. The equivalence between conditions (1) and (2) is [AGH24, Cor. 5.2.12]. Moreover, (1) is seen by [AGH24, Prop. 5.2.2] to correspond with existence of a $(\infty,2)$-right adjoint $\mathrm{Tot}\, R$ over $\mathcal{T}$ whose unit lies over an equivalence and such that $\mathrm{Tot}\, R$ lies in $\mathrm{Cat}_{\mathcal{T},(\infty,2)} \subset \mathrm{Cat}_{(\infty,2),/\mathcal{T}}$; this is precisely the remaining stipulated condition. □

In particular, we find a familiar notion of $\mathcal{T}$-adjunctions here.

Corollary I.1.65. *A $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-left adjoint if and only if it is a left adjoint in $\mathrm{CAT}_{\mathcal{T}}$.*

Proof. Since $\mathrm{CAT}_{\mathcal{T}} \hookrightarrow \mathrm{CAT}_{\mathcal{T},(\infty,2)}$ is full, we may instead prove this in $\mathrm{CAT}_{\mathcal{T},(\infty,2)}$; then, we may simply identify condition Proposition I.1.64 with Definition I.1.27. □

Now, we may use this to identify this in terms of representability.

Corollary I.1.66. *A functor of $\mathcal{T}$-$(\infty,2)$-categories $L\colon \mathbb{A} \to \mathbb{B}$ is a left adjoint if and only if for every $V \in \mathcal{T}$ and $A \in \mathbb{B}_V$, the V-presheaf*

$$\underline{\mathrm{Fun}}_{\mathrm{Res}^{\mathcal{T}}_V \mathbb{B}}(L-, B)\colon \mathrm{Res}^{\mathcal{T}}_V \mathbb{B}^{\mathrm{op}} \longrightarrow \underline{\mathrm{CAT}}_V$$

is representable, i.e. equivalent to $\text{よ}(R_V B)$ for some $R_V B \in \mathbb{B}_V$.

Proof. First, suppose this holds, and let $R_V B \in \mathbb{B}_V$ be the representing object. Then, composing with fixed points yields a natural equivalence

$$\mathrm{Fun}_{\mathbb{B}_V}(L-, B) \simeq \mathrm{Fun}_{\mathbb{A}_V}(-, R_V B),$$

so in particular, $R^V B$ is the value of a right adjoint to L^V for each $V \in \mathcal{T}$ by [AGH24, Cor. 5.1.2]. Now, naturality of the above natural equivalence in V moreover shows that the mate transformation $\mathrm{Res}^V_U R^V \to R^U$ is an equivalence, so these indeed assemble to a $\mathcal{T}$-right adjoint R to L. The converse follows by reversing the above argument. □

In particular, this provides a natural equivalence of functors of $\mathcal{T}$-$(\infty,2)$-categories

$$\underline{\mathrm{Fun}}_{\mathbb{B}}(L-,-) \simeq \underline{\mathrm{Fun}}_{\mathbb{A}}(-,R-)\colon \mathbb{A}^{\mathrm{op}} \times \mathbb{B} \longrightarrow \underline{\mathrm{CAT}}_{\mathcal{T}},$$

which we will find generally useful.

Lastly, we will need to describe the *relative* situation in the same way: following [Sha22, Def. 4.13], refer to the following notion as *$\mathcal{B}$-relative $\mathcal{T}$-adjunctions.*

Corollary I.1.67. *Given a pair of $\mathcal{T}$-functors $F\colon \mathcal{X} \leftrightarrows \mathcal{Y}\colon G$ over a $\mathcal{T}$-∞-category $\mathcal{B}$, $\mathrm{Tot}\, F$ is a $\mathrm{Tot}\,\mathcal{B}$-relative left adjoint to $\mathrm{Tot}\, G$ if and only if F is a left adjoint to G in $\mathrm{CAT}_{\mathcal{T},/\mathcal{B}}$.*

Proof. Since $\mathrm{CAT}_{\mathcal{T},/\mathcal{B}} \to \mathrm{CAT}_{/\mathrm{Tot}\,\mathcal{B}}$ is locally fully faithful, it suffices to prove this in the nonequivariant case; this is a specialization of [AGH24, Prop. 5.2.2] (c.f. [HA, Prop. 7.3.2.1]). □

I.1.3 Atomic orbital subcategories and admissible maps

Definition I.1.68. Given a small ∞-category $\mathcal{T}$, the ∞-category of *small $\mathcal{T}$-∞-categories* is given by

$$\mathrm{Cat}_{\mathcal{T}} = \mathsf{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$$

and the ∞-category of *$\mathcal{T}$-spaces* is the presheaf category

$$\mathcal{S}_{\mathcal{T}} = \mathsf{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S}).$$

We refer to the objects in the image of the Yoneda embedding $\mathcal{T} \to \mathcal{S}_{\mathcal{T}}$ as *orbits*. The category of *orbit-finite $\mathcal{T}$-spaces* is the coproduct completion

$$\mathbb{F}_{\mathcal{T}} = \left\{ \coprod_{i=0}^{n} U_i \mid U_i \in \mathcal{T}, n \in \mathbb{N} \right\} \subset \mathsf{Fun}(\mathcal{T}^{\mathrm{op}}, \mathcal{S}).$$

It is routine to verify that this is disjunctive, i.e. it satisfies the condition that

$$\mathbb{F}_{\mathcal{T}/S \sqcup T} \simeq \mathbb{F}_{\mathcal{T}/S} \times \mathbb{F}_{\mathcal{T}/T},$$

Given a wide subcategory $\mathcal{R} \subset \mathcal{T}$, we define the wide subcategory $\mathbb{F}_{\mathcal{R}} \subset \mathbb{F}_{\mathcal{T}}$ spanned by the morphisms of the form

$$\left\{ \coprod_{i \in I} \left(\coprod_{j \in J_i} U_j \to V_i \right) \mid \forall i, j \in J_i, \quad U_j \to V_i \in \mathcal{R} \right\};$$

that is, all maps of orbit-finite $\mathcal{T}$-spaces are induced under disjunctiveness and the universal property of coproducts by maps between orbits, and $\mathbb{F}_{\mathcal{R}}$ collects those built out of orbit maps lying in $\mathcal{R}$.

We say that a morphism of $\mathcal{T}$-spaces $f \colon X \to Y$ is *$\mathcal{R}$-admissible* if, for each fiber square

$$\begin{array}{ccc} f^{-1}(S) & \longrightarrow & Y \\ {\scriptstyle f_S}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ S & \xrightarrow{\iota} & X \end{array}$$

such that S is a orbit-finite $\mathcal{T}$-space, f_S is in $\mathbb{F}_{\mathcal{R}}$.

A wide subcategory $\mathcal{R} \subset \mathcal{T}$ is said to be an *orbital subcategory* if $\mathbb{F}_{\mathcal{R}} \subset \mathbb{F}_{\mathcal{T}}$ consists precisely of the $\mathcal{R}$-admissible maps; $\mathcal{R}$ is additionally *atomic* if all maps in $\mathcal{R}$ which admit a section in $\mathcal{T}$ are equivalences.

A category $\mathcal{T}$ is *orbital* (resp. *atomic orbital*) if $\mathcal{T} \subset \mathcal{T}$ is an orbital (atomic orbital) subcategory. ◁

Given a $\mathcal{T}$-category $\mathcal{C} \in \mathsf{Fun}(\mathcal{T}^{\mathrm{op}}, \mathrm{Cat})$ and an orbit $T \in \mathcal{T}$, we write $\mathcal{C}^T$ for the value of $\mathcal{C}$ at T.

Remark I.1.69. Choosing ι to be the identity, it follows that all $\mathcal{R}$-admissible maps between orbit-finite $\mathcal{T}$-spaces lie in $\mathbb{F}_{\mathcal{R}}$ regardless of orbitality; in particular, our definition recovers [CLL23, Def. 4.2.2] by [CLL23,

Lem. 4.3.2]. That is, $\mathcal{R} \subset \mathcal{T}$ is orbital if and only if, for all $V \subset W$ $\mathbb{F}_{\mathcal{T},/V} \to \mathcal{S}_{\mathcal{T},/W}$ takes values in $\mathbb{F}_{\mathcal{T},/W}$ and moreover for all $\mathcal{T}$-diagrams

$$\begin{array}{ccc} U' & \longrightarrow & U \\ \downarrow & & \downarrow f \\ V' & \longrightarrow & V \end{array}$$

with f in $\mathcal{T}$ and the induced map $U' \to f^*U$ a summand inclusion, $U' \to V'$ lies in $\mathcal{R}$. ◁

Example I.1.70. For G a compact Lie group, we can choose $\mathcal{T}$ to be the orbit category $\mathcal{O}_G$ of G and $\mathcal{R}$ to be the wide subcategory $\mathcal{O}_G^{f.i.}$ the class of maps with finite fibers. This is an atomic orbital subcategory, and an example of primary interest. ◁

Now, we review a variant of a characterization of atomic orbitality due to Cnossen-Lenz-Linskens.

Lemma I.1.71. *An orbital subcategory $\mathcal{R} \subset \mathcal{T}$ is atomic if and only if, for all composable sequences of maps $K \xrightarrow{g} T \xrightarrow{f} S$ between orbit-finite $\mathcal{T}$-spaces such that f is $\mathcal{R}$-admissible, the associated map ι*

$$\begin{array}{ccccc} K & & \xrightarrow{g} & & \\ & \searrow^{\iota} & & & \\ & & K \times_S T & \longrightarrow & T \\ & & \downarrow \pi & \lrcorner & \downarrow f \\ & & K & \xrightarrow{gf} & S \end{array}$$

is a summand inclusion, with pullback formed in $\mathcal{S}_{\mathcal{T}}$; moreover, it suffices to verify this in the case g is an equivalence.

Proof. [CLL23, Lem. 4.3.2] shows that atomic orbitality is equivalent to the condition in the case g is an equivalence, so it suffices to show that atomic orbitality implies this condition.

In the case $K = \varnothing$, the statement is obvious, so we may assume K is nonempty. In this case the trick is that colimits in $\mathcal{S}_{\mathcal{T}}$ are universal, as it is a topos [HTT, Thm. 6.1.0.6]; in particular, given a coproduct expression $K = K' \sqcup K''$, the lefthand triangle decomposes as a direct sum

$$\begin{array}{ccccc} K & \xrightarrow{\quad\iota\quad} & K \times_S T & \xrightarrow{\quad\pi\quad} & K \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ K' \sqcup K'' & \xrightarrow{\iota' \sqcup \iota''} & K' \times_S T \sqcup K'' \times_S T & \xrightarrow{\iota' \sqcup \iota''} & K' \sqcup K'' \end{array}$$

This shows that the statement for ι' and ι'' implies that of ι, so we may assume that K has one summand, i.e. it lies in the Yoneda image $\mathcal{T}^{\mathrm{op}} \subset \mathcal{S}_{\mathcal{T}}$. In particular, letting $K' \subset K \times_S T$ be the orbit in the image of ι, the lefthand triangle restricts to a sequence

$$K \xrightarrow{\bar{\iota}} K' \xrightarrow{\overline{\pi}} K$$

whose composite is the identity; by atomicity, $\overline{\pi}$ is an equivalence, so $\bar{\iota}$ is as well, implying that ι is a summand inclusion. □

Remark I.1.72. Said another way, Lemma I.1.71 characterizes atomicity equivalently as *0-truncatedness* of $\mathcal{R}$-admissible maps between orbit-finite $\mathcal{T}$-spaces. But given an arbitrary $\mathcal{R}$-admissible map $f : X \to Y$ Pulling back along the canonical effective epimorphism $\coprod_{\substack{V \to S \\ V \in \mathcal{T}}}$ and applying [HTT, Prop. 6.2.3.17] shows that f is 0-truncated if and only if its fibers along maps from orbits are 0-truncated. Applying the lemma, we find that an orbital subcategory is atomic if and only if every $\mathcal{R}$-admissible map is 0-truncated. ◁

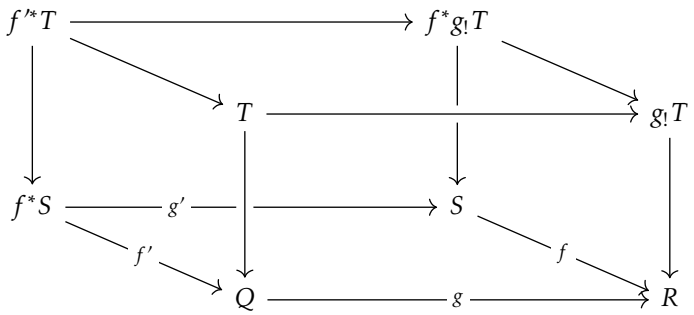
Using this, we may identify the *universal* atomic orbital subcategory.

Proposition I.1.73. *Fix $\mathcal{T}$ an ∞-category.*

(1) Let $\widetilde{\mathfrak{o}}(\mathcal{T}) \subset \mathcal{T}$ be the wide subcategory spanned by arrows $f : V \to W$ such that the postcomposition arrow $f^* : \mathbb{F}_{\mathcal{T},/W} \to \mathcal{S}_{\mathcal{T},/V}$ takes values in $\mathbb{F}_{\mathcal{T},/V}$. Then, $\widetilde{\mathfrak{o}}(\mathcal{T}) \subset \mathcal{T}$ is the terminal object within the poset of orbital subcategories of $\mathcal{T}$.

(2) Let $\mathfrak{o}(\mathcal{T}) \subset \widetilde{\mathfrak{o}}(\mathcal{T})$ be the wide subcategory of maps $f : V \to W$ which are 0-truncated as maps lying in the ∞-topos $\mathcal{S}_{\mathcal{T}}$. Then, $\mathfrak{o}(\mathcal{T}) \subset \mathcal{T}$ is an atomic orbital subcategory, and $\mathcal{R} \subset \mathcal{T}$ an orbital subcategory is atomic if and only if $\mathcal{R} \subset \mathcal{T}$.

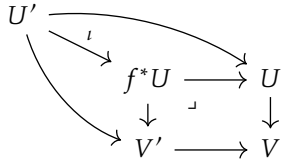
Proof. If $\widetilde{\mathfrak{o}}(\mathcal{T}) \subset \mathcal{T}$ is orbital, then it is obviously terminal. To show that it is orbital, by Remark I.1.69 we are tasked with proving that $\mathbb{F}_{\widetilde{\mathfrak{o}}(\mathcal{T})} \subset \mathcal{S}_{\mathcal{T}}$ is pullback-stable along maps from orbit-finite $\mathcal{T}$-spaces. To do so, draw a cube in $\mathcal{S}_{\mathcal{T}}$



now, the bottom and right faces are defined in order to be cartesian, so the left face is cartesian if and only if the top is. The left face is also cartesian, so the top is as well; but $T \to g_! T$ is simply an equivalence in $\mathcal{S}_{\mathcal{T}}$, so $f'^* T \to f^* g_! T$ is an equivalence in $\mathcal{S}_{\mathcal{T}}$.

Now, if f lies in $\mathbb{F}_{\widetilde{\mathfrak{o}}(\mathcal{T})}$, and Q, T in $\mathbb{F}_{\mathcal{T}}$, we have $g_! T \in \mathbb{F}_{\mathcal{T},/R}$ so $f^* g_! T \in \mathbb{F}_{\mathcal{T},/S}$, i.e. $f'^T \in \mathbb{F}_{\mathcal{T},/f^*S}$.

For (2), using Remark I.1.69 and (1), to show that $\mathfrak{o}(T)$ is orbital it suffices to prove given a square



such that f is 0-truncated and ι is a summand inclusion, the composite map $U' \to V'$ is 0-truncated; but summand inclusions are (-1)-truncated (and hence 0-truncated), so this is just composition-stability of truncatedness. Now, to show that it represents atomicity is simply Lemma I.1.71. □

Example I.1.74. $\mathcal{T}^{\simeq} \subset \mathcal{T}$ is the minimal orbital subcategory, and it is atomic. ◁

Example I.1.75. If G is a compact lie group, then $\widetilde{\mathfrak{o}}(\mathcal{O}_G)$ is the wide subcategory $\mathcal{O}_G^{\text{f.i.}} \subset \mathcal{O}_G$ of maps $G/K \to G/H$ with finite fibers, i.e. maps whose closed subconjugacy relations that admit a model $K \subset H$ which is finite index; indeed, this follows by the double coset formula. Moreover, every surjective closed subgroup inclusion is an isomorphism, so $\mathcal{O}_G^{f.i.}$ is atomic, i.e. $\mathcal{O}_G^{\text{f.i.}} = \mathfrak{o}(\mathcal{O}_G)$ as well.

Similarly, let Glo be the global indexing ∞-category, e.g. as recalled in [CLL23, Ex. 4.2.7]; its objects are compact lie groups G and its morphism spaces are $\mathsf{Hom}_{\mathsf{TopGrp}}(G,H)_{hH}$, where H acts via conjugation. Let $\mathsf{Orb} \subset \mathsf{Glo}$ be the wide subcategory of finite-index injective homomorphisms. We claim that $\mathsf{Orb} \simeq \mathfrak{o}(\mathsf{Glo})$. Indeed, the same argument as [CLL23, Ex. 4.3.3] shows that all $\mathcal{S}_{\mathsf{Glo}}$-0-truncated morphisms between representables classify closed subgroup inclusions, and the previous example shows that orbitality requires them to be finite-index.

This specializes to finite groups well: the full subcategory $\mathsf{Glo}^{\mathsf{fin}} \subset \mathsf{Glo}$ spanned by finite groups similarly has $\mathfrak{o}\left(\mathsf{Glo}^{\mathsf{fin}}\right) = \mathsf{Orb}^{\mathsf{fin}}$ the wide subcategory of injective group homomorphisms, and $\mathcal{O}_G$ itself is atomic orbital whenever G is a finite group. ◁

Notation I.1.76. We write $\mathbb{F}_{\mathcal{R}}^V \subset \mathcal{S}_V$ for the full subcategory spanned by $\mathcal{R}$-admissible maps $X \to V$. ◁

We will need several niceness properties of $\mathcal{R}$-admissible maps, so we establish them now.

Lemma I.1.77. *Let $\mathcal{R} \subset \mathcal{T}$ be an orbital subcategory.*

(1) Equivalences are $\mathcal{R}$-admissible.

(2) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a composable sequence of $\mathcal{R}$-admissible maps, then gf is $\mathcal{R}$-admissible.

(3) Given a pullback diagram in $\mathcal{S}_{\mathcal{T}}$

$$\begin{array}{ccc} Z \times_X Y & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ Z & \longrightarrow & X \end{array}$$

such that f is $\mathcal{R}$-admissible, f' is $\mathcal{R}$-admissible.

(4) The inclusions $\mathbb{F}^V_{\mathcal{R}} \subset \mathcal{S}_V$ assemble to a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_{\mathcal{R}} \subset \underline{\mathcal{S}}_{\mathcal{T}}$.

(5) $\underline{\mathbb{F}}_{\mathcal{R}}$ is a $\mathcal{T}$-1-category if and only if $\mathcal{R}$ is atomic.

(6) $\underline{\mathbb{F}}_{\mathcal{R}} \subset \underline{\mathcal{S}}_{\mathcal{T}}$ is closed under binary products.

(7) If $f\colon X \to Y$ and $f'\colon X' \to Y'$ are $\mathcal{R}$-admissible, then the external product

$$f \times f'\colon X \times X' \longrightarrow Y \times Y'$$

is $\mathcal{R}$-admissible. Hence $\mathcal{R}$-admissible maps form a product-closed wide subcategory $\mathcal{S}^{\mathcal{R}}_{\mathcal{T}} \subset \mathcal{S}_{\mathcal{T}}$.

(8) $\underline{\mathbb{F}}_{\mathcal{R}} \subset \underline{\mathcal{S}}_{\mathcal{T}}$ is closed under $\underline{\mathbb{F}}_{\mathcal{R}}$-indexed coproducts. Hence $\underline{\mathbb{F}}_{\mathcal{R}}$ admits finite indexed coproducts.

Proof. Statement (1) follows by noting that each f_S is an equivalence and $\mathcal{R}$ contains all equivalences between orbit-finite $\mathcal{T}$-spaces by assumption. Statement (2) follows by compositing pullback squares; indeed we have $(gf)_S = g_{f^{-1}(S)} f_S$, and $f^{-1}(S)$ is an orbit finite $\mathcal{T}$-space by assumption, so $(gf)_S$ is a composition of maps in $\mathbb{F}_{\mathcal{R}}$, implying it lies in $\mathbb{F}_{\mathcal{R}}$. Statement (3) also follows by pasting pullback squares, noting that $f'_S = f_S$. Statement (4) follows immediately from Statement (3); Note that the restriction functor $\mathbb{F}^V_{\mathcal{R}} \to \mathbb{F}^U_{\mathcal{R}}$ is given by pulling back along the morphism $U \to V$. For Statement (5), this follows by Remark I.1.72, as each is equivalent to each object in each $\mathbb{F}^V_{\mathcal{R}}$ being 0-truncated.

For Statement (6), note that the proof of [NS22, Prop. 3.25] goes through without modification to show that non-parameterized products in $\underline{\mathcal{S}}[\mathcal{T}]$ preserve $\mathcal{T}$-colimits separately in each variable. In particular, given $S, T \in \mathbb{F}^V_{\mathcal{R}}$, we have

$$S \times T \simeq S \times \coprod_{U \in \mathrm{Orb}(T)} U \simeq \coprod_{U \in \mathrm{Orb}(T)} \mathrm{Ind}^V_U \mathrm{Res}^V_U S$$

Now, $\underline{\mathbb{F}}_{\mathcal{R}} \subset \underline{\mathcal{S}}[\mathcal{T}]$ is closed under restrictions and coproducts by fiat, and closure under induction along $\mathcal{R}$-admissible maps follows by unwinding definitions. This implies that $S \times T$ lies in $\mathbb{F}^V_{\mathcal{R}}$, proving the claim.

For Statement (7), note that each fiber square extends to a pair of fiber squares

$$\begin{array}{ccccc} (f\times f')^{-1}(S) & \longrightarrow & f^{-1}(S)\times f'^{-1}(S) & \longrightarrow & Y\times Y' \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f_S\times f_{S'}} & \lrcorner & \downarrow \\ S & \xrightarrow{\Delta} & S\times S & \xrightarrow{(\pi_X i, \pi_{X'} i)} & X\times X' \end{array}$$

It suffices to prove that $f_S \times f_{S'}$ lies in $\mathbb{F}_{\mathcal{R}}$. In particular, picking two orbits $U, V \subset S$ and applying disjunctiveness shows that it suffices to prove that the left hand arrow of the following lies in $\mathbb{F}_R$ for all maps $\iota_{WU}\colon W \to U$ and $\iota_{WV}\colon W \to V$.

$$\begin{array}{ccccc} \iota^*_{WU} f^{-1}(U) \times_W \iota^*_{WV} f'^{-1}(V) & \longrightarrow & f^{-1}(U)\times f'^{-1}(V) & \longrightarrow & Y\times Y' \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ W & \longrightarrow & U\times V & \longrightarrow & X\times X' \end{array}$$

Now, $f^{-1}(U)$ and $f^{-1}(W)$ lie in $\underline{\mathbb{F}}_{\mathcal{R}}$ by fiat, so $\iota_{WU}^* f^{-1}(U)$ and $\iota_{WV}^* f'^{-1}(V)$ lie in $\underline{\mathbb{F}}_{\mathcal{R}}$ by Statement (4). The product lies in $\underline{\mathbb{F}}_{\mathcal{R}}$ by Statement (6), and unwinding definitions, this says that the lefthand arrow lies in $\mathbb{F}_{\mathcal{R}}$.

For Statement (8), note that every object in $\underline{\mathbb{F}}_{\mathcal{R}}$ is a finite index of orbits; in particular, it suffices to prove that $\underline{\mathbb{F}}_{\mathcal{R}} \subset \underline{\mathcal{S}}_{\mathcal{T}}$ is closed under finite coproducts and induction along $\mathcal{R}$-maps. For finite coproducts, this follows immediately by definition. For induction along $\mathcal{R}$-maps, note that induction $\mathcal{S}_U \simeq \mathcal{S}_{V,/U} \to \mathcal{S}_V$ is simply postcomposition along the map $U \to V$; if $S \to U$ is $\mathcal{R}$-admissible and $U \to V$ is $\mathcal{R}$-admissible, then $S \to V$ is as well by Statement (2), i.e. $\mathrm{Ind}_U^V S \in \mathbb{F}_R^V$. □

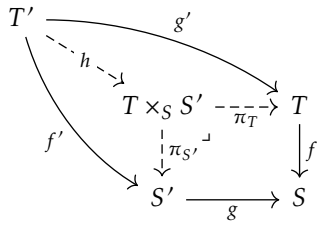
From this, we set some important notation.

Definition I.1.78. The $\mathcal{T}$-∞-*category of finite* $\mathcal{T}$-*sets* is $\underline{\mathrm{Fin}}_{\mathcal{T}} \coloneqq \underline{\mathbb{F}}_{\mathfrak{o}(\mathcal{T})}$. ◁

An occasionally useful lemma is the following.

Lemma I.1.79. *Suppose $\mathcal{R} \subset \mathcal{T}$ is an atomic orbital subcategory. Then, $\mathcal{R}$ is atomic orbital as an ∞-category.*

Proof. First, assume we have a square in $\mathbb{F}_{\mathcal{R}}$; since $\mathcal{R} \subset \mathcal{T}$ is an orbital subcategory, we may extend our square to a pullback diagram $\mathbb{F}_{\mathcal{T}}$



To prove that $\mathcal{R}$ is orbital, it suffices to verify that the inner square is a pullback diagram lying in $\mathcal{R}$; to check that it lies in $\mathcal{R}$ we are tasked with verifying that $\pi_{S'}$ and $\pi_{\mathcal{T}}$ are in $\mathcal{R}$ and to check that it's a pullback we are tasked with verifying that h lies in $\mathcal{R}$. In fact, $\pi_{S'}$ and π_T are in $\mathcal{R}$ since $\mathcal{R} \subset \mathcal{T}$ is an orbital subcategory; h is then in $\mathcal{R}$ since atomic orbital subcategories are left cancellable by [CLL23, Lem. 4.3.5].

We've proved that $\mathcal{R}$ is orbital. Moreover, atomicity follows from atomicity in $\mathcal{T}$. □

Example I.1.80. Given X a space considered as an ∞-category, X is atomic orbital; by [Gla18, Thm 2.13], the associated stable ∞-category is the Ando-Hopkins-Rezk ∞-category of parameterized spectra over X (c.f. [ABGHR14]). In particular, for $X = BG$, this recovers *spectra with G-action*. ◁

I.1.4 Weak indexing systems and weak indexing categories

I.1.4.1 Definitions We say that a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathrm{Fin}}_{\mathcal{T}}$ is *closed under self-indexed coproducts* if it is closed under $\mathcal{C}$-indexed coproducts.

Definition I.1.81. A $\mathcal{T}$*-weak indexing system* is a full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathsf{Fin}}_{\mathcal{T}}$ with V-values $\mathbb{F}_I^V := (\underline{\mathbb{F}}_I)^V$ satisfying the following conditions:

(IS-a) whenever $\mathbb{F}_I^V \neq \varnothing$, we have $*_V \in \mathbb{F}_I^V$; and

(IS-b) $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts.

We denote by $\mathsf{wIndex}_{\mathcal{T}} \subset \mathsf{FullSub}_{\mathcal{T}}(\underline{\mathsf{Fin}}_{\mathcal{T}})$ the embedded sub-poset spanned by weak indexing systems. Moreover, we say that a I *has one color* if it satisfies the following condition:

(IS-i) for all $V \in \mathcal{T}$, we have $\mathbb{F}_I^V \neq \varnothing$;

We say that a I-weak is *almost-unital* if it has one color and satisfies the following condition:

(IS-ii) for all V-sets $S \sqcup S' \in \mathbb{F}_I^V$ such that $S, S' \neq \varnothing_V$, we have $S, S' \in \mathbb{F}_I^V$.

We say that I is *unital* if it has one color and satisfies the following condition"

(IS-iii) for all V-sets $S \sqcup S' \in \mathbb{F}_I^V$, we have $S, S' \in \mathbb{F}_{I,V}$.

Lastly, we say a I is an *indexing system* if it satisfies the following condition:

(IS-iv) $\mathbb{F}_I^V \subset \mathbb{F}_{\mathcal{R}}^V$ is closed under finite coproducts for all $V \in \mathcal{T}$.

We denote the resulting posets by

$$\mathsf{Index}_{\mathcal{T}} \subset \mathsf{wIndex}_{\mathcal{T}}^{\mathsf{uni}} \subset \mathsf{wIndex}_{\mathcal{T}}^{a\mathsf{uni}} \subset \mathsf{wIndex}_{\mathcal{T}}^{\mathsf{oc}} \subset \mathsf{wIndex}_{\mathcal{T}}. \qquad \triangleleft$$

Remark I.1.82. The indexing systems of [BH15] are seen to be equivalent to ours when $\mathcal{T} = \mathcal{O}_G$ by unwinding definitions and noting that $\mathcal{O}_G$ itself is atomic orbital. The weak indexing systems of [BP21; Per18] are equivalent to our *unital* weak indexing systems when $\mathcal{T} = \mathcal{O}_G$ by [Per18, Rem 9.7] and [BP21, Rem 4.60]. $\triangleleft$

Observation I.1.83. Denote by $\mathsf{Ind}_V^{\mathcal{T}} S \to V$ the map corresponding with a finite V-set S under the equivalence $\mathbb{F}_V \simeq \mathbb{F}_{\mathcal{T}/V}$. This equivalence implies that a full $\mathcal{T}$-subcategory $\mathcal{C} \subset \underline{\mathsf{Fin}}_{\mathcal{T}}$ is determined by the subgraph

$$\mathtt{I}(\mathcal{C}) := \left\{ \coprod_i \mathsf{Ind}_{V_i}^{\mathcal{T}} S_i \to V_i \;\middle|\; \forall i, \quad S \in \mathcal{C}_{V_i} \right\} \subset \mathbb{F}_{\mathcal{T}}.$$

In other words, the construction $\mathtt{I}$ yields an embedding of posets

$$\mathtt{I}(-) \colon \mathsf{wIndex}_{\mathcal{T}} \hookrightarrow \mathsf{Sub}_{\mathsf{graph}}(\mathsf{ho}\mathbb{F}_{\mathfrak{o}(\mathcal{T})}). \qquad \triangleleft$$

Our main result about this construction is the following.

Proposition I.1.84. *The image of* $\mathtt{I}(-)$ *consists of the subcategories* $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ *satisfying the following conditions*

(IC-a) (restrictions) I *is stable under arbitrary pullbacks in* $\mathbb{F}_{\mathcal{T}}$*, and all such pullbacks exist;*

(IC-b) (segal condition) the pair $T \to S$ *and* $T' \to S'$ *are in* I *if and only if* $T \sqcup T' \to S \sqcup S'$ *is in* I*; and*

Moreover, for all numbers n, condition (IS-n) of Definition I.1.81 is equivalent to condition (IC-n) below:

(IC-i) (one color) I is wide; equivalently, I contains $\mathbb{F}_{\mathcal{T}}^{\simeq}$.

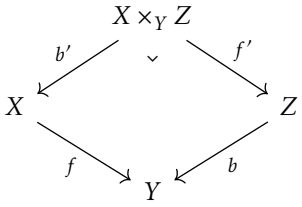
(IC-ii) (almost unital) I is wide, and if S is non-contractible then $\varnothing \to S$ is in I.

(IC-iii) (unital) for all S, $\varnothing \to S$ is in I.

(IC-iv) (indexing category) the fold maps $n \cdot V \to V$ are in I for all $n \in \mathbb{N}$ and $V \in \mathcal{T}$.

We refer to the image of $\mathfrak{I}(-)$ as the *weak indexing categories* $\mathsf{wIndexCat}_{\mathcal{T}} \subset \mathsf{Sub}_{\mathsf{Cat}}(\mathbb{F}_{\mathfrak{o}(\mathcal{T})})$. In general, we will refer to a generic weak indexing category as I and its corresponding weak indexing system as $\underline{\mathbb{F}}_I$.

Definition I.1.85 ([Bar14, Def. 5.2])**.** An *adequate triple* is a pair of wide subcategory inclusion $\mathcal{X}^b, \mathcal{X}^f \subset \mathcal{X}$ subject to the condition that all cospans $X \xrightarrow{f} Y \xleftarrow{b} Z$ with $f \in \mathcal{X}^f$ and $b \in \mathcal{X}^b$ extend to $\mathcal{X}$-pullback diagrams



such that $f' \in \mathcal{X}^f$ and $b' \in \mathcal{X}^b$. In the case $\mathcal{X} = \mathcal{X}^b$, following [CHLL24b; EH23] we call $\left(\mathcal{X}^f, \mathcal{X}\right)$ a *span pair*. ◁

Remark I.1.86. Condition (IC-a) is equivalent to the condition that $(\mathbb{F}_{\mathcal{T}}, I)$ is a span pair, and Condition (IC-b) is easily seen to be equivalent to the conditions of [CHLL24b, Def. 2.2.1] in the case $\mathcal{C} = \mathbb{F}_{\mathcal{T}}$, so our weak indexing *categories* are equivalent to *weakly extensive span pairs* $(\mathcal{C}, I)$ such that $\mathcal{C} = \mathbb{F}_{\mathcal{T}}$ and $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$. Moreover, we will see in Lemma I.1.96 shows that indexing categories are equivalent to *extensive span pairs* as defined in *loc. cit.* ◁

The following observations motivate and help to prove Proposition I.1.84.

Observation I.1.87. By a basic inductive argument, Condition (IC-b) is equivalent to the following condition:

(IC-b') $T \to S$ is in I if and only if $T_U = T \times_S U \to U$ is in I for all $U \in \mathsf{Orb}(S)$.

In particular, I is uniquely determined by the maps to orbits. ◁

Observation I.1.88. By Observation I.1.87, in the presence of Condition (IC-b), Condition (IC-a) is equivalent to the following condition:

(IC-a') for all Cartesian diagrams in $\mathbb{F}_{\mathcal{T}}$

$$\begin{array}{ccc} T \times_V U & \longrightarrow & T \\ \downarrow{\scriptstyle \alpha'} & \lrcorner & \downarrow{\scriptstyle \alpha} \\ U & \longrightarrow & V \end{array} \tag{2}$$

with $U, V \in \mathcal{T}$ and $\alpha \in I$, we have $\alpha' \in I$. ◁

Now, given $\mathcal{D}$ a $\mathcal{T}$-∞-category, we use the term *collection* to mean the data of, for each $V \in \mathcal{T}$, a full subcategory $\mathcal{C}^V \subset \mathcal{D}^V$, without compatibility assumptions. Our candidate inverse to $\mathbb{I}$ is the following.

Construction I.1.89. Given $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ a subcategory, define the class of *I-admissible objects*

$$\mathbb{F}_I^V \coloneqq \left\{ S \;\middle|\; \mathrm{Ind}_V^{\mathcal{T}} S \to V \in I \right\} \subset \mathrm{Fin}_V.$$

Taken altogether, we refer to the associated collection as $\underline{\mathbb{F}}_I \subset \underline{\mathrm{Fin}}_{\mathcal{T}}$. ◁

Observation I.1.90. Given $\mathcal{C} \subset \underline{\mathrm{Fin}}_{\mathcal{T}}$ a collection, we have $\underline{\mathbb{F}}^V_{\mathbb{I}(\mathcal{C})} \simeq \mathcal{C}$; conversely, if a subcategory $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ satisfies Condition (IC-b), then $\mathbb{I}(\underline{\mathbb{F}}_I) = I$. ◁

Observation I.1.91. By definition, the restriction functor $\mathrm{Res}_V^W \colon \mathbb{F}_W \to \mathbb{F}_V$ is implemented by the pullback

$$\begin{array}{ccc} \mathrm{Ind}_V^{\mathcal{T}} \mathrm{Res}_V^W S & \longrightarrow & \mathrm{Ind}_W^{\mathcal{T}} S \\ \downarrow & \lrcorner & \downarrow \\ V & \longrightarrow & W \end{array}$$

thus I satisfies Condition (IC-a') if and only if $\mathrm{Res}_V^W \mathbb{F}_I^W \subset \mathbb{F}_I^V$ for all orbit maps $V \to W$; in particular, in this case, $\left\{\mathbb{F}_I^V\right\}_{V \in \mathcal{T}}$ corresponds with a unique full $\mathcal{T}$-subcategory $\underline{\mathbb{F}}_I \subset \underline{\mathbb{F}}_{\mathcal{R}}$. ◁

The following is fundamental to passing between weak indexing categories and weak indexing systems.

Observation I.1.92. Let $(T_U) \in \mathbb{F}_{\mathcal{R}}^S$ be an S-tuple of elements of $\underline{\mathbb{F}}_{\mathcal{R}}$ for some $S \in \mathbb{F}_{\mathcal{R}}^V$. Then, the indexed coproduct of (T_U) corresponds with the composite arrow

$$\mathrm{Ind}_V^{\mathcal{T}} \coprod_U^S T_U = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_V^{\mathcal{T}} \mathrm{Ind}_U^V T_U = \coprod_{U \in \mathrm{Orb}(S)} \mathrm{Ind}_U^{\mathcal{T}} T_U \to \mathrm{Ind}_V^{\mathcal{T}} S \to V; \tag{3}$$

in particular, if $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is a subcategory satisfying Condition (IC-b) and (T_U) and S are I-admissible, both arrows in Eq. (3) lie in I, so the structure map of $\coprod_U^S T_U$ is in I, i.e. $\coprod_U^S T_U \in \underline{\mathbb{F}}_I$. In other words, Condition (IC-b) and the condition that $I \subset \mathbb{F}_{\mathcal{R}}$ is a subcategory together imply that $\underline{\mathbb{F}}_I$ satisfies Condition (IS-b).

On the other hand, if $I \subset \mathbb{F}_{\mathcal{O}(\mathcal{T})}$ is a collection of arrows satisfying Condition (IC-b) such that $\underline{\mathbb{F}}_I$ satisfies Condition (IS-b), then taking coproducts of Eq. (3) shows that I is closed under composition. If I additionally is closed under equivalence (such as being closed under pullback), this implies that $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is a subcategory. ◁

We are now ready to verify that $\mathbb{I}(-)$ and $\underline{\mathbb{F}}_{(-)}$ restrict to maps between $\mathsf{wIndex}_{\mathcal{T}}$ and $\mathsf{wIndexCat}_{\mathcal{T}}$.

Lemma I.1.93. *If $\mathcal{C} \subset \underline{\mathrm{Fin}}_{\mathcal{T}}$ is a weak indexing system, then $\mathbb{I}(\mathcal{C})$ is a weak indexing category.*

PROOF. By Observations I.1.87 and I.1.88, it suffices to verify that $\mathbb{I}(\mathcal{C}) \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is a subcategory satisfying Conditions (IC-a') and (IC-b'). Condition (IC-a') is verified by Observation I.1.91; Condition (IC-b') follows immediately from construction; these together with Observation I.1.92 verify that $\mathbb{I}(\mathcal{C}) \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is a subcategory. □

Lemma I.1.94. *If $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is a weak indexing category, then $\underline{\mathbb{F}}_I$ is a weak indexing system.*

PROOF. Observation I.1.91 verifies that $\underline{\mathbb{F}}_I \subset \underline{\mathsf{Fin}}_{\mathcal{T}}$ is a full $\mathcal{T}$-subcategory, and the fact that the identity arrow on V corresponds with the contractible V-set implies that whenever $\underline{\mathbb{F}}_I^V \neq \varnothing$ (i.e. $V \in I$), $*_V \in \underline{\mathbb{F}}_I^V$. Thus it suffices to verify that $\underline{\mathbb{F}}_I$ is closed under self-indexed coproducts; this is Observation I.1.92. □

Having done this, we're poised to conclude that $\mathbb{I}(-)$ and $\underline{\mathbb{F}}_-$ are inverse equivalences. To handle the conditions, we introduce some invariants. Write

$$n \cdot S \coloneqq \overbrace{S \sqcup \cdots \sqcup S}^{n\text{-fold}}.$$

Lemma I.1.95. *Given $\underline{\mathbb{F}}_I$ a $\mathcal{T}$-weak indexing system, the following are $\mathcal{T}$-families:*

$$c(I) \coloneqq \left\{V \in \mathcal{T} \mid *_V \in \mathbb{F}_I^V\right\}$$
$$\upsilon(I) \coloneqq \left\{V \in \mathcal{T} \mid \varnothing_V \in \mathbb{F}_I^V\right\}$$
$$\nabla(I) \coloneqq \{V \in \mathcal{T} \mid 2 \cdot *_V \in \mathbb{F}_I\}$$

PROOF. This follows by noting that $\mathrm{Res}^V_U n \cdot *_V = n \cdot *_U$. □

We call $c(I)$ the *color family of* I, $\upsilon(I)$ the *unit family*, and $\nabla(I)$ the *fold map family*. Note that $c(I) \leq \upsilon(I) \cap \nabla(I)$; that is, Condition (IS-a) implies that whenever $\underline{\mathbb{F}}_I$ prescribes a unit or a fold map over V, it possesses a color over V. We will use the following lemma ubiquitously.

Lemma I.1.96. *Let $\underline{\mathbb{F}}_I$ be a $\mathcal{T}$-weak indexing system.*

(1) $\underline{\mathbb{F}}_I$ has one color if and only if $c(I) = \mathcal{T}$.

(2) $\underline{\mathbb{F}}_I$ is unital if and only if $\upsilon(I) = \mathcal{T}$.

(3) $\underline{\mathbb{F}}_I$ is an indexing system if and only if $\upsilon(I) \cap \nabla(I) = \mathcal{T}$.

PROOF. (1) follows immediately by unwinding definitions. For (2), if $\underline{\mathbb{F}}_I$ is unital, then choosing $\varnothing_V \sqcup *_V \in \mathbb{F}_I^V$ yields $\varnothing_V \in \mathbb{F}_I^V$. Conversely, if $\upsilon(I) = \mathcal{T}$ and $S \sqcup S' \in \mathbb{F}_I^I$, then

$$S = \coprod_U^{S \sqcup S'} \chi_S(U), \qquad \text{where } \chi_S(U) \coloneqq \begin{cases} *_U & U \in S \\ \varnothing_U & U \notin S \end{cases}$$

so $S \in \underline{\mathbb{F}}_I$, i.e. $\underline{\mathbb{F}}_I$ is unital.

For (3), note that $\underline{\mathbb{F}}_I$ an indexing system implies that $\upsilon(I) \cap \nabla(I) = \mathcal{T}$ by taking nullary and binary coproducts of $*_V \in \mathbb{F}_{I,V}$. Conversely, if $\upsilon(I) \cap \nabla(I) = \mathcal{T}$, then by iterating binary coproducts $(n-1)$-times, we find that $n \cdot *_V = (*_V \sqcup (n-1) \cdot *_V) \in \mathbb{F}_{I,V}$ for all $V \in \mathcal{T}$ and $n \in \mathbb{N}$. Applying Condition (IS-b), we find that $\mathbb{F}_{I,V}$ is closed under n-ary coproducts for all $n \in \mathbb{N}$, i.e. $\underline{\mathbb{F}}_I$ is an indexing system. □

PROOF OF PROPOSITION I.1.84. By Lemmas I.1.93 and I.1.94, $\mathfrak{I} \colon \mathsf{wIndex}_{\mathcal{T}} \rightleftarrows \mathsf{wIndexCat}_{\mathcal{T}} : \underline{\mathbb{F}}_{(-)}$ are well defined monotone maps; by Observation I.1.90, they are inverse to each other, so they are equivalences.

What remains is to verify that (IC-n) is equivalent to (IS-n) in Definition I.1.81 and Proposition I.1.84. For $n = i$, this follows immediately by noting that $V \in I \iff \mathrm{id}_V \in I \iff *_V \in \mathbb{F}_I^V \iff \mathbb{F}_I^V \neq \varnothing$. For $n = ii$ and $n = iii$, this follows by unwinding definitions using Condition (IC-b'). For $n = iv$, this follows by noting that the fold map $n \cdot V \to V$ corresponds with the element $n \cdot *_V \in \mathbb{F}_V$. □

In the case of indexing systems, I bears significant resemblance to $\mathbb{F}_{\mathcal{R}}$. We bear this out in the following.

I.1.4.2 Example: transfer systems Given I a weak indexing category, we define $\mathfrak{R}(I) \coloneqq I \cap \mathcal{T}$

Proposition I.1.97. *$\mathfrak{R}$ restricts to a map $\mathsf{wIndexCat}_{\mathcal{T}}^{\mathrm{uni}} \to \mathsf{AtmOrbSub}_{\mathcal{T}}$ with fully faithful right adjoint given by the assignment $\mathcal{R} \mapsto \mathbb{F}_{\mathcal{R}}$; moreover, the essential image of this adjoint consists of the indexing categories.*

To prove this, we use the following important lemma about indexing categories.

Lemma I.1.98. *If $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is an indexing category, then $f \colon T \to S$ lies in I if and only if, for each orbit $U \in T$, the induced map $U \to f(U)$ lies in $\mathfrak{R}(I)$; moreover, any one-color weak indexing category satisfying this is an indexing category.*

PROOF. Given $f \colon T \to S$ lying in I, and an orbit $U \subset T$, unitality guarantees that the composition $U \simeq U \sqcup \varnothing \to U \sqcup (T - U) \to S$ lies in I, and Condition (IC-b) then implies that $U \to f(U)$ lies in I. The converse follows by noting that there is a composition $T \to \coprod_{U \in T} f(U) \to S$, whose second map is a coproduct of fold maps and hence lies in I. The converse follows simply by noting that each fold map satisfies the above condition. □

PROOF OF PROPOSITION I.1.97. First we must show that $\mathfrak{K}(I) \subset \mathcal{T}$ is atomic orbital; by definition, if it is orbital, it is atomic. Given a pullback square in $\mathcal{S}_{\mathcal{T}}$

$$\begin{array}{ccc} S & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow f \\ W & \longrightarrow & V \end{array}$$

such that f lies in $I \cap \mathfrak{K}$ and an orbital summand $X \subset S$, we're tasked with proving that the composite arrow $X \to W$ lies in I; this is precisely unitality.

Now, the fact that $\mathbb{F}_{\mathcal{R}}$ is an indexing category follows directly by definition, as does the fact that $\mathcal{R} = \mathbb{F}_{\mathcal{R}} \cap \mathcal{T}$; moreover, it follows fairly straightforwardly by combining Proposition I.1.84 and Lemma I.1.98 that $I = \mathbb{F}_{I \cap \mathcal{T}}$ if and only if I is an indexing category.

To verify that $\mathbb{F}_{(-)}$ is fully faithful right adjoint with specified image, all that is left is to prove that it's a right adjoint, i.e. $I \subset \mathbb{F}_{\mathcal{R}}$ if and only if $\mathcal{T} \cap I \subset \mathbb{F}_{\mathcal{R}}$. The forward direction is obvious, so assume $\mathcal{T} \cap I \subset \mathbb{F}_{\mathcal{R}}$; applying Lemma I.1.98 directly shows $I \subset \mathbb{F}_{\mathcal{R}}$. □

Remark I.1.99. These are not apriori equivalent to Rubin and Balchin-Barnes-Roitzheim's transfer systems [BBR21; Rub21]; however, Proposition I.1.97 confirms that they recover equivalent posets. ◁

I.1.4.3 Joins of weak indexing systems The following construction will be used often.

Construction I.1.100. Given collections $\mathcal{D}, \mathcal{C} \subset \underline{\mathrm{Fin}}_{\mathcal{T}}$, we inductively define $\mathrm{Cl}_{\mathcal{D},0}(\mathcal{C}) \coloneqq \mathcal{C}$ and

$$\mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})_V = \left\{ \coprod_U^S T_U \;\middle|\; (T_U) \in \mathrm{Cl}_{n-1}(\mathcal{C})_S,\ S \in \mathcal{D} \right\},$$

with $\mathrm{Cl}_{\mathcal{D},\infty}(\mathcal{C}) \coloneqq \bigcup_n \mathrm{Cl}_{\mathcal{D},n}(\mathcal{C})$ and $\mathrm{Cl}_n(\mathcal{C}) \coloneqq \mathrm{Cl}_{\mathcal{C},n}(\mathcal{C})$. We call this the *$n$-step closure of $\mathcal{C}$ under $\mathcal{D}$-indexed coproducts*, or just the *closure of $\mathcal{C}$ under $\mathcal{D}$-indexed coproducts* when $n = \infty$. ◁

Proposition I.1.101. $\mathrm{wIndex}_{\mathcal{T}}^{\mathrm{oc}}$ *is a lattice; the meets are intersections, and the joins are*

$$\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \bigcup_{n \in \mathbb{N}} \overbrace{\mathrm{Cl}_I \mathrm{Cl}_J \cdots \mathrm{Cl}_I \mathrm{Cl}_J}^{2n\text{-fold}} \{*_{\mathcal{T}}\}.$$

Remark I.1.102. If $\mathcal{D}$ is a weak indexing system, then the dual statement to Eq. (4) the canonical inclusion

$$\mathrm{Cl}_{\mathcal{D},1}(\mathcal{C}) \subset \mathrm{Cl}_{\mathcal{D}}(\mathcal{C})$$

is an equality for all collections $\mathcal{C}$; in particular each of the closures in Proposition I.1.101 can be equivalently replaced with $\mathrm{Cl}_{I,1}$ or $\mathrm{Cl}_{J,1}$. ◁

Remark I.1.103. Proposition I.1.101 holds without the one-color assumption, but the expression of closures is marginally more complicated, so we omit it. See [Ste24]. ◁

Observation I.1.104. If $\mathcal{D}$ is a one-color collection, then by taking $*_V$-indexed coproducts for all $V \in c(\mathcal{C})$, we find that $\mathcal{C} \subset \mathrm{Cl}_{\mathcal{D},1}(\mathcal{C})$. Similarly, if $\mathcal{C}$ is a one-color collection, by taking indexed coproducts of $(*_U)$, we find that $\mathcal{C} \subset \mathrm{Cl}_{\mathcal{C},1}(\mathcal{D})$. ◁

Let $\mathsf{FullSub}^*_{\mathcal{T}}(\underline{\mathsf{Fin}}_{\mathcal{T}}) \subset \mathsf{FullSub}_{\mathcal{T}}(\underline{\mathsf{Fin}}_{\mathcal{T}})$ denote the full subposet of one color full subcetegories.

Lemma I.1.105. *The fully faithful map* $\iota\colon \mathsf{wIndex}^{\mathrm{oc}}_{\mathcal{T}} \hookrightarrow \mathsf{FullSub}^*_{\mathcal{T}}(\underline{\mathsf{Fin}}_{\mathcal{T}})$ *is right adjoint to* Cl_∞.

Proof. If $\mathrm{Cl}_\infty(\mathcal{C})$ is a one-color weak indexing system, then it is clearly minimal among those containing $\mathcal{C}$, so it suffices to prove that it's a one-color weak indexing system. By Observation I.1.104, $\mathrm{Cl}_\infty(\mathcal{C})$ has one color so it suffices to verify that it's closed under self-indexed coproducts.

In fact, by the dual to Eq. (4), we find that $\mathrm{Cl}_i(\mathcal{C})$-indexed coproducts of elements of $\mathrm{Cl}_j(\mathcal{C})$ are $\mathrm{Cl}_{i+1}(\mathcal{C})$-indexed coproducts of elements of $\mathrm{Cl}_{j-1}(\mathcal{C})$; applying this j-many times, we find that $\mathrm{Cl}_i(\mathcal{C})$-indexed coproducts of elements in $\mathrm{Cl}_j(\mathcal{C})$ are in $\mathrm{Cl}_\infty(\mathcal{C})$, and the result follows. □

Proof of Proposition I.1.101. By Lemma I.1.105, $\mathsf{wIndex}_{\mathcal{T}}$ has meets computed in $\mathsf{FullSub}^*_{\mathcal{T}}(\underline{\mathbb{F}}_{\mathcal{T}})$, which are clearly given by intersections. Furthermore, Lemma I.1.105 implies that $\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J = \mathrm{Cl}_\infty(\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J)$. Note that, for arbitrary $\mathcal{C},\mathcal{D},\mathcal{E}$, we have

$$\mathrm{Cl}_{\mathcal{C}\cup\mathcal{D}}(\mathcal{E}) = \bigcup_{n\in\mathbb{N}} \overbrace{\mathrm{Cl}_{\mathcal{C}}\mathrm{Cl}_{\mathcal{D}}\cdots\mathrm{Cl}_{\mathcal{C}}\mathrm{Cl}_{\mathcal{D}}}^{2n\text{-fold}}(\mathcal{E}),$$

and set $\mathcal{C} = \underline{\mathbb{F}}_I$, $\mathcal{D} = \underline{\mathbb{F}}_J$, and $\mathcal{E} = \underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J$; the result then follows by noting that $\underline{\mathbb{F}}_I \cup \underline{\mathbb{F}}_J \subset \mathrm{Cl}_I\mathrm{Cl}_J(*_{\mathcal{T}})$ by Observation I.1.104. □

Remark I.1.106. Since $\mathfrak{R}\colon \mathsf{wIndexCat}^{\mathrm{uni}}_{\mathcal{T}} \to \mathsf{AtmOrbSub}_{\mathcal{T}}$ is a left adjoint, it preserves joins; moreover,

$$\mathfrak{R}\left(\mathrm{Cl}_{\mathcal{C}}(\mathcal{D})\right) \simeq \mathfrak{R}\Big(\mathrm{Cl}_{\mathfrak{R}(\mathcal{C})}\mathfrak{R}(\mathcal{D})\Big),$$

as an indexed coproduct of nonempty $\mathcal{T}$-spaces is an orbit iff its indexing $\mathcal{T}$-space is an orbit and the input is also an orbit. Lastly, note that indexing categories are closed under joins. Said altogether, the join of atomic orbital subcategories $\mathcal{P},\mathcal{Q}$ is given by the wide subcategory of $\mathcal{T}$ spanned by alternating compositions $V_0 \xrightarrow{p_0} V_1 \xrightarrow{q_0} \cdots \xrightarrow{q_n} V_n$ such that each p_i lies in $\mathcal{P}$ and each q_i lies in $\mathcal{Q}$; moreover, $\mathfrak{R}(\underline{\mathbb{F}}_I \vee \underline{\mathbb{F}}_J) \simeq \mathfrak{R}\underline{\mathbb{F}}_I \vee \mathfrak{R}\underline{\mathbb{F}}_J$ and $\mathbb{F}_{\mathcal{P}\vee\mathcal{Q}} \simeq \mathbb{F}_{\mathcal{P}} \vee \mathbb{F}_{\mathcal{Q}}$. ◁

I.1.4.4 Indexed semiadditivity One central source of weak indexing categories is *indexed semiadditivity*, which only makes sense to evaluate in the pointed setting.

Definition I.1.107. Given $\mathcal{F} \subset \mathcal{T}$ a $\mathcal{T}$-family, we say that $\mathcal{D}$ is $\mathcal{F}$*-pointed* if $\mathcal{D}_V$ is pointed for all $V \in \mathcal{F}$. ◁

Given $S \in \mathrm{Fin}_V$ a finite V-set with a distinguished orbit $W \in \mathrm{Orb}(S)$, $\mathcal{D}$ a $\mathcal{T}_{\leq V}$-pointed $\mathcal{T}$-∞-category admitting S-indexed products and coproducts, and $(X_U) \in \mathcal{D}_S$ a S-tuple in $\mathcal{D}$, [Nar16, Cons 5.2] constructs a map

$$\chi_W \colon \mathrm{Res}^V_W \coprod_U^S X_U \longrightarrow X_W$$

by distinguishing an "identity coset" X_W-summand on the left hand side and dictating the map to be the indentity on this summand and 0 elsewhere; then, the *Wirthmüller map*

$$W_{S,(X_U)} \colon \coprod_U^S X_U \longrightarrow \prod_U^S X_W$$

has projected map $\coprod_U^S X_U \longrightarrow \mathrm{CoInd}^V_W X_W$ adjunct to χ_W. Schematically, this is a "diagonal matrix"

$$W_{\sqcup_{i=1}^n U_i,(X_{U_i})} = \begin{bmatrix} W_{U_1,X_1} & 0 & \cdots & 0 \\ 0 & W_{U_2,X_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_{U_n,X_n} \end{bmatrix} \colon \coprod_{U_i}^S X_i \to \prod_{U_i}^S X_i$$

where $W_{U_i,X_i} \colon \mathrm{Ind}^V_{U_i} X_i \to \mathrm{CoInd}^V_{U_i} X_i$ is the "norm map" comparing induction and coinduction.

Definition I.1.108. Given $\mathcal{D}$ a $\mathcal{T}$-∞-category and $S \in \mathrm{Fin}_V$ a finite V-set, we say that S is $\mathcal{D}$*-ambidextrous* if $\mathcal{D}$ admits S-indexed products and coproducts, is $\mathcal{T}_{\leq V}$-pointed, and for all $(X_U) \in \mathcal{D}_S$, the norm map is an equivalence

$$\coprod_U^S X_U \xrightarrow{\sim} \prod_U^S X_U.$$

Given I a weak indexing category, we say that $\mathcal{D}$ is I*-semiadditive* if S is $\mathcal{D}$-ambidextrous for all $S \in \underline{\mathbb{F}}_I$. ◁

Remark I.1.109. We've given an elementary presentation of this notion; this has been generalized to encapsulate Hopkins-Lurie's *higher semiadditivity* in [CLL24]. In particular, we find that $T \to S$ is $\mathcal{D}$-ambidextrous in the sense of [CLL24] if and only if the U-set $T \times_S U$ is $\mathcal{D}$-ambidextrous for all orbits $U \subset S$, so we adpot their language for *ambidextrous maps*. In particular, by [Cno23, Prop 3.13, Prop 3.16], ambidextrous maps are closed under composition and base change. ◁

Given $\mathcal{D}$ a $\mathcal{T}$-∞-category, we define the *semiadditive locus*

$$\mathfrak{s}(\mathcal{D}) = \{f \colon T \to S \mid f \text{ is } \mathcal{D}\text{-ambidextrous}\} \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}.$$

This is closed under composition by Remark I.1.109; furthermore, it's clear that an equivalence $T \simeq S$ is $\mathcal{D}$-ambidextrous if and only if $\mathcal{D}$ is $\mathcal{T}_{\leq V}$-pointed, so $\mathfrak{s}(\mathcal{D}) \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is a subcategory. In fact, we may say more.

Proposition I.1.110. *$\mathfrak{s}(\mathcal{D})$ is a weak indexing category and $\mathcal{D}$ is I-semiadditive if and only if $I \leq \mathfrak{s}(\mathcal{D})$.*

Proof. By Observation I.1.87 and Remark I.1.109, $\mathfrak{s}(\mathcal{D})$ satisfies Condition (IC-b). In fact, by Remark I.1.109, ambidextrous maps are closed under base change, i.e. $\mathfrak{s}(\mathcal{D})$ satisfies Condition (IC-a). We're left with verifying that $\mathcal{D}$ is I-semiadditive if and only if $I \leq \mathfrak{s}(\mathcal{D})$, but this follows immediately by unwinding definitions. □

Corollary I.1.111. *$\mathcal{D}$ is $I \vee J$-semiadditive if and only if it is I-semiadditive and J-semiadditive.*

I.1.4.5 Restrictions of weak indexing categories

Construction I.1.112. Given $\mathcal{C} \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ a subcategory and $V \in \mathcal{T}$ an object, we write

$$\mathcal{C}_V := \left\{ f : \begin{array}{ccc} S & \xrightarrow{\widetilde{f}} & T \\ & \searrow \quad \swarrow & \\ & V & \end{array} \;\middle|\; \widetilde{f} \in \mathcal{C} \right\} \subset \mathbb{F}_V;$$

that is, maps in $\mathcal{C}_V$ are maps over V whose underlying map in $\mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ lies in $\mathcal{C}$. For every map $V \to W$, this yields a map $(-)_V : \mathsf{Sub}_{\mathsf{Cat}_W}(\mathbb{F}_W) \to \mathsf{Sub}_{\mathsf{Cat}_V}(\mathbb{F}_V)$, compatibly with composition. We let $\underline{\mathsf{Sub}}_{\underline{\mathsf{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$ be the resulting $\mathcal{T}$-poset ◁

Proposition I.1.113. *If $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is a $\mathcal{T}$-weak indexing category, then $I_V \subset \mathbb{F}_V$ is a $\mathcal{T}_{/V}$-weak indexing category.*

Proof. Condition (IC-b) for I_V follows by unwinding definitions, noting that $\mathsf{Ind}_V^{\mathcal{T}} : \mathbb{F}_V \to \mathbb{F}_{\mathcal{T}}$ is coproduct-preserving; Condition (IC-a) follows by unwinding definitions, noting that the pullback functor $\mathbb{F}_V \to \mathbb{F}_W$ is pullback-preserving for each $W \to V$. □

Proposition I.1.113 lifts $\mathsf{wIndexCat}_{\mathcal{T}} \subset \mathsf{Sub}_{\mathsf{Cat}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$ to an embedded $\mathcal{T}$-subposet

$$\underline{\mathsf{wIndexCat}}_{\mathcal{T}} \subset \underline{\mathsf{Sub}}_{\underline{\mathsf{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}}).$$

Given a $\mathcal{T}$-poset $P : \mathcal{T}^{\mathrm{op}} \to \mathsf{Poset}$, we denote by $\Gamma^{\mathcal{T}} P$ the associated limit. There is a monotone map

$$\widetilde{\gamma} : \mathsf{Sub}_{\mathsf{Cat}}(\mathbb{F}_{\mathcal{T}}) \to \Gamma \underline{\mathsf{Sub}}_{\underline{\mathsf{Cat}}_{\mathcal{T}}}(\underline{\mathbb{F}}_{\mathcal{T}})$$

defined by $\widetilde{\gamma}(\mathcal{C})_V := \mathcal{C}_V$. We may use $\widetilde{\gamma}$ to recover $\mathsf{wIndexCat}_{\mathcal{T}}$ from $\underline{\mathsf{wIndexCat}}_{\mathcal{T}}$.

Proposition I.1.114. *$\widetilde{\gamma}$ restricts to an equivalence*

$$\gamma : \mathsf{wIndexCat}_{\mathcal{T}} \xrightarrow{\sim} \Gamma \underline{\mathsf{wIndexCat}}_{\mathcal{T}}$$

PROOF. Proposition I.1.113 implies that $\widetilde{\gamma}$ restricts to a monotone map of posets $\gamma\colon \mathsf{wIndexCat}_{\mathcal{T}} \to \Gamma^{\mathcal{T}}\underline{\mathsf{wIndexCat}}_{\mathcal{T}}$, so it suffices to prove that this is fully faithful and essentially surjective. Given $\gamma I \subset \gamma J$, then for a map $f\colon T \to V$, the canonical $\mathcal{T}_{/V}$-map $T \to *_V$ lying in I_V must lie in J_V, so f lying in I implies that it if it lies in J; thus Condition (IC-b') implies that $I \subset J$, so γ is fully faithful.

It remains to prove that γ is surjective, so we fix $I_\bullet \in \Gamma^{\mathcal{T}}\underline{\mathsf{wIndexCat}}_{\mathcal{T}}$. Define the subcategory

$$I \coloneqq \{T \to S \mid \forall U \in \mathrm{Orb}(S),\ \ T \times_S U \to U \in I_U\} \subset \mathbb{F}_{\mathfrak{v}(T)}.$$

By definition, $\gamma I = I_\bullet$, so it suffices to verify that I is a weak indexing category. First note that I satisfies Condition (IC-b') by definition. Moreover, Condition (IC-a') is precisely the condition that $I_{(-)}$ is an element of $\underline{\mathsf{wIndexCat}}_{\mathcal{T}}$. Hence I is a $\mathcal{T}$-weak indexing system, proving that γ is an isomorphism. □

I.1.4.6 Parameterized commutative monoids Let $\underline{I} \subset \underline{\mathsf{Fin}}_{\mathcal{T}}$ be the γ-preimage of I. The wide $\mathcal{T}$-subcategory inclusion $\underline{I} \subset \underline{\mathbb{F}}_{\mathcal{T}}$ is fiberwise given by a (one object) weak indexing category so the inclusion corresponds with a functor $\mathcal{T}^{\mathrm{op}} \to \mathsf{SpanPair}$ under Remark I.1.86. We define the composite $\mathcal{T}$-category

$$\underline{\mathsf{Span}}_I(\underline{\mathbb{F}}_{\mathcal{T}})\colon \mathcal{T}^{\mathrm{op}} \xrightarrow{\underline{I}\subset\underline{\mathbb{F}}_{\mathcal{T}}} \mathsf{SpanPair} \xrightarrow{\mathsf{Span}} \mathsf{Cat},$$

where Span denotes Barwick's *effective Burnside category* construction [Bar14].

Example I.1.115. If $\mathcal{C}$ is a $\mathcal{T}$-∞-category admitting I-indexed products, then the *$\mathcal{T}$-∞-category of I-commutative monoids in $\mathcal{C}$* is

$$\underline{\mathsf{CMon}}_I(\mathcal{C}) \coloneqq \underline{\mathrm{Fun}}_{\mathcal{T}}^{I\text{-}\times}\left(\underline{\mathsf{Span}}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C}\right).$$

The *∞-category of I-commutative monoids* is $\mathsf{CMon}_I(\mathcal{C}) \coloneqq \Gamma^{\mathcal{T}}\underline{\mathsf{CMon}}(\mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{T}}^{I\text{-}\times}\left(\underline{\mathsf{Span}}_I(\underline{\mathbb{F}}_{\mathcal{T}}), \mathcal{C}\right)$.

It is shown in [CLL24, Thm B] that this is *universal* for the property that it is I-semiadditive and its forgetful $\mathcal{T}$-functor to $\mathcal{C}$ is I-product preserving; we will (mostly) independently reprove this, In particular, we will conclude in Remark III.1.11 that

$$\mathfrak{s}\left(\underline{\mathsf{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})\right) = I;$$

hence $\mathfrak{s}$ *surjects* onto one-color weak indexing categories. We may read this as saying that $\mathsf{wIndexCat}_{\mathcal{T}}$ *is the poset of all notions of ≤ 0-semiadditivty for $\mathcal{T}$-categories determined by push-pull adjunctions between categories of finite $\mathcal{T}$-sets.* ◁

For the rest of the paper, we will take the following innocuous proposition as input.[6]

[6] Strictly speaking, [CLL23, Prop. 4.6.11] only covers the case that I is an indexing category; alternatively, together [CLL24, Thm. 6.9] and [CLL24, Thm. 7.4] do cover this result, but they are quite strong results to assume. For the sake of completeness, we will not assume these, and instead we suggest to the reader that the proof of [CLL23, Prop. 4.6.11] applies to the case of a general almost-unital weak indexing category mutatis-mutandis. Indeed, since I-semiadditivity is detected under restriction to $\upsilon(I)$, we may restrict to the

Proposition I.1.116 ([CLL23, Prop. 4.6.11])**.** *If $\mathcal{C},\mathcal{D}$ are $\mathcal{T}$-categories with I-indexed products and I-indexed coproducts, then $\underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\oplus}(\mathcal{C},\mathcal{D})$ is I-semiadditive.*

I.1.4.7 Compatible pairs of weak indexing systems We finish the section with a discussion of *compatible pairs of weak indexing systems*, generalizing the setting of [BH22].

Definition I.1.117. A pair of one-object weak indexing categories (I_a, I_m) is *compatible* if $\underline{\mathbb{F}}_{I_a} \subset \underline{\mathcal{S}}_{\mathcal{T}}$ is closed under I_m-indexed products. ◁

We'd like to compare these with the notions from [CHLL24b], beginning with the following.

Observation I.1.118. Orbitality implies that $(\mathbb{F}_{\mathcal{T}}, \mathbb{F}_{\mathcal{R}})$ is *extensive* in the sense of [CHLL24b, Def. 2.2.1]. Furthermore, a subcategory $I \subset \mathbb{F}_{\mathcal{R}}$ furnishes a *span pair* $(\mathbb{F}_{c(I)}, I)$ if and only if it satisfies Condition (IC-a); thus a span pair $(\mathbb{F}_{c(I)}, I)$ with $I \subset \mathbb{F}_{\mathcal{R}}$ is *weakly extensive* in the sense of [CHLL24b, Def. 2.2.1] if and only if I is a weak indexing category. Furthermore, by Lemma I.1.96, a weakly extensive pair $(\mathbb{F}_{c(I)}, I)$ with $I \subset \mathcal{R}$ is *extensive* if and only if I is an indexing category. ◁

They have their own notion of compatibility, which generalizes ours.

Observation I.1.119. A bispan triple $(\mathcal{C}, I_m, I_a)$ with $\mathcal{C}$ extensive (i.e. disjunctive) and whose span pairs are weakly extensive is called a *semiring context* in [CHLL24b, Def. 4.1.1] when the right adjoint $f_*\colon \mathbb{F}_{\mathcal{T},/X} \to \mathbb{F}_{\mathcal{T},/Y}$ to pullback along a map $f\colon X \to Y$ in I_m exists and preserves morphisms whose image in $\mathbb{F}_{\mathcal{R}}$ lies in I_a; unwinding definitions, this is precisely the condition that (I_a, I_m) is a compatible pair of one-object weak indexing systems. ◁

Note that (I_a, I_m) is a compatible pair of *indexing categories* in the sense of [BH22, Def 3.4] if and only if it is a compatible pair of weak indexing categories such that I_a and I_m are both indexing categories. In this setting, we have argued that the triple $(\mathbb{F}_{\mathcal{T}}, I_m, I_a)$ is a semiring context in the sense of [CHLL24a]. This is useful, as [CHLL24a, Thm 4.2.4] yields an operadic presentation for the associated theory of *bi-incomplete Tambara functors* valued in cocomplete cartesian closed ∞-categories.

Our main contribution to this is to concretely characterize the terminal (weak) indexing category $m(I)$ such that $(I, m(I))$ is a compatible pair, generalizing [BH22, Cor 6.19]:

Construction I.1.120. Given $I \subset \mathbb{F}_R$ a weak indexing category, we define the collection

$$\mathbb{F}^V_{\mathfrak{m}(I)} \coloneqq \left\{ S \in \mathbb{F}_V \mid \underline{\mathbb{F}}_I \subset \underline{\mathcal{S}}_{\mathcal{T}} \text{ is closed under } S\text{-indexed products} \right\};$$

we define the *multiplicative hull* $\mathfrak{m}(I) \coloneqq \mathrm{I}\left(\underline{\mathbb{F}}_{\mathfrak{m}(I)}\right)$.

unital case, in which case I-semiadditivity implies pointeness; once one has that, the proof of [CLL23, Prop. 4.6.11] and its antecedents apply verbatim after replacing the notation $\mathbb{F}^{\mathcal{P}}_{\mathcal{T}}$ with I, and mostly amount to painstaking diagram chases.

Moreover, given I a one-object weak indexing category, we say that a map of $\mathcal{T}$-spaces $f\colon X \to Y$ is *I-admissible* if, for all maps $g\colon K \to Y$ with K orbit-finite, the map $g^*X \to K$ lies in I. We write $\mathcal{S}_I \subset \mathcal{S}_{\mathcal{R}}$ for the wide subcategory of I-admissible maps.

We say that an $\mathcal{R}$-admissible map $f\colon T \to S$ of orbit-finite $\mathcal{T}$-spaces is *compatible with I* if, for all compositions $Y \xrightarrow{g} Z \to T$ with g I-admissible, the map $f_*Y \to Z$ is I-admissible; the composition-stability and automorphism-inclusion of Lemma I.1.77 implies that these span a wide subcategory $\underline{\mathfrak{m}}(I) \subset \mathbb{F}_{\mathcal{R}}$, which we call the *big multiplicative hull*. ◁

Proposition I.1.121 (Multiplicative hull)**.** *Fix I a one-object weak indexing category.*

(1) $\mathfrak{m}(I)$ is a unital weak indexing category.

(2) $(I, \mathfrak{m}(I))$ is compatible.

(3) given $J \in \mathsf{wIndexCat}^{\mathcal{R}}_{\mathcal{T}}$, (I,J) is compatible if and only if $J \subset \mathfrak{m}(I)$.

(4) $\underline{\mathfrak{m}}(I)$ is an indexing category.

(5) $\left(\mathcal{S}_{\mathcal{T}}, \mathcal{S}_I, \mathcal{S}_{\underline{\mathfrak{m}}(I)}\right)$ is a semiring context.

(6) given $J \in \mathsf{wIndexCat}^{\mathcal{R}}_{\mathcal{T}}$, the triple $\left(\mathcal{S}_{\mathcal{T}}, \mathcal{S}_I, \mathcal{S}_J\right)$ is a semiring context if and only if $J \subset \underline{\mathfrak{m}}(I)$.

Proof. First note the $*_V$-indexed product functor is the identity, so $*_V \in \mathbb{F}^V_{\underline{\mathfrak{m}}(I)} \cap \mathbb{F}^V_{\underline{\mathfrak{m}}(I)}$ for all V. Moreover, empty products are terminal objects (i.e. $*_V$), so $\varnothing_V \in \mathbb{F}^V_{\mathfrak{m}(I)} \cap \mathbb{F}^V_{\underline{\mathfrak{m}}(I)}$ for all $V \in \mathcal{T}$. Now, for (1), we claim that there exists a natural equivalence

$$\prod_U^{\coprod_W^S R_W} T_U \simeq \prod_W^S \prod_U^{R_W} T_U; \tag{4}$$

indeed this is obvious for binary coproducts and on induction this follows from

$$\begin{aligned}
\prod_U^{\mathrm{Ind}_W^V S} T_U &= \prod_{U \in \mathrm{Orb}(\mathrm{Ind}_W^V S)} \mathrm{CoInd}_U^V T_U \\
&= \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_W^V \mathrm{CoInd}_U^W T_U \\
&= \mathrm{CoInd}_W^V \prod_{U \in \mathrm{Orb}(S)} \mathrm{CoInd}_U^W T_U \\
&= \mathrm{CoInd}_W^V \prod_U^S T_U.
\end{aligned}$$

Now if S and each R_W lie in $\underline{\mathbb{F}}_{\mathfrak{m}(I)}$ and each T_U in $\underline{\mathbb{F}}_I$, applying the aforementioned compatability twice shows that $\prod_W^S \prod_U^{R_W} T_U$ lies in $\underline{\mathbb{F}}_I$, i.e. $\coprod_W^S R_W \in \underline{\mathbb{F}}_{\mathfrak{m}(I)}$, proving (1). (2) and (3) follow directly by unwinding definitions.

For (4), note that $2 \cdot *_V \in \underline{\mathbb{F}}_{\overline{m}(I)}$ by Lemma I.1.77, so it suffices to show that $\overline{m}(I)$ is closed under self-induction. This is similar to (1), using Eq. (4), i.e.

$$\psi_{\mathrm{Ind}_W^V S,*} \simeq \mathrm{CoInd}_W^V \psi_{S*};$$

in particular, if $S \to V$ and $W \to V$ lie in $\overline{m}(I)$, this expresses that $\psi_{\mathrm{Ind}_W^V S,*}$ preserves I-admissible maps, which is exactly closure under self-induction. (5) and (6) similarly follow simply by unwinding definitions. □

I.2 Algebraic patterns

An algebraic pattern is a collection of data encoding *Segal conditions* for the purpose of homotopy-coherent algebra. Given an algebraic pattern $\mathcal{O}$ and a complete ∞-category $\mathcal{C}$, there is an ∞-category of *Segal $\mathcal{O}$-objects in $\mathcal{C}$*, which we view as some analog of "$\mathcal{O}$-monoids in $\mathcal{C}$" generalizing [HA, Def. 2.4.2.1]; these are presented as functors $\mathcal{O} \to \mathcal{C}$ satisfying a Segal condition that an "inert" subdiagram is right Kan extended from an "elementary" further subdiagram.

We may view Segal $\mathcal{O}$-objects in Cat (aka Segal $\mathcal{O}$-∞-categories) as $\mathcal{O}$-monoidal ∞-categories; these straighten to cocartesian fibrations over $\mathcal{O}$ satisfying conditions. As in [HA, § 2], the condition of *being a cocartesian fibration* may be relaxed to construct a form of operads parameterized by $\mathcal{O}$, called *fibrous $\mathcal{O}$-patterns*.

We summarize some elements of the theory of algebraic patterns and fibrous patterns in Section I.2.1. The author would like to emphasize that the program surrounding algebraic patterns has achieved many results not mentioned here; for a significantly more thorough and elegant treatment, we recommend [BHS22; CH21].

Those familiar with [HA] will likely notice similarity with the model structure for fibrations over a categorical pattern; in contrast to the categorical patterns of [HA, § B], algebraic patterns are manifestly ∞-categorical and their Segal conditions derived canonically from their definitional "inert-active" factorization system and elementary objects. Nevertheless, we may view algebraic patterns as a functorial presentation of particularly nice categorical patterns; in Section I.2.2, we characterize of maps of categorical patterns to algebraic patterns which induce Quillen equivalences from Lurie's model structure to a model structure on Segal ∞-categories or fibrous patterns, therein called *sufficient sets of generalized (fibrous) Segal conditions*. [7]

Moving on, in Section I.2.3, we translate Lurie's result on push-pull Quillen adjunctions to algebraic patterns. Last, in Section I.2.4, we use algebraic patterns to classify the monomorphisms of $\mathcal{T}$-$(\infty,2)$-categories, i.e. to classify the *parameterized* $(\infty,2)$*-subcategories*.

[7] the name "fibrous pattern" itself is derived from Lurie's fibrous maps $\overline{\mathcal{E}} \to \mathcal{E}$, where $\mathcal{E}$ has a distinguished approximation map to an ∞-operad $\mathcal{O}^{\otimes}$; see the proof of [HA, Prop. 2.3.3.30].

I.2.1 Recollections of algebraic patterns The following definition is central.

Definition I.2.1. An *algebraic pattern* is a triple $(\mathfrak{B}, (\mathfrak{B}^{\mathrm{in}}, \mathfrak{B}^{\mathrm{act}}), \mathfrak{B}^{\mathrm{el}})$, where $(\mathfrak{B}^{\mathrm{in}}, \mathfrak{B}^{\mathrm{act}})$ is a factorization system on $\mathfrak{B}$ and $\mathfrak{B}^{\mathrm{el}} \subset \mathfrak{B}^{\mathrm{in}}$ is a full subcategory.[8] The ∞-category $\mathsf{AlgPatt} \subset \mathsf{Fun}(\mathbf{Q}, \mathsf{Cat})$ is the full subcategory spanned by algebraic patterns, where

$$\mathbf{Q} := \bullet \to \bullet \to \bullet \leftarrow \bullet. \tag{5}$$

◁

We refer to the morphisms in $\mathfrak{B}^{\mathrm{in}}$ as "inert morphisms," morphisms in $\mathfrak{B}^{\mathrm{act}}$ as "active morphisms," and objects in $\mathfrak{B}^{\mathrm{el}}$ as "elementary objects." When it is clear from context, we will abusively refer to the quadruple $\left(\mathfrak{B}, (\mathfrak{B}^{\mathrm{in}}, \mathfrak{B}^{\mathrm{act}}), \mathfrak{B}^{\mathrm{el}}\right)$ simply as $\mathfrak{B}$.

I.2.1.1 Segal objects Algebraic patterns provide a general framework for algebraic structures satisfying the associated *Segal conditions*, which are encoded in the notion of *Segal objects*.

Definition I.2.2. Let $\mathcal{C}$ be a complete ∞-category and $\mathcal{O}$ an algebraic pattern. Then, the ∞-category of *Segal $\mathcal{O}$-objects in $\mathcal{C}$* is the full subcategory $\mathsf{Seg}_{\mathcal{O}}(\mathcal{C}) \subset \mathsf{Fun}(\mathcal{O}, \mathcal{C})$ consisting of functors $F \colon \mathcal{O} \to \mathcal{C}$ such that, for every object $O \in \mathcal{O}$, the natural map

$$F(O) \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} F(E)$$

is an equivalence, where $\mathcal{O}^{\mathrm{el}}_{O/} := \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{in}}, \mathrm{ev}_1} \mathcal{O}^{\mathrm{in}}_{O/}$ is the ∞-category whose objects consist of inert morphisms from O to an elementary object. We will call Segal $\mathcal{O}$-objects in $\mathcal{S}$ and Cat Segal *spaces* and *∞-categories*, and identify such objects with their unstraightenings in $\mathsf{Cat}_{\mathcal{O}}$ whenever doing so will not cause confusion. ◁

Remark I.2.3. By [CH21, Lem. 2.9], a functor $F \colon \mathcal{O} \to \mathcal{C}$ is a Segal $\mathcal{O}$-object if and only if the associated functor $F|_{\mathcal{O}^{\mathrm{int}}}$ is right Kan extended from $F|_{\mathcal{O}^{\mathrm{el}}}$ along the inclusion $\mathcal{O}^{\mathrm{el}} \to \mathcal{O}^{\mathrm{int}}$. ◁

We include the following example in order to convince the reader, but we will use none of its cases.

Example I.2.4. Each of the following ∞-categories admit pattern structures with the indicated Segal objects:

- Δ^{op} admits a pattern structure whose Segal objects are monoid objects [CH21, Ex. 3.3];
- Δ^{op} admits another pattern structure whose Segal objects are traditional Segal objects [CH21, Ex. 3.3];
- Rezk's category Θ_n admits a pattern structure whose Segal spaces are Θ_n-spaces, i.e. (∞, n)-categories [CH21, Ex. 3.5];
- Moerdijk-Weiss' dendroidal category admits a pattern structure whose Segal objects are dendroidal Segal spaces [CH21, Ex. 3.7];

[8] Throughout this paper, we adopt the definition of *factorization system* used in [CH21, Rmk. 2.2], which does not assert any lifting properties; that is, a factorization system on $\mathcal{C}$ is a pair of wide subcategories $\mathcal{C}^L, \mathcal{C}^R \subset \mathcal{C}$ satisfying the condition that, for all maps $X \xrightarrow{f} X'$, the space of factorizations $X \xrightarrow{l} Y \xrightarrow{r} X'$ with $l \in \mathcal{C}^L$ and $r \in \mathcal{C}^R$ is contractible.

- the category of correspondences between m-finite spaces with $(m-1)$-truncated backwards leg $\mathsf{Span}_{m-1}(\mathcal{S}_m) = \mathcal{S}_m^{m-1}$ admits a pattern structure whose Segal objects are the m-commutative monoids of [Har20, Def. 5.10];
- the category of correspondences between m-finite spaces $\mathsf{Span}(\mathcal{S}_m)$ admits a pattern structure whose Segal objects are the m-finite limit preserving functors; and
- there will be several algebraic patterns which present I-commutative monoids (see Section I.4).

The penultimate and antepenultimate examples present equivalent Segal objects [Har20, Prop. 5.14]; this will eventually fit in a general *Morita theory* for patterns (see Example I.2.18 and Corollary I.4.10). ◁

One benefit of the framework of Segal objects is the following monadicity result.

Proposition I.2.5 ([CH21, Cor. 8.2])**.** *If $\mathcal{O}$ is an algebraic pattern and $\mathcal{C}$ a presentable ∞-category, then the forgetful functor*

$$U\colon \mathsf{Seg}_{\mathcal{O}}(\mathcal{C}) \to \mathsf{Fun}(\mathcal{O}^{\mathrm{el}}, \mathcal{C})$$

is monadic; in particular, it is conservative.

Another benefit of Segal objects is a rich framework for functoriality.

Definition I.2.6. Suppose $\mathcal{P}, \mathcal{O}$ are algebraic patterns. A functor $f\colon \mathcal{P} \to \mathcal{O}$ is *compatible with Segal objects* if it preserves the inert-active factorization system and $f^*\colon \mathsf{Fun}(\mathcal{O}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{P}, \mathcal{C})$ preserves Segal objects in any complete ∞-category $\mathcal{C}$. Moreover, a morphism of algebraic patterns $f\colon \mathcal{P} \to \mathcal{O}$ is a called a:

- *Segal morphism* if it is compatible with Segal objects, and a
- *strong Segal morphism* if the associated functor $f^{\mathrm{el}}_{X/}\colon \mathcal{P}^{\mathrm{el}}_{X/} \to \mathcal{O}^{\mathrm{el}}_{f(X)/}$ is initial for all $X \in \mathcal{P}$. ◁

Observation I.2.7. The conditions for Segal morphisms and strong Segal morphisms are each compatible with compositions and equivalences; that is, there are wide subcategories

$$\mathsf{AlgPatt}^{\mathrm{Seg}}, \mathsf{AlgPatt}^{\mathrm{Strong-Seg}} \subset \mathsf{AlgPatt}$$

whose morphisms are the Segal morphisms and strong Segal morphisms, respectively. ◁

Remark I.2.8. [CH21, Lem. 4.5] concludes that f is a Segal morphism if f^* preserves Segal *spaces*. ◁

Limits of patterns construct a large number of examples according to the following lemma.

Lemma I.2.9 ([CH21, Cor. 5.5])**.** $\mathsf{AlgPatt} \subset \mathsf{Fun}(\mathbf{Q}, \mathsf{Cat})$ *is a localizing subcategory; in particular,* $\mathsf{AlgPatt}$ *has small limits.*

Example I.2.10. By [CH21, Ex. 5.7], the binary products in AlgPatt have Segal objects

$$\mathrm{Seg}_{\mathfrak{B}\times\mathfrak{B}'}(\mathcal{C}) \simeq \mathrm{Seg}_{\mathfrak{B}}\mathrm{Seg}_{\mathfrak{B}'}(\mathcal{C}).$$ ◁

Cartesian products of patterns also play nicely with well-structured maps of patterns.

Lemma I.2.11. *Suppose $f\colon \mathfrak{O} \to \mathfrak{P}$ and $f'\colon \mathfrak{O}' \to \mathfrak{P}'$ are (resp. strong) Segal morphisms. Then,*

$$f \times f'\colon \mathfrak{O}\times\mathfrak{O}' \to \mathfrak{P}\times\mathfrak{P}'$$

is a (strong) Segal morphism.

Proof. The case of Segal morphisms follows immediately from Example I.2.10, so we assume that f, f' are strong Segal. Then, the induced map

$$f^{\mathrm{el}}_{X/} \times f'^{\mathrm{el}}_{X'/} = (f\times f')^{\mathrm{el}}_{(X,X')/} : (\mathfrak{O}\times\mathfrak{O}')^{\mathrm{el}}_{(X,X')/} \to (\mathfrak{P}\times\mathfrak{P}')^{\mathrm{el}}_{(fx,fx')/}$$

is a product of initial maps; it follows that it is initial, since limits in product categories are computed pointwise. □

I.2.1.2 An interlude on soundness and extendability We will move on to describe the theory of operads corresponding with an algebraic pattern, but to do so, we make some technical assumptions. Let $\mathfrak{O}$ be an algebraic pattern and $\omega\colon X \to Y$ an active map. Define the pullback square

$$\begin{array}{ccc} \mathfrak{O}^{\mathrm{el}}(\omega) & \longrightarrow & \mathrm{Ar}(\mathfrak{O}^{\mathrm{int}}_{X/}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ \mathfrak{O}^{\mathrm{el}}_{Y/}\times\mathfrak{O}^{\mathrm{el}}_{X/} & \xrightarrow{(\omega_{(-)},\mathrm{id})} & \mathfrak{O}^{\mathrm{int}}_{X/}\times\mathfrak{O}^{\mathrm{int}}_{X/} \end{array}$$

where $\omega_{(-)}\colon \mathfrak{O}^{\mathrm{el}}_{Y/} \to \mathfrak{O}^{\mathrm{int}}_{X/}$ sends $\alpha : Y \to E$ to the inert map ω_a of the inert-active factorization of $X \xrightarrow{\omega} Y \xrightarrow{a} E$.

Definition I.2.12. $\mathfrak{O}$ is *sound* if, for all $\omega\colon X \to Y$ active, the associated map $\mathfrak{O}^{\mathrm{el}}(\omega) \to \mathfrak{O}^{\mathrm{el}}_{X/}$ is initial. A sound pattern $\mathfrak{O}$ is *soundly extendable* if $\mathscr{A}_{\mathfrak{O}} \coloneqq \mathrm{Ar}^{\mathrm{act}}(\mathfrak{O}) \xrightarrow{t} \mathfrak{O}$ is a Segal $\mathfrak{O}$-∞-category, where $\mathrm{Ar}^{\mathrm{act}}(\mathfrak{O}) \subset \mathrm{Ar}(\mathfrak{O}) = \mathsf{Fun}(\Delta^1, \mathfrak{O})$ is the full subcategory spanned by active arrows. ◁

Soundness as a condition allows one to simplify Segal conditions. A condition of *extendability* was originally introduced in [CH21, Def. 8.5] for the sake of explicit formulas for the free Segal $\mathfrak{O}$-object monad, and is equivalent to sound extendability in the presence of soundness [BHS22, Rmk. 3.3.17]. Now, sound extendibility reduces many instances of *relative Segal objects* in the sense [BHS22, Def. 3.1.8] to a morphism with Segal domain by [BHS22, Obs. 3.1.9].

To transfer soundness, around, we will use a strengthening of [BHS22, Lem. 3.3.7].

Proposition I.2.13. *Suppose $\varphi\colon \mathcal{O} \to \mathcal{P}$ is a strong Segal morphism.*

(1) If φ is iso-Segal and $\mathcal{P}$ is sound, then $\mathcal{O}$ is sound.

(2) If φ is essentially surjective, each map $\varphi^{\mathrm{act}}_{/X}\colon \mathcal{O}^{\mathrm{act}}_{/X} \to \mathcal{P}^{\mathrm{act}}_{/\varphi X}$ is essentially surjective, and $\mathcal{O}$ is sound, then $\mathcal{P}$ is sound.

To prove this, we import an observation of Barkan-Haugseng-Steinebrunner.

Lemma I.2.14 ([BHS22, Obs. 3.3.6])**.** *The cartesian fibration $\mathrm{ev}_0\colon \mathcal{O}^{\mathrm{el}}(\omega) \to \mathcal{O}^{\mathrm{el}}_{Y/}$ has unstraightening given by the composite $\left(\mathcal{O}^{\mathrm{el}}_{Y/}\right)^{\mathrm{op}} \xrightarrow{\mathcal{O}^{\mathrm{el}}_{\omega_{(-)}(-)}} \mathrm{Cat}$.*

Proof of Proposition I.2.13. Claim (1) is [BHS22, Lem. 3.3.7]; we must give our own proof of the converse, as we have strengthened it. By the dual of [Sha23, Rmk. 6.10], a morphism of cartesian fibrations whose straightening is pointwise-initial is initial; applying this and Lemma I.2.14 shows that the induced functor $\varphi^{\mathrm{el}}(\omega)\colon \mathcal{O}(\omega) \to \mathcal{P}(\varphi(\omega))$ is initial. Thus we may draw the diagram

$$\begin{array}{ccc} \mathcal{O}^{\mathrm{el}}(\omega) & \xrightarrow{\varphi^{\mathrm{el}}(\omega)} & \mathcal{P}^{\mathrm{el}}(\varphi\omega) \\ {\scriptstyle \mathrm{ev}_0}\downarrow & & {\scriptstyle \mathrm{ev}_0}\downarrow \\ \mathcal{O}^{\mathrm{el}}_{X/} & \xrightarrow{\varphi^{\mathrm{el}}_{X/}} & \mathcal{P}^{\mathrm{el}}_{\varphi X/} \end{array}$$

whose top, bottom, and left arrow are initial in the situation of claim (2); initiality of the righthand arrow then follows by cancellation for initial functors [kerodon, Tag 02N6]. Our essential surjectivity assumptions guarantee that all active arrows in $\mathcal{P}$ arise in such away, giving claim (2). □

I.2.1.3 Span patterns The following is our primary source of examples.

Construction I.2.15 ([BHS22, Def. 3.2.6])**.** An *adequate quadruple* is the data of an adequate triple $\mathcal{X}^b, \mathcal{X}^f \subset \mathcal{X}$ in the sense of Definition I.1.85 together with a full subcategory $\mathcal{X}_0 \subset \mathcal{X}$; the *$\infty$-category of adequate quadruples* is the full subcategory

$$\mathrm{Quad}^{\mathrm{adeq}} \subset \mathrm{Fun}(\mathbf{Q}, \mathrm{Cat})$$

spanned by adequate quadruples, where $\mathbf{Q}$ is defined by Eq. (5).

Given an adequate quadruple $\left(\mathcal{X}, \mathcal{X}^f, \mathcal{X}^b; \mathcal{X}_0\right)$ let $\mathcal{X}^{b,\mathrm{op}} \subset \mathrm{Span}_{b,f}(\mathcal{X})$ be the wide subcategory spanned by the spans $X \xleftarrow{\psi_b} R \xrightarrow{\psi_f} Y$ with ψ_f an equivalence, and similarly $\mathcal{X}^f \subset \mathrm{Span}_{b,f}(\mathcal{X})$ the wide subcategory of spans with ψ_b an equivalence.

This yields a factorization system [HHLN23, Prop. 4.9]

$$\mathcal{X}^{b,\mathrm{op}} \hookrightarrow \mathrm{Span}_{b,f}(\mathcal{X}) \hookleftarrow \mathcal{X}^f.$$

We define the span pattern $\mathrm{Span}_{b,f}\left(\mathcal{X};\mathcal{X}_0^{\mathrm{op}}\right)$ via the data

- underlying ∞-category $\mathrm{Span}_{b,f}(\mathcal{X})$,
- inert morphisms $\mathcal{X}^{b,\mathrm{op}} \subset \mathrm{Span}(\mathcal{X})$,
- active morphisms $\mathcal{X}^{f} \subset \mathrm{Span}(\mathcal{X})$, and
- elementary objects $\mathcal{X}_0^{b,\mathrm{op}} = \mathcal{X}_0^{\mathrm{op}} \cap \mathcal{X}^{b,\mathrm{op}} \subset \mathcal{X}^{b,\mathrm{op}}$.

Given a map of adequate quadruples $\left(\mathcal{X},(\mathcal{X}^b,\mathcal{X}^f),\mathcal{X}_0\right) \to \left(\mathcal{Y},(\mathcal{Y}^b,\mathcal{Y}^f),\mathcal{Y}_0\right)$ the associated functor $\mathrm{Span}_{b,f}(\mathcal{X}) \to \mathrm{Span}_{b,f}(\mathcal{Y})$ preserves inert morphisms, active morphisms, and elementary objects by definition; hence the functor $\mathrm{Span}_{_,_}(-;-)\colon \mathrm{Quad}^{\mathrm{adeq}} \to \mathrm{Fun}(\mathbf{Q},\mathrm{Cat})$ descends to a functor

$$\mathrm{Span}_{_,_}(-;-)\colon \mathrm{Quad}^{\mathrm{adeq}} \to \mathrm{AlgPatt}. \qquad \triangleleft$$

Barkan-Haugseng-Steinebrunner supply conditions for span patterns to be sound(ly extendable).

Proposition I.2.16 ([BHS22, Prop. 3.3.23])**.** *If $(\mathcal{X},\mathcal{X}^b,\mathcal{X}^f;\mathcal{X}_0)$ is an adequate quadruple. Then, $\mathrm{Span}_{b,f}(\mathcal{X};\mathcal{X}_0)$ is:*

(1) sound if $\mathcal{X}^b_{/x} \hookrightarrow \mathcal{X}_{/x}$ is fully faithful and the inclusion $\mathcal{X}^b_{0,/x} \hookrightarrow \mathcal{X}_{0,/x}$ is final for all $x \in \mathcal{X}$, and

(2) soundly extendable if and only if it is sound and the functor $\mathcal{X}^f_{/(-)}\colon \mathcal{X}^{b,\mathrm{op}} \to \mathrm{Cat}$ is right Kan extended from $\mathcal{X}_0^{b,\mathrm{op}}$.

In particular, it is sound if $\mathcal{X}^b = \mathcal{X}$, and soundly extendable if $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{X}^b_{/x} \hookrightarrow \mathcal{X}_{/x}$ is fully faithful (e.g. if $\mathcal{X}_0 = \mathcal{X}^b = \mathcal{X}$).

Now, we will often construct maps of patterns via maps of adequate quadruples, for which the following lemma is useful, and becomes obvious by unwinding definitions.

Lemma I.2.17. *Fix $\widetilde{\varphi}\colon \left(\mathcal{X},\mathcal{X}^b,\mathcal{X}^f;\mathcal{X}_0\right) \to \left(\mathcal{Y},\mathcal{Y}^b,\mathcal{Y}^f;\mathcal{Y}_0\right)$ a map of adequate quadruples classifying an algebraic pattern map $\varphi\colon \mathrm{Span}_{b,f}(\mathcal{X};\mathcal{X}_0) \to \mathrm{Span}_{b,f}(\mathcal{Y};\mathcal{Y}_0)$. Then, φ is a strong Segal morphism if and only if the induced functor $\mathcal{X}^b_{0,/X} \to \mathcal{Y}^b_{0,/\varphi X}$ is final for all $X \in \mathcal{X}$, and an iso-Segal morphism if and only if this functor is an equivalence.*

Example I.2.18. Let $\mathcal{S}_m \subset \mathcal{S}$ be the full subcategory of m-finite spaces for some $m \leq \infty$, where $m = \infty$ is interpreted as π-finite spaces. Let $\mathcal{S}_{m,n} \subset \mathcal{S}_m$ be the wide subcategory of n-truncated maps. The algebraic pattern $\mathrm{Span}_n(\mathcal{S}_m) \coloneqq \mathrm{Span}_{all,\mathcal{S}_{m,n}}(\mathcal{S}_m;*)$ is described implicitly in [Har20]. In particular, in the case $n \in \{m-1,m\}$, it has Segal $\mathcal{C}$-objects the functors $\mathrm{Span}_n(\mathcal{S}_m) \to \mathcal{C}$ whose subdiagram $X\colon \mathcal{S}_m^{\mathrm{op}} \to \mathcal{C}$ whose coassembly map

$$X(A) \longrightarrow \lim_{* \to A} X(*) \tag{6}$$

is an equivalence by the pointwise formula for right Kan extension, as we have that $* \to A$ is n-truncated if and only if A is $(n+1)$-truncated).

In the case that $m = n$, this has a more elementary presentation: by the dual to [Har20, Cor. 2.16] a pre-Segal $\mathcal{C}$-object $\mathsf{Span}(\mathcal{S}_m)^{\mathrm{op}} \simeq \mathsf{Span}(\mathcal{S}_m) \to \mathcal{C}$ is m-finite limit preserving if and only if its restriction to $\mathcal{S}_m^{\mathrm{op}}$ is m-finite limit preserving, which by the dual to [Har20, Lem. 2.16] is equivalent to Eq. (6) being an equivalence for each A. ◁

To illustrate the application of Proposition I.2.16, we may characterize this further.

Corollary I.2.19. $\mathsf{Span}(\mathcal{S}_m)$ *is soundly extendable and* $\mathsf{Span}_{m-1}(\mathcal{S}_m) \hookrightarrow \mathsf{Span}(\mathcal{S}_m)$ *an iso-Segal morphism.*

Proof. It has all backwards maps, so it is automatically sound. For extendability, we are tasked with proving that the coassembly functor

$$\mathcal{S}_{/X} \longrightarrow \prod_{x \in \mathcal{X}} \mathcal{S}_{/*} \simeq \mathsf{Fun}(X, \mathcal{S})$$

is an equivalence. This may be identified with the functor taking fibers, i.e. the *straightening* functor, which is an equivalence. The iso-Segal condition follows by noting that every map $* \to A$ for A m-finite is automatically $(m-1)$-truncated. □

I.2.1.4 Fibrous patterns Unstraightening realizes $\mathsf{Seg}_{\mathfrak{O}}(\mathsf{Cat})$ as a non-full subcategory of $\mathsf{Cat}_{/\mathfrak{O}}$ consisting of cocartesian fibrations satisfying Segal conditions; we relax this for the following definition, which is equivalent to the original definition stated in [BHS22, Def. 4.1.2] by [BHS22, Prop. 4.1.6].

Definition I.2.20. Let $\mathfrak{B}$ be a sound algebraic pattern. A *fibrous* $\mathfrak{B}$*-pattern* is a functor $\pi : \mathfrak{O} \to \mathfrak{B}$ such that

(1) (inert morphisms) $\mathfrak{O}$ has π-cocartesian lifts for inert morphisms of $\mathfrak{B}$,

(2) (Segal condition for colors) For every active morphism $\omega \colon V_0 \to V_1$ in $\mathfrak{B}$, the functor

$$\mathfrak{O}_{V_0} \to \lim_{\alpha \in \mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathfrak{O}_{\omega_{\alpha,!} V_1}$$

induced by cocartesian transport along ω_α is an equivalence, where $\omega_{(-)} \colon \mathfrak{B}^{\mathrm{el}}_{Y/} \to \mathfrak{B}^{\mathrm{int}}_{X/}$ is the inert morphism appearing in the inert-active factorization of $\alpha \circ \omega$, and

(3) (Segal condition for multimorphisms) for every pair of objects $V_1, V_2 \in \mathfrak{B}$ and colors $X_i \in \mathcal{O}_{V_i}$, the commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{O}}(X_0, X_1) & \longrightarrow & \lim_{\alpha\colon V_1 \to E \in \mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathrm{Map}_{\mathcal{O}}(X_0, \alpha_! X_1) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathfrak{B}}(V_0, V_1) & \longrightarrow & \lim_{\alpha\colon V_1 \to E \in \mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathrm{Map}_{\mathfrak{B}}(V_0, E) \end{array}$$

is cartesian.

We denote by $\mathrm{Fbrs}(\mathfrak{B}) \subset \mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathfrak{B}}$ the full subcategory spanned by the fibrous $\mathfrak{B}$-patterns, where the latter category has objects the functors to $\mathfrak{B}$ possessing cocartesian lifts over inert morphisms and morphisms the functors preserving such cocartesian lifts. ◁

Remark I.2.21. As noted in [BHS22, Rmk. 4.1.8], in the presence of condition (3) above, condition (2) may be weakened to assert that the functor $\mathcal{O}_{V_0} \to \lim_{\alpha \in \mathfrak{B}^{\mathrm{el}}_{V_1/}} \mathcal{O}_{\omega_{\alpha,!} V_1}$ is a π_0-surjection without changing the resulting notion. To match [BHS22, Prop. 4.1.6], we may even take the intermediate assumption that this functor induces an equivalence on cores.

There are a few more equivalent statements of conditions (2) and (3); first, [BHS22, Obs. 4.1.3] and [BHS22, Prop. 4.1.7] show that they are equivalent to the condition that the envelope map $\mathrm{Env}_{\mathfrak{B}}(\mathcal{O}) \coloneqq \mathcal{O} \times_{\mathfrak{B}} \mathscr{A}_{\mathfrak{B}} \to \mathscr{A}_{\mathfrak{B}}$ is unstraightened from a so-called *relative Segal* $\mathfrak{B}$*-category*, i.e. the following square is cartesian:

In the case that $\mathfrak{B}$ is soundly extendable, [BHS22, Rmk. 4.1.4] notes that functors to Segal $\mathcal{O}$-categories are relative Segal $\mathcal{O}$-categories if and only if their domains are Segal $\mathcal{O}$-categories; that is, in the soundly extendable case, conditions (2) and (3) are equivalent to the fact that $\mathcal{O} \times_{\mathfrak{B}} \mathscr{A}_{\mathfrak{B}}$ is a Segal $\mathcal{O}$-category, i.e. the functor $\mathcal{P} \times_{\mathcal{O}} \mathcal{O}^{\mathrm{act}}_{/O} \to \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O/}} \mathcal{O}^{\mathrm{act}}_{/E}$ is an equivalence for all $O \in \mathcal{O}$. ◁

Example I.2.22. Fibrous $\mathbb{F}_* = \mathrm{Span}_{-1}(\mathcal{S}_0)$-patterns are equivalent to ∞-operads (c.f. [HA, Rmk. 2.1.1.14]). ◁

Remark I.2.23. [BHS22, Obs. 4.1.10] shows that a cocartesian fibration to $\mathfrak{B}$ is a fibrous pattern if and only if it is a Segal $\mathfrak{B}$-∞-category. ◁

The fully faithful functor $U\colon \mathrm{Fbrs}(\mathfrak{B}) \to \mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathfrak{B}}$ is a reflective subcategory inclusion.

Proposition I.2.24 ([BHS22, Cor. 4.2.3])**.** *U participates in an adjunction*

$$\mathrm{Cat}^{\mathrm{int-cocart}}_{/\mathfrak{B}} \underset{U}{\overset{L_{\mathrm{Fbrs}}}{\rightleftarrows}} \mathrm{Fbrs}(\mathfrak{B}) \qquad (\dashv)$$

Construction I.2.25. We write define the ∞-category of *prefibrous* $\mathcal{B}$*-patterns*

$$P\mathrm{Fbrs}(\mathcal{B}) \coloneqq \mathrm{Cat}_{/\mathcal{B}}^{\mathrm{int-cocart}},$$

so that $\mathrm{Fbrs}(\mathcal{B}) \subset P\mathrm{Fbrs}(\mathcal{B})$ is the localizing subcategory of prefibrous $\mathcal{B}$-patterns satisfying the fibrous Segal conditions. Given $\mathcal{O} \to \mathcal{B}$ a prefibrous $\mathcal{B}$-pattern, we define the swide subcategory $\mathcal{O}^{\mathrm{int}} \subset \mathcal{O}$ spanned by cocartesian lifts of inert arrows, $\mathcal{O}^{\mathrm{el}} \subset \mathcal{O}^{\mathrm{int}}$ the full subcategory of arbitrary lifts of elementary objects and $\mathcal{O}^{\mathrm{act}} \subset \mathcal{O}$ the wide subcategory of arbitrary lifts of active arrows. We refer to this as the $\mathcal{B}$*-prefibrous pattern structure on* $\mathcal{O}$. ◂

Now, the following lemma is straightforward.

Lemma I.2.26. *The* $\mathcal{B}$*-prefibrous pattern structure lifts to a functor*

$$P_{\mathcal{B}} \colon P\mathrm{Fbrs}(\mathcal{B}) \to \mathrm{AlgPatt}_{/\mathcal{B}},$$

which is natural in prefibrous patterns $\mathcal{O} \to \mathcal{B}$ *which are Segal morphisms.*

Proof. It follows by definition that morphisms of prefibrous $\mathcal{B}$-patterns preserve the pattern structure, so this functor exists. Given $\varphi \colon \mathcal{O} \to \mathcal{B}$ a prefibrous $\mathcal{B}$-pattern which is a Segal morphism, we're left with constructing a lift making the following diagram commute:

$$\begin{array}{ccc} P\mathrm{Fbrs}(\mathcal{B}) & \xrightarrow{\varphi^*} & P\mathrm{Fbrs}(\mathcal{O}) \\ \downarrow & & \downarrow \\ \mathrm{AlgPatt}_{/\mathcal{B}} & \xrightarrow{\varphi^*} & \mathrm{AlgPatt}_{/\mathcal{O}} \\ \downarrow & & \downarrow \\ \mathrm{Cat}_{/\mathcal{B}} & \longrightarrow & \mathrm{Cat}_{/\mathcal{O}} \end{array}$$

Unwinding definitions, it is equivalent to prove that $\varphi^*\mathcal{B}^{\mathrm{int}}$ consists of the cocartesian lifts for inert $\mathcal{O}$-morphisms and that $\varphi^*\mathcal{B}^{\mathrm{act}}\varphi^*\mathcal{B}^{\mathrm{el}}$ the arbitrary lifts of active morphisms and elementary objects. The latter is immediate, and the former follows by is cancellation for cocartesian arrows [kerodon, Tag 01TU]. □

Importantly, in the fibrous case, these are soundly extendable.

Proposition I.2.27 ([BHS22, Lem. 4.1.15])**.** *If* $\mathcal{O}$ *is a fibrous* $\mathcal{B}$ *pattern and* $\mathcal{B}$ *is sound (resp. soundly extendable), then* $\mathcal{O}$ *is sound (soundly extendable).*

Now, we may explicitly describe the fibrous patterns for this structure in this case.

Proposition I.2.28 ([BHS22, Cor. 4.1.7])**.** *Given* $\mathcal{O} \to \mathcal{B}$ *a fibrous* $\mathcal{B}$*-pattern,* P *induces an equivalence*

$$\mathrm{Fbrs}(\mathcal{O}) \simeq \mathrm{Fbrs}(\mathcal{B})_{/\mathcal{O}}.$$

Now, we conclude quite a lot of naturality as follows.

Proposition I.2.29 ([BHS22, Obs. 4.1.14])**.** *Fibrous patterns are strong Segal morphisms.*

Lastly, we note that fibrous patterns are quite functorial in general.

Proposition I.2.30. *Suppose $\varphi\colon \mathcal{O} \to \mathfrak{P}$ is functor which is compatible with the inert-active factorization system, and $\mathfrak{P}$ is soundly extendable. Then,*

(1) If the precomposition functor $\varphi^*\colon \mathsf{Fun}(\mathfrak{P},\mathsf{Cat}) \to \mathsf{Fun}(\mathcal{O},\mathsf{Cat})$ preserves Segal objects, then the pullback functor $\varphi^*\colon \mathsf{Cat}_{/\mathfrak{P}} \to \mathsf{Cat}_{/\mathcal{O}}$ preserves fibrous patterns.

(2) If φ is an inert-cocartesian fibration and the left Kan extension functor $\varphi_!\colon \mathsf{Fun}(\mathcal{O},\mathsf{Cat}) \to \mathsf{Fun}(\mathcal{P},\mathsf{Cat})$ preserves Segal objects, then postcomposition $\varphi_!\colon \mathsf{Cat}_{/\mathcal{O}} \to \mathsf{Cat}_{/\mathfrak{P}}$ preserves fibrous patterns.

In particular, if φ is an inert-cocartesian Segal morphism with soundly extendable codomain whose left Kan extension preserves Segal categories, then pullback and postcomposition restrict to an adjunction on fibrous patterns

$$\varphi_!\colon \mathsf{Fbrs}(\mathcal{O}) \rightleftarrows \mathsf{Fbrs}(\mathfrak{P})\colon \varphi^*$$

Proof. Our argument is only a minor variation of [BHS22, Lem. 4.1.19], so we temporarily assume close familiarity with the language used In either case, the property of being an inert-cocartesian fibration is always preserved, either by assumption or by [BHS22, Obs. 2.2.6].

We prove statement (1) first. Fixing $\mathscr{F} \in \mathsf{Fbrs}(\mathfrak{P})$, by [BHS22, Obs. 4.1.3], it suffices to prove that the left vertical arrow in the following pullback diagram is a relative Segal $\mathcal{O}$-∞-category.

$$\begin{array}{ccc} \mathrm{St}^{\mathrm{int}}_{\mathcal{O}}(\varphi^*\mathscr{F}) & \longrightarrow & \varphi^*\mathrm{St}^{\mathrm{int}}_{\mathfrak{P}}\mathscr{F} \\ \downarrow & \lrcorner & \downarrow \\ \mathscr{A}_{\mathcal{O}} & \longrightarrow & \varphi^*\mathscr{A}_{\mathfrak{P}} \end{array}$$

By [BHS22, Lem. 3.1.10], relative Segal $\mathcal{O}$-∞-categories are pullback-stable, so it suffices to prove that the right vertical arrow is a relative Segal $\mathcal{O}$-∞-category. By sound extendability $\mathscr{A}_{\mathfrak{P}}$ is a Segal $\mathfrak{P}$-∞-category, and since φ^* preserves Segal ∞-categories, $\varphi^*\mathscr{A}_{\mathfrak{P}}$ is a Segal $\mathcal{O}$-∞-category; by [BHS22, Obs. 3.1.18], it then suffices to prove that $\varphi^*\mathrm{St}^{\mathrm{int}}_{\mathfrak{P}}\mathscr{F}$ is a Segal $\mathcal{O}$-∞-category. Since φ^* preserves Segal ∞-categories, it suffices to prove that $\mathrm{St}^{\mathrm{int}}_{\mathfrak{P}}\mathscr{F}$ is a Segal $\mathfrak{P}$-category, which follows by the assumption that $\mathscr{F}$ is a fibrous pattern.

Statement (2) is similar; this time, by taking left adjoints to the commutative square of [BHS22, Prop. 4.2.5], it suffices to prove that the composition

$$\varphi_!\mathrm{St}^{\mathrm{int}}_{\mathcal{O}}\mathscr{F} \to \varphi_!\mathscr{A}_{\mathcal{O}} \to \mathscr{A}_{\mathfrak{P}}$$

is relative Segal; since $\mathfrak{P}$ is soundly extendable, [BHS22, Obs. 3.1.8] again reduces this to verifying that $\varphi_! \mathrm{St}^{\mathrm{int}}_{\mathcal{O}} \mathscr{F}$ is Segal; this follows from the facts that $\mathscr{F}$ is a fibrous pattern and $\varphi_!$ preserves Segal ∞-categories. □

I.2.1.5 The Segal envelope We would like to describe adjunctions relating fibrous patterns to Segal objects, but to do so, we need a few constructions.

Definition I.2.31. Given $\mathcal{O} \to \mathcal{B}$ a pattern map, the *Segal envelope of* $\mathcal{O}$ is the horizontal composite

$$\begin{array}{ccccc} \mathrm{Env}_{\mathcal{B}}\mathcal{O} & \longrightarrow & \mathrm{Ar}^{\mathrm{act}}(\mathcal{B}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{B} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_0} & & \\ \mathcal{O} & \longrightarrow & \mathcal{B} & & \end{array}$$

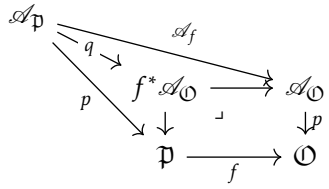
where $\mathrm{Ar}^{\mathrm{act}}(\mathcal{B}) \subset \mathrm{Ar}(\mathcal{B}) = \mathsf{Fun}(\Delta^1, \mathcal{B})$ is the full subcategory spanned by active arrows. We denote the envelope of the terminal $\mathcal{B}$-pattern as

$$\mathscr{A}_{\mathcal{B}} := \mathrm{Ar}^{\mathrm{act}}(\mathcal{B}) \xrightarrow{\mathrm{ev}_0} \mathcal{B}. \qquad \triangleleft$$

Given $f : \mathfrak{P} \to \mathcal{O}$ a Segal morphism, we define the composite functor

$$f^{\circledast} : \mathrm{Seg}_{\mathcal{O}}^{/\mathscr{A}_{\mathcal{O}}} \xrightarrow{f^*} \mathrm{Seg}_{\mathcal{O}}^{/f^*\mathscr{A}_{\mathcal{O}}} \xrightarrow{q^*} \mathrm{Seg}_{\mathcal{O}}^{/\mathscr{A}_{\mathfrak{P}}}$$

where q is the map fitting into the following diagram:



This participates in the following theorem, which was proved under a *strong Segal* assumption which is rendered unnecessary by Proposition I.2.30.

Theorem I.2.32 ([BHS22, Prop 4.2.1, Prop 4.2.5, Thm 4.2.6, Rem 4.2.8])**.** *Let* $\mathcal{O}$ *be a soundly extendable pattern. Then,* $\mathrm{Env}_{\mathcal{O}}$ *participates in an adjunction*

$$\mathrm{Fbrs}(\mathcal{O}) \underset{\mathrm{Un}}{\overset{\mathrm{Env}_{\mathcal{O}}}{\rightleftarrows}} \mathrm{Seg}_{\mathcal{O}}(\mathrm{Cat}).$$

By taking slice categories, this induces an adjunction

$$\mathrm{Fbrs}(\mathcal{O}) \underset{}{\overset{\mathrm{Env}_{\mathcal{O}}^{/\mathscr{A}_{\mathcal{O}}}}{\rightleftarrows}} \mathrm{Seg}_{\mathcal{O}}(\mathrm{Cat})$$

whose left adjoint is fully faithful. Furthermore, if $f : \mathcal{O} \to \mathcal{P}$ is a Segal morphism between soundly extendable patterns, the following diagram commutes:

$$\begin{array}{ccccccc}
\mathrm{Seg}_{\mathcal{O}}(\mathrm{Cat}_\infty) & \xrightarrow{\mathrm{Un}} & \mathrm{Fbrs}(\mathcal{O}) & \xrightarrow{\mathrm{Env}_{\mathcal{O}}^{/\mathscr{A}_{\mathcal{O}}}} & \mathrm{Seg}_{\mathcal{O}}(\mathrm{Cat}_\infty)_{/\mathscr{A}_{\mathcal{O}}} & \xrightarrow{\mathrm{Un}} & \mathrm{Fbrs}(\mathcal{O}) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^{\circledast}} & & \downarrow{\scriptstyle f^*} \\
\mathrm{Seg}_{\mathcal{P}}(\mathrm{Cat}_\infty) & \xrightarrow[\mathrm{Un}]{} & \mathrm{Fbrs}(\mathcal{P}) & \xrightarrow[\mathrm{Env}_{\mathcal{P}}^{/\mathscr{A}_{\mathcal{P}}}]{} & \mathrm{Seg}_{\mathcal{P}}(\mathrm{Cat}_\infty)_{/\mathscr{A}_{\mathcal{P}}} & \xrightarrow[\mathrm{Un}]{} & \mathrm{Fbrs}(\mathcal{P})
\end{array}$$

Remark I.2.33. The lefthand commutative square of Theorem I.2.32 induces a Beck-Chevalley condition

$$\beta \colon \mathrm{Env}_{\mathcal{O}} f^* \implies f^* \mathrm{Env}_{\mathcal{P}}$$

which is not in general an equivalence. Indeed, in sight of horizontal left-adjointability of the righthand square, we instead find that β is the transformation

$$\mathrm{Env}_{\mathcal{O}} f^* \mathcal{Q} \simeq f^* \mathrm{Env}_{\mathcal{P}} \mathcal{Q} \times_{f^* \mathscr{A}_{\mathcal{O}}, q} \mathscr{A}_{\mathcal{P}} \xrightarrow{\pi} f^* \mathrm{Env}_{\mathcal{P}} \mathcal{Q};$$

for β to be an equivalence, it suffices to check the nontrivial condition that q is an equivalence. ◁

Under the following language, the square is left-adjointable if α is strongly BC.

Definition I.2.34. A strong Segal morphism $\varphi : : \mathcal{O} \to \mathcal{P}$ is *strongly BC* if the Beck-Chevalley transformation $q \colon \mathscr{A}_{\mathcal{O}} \to' \varphi^* \mathscr{A}_{\mathcal{P}}$ is an equivalence. ◁

We have a quick but useful result of this.

Proposition I.2.35. *If $\varphi \colon \mathcal{O} \to \mathcal{P}$ is strongly BC, $\mathcal{O}$ is sound, and $\mathcal{P}$ is soundly extendable, then $\mathcal{O}$ is soundly extendable.*

Proof. φ is a Segal morphism so $\mathscr{A}_{\mathcal{O}} \simeq \varphi^* \mathscr{A}_{\mathcal{P}}$ is a Segal $\mathcal{O}$-∞-category. □

We will make frequent use of product patterns, so we observe their interaction with Segal envelopes.

Observation I.2.36. If $\mathcal{O}, \mathcal{P}$ are fibrous $\mathcal{B}$-patterns, then their Segal envelopes satisfy

$$\begin{aligned}
\mathrm{Env}_{\mathcal{B}\times\mathcal{B}}(\mathcal{O} \times \mathcal{P}) &\simeq (\mathcal{O} \times \mathcal{P}) \times_{\mathcal{B}\times\mathcal{B}} \mathrm{Ar}^{\mathrm{act}}(\mathcal{B} \times \mathcal{B}) \\
&\simeq \left(\mathcal{O} \times_{\mathcal{B}} \mathrm{Ar}^{\mathrm{act}}(\mathcal{B})\right) \times \left(\mathcal{P} \times_{\mathcal{B}} \mathrm{Ar}^{\mathrm{act}}(\mathcal{B})\right) \\
&\simeq \mathrm{Env}_{\mathcal{B}}(\mathcal{O}) \times \mathrm{Env}_{\mathcal{B}}(\mathcal{P})
\end{aligned}$$

◁

I.2.1.6 Equifibrations We will frequently make use of an explicit identification of the essential image of $\mathrm{Env}_{\mathcal{O}}^{/\mathscr{A}_{\mathcal{O}}}$ in terms of Barkan-Haugseng-Steinebrunner's theory of *equifibrations*.

Definition I.2.37. A natural transformation $\alpha\colon F \Longrightarrow G$ of functors $\mathcal{C} \to \mathcal{D}$ is an *equifibration* if each naturality square is cartesian:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow{\scriptstyle \alpha_X} & \lrcorner & \downarrow{\scriptstyle \alpha_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

More generally, given $\mathcal{B} \subset \mathcal{C}$ a class of morphisms, we say that α is an *equifiberation over* $\mathcal{B}$ if naturality squares are cartesian for all $f \in \mathcal{B}$.

A morphism of Segal $\mathfrak{B}$-objects in $\mathcal{C}$ is an *equifibration* if its corresponding natural transformation of functors $\mathfrak{B} \to \mathcal{C}$ is equifibered over $\mathfrak{B}^{\mathrm{act}}$. ◁

Proposition I.2.38 ([BHS22, Prop. 4.2.1])**.** *The essential image of* $\mathrm{Env}^{/}\colon \mathrm{Fbrs}(\mathfrak{B}) \to \mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})_{/\mathscr{A}_{\mathfrak{B}}}$ *is the equifibrations.*

To identify these and study envelopes of fibrous pattern maps, we note the following easy properties.

Proposition I.2.39. *Fix* $\mathcal{B} \subset \mathcal{C}$ *a class of arrows and* $\alpha\colon F \Longrightarrow G$ *a natural transformation of functors* $\mathcal{C} \to \mathcal{D}$*.*

(1) If $\mathcal{B}' \subset \mathcal{B}$ *is a subclass which generates* $\mathcal{B}$ *under composition, then then* α *is an equifibration over* $\mathcal{B}'$ *if and only if it is equifibered over* $\mathcal{B}$*.*

(2) If $F \xRightarrow{\alpha} G \xRightarrow{\beta} H$ *is a composable sequence of natural transformations such that* β *is an equifibration over* $\mathcal{B}$*, then* α *is an equifibration over* $\mathcal{B}$ *if and only if* $\beta \circ \alpha$ *is.*

Proof. These arguments are simple and already appear in [BHS22]; indeed, we may draw the diagram

$$\begin{array}{ccccc} F(X) & \xrightarrow{F(f)} & F(Y) & \xrightarrow{F(g)} & F(Z) \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_Z} \\ G(X) & \xrightarrow{G(f)} & G(Y) & \xrightarrow{F(g)} & G(Z) \\ \downarrow{\scriptstyle \beta_X} & & \downarrow{\scriptstyle \beta_Y} & & \\ H(X) & \xrightarrow{H(f)} & H(Y) & & \end{array}$$

In situation (1), if f and g lie in $\mathcal{B}'$, then the top two squares are cartesian, so the top outer rectangle is cartesian by pullback pasting; hence the locus of arrows over which α is an equifibration is closed under composition, and the claim follows.

In situation (2), the bottom square is cartesian, so the top is if and only if the outer square is by pullback pasting; quantifying over all morphisms f yields the claim. □

Corollary I.2.40. *Fix* $F\colon \mathfrak{O} \to \mathfrak{P}$ *a Segal* $\mathfrak{B}$*-functor.*

(1) If $\mathfrak{O}, \mathfrak{P}$ *arise as envelopes of fibrous* $\mathfrak{B}$*-patterns, then every Segal* $\mathfrak{B}$*-functor* $\mathfrak{O} \to \mathfrak{P}$ *over* $\mathscr{A}_{\mathfrak{B}}$ *is equifibered.*

(2) F is an equifibration if and only if it is an equifibration over all active arrows with elementary codomain.

PROOF. Claim (1) is gotten by applying the two out of three claim of Proposition I.2.39 to the image expression of Proposition I.2.38. For Claim (2) fix an active arrow $F\colon X \to Y$. Under the equivalence

$$\mathcal{O}_Y \simeq \lim_{E \in \mathcal{O}^{\mathrm{el}}_{Y/}} \mathcal{O}_E,$$

cartesian squares taken equivalently onto diagrams of cartesian squares. Unwinding definitions, the naturality square is taken onto a diagram of naturality squares, which are cartesian by assumption, yielding the claim. □

I.2.2 Generalized Segal conditions and categorical patterns For this section, fix $\mathfrak{C}$ an algebraic pattern. We will need flexibility in what precisely we mean by *Segal conditions* for Segal objects and fibrous patterns. To discuss these for Segal objects, we use the following notion.

Definition I.2.41. A diagram $\overline{p}\colon K^{\triangleleft} \to \mathfrak{C}^{\mathrm{el}}_{C/}$ extending p is a *generalized $\mathfrak{C}$-Segal condition* if for all $\mathfrak{C}$-Segal spaces $F\colon \mathfrak{C} \to \mathcal{S}$, $F \circ \overline{p}$ is an $F \circ p$-limit diagram. A set of generalized $\mathfrak{C}$-Segal conditions $\{\overline{p_\alpha}\}$ is *sufficient* if every functor $F\colon \mathfrak{C} \to \mathcal{S}$ sending $F \circ \overline{p}_\alpha$ to $\overline{p}$-limit diagrams is a Segal space. ◁

The conventional Segal conditions are the following.

Example I.2.42. Any set of generalized $\mathfrak{C}$-Segal conditions containing a sufficient set is sufficient, and the set of diagrams $\mathfrak{C}^{\mathrm{el}}_{C/} \to \mathfrak{C}$ is sufficient; hence the set of *all* generalized $\mathfrak{C}$-Segal conditions is sufficient. ◁

Segal spaces were chosen for concreteness; we can choose other value categories.

Lemma I.2.43. *Fix $\mathfrak{C}$ an algebraic pattern.*

(1) If $\overline{p}\colon K^{\triangleleft} \to \mathfrak{C}^{\mathrm{int}}$ is a generalized $\mathfrak{C}$-Segal condition and $\mathcal{C}$ is a $\mathfrak{C}$-complete ∞-category, then every Segal $\mathfrak{C}$-object $F\colon \mathfrak{C} \to \mathcal{C}$ has $F \circ \overline{p}$ an $F \circ p$-limit diagram.

(2) If $\left\{\overline{p}_\alpha\colon K_\alpha^{\triangleleft} \to \mathfrak{C}^{\mathrm{int}}\right\}$ is a sufficient set of generalized $\mathfrak{C}$-Segal conditions and $F\colon \mathfrak{C} \to \mathcal{C}$ satisfies the condition that each $F \circ \overline{p}_\alpha$ is an $F \circ p_\alpha$-limit diagram, then F is a Segal $\mathfrak{C}$-object.

PROOF. Each statement is a straightforward modification of the proof of [BHS22, Prop. 2.19]; indeed, Yoneda's lemma shows that it suffices to prove this for $\mathcal{C}$ a presheaf topos, after which the result follows by identifying $\mathrm{Seg}_{\mathfrak{C}}(\mathsf{Fun}(\mathcal{B},\mathcal{S})) \simeq \mathsf{Fun}\left(\mathcal{B}, \mathrm{Seg}_{\mathfrak{C}}(\mathcal{S})\right)$ and using the pointwise expression for limits in functor categories. □

We will mostly need this to identify fibrous patterns. Unfortunately to utter sentences such as "to check that $\mathcal{O} \to \mathfrak{C}$ we may check on a set of conditions satisfying...," we need a more careful notion than what we've just defined:

Definition I.2.44. A functor $p\colon K^{\triangleleft} \to \mathfrak{C}^{\mathrm{int}}$ is a *generalized fibrous $\mathfrak{C}$-Segal condition* if for all *relative* $\mathcal{C}$-Segal objects $F \to G$, the associated square

$$\begin{array}{ccc} F(p(V)) & \longrightarrow & \lim_{W\in K} F(W) \\ \downarrow & & \downarrow \\ G(p(V)) & \longrightarrow & \lim_{W\in K} G(W) \end{array}$$

is cartesian. A set of generalized fibrous $\mathfrak{C}$-Segal conditions $\left\{\overline{p}_\alpha\colon K_\alpha^{\triangleleft} \to \mathfrak{C}^{\mathrm{int}}\right\}$ is *sufficient* if:

(GS-a) it is a sufficient as a set of generalized $\mathfrak{C}$-Segal conditions,[9]

(GS-b) each $\omega_!\overline{p}_\alpha$ is a generalized $\mathfrak{C}$-Segal condition, and

(GS-c) given objects $C, D \in \mathfrak{C}$, active maps $\omega \in \mathfrak{C}^{\mathrm{act}}_{/C}$, and inert-cocartesian fibration $\mathcal{O} \to \mathfrak{C}$ subject to the condition that for all for all cocartesian sections s_α of p_α, the corresponding square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{O}}(D, C) & \longrightarrow & \lim_{W\in K} \mathrm{Map}(D, s_\alpha(W)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathfrak{C}}(\pi D, \pi C) & \longrightarrow & \lim_{W\in K} \mathrm{Map}(\pi D, p_\alpha(W)) \end{array}$$

is cartesian, we have for each cocartesian section $s_{\omega,C}$ of $\omega_!\colon (\mathfrak{C}^{\mathrm{el}}_{C/} \times_{\mathfrak{C}} \mathcal{O} \to \mathfrak{C})$ that the square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{O}}(D, C) & \longrightarrow & \lim_{E\in\mathfrak{C}^{\mathrm{el}}_{C/}} \mathrm{Map}(D, s_{\omega,C}E) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathfrak{C}}(\pi D, \pi C) & \longrightarrow & \lim_{E\in\mathfrak{C}^{\mathrm{el}}_{C/}} \mathrm{Map}(\pi D, \omega_!E) \end{array}$$

is also cartesian. ◁

Example I.2.45. If $\mathfrak{C}$ itself is a sound pattern, then [BHS22, Prop. 3.3.8] shows that $\left\{\mathfrak{C}^{\mathrm{el}}_{C/}\right\}$ is a sufficient set of generalized fibrous $\mathfrak{C}$-Segal conditions. ◁

The proof of [BHS22, Prop. 4.1.7], extends easily to the following.

Proposition I.2.46 (c.f. [HA, Def. B.0.19])**.** *Suppose $\left\{\overline{p}_\alpha\colon K^{\triangleleft} \to \mathfrak{C}^{\mathrm{int}}\right\}$ is a sufficient set of generalized fibrous $\mathfrak{C}$-Segal conditions and $\mathcal{O} \to \mathfrak{C}$ is an inert cocartesian functor. Then, $\mathcal{O}$ is a fibrous pattern if and only if the following conditions hold for all α:*

(b) the unstraightening of $\mathcal{O}^{\mathrm{int}} \to \mathfrak{C}^{\mathrm{int}}$ composes to a limit diagram $K_\alpha^{\triangleleft} \to \mathfrak{C}^{\mathrm{int}} \to \mathrm{Cat}$, and

(c) for all cocartesian sections $s_C\colon K_\alpha^{\triangleleft} \to K_\alpha^{\triangleleft} \times_{\mathfrak{C}} \mathcal{O}$, the composite functor $K_\alpha^{\triangleleft} \to \mathcal{O}$ is a $\pi_{\mathfrak{C}}$-limit diagram.

Proof. By [BHS22, Obs. 4.1.3], we find that $\rho_{\mathfrak{C}}\colon \mathrm{Env}_{\mathfrak{C}}(\mathcal{O}) \to \mathrm{Env}_{\mathfrak{C}}\mathrm{Comm}_{\mathfrak{C}}$ is a relative Segal $\mathcal{O}$-category if and only if $\mathcal{O}$ is a fibrous pattern; the same proof as [BHS22, Prop. 4.1.6] reduces the relative p-Segal condition for $\rho_{\mathfrak{C}}$ to the disjunction of

[9] To see that generalized fibrous Segal conditions are Segal conditions, simply fix $G = *$.

(b') the absolute $\omega_! p$-Segal condition for $\mathcal{O}^{\simeq}$ for all active ω, and

(c') a relative $\omega_! p$-Segal condition for $\mathrm{Map}_{\mathcal{O}}(Y, s_\alpha(-)) \Longrightarrow \mathrm{Map}_{\mathcal{C}}(\pi_{\mathcal{O}} Y, \pi_{\mathcal{O}} p_\alpha(-))$ for each cocartesian section $s_{\omega,C}$.

Conditions (GS-a) and (GS-b) identify conditions (b) and (b'); Condition (GS-c) identifies (c) with condition (c') applied to all of the canonical Segal conditions, and the proposition follows. □

Corollary I.2.47. *If $\overline{p}\colon K^{\triangleleft} \to \mathcal{C}^{\mathrm{int}}$ is a fibrous generalized $\mathcal{C}$-Segal condition such that $\omega_! \overline{p}_\alpha$ is a generalized $\mathcal{C}$-Segal condition and $\mathcal{O} \to \mathcal{C}$ is a fibrous pattern over a sound algebraic pattern, then the conditions of Proposition I.2.46 hold for $\overline{p}$.*

Proof. Apply Proposition I.2.46 to the sufficient set $\{\overline{p}\} \cup \left\{\mathcal{C}^{\mathrm{el}}_{/C} \mid C \in \mathcal{C}\right\}$. □

Now, our main tool to construct these sufficient sets is via *pasting*.

Lemma I.2.48 (Segal condition pasting)**.** *Fix $\left\{\overline{p}_\alpha\right\}$ and $\left\{\overline{q_\beta}\right\}$ are sets of functors to $\mathcal{C}^{\mathrm{int}}$. Suppose for all indices β and active maps $W \to q_\beta(v_\beta)$ that there exists exists an index α with functor $\gamma\colon K_\beta \to K_\alpha$ as well as for each $k \in K_\alpha$ an index α_k with functor $\delta\colon K^{\triangleleft}_{\alpha_k} \to K_\alpha$ such that the $\omega_! p_\beta$-assembly map factors naturally as a composition of limits of α and α_k-coassembly maps*

$$\begin{array}{ccccc} F(v_\beta) & & \xrightarrow{\quad c_{\omega_!\beta} \quad} & & \lim_{b \in K_\beta} F(\omega_! b) \\ \wr\uparrow & & & & \downarrow\sim \\ \gamma^* F(v_\alpha) & \xrightarrow{c_\alpha} & \lim_{k \in K_\alpha} \gamma^* F(k) & \xrightarrow{\lim_k c_{\alpha_k}} & \lim_{k \in K_\alpha} \gamma^* F(v_{\alpha_K}) \end{array}$$

(1) If $\left\{\overline{p}_\alpha\right\}$ is a set of generalized $\mathcal{C}$-Segal conditions, then so is $\left\{\overline{q}_\beta\right\}$.

(2) If $\left\{\overline{p}_\alpha\right\}$ is a set of generalized fiborous $\mathcal{C}$-Segal conditions, then so is $\left\{\overline{q}_\beta\right\}$.

(3) If $\left\{\overline{q}_\beta\right\}$ is sufficient as generalized Segal conditions, then so is $\left\{\overline{p}_\alpha\right\}$.

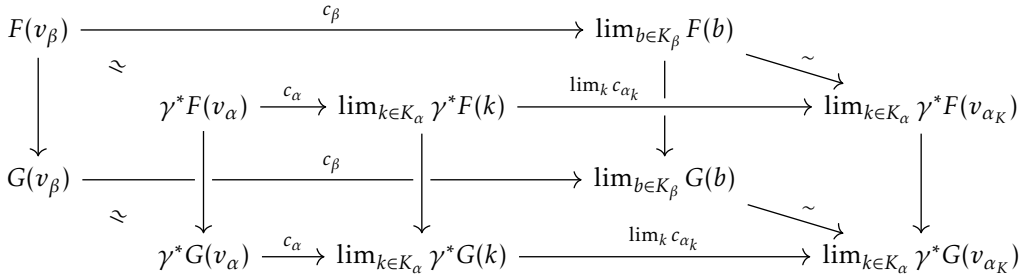
(4) If $\left\{\overline{q}_\beta\right\}$ is sufficient as generalized fibrous Segal conditions, then so is $\left\{\overline{p}_\alpha\right\}$.

Now, suppose instead that each $\alpha_k \in B$, i.e. the righthand coassmebly maps correspond with $\left\{\overline{q}_\beta\right\}$-Segal conditions.

(1') If $\left\{\overline{q}_\beta\right\}$ is a set of generalized $\mathcal{C}$-Segal conditions, then so is $\left\{\overline{p}_\alpha\right\}$.

(2') If $\left\{\overline{q}_\beta\right\}$ is a set of generalized fibrous $\mathcal{C}$-Segal conditions, then so is $\left\{\overline{p}_\alpha\right\}$.

PROOF. Given a natural transformation $\eta\colon F \Longrightarrow G\colon K_\alpha^\triangleleft \to \mathcal{C}$, naturality of everything in sight allows to construct a commutative diagram



In particular, if the front right square is cartesian, then the back face is cartesian if and only if the front left square is by pullback pasting. Said another way, if η's precomposed natural transformations already satisfy the α_k-Segal conditions, then it satisfies the β-relative-Segal condition if and only if it satisfies the α-relative-Segal condition.

Now, if this underlies a natural transformation of $\mathfrak{C}$-Segal objects with $G = *$, then each c_α and c_{α_k} are equivalences, so c_β is an equivalence as well, giving (1). Sufficiency is similar; if F satisfies each α-Segal condition then by the above it must satisfy each β-Segal condition, so it must be a Segal object. The fibrous cases and apostrophe cases follow *mutatis mutandis*. □

Remark I.2.49. If $\left\{\overline{q}_\beta\right\}$ are closed under the $\omega_!$ construction, then the condition of Lemma I.2.48 may be specialized to $\omega = \mathsf{id}$; this is a strict *soundness* condition on $\left\{\overline{q}_\beta\right\}$, related to sound algebraic patterns. ◁

Remark I.2.50. Suppose $\{p_\alpha \to q_\alpha\}$ is a set of initial functors over $\mathfrak{C}^{\mathrm{int}}$. Then, the conditions of Lemma I.2.48 are satisfied for $\{p_\alpha\}$ and $\{q_\alpha\}$. For instance, given a G-operad $\mathcal{O}^\otimes$, the initial subcategory inclusions $S^\triangleleft \hookrightarrow \mathrm{Tot}\,\mathcal{O}^{\mathrm{el}}_{/S}$ or $T^\triangleleft \hookrightarrow \mathrm{Tot}\,\mathrm{Tot}_G\,\mathcal{O}^{\mathrm{el}}_{/S}$ give the sets of generalized fibrous Segal conditions considered in [BHS22]. One thus view Lemma I.2.48 as an elaboration on Barkan-Haugseng-Steinebrunner's fibrous pattern model for G-operads expression of their Segal objects with the variation of Lurie's $\mathcal{O}$-monoids used in [Ste25c]. ◁

We develop this notion in order to utilize Lurie's notion of *categorical patterns*.

Definition I.2.51 ([HA, Def. B.0.19])**.** A *categorical pattern with all 2-cells* $\mathfrak{P}$ is a simplicial set S, a collection of 1-cells $M_S \subset S_1$, and a collection of diagrams $K_\alpha^\triangleleft \to S$ whose 1-cells land in M_S. Given such a categorical pattern $\mathfrak{P}$, a $\mathfrak{P}$-*fibered object* is a map of marked simplicial sets $(X, M) \to (S, M_S)$ such that:

(1) the underlying map $f\colon X \to S$ is an inner fibration.

(2) Every element of M is f-cocartesian, and every f-cocartesian lift of an edge of M_S lies in M.

(5) For all α, the induced map $f_\alpha\colon X \times_S K_\alpha^\triangleleft \to K_\alpha^\triangleleft$ unstraightens to a limit diagram $K_\alpha^\triangleleft \to \mathsf{Cat}$.

(6) For every index α and cocartesian section s of f_α, s is an f-limit diagram in X. ◁

We find many examples in our study of algebraic patterns.

Construction I.2.52. An algebraic pattern $\mathfrak{B}$ has an associated categorical pattern

$$\mathsf{CatPatt}(\mathfrak{B}) \coloneqq \left(\mathfrak{B}, \mathsf{inert}, \mathsf{all}, \left\{\mathfrak{B}^{\mathrm{el}}_{O/}\right\}_{O\in\mathfrak{B}}\right)$$

Given $\left\{\overline{p}_\alpha \colon K_\alpha^\triangleleft \to \mathfrak{C}^{\mathrm{int}}\right\}$ a set of generalized $\mathfrak{C}$-Segal conditions, we define the categorical pattern structure $\mathfrak{P}$ on $\mathfrak{C}$ to have the same inert arrows as $\mathsf{CatPatt}(\mathfrak{C})$, all 2-cells, and Segal diagrams exactly $\left\{\overline{p}_\alpha\right\}$. ◁

Lurie uses this notion to construct a model category.

Proposition I.2.53 ([HA, Thm. B.0.20])**.** *There exists a left-proper combinatorial simplicial model structure on* $\mathsf{sSet}^+_{/(X,M)}$ *uniquely characterized by the fact that its cofibrations are the monomorphisms and its fibrant objects are the* $\mathfrak{P}$*-fibered objects.*

This may be viewed as a joint generalization of the model structure on ∞-preoperads and the cocartesian model structure, and it often presents a familiar ∞-category:

Proposition I.2.54. *Lurie's model structure on* $\mathsf{sSet}^+_{/\mathsf{CatPatt}(\mathfrak{B})}$ *presents* $\mathsf{Fbrs}(\mathfrak{B})$.

Now, this follows immediately from a more general proposition.

Proposition I.2.55. *Suppose* $\mathfrak{P}$ *is a categorical pattern with all* 2*-simplices such that* M_S *is closed under equivalence, and hence defines a wide subcategory* $S^{\mathrm{int}} \hookrightarrow S$*. Then, the identity underlies a left Quillen functor*

$$\mathsf{id}_! \colon \mathsf{sSet}_{/S} \to \mathsf{sSet}^+_{/\mathfrak{P}},$$

whose right adjoint has total derived functor a subcategory inclusion whose objects are the $\mathfrak{P}$*-fibered objects (up to homotopy) and whose morphisms are the functors over* N^{hc} *preserving inert-cocartesian lifts.*

Proof. That $\mathsf{id}_!$ is a left Quillen functor and id^* a right Quillen functor follows from [HA, Prop. B.2.9]. Now, to prove that $N^{hc}\,\mathsf{id}^*$ is a subcategory inclusion, we must prove that it is acts on mapping spaces between fibrant object as a summand inclusion, which surjects onto the path components of equivalences; but this follows by unwinding the fact that the marked arrows are precisely the inert-cocartesian lifts. □

We would like more general presentations of our categories, for which we use the following.

Lemma I.2.56. *Suppose* $\overline{p}_\alpha$ *is a soundly sufficient set of generalized* $\mathfrak{C}$*-Segal conditions for* $\mathfrak{C}$ *a soundly extendable algebraic pattern. Then, a functor* $\pi \colon \mathfrak{O} \to \mathfrak{C}$ *is a fibrous pattern if and only if it possesses a (nessentially unique) model as a map of marked simplicial sets making it* $\mathfrak{P}$*-fibered.*

PROOF. Condition (1) is assured by the word "functor;" condition (2) is equivalent to π being an inert-cocartesian fibration. We're left with showing that (5) and (6) together are equivalent to fibrousnes; this is Proposition I.2.46. □

From this, we find categorical pattern versions of *Segal morphisms* and *Morita equivalences*.

Proposition I.2.57. *Let $\mathfrak{C}$ be a soundly extendable algebraic pattern and $\left\{\overline{p}_\alpha\right\}$ a set of diagrams.*

(1) if $\left\{\overline{p}_\alpha\right\}$ is a set of generalized fibrous Segal conditions then the identity on a model for $\mathfrak{C}$ underlies a left Quillen functor $\mathrm{id}_!\colon \mathrm{sSet}^+_{/\mathfrak{P}} \to \mathrm{sSet}^+_{/\mathrm{CatPatt}(\mathfrak{C})}$, and

(2) if $\left\{\overline{p}_\alpha\right\}$ is sufficient then $\mathrm{id}_!$ is an equivalence of model categories.

PROOF. The identity always preserves and reflects monomorphisms of simplicial sets, i.e. cofibrations; hence statement (2) follows from Lemma I.2.56.

Now, in the situation of (1), let $\left\{\overline{q}_\beta\right\}$ be the set of *all* sound generalized $\mathfrak{C}$-Segal conditions (which is sufficient by Examples I.2.42 and I.2.45si=pkOkRhagZKpimp8r) and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{sSet}_{/\mathfrak{P}} & \longrightarrow & \mathrm{sSet}_{/\mathrm{CatPatt}(\mathfrak{C})} \\ & \searrow & \downarrow \\ & & \mathrm{sSet}_{/\mathfrak{Q}} \end{array}$$

where each arrow is the identity on underlying functors. The vertical arrow is an equivalence of model categories by (2), so the diagonal arrow is a left Quillen functor if and only if the horizontal arrow is; [HA, Prop. B.2.9] affirms this for the diagonal arrow. □

Now, we will need a variation of this purely for categorical patterns. Given $\mathfrak{P} = (S, M)$ a categorical pattern and $\overline{p}\colon K^\triangleleft \to M$, we say that $\overline{p}$ is a $\mathfrak{P}$-*fibered Segal condition* if the conditions of Definition I.2.51 for $\overline{p}$ hold for all $\mathfrak{P}$-fibered Segal objects. i.e. the modified algebraic pattern $\mathfrak{P}''$ on the same S and M with $K^\triangleleft \to S$ an added Segal condition, the identity functor $\mathrm{sSet}^+_{\mathfrak{P}''} \to \mathrm{sSet}^+_{\mathfrak{P}}$ is an equivalence of model categories.

Lemma I.2.58. *Suppose $\varphi\colon \mathfrak{P}' \to \mathfrak{P}$ is a map of categorical patterns with all 2-cells such that for all $\alpha \in A$, the composite functor*

$$\overline{q}_\alpha\colon K_\alpha^\triangleleft \to S' \to S$$

is a $\mathfrak{P}$-fibered Segal condition. Then, composition with f induces a left Quillen functor $f\colon \mathrm{sSet}^+_{/\mathfrak{P}} \longrightarrow \mathrm{sSet}^+_{/\mathfrak{P}'}$.

PROOF. After replacing $\mathfrak{P}$ with the modified categorical pattern $\mathfrak{P}''$ on (M, S) defined above (which adds all $\overline{q}_\alpha$) and composing with the equivalence of model categories $\mathrm{sSet}^+_{/\mathfrak{P}} \simeq \mathrm{sSet}^+_{/\mathfrak{P}''}$, this follows from [HA, Prop. B.2.9]. □

I.2.3 Push-pull adjunctions between algebraic patterns We now make Lemma I.2.58 right-handed.

Definition I.2.59. A span of algebraic pattern maps $\mathfrak{B} \xleftarrow{\varphi} \mathcal{O} \xrightarrow{\pi} \mathfrak{C}$ is *pairable* if φ is an exponentiable functor and the functor $\pi^*\varphi_* \colon \mathsf{Cat}_{/\mathfrak{C}} \to \mathsf{Cat}_{/\mathfrak{B}}$ restricts to a right adjoint $\pi^*\varphi_*\mathsf{Fbrs}(\mathfrak{C}) \to \mathsf{Fbrs}(\mathfrak{B})$ to $L_{\mathsf{Fbrs}}\pi_!\varphi^*$. In the case that π is an equivalence, we call φ *exponentiable*, and in the case that φ is an equivalence, note that this is equivalent to π being a *Segal morphism*. ◁

Proposition I.2.60. *Suppose $\mathfrak{B} \xleftarrow{\varphi} \mathcal{O} \xrightarrow{\psi} \mathfrak{C}$ is a span of functors such that $\mathfrak{B}$ and $\mathfrak{C}$ are algebraic patterns and $\left\{\overline{p}_\alpha\right\}$ is a sufficient set of generalized fibrous $\mathfrak{B}$-Segal conditions such that the following conditions hold:*

(p-1) The functor $\varphi \colon \mathcal{O} \to \mathfrak{B}$ is exponentiable.

(p-4) The functor $\varphi \colon \mathcal{O} \to \mathfrak{B}$ is an inert locally-cartesian fibration.

(p-5) For each $\alpha \in A$, the projection $\varphi'_\alpha \colon K_\alpha^\triangleleft \times_{\mathfrak{B}} \mathcal{O} \to K_\alpha^\triangleleft$ is a cocartesian fibration.

(p-6) For each $\alpha \in A$ and cocartesian section s_α of φ'_α, the composition

$$K_\alpha^\triangleleft \xrightarrow{s_\alpha} K_\alpha^\triangleleft \times_{\mathfrak{B}} \mathcal{O} \xrightarrow{\pi} \mathcal{O} \xrightarrow{\psi} \mathfrak{C}$$

is a generalized fibrous $\mathfrak{C}$-Segal condition.

(p-7) Given a composable pair of arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{O}$ such that g is locally $\pi_{\mathcal{O}}$-cartesian, $\psi(g)$ is inert, and $\psi(f)$ is an equivalence, f is inert if and only if gf is inert.

(p-8) Given an index $\alpha \in A$ and a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $K_\alpha^\triangleleft \times_{\mathfrak{B}} \mathcal{O}$ such that f is φ'_α-cocartesian and $\varphi'_\alpha(g)$ is an equivalence, $\pi(g)$ is inert if and only if $\pi(gf)$ is.

Then, the span $\mathfrak{B} \xleftarrow{\varphi} \mathcal{O} \xrightarrow{\psi} \mathfrak{C}$ is pairable.

This mostly boils down a variation on Lurie's categorical pattern pairing:

Lemma I.2.61. *Suppose $\mathfrak{P}$ and $\mathfrak{P}'$ are categorical patterns on ∞-categories and we have a span of marked simplicial sets*

$$(\mathcal{C}, M_{\mathcal{C}}) \xleftarrow{\pi} (\mathcal{K}, M) \xrightarrow{\pi'} (\mathcal{C}', M_{\mathcal{C}'})$$

satisfying the conditions of [HA, Thm. B.4.2], except that condition (6) is replaced with the condition that for all $\alpha \in A$ and cocartesian section s of π_α, the composite functor

$$K_\alpha^\triangleleft \xrightarrow{s_\alpha} K_\alpha^\triangleleft \times_{\mathcal{C}} \mathcal{K} \longrightarrow \mathcal{K} \xrightarrow{\pi} \mathcal{C}'$$

is a $\mathfrak{P}'$-fibered Segal condition. Then, the construction $(-) \times_{(\mathcal{C}, M_{\mathcal{C}})} (\mathcal{K}, M)$ determines a left Quillen functor $\mathsf{sSet}^+_{/\mathfrak{P}} \longrightarrow \mathsf{sSet}^+_{/\mathfrak{P}'}$.

PROOF. The same proof as [HA, Thm. B.4.2] applies; the only modification is to cite Lemma I.2.58 instead of [HA, Lem. B.2.9], □

PROOF OF PROPOSITION I.2.60. Let $\mathfrak{P}_{\mathfrak{B}}$ be the categorical pattern induced by the indicated generalized Segal conditions under Construction I.2.52 and let $\mathfrak{P}_{\mathfrak{O}}$ be the categorical pattern structure on $\mathfrak{O}$ from the proof of [HA, Thm. B.4.2]. We know from Proposition I.2.57 that $\mathsf{id}_!\colon \mathsf{sSet}^+_{/\mathfrak{P}_{\mathfrak{B}}} \to \mathsf{sSet}^+_{/\mathsf{CatPatt}(\mathfrak{B})}$ is an equivalence of model categories, so to conclude by composition of left Quillen functors, it suffices to prove that each of the following functors are left Quillen functors

$$\mathsf{sSet}^+_{/\mathfrak{P}_{\mathfrak{B}}} \xrightarrow{\varphi^*} \mathsf{sSet}^+_{/\mathfrak{P}_{\mathfrak{O}}} \xrightarrow{\psi_!} \mathsf{sSet}^+_{/\mathsf{CatPatt}(\mathfrak{C})}$$

Now, we have matched up numbers so that every condition becomes obviously equivalent to a condition of [HA, Thm. B.4.2] applied to this diagram, with the exception of Condition (p-6), which is matched with the modified condition of Lemma I.2.61; identifying these conditions is straightforward by unwinding definitions. Moreover, (2) compiles down to verifying that inert morphisms are a wide subcategory, and (3) is automatic as T and T' contain all 2-simplices. □

Remark I.2.62. In the case that φ is an equivalence, all of the above conditions except for (p-6) become automatic; (p-6) then becomes a condition lying in between the conditions that ψ is *Segal* or *strong Segal*. Note that the condition for nontrivial φ does not imply the same for $\varphi = \mathsf{id}$, e.g. ψ need not be strong Segal for a pattern structure on the middle to apply the pairability result; likewise φ need not be exponentiable for a pattern structure on the middle for the span to be pairable. ◁

Remark I.2.63. By definition the image of every morphism in $K^{\triangleleft}_{\alpha}$ is inert; the description of preservation of cocartesian arrows under pullback of [kerodon, 01UF] then reduces Condition (p-5) to the following stronger condition:

(p-5') φ is an inert-cocartesian fibration. ◁

Observation I.2.64. Suppose $\mathfrak{B}$, $\mathfrak{B}'$ are soundly extendable algebraic patterns. A variation on this concludes that cartesian products furnish a distributive bifunctor

$$\mathsf{Fbrs}(\mathfrak{B}) \times \mathsf{Fbrs}(\mathfrak{B}') \to \mathsf{Fbrs}(\mathfrak{B} \times \mathfrak{B}');$$

the restriction $\mathsf{Fbrs}(\mathfrak{B}) \simeq \mathsf{Fbrs}(\mathfrak{B}) \times \{\mathfrak{B}'\} \to \mathsf{Fbrs}(\mathfrak{B} \times \mathfrak{B}')$ is then seen to be both equivalent to pullback along the projection $p\colon \mathfrak{B} \times \mathfrak{B}' \to \mathfrak{B}$ and a left adjoint [HA, Rmk. B.2.5]. We will write p_* for its right adjoint. The

same result applies for Segal $\mathfrak{B}$-∞-categories using the pattern

$$\mathrm{CatPatt}^{\otimes}(\mathfrak{B}) := \left(\mathfrak{B}, \mathrm{all}, \mathrm{all}, \left\{\mathfrak{B}^{\mathrm{el}}_{O/}\right\}_{O\in\mathfrak{B}}\right). \qquad \triangleleft$$

I.2.4 Digression: parameterized $(\infty,2)$-subcategories We now use Segal categories as a convenient ambient framework to describe parameterized $(\infty,2)$-subcategories.

I.2.4.1 $(\infty,2)$*-subcategories* First, the non-parameterized story.

Proposition I.2.65. *Let $F\colon \mathbb{C}\to\mathbb{D}$ be a functor of $(\infty,2)$-categories. Then, F is a monomorphism in $\mathrm{Cat}_{(\infty,2)}$ if and only if the following conditions hold:*

(a) $F^{\simeq}\colon \mathbb{C}^{\simeq}\to\mathbb{D}^{\simeq}$ is a summand inclusion,

(b) For all objects $C,C'\in\mathbb{C}$, the induced map $\mathrm{Fun}_{\mathbb{C}}(C,C')^{\simeq}\to\mathrm{Fun}_{\mathbb{D}}(FC,FC')^{\simeq}$ is a summand inclusion.

(c) For all objects $C,C';\in\mathbb{C}$ and $f,g\in\mathrm{Fun}_{\mathcal{C}}(C,C')$, the induced map

$$\mathrm{Fun}_{\mathbb{C}}(C,C')(f,g)^{\simeq}\to\mathrm{Fun}_{\mathbb{D}}(FC,FC')(Ff,Fg)^{\simeq}$$

is a summand inclusion.

To prove this, we note the following.

Lemma I.2.66. *Suppose $\mathfrak{B}$ is an algebraic pattern and $F\colon\mathcal{C}\to\mathcal{D}$ a morphism of Segal $\mathfrak{B}$-objects in $\mathcal{E}$. Then, F is a monomorphism in $\mathrm{Seg}_{\mathfrak{B}}(\mathcal{E})$ if and only if, for all $B\in\mathfrak{B}^{\mathrm{el}}$, the induced map $\mathcal{C}(B)\to\mathcal{D}(B)$ is a monomorphism.*

Proof. Since $\mathrm{Seg}_{\mathfrak{B}}(\mathcal{E})\xrightarrow{(\mathrm{ev}_B)}\prod_{B\in\mathfrak{B}^{\mathrm{el}}}\mathcal{E}$ is a conservative limit-preserving functor, it preserves and reflects monomorphisms, and these are measured elementwise in product ∞-categories. □

Proof of Proposition I.2.65. Using the pattern $\Delta_2^{\mathrm{op},\natural}$ of [BHS22, Ex. 3.2.1], we find a full subcategory embedding $\mathrm{Cat}_{(\infty,2)}\hookrightarrow\mathrm{Seg}_{\Delta^{\mathrm{op},\natural}\times\Delta^{\mathrm{op},\natural}}(\mathrm{Cat})$ spanned by those $\mathbb{C}_{\bullet\bullet}$ such that $\mathbb{C}_{n,\bullet}$ and $\mathbb{C}_{\bullet,m}$ are complete for all $n,m\geq 0$ and $\mathbb{C}_{0,\bullet}$ is a constant simplicial space [BB25, Rmk. A.1.11]; the constant value of this simplicial space is identified with $\mathbb{C}^{\simeq}$. This condition is closed under limits, so applying Lemma I.2.66, it suffices to show that the conditions are equivalent to showing that $\mathbb{C}_{ij}\to\mathbb{D}_{ij}$ is a summand inclusion for $0\leq i,j\leq 1$.

Now, the cases $i=0$ is identified in an obvious way with (a), so we may assume $I=1$; then, [BB25, Rmk. A.1.26] provides a natural pullback diagram

$$\begin{array}{ccc}\mathrm{Fun}_{\mathbb{C}}(C,C')_{\bullet} & \longrightarrow & \mathbb{C}_{1,\bullet}\\ \downarrow & \lrcorner & \downarrow\\ \{(C,C')\} & \longrightarrow & \mathbb{C}^2_{0,\bullet}\end{array}$$

so under (a), being monic is equivalent to all of the functors $\mathsf{Fun}_{\mathbb{C}}(C,C') \to \mathsf{Fun}_{\mathbb{D}}(FC,FC')$ being monic for all C,C'; if the claim holds for ∞-categories, then this is identified with (b) and (c). That is, we may reduce to the ∞-categorical case.

In this case, the same argument reduces to showing that the corresponding natural transformation of simplicial spaces is monic in degree 0 and 1; (a) is still equivalent to being monic in degree 0, and the same natural pullback square (with $\bullet = 0$) then shows that being monic in degree 1 is the same as inducing summand inclusions on mapping spaces, as desired. □

I.2.4.2 Parameterized $(\infty,2)$-subcategories Applying Lemma I.2.66 for any pattern structure on $\mathcal{T}^{\mathrm{op}}$ with all objects elementary yields the following.

Corollary I.2.67. *A $\mathcal{T}$-$(\infty,2)$-functor $F\colon \mathbb{C} \to \mathbb{D}$ is a monomorphism if and only if, for each $V \in \mathcal{T}$, the $(\infty,2)$-functor $F^V\colon \mathbb{C}^V \to \mathbb{D}^V$ is a monomorphism.*

We say that F^V is *locally fully faithful* if it is a monomorphism and condition (c)'s maps are equivalences, and *full* if additionally condition (b)'s maps are equivalences; we say that F has these properties if F^V has these properties for all $V \in \mathcal{T}$. Now, the following follows by unwinding definitions using Proposition I.2.65.

Corollary I.2.68. *Given $\mathbb{C} \in \mathsf{Cat}_{\mathcal{T},(\infty,2)}$, the following maps are equivalences of posets*

$$\begin{array}{ccccc} \mathsf{LocFullSub}(\mathbb{C}) & \longrightarrow & \mathsf{Sub}\left(\mathbb{C}^{\leq 1}\right) & \longrightarrow & \mathsf{Sub}\left(\mathrm{ho}\mathbb{C}^{\leq 1}\right) \\ \mathsf{FullSub}(\mathbb{C}) & \longrightarrow & \mathsf{FullSub}\left(\mathbb{C}^{\leq 1}\right) & \longrightarrow & \mathsf{Sub}\left(\mathbb{C}^{\simeq}\right) \longrightarrow \mathscr{P}(\pi_0\mathbb{C}) \end{array}$$

The lower row consists of subsets $D^V \subset \pi_0\mathbb{C}^V$ for all $V \in \mathcal{T}$ satisfying the condition that $\mathrm{Res}^V_U D^V \subset D^U$ for all maps $U \to V$; the upper row consists of those data together with subsets $D^V_{X,Y} \subset \pi_0\mathsf{Map}(X,Y)$ for all $X,Y \in D^V$ and $V \in \mathcal{T}$, subject to the condition that $D^V_{Y,Z} \circ D^V_{X,Y} \subset D^V_{X,Z}$.

Now, we point out these properties primarily for the reason of the following observations.

Observation I.2.69. Let f be an arrow lying in a locally fully faithful $(\infty,2)$-subcategory $\mathbb{C} \hookrightarrow \mathbb{D}$. Then, the following conditions are equivalent:

(1) f admits an internal right adjoint in $\mathbb{C}$.

(2) f admits an internal right adjoint g in $\mathbb{D}$ such that g lies in $\mathbb{C}$. ◁

Observation I.2.70. Given a $\mathcal{T}$-$(\infty,2)$-functor $F\colon \mathbb{B} \to \mathbb{D}$ and a locally fully faithful $\mathcal{T}$-$(\infty,2)$-subcategory $\mathbb{C} \hookrightarrow \mathbb{D}$, F factors through $\mathbb{C}$ if and only if $F^{\leq 1}$ factors through $\mathbb{C}^{\leq 1}$; moreover, this is true if and only if $\mathrm{ho}F^{\leq 1}$ factors through $\mathrm{ho}\mathbb{C}^{\leq 1}$, i.e. if F takes objects to $\mathbb{C}$-objects and morphisms to $\mathbb{C}$-morphisms. ◁

I.2.4.3 Relative $\mathcal{T}$-subcategories For discussions around *suboperads*, we point out a fiberwise perspective on relative subcategories. First, a factorization fact.

Observation I.2.71. Given a subcategory inclusion $F\colon \mathcal{C} \hookrightarrow \mathcal{D}$, F admits an essentially unique factorization $F\colon \mathcal{C} \hookrightarrow \mathrm{EssIm}(\mathcal{C}) \hookrightarrow \mathcal{D}$ of a wide subcategory inclusion followed by a full subcategory inclusion; moreover, all such compositions are subcategory inclusions.

If instead F is a $\mathcal{T}$-functor, note that the full subcategories $\mathrm{EssIm}(\mathcal{C}^V) \subset \mathcal{D}^V$ are closed under restriction, so this defines a full $\mathcal{T}$-subcategory inclusion; that is, $\mathcal{T}$-subcategory inclusions are precisely the compositions of wide $\mathcal{T}$-subcategory inclusions with full $\mathcal{T}$-subcategory inclusions. ◁

As such, to simplify the discussion of relative $\mathcal{T}$-subcategories, we may break it down into the wide and full case. The full case is relatively simple:

Proposition I.2.72. *Fix $F\colon \mathcal{C} \to \mathcal{B}$ a $\mathcal{T}$-funtcor. Taking fibers yields an equivalence*

$$\mathrm{FullSub}_{\mathcal{T}}(\mathcal{C}) \simeq \lim_{V\in\mathcal{T}} \prod_{B\in\pi_0\mathcal{B}^V} \mathrm{FullSub}\left(\mathcal{C}_B^V\right).$$

Proof. In sight of Corollary I.2.68, we may assume that $\mathcal{C}$ is a (large) $\mathcal{T}$-set; passing through the core adjunction, we may also assume that $\mathcal{B}$ is a (large) $\mathcal{T}$-set. Now, of course we have $\mathscr{P}(\mathcal{C}) \simeq \lim_{V\in\mathcal{T}} \mathscr{P}(\mathcal{C}^V)$, so we may pass to the case $\mathcal{T} = *$; that is, we're left with accepting the obvious fact that subsets of $\mathcal{C}$ are in bijection with products of subsets of its fibers over $\mathcal{B}$. □

The wide case is slightly more complicated, but still essentially obvious.

Proposition I.2.73. *Fix $F\colon \mathcal{C} \to \mathcal{B}$ a $\mathcal{T}$-functor. Taking mapping fibers yields a fully faithful embedding*

$$\mathrm{WideSub}_{\mathcal{T}}(\mathcal{C}) \hookrightarrow \lim_{V\in\mathcal{T}} \prod_{f\colon B\to B'\in\pi_0\mathrm{Ar}(\mathcal{B}^V)} \prod_{(C,C')\in\mathcal{C}_B^V\times\mathcal{C}_{B'}^V} \mathscr{P}\left(\pi_0\mathrm{Map}_{\mathcal{C}}^f(C,C')\right);$$

the essential image consists of those subsets $\left\{\mathrm{Map}_{\mathcal{D}^V}^f(C,C')\right\}$ which are closed under restriction, contain all equivalences, and admit factorizations

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{D}^V}^f(C,C')\times\mathrm{Map}_{\mathcal{D}^V}^g(C',C'') & \dashrightarrow & \mathrm{Map}_{\mathcal{D}^V}^{g\circ f}(C,C'') \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{C}^V}^f(C,C')\times\mathrm{Map}_{\mathcal{C}^V}^g(C',C'') & \xrightarrow{\circ} & \mathrm{Map}_{\mathcal{C}^V}^f(C,C')\times\mathrm{Map}_{\mathcal{C}^V}^{g\circ f}(C',C'') \end{array}$$

for all $B \xrightarrow{f} B' \xrightarrow{g} B''$ and $(C,C',C'') \in \mathcal{C}_B^V \times \mathcal{C}_{B'}^V \times \mathcal{C}_{B''}^V$.

PROOF. The same reduction as Proposition I.2.72 allows us moreover to assume that $\mathcal{T} \simeq *$. Now, we certainly have a full subcategory inclusion

$$\mathsf{WideSub}(\mathcal{C}) \hookrightarrow \prod_{f\colon B \to B' \in \pi_0 \mathrm{Ar}(\mathcal{B})} \prod_{(C,C') \in \mathcal{C}_B \times \mathcal{C}_{B'}} \mathscr{P}\pi_0 \left(\mathrm{Map}_{\mathcal{C}}(C,C') \times_{\mathrm{Map}_{\mathcal{B}}(B,B')} B\mathrm{Aut}(f) \right);$$

as we are simply decomposing over the summand decomposition of the mapping spaces $\mathrm{Map}_{\mathcal{B}}(B,B')$. Thus we may reduce to proving that, given a map of spaces $X \to Y$ with Y connected, subsets of $\pi_0 X$ are in bijection with subsets of $X \times_Y \{*\}$, i.e. $\pi_0(X \times_Y \{*\}) \to \pi_0 X$ is a bijection. This is pullback-stability of 0-connected maps. □

Together, these characterize the monomorphisms in $\mathsf{Cat}_{\mathcal{T},/\mathcal{B}}$. We care mostly for the sake of factorization, so we will explicitly describe that here.

Corollary I.2.74. *Suppose $\mathcal{D} \hookrightarrow \mathcal{C}$ is a $\mathcal{B}$-relative $\mathcal{T}$-subcategory. Then, a $\mathcal{T}$-functor $F\colon \mathcal{A} \to \mathcal{C}$ over $\mathcal{B}$ factors through $\mathcal{D}$ if and only if the following conditions hold for all $V \in \mathcal{T}$:*

(a) For all $A \in \mathcal{A}^V$, $F(A) \in \mathcal{D}^V$, and

(b) for all maps $f\colon B \to B'$ in $\mathcal{B}^V$ and lifts $(A,A') \in \mathcal{A}^V_B \times \mathcal{A}^V_{B'}$, there exists a factorization

$$\begin{array}{ccc} & & \mathrm{Map}^f_{\mathcal{D}^V}(FA,FA') \\ & \nearrow & \downarrow \\ \mathrm{Map}^f_{\mathcal{A}^V}(A,A') & \longrightarrow & \mathrm{Map}^f_{\mathcal{C}^V}(FA,FA') \end{array}$$

Moreover, this factorization may be checked on path-components.

PROOF. By taking a wide-full factorization, we may prove this in either the wide or full case. The full case is obvious, so we pass to the wide case, where condition (a) is automatic.

Note that, if F^V factors through $\mathcal{D}^V$ for all $V \in \mathcal{T}$, then F factors through $\mathcal{D}$; hence we may pass to the case $\mathcal{T} = *$. That is, we're tasked with proving that there exists the desired factorization if and only if there exists a factorization

$$\begin{array}{ccc} & & \mathrm{Map}_{\mathcal{C}^V}(FD,FD') \\ & \nearrow & \downarrow \\ \mathrm{Map}_{\mathcal{D}^V}(D,D') & \longrightarrow & \mathrm{Map}_{\mathcal{B}^V}(B,B') \end{array}$$

i.e. factorizations through relative subspaces may be checked fiberwise; but this is obvious by unwinding definitions. □

I.3 Parameterized algebraic patterns

Many *Segal object* descriptions of constructions in parameterized higher algebra will occur more naturally in terms of *parameterized right Kan-extensions*. For this reason, in Section I.3.1 we generalize the rudiments of some the above work to a version of algebraic patterns living within parameterized higher category theory, as well as developing some parameterized $(\infty,2)$-categorical aspects of the framework.

As this will frequently be useful, we develop essentially all of our new perspectives on fibrous patterns directly in the parameterized setting; non-parameterized results are immediate corollaries. First, in Section I.3.2 we lift Barkan and Barkan-Haugseng-Steinebrunner's *Morita theory for patterns* to parameterized patterns, incorporating some perspectives of Harpaz [Har] and Lurie [HA, § 2.3.3] along the way, chiefly centered on a notion of *Morita approximation* and *strong Morita equivalence*, the latter inducing equivalences of parameterized $(\infty,2)$-categories of Segal ∞-categories and fibrous patterns.

Following this, in Section I.3.3 we use parameterized pattern Morita theory to give a new perspective on Lurie's theory of disintegration of generalized ∞-operads, working on the level of so-called *disintegrable* $\mathcal{T}$-algebraic patterns. Similarly, in Section I.3.4 we characterize the notion of *monomorphisms* in the $(\infty,2)$-category of fibrous patterns, and in Section I.3.6 we characterize an internal variant of *localizing subcategories* in the $(\infty,2)$-category of Segal categories. Along the way, we develop a doctrinal adjunction in Section I.3.5 We finish with a useful construction: in Section I.3.7 we develop the rudiments for a theory of *Boardman-Vogt tensor products of fibrous patterns*, depending on a magma structure on the underlying $\mathcal{T}$-pattern.

I.3.1 Rudiments of parameterized algebraic patterns

I.3.1.1 Basic definitions The following definition is central.

Definition I.3.1. A *$\mathcal{T}$-algebraic pattern* is a functor $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{AlgPatt}$. Given a $\mathcal{T}$-algebraic pattern $\mathfrak{B}$ and a complete $\mathcal{T}$-∞-category $\mathcal{C}$, we define the *$\mathcal{T}$-∞-category of $\mathfrak{B}$-Segal objects in $\mathcal{C}$*

$$\underline{\mathrm{Seg}}_{\mathfrak{B}}(\mathcal{C}) \coloneqq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathfrak{B},\mathcal{C}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathfrak{B}^{\mathrm{int}},\mathcal{C}),RKE} \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathfrak{B}^{\mathrm{el}},\mathcal{C}\right);$$

we also write $\mathrm{Seg}_{\mathfrak{B}}(\mathcal{C}) \coloneqq \Gamma^{\mathcal{T}} \underline{\mathrm{Seg}}_{\mathfrak{B}}(\mathcal{C})$. ◁

Our central motivation for this formalism is the following easy proposition.

Proposition I.3.2. $\mathrm{AlgPatt}_{\mathcal{T}} \subset \mathrm{Fun}\left(\Lambda^2_2, \mathrm{Cat}_{\mathcal{T}}\right) \times_{\mathrm{Cat}_{\mathcal{T}}} \mathrm{Ar}(\mathrm{Cat}_{\mathcal{T}})$ *is closed under products, and the product pattern has Segal V-objects*

$$\underline{\mathrm{Seg}}_{\mathfrak{B}\times\mathfrak{B}'}(\mathcal{C}) \simeq \underline{\mathrm{Seg}}_{\mathfrak{B}}\underline{\mathrm{Seg}}_{\mathfrak{B}'}(\mathcal{C}).$$

PROOF. Closure under products follows from the observation that wide and full $\mathcal{T}$-subcategory inclusions are closed under products. Now, of course, we acquire a natural fully faithful $\mathcal{T}$-functor

$$\underline{\mathrm{Seg}}(\mathfrak{B}\times\mathfrak{B}')(\mathcal{C}) \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathfrak{B}\times\mathfrak{B},\mathcal{C}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathfrak{B},\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathfrak{B}',\mathcal{C}));$$

to see that the essential image agrees with $\underline{\mathrm{Seg}}_{\mathfrak{B}}\underline{\mathrm{Seg}}_{\mathfrak{B}'}(\mathcal{C})$, apply the pointwise presentation for $\mathcal{T}$-limits of Proposition I.1.57 exactly as in [CH21, Ex. 5.7]. □

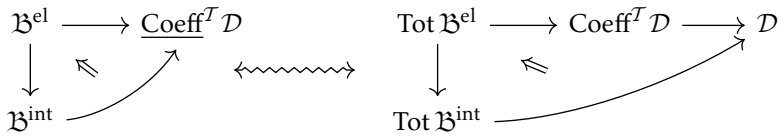
Construction I.3.3. Given $\mathfrak{B}$ a $\mathcal{T}$-algebraic pattern, we define its *totalization pattern* $\mathrm{Tot}\,\mathfrak{B}$ to have underlying category $\mathrm{Tot}\,\mathfrak{B}$, inert arrows the arrows $f\colon (U,X)\to(V,Y)$ who correspond with inert V-morphisms under cocartesian transport, active arrows the arrows $(U,X)\to(V,Y)$ whose corresponding orbit-map is an equivalence $V\simeq U$ and whose V-map $X\to Y$ is active, and elementary objects those lying over elementary V-objects for some V. ◁

Note that $(\mathrm{Tot}\,\mathfrak{B})^{\mathrm{int}} \simeq \mathrm{Tot}\,\mathfrak{B}^{\mathrm{int}}$, and the same for the elementary subcategory. The same is not true of active arrows. This allows one to prove the following.

Proposition I.3.4. *Postcomposition with fixed points furnishes an equivalence*

$$\mathrm{Seg}_{\mathfrak{B}}\left(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}\right) \xrightarrow{\sim} \mathrm{Seg}_{\mathrm{Tot}\,\mathfrak{B}}(\mathcal{D})$$

PROOF. In sight of [Sha23, Prop. 3.10], we're tasked with proving that the left diagram is a $\mathcal{T}$-right Kan extension if and only if the right is a right Kan extension:



Indeed, this follows by the same proof as [Sha23, Prop. 10.10], using [Sha23, Prop. 5.6] in place of [Sha23, Prop. 5.5]. □

This is functorial, in the following sense.

Lemma I.3.5. *Given a map of orbits $U\to V$, the functor $\mathrm{Tot}^{U}\mathfrak{B}\to\mathrm{Tot}^{V}\mathfrak{B}$ is a Segal morphism.*

PROOF. Apply restriction-stability of $\mathcal{T}$-right Kan extensions [Sha23, Rmk. 10.2]. □

In particular, restricted totalization yields a functor making the following diagram commute

$$\begin{array}{ccc} \mathcal{T}^{\mathrm{op}} & \xrightarrow{\underline{\mathrm{Seg}}_{\mathfrak{B}}(\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C})} & \mathrm{Cat} \\ & {\scriptstyle \mathrm{Tot}^{\bullet}\mathfrak{B}} \searrow \quad \nearrow {\scriptstyle \mathrm{Seg}_{\bullet}(\mathcal{C})} & \\ & \left(\mathrm{AlgPatt}^{\mathrm{Seg}}\right)^{\mathrm{op}} & \end{array}$$

We would like to apply other similar constructions, for which we need a naming meta-convention.

Convention I.3.6. Suppose P denotes a property of algebraic patterns. A $\mathcal{T}$-algebraic pattern $\mathfrak{B}$ *has property* P if each Tot^{V} has property P, and it *fiberwise has property* P if each value $\mathfrak{B}^{V} \in \mathrm{AlgPatt}$ has property P. Properties and fiberwise-properties of maps are defined similarly. ◁

In particular, we define *sound* and *soundly extendable* $\mathcal{T}$-algebraic patterns to have each $\mathrm{Tot}^{V}(-)$ sound and soundly extendable.

Construction I.3.7. Given $\mathfrak{B}$ a soundly extendable $\mathcal{T}$-algebraic pattern, the $\mathcal{T}$-category of fibrous $\mathfrak{B}$-patterns is the composite functor

$$\underline{\mathrm{Fbrs}}(\mathfrak{B})\colon \mathcal{T}^{\mathrm{op}} \xrightarrow{\mathrm{Tot}^{\bullet}\mathfrak{B}} \mathrm{Fbrs}(\bullet).$$ ◁

I.3.1.2 Envelopes These admit an equifibered perspective, which follows formally from that of [BHS22].

Corollary I.3.8. *Given $\mathfrak{B}$ a soundly extendable $\mathcal{T}$-algebraic pattern, there exists a $\mathcal{T}$-adjunction*

$$\mathrm{Env}\colon \underline{\mathrm{Fbrs}}(\mathfrak{B}) \rightleftarrows \underline{\mathrm{Seg}}_{\mathfrak{B}}(\underline{\mathrm{Cat}}_{\mathcal{T}})\colon \iota$$

whose associated sliced $\mathcal{T}$-adjunction over the terminal object $\mathrm{Comm}_{\mathrm{Tot}^{V}\mathfrak{B}} \in \mathrm{Fbrs}^{V}(\mathfrak{B})$ exhibits the lefthand side as a colocalizing $\mathcal{T}$-subcategory

$$\underline{\mathrm{Fbrs}}(\mathfrak{B}) \rightleftarrows \underline{\mathrm{Seg}}_{\mathfrak{B}}(\underline{\mathrm{Cat}}_{\mathcal{T}})_{/\mathrm{Env}^{\otimes}_{\mathfrak{B}}\mathrm{Comm}_{\mathrm{Tot}^{B}\mathfrak{B}}}$$

identified with the $\mathfrak{B}$-Segal V-functors $\mathcal{C}^{\otimes} \to \mathrm{Env}^{\otimes}_{\mathfrak{B}}\mathrm{Comm}_{\mathrm{Tot}^{B}\mathfrak{B}}$ whose totalizations are $\mathrm{Tot}^{V}\mathfrak{B}$-equifibrations.

Now, to prove this, we need a Beck-Chevalley condition.

Lemma I.3.9. *The morphism $\psi\colon \mathrm{Tot}^{U}\mathfrak{B} \to \mathrm{Tot}^{V}\mathfrak{B}$ induces an equivalence*

$$q\colon \mathscr{A}_{\mathrm{Tot}^{U}\mathfrak{B}} \xrightarrow{\sim} \psi^{*}\mathscr{A}_{\mathrm{Tot}^{V}\mathfrak{B}}$$

PROOF. q is the coassembly functor for the top square of the following:

$$\begin{array}{ccc} \mathrm{Ar}^{\mathrm{Act}}\left(\mathrm{Tot}^U \mathfrak{B}\right) & \longrightarrow & \mathrm{Ar}^{\mathrm{Act}}\left(\mathrm{Tot}^V \mathfrak{B}\right) \\ \downarrow & & \downarrow \\ \mathrm{Tot}^U \mathfrak{B} & \longrightarrow & \mathrm{Tot}^V \mathfrak{B} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{T}_{/U} & \longrightarrow & \mathcal{T}_{/V} \end{array}$$

In particular, by pullback pasting it suffices to prove that the outer rectangle is cartesian. Now, the coassembly arrow $\mathrm{Ar}^{\mathrm{act}}\left(\mathrm{Tot}^U \mathfrak{B}\right) \to \mathrm{Res}^V_U \mathrm{Ar}^{\mathrm{act}}\left(\mathrm{Tot}^V \mathfrak{B}\right)$ is a functor of cocartesian fibrations over $\mathcal{T}_{/U}$, i.e. a U-functor, so we may check that it is an equivalence fiberwise. The result follows by noting that the fibers of each side over W admit compatible equivalences with the full subcategory of $\mathrm{Ar}\left(\mathfrak{B}^V\right)$ spanned by active arrows. □

PROOF OF COROLLARY I.3.8. Of course, we have an unstraightening $\mathcal{T}$-functor

$$\underline{\mathrm{Seg}}_{\mathfrak{B}}\left(\underline{\mathrm{Cat}}_{\mathcal{T}}\right) \longrightarrow \underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{Tot}^\bullet \mathfrak{B}};$$

this fiberwise-factors through the full subcategory $\mathbf{Fbrs}(\mathrm{Tot}^V \mathfrak{B}) \hookrightarrow \mathrm{Cat}^{\mathrm{int-cocart}}_{V,/\mathrm{Tot}^V \mathfrak{B}}$ as Segal objects unstraighten to fibrous patterns, so it factors through the associated full $\mathcal{T}$-subcategory $\underline{\mathrm{Fbrs}}(\mathrm{Tot}^V \mathfrak{B}) \hookrightarrow \underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{Tot}^\bullet \mathfrak{B}}$. To construct a left $\mathcal{T}$-adjoint to ι, in light of the dual to Proposition I.1.28 it suffices to prove that ι pointwise admits left adjoints and they Beck-Chevalley natural transformation

$$\mathrm{Env} \circ \mathrm{Res}^V_U \implies \mathrm{Res}^V_U \circ \mathrm{Env}$$

is an equivalence. The pointwise left adjoints are given by Theorem I.2.32, and Remark I.2.33 shows that Lemma I.3.9 implies this Beck-Chevalley condition.

This leads to a sliced $\mathcal{T}$-adjunction under Proposition I.1.33. To verify that the right adjoint is fully faithful, it suffices to prove that it is fully faithful fiberwise, which follows from Theorem I.2.32. □

Now, note that this implies presentability of $\underline{\mathrm{Fbrs}}$ by the same argument as [BHS22, Cor. 4.2.2].

Corollary I.3.10. *$\underline{\mathrm{Fbrs}}(\mathfrak{B})$ is a presentable $\mathcal{T}$-∞-categrory, and so is $\underline{\mathrm{Seg}}_{\mathfrak{B}}(\mathcal{C})$ whenever $\mathcal{C}$ is presentable. In particular, they are $\mathcal{T}$-complete and $\mathcal{T}$-cocomplete.*

PROOF. First, note for all $\mathcal{T}$-functors $X\colon \mathfrak{B} \to \mathcal{C}$ that the fully faithful inclusion $\mathrm{Seg}_{\mathfrak{B}}(\mathcal{C}) \subset \mathrm{Fun}_{\mathcal{T}}(\mathcal{C}, \mathfrak{B})$ admits an accessible left adjoint by the same argument as [BHS22, Lem. 3.1.11]; in particular, $\iota\colon \underline{\mathrm{Seg}}_{\mathfrak{B}}(\mathcal{C}) \subset \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \mathfrak{B})$ fiberwise admits an acessible left adjoint. The codomain is presentable by Example I.1.51, so to prove that $\underline{\mathrm{Seg}}_{\mathfrak{B}}(\mathcal{C})$, it suffices to prove that the fiberwise left adjoints satisfy the Beck-Chevalley condition; this follows by restriction-stability of the collection of morphisms appearing in [BHS22, Lem. 3.1.11]. In

particular, by Example I.1.51, we find that $\underline{\mathrm{Seg}}_{\mathfrak{B}}\big(\underline{\mathrm{Cat}}_{\mathcal{T}}\big)_{/\mathscr{A}_{\mathfrak{B}}}$ is presentable. A similar argument, this time lifting [BHS22, Prop. 2.3.7], shows that the inclusion $\underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathfrak{B}} \hookrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}\big(\mathfrak{B}, \underline{\mathrm{Cat}}_{\mathcal{T}}\big)_{/\mathscr{A}_{\mathfrak{B}}}$ is an accessible localizing $\mathcal{T}$-subcategory. Now, applying [BHS22, Cor. 4.2.2] fiberwise yields a cartesian square of fully faithful accessible $\mathcal{T}$-functors

$$\begin{array}{ccc} \underline{\mathrm{Fbrs}}(\mathfrak{B}) & \longrightarrow & \underline{\mathrm{Seg}}_{\mathfrak{B}}\big(\underline{\mathrm{Cat}}_{\mathcal{T}}\big)_{/\mathscr{A}_{\mathfrak{B}}} \\ \downarrow & \lrcorner & \downarrow \\ \underline{\mathrm{Cat}}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathfrak{B}} & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}\big(\mathfrak{B}, \underline{\mathrm{Cat}}_{\mathcal{T}}\big)_{/\mathscr{A}_{\mathfrak{B}}} \end{array}$$

In fact, this square lives in $\mathrm{Pr}^{R}_{\mathcal{T}}$ by [MW22, Prop. 2.4.4.11], yielding the claim. □

Variant I.3.11. Following the results of [BHS22, § 5.3], we may indeed upgrade the formation of Segal objects and fibrous patterns to a pair of functors valued in Cat-modules, and hence $(\infty,2)$-categories

$$\mathrm{SEG}(\mathrm{CAT}), \mathrm{FBRS}(-) \colon \mathrm{AlgPatt}^{\mathrm{SE,Seg,op}} \to \widehat{\mathrm{Cat}}_{(\infty,2)};$$

In particular, our constructs lift to a pair of functors

$$\underline{\mathrm{SEG}}_{(-)}(\mathrm{CAT}_{\mathcal{T}}), \underline{\mathrm{FBRS}}(-) \colon \mathrm{AlgPatt}^{\mathrm{SE,Seg,op}}_{\mathcal{T}} \longrightarrow \mathrm{Cat}_{(\infty,2),\mathcal{T}};$$

these are connected by a $\mathcal{T}$-$(\infty,2)$-natural transformation, i.e. a functor

$$\underline{HA}(-) \colon \mathrm{AlgPatt}^{\mathrm{SE,Seg,op}}_{\mathcal{T}} \longrightarrow \mathrm{ArCat}_{(\infty,2),\mathcal{T}}$$

taking $\mathfrak{B}$ to the $\mathcal{T}$-$(\infty,2)$-functor $\underline{\mathrm{SEG}}_{\mathfrak{B}}(\mathrm{CAT}) \to \underline{\mathrm{FBRS}}(\mathfrak{B})$. We will use the special notation

$$\underline{\mathrm{Alg}}_{(-)/\mathfrak{B}}(-) \coloneqq \underline{\mathrm{Fun}}_{\underline{\mathrm{Fbrs}}(\mathfrak{B})}(-,-); \qquad \underline{\mathrm{Fun}}^{\otimes}_{\mathfrak{B}}(-,-) \coloneqq \underline{\mathrm{Fun}}_{\underline{\mathrm{SEG}}_{\mathfrak{B}}(\mathrm{CAT})}(-,-).$$

We will omit the $\mathfrak{B}$ from the lefthand notation when it is clear. ◁

For our last variation, we may lift Theorem I.2.32 to a parameterized $(\infty,2)$-adjunction.

Corollary I.3.12. *Unstraightening underlies the right adjoint in a $\mathcal{T}$-$(\infty,2)$-adjunction*

$$\underline{\mathrm{FBRS}}(\mathfrak{B}) \rightleftarrows \underline{\mathrm{SEG}}_{\mathfrak{B}}(\mathrm{CAT}),$$

in particular, it yields a natural equivalence of $\mathcal{T}$-$(\infty,2)$-functors $\underline{\mathrm{FBRS}}(\mathfrak{B})^{\mathrm{op}} \times \underline{\mathrm{SEG}}_{\mathfrak{B}}(\mathrm{CAT}) \to \underline{\mathrm{CAT}}_{\mathcal{T}}$:

$$\underline{\mathrm{Alg}}_{\mathcal{O}/\mathfrak{B}}(\mathcal{C}) \simeq \underline{\mathrm{Fun}}^{\otimes}_{\mathfrak{B}}(\mathrm{Env}_{\mathfrak{B}}\mathcal{O}, \mathcal{C}).$$

PROOF. Again, we may construct the $\mathcal{T}$-$(\infty,2)$-adjunction fiberwise and verify the Beck-Chevalley condition by virtue of Proposition I.1.64. Fiberwise, this occurs by taking the $(\infty,2)$-categorical adjunction arising from the lax Cat-tensored adjunction of [BHS22, Cor. 5.3.13] under [Hei23, Thm. 1.8]. The Beck-Chevalley conditions follow from the same argument as Corollary I.3.8. □

I.3.1.3 Prefibrous structures Now, we note a souped up version of the identification of fibrous patterns over prefibrous structures.

Construction I.3.13. As in Construction I.2.25, given $\mathcal{O} \to \mathrm{Tot}^{\mathcal{T}} \mathfrak{B}$ a prefibrous pattern, define the *prefibrous $\mathcal{T}$-pattern structure on* $\mathcal{O}$ to have inert V-arrows the cocartesian lifts of inert $\mathfrak{B}^V$-arrows and active V-arrows and elementary V-objects the arbitrary lifts of active V-arrows and elementary V-objects. ◁

Note that there exists a canonical locally fully faithful $\mathcal{T}$-$(\infty,2)$-functor $\underline{\mathrm{FBRS}}(\mathfrak{B}) \hookrightarrow \underline{\mathrm{CAT}}_{\mathcal{T},/\mathfrak{B}}$. Using this, we can phrase a *unique* identification of fibrous patterns over the prefibrous structure in the fibrous case.

Proposition I.3.14. *If $\mathcal{O} \to \mathfrak{B}$ is a fibrous $\mathfrak{B}$-pattern there exists an essentially unique equivalence*

$$\underline{\mathrm{FBRS}}(\mathfrak{B})_{/\mathcal{O}} \simeq \underline{\mathrm{FBRS}}(\mathcal{O})$$

lying over the equivalence $(\mathrm{CAT}_{\mathcal{T},/\mathfrak{B}})_{/\mathcal{O}} \simeq \mathrm{CAT}_{\mathcal{T},/\mathcal{O}}$.

Of course, we will prove this by reduction to the nonequivariant case. We use the following lemma.

Lemma I.3.15. $\mathrm{Tot}^{\mathcal{T}} \mathcal{O} \to \mathrm{Tot}^{\mathcal{T}} \mathfrak{B}$ *is a prefibrous $\mathfrak{B}$-pattern, and* $\mathrm{Tot}^{\mathcal{T}} \mathcal{O}$ *has the prefibrous pattern structure.*

PROOF. Note that the inert morphisms in $\mathrm{Tot}^{\mathcal{T}} \mathfrak{B}$ are precisely the compositions of a $\mathcal{T}^{\mathrm{op}}$-cocartesian arrow and an inert arrow in some fiber $\mathfrak{B}^V \subset \mathrm{Tot}^{\mathcal{T}} \mathfrak{B}$. In particular, to prove that $\mathrm{Tot}^{\mathcal{T}} \mathcal{O}$ is a prefibrous $\mathfrak{B}$-pattern, it suffices to prove that each of these have cocartesian lifts; cocartesian lifts over $\mathcal{T}^{\mathrm{op}}$ are due to the fact that our functor is unstraightened from a $\mathcal{T}$-functor, and fiberwise inert arrows by definition, so the result follows by composition-stability of cocartesian arrows.

Moreover, this argument shows that the cocartesian lifts of inert morphisms in $\mathrm{Tot}^{\mathcal{T}} \mathfrak{B}$ are *precisely* the compositions of cocartesian lifts from fibers of $\mathfrak{B}^V$ and $\mathcal{T}^{\mathrm{op}}$-cocartesian arrows; these are precisely the inert arrows for the prefibrous pattern structure. The active arrows and elementary objects are defined fiberwise, so the claim follows immediately for them as well. □

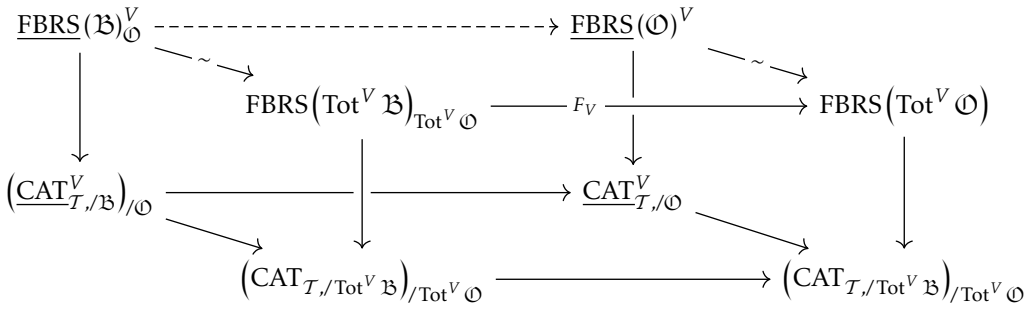
Lemma I.3.16. $\mathrm{Tot}^{U} \mathfrak{B} \to \mathrm{Tot}^{V} \mathfrak{B}$ *is a prefibrous $\mathfrak{B}$-pattern and* $\mathrm{Tot}^{U} \mathfrak{B}$ *has the prefibrous pattern structure.*

PROOF. We have a pullback square

$$\begin{array}{ccc} \mathrm{Tot}^U \mathfrak{B} & \longrightarrow & \mathrm{Tot}^V \mathfrak{B} \\ \downarrow & \lrcorner & \downarrow \\ (\mathcal{T}_{/U})^{\mathrm{op}} & \longrightarrow & (\mathcal{T}_{/V})^{\mathrm{op}} \end{array}$$

and the bottom arrow is a cocartesian fibration, so the top is as well. The result then follows by unwinding definitions using the description of cocartesian arrows in pullbacks. □

PROOF OF PROPOSITION I.3.14. It suffices to construct an equivalence F_V, natural in $V \in \mathcal{T}$, making the solid subdiagram of the following commute:



But by Lemma I.3.15, it suffices to lift the nonparameterized equivalence of Theorem I.2.32 to the $(\infty,2)$-categorical setting and show it to be natural under the morphisms $\mathrm{Tot}^U \mathfrak{B} \to \mathrm{Tot}^V \mathfrak{B}$. Now, by Lemma I.3.16, these morphisms exhibit $\mathrm{Tot}^U \mathfrak{B}$ as having the $\mathfrak{B}$-prefibrous structure of $\mathrm{Tot}^V \mathfrak{B}$, so it suffices to lift Proposition I.2.28 to the $(\infty,2)$-categorical setting with the *same* functoriality; but indeed, this follows by noting that the implicit argument of [BHS22, Cor. 4.1.7] indeed induces equivalences on mapping ∞-categories. □

I.3.1.4 Push-pull adjunctions, upgraded

Proposition I.3.17. *Suppose $\mathfrak{B} \xleftarrow{\varphi} \mathfrak{O} \xrightarrow{\psi} \mathfrak{C}$ is a span of $\mathcal{T}$-functors with distinguished $\mathcal{T}$-pattern structures on $\mathfrak{B}$ and $\mathfrak{C}$ such that the* Tot*-span of patterns is pairable. Then, there exists a restricted $\mathcal{T}$-$(\infty,2)$-adjunction*

$$\begin{array}{ccc} \underline{\mathrm{FBRS}}(\mathfrak{B}) & \underset{\psi^*\varphi_*}{\overset{\varphi^*\psi_!}{\rightleftarrows}} & \underline{\mathrm{FBRS}}(\mathfrak{C}) \\ \downarrow & & \downarrow \\ \underline{\mathrm{CAT}}_{\mathcal{T},/\mathrm{Tot}\,\mathfrak{B}} & \underset{\psi^*\varphi_*}{\overset{\varphi^*\psi_!}{\rightleftarrows}} & \underline{\mathrm{CAT}}_{\mathcal{T},/\mathrm{Tot}\,\mathfrak{C}} \end{array}$$

In particular, it induces a natural equivalence of $\mathcal{T}$-∞-categories

$$\underline{\mathrm{Alg}}_{\varphi^*\psi_!\mathfrak{O}/\mathfrak{C}}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}_{\mathfrak{O}/\mathfrak{B}}(\psi^*\varphi_*\mathfrak{C}).$$

PROOF. By assumption, $\mathrm{Tot}\,\varphi$ is exponentiable, so the bottom adjunction indeed exists. Moreover to show that the top left $\mathcal{T}$-$(\infty,2)$-functor exists, since the righthand vertical arrow is locally fully faithful, we

only need verify that a factorization exists on objects and equivalence classes in functor categories; this may be verified fiberwise, where it follows from pairability. The same argument constructs $\psi^*\varphi_*$ as a $\mathcal{T}$-$(\infty,2)$-functor, and the adjunction data comes for free by local fully-faithfulness of both vertical arrows. □

I.3.1.5 Parameterized overcategory patterns Fix $\mathfrak{C}$ a fibrous $\mathfrak{B}$-pattern. Given a functor $p\colon \mathcal{K} \to \mathrm{Alg}_{/\mathfrak{B}}(\mathfrak{C})$ we define a functor $\mathfrak{C}_{/p_{\mathfrak{B}}} \to \mathrm{Tot}\,\mathfrak{B}$ to have the universal property

$$\mathrm{Fun}_{/\mathrm{Tot}\,\mathfrak{B}}\left(Y, \mathfrak{C}_{/p_{\mathfrak{B}}}\right) \simeq \left\{ \text{lifts} \quad \begin{array}{ccccc} Y\times\mathcal{K} & \hookrightarrow & Y\times\mathcal{K}^{\triangleright} & \xrightarrow{\mathrm{pr}_1} & Y \\ {\scriptstyle \pi_Y\times\mathrm{id}}\downarrow & & \downarrow & & \downarrow \\ \mathrm{Tot}\,\mathfrak{B}\times\mathcal{K} & \xrightarrow[\widetilde{p}]{} & \mathrm{Tot}\,\mathfrak{C} & \longrightarrow & \mathrm{Tot}\,\mathfrak{B} \end{array} \right\}$$

where $\widetilde{p}\colon \mathcal{K} \to \mathrm{Tot}\,\mathfrak{B} \to \mathrm{Tot}\,\mathfrak{C}$ is mate to the functor $\mathcal{K} \to \mathrm{Fun}_{/\mathrm{Tot}\,\mathfrak{B}}(\mathrm{Tot}\,\mathfrak{B}, \mathrm{Tot}\,\mathfrak{C})$ underlying p; this exists by [HA, Lem. 2.2.2.6]. The proof of the following theorem will involve essentially no new ideas over [HA, Thm. 2.2.2.4].

Proposition I.3.18. *q'' exhibits $\mathfrak{C}_{/p_{\mathfrak{O}}}$ as a fibrous $\mathfrak{B}$ pattern, which is Segal if $\mathfrak{C}$ is.*

Proof. We apply [HA, Lem 2.2.2.7-9] on opposite categories. In particular, given an arrow $g\colon X \to q''(Y)$ in $\mathrm{Tot}\,\mathfrak{B}$ over which $\mathrm{Tot}\,\mathfrak{C}$ has a cocartesian lift, [HA, Lem. 2.2.2.8] supplies a relative colimit

$$\begin{array}{ccc} * & \xrightarrow{\{Y\}} & \mathrm{Tot}\,\mathfrak{C}_{/p_{\mathfrak{O}}} \\ \downarrow & \overset{\overline{g}}{\nearrow} & \downarrow{\scriptstyle q''} \\ *^{\triangleright} & \xrightarrow[g]{} & \mathrm{Tot}\,\mathfrak{B} \end{array}$$

i.e. by [HA, Lem. 2.2.2.7] a q''-cocartesian lift of g. Since $\mathrm{Tot}\,\mathfrak{C}$ has inert-cocartesian lifts, so does $\mathrm{Tot}\,\mathfrak{C}_{/p_{\mathfrak{O}}}$, and when q is a cocartesian fibration, so is q''.

We're left with verifying the Segal conditions. As in [HA, Thm. 2.2.2.4], it follow from a simple application of [HA, Lem. 2.2.2.9] that each of the relevant diagrams are relative limit diagrams, as they project to relative limit diagrams in $\mathfrak{C}$. □

Remark I.3.19. As in [HA, Thm 2.2.2.4.(2)], it follows from the above diagram that, given an arrow f in $\mathfrak{C}_{/p_{\mathfrak{B}}}$ such that $\mathfrak{C}$ admits a $q''(f)$-cocartesian arrow, f is q''-cocartesian if and only if its image in $\mathfrak{C}$ is q-cocartesian. In particular the inert arrows in $\mathfrak{C}_{/p_{\mathfrak{B}}}$ are the preimages of the inert morphisms of $\mathfrak{C}$, and if $\mathfrak{C}$ is Segal, then the cocartesian arrows in $\mathfrak{C}_{/p_{\mathfrak{B}}}$ are the preimages of cocartesian arrows in $\mathfrak{C}$. ◁

It is worthwhile to explicitly record following immediate corollary of Remark I.3.19, in part because it establishes the "pointwise" nature of the coherences for slice $\mathfrak{O}$-monoidal structure.

Corollary I.3.20. *If $\mathfrak{C}$ is a Segal $\mathfrak{B}$-∞-category, then the unslicing functor $\mathfrak{C}_{/p_{\mathfrak{O}}} \to \mathfrak{C}$ is a Segal $\mathfrak{B}$-functor.*

Note that this recovers ordinary slice $\mathcal{T}$-categories by [HA, Rmk. 2.2.2.2].

I.3.2 Approximations and Morita theory for parameterized patterns We may now say our aim with Morita theory: characterize $\underline{\mathrm{Seg}}_{\bullet}(\mathcal{C})$-equivalences of $\mathcal{T}$-patterns (called Morita equivalences), its strengthening to $\underline{HA}$-equivalences (called fibrous Morita equivalences), its further strengthening to fibrous-pullback stable $\underline{HA}$-equivalences (called strong Morita equivalences), and its weakening to the internal notion of localization (called Morita $\mathcal{T}$-localization).

I.3.2.1 Morita equivalences, fibrous Morita equivalences, strong Morita equivalences

Definition I.3.21. A Segal morphism of $\mathcal{T}$-patterns $\varphi\colon \mathfrak{O} \to \mathfrak{P}$ is a *Morita equivalence* if the pullback functor

$$\underline{\mathrm{Seg}}_{\mathfrak{P}}\left(\underline{\mathcal{S}}_{\mathcal{T}}\right) \to \underline{\mathrm{Seg}}_{\mathfrak{O}}\left(\underline{\mathcal{S}}_{\mathcal{T}}\right)$$

is an equivalence. If $\mathfrak{O}, \mathfrak{P}$ are soundly extendable, we say φ is a *fibrous Morita equivalence* if the pullback

$$\underline{\mathrm{FBRS}}(\mathfrak{P}) \to \underline{\mathrm{FBRS}}(\mathfrak{O})$$

is also an equivalence of $\mathcal{T}$-$(\infty,2)$-categories. Finally, we say that φ is a *strong Morita equivalence* if, for all fibrous $\mathfrak{P}$-patterns $\mathfrak{Q}$, the canonical map $\varphi^*\mathfrak{Q} \to \mathfrak{Q}$ is a fibrous Morita equivalence. ◁

Remark I.3.22. If φ is a fibrous Morita equivalence, $\varphi'\colon \varphi^*\mathfrak{Q} \to \mathfrak{Q}$ may not be a strong Morita equivalence for $\mathfrak{Q}$ a fibrous $\mathfrak{P}$-pattern. Nevertheless, the equivalence of Proposition I.3.14 shows that it yields a pullback on $\underline{\mathrm{FBRS}}(-)$, so for φ to be a strong Morita equivalence, it suffices to prove that it is a fibrous Morita equivalence such that each φ' is a Morita equivalence. ◁

Remark I.3.23. It follows from Proposition I.3.4 that φ is a (resp. fibrous, strong) Morita equivalence if and only if each totalized pattern map $\mathrm{Tot}^V \varphi\colon \mathrm{Tot}^V \mathfrak{O} \to \mathrm{Tot}^V \mathfrak{P}$ is a (strong, fibrous) Morita equivalence. ◁

A key property of Morita equivalences is that they lift to equivalences on Segal objects in $\mathcal{T}$-∞-categories which are *not* described by Proposition I.3.4:

Proposition I.3.24. *If φ is a Morita equivalence, then, for all complete $\mathcal{T}$-∞-categories $\mathcal{C}$, pullback*

$$\underline{\mathrm{Seg}}_{\mathfrak{P}}(\mathcal{C}) \longrightarrow \underline{\mathrm{Seg}}_{\mathfrak{O}}(\mathcal{C})$$

is an equivalence; moreover, the pullback $\mathcal{T}$-$(\infty,2)$-functor

$$\varphi^*_{(\infty,2)}\colon \underline{\mathrm{SEG}}_{\mathfrak{P}}(\mathrm{CAT}_{\mathcal{T}}) \longrightarrow \underline{\mathrm{SEG}}_{\mathfrak{O}}(\mathrm{CAT}_{\mathcal{T}})$$

is an equivalence.

PROOF. For the first statement, we use Barkan's proof [Bar23a, Prop. 2.19]; this time, the equivalence

$$\underline{\mathrm{Seg}}_{\bullet}\left(\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathcal{A},\underline{\mathcal{S}}_{\mathcal{T}}\right)\right)\simeq\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathcal{A},\underline{\mathrm{Seg}}_{\bullet}\left(\mathcal{A},\underline{\mathcal{S}}_{\mathcal{T}}\right)\right)$$

shows the claim in the case that $\mathcal{C}$ is a presheaf category, and since the Yoneda embedding

$$\text{ɟ}:\mathcal{A}\hookrightarrow\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathcal{A}^{\mathrm{op}},\underline{\mathcal{S}}_{\mathcal{T}}\right)$$

is fully faithful and creates $\mathcal{T}$-limits (by Proposition I.1.41), the equivalence for $\underline{\mathrm{Seg}}_{\bullet}\left(\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathcal{A}^{\mathrm{op}},\underline{\mathcal{S}}_{\mathcal{T}}\right)\right)$ preserves and reflects the ɟ-essential image of Segal objects; the claim follows.

Now, for the $\mathcal{T}$-$(\infty,2)$-categorical statement, morita equivalence may be checked totwise via Remark I.3.23; moreover, there exists a factorization of inclusions

$$\begin{array}{ccc}\underline{\mathrm{SEG}}_{\mathfrak{P}}(\mathrm{CAT}_{\mathcal{T}})^{V} & \xrightarrow{\varphi^{*,V}_{(\infty,2)}} & \underline{\mathrm{SEG}}_{\mathfrak{O}}(\mathrm{CAT}_{\mathcal{T}})^{V}\\ \downarrow & & \downarrow\\ \mathrm{CAT}^{\mathrm{int-cocart}}_{/\mathrm{Tot}_{V}\mathfrak{P}} & \xrightarrow{\mathrm{Tot}_{V}\varphi^{*}} & \mathrm{CAT}^{\mathrm{int-cocart}}_{/\mathrm{Tot}_{V}\mathfrak{O}}\end{array}$$

Indeed, $\mathrm{Tot}_V\,\varphi^*$ is an equivalence, since $\mathrm{Tot}_V\,\varphi^*$ is an equivalence on $(-)^{\leq 1}$ by the first claim and each $(\infty,2)$-category comes from Cat-cotensoring, giving the result. □

In some higher-algebraic settings, we will find these notions of Morita equivalence quite well-behaved. To say how, say that $\mathfrak{B}$ *admits a cartesian structure on* $\mathcal{C}$ (or more accurately, its fibrous Morita equivalence class does) if the composite $\mathcal{T}$-functor

$$\underline{\mathrm{Fbrs}}(\mathfrak{B})^{\mathrm{op}}\longrightarrow\underline{\mathrm{AlgPatt}}^{\mathrm{Seg,SE,op}}_{\mathcal{T}}\xrightarrow{\underline{\mathrm{Seg}}_{(-)}(\mathcal{C})}\underline{\mathrm{Cat}}_{\mathcal{T}}.$$

is representable by an object $\mathcal{C}^{\mathfrak{B}-\times}_{\mathcal{T}}\in\underline{\mathrm{Fbrs}}(\mathfrak{B})$. Given $\varphi\colon\mathfrak{O}\to\mathfrak{P}$ a map of fibrous $\mathfrak{B}$-patterns, we find

$$\underline{\mathrm{Alg}}_{\mathfrak{P}/\mathfrak{O}}\left(\pi^{*}_{\mathfrak{O}}\underline{\mathcal{S}}^{\mathfrak{B}-\times}_{\mathcal{T}}\right)\simeq\underline{\mathrm{Alg}}_{\mathfrak{P}/\mathfrak{B}}\left(\underline{\mathcal{S}}^{\mathfrak{B}-\times}_{\mathcal{T}}\right)\simeq\underline{\mathrm{Seg}}_{\mathfrak{P}}\left(\underline{\mathcal{S}}_{\mathcal{T}}\right).$$

In particular, we find the following.

Proposition I.3.25. *If $\mathfrak{B}$ admits a cartesian structure on $\underline{\mathcal{S}}_{\mathcal{T}}$ and $\mathfrak{O}$ is a fibrous $\mathfrak{B}$-pattern, then $\mathfrak{O}$ admits a cartesian structure on $\underline{\mathcal{S}}_{\mathcal{T}}$ given by $\underline{\mathcal{S}}^{\mathfrak{O}-\times}_{\mathcal{T}}\simeq\pi^{*}_{\mathfrak{O}}\underline{\mathcal{S}}^{\mathfrak{B}-\times}_{\mathcal{T}}$.*

We may conclude the following.

Corollary I.3.26. *Suppose $\varphi\colon\mathfrak{O}\to\mathfrak{P}$ is a fibrous Morita equivalence and $\mathfrak{P}$ admits a cartesian structure on $\underline{\mathcal{S}}_{\mathcal{T}}$. Then, φ is a strong Morita equivalence.*

PROOF. Since the property of admitting a cartesian structure on $\mathcal{T}$-spaces is a fibrous Morita invariant, $\mathcal{O}$ admits this as well; given $\mathcal{Q} \in \mathsf{Fbrs}(\mathcal{O})$, Proposition I.3.25 then shows that $\mathcal{Q}$ and $\varphi^*\mathcal{Q}$ do as well. Now, the equivalence $\underline{\mathrm{FBRS}}(\mathcal{Q}) \simeq \underline{\mathrm{FBRS}}(\mathcal{O})_{/\mathcal{Q}}$ of Proposition I.3.14 shows that $\varphi' \colon \varphi^*\mathcal{Q} \to \mathcal{Q}$ is a $\underline{\mathrm{FBRS}}(-)$-equivalence. But since $\underline{\mathrm{Seg}}_-(\underline{\mathcal{S}}_{\mathcal{T}})$ is representable in each category, this implies that φ' is a Segal equivalence, completing the claim. □

Remark I.3.27. Corollary I.3.26 is a form of rigidity result on a fibrous Morita class of pattern, which indeed identifies its fibrous and strong Morita classes; we will find in Theorem II.2.1 that this applies to the parameterized fibrous Morita class appearing in equivariant homotopy theory. Indeed, these satisfy the additional assumption that the composite functor $\underline{\mathrm{Seg}}_{(-)}(\underline{\mathcal{S}}_{\mathcal{T}}) \colon \underline{\mathsf{Fbrs}}(\mathfrak{B})^{\mathrm{op}} \longrightarrow \underline{\mathrm{Cat}}_{\mathcal{T}}$ is *conservative*; under this assumption, Morita equivalences of fibrous $\mathfrak{B}$-patterns are automatically equivalences (and hence strong Morita equivalences). ◁

While fibrous Morita equivalences induce equivalences under $\underline{HA}(-)$, even strong Morita equivalences do not necessarily induce equivalences under $P\mathsf{Fbrs}(-)$. This is a feature, rather than a bug; a model $\mathfrak{B}$ for the Morita class $[\mathfrak{B}]$ gives rise to a homotopy theory of prefibrous $\mathfrak{B}$-patterns enveloping fibrous $\mathfrak{B}$-patterns which is sensitive to the *pattern equivalence class* of $\mathfrak{B}$, and models $\mathfrak{B}$ will give rise to such homotopy theories which are convenient in some settings and inconvenient in some others. In particular, in the setting of Remark I.3.27, one may produce an equivalence of fibrous $\mathfrak{B}$-patterns $\mathcal{O} \to \mathfrak{P}$ by instead resolving each $\mathcal{O}$ and $\mathfrak{P}$ by prefibrous $\mathfrak{B}$-patterns, and showing the morita equivalence there.

This strategy will be central to our eventual strategy for equivariant Dunn additivity. In that case, these resolutions will not end up being a Morita equivalence, but rather a Morita-theoretic version of *reflective localizations* for prefibrous patterns. As such, we study this now for prefibrous patterns over a general parameterized algebraic pattern.

I.3.2.2 Weak approximations for $\mathcal{T}$-algebraic patterns We recognize a form of exponentiability.

Definition I.3.28. We say that a strong Segal morphism of $\mathcal{T}$-algebraic patterns $\alpha \colon \mathfrak{P} \to \mathfrak{E}$ is *BC in the sense of Barkan* if, for all $X \in \underline{\mathrm{Seg}}_{\mathcal{O}}(\mathcal{C})$ with $\mathcal{C}$ complete, the Beck-Chevalley map

$$(\alpha_* X)\,|_{\mathfrak{E}^{\mathrm{int}}} \longrightarrow \alpha_*^{\mathrm{int}}\left(X\,|_{\mathfrak{P}^{\mathrm{int}}}\right)$$

is an equivalence. ◁

Proposition I.3.29. *If α is BC in the sense of Barkan, then α_* preserves Segal $\mathcal{T}$-objects; if moreover α^{el} admits a fully faithful right adjoint β, then α^* is the inclusion of the coreflective $\mathcal{T}$-subcategory of Segal $\mathfrak{P}$-objects $Y \colon \mathfrak{P} \to \mathcal{C}$ whose elementary $\mathcal{T}$-functor $Y^{\mathrm{el}} \colon \mathfrak{P}^{\mathrm{el}} \to \mathcal{C}$ factors through β.*

We refer to such Segal $\mathfrak{P}$-objects as *α-locally constant*; we say that an arbitrary $\mathcal{T}$-pattern map α is a *Morita $\mathcal{T}$-localization* if the conclusion of Proposition I.3.29 holds for α.

PROOF. For compatibility of α_* with Segal objects, the same proof as [CH21, Prop. 6.3] works: given a Segal $\mathcal{T}$-object $X\colon \mathfrak{E} \to \mathcal{C}$, we have a chain of natural equivalences

$$\begin{aligned}(\alpha_* X)\,|_{\mathfrak{E}^{\mathrm{int}}} &\xrightarrow{\sim} \alpha_*^{\mathrm{int}}\left(X\,|_{\mathfrak{P}^{\mathrm{int}}}\right)\\ &\simeq \alpha_*^{\mathrm{int}} i_{\mathfrak{P}*}\left(X\,|_{\mathfrak{P}^{\mathrm{el}}}\right)\\ &\simeq i_{\mathfrak{E}*}\alpha_*^{\mathrm{el}}\left(X\,|_{\mathfrak{P}^{\mathrm{el}}}\right)\\ &\simeq i_{\mathfrak{E}*}\left((\alpha_* X)\,|_{\mathfrak{E}^{\mathrm{el}}}\right),\end{aligned}$$

so $\alpha_* X$ is a Segal $\mathcal{T}$-object. This yields a fiberwise left adjoint to α^*, and the Beck-Chevalley condition follows from the same for the slice category push-pull adjunction; hence we have a $\mathcal{T}$-adjunction

$$\alpha^*\colon \underline{\mathrm{Seg}}_{\mathfrak{E}}(\mathcal{C}) \rightleftarrows \underline{\mathrm{Seg}}_{\mathfrak{P}}(\mathcal{C})\colon \alpha_*.$$

The remaining statements are checked fiberwise, so we may replace $\underline{\mathrm{Seg}}$ with Seg. We must show that the unit of the adjunction on $\mathrm{Seg}_{(-)}(\mathcal{C})$ is always an equivalence, and the counit an equivalence precisely on locally constant Segal objects.

Now, note by the BC assumption that the unit of this adjunction restricts to elementaries as

$$\begin{aligned}X^{\mathrm{el}} &\longrightarrow (\alpha_*\alpha^* X)^{\mathrm{el}}\\ &\simeq \alpha_*^{\mathrm{el}}(\alpha^* X)^{\mathrm{el}}\\ &\simeq \alpha_*^{\mathrm{el}}\alpha^{\mathrm{el},*} X^{\mathrm{el}}\\ &\simeq \beta^*\alpha^{\mathrm{el},*} X^{\mathrm{el}}\\ &\simeq \left(\alpha^{\mathrm{el}}\circ\beta\right)^* X^{\mathrm{el}}\end{aligned}$$

with composite transformation given by functoriality of pullbacks along the counit of the adjunction $\alpha^{\mathrm{el}} \dashv \beta$; this is an equivalence by fully-faithfulness of β, and $(-)^{\mathrm{el}}$ is conservative, so the unit is an adjunction, implying that α^* is fully faithful.

Now, given Y a Segal $\mathfrak{P}$-object, a similar argument shows that the counit of the adjunction $\alpha^* \dashv \alpha_*$ restricts on elementaries to an arrow equivalent to the transformation $\left(\beta\circ\alpha^{\mathrm{el}}\right)^* Y^{\mathrm{el}} \to Y^{\mathrm{el}}$ given by pullback along the unit of the adjunction $\alpha^{\mathrm{el}} \dashv \beta$; this is an equivalence if and only if Y^{el} factors through β, as desired. □

We may recognize this condition via active arrow $\mathcal{T}$-categories.

Proposition I.3.30 (Unique lifting for active morphisms)**.** *Suppose $\alpha\colon \mathfrak{P} \to \mathfrak{E}$ is a strong Segal morphism of $\mathcal{T}$-algebraic patterns. The following conditions are equivalent and imply that α is BC in the sense of Barkan.*

(a) for every $P \in \mathfrak{P}$, the homotopy fibers of

$$\alpha^{\mathrm{act}}_{/P}\colon \mathfrak{P}^{\mathrm{act}}_{/P} \longrightarrow \mathfrak{E}^{\mathrm{act}}_{/\alpha P}$$

are weakly contractable, where $\mathfrak{P}^{\mathrm{act}}_{/P} = \mathrm{Ar}^{\mathrm{act}}(\mathfrak{P}) \times_{\mathfrak{P}} \{P\}$.

(b) The inclusion $\mathfrak{P}^{\mathrm{int}}_{/E} \hookrightarrow \mathfrak{P}_{/E}$ is initial, and

(c) for every $P \in \mathfrak{P}$, the map

$$\alpha^{\mathrm{act},\simeq}_{/P}\colon \left(\mathfrak{P}^{\mathrm{act}}_{/P}\right)^{\simeq} \longrightarrow \left(\mathfrak{E}^{\mathrm{act}}_{/\alpha P}\right)^{\simeq}$$

is an equivalence.

To be consistent with [CH21, Def. 6.1], we will say that α has *unique lifting for active morphisms.*

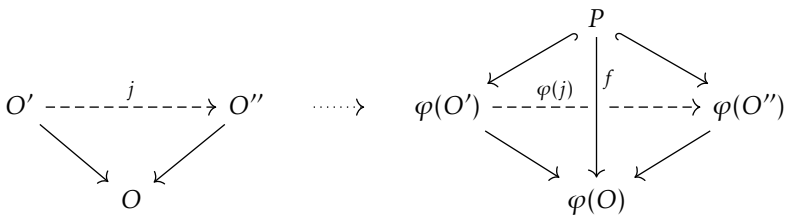
Proof. Note that the condition that α is BC in the sense of Barkan is detected fiberwise, as is condition (c); conditions (a) and (b) are as well by [Sha23, Thm. 6.7] and the dual to [kerodon, Tag 03LR]. We may henceforth pass to the nonequivariant case. In this case, the implications (b) $\iff$ (c) are [CH21, Prop. 6.3].

Now, condition (a) is equivalent to the condition that the functor $\mathfrak{O} \times_{\mathfrak{P}} \mathfrak{P}^{\mathrm{int}}_{P/} \to \mathfrak{O} \times_{\mathfrak{P}} \mathfrak{P}_{P/}$ is initial; Joyal's Theorem A [kerodon, Tag 02NY] shows that this is moreover equivalent to proving that, for all $O \in \mathfrak{O}$ and $f\colon P \to \varphi(O)$, the corresponding ∞-category

$$\mathcal{X}_f = \mathfrak{O}_{/O} \times_{\mathfrak{P}_{/\varphi(O)}} \left(\mathfrak{P}^{\mathrm{int}}_{P/}\right)_{/f}$$

is weakly contractible. This fact that $\mathcal{X}_f$ is weakly contractible under (a) is shown under unnecessary assumptions in [Har, Prop. 4.2.19], so we repeat a version of their proof here.

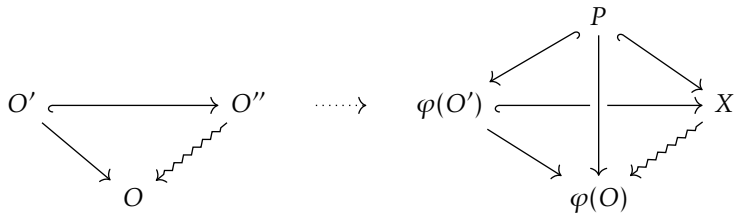
Now, an object of $\mathcal{X}_f$ may be viewed as a morphism $g_{\mathfrak{O}}\colon O' \to O$ together with a factorization $f \sim g_{\mathfrak{P}} \circ h$, where $g_{\mathfrak{P}}$ is inert; a morphism in $\mathcal{X}_f$ of such things is the data of a diagram



whose hooked arrows are inert. In particular, unwinding definitions we find that the full subcategory inclusion

$$\mathcal{X}'_f := \mathcal{O}^{\mathrm{act}}_{/O} \times_{\mathcal{P}_{/\varphi(O)}} \left(\mathcal{P}^{\mathrm{int}}_{P/}\right)_{/f} \subset \mathcal{O}_{/O} \times_{\mathcal{P}_{/\varphi(O)}} \left(\mathcal{P}^{\mathrm{int}}_{P/}\right)_{/f} = \mathcal{X}_f$$

admits a left adjoint given by the essentially unique diagram



whose squiggly arrows are active and hooked arrows are inert; hence it is a weak equivalence. Thus it suffices to show that $\mathcal{X}'_f$ is weakly contractible. Now, we have a commutative diagram

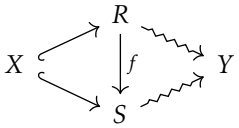
$$\begin{array}{ccc} \mathcal{X}'_f & \longrightarrow & \mathcal{P}^{\mathrm{act}}_{/P} \\ \downarrow & & \downarrow \\ \left(\mathcal{E}^{\mathrm{int}}_{/\varphi(P)}\right)_{/f} \times_{\mathcal{E}_{E/}} \mathcal{E}^{\mathrm{act}}_{E/} & \longrightarrow & \mathcal{E}^{\mathrm{act}}_{/\varphi(P)} \end{array}$$

where the bottom left term consists of the *inert-active factorizations of* f. We make two claims:

Claim i $\left(\mathcal{E}^{\mathrm{int}}_{/\varphi(P)}\right)_{/f} \times_{\mathcal{E}_{E/}} \mathcal{E}^{\mathrm{act}}_{E/}$ is contractible, and
Claim ii the above square is cartesian.

Indeed, combining the two will show that $\mathcal{X}'_f$ is equivalent to a homotopy fiber of $\mathcal{P}^{\mathrm{act}}_{/P} \to \mathcal{E}^{\mathrm{act}}_{/E}$; this will show that (a) $\implies$ (b). Indeed, also, in the case of (c), we find that $\alpha^{\mathrm{act}}_{/P}$ is essentially surjective, so in particular each element of $\mathcal{E}^{\mathrm{act}}_{/\alpha P}$ admits a lift $P' \to P''$, which projects to a morphism $\varphi(P') \to \alpha(P'') = \alpha(P)$; thus all homotopy fibers arise in this way, implying that (c) $\implies$ (a). Thus we're left with verifying the claims. Luckily, for *Claim ii* this follows simply by unwinding definitions.

The assumption that $\mathcal{E}^{\mathrm{int}}, \mathcal{E}^{\mathrm{act}} \subset \mathcal{E}$ is a factorization system is *Claim i after taking cores*, i.e. we are tasked with proving that the given category is a space; unwinding definitions, we're tasked with proving that all diagrams



whose squiggly arrows are active and hooked arrows are inert must have f an equivalence. Cancellation on the lefthand triangle shows that f is inert and cancellation on the righthand triangle shows that f is active, so in fact f is an equivalence, showing the claim. □

We acquire a corollary recognizing Morita localizations and Morita equivalences, which in the nonequivariant case is essentially due to Barkan [Bar23a, Cor. 2.64] and Barkan-Haugseng-Steinebrunner [BHS22, Thm 5.1.1], generalizing and strengthening Harpaz [Har, Prop. 4.2.18], generalizing Lurie [HA, Thm 2.3.3.23].

Corollary I.3.31. *Suppose $\alpha\colon \mathfrak{P} \to \mathfrak{E}$ is a strong Segal morphism of $\mathcal{T}$-algebraic patterns such that*

(ML-a) α has unique lifting for active morphisms, and

(ML-b) $\alpha^{\mathrm{el}}\colon \mathfrak{P}^{\mathrm{el}} \to \mathfrak{E}^{\mathrm{el}}$ is a reflective $\mathcal{T}$-localization.

Then α is a Morita $\mathcal{T}$-localization. In particular, if α^{el} is an equivalence, then α is a Morita equivalence; if moreover either of $\mathfrak{P}$ or $\mathfrak{E}$ is extendable and any of the following conditions hold, then both $\mathfrak{P}$ and $\mathfrak{E}$ are soundly extendable and α is a strong Morita equivalence:

(i) $\mathfrak{P}$ is sound and α is essentially surjective;

(ii) $\mathfrak{E}$ is sound and α is iso-Segal; or

(iii) both of $\mathfrak{P}$ and $\mathfrak{E}$ are sound.

We will do this via an analogous equifibered proof to [BHS22, Thm. 5.1.1].

Lemma I.3.32 ([BHS22, Lem. 5.1.4])**.** *Under Conditions (ML-a) and (ML-b) α is strongly BC.*

Proof. Everything may be checked fiberwise, so we may check this in the nonequivariant case, which is [BHS22, Lem. 5.1.4]. □

Lemma I.3.33. *Under Conditions (ML-a) and (ML-b) $\alpha^{\circledast}$ reflects equifibrations.*

Proof. Equifibrations are determined fiberwise, so this may be verified in the nonequivariant case; this is the last paragraph of the proof of [BHS22, Thm. 5.1.1]. □

For fibrousness of the Morita equivalence, we will use the following.

Proposition I.3.34. *If $\varphi\colon \mathcal{O} \to \mathfrak{P}$ is a Morita equivalence which is strongly BC and $\mathfrak{P}$ is soundly extendable, then φ is a fibrous Morita equivalence if and only if $\varphi^{\circledast}$ reflects equifibrations.*

Proof. For general φ we have a commutative diagram

$$\begin{array}{ccccc}
\underline{\mathrm{FBRS}}(\mathfrak{P}) & = & \underline{\mathrm{FBRS}}(\mathfrak{P}) & \xrightarrow{\varphi^*} & \underline{\mathrm{FBRS}}(\mathcal{O}) \\
\downarrow{\scriptstyle \mathrm{Env}^{\surd}_{\mathfrak{P}}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{Env}^{\surd}_{\mathcal{O}}} \\
\underline{\mathrm{SEG}}_{\mathfrak{P}}\big(\underline{\mathrm{CAT}}_{\mathcal{T}}\big)_{/\mathscr{A}_{\mathfrak{P}}} & \xrightarrow{\varphi^*_{/\mathscr{A}_{\mathfrak{P}}}} & \underline{\mathrm{SEG}}_{\mathcal{O}}\big(\underline{\mathrm{CAT}}_{\mathcal{T}}\big)_{/\varphi^*\mathscr{A}_{\mathfrak{P}}} & \xrightarrow{q^*} & \underline{\mathrm{SEG}}_{\mathcal{O}}\big(\underline{\mathrm{CAT}}_{\mathcal{T}}\big)_{/\mathscr{A}_{\mathcal{O}}}
\end{array}$$

When φ is strongly BC, q^* is an equivalence, so by cancellation of fully faithful $\mathcal{T}$-$(\infty,2)$-functors, φ^* is fully faithful. Essential surjectivity is then equivalent to the condition that the essential image of $\mathrm{Env}^{\surd}_{\mathcal{O}}$

has preimages in the essential image of $\mathrm{Env}^{\surd}_{\mathfrak{P}}$; Corollary I.3.8 shows that this is equivalent to reflecting equifibrations. □

Proof of Corollary I.3.31. The fact that α is a Morita equivalence follows from Proposition I.3.30; Lemmas I.3.32 and I.3.33 and Proposition I.3.34 then shows that it is a fibrous Morita equivalence. To show sound extendability under our additional assumptions, note by Propositions I.2.13 and I.2.35 and Lemma I.3.32 that all patterns in sight are sound, and if $\mathfrak{E}$ is soundly extendable, then so is $\mathfrak{P}$. For the converse, note that $\mathscr{A}_{\mathfrak{E}} \simeq \alpha_* \alpha^{\circledast} \mathscr{A}_{\mathfrak{E}} \simeq \alpha_* \mathscr{A}_{\mathfrak{P}}$ under our assumptions; α_* preserves Segal objects by Proposition I.3.29, so this implies that $\mathscr{A}_{\mathfrak{E}}$ is a Segal object, yielding sound extendability.

Now to show that φ is a strong Morita equivalence, it suffices to show that, given a fibrous $\mathfrak{P}$-pattern $\mathfrak{Q}$, the associated morphism $\varphi' \colon \varphi^* \mathfrak{Q} \to \mathfrak{Q}$ satisfies the conditions of Corollary I.3.31. Since each condition is verified fiberwise, and pullbacks are computed fiberwise, we may henceforth pass to the nonequivariant case. Now, Condition (ML-b) for φ; follows by pullback stability of equivalence under the observation

$$\varphi^* (\mathfrak{Q})^{\mathrm{el}} \simeq \varphi^{\mathrm{el},*} \mathfrak{Q}.$$

Note that $\left(\mathfrak{Q}^{\mathrm{act}}_{/X}\right)^{\simeq}$ is the space of $\mathfrak{Q}$-active arrows $Y \to X$, and $\left(\mathfrak{Q}^{\mathrm{act}}\right)_{/X'}$ the space of pairs of active $\mathcal{P}$-arrows $\mu \colon P \to \pi_{\varphi^* \mathfrak{Q}} X$ and lifts of $\varphi\mu$ to an active $\mathfrak{Q}$-arrow $Y' \to X$. In particular, Condition (ML-a) implies that the fiber of this over the active $\mathfrak{Q}$-arrow $Y' \to X$ is contractible, and this combined with Condition (ML-b) implies that it is π_0-surjective, so we are done. □

I.3.3 Disintegration for parameterized patterns As found throughout [HA], we will find it occasionally useful to have a pattern presentation for *families* of fibrous patterns or Segal categories. In *loc. cit.*, allowing the "shape" of the family to vary, this was implemented via the theory of *generalized* ∞*-operads*.

Now, sometimes this shape-varying formalism is expedient, and some other times it pays instead to fix the shape and work with a simpler formalism there. The meta-procedure of passing from the former to the latter will be called *disintegration*, and is the subject of this section.

I.3.3.1 The pattern structures for families We begin with the fixed-shape theory for general $\mathcal{T}$-patterns.

Example I.3.35. Given $\mathcal{C}$ a $\mathcal{T}$-category, we define the *trivial* $\mathcal{T}$*-algebraic pattern structure on* $\mathcal{C}$

$$\mathrm{Patt}_{\mathcal{T}}(\mathcal{C})^{\mathrm{el}} = \mathrm{Patt}_{\mathcal{T}}(\mathcal{C})^{\mathrm{int}} = \mathcal{C}^{\simeq}; \qquad \mathrm{Patt}_{\mathcal{T}}(\mathcal{C})^{\mathrm{act}} = \mathcal{C}.$$

Similarly, given $\mathcal{C}$ a $\mathcal{T}$-category, we define the *cocartesian* $\mathcal{T}$*-algebraic pattern structure on* $\mathcal{C}$

$$\mathrm{Patt}^{cc}_{\mathcal{T}}(\mathcal{C})^{\mathrm{el}} = \mathrm{Patt}^{cc}_{\mathcal{T}}(\mathcal{C})^{\mathrm{int}} = \mathcal{C}; \qquad \mathrm{Patt}^{cc}_{\mathcal{T}}(\mathcal{C})^{\mathrm{act}} = \mathcal{C}^{\simeq}.$$ ◁

Proposition I.3.36. *Fix $\mathcal{C}$ a $\mathcal{T}$-∞-category.*

(1) There exists a natural equivalence $\mathrm{Tot}^{\bullet}\,\mathrm{Patt}^{cc}_{\mathcal{T}}(\mathcal{C}) \simeq \mathrm{Patt}^{cc}_{*}(\mathrm{Tot}^{\bullet}\,\mathcal{C})$.

(2) The forgetful $\mathcal{T}$-functor $\underline{\mathrm{Seg}}_{\mathrm{Patt}^{cc}_{\mathcal{T}}(\mathcal{C})}(\mathcal{D}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ *is an equivalence.*

(3) $\mathrm{Patt}^{cc}_{\mathcal{T}}(\mathcal{C})$ *is soundly extendable.*

(4) The associated inclusion $\underline{\mathrm{SEG}}_{\mathrm{Patt}^{cc}_{\mathcal{T}}(\mathcal{C})}(\underline{\mathrm{CAT}}_{\mathcal{T}}) \simeq \underline{\mathrm{CAT}}^{\mathrm{cocart}}_{\mathcal{T},/\mathcal{C}} \hookrightarrow \underline{\mathrm{FBRS}}(\mathrm{Patt}^{cc}_{\mathcal{T}}(\mathcal{C}))$ *is an equivalence.*

(5) There exists an equivalence of patterns $\mathrm{Tot}^{V}\,\mathrm{Patt}_{\mathcal{T}}(\mathcal{C}) \simeq \mathrm{Span}_{cocart,\pi_{\mathcal{C}}-dgen}\left(\mathrm{Tot}^{V}\,\mathcal{C};\mathrm{Tot}^{V}\,\mathcal{C}\right)$.

(6) $\mathrm{Patt}_{\mathcal{T}}(\mathcal{C})$ *is soundly extendable.*

(7) The forgetful $\mathcal{T}$-functor $\underline{\mathrm{Seg}}_{\mathrm{Patt}_{\mathcal{T}}(\mathcal{C})} \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ *is an equivalence.*

(8) The forgetful $\mathcal{T}$-$(\infty,2)$-functor $\underline{\mathrm{FBRS}}(\mathrm{Patt}_{\mathcal{T}}(\mathcal{C})) \to \underline{\mathrm{CAT}}_{\mathcal{T},/\mathcal{C}}$ *is an equivalence.*

Proof. Statement (1) follows by noting that each have all morphisms inert, all objects elementary, and only equivalences active. Statement (2) follows straightforwardly by unwinding definitions. Now, Statement (1) reduces Statement (3) to the case $\mathcal{T} = *$; in this case, sound extendability follows under the equivalence $\mathcal{C} \simeq \mathrm{Span}_{all,\simeq}(\mathcal{C};\mathcal{C})$ from the sound extendability criterion of Proposition I.2.16.

For Statement (4), we may again reduce to the nonequivariant case using Statement (1); then, it follows by noting that all inert-cocartesian fibrations to $\mathrm{Patt}^{cc}_{\mathcal{T}}(\mathcal{C})$ are cocartesian and applying Remark I.2.23.

Statements (5) and (7) follow by unwinding definitions. From this, for Statement (6) we are tasked with proving that $\mathrm{Span}_{cocart,\pi_{\mathcal{C}}-dgen}\left(\mathrm{Tot}^{V}\,\mathcal{C};\mathrm{Tot}^{V}\,\mathcal{C}\right)$ is soundly extendable. Under Proposition I.2.16 it suffices to note that whenever $g \circ f$ is cocartesian and f is cocartesian, g is cocartesian [kerodon, Tag 01TS].

Finally, for Statement (8), since every inert morphism of $\mathrm{Patt}_{\mathcal{T}}(\mathcal{C})$ is an equivalence, every $\mathcal{T}$-functor to $\mathcal{C}$ is an inert-cocartesian fibration and all functors over $\mathcal{C}$ are functors of inert-cocartesian fibrations [kerodon, Tag 01TT], so this functor is fully faithful; to see that it is an equivalence, note that a fibrous pattern over a strongly extendable $\mathcal{T}$-pattern $\mathfrak{B}$ is precisely an inert-cocartesian fibration whose unstraightening is a Segal $\mathfrak{B}$-category by Remark I.2.23 and Proposition I.3.4. But this condition is automatic by Statement (7). □

Remark I.3.37. There exists an equivalence $\mathrm{Patt}^{cc}(\mathcal{C}) \simeq \mathrm{Patt}(\mathcal{C})$ if and only if $\mathcal{C}$ is a $\mathcal{T}$-space; on fibrous patterns, this lifts the fact that *all $\mathcal{T}$-functors to $\mathcal{C}$ are cocartesian fibrations if and only if $\mathcal{C}$ is a $\mathcal{T}$-space.* ◁

We acquire a kind of straightening-unstraightening equivalence for fibrous patterns:

Proposition I.3.38. *Let $\mathfrak{B}$ be a $\mathcal{T}$-algebraic pattern and $\mathcal{C}$ a $\mathcal{T}$-∞-category. The $\mathcal{T}$-$(\infty,2)$-functor*

$$\underline{\mathrm{FBRS}}(\mathrm{Patt}^{cc}(\mathcal{C}) \times \mathfrak{B}) \xrightarrow{\text{pullback}} \underline{\mathrm{FUN}}_{\mathcal{T}}(\mathcal{C}, \underline{\mathrm{FBRS}}(\mathfrak{B}))$$

is an equivalence.

We prove this via the equifibered perspective, for which we explicitly use the following.

Lemma I.3.39. *There is an equivalence $\mathscr{A}_{\mathrm{Patt}^{cc}(\mathcal{C})\times\mathfrak{B}} \simeq \mathcal{C}\times\mathscr{A}_{\mathfrak{B}}$ over $\mathcal{C}\times\mathfrak{B}$, under which the $\mathcal{T}$-$(\infty,2)$-functor*

$$P\colon \underline{\mathrm{SEG}}_{\mathfrak{B}\times\mathrm{Patt}^{cc}(\mathcal{C})}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right)^{\mathrm{equi}}_{/\mathscr{A}_{\mathrm{Patt}(\mathcal{C})\times\mathfrak{B}}} \longrightarrow \underline{\mathrm{FUN}}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{SEG}}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right)_{/\mathscr{A}_{\mathfrak{B}}}\right)$$

is fully faithful with image the $\mathcal{T}$-functors with equifibered values.

Proof. We may construct the equivalence via a string of relative equivalences

$$\begin{aligned}\mathscr{A}_{\mathrm{Patt}^{cc}(\mathcal{C})\times\mathfrak{B}} &\simeq \mathrm{Ar}^{\mathrm{act}}\left(\mathrm{Patt}^{cc}(\mathcal{C})\times\mathfrak{B}\right)\\ &\simeq \mathrm{Ar}^{\mathrm{act}}\left(\mathrm{Patt}^{cc}(\mathcal{C})\right)\times\mathrm{Ar}^{\mathrm{act}}\left(\mathfrak{B}\right)\\ &\simeq \mathcal{C}\times\mathrm{Ar}^{\mathrm{act}}(\mathfrak{B}).\end{aligned}$$

Now, applying Proposition I.3.2 and Proposition I.3.36 shows that P is fully faithful; we're left with showing that a cocartesian $\mathcal{T}$-fibration of Segal $\mathfrak{B}\times\mathrm{Patt}^{cc}(\mathcal{C})$-$\infty$-categories $\mathcal{D}\to\mathcal{C}\times\mathscr{A}_{\mathfrak{B}}$ is an equifibration if and only if its fibers over $\mathcal{C}$ are equifibrations. That is, we're tasked with proving that every square on the left is cartesian if and only if every square on the right is cartesian:

$$\begin{array}{ccc}\mathcal{D}\times_{\mathcal{C}\times\mathfrak{B}}(\mathcal{C}\times\{X\}) & \longrightarrow & \mathcal{D}\times_{\mathcal{C}\times\mathfrak{B}}(\mathcal{C}\times\{Y\})\\ \downarrow & & \downarrow\\ \mathcal{C}\times\mathfrak{B}^{\mathrm{act}}_{/X} & \xrightarrow{\omega} & \mathcal{C}\times\mathfrak{B}^{\mathrm{act}}_{/Y}\end{array}\qquad\qquad \begin{array}{ccc}\mathcal{D}\times_{\mathcal{C}\times\mathfrak{B}}(\{C\}\times\{X\}) & \longrightarrow & \mathcal{D}\times_{\mathcal{C}\times\mathfrak{B}}(\{C\}\times\{Y\})\\ \downarrow & & \downarrow\\ \{C\}\times\mathfrak{B}^{\mathrm{act}}_{/X} & \xrightarrow{\omega} & \{C\}\times\mathfrak{B}^{\mathrm{act}}_{/Y}\end{array}$$

Here, the left side quantifies over active $\mathfrak{B}$ morphsims $X\to Y$, and the right side additionally over V-objects $C\in\mathcal{C}$. But the righthand side is just the fibers of the lefthand side over orbits, so the claim follows. □

Proof of Proposition I.3.38. Unwinding definitions, the following diagram commutes:

$$\begin{array}{ccc}\underline{\mathrm{FBRS}}(\mathfrak{B}\times\mathcal{C}) & \longrightarrow & \underline{\mathrm{FUN}}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{FBRS}}(\mathfrak{B})\right)\\ \downarrow & & \downarrow\\ \underline{\mathrm{SEG}}_{\mathfrak{B}\times\mathcal{C}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right)_{/\mathrm{EnvComm}_{\mathfrak{B}\times\mathcal{C}}} & \longrightarrow & \underline{\mathrm{FUN}}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{SEG}}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right)_{/\mathrm{EnvComm}_{\mathfrak{B}}}\right)\end{array}$$

To conclude, Corollary I.3.12 shows that the vertical functors are fully faithful and Lemma I.3.39 shows that the bottom horizontal functor takes the essential image of $\underline{\mathrm{FBRS}}(\mathcal{C}\times\mathfrak{B})$ equivalently onto the essential image of $\underline{\mathrm{FUN}}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{FBRS}}(\mathfrak{B})\right)$, so we are done. □

I.3.3.2 Disintegrable parameterized patterns We would like to internalize the previous study to the category of fibrous $\mathfrak{B}$-patterns, i.e. give a fibrous $\mathfrak{B}$-pattern representing the fibrous Morita class of $\mathfrak{B}\times\mathrm{Patt}^{cc}(\mathcal{C})$. We'll do so in the following setting.

Definition I.3.40. A $\mathcal{T}$-algebraic pattern $\mathfrak{B}$ is *weakly disintegrable* if it is soundly extendable and $\mathfrak{B}^{\mathrm{int}}$ has an initial $\mathcal{T}$-object which lies in $\mathfrak{B}^{\mathrm{el}}$ and receives empty spaces of inert V-morphisms from any non-initial elementary V-objects; $\mathfrak{B}$ is *disintegrable* if this inert-initial $\mathcal{T}$-object is terminal in $\mathfrak{B}^{\mathrm{act}}$. ◁

We write $\varnothing_{\bullet,+} \in \mathfrak{B}^{\mathrm{int}}$ for the inert-initial $\mathcal{T}$-object. Our fibrous patterns for families are as follows.

Proposition I.3.41. *Given a weakly disintegrable $\mathcal{T}$-pattern $\mathfrak{B}$, $\varnothing_{\bullet,+}$ is classified by an Segal morphism*

$$\iota \colon \mathsf{Patt}(*_{\mathcal{T}}) \to \mathfrak{B}$$

whose pullback $\mathcal{T}$-$(\infty,2)$-functor $\mathsf{sh}_{\mathfrak{B}} \coloneqq \iota^* \colon \underline{\mathrm{FBRS}}(\mathfrak{B}) \longrightarrow \underline{\mathrm{CAT}}_{\mathcal{T}}$ has fully faithful right $\mathcal{T}$-adjoint $\mathsf{const}_{\mathfrak{B}}(\mathcal{C})$; moreover, there exists a natural equivalence of $\mathcal{T}$-patterns $\mathsf{const}_{\mathfrak{B}}(\mathcal{C}) \simeq \mathsf{Patt}(\mathcal{C}) \times \mathfrak{B}$ over $\mathfrak{B}$.

Now, the categorical reason that the const construction works is the following pair of lemmas.

Lemma I.3.42. *Suppose $\mathfrak{O}$ is a soundly extendable $\mathcal{T}$-algebraic pattern such that, for all $O \in \mathfrak{O}_V$, the category $\left(\mathfrak{O}_V^{\mathrm{el}}\right)_{O/}$ is weakly contractible. Then, the projection $\mathsf{const}_{\mathfrak{O}}(\mathcal{C}) \coloneqq \mathcal{C} \times \mathfrak{O} \to \mathfrak{O}$ is a fibrous pattern, and the induced $\mathcal{T}$-pattern map is equivalent to $\mathsf{Patt}(\mathcal{C}) \times \mathfrak{O} \to \mathfrak{O}$.*

Proof. $\mathsf{const}_{\mathfrak{O}}(\mathcal{C}) \to \mathfrak{O}$ is obviously a cocartesian fibration, so we're left with verifying the Segal condition. Now, given $O \in \mathfrak{O}_V$, note that $\left(\mathfrak{O}_V^{\mathrm{el}}\right)_{O/} \subset \left(\mathrm{Tot}^V \mathfrak{O}\right)^{\mathrm{el}}_{O/}$ is initial, so we may rewrite Remark I.2.21 as stating that the composite arrow is an equivalence

$$\mathrm{Tot}\,\mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Tot}\,\mathfrak{O}^{\mathrm{act}}_{/O} \longrightarrow \mathrm{Tot}\,\mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} \lim_{E \in \left(\mathfrak{O}_V^{\mathrm{el}}\right)_{O/}} \mathrm{Tot}\,\mathfrak{O}^{\mathrm{act}}_{/E} \xrightarrow{c} \lim_{E \in \left(\mathfrak{O}_V^{\mathrm{el}}\right)_{O/}} \mathrm{Tot}\,\mathcal{C} \times_{\mathcal{T}^{\mathrm{op}}} \mathrm{Tot}\,\mathfrak{O}^{\mathrm{act}}_{/E},$$

where c is coassembly. Now, since $\left(\mathfrak{O}_V^{\mathrm{el}}\right)_{O/}$ is weakly contractible, commutation of limits with limits precisely says that c is an equivalence, and the lefthand arrow is an equivalence since $\mathfrak{O}$ is extendable.

What's left is to understand the $\mathcal{T}$-pattern structure. Note that the prefibrous inert V-arrows in $\mathcal{C} \times \mathfrak{O}$ are precisely cocartesian lifts of inert V-arrows of $\mathfrak{O}$, and lifts of V-arrows are cocartesian if and only if their projection to $\mathcal{C}$ is an equivalence;

$$(\mathcal{C} \times \mathfrak{O})^{\mathrm{int}} \simeq \mathcal{C}^{\mathrm{int}} \times \mathfrak{O}^{\mathrm{int}} \simeq \mathsf{Patt}(\mathcal{C})^{\mathrm{int}} \times \mathfrak{O}^{\mathrm{int}} \simeq (\mathsf{Patt}(\mathcal{C}) \times \mathfrak{O})^{\mathrm{int}}$$

for active V-arrows and elementary V-objects, it suffices to note that the projection $\mathcal{C} \times \mathfrak{O} \to \mathfrak{O}$ preserves and reflects each property for each pattern structure. □

Note that existence of the (elementary) initial $\mathcal{T}$-object in $\mathfrak{B}^{\mathrm{int}}$ implies that each $\left(\mathfrak{B}_V^{\mathrm{el}}\right)_{O/}$ is weakly contractible, so Lemma I.3.42 constructs the $\mathcal{T}$-functor $\mathsf{const}_{\mathfrak{B}} \colon \underline{\mathrm{Cat}}_{\mathcal{T}} \longrightarrow \underline{\mathrm{Fbrs}}(\mathfrak{B})$.

Lemma I.3.43. *Suppose $\mathfrak{B}$ is a weakly disintegrable $\mathcal{T}$-pattern, $\mathfrak{O}$ a fibrous $\mathrm{Tot}^{\mathcal{T}}\mathfrak{B}$-pattern, and $\mathcal{C}$ is a $\mathcal{T}$-∞-category. The following conditions on a $\mathcal{T}$-functor $\varphi\colon \mathfrak{O} \to \mathcal{C}$ are equivalent:*

(a) φ carries inert arrows to equivalences,

(b) for all $O \in \mathfrak{O}_V$, φ carries the inert V-arrow $O_{\varnothing_V} \to O$ with $\pi_{\mathfrak{O}} O_{\varnothing_V} = \varnothing_{V,+}$ to an equivalence, and

(c) φ is right Kan-extended from $\mathsf{sh}_{\mathfrak{B}}\mathfrak{O}$.

Moreover, every $\mathcal{T}$-functor $\mathsf{sh}_{\mathfrak{B}}\mathfrak{O} \to \mathcal{C}$ admits a $\mathcal{T}$-right Kan extension along $\mathsf{sh}_{\mathfrak{B}}\mathfrak{O} \hookrightarrow \mathfrak{O}$.

Proof. Condition (a) obviously implies Condition (b); the converse follows for a $\mathfrak{B}_V^{\mathrm{int}}$-map $B \to C$ and V-object $O \in \pi_{\mathfrak{O}}^{-1}(C)$ by two-out-of-three applied to the factorization

$$\begin{array}{ccc} O_{\varnothing_V} & \longrightarrow & O_B \\ & \searrow & \downarrow \\ & & O \end{array}$$

where every morphism is cocartesian. Now, note that there is an initial V-object

$$\left(O_{\varnothing_V} \xrightarrow{!} O\right) \in \left(\mathsf{sh}\,\mathfrak{O} \times_{\mathfrak{O}} \left(\mathfrak{O}_V^{\mathrm{int}}\right)_{/O}\right)_V;$$

the latter is precisely the $\mathcal{T}$-∞-category appearing in the pointwise formula for $\mathcal{T}$-right Kan extension along $\mathsf{sh}\,\mathfrak{O} \to \mathfrak{O}$ of Theorem I.1.37, so the equivalence of Conditions (b) and (c) and existence of such right $\mathcal{T}$-Kan extensions follows directly from Theorem I.1.37 in sight of Corollary I.1.21. □

Proof of Proposition I.3.41. Lemma I.3.42 constructs the $\mathcal{T}$-$(\infty,2)$-functor $\mathsf{const}_{\mathfrak{B}}$ and verifies the relative natural equivalence of $\mathcal{T}$-patterns $\mathsf{Patt}(\mathcal{C}) \times \mathfrak{B}$. In order to prove that $\mathsf{const}_{\mathfrak{B}}$ is a fully faithful right $\mathcal{T}$-adjoint to $\mathsf{sh}_{\mathfrak{B}}$, we may check this on V-value category using Proposition I.1.64; by restriction, we only need to prove that $\mathsf{const}_{\mathfrak{B}}\colon \mathrm{CAT}_{\mathcal{T}} \longrightarrow \mathrm{FBRS}(\mathfrak{B})$ is a fully faithful right adjoint to $\mathsf{sh}_{\mathfrak{B}}$.

Now, the standard natural equivalence $\mathcal{C} \times \mathfrak{B} \times_{\mathfrak{B}} *_{\mathcal{T}} \simeq \mathcal{C}$ yields a natural equivalence

$$\eta\colon \mathsf{sh}_{\mathfrak{B}}\mathsf{const}_{\mathfrak{B}}(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}.$$

It suffices to prove that composition with this yields a natural equivalence

$$\underline{\mathrm{Alg}}_{\mathfrak{P}/\mathfrak{B}}(\mathsf{const}_{\mathfrak{B}}(\mathcal{C})) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathsf{sh}\,\mathfrak{P}, \mathsf{sh}\,\mathsf{const}_{\mathfrak{B}}(\mathcal{C})) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathsf{sh}\,\mathfrak{P}, \mathcal{C});$$

again, we may check this on V-values, and hence remove the underline. Then, this follows directly from Lemma I.3.43. □

I.3.3.3 Disintegration of families When X is a $\mathcal{T}$-space, we've constructed a natural equivalence

$$\underline{\mathrm{FBRS}}(\mathfrak{B})_{/\mathrm{const}_{\mathfrak{B}}(X)} \simeq \underline{\mathrm{FUN}}_{\mathcal{T}}(X, \underline{\mathrm{FBRS}}(\mathfrak{B})). \tag{7}$$

Now, $\mathcal{T}$-space shaped fibrous $\mathfrak{B}$-patterns come equipped with *canonical* morphisms down to some $\mathsf{const}_{\mathfrak{B}}(X)$, which may be recognized as the sh-equivalences with codomain lying in the essential image of $\mathsf{const}_{\mathfrak{B}}|_{\underline{\mathcal{S}}_{\mathcal{T}}}$. We would like to unstraighten the corresponding restriction Eq. (7) to some explicit $\mathcal{T}$-$(\infty,2)$-category of *families of fibrous patterns* on the righthand side. To do so, we define another $\mathcal{T}$-pattern, for which we will begin the convention of writing $\mathfrak{B}^{\natural}$ for our disintegrable pattern.

Construction I.3.44. Now, note that the initial object in $\mathfrak{B}^{\natural,\mathrm{el}}$ exhibits an equivalence

$$\mathfrak{B}^{\natural,\mathrm{el}} \simeq \left(\mathfrak{B}^{\flat,\mathrm{el}}\right)^{\triangleleft}$$

for $\mathfrak{B}^{\flat,\mathrm{el}} \subset \mathfrak{B}^{\natural,\mathrm{el}}$ the full $\mathcal{T}$-subcategory of non-initial V-objects. Let $\mathfrak{B}^{\flat} \hookrightarrow \mathfrak{B}^{\natural}$ be the $\mathcal{T}$-subpattern with the same inert and active $\mathcal{T}$-categories and elementary $\mathcal{T}$-objects $\mathfrak{B}^{\flat}$. ◁

Lemma I.3.45. *Suppose $\mathfrak{B}^{\natural}$ is disintegrable. An inert-cocartesian $\mathcal{T}$-fibration $\mathcal{O} \to \mathfrak{B}$ is a fibrous $\mathfrak{B}^{\flat}$-pattern if and only if it is a fibrous $\mathfrak{B}^{\natural}$-pattern such that $\mathsf{sh}(\mathcal{O})$ is contractible; this yields an equivalence*

$$\underline{\mathrm{FBRS}}\left(\mathfrak{B}^{\flat}\right) \xrightarrow{\sim} \underline{\mathrm{FBRS}}\left(\mathfrak{B}^{\natural}\right) \times_{\mathsf{sh},\underline{\mathrm{Cat}}_{\mathcal{T}}} \{*_{\mathcal{T}}\}.$$

Proof. To see that the equivalence follows from the statement before, it suffices to note that $\mathfrak{B}$ and $\mathfrak{B}^{\flat}$ have the same inert morphisms. We now prove the first clause.

Now, for an object $W \in \mathcal{T}$, our assumptions guarantee that the category $\mathrm{Tot}^{V} \mathfrak{B}^{\flat,\mathrm{el}}_{/(W,\varnothing_W)}$ is empty. In particular, if $\mathcal{O}$ is a fibrous $\mathfrak{B}^{\flat}$-pattern, then $\mathcal{O}_{\varnothing_V}$ is contractible, i.e. $\mathsf{sh}(\mathcal{O})$ is contractible.

From this, in either case, Proposition I.1.22 implies that the coassembly maps

$$\lim_{E \in \mathfrak{B}^{\natural,\mathrm{el}}_{/V}} \mathcal{O}_E \longrightarrow \lim_{E \in \mathfrak{B}^{\flat,\mathrm{el}}_{/V}} \mathcal{O}_E$$

is an equivalence, so the Segal condition for colors are equivalent to each other. Now, moreover, our assumptions ensure that every morphism to $\varnothing_V$ is inert, so in particular given a prefibrous $\mathfrak{B}$-pattern $\mathcal{O}$ satisfying $\mathcal{O}_{\varnothing_V} = *$, we find

$$\mathrm{Map}_{\mathcal{O}^{V}}(X, \varnothing_V) \simeq \mathrm{Map}_{\mathcal{O}^{\mathrm{int},V}}(X, \varnothing_V) \simeq \mathrm{Map}_{\mathfrak{B}^{\natural,\mathrm{int}}}(\pi X, \varnothing_V) \simeq *$$

where $\varnothing_V$ is used to refer to the element of $\mathcal{O}_{\varnothing_V}$; that is, the two Segal conditions for multimorphisms are equivalent. □

Example I.3.46. Given $\mathcal{B}^\natural$ a weakly disintegrable pattern and $\mathcal{O} \to \mathcal{B}^\flat$ a fibrous pattern, we claim that the associated fibrous pattern structure $\mathcal{O}^\natural \to \mathcal{B}^\natural$ is weakly disintegrable. Indeed, Lemma I.3.45 shows that $\mathsf{sh}(\mathcal{O}) = *_{\mathcal{T}}$; it thus follows from the fact that inert arrows in $\mathcal{O}$ are cocartesian over inert arrows in $\mathcal{B}$ and $\varnothing_\bullet$ is inert-initial that the unique lift $\varnothing_\bullet \colon *_{\mathcal{T}} \to \mathcal{B}^{\mathrm{int}}$ is also an initial $\mathcal{T}$-object. The fact that it receives empty spaces of inert V-morphisms from non-initial elementary V-objects is automatic, as such spaces live over the empty space of such maps in $\mathcal{B}$.

Moreover, a similar argument shows that $\mathcal{O}^\natural$ is disintegrable whenever $\mathcal{B}^\natural$ is (c.f. the mapping space triviality in the proof of Lemma I.3.45), and $\mathcal{O}^\flat \simeq \mathcal{O}$ as patterns. ◁

Construction I.3.47. Under Proposition I.3.36, we have a composite $\mathcal{T}$-$(\infty,2)$-functor

$$\mathsf{sh}^{-1}(\mathcal{C}) \hookrightarrow \underline{\mathrm{FBRS}}(\mathcal{B})_{/\mathrm{const}_{\mathcal{B}}(\mathcal{C})} \longrightarrow \underline{\mathrm{FBRS}}(\mathcal{B})_{/\mathrm{const}_{\mathcal{B}}(\mathcal{C}^{\simeq})} \longrightarrow \underline{\mathrm{FUN}}_{\mathcal{T}}\left(\mathcal{C}^{\simeq}, \underline{\mathrm{FBRS}}\left(\mathcal{B}\right)\right);$$

moreover, $\mathsf{sh}_{\mathcal{B}}$ is compatible with pullback, so this in fact lands in $\underline{\mathrm{FUN}}_{\mathcal{T}}\left(\mathcal{C}^{\simeq}, \mathsf{sh}^{-1}(*_{\mathcal{T}})\right) \simeq \underline{\mathrm{FUN}}_{\mathcal{T}}\left(\mathcal{C}^{\simeq}, \underline{\mathrm{FBRS}}(\mathcal{B}^\flat)\right)$. We refer to the associated $\mathcal{T}$-$(\infty,2)$-functor as *disintegration*

$$\delta \colon \mathsf{sh}^{-1}(\mathcal{C}) \longrightarrow \underline{\mathrm{FUN}}_{\mathcal{T}}\left(\mathcal{C}^{\simeq}, \underline{\mathrm{FBRS}}\left(\mathcal{B}^\flat\right)\right).$$ ◁

This yields a disintegration equivalence extending [KK24, Thm. 2.2] and [HA, § 2.3].

Proposition I.3.48. *If $\mathcal{B}$ is a disintegrable $\mathcal{T}$-pattern, then disintegration yields an equivalence of cartesian $\mathcal{T}$-fibrations over $\underline{\mathcal{S}}_{\mathcal{T}}$,*

$$\underline{\mathrm{Fbrs}}(\mathcal{B}) \times_{\mathsf{sh},\underline{\mathrm{Cat}}_{\mathcal{T}}} \underline{\mathcal{S}}_{\mathcal{T}} \simeq \int_{X \in \underline{\mathcal{S}}_{\mathcal{T}}} \underline{\mathrm{Fun}}_{\mathcal{T}}\left(X, \underline{\mathrm{Fbrs}}\left(\mathcal{B}^\flat\right)\right).$$

To prove this, the author tried and failed to find the following lemma recorded somewhere.

Lemma I.3.49. *Suppose $L \colon \mathcal{C} \to \mathcal{D}$ is a localization functor with the condition that localization maps $X \to \iota LX$ are closed under pullback between morphisms in $\mathcal{D}$. Then, L is a cartesian fibration with fibers described by the full subcategories*

$$L^{-1}(d) \subset \mathcal{C}_{/\iota(d)}$$

spanned by localization maps. In particular, this holds if L is compatible with pullbacks.

Proof. Now, first note that we have an essentially unique factorization

$$\begin{array}{ccc} \mathrm{Ar}(\mathcal{C}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{C} \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \iota} \\ \mathcal{C} & \xrightarrow[L]{} & \mathcal{D} \end{array}$$

where i fully faithfully embeds $\mathcal{C}$ as the localization maps. Our assumption shows that, given an object $C \in \mathcal{C}$ and a $\mathcal{D}$-map $D \to LC$, the ev_1-cartesian transport map $f' \colon f^*C \to C$ lies in i; it is then straightforward to verify that it satisfies the universal property for L-cartesian arrows. The fiber computation similarly follows. □

Proof of Proposition I.3.48. First note that $\mathrm{sh}_{\mathfrak{B}}$ is compatible with pullback, and so it satisfies the conditions of Lemma I.3.49. In particular, after straightening, the proposition reduces to proving that the natural assignment

$$\mathrm{sh}^{-1}(X) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}\left(X, \underline{\mathrm{Fbrs}}\left(\mathfrak{B}^{\flat}\right)\right)$$

is an equivalence when X is a $\mathcal{T}$-space. But in fact this follows by combining Propositions I.3.38 and I.3.41, Remark I.3.37, and Lemma I.3.45. □

I.3.3.4 Disintegrable fibrous Segal conditions Before moving on, we point out a mild generalization to the argument of Lemma I.3.45, phrased in terms of generalized fibrous Segal conditions. The proof is identical, so we omit it.

Proposition I.3.50. *Suppose $\mathfrak{B}^{\natural}$ is disintegrable.*

1. *If $\overline{p} \colon \mathcal{K}^{\triangleleft} \to \mathfrak{B}^{\mathrm{int}}$ is a generalized fibrous $\mathfrak{B}^{\natural}$-Segal condition and $\mathcal{L} \subset \mathcal{K}$ a terminal full subcategory such that $\overline{p}|_{\mathcal{L}}$ is constant on $\varnothing_{\bullet}$, and $\mathcal{K}' \subset \mathcal{K}$ is a complementary full $\mathcal{T}$-subcategory, then $\overline{p}' = \overline{p}|_{\mathcal{K}'^{\triangleleft}}$ is a generalized $\mathfrak{B}^{\flat}$-Segal condition.*
2. *If $\left\{\overline{p}_{\alpha}\right\}$ is a sufficient set of generalized fibrous $\mathfrak{B}^{\natural}$-Segal conditions of the above form, then $\left\{\overline{p}'_{\alpha}\right\} \cup \left\{\{\varnothing_V\} \hookrightarrow \mathfrak{B}^{\mathrm{int}}\right\}$ is a sufficient set of generalized Segal $\mathfrak{B}^{\flat}$-conditions.*

I.3.4 Parameterized fibrous subpatterns Fix $\mathfrak{B}$ a soundly extendable $\mathcal{T}$-algebraic pattern. For convenience, we collect several facts about parameterized subpatterns into one theorem.

Theorem I.3.51. *Let $\varphi \colon \mathfrak{O} \to \mathfrak{P}$ be a morphism of fibrous $\mathfrak{B}$-patterns.*

1. *φ is a monomorphism in $\mathrm{Fbrs}(\mathfrak{B})$ if and only if its corresponding $\mathcal{T}$-functor $\mathrm{Tot}_{\mathcal{T}} \mathfrak{O} \to \mathrm{Tot}_{\mathcal{T}} \mathfrak{P}$ is a $\mathcal{T}$-subcategory inclusion.*
2. *The associated poset map $\mathrm{Sub}_{\mathrm{Fbrs}(\mathfrak{B})}(\mathfrak{P}) \to \mathrm{Sub}_{\mathrm{Cat}_{\mathcal{T}}}(\mathrm{Tot}_{\mathcal{T}} \mathfrak{P})$ is fully faithful with image those $\mathcal{T}$-subcategories $\mathrm{Tot}_{\mathcal{T}} \mathfrak{O} \subset \mathrm{Tot}_{\mathcal{T}} \mathfrak{P}$ satisfying the following conditions:*
   (a) *every inert V-arrow of $\mathrm{Tot}_{\mathcal{T}} \mathfrak{P}$ with domain in $\mathfrak{O}$ lies in $\mathfrak{O}$, and*
   (b) *the composite functor $\mathrm{Tot}_{\mathcal{T}} \mathfrak{O} \to \mathfrak{B}$ is a fibrous pattern.*
3. *If φ is a monomorphism, then for all $\mathfrak{Q} \in \mathrm{Fbrs}(\mathfrak{B})$, the induced $\mathcal{T}$-subcategory inclusion*

$$\underline{\mathrm{Alg}}_{\mathfrak{Q}/\mathfrak{B}}(\mathfrak{O}) \hookrightarrow \underline{\mathrm{Alg}}_{\mathfrak{Q}/\mathfrak{B}}(\mathfrak{P})$$

has image with

- *objects the algebras* $\mathcal{Q} \to \mathcal{P}$ *whose associated* $\mathcal{B}^{\mathrm{el,op}}$*-functor* $\mathcal{Q}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ *factors through* $\mathcal{O}^{\mathrm{el}}$ *and such that, for all active* $\mathcal{B}$*-arrows* $\omega \colon X \to E$ *with elementary codomain and pairs* $(O,P) \in \mathcal{Q}_X \times \mathcal{Q}_E$ *there exists a factorization*

$$\begin{array}{ccc} & & \mathrm{Map}^{\omega}_{\mathcal{O}}(\varphi O, \varphi P) \\ & \overset{\exists}{\nearrow} & \big\downarrow \\ \mathrm{Map}^{\omega}_{\mathcal{Q}}(O,P) & \xrightarrow{\varphi} & \mathrm{Map}^{\omega}_{\mathcal{P}}(\varphi O, \varphi P) \end{array}$$

  where $\mathrm{Map}^{\omega}_{\mathcal{Q}}(O,P) \coloneqq \mathrm{Map}_{\mathcal{Q}}(O,P) \times_{\mathrm{Map}_{\mathcal{B}}(X,E)} \{\omega\}$*, and*

- *morphisms the natural transformations* $X \Longrightarrow Y \colon \mathcal{Q} \to \mathcal{P}$ *whose elementary natural transformations* $X^{\mathrm{el}} \Longrightarrow Y^{\mathrm{el}}$ *land pointwise in* $\mathcal{O}^{\mathrm{el}}$*.*

(4) If $\mathcal{O}$ *and* $\mathcal{P}$ *are Segal* $\mathcal{B}$*-*∞*-categories and* φ *lifts to* $\mathrm{Seg}_{\mathcal{B}}(\mathrm{Cat})$*, then is a monomorphism in* $\mathrm{Seg}_{\mathcal{B}}(\mathrm{Cat})$ *if and only if it is a monomorphism in* $\mathrm{Fbrs}(\mathcal{B})$*.*

(5) The associated poset map $\mathrm{Sub}_{\mathrm{Seg}(\mathcal{B})}(\mathcal{P}) \longrightarrow \mathrm{Sub}_{\mathrm{Cat}_{\mathcal{B}^{\mathrm{el,op}}}}\left(\mathcal{P}^{\mathrm{el}}\right)$ *is fully faithful, with essential image those* $\mathcal{B}^{\mathrm{el,op}}$*-subcategories which, for all active* $\mathcal{B}$*-arrows* $\omega \colon X \to E$ *with elementary codomain, the corresponding functor admits a factorization*

$$\begin{array}{ccccc} \lim_{F \in \mathcal{B}^{\mathrm{el}}_{X/}} \mathcal{O}_F & & \dashrightarrow & & \mathcal{O}_E \\ \big\downarrow & & & & \big\downarrow \\ \lim_{F \in \mathcal{B}^{\mathrm{el}}_{X/}} \mathcal{P}_F & \simeq & \mathcal{P}_X & \xrightarrow{\omega_{\otimes}} & \mathcal{P}_E \end{array}$$

(6) If φ *lifts to a monomorphism in* $\mathrm{Seg}_{\mathcal{B}}(\mathrm{Cat})$*, then for all* $\mathcal{C} \in \mathrm{Seg}_{\mathcal{B}}(\mathrm{Cat})$*, the* $\mathcal{T}$*-subcategory*

$$\underline{\mathrm{Fun}}^{\otimes}_{\mathcal{B}}(\mathcal{C},\mathcal{O}) \hookrightarrow \underline{\mathrm{Fun}}^{\otimes}_{\mathcal{B}}(\mathcal{C},\mathcal{P})$$

has

- *objects the* $\mathcal{C} \to \mathcal{P}$ *whose associated* $\mathcal{B}^{\mathrm{el,op}}$*-functor* $\mathcal{Q}^{\mathrm{el}} \to \mathcal{P}^{\mathrm{el}}$ *factors through* $\mathcal{O}^{\mathrm{el}}$*, and*
- *morphisms the natural transformations* $X \Longrightarrow Y \colon \mathcal{C} \to \mathcal{P}$ *whose elementary natural transformations* $X^{\mathrm{el}} \Longrightarrow Y^{\mathrm{el}}$ *land pointwise in* $\mathcal{O}^{\mathrm{el}}$*.*

We call these *fibrous* $\mathcal{B}$*-subpatterns* and *Segal* $\mathcal{B}$*-subcategories*. We also study the full case.

Definition I.3.52. A fibrous subpattern $\varphi \colon \mathcal{O} \to \mathcal{P}$ is *full* if $\mathrm{Tot}_{\mathcal{O}}\mathcal{O} \subset \mathrm{Tot}_{\mathcal{T}}\mathcal{P}$ is full. A Segal $\mathcal{B}$-subcategory is full if its underlying fibrous subpattern is full; equivalently, $\mathcal{O}^{\mathrm{el}} \hookrightarrow \mathcal{P}^{\mathrm{el}}$ is full. ◁

These are classified in analogy to [HA, § 2.2.1].

Corollary I.3.53. *The poset map*

$$(-)^{\mathrm{el}} \colon \mathsf{FullSub}_{\mathrm{Fbrs}(\mathfrak{B})}(\mathfrak{P}) \to \mathsf{FullSub}_{\mathfrak{B}^{\mathrm{el},\mathrm{op}}}\left(\mathfrak{P}^{\mathrm{el}}\right)$$

is an equivalence. If $\mathfrak{P}$ is a Segal $\mathfrak{B}$-∞-category, then an object in $\mathsf{FullSub}_{\mathfrak{B}^{\mathrm{el}}}\left(\mathfrak{P}^{\mathrm{el}}\right)$ lifts to a Segal full $\mathfrak{B}$-subcategory if and only if, for all objects $O \in \mathfrak{O}_X$ and active arrows $\omega \colon X \to E$ active with E elementary, $\omega_{\otimes} O \in \mathfrak{O}_E$.

Proof. For the first claim, we begin by constructing a section; indeed, given $\mathfrak{O}^{\mathrm{el}} \subset \mathfrak{P}^{\mathrm{el}}$ a full $\mathfrak{B}^{\mathrm{el}}$-subcategory, we define the full $\mathfrak{B}$-relative $\mathcal{T}$-subcategory $S(\mathfrak{O}) \hookrightarrow \mathsf{Tot}_{\mathcal{T}} \mathfrak{P}$ by the fiber over $X \in \mathfrak{B}^{\mathrm{el}}$

$$S(\mathfrak{O})_X \coloneqq \lim_{F \in \mathfrak{B}^{\mathrm{el}}_{X/}} \mathfrak{O}_E \subset \mathfrak{P}_X.$$

The claim that this is closed under inert-cocartesian transport (i.e. contains all inert arrows of $\mathsf{Tot}_{\mathcal{T}} \mathfrak{P}$ with codomain in $\mathfrak{O}$) reduces to observing this in each element of the limit diagram, where this is the observation that $\mathsf{Tot}\, \mathfrak{O}^{\mathrm{el}} \to \mathsf{Tot}\, \mathfrak{P}^{\mathrm{el}}$ is a cocartesian fibration; the Segal condition for colors follows immediately by definition, and the Segal condition for multimorphisms is simply a specialization of the same condition for $\mathfrak{P}$. The fact that $\mathfrak{O}^{\mathrm{el}}$ recovers our input follows by the fact that $E \in \mathfrak{B}^{\mathrm{el}}_{E/}$ is initial.

For the first claim, given $\mathfrak{O} \subset \mathfrak{P}$, we are thus reduced to proving that $S(\mathfrak{O}^{\mathrm{el}})$ and $\mathfrak{O}$ resent equal full $\mathcal{T}$-subcategories of $\mathsf{Tot}_{\mathcal{T}} \mathcal{O}$, i.e. that

$$\mathfrak{O}_X = \lim_{F \in \mathfrak{B}^{\mathrm{el}}_{X/}} \mathfrak{O}_E \subset \mathfrak{P}_X;$$

this is simply the fact that the inclusion $\mathfrak{O} \subset \mathfrak{P}$ is a fibrous pattern map.

For the second claim, note that the lift to a Segal full $\mathfrak{B}$-subcategory is equivalent to the condition that $\mathsf{Tot}_{\mathcal{T}} \mathfrak{O} \subset \mathsf{Tot}_{\mathcal{T}} \mathfrak{P}$ is closed under cocartesian transport; the Segal condition for colors reduces this to transport with elementary codomain, giving the claim. □

Full sub-patterns moreover recieve the expected algebras.

Corollary I.3.54. *If $\varphi \colon \mathfrak{O} \to \mathfrak{P}$ is a fibrous full $\mathfrak{B}$-subpattern, then the inclusion*

$$\underline{\mathrm{Alg}}_{\mathfrak{Q}/\mathfrak{B}}(\mathfrak{O}) \hookrightarrow \underline{\mathrm{Alg}}_{\mathfrak{Q}/\mathfrak{B}}(\mathfrak{P})$$

is fully faithful, with image those $\mathfrak{Q}$-algebras X such that, for all $Q \in \mathfrak{Q}^{\mathrm{el}}$, $X(Q) \in \mathfrak{P}^{\mathrm{el}}$ lies in $\mathfrak{O}^{\mathrm{el}}$. If moreover φ is a Segal full $\mathfrak{B}$-subcategory, then

$$\underline{\mathrm{Fun}}^{\otimes}_{\mathfrak{B}}(\mathfrak{C}, \mathfrak{O}) \hookrightarrow \underline{\mathrm{Fun}}^{\otimes}_{\mathfrak{B}}(\mathfrak{C}, \mathfrak{P})$$

is fully faithful with image those Segal $\mathfrak{B}$-functors satisfying the same condition.

PROOF. We may check each condition fiberwise, i.e. remove the underline. For fully faithfulness, simply note that the condition that a natural transformation lands in $\mathfrak{O}^{\mathrm{el}}$ is automatic, since $\mathfrak{O} \hookrightarrow \mathfrak{P}$ is full. Our image condition follows by Theorem I.3.51, noting that our objectwise factorization condition is equivalent to the factorization condition of Theorem I.3.51 by fully faithfulness. □

We now move back to the general case and prove Theorem I.3.51. First, an obvious lemma.

Lemma I.3.55. *Suppose $\mathbb{C}$ is a $\mathcal{T}$-$(\infty,2)$-category such that each $\mathbb{C}^V$ admits tensors by Δ^1 in the sense of [BB25, Def. A.2.22] and $f\colon X \to Y$ a monomorphism in $\Gamma^{\mathcal{T}}\mathbb{C}^{\leq 1}$; for instance, $\mathbb{C} = \underline{\mathrm{FBRS}}(\mathfrak{B})$ or $\underline{\mathrm{SEG}}_{\mathfrak{B}}(\underline{\mathrm{CAT}}_{\mathcal{T}})$. Then, postcomposition along f yields a natural $\mathcal{T}$-subcategory inclusion*

$$f_*\colon \underline{\mathrm{Fun}}_{\mathbb{C}}(Z,X) \hookrightarrow \underline{\mathrm{Fun}}_{\mathbb{C}}(Z,Y).$$

PROOF. After Corollary I.2.67, we may pass to fixed points and prove this for $\mathcal{T} = *$. We see that f_* is a monomorphism on cores by the defining property of nomomorphisms, and that f_* is a monomorphism on $\mathrm{Ar}(\mathrm{Fun}_{\mathcal{C}}(Z,-))^{\simeq}$ by replacing Z with $\Delta^1 \otimes Z$, so the result follows by Proposition I.2.65.

To see that $\underline{\mathrm{FBRS}}(\mathfrak{B})$ and $\underline{\mathrm{SEG}}_{\mathfrak{B}}$ are tensored over Δ^1, note that this is a straighforward consequence of the fact that they *come from* Cat-modules. □

PROOF OF THEOREM I.3.51. We perform proof by a thousand cuts. Statement (1) follows by noting that $\mathsf{Tot}_{\mathcal{T}}\colon \mathsf{Fbrs}(\mathfrak{B}) \to \mathsf{Cat}_{\mathcal{T},/\mathfrak{B}} \to \mathsf{Cat}_{\mathcal{T}}$ is a limit-preserving subcategory inclusion conservative right adjoint[10], so it preserves and reflects monomorphisms, and Corollary I.2.67 identifies monomorphisms in $\mathsf{Cat}_{\mathcal{T}}$ with $\mathcal{T}$-subcategory inclusions. Statement (2) then follows by observing that a $\mathcal{T}$-subcategory inclusion $\mathsf{Tot}_{\mathcal{T}}\mathfrak{O} \subset \mathsf{Tot}_{\mathcal{T}}\mathfrak{P}$ is a fibrous pattern map if and only if $\mathsf{Tot}_{\mathcal{T}}\mathfrak{O}$ is a pattern whose inert-cocartesian lifts agree with that of $\mathsf{Tot}_{\mathcal{T}}\mathfrak{P}$, i.e. the two conditions hold.

For Statement (3), after Lemma I.3.55 since $\underline{\mathrm{FBRS}}(\mathfrak{B}) \hookrightarrow \underline{\mathrm{CAT}}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathfrak{B}}$ is fully faithful, applying transitivity of inert-cocartesian fibrations [kerodon, Tag 01TU], for objects it suffices to prove that such $\mathfrak{Q} \to \mathfrak{P}$ factors through $\mathfrak{O}$ if and only if the above conditions hold. They are certainly necessary; that they are sufficient is implied by applying the Segal conditions of Definition I.2.20 together with Corollary I.2.74. For morphisms the same argument applies, this time noting that the data of the natural transformation must itself lie in the fibers, hence the Segal condition for colors reduces them to fibers over elementary objects.

[10] This follows by applying [BHS22, Cor. 2.1.5] to the collection of inert lifts of $\mathcal{T}^{\mathrm{op}}$-maps to $\mathfrak{B}$ for the first functor, and taking $\mathcal{T}$-limits for the second functor.

Statement (4) follows from the fact that $\mathsf{Seg}_{\mathfrak{B}}(\underline{\mathsf{Cat}}_{\mathcal{T}}) \to \mathsf{Fbrs}(\mathfrak{B})$ is a conservative right adjoint; moreover, the functor $\mathsf{Seg}_{\mathfrak{B}}(\underline{\mathsf{Cat}}_{\mathcal{T}}) \to \mathsf{Cat}_{\mathfrak{B}^{\mathrm{el}}}$ is a conservative right adjoint, so it preserves and reflects monomorphisms as well, yielding the desired poset map. Thus for Statement (5), we're tasked with proving that

Claim i if Segal $\mathfrak{B}$-subcategories $\mathcal{O}, \mathcal{O}' \subset \mathcal{P}$ have $\mathcal{O}^{\mathrm{el}} \subset \mathcal{O}'^{\mathrm{el}}$, then $\mathcal{O} \subset \mathcal{O}'$, and

Claim ii the essential image is as described.

For Claim i, note that factorizations $\mathcal{O} \subset \mathcal{O}'$ in $\mathsf{Seg}_{\mathfrak{B}}(\underline{\mathsf{Cat}}_{\mathcal{T}})$ are detected in $\mathsf{Cat}^{\mathrm{cocart}}_{/\mathrm{Tot}\,\mathfrak{B}}$, where they are described as fiberwise factorization of subobjects; the Segal condition for colors reduces this to factorization of fibers over elementary objects, which by reversing this argument is equivalent to factorization $\mathcal{O}^{\mathrm{el}} \subset \mathcal{O}'^{\mathrm{el}}$. For Claim ii, our condition is certainly necessary (for the inclusion $\mathcal{O} \to \mathcal{P}$ to be a Segal $\mathfrak{B}$-functor), and the Segal condition for colors shows that it is sufficient.

Lastly, for Statement (6), since every cocartesian arrow of $\mathcal{P}$ with domain lying in $\mathcal{O}$ lies in $\mathcal{O}$, given a factorization $\mathcal{C} \xrightarrow{F} \mathcal{O} \xrightarrow{\iota} \mathcal{P}$, of lax Segal $\mathfrak{B}$-functors, F is a Segal $\mathfrak{B}$-functor if and only if $\iota \circ F$ is; hence the essential image in question is precisely those $\mathcal{C} \to \mathcal{O}$ which factor through $\mathcal{Q}$ as lax Segal $\mathfrak{B}$-functors, i.e. those satisfying the conditions of Statement (3). But the mapping space factorization is transported to a mapping space factorization for mapping spaces in a fiber over $\mathfrak{B}^{\mathrm{el,op}}$ by cocartesian transport in our case, so it is implied by the condition that F^{el} factors through $\mathcal{O}^{\mathrm{el}}$, yielding the essential image. The morphism description is automatic from Statement (3). □

I.3.5 The doctrinal adjunction for parametrized patterns For convenience, we set the notation

$$\underline{\mathsf{SEG}}^{\mathrm{lax}}_{\mathfrak{B}}\left(\underline{\mathsf{CAT}}_{\mathcal{T}}\right) \coloneqq \mathsf{EssIm}\left(\underline{\mathsf{SEG}}_{\mathfrak{B}}\left(\underline{\mathsf{Cat}}_{\mathcal{T}}\right) \hookrightarrow \underline{\mathsf{FBRS}}(\mathfrak{B})\right).$$

Now, note that the theory of Segal $\mathfrak{B}$-functors has a *vertical opposite* involution coming from a factorization

$$\begin{array}{ccc} \underline{\mathsf{SEG}}_{\mathfrak{B}}\left(\underline{\mathsf{CAT}}_{\mathcal{T}}\right) & \longhookrightarrow & \underline{\mathsf{FUN}}_{\mathcal{T}}\left(\mathfrak{B}, \underline{\mathsf{CAT}}_{\mathcal{T}}\right) \\ \downarrow{\scriptstyle \mathrm{op}} & & \downarrow{\scriptstyle \mathrm{op}_*} \\ \underline{\mathsf{SEG}}_{\mathfrak{B}}\left(\underline{\mathsf{CAT}}_{\mathcal{T}}\right) & \longhookrightarrow & \underline{\mathsf{FUN}}_{\mathcal{T}}\left(\mathfrak{B}, \underline{\mathsf{CAT}}_{\mathcal{T}}\right) \end{array}$$

We likewise define the *oplax* $\mathcal{T}$*-*$(\infty, 2)$*-category* For convenience, we set the notation

$$\underline{\mathsf{SEG}}^{\mathrm{oplax}}_{\mathfrak{B}}\left(\underline{\mathsf{CAT}}_{\mathcal{T}}\right) \coloneqq \mathsf{EssIm}\left(\underline{\mathsf{SEG}}_{\mathfrak{B}}\left(\underline{\mathsf{CAT}}_{\mathcal{T}}\right) \xrightarrow{\mathrm{op}} \underline{\mathsf{SEG}}_{\mathfrak{B}}\left(\underline{\mathsf{CAT}}_{\mathcal{T}}\right) \hookrightarrow \underline{\mathsf{FBRS}}(\mathfrak{B})\right).$$

Given $F\colon \mathcal{C} \to \mathcal{D}$ a lax Segal $\mathfrak{B}$-functor $f\colon P \to O$ an active V-arrow in $\mathfrak{B}$, and $X \in \mathcal{C}_P$ a P-object in $\mathcal{C}$, we refer to the dashed arrow

$$\begin{array}{ccc} FX & & \\ {\scriptstyle c_{f,FX}}\downarrow & \searrow^{Fc_{f,X}} & \\ f_{\otimes}F_X & \xrightarrow{\ell_{F,f,X}} & F f_{\otimes} X \end{array}$$

supplied by the universal property for cocartesian arrows as the *f-indexed lax structure map of F at X*; here, $f_\otimes$ is cocartesian transport along f and $c_{f,Y}$ is the cocartesian arrow lifting f with domain Y. Note that the $\mathcal{T}$-$(\infty,2)$-functor

$$\underline{\mathrm{SEG}}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right) \longrightarrow \underline{\mathrm{SEG}}^{\mathrm{lax}}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right)$$

is essentially surjective and locally fully faithful, with image consisting of the lax Segal $\mathfrak{B}$-functors such that $\ell_{F,f,X}$ is an equivalence for all f, X. Likewise, an oplax Segal $\mathfrak{B}$-functor $F\colon \mathfrak{C} \to \mathfrak{D}$ is a Segal $\mathfrak{B}$-functor if and only if the f-indexed oplax structure map of F at X, $o_{F,f,X}\colon Ff_\otimes X \longrightarrow f_\otimes FX$, is an equivalence for all f, X.

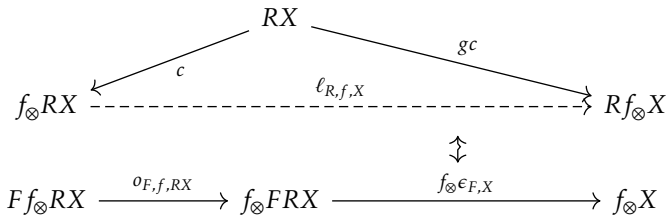
We refer to adjunctions in $\mathrm{SEG}^{\mathrm{lax}}_{\mathfrak{B}}(\underline{\mathrm{CAT}}_{\mathcal{T}})$ as *lax Segal $\mathfrak{B}$-adjunctions*, and oplax Segal $\mathfrak{B}$-adjunctions and Segal $\mathfrak{B}$-adjunction similarly. These occur quite frequently:

Proposition I.3.56 (Doctrinal adjunction for parameterized patterns)**.** *Suppose $F\colon \mathfrak{D} \to \mathfrak{C}$ is an oplax Segal $\mathfrak{B}$-functor between Segal $\mathfrak{B}$-∞-categories whose underlying $\mathfrak{B}^{\mathrm{el,op}}$-functor F map admits an $\mathfrak{B}^{\mathrm{el,op}}$-right adjoint R^{el}. Then, there is a $\mathfrak{B}$-relative right adjoint $R \vdash F$ lifting R^{el} which is oplax, and such that*

(a) For all $O \in \mathfrak{O}$, the following commutes;

$$\begin{array}{ccc} \mathfrak{C}_O & \xrightarrow{\quad R_O \quad} & \mathfrak{D}_O \\ \wr\downarrow & & \wr\downarrow \\ \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{/O}} \mathfrak{C}_E & \xrightarrow[\lim_E R^{\mathrm{el}}_E]{} & \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{/O}} \mathfrak{D}_E \end{array}$$

(b) given $f\colon P \to O$ in $\mathfrak{O}^{\mathrm{act}}$, the f-indexed lax structure map of R at $X \in \mathfrak{C}_P$ is an adjunct arrow



where $\epsilon_{F,X}$ is the counit of the adjunction $R_P \vdash F_P$ and $o_{F,f,RX}$ is the oplax structure map of F.

Moreover, if F is a Segal $\mathfrak{B}$ functor then this is a lax Segal $\mathfrak{B}$-adjunction, and if R is also a Segal $\mathfrak{B}$-functor, then this is a Segal $\mathfrak{B}$-adjunction.

Proof. Again, since $\mathrm{SEG}^{\mathrm{lax}}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right) \to \mathrm{CAT}^{\mathrm{int-cocart}}_{/\mathrm{Tot}\,\mathfrak{B}} \hookrightarrow \mathrm{CAT}_{/\mathrm{Tot}\,\mathfrak{B}}$ is full and locally fully-faithful it suffices to construct a $\mathfrak{B}$-relative right adjoint to F which preserves inert-cocartesian arrows. Now, our formula from condition (a) does supply fiberwise right adjoints with unit given by $\lim_E \eta_E$; these units do lie in the fiber, yielding a relative right adjoint satisfying (a). Moreover, unwinding the definition shows that this is compatible with *inert*-cocartesian transport, i.e. it is a lax Segal $\mathfrak{B}$-functor.

To prove condition (b), it suffices to note that the specified mate indeed makes the triangle diagram commute; the universal property for cocartesian arrows then constructs a homotopy between $\ell_{R,f,X}$ and the adjunct to $f_{\otimes}, \epsilon_{F,X}$.

The remaining properties follow by the fact that the inclusions

$$\mathrm{FUN}_{\mathcal{T}}\left(\mathfrak{B}, \underline{\mathrm{CAT}}_{\mathcal{T}}\right) \subset \mathrm{SEG}^{\mathrm{lax}}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right), \mathrm{SEG}^{\mathrm{oplax}}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right) \subset \mathrm{SEG}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right)$$

are locally fully faithful, so lifting adjunctions along them is equivalent to lifting each adjoint. □

The oplax case follows immediately by applying op. We acquire the following.

Corollary I.3.57. *If $\iota\colon \mathcal{C} \hookrightarrow \mathcal{D}$ is a full Segal $\mathfrak{B}$-subcategory, which is localizing as a $\mathfrak{B}^{\mathrm{el,op}}$-category, then its localization $\mathfrak{B}^{\mathrm{el,op}}$-functor admits an essentially unique oplax Segal $\mathfrak{B}$-functor making it left adjoint to ι.*

I.3.6 Localizing parameterized Segal subcategories Often, full Segal $\mathfrak{B}$-subcategories come equipped with a *localization* adjoint, in which case they have an approachable mapping-out property.

Theorem I.3.58. *Suppose $\mathfrak{P}$ is a Segal $\mathfrak{B}$-∞-category and $\varphi\colon \mathcal{O} \hookrightarrow \mathfrak{P}$ a fibrous full $\mathfrak{B}$-subpattern inclusion. Then, the following conditions are equivalent:*

(a) $\mathcal{O}^{\mathrm{el}} \subset \mathfrak{P}^{\mathrm{el}}$ admits a $\mathfrak{B}^{\mathrm{el,op}}$-left adjoint L^{el} such that, for all morphisms f in $\lim_{F \in \mathfrak{P}^{\mathrm{el}}_{X/}} \mathcal{O}_F$ whose component f_F is an L^{el}-equivalence for all F and active $\mathfrak{B}$-maps $\omega\colon X \to E$ with elementary codomain, $\omega_{\otimes} f$ is an L^{el}-equivalence.

(b) $\mathfrak{P}$ admits a Segal $\mathfrak{B}$-∞-category structure and a Segal $\mathfrak{B}$-functor such that $L\colon \mathfrak{P} \to \mathcal{O}$ which is left adjoint to φ in $\mathrm{FBRS}(\mathfrak{B})$.

In this case, the image of the inclusions

$$L^*\colon \underline{\mathrm{Alg}}_{\mathcal{O}/\mathfrak{B}}(\mathcal{C}) \hookrightarrow \underline{\mathrm{Alg}}_{\mathfrak{P}/\mathfrak{B}}(\mathfrak{C})$$

$$L^*\colon \underline{\mathrm{Fun}}^{\otimes}_{\mathfrak{B}}(\mathcal{O}, \mathfrak{C}) \hookrightarrow \underline{\mathrm{Fun}}^{\otimes}_{\mathfrak{B}}(\mathfrak{P}, \mathfrak{C})$$

is those $\mathfrak{P}$-algebras (resp. Segal $\mathfrak{B}$-functor maps) $\mathfrak{P} \to \mathfrak{C}$ whose underlying $\mathfrak{B}^{\mathrm{el,op}}$-functor factors through L^{el}. If φ is additionally a Segal $\mathfrak{B}$-subcategory inclusion, then the induced structure on $\mathcal{O}$ agrees with the localization structure, and this adjunction lifts to $\mathrm{SEG}_{\mathfrak{B}}(\underline{\mathrm{CAT}}_{\mathcal{T}})$.

We refer to Segal $\mathfrak{B}$-functors L arising in this way as *Segal $\mathfrak{B}$-localizations*, and when φ satisfies these conditions and is a full Segal $\mathfrak{B}$-subcatory, we call $\mathcal{O}$ a *localizing Segal $\mathfrak{B}$-subcategory.*

Proof. First assume (b). Since $\mathrm{FBRS}(\mathfrak{B}) \to \mathrm{CAT}_{\mathcal{T},/\mathfrak{B}}$ is locally fully-faithful, this adjunction takes place equivalently in $\mathrm{CAT}_{\mathcal{T},/\mathfrak{B}}$, and under Corollary I.1.67, this is equivalent to a relative adjunction of ∞-categories

over $\mathrm{Tot}\,\mathcal{B}$. Applying this in reverse upgrades this to a relative $\mathcal{B}^{\mathrm{el},\mathrm{op}}$-adjunction over $\mathcal{B}$, which restricts to the desired parameterized adjunction on $\mathbf{el}$.

To finish proving (a), it suffices to prove that L-equivalences are closed under cocartesian transport; but this follows directly by the fact that L is compatible with cocartesian transport.

Now assume condition (a) holds. We first construct a $\mathrm{Tot}\,\mathcal{B}$-relative left adjoint $\mathrm{Tot}\,L$ to $\mathrm{Tot}\,\varphi$ over $\mathrm{Tot}\,\mathcal{B}$. Indeed in order to do so, applying the well known-variant to [HTT, Prop. 5.2.7.8], it suffices to prove for all $P \in \mathcal{P}$ that that there exists a localization map $f\colon P \to LP$, in the sense that $LP \in \mathcal{O}$ and this induces an equivalence

$$\mathrm{Map}_{\mathcal{O}}(LP, O) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{P}}(P, \varphi O) \tag{8}$$

for all $O \in \mathcal{O}$. Such maps exist when $P \in \mathcal{P}^{\mathrm{el}}$; moreover, the Segal condition for colors for $\mathcal{O}$ and $\mathcal{P}$ guarantee, that under the equivalence $\mathcal{P}_P \simeq \lim_{E \in \mathcal{P}^{\mathrm{el}}_{P/}} \mathcal{P}_E$, a diagram of localization maps corresponds with a single map such that Eq. (8) is an equivalence whenever $\pi O = \pi P$.

The Segal condition for multimorphisms in this case guarantees that to check Eq. (8), for all $O \in \mathcal{O}$, it suffices to check in the elementary case; using the universal property for cocartesian arrows, it then suffices to check given an arrow $\omega\colon P \to E$ that $\omega_! P \to \omega_! LP$ is an L-equivalence. Writing ω as a composition of an inert and active arrow, the inert case follows by assumption that L^{el} is a $\mathcal{B}^{\mathrm{el},\mathrm{op}}$-left adjoint, and the active case by the assumed closure property of L^{el}-equivalences under active cocartesian transport. Hence the $\mathrm{Tot}\,\mathcal{B}$-relative left adjoint $\mathrm{Tot}\,L \dashv \mathrm{Tot}\,\varphi$ exists.

Now, since L preserves fibers, L-equivalences are $\mathcal{B}$-trivial, so [HA, Lem. 2.2.1.11] (for trivial localization on $\mathcal{B}$) guarantees that $\mathrm{Tot}\,\mathcal{O} \to \mathrm{Tot}\,\mathcal{B}$ is a cocartesian fibration such that $\mathrm{Tot}\,L$ preserves cocartesian edges. That is, $\mathcal{O}$ is a Segal $\mathcal{B}$-∞-category and $\mathrm{Tot}\,L$ lifts to a $\mathcal{B}$-relative $\mathcal{T}$-left adjoint to $\mathrm{Tot}_{\mathcal{T}}\,L \dashv \mathrm{Tot}_{\mathcal{T}}\,\mathcal{C}$, which is additionally a Segal $\mathcal{B}$-functor. Since $\mathrm{FBRS}(\mathcal{B}) \to \mathrm{CAT}_{\mathcal{T},/\mathcal{B}}$ is locally fully-faithful, under Corollary I.1.67, this adjunction takes place in $\mathrm{FBRS}(\mathcal{B})$, i.e. we have shown (b).

To describe the essential images, note by $\mathcal{T}$-$(\infty,2)$-functoriality of pullback along the inclusion $\mathcal{B}^{\mathrm{el}} \hookrightarrow \mathcal{B}$ that the $\mathcal{B}^{\mathrm{el},\mathrm{op}}$-functor condition is necessary in both cases. We're tasked with proving that it is *sufficient*, i.e. that $\mathcal{P}$-algebras X satisfying this condition admit an equivalence $X \simeq L^*\varphi^* X$. Now, of course, we may pullback along the counit natural transformation $L\varphi \Longrightarrow \mathrm{id}$, and we claim that this is an equivalence if and only if X satisfies this factorization condition; to see this, simply note that this performs L^{el}-localization on each color.

Finally, since $\mathrm{SEG}_{\mathfrak{B}}(\underline{\mathrm{CAT}}_{\mathcal{T}}) \hookrightarrow \mathrm{FBRS}(\mathfrak{B})$ is locally fully faithful, the assumed lift of φ to $\mathrm{SEG}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right)$ lifts the entire adjunction. In particular, this exhibits φ as a split monomorphism in $\mathrm{SEG}_{\mathfrak{B}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right)$, so agreement with the Segal full $\mathfrak{B}$-subcategory structure follows by the classification of Corollary I.3.53. □

I.3.7 Boardman-Vogt tensor products of fibrous patterns

Definition I.3.59. A *magmatic $\mathcal{T}$-pattern* is the data of a soundly extendable $\mathcal{T}$-algebraic pattern $\mathfrak{B}$ together with a Segal morphism $\wedge\colon \mathfrak{B}\times\mathfrak{B} \to \mathfrak{B}$. ◁

Construction I.3.60. Let $(\mathfrak{B},\wedge)$ be a magmatic $\mathcal{T}$-pattern. Then, the $\mathfrak{B}$-*Boardman-Vogt tensor product* is the bifunctor $- \overset{\mathrm{BV}}{\otimes} -\colon \mathsf{Fbrs}(\mathfrak{B})\times\mathsf{Fbrs}(\mathfrak{B}) \to \mathsf{Fbrs}(\mathfrak{B})$ defined by

$$\mathfrak{O} \overset{\mathrm{BV}}{\otimes} \mathfrak{P} \coloneqq L_{\mathrm{Fbrs}}\left(\mathfrak{O}\times\mathfrak{P} \to \mathfrak{B}\times\mathfrak{B} \xrightarrow{\wedge} \mathfrak{B}\right).$$ ◁

We defined this in order to have a mapping out property with respect to the following construction.

Definition I.3.61. Let $(\mathfrak{B},\wedge)$ be a magmatic $\mathcal{T}$-pattern and $\mathfrak{O},\mathfrak{P},\mathfrak{Q}$ fibrous $\mathfrak{B}$-patterns. Then, a *bifunctor of fibrous $\mathfrak{B}$ patterns* $\mathfrak{O}\times\mathfrak{P}\to\mathfrak{Q}$ is a commutative diagram in $\mathsf{Cat}_{\mathcal{T}}$

$$\begin{array}{ccc} \mathfrak{O}\times\mathfrak{P} & \longrightarrow & \mathfrak{Q} \\ \downarrow & & \downarrow \\ \mathfrak{B}\times\mathfrak{B} & \xrightarrow{\wedge} & \mathfrak{B} \end{array}$$

whose top horizontal arrow lies in $\mathsf{AlgPatt}$,[11] where $\mathfrak{O}\times\mathfrak{P}\to\mathfrak{B}\times\mathfrak{B}$ is induced by the structure maps of $\mathfrak{O}$ and $\mathfrak{P}$. The collection of bifunctors fits into a full subcategory

$$\mathsf{BiFun}_{\mathfrak{B}}(\mathfrak{O},\mathfrak{P};\mathfrak{Q}) \subset \mathsf{Fun}(\Delta^1\times\Delta^1,\mathsf{Cat}).$$ ◁

Example I.3.62. Let $\mathfrak{O},\mathfrak{P}$ be fibrous $\mathfrak{B}$-patterns, and consider $\mathfrak{B}$ to be a fibrous $\mathfrak{B}$-pattern via the identity. Then, the ∞-category of bifunctors $\mathfrak{O}\times\mathfrak{P}\to\mathfrak{B}$ is contractible, as it is equivalent to composite arrows $\mathfrak{O}\times\mathfrak{P}\to\mathfrak{B}\times\mathfrak{B}\to\mathfrak{B}$. ◁

Observation I.3.63. There are natural equivalences

$$\begin{aligned} \mathsf{BiFun}_{\mathfrak{B}}(\mathfrak{O},\mathfrak{P};\mathfrak{Q}) &\simeq \mathsf{Fun}^{\mathrm{int-cocart}}_{\mathcal{T}_{/\mathfrak{B}\times\mathfrak{B}}}(\mathfrak{O}\times\mathfrak{P},\wedge^*\mathfrak{Q}) \\ &\simeq \mathsf{Fun}^{\mathrm{int-cocart}}_{\mathcal{T}_{/\mathfrak{B}}}(\wedge_!(\mathfrak{O}\times\mathfrak{P}),\mathfrak{Q}) \\ &\simeq \mathsf{Fun}^{\mathrm{int-cocart}}_{\mathcal{T}_{/\mathfrak{B}}}(\mathfrak{O}\overset{\mathrm{BV}}{\otimes}\mathfrak{P},\mathfrak{Q}). \end{aligned}$$ ◁

[11] The lift to AlgPatt is unique, since each structure map in an algebraic pattern is a replete subcategory inclusion, hence a monomorphism in Cat.

As in [BV73, Prop 2.19] and the variety of recontextualizations of their ideas (e.g. [HA; Wei11]), we recognize this as $\mathfrak{O}$-*algebras in* $\mathfrak{P}$-*algebras*, making $\overset{\mathrm{BV}}{\otimes}$ into a closed tensor product, using the following.

Construction I.3.64. Fix $(\mathfrak{B},\wedge)$ a magmatic $\mathcal{T}$-pattern, let $F\colon \mathfrak{O}\times\mathfrak{P}\to\mathfrak{Q}$ be a bifunctor of fibrous $\mathfrak{B}$-patterns, and let $\mathfrak{C}$ be a fibrous $\mathfrak{Q}$-pattern. We have a diagram

$$\mathfrak{O} \xleftarrow{p} \mathfrak{O}\times\mathfrak{P} \xrightarrow{F} \mathfrak{Q};$$

admitting push-pull adjunctions $p^*\dashv p_*$ and $L_{\mathrm{Fbrs}}F_!\dashv F^*$ on fibrous patterns, with compatible adjunctions on Segal objects by Propositions I.2.28 and I.2.29 and Observation I.2.64. We define the pattern

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{Q}}(\mathfrak{C}) \coloneqq p_*F^*\mathfrak{C} \in \mathrm{Fbrs}(\mathfrak{O});$$

this is the *fibrous* $\mathfrak{O}$-*pattern of* $\mathfrak{P}$-*algebras in* $\mathfrak{C}$ *over* $\mathfrak{Q}$. In most cases, we will have $\mathfrak{Q}=\mathfrak{O}=\mathfrak{B}$, in which case the information of a bifunctor $\mathfrak{B}\times\mathfrak{P}\to\mathfrak{B}$ is simply that of a fibrous $\mathfrak{B}$-pattern $\mathfrak{P}$ by Example I.3.62. In this case, we simply write

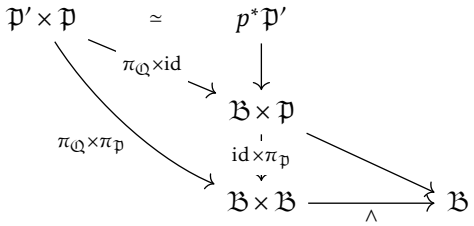
$$\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}}(\mathfrak{C}) \coloneqq \underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathfrak{B}}(\mathfrak{C}) \in \mathrm{Fbrs}(\mathfrak{B});$$

this is the *fibrous* $\mathfrak{B}$-*pattern of* $\mathfrak{P}$-*algebras in* $\mathfrak{C}$. ◁

In the case $\mathfrak{Q}=\mathfrak{O}=\mathfrak{B}$, the above diagram refines to

$$\mathfrak{B} \xleftarrow{p} \mathfrak{B}\times\mathfrak{P} \xrightarrow{\mathrm{id}\times\pi} \mathfrak{B}\times\mathfrak{B} \xrightarrow{\wedge} \mathfrak{B},$$

so the functor $\mathfrak{P}\mapsto\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}}(\mathfrak{C})$ has a left adjoint computed by $L_{\mathrm{Fbrs}}\wedge_!(\mathrm{id}\times\pi)_!p^*$; explicitly, this is computed on $\mathfrak{P}'$ by the fibrous localization of the diagonal composite



By definition, this is precisely $\mathfrak{P}'\otimes^{\mathrm{BV}}\mathfrak{P}$, so we've proved the following.

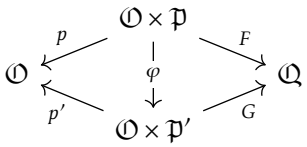
Proposition I.3.65. *The functor* $(-)\overset{BV}{\otimes}\mathfrak{O}\colon \mathrm{Fbrs}(\mathfrak{B})\to\mathrm{Fbrs}(\mathfrak{B})$ *is left adjoint to* $\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{O}}(-)$.

We additionally spell out a few useful characteristics $\overset{\mathrm{BV}}{\otimes}$ and $\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{O}}(-)$ here. First, functoriality.

Observation I.3.66. Fix the fibrous $\mathfrak{B}$-pattern $\mathfrak{Q}$. Suppose we have bifunctors of fibrous $\mathfrak{B}$-patterns

$$F\colon \mathfrak{O}\times\mathfrak{P}\longrightarrow\mathfrak{Q}\longleftarrow\mathfrak{O}\times\mathfrak{P}'\colon G$$

together with a morphism of fibrous $\mathfrak{B}$-patterns $\varphi\colon \mathfrak{P} \to \mathfrak{P}'$ making the following diagram commute:



The left triangle possesses a Beck-Chevalley transformation

$$p^* \varphi_! \implies \mathrm{id}_! \, p'^* = p'^*,$$

which possesses a mate natural transformation $p'_* \implies p_* \varphi^*$; precomposing with G^*, this yields a "pullback" natural transformation

$$\mathrm{Alg}^{\otimes}_{\mathfrak{P}'/\mathcal{Q}}(-) \implies \mathrm{Alg}^{\otimes}_{\mathfrak{P}/\mathcal{Q}}(-). \qquad \triangleleft$$

.We observe that, in all of the work above, we may have instead assumed that $\mathcal{C} \in \mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})$, in which case all of our constructions land in $\mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})$. Spelled out, this yields the following.

Proposition I.3.67. *Fix $\mathcal{O}, \mathfrak{P}, \mathcal{Q}, \mathcal{C}$ as in Construction I.3.64. Then*

(1) if $\mathcal{C}$ is a Segal $\mathcal{Q}$-∞-category, then $\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathcal{Q}}(\mathcal{C})$ is a Segal $\mathcal{O}$-∞-category;

(2) if $\mathcal{C} \to \mathcal{D}$ is a morphism of Segal $\mathcal{Q}$-∞-categories, then the induced map $\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathcal{Q}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathcal{Q}}(\mathcal{D})$ is a morphism of Segal $\mathcal{O}$-∞-categories, i.e. it preserves cocartesian arrows; and

(3) if $\mathfrak{P} \to \mathfrak{P}'$ is a morphism of fibrous $\mathfrak{B}$-patterns and $\mathcal{C}$ is a Segal $\mathcal{Q}$-∞-category, then the induced map of fibrous patterns

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}'/\mathcal{Q}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathfrak{P}/\mathcal{Q}}(\mathcal{C})$$

is a functor of Segal $\mathcal{O}$-∞-categories.

In analogy to [BS24a] we show that this tensor product is compatible with Segal envelopes.

Proposition I.3.68. *The following diagram commutes*

$$\begin{array}{ccccc} \mathrm{Fbrs}(\mathfrak{B})^2 & & \xrightarrow{\overset{BV}{\otimes}} & & \mathrm{Fbrs}(\mathfrak{B}) \\ \downarrow{\scriptstyle \mathrm{Env}} & & & & \downarrow{\scriptstyle \mathrm{Env}} \\ \mathrm{Fun}_{\mathcal{T}}(\mathfrak{B}, \underline{\mathrm{Cat}}_{\mathcal{T}})^2 & \xrightarrow{\circledast} & \mathrm{Fun}_{\mathcal{T}}(\mathfrak{B}, \underline{\mathrm{Cat}}) & \xrightarrow{L_{\mathrm{Seg}}} & \mathrm{Seg}_{\mathfrak{B}}(\underline{\mathrm{Cat}}_{\mathcal{T}}) \end{array}$$

PROOF. Fix $\mathfrak{C}$ a Segal $\mathfrak{B}$-∞-category. Then, there are natural equivalences

$$\begin{aligned}
\mathrm{Fun}_{\mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})}\left(\mathrm{Env}\left(\mathcal{O} \overset{\mathrm{BV}}{\otimes} \mathcal{P}\right), \mathcal{C}\right) &\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B}}\left(\wedge_! \mathcal{O} \times \mathcal{P}, \mathcal{C}\right) \\
&\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathfrak{B}\times\mathfrak{B}}\left(\mathcal{O} \times \mathcal{P}, \wedge^* \mathcal{C}\right) \\
&\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{B}\times\mathfrak{B}}\left(\mathrm{Env}_{\mathfrak{B}\times\mathfrak{B}}(\mathcal{O} \times \mathcal{P}), \wedge^* \mathcal{C}\right) \\
&\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{B}\times\mathfrak{B}}\left(\mathrm{Env}_{\mathfrak{B}}(\mathcal{O}) \times \mathrm{Env}_{\mathfrak{B}}(\mathcal{P}), \wedge^* \mathcal{C}\right) \qquad (9) \\
&\simeq \mathrm{Fun}^{\mathrm{cocart}}_{/\mathfrak{B}}\left(L_{\mathrm{Seg}} \wedge_! \left(\mathrm{Env}_{\mathfrak{B}}(\mathcal{O}) \times \mathrm{Env}_{\mathfrak{B}}(\mathcal{P})\right), \mathcal{C}\right) \\
&\simeq \mathrm{Fun}_{\mathrm{Seg}_{\mathfrak{B}}(\mathrm{Cat})}\left(L_{\mathrm{Seg}}\left(\mathrm{Env}_{\mathfrak{B}}(\mathcal{O}) \circledast \mathrm{Env}_{\mathfrak{B}}(\mathcal{P})\right), \mathcal{C}\right) \qquad (10)
\end{aligned}$$

Equivalence Eq. (9) is Observation I.2.36; Eq. (10) follows by symmetric monoidality of the Grothendieck construction [Ram22, Thm B]. The result then follows by Yoneda's lemma. □

We derive a uniqueness statement for $\overset{\mathrm{BV}}{\otimes}$ by an analogous argument to [BS24a].

Corollary I.3.69. *$\overset{BV}{\otimes}$ is the unique bifunctor on $\mathrm{Fbrs}(\mathfrak{B})$ making the following diagram commute:*

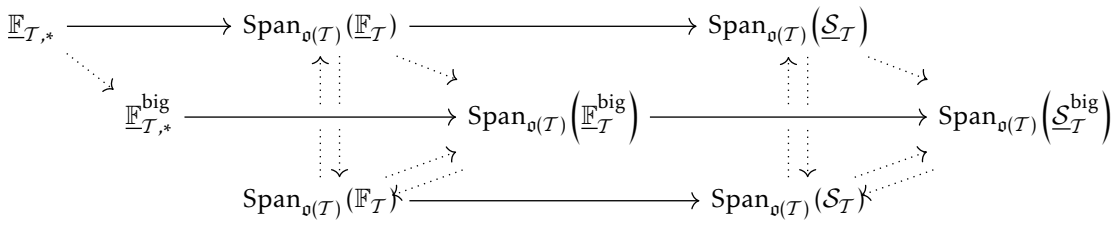
$$\begin{array}{ccccccc}
\mathrm{Fbrs}(\mathfrak{B})^2 & & \xrightarrow{\qquad\qquad \overset{BV}{\otimes} \qquad\qquad} & & & & \mathrm{Fbrs}(\mathfrak{B}) \\
{\scriptstyle \mathrm{Env}^2}\downarrow & & & & & & \downarrow{\scriptstyle \mathrm{Env}} \\
\left(\mathrm{Fun}(\mathfrak{B}, \mathrm{Cat})_{/\mathscr{A}_{\mathfrak{B}}}\right)^2 & \xrightarrow[\circledast]{} & \mathrm{Fun}_{\mathcal{T}}(\mathfrak{B}, \underline{\mathrm{Cat}}_{\mathcal{T}})_{/\mathscr{A}_{\mathfrak{B}}\circledast\mathscr{A}_{\mathfrak{B}}} & \xrightarrow[\mathrm{Env}(\wedge)_!]{} & \mathrm{Fun}_{\mathcal{T}}(\mathfrak{B}, \underline{\mathrm{Cat}}_{\mathcal{T}})_{/\mathscr{A}_{\mathfrak{B}}} & \xrightarrow[L_{\mathrm{Seg}}]{} & \mathrm{Seg}(\mathfrak{B})_{/\mathscr{A}_{\mathfrak{B}}}
\end{array}$$

PROOF. Commutativity of the diagram follows by Proposition I.3.67 and uniqueness of $\overset{\mathrm{BV}}{\otimes}$ follows from the fact that the right vertical functor is fully faithful, hence a monomorphism in Cat. □

I.4 The algebraic patterns of equivariant homotopy theory

At this point, we are finally ready to specialize to our distinguished theory of *parameterized higher algebra*. From here, pattern Morita theory awards us with as many perspectives as elements we can construct in the graph of *patterns modeling $\mathcal{T}$-operads and strong Morita equivalences*. We draw an expansive graph:

(11)



Theorem I.4.1. *All of the algebraic patterns in Eq. (11) are soundly extendable, and all of the morphisms are strong Morita equivalences after applying* Tot*; moreover, the top row consists of strong $\mathcal{T}$-Morita equivalences and the middle row strong $\mathbb{F}_{\mathcal{T}}$-Morita equivalences.*

The pattern $\mathrm{Tot}\,\underline{\mathbb{F}}_{\mathcal{T},*}$ describes $\mathcal{T}$-operads in the sense of [NS22] in the case that $\mathcal{T}$ itself is atomic orbital, and the leftmost diagonal arrow totalizing to a Morita equivalence is due to Nardin-Shah in that case. The composite Morita equivalence $\mathrm{Tot}\,\underline{\mathbb{F}}_{\mathcal{T},*} \to \mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$ is due to [BHS22] in the case $\mathcal{T} = \mathcal{O}_G$ for G finite. Now, we will see the following corollaries.

Corollary I.4.2. *The pullback natural transformations*

$$\underline{\mathrm{Fun}}^{I-\times}_{\mathcal{T}}\left(\mathrm{Span}_I\left(\underline{\mathbb{F}}_{\mathcal{T}}\right),-\right) \longrightarrow \underline{\mathrm{Fun}}^{I-\oplus}_{\mathcal{T}}\left(\underline{\mathbb{F}}_{\mathcal{T},*},-\right)$$

are natural equivalences; hence $\underline{\mathrm{CMon}}_I(\mathcal{C})$ *is I-semiadditive.*

On one hand, this cements I-commutative monoids as $\mathcal{T}$-parameterized sheaves with I-admissible transfers, and on the other, as an I-indexed and $\mathcal{T}$-parameterized version of *special* Γ*-objects*.

We prove a number of general strong Morita equivalences in Section I.4.1, preparing for Theorem I.4.1; then, in Section I.4.2 we pause to characterize fibrous subpatterns of span patterns, giving a significant generalization of Blumberg-Hill's $\mathcal{N}_\infty$-operads [BH20], Nardin-Shah and the author's generalizations [NS22; Ste25b], and Barrero's global $\mathcal{N}_\infty$-operads [Bar23b]. We finally prove Theorem I.4.1 in Section I.4.3. Then, we describe modifications of these patterns for *generalized* $\mathcal{T}$-operads in Section I.4.4, ultimately describing generalized fibrous Segal conditions for that pattern, and acquiring generalized fibrous Segal conditions for $\mathcal{T}$-operads along the way.

I.4.1 Strong Morita equivalences between span patterns We now repeatedly apply Corollary I.3.31 to span $\mathcal{T}$-patterns, forming a general basis for Theorem I.4.1.

I.4.1.1 Fibers of span $\mathcal{T}$-patterns Often we may change base of parameterization via fibers.

Definition I.4.3. We say that an adequate $\mathcal{T}$-quadruple $(\mathcal{X},\mathcal{X}^b,\mathcal{X}^f;\mathcal{X}_0)$ *admits induction* if $\mathcal{X}$ admits transitive $\mathcal{T}$-colimits; we say that it *admits inert-induction* if moreover $\mathcal{X}^b \subset \mathcal{X}$ is closed under induction and *strongly admits inert-induction* if the mate $\mathrm{Ind}^V_U X \to Y$ to an inert map $X \to \mathrm{Res}^V_U Y$ is inert. We define *admits active-induction* similarly. ◁

Construction I.4.4. We define the *fiber* pattern structure on $\mathrm{Span}_{b,f}\left(\mathcal{X}^V\right)$ to have inert and active morphisms the elementary and active V-arrows, and elementary objects the *inductions* of elementary V-objects. ◁

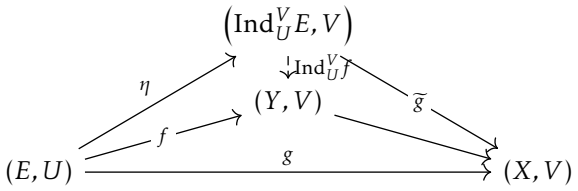
Proposition I.4.5. *If $(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f, \mathcal{X}_0)$ strongly admits inert-induction, then for all $V \in \mathcal{T}$, the inclusion*

$$\iota\colon \mathrm{Span}_{b,f}\left(\mathcal{X}^V\right) \to \mathrm{Tot}^V \mathrm{Span}_{b,f}(\underline{\mathcal{X}})$$

is a Morita equivalence; moreover, both patterns are soundly extendable under the assumptions of Conditions (i) to (iii), in which case ι is a strong Morita equivalence.

Proof. We verify the conditions of Corollary I.3.31 in parts, after which the claim will follow.

- To show that ι is a strong Segal morphism, we're tasked verifying that $\mathcal{X}^{b,V}_{0,/X} \to \left(\mathrm{Tot}^V \mathcal{X}^b\right)_{0,/\iota X}$ is final; by Joyal's Theorem A [kerodon, Tag 02NY] this is in turn equivalent to verifying for all maps $E \to \iota X$ in $\mathrm{Tot}^V \mathcal{X}^b$ with E elementary, the slice category $\mathcal{X}^{b,V}_{0,E//X}$ is weakly contractible. We claim that, if $E \in \mathcal{X}^{b,U}$, then $\mathrm{Ind}^V_U E \in \mathcal{X}^{b,V}_{0,E//X}$ is initial, i.e. $\mathrm{Ind}^V_U f$ is the unique map making the following diagram commute



  where $f\colon E \to \mathrm{Res}^V_U Y$ is a map, η is the unit of the induction adjunction, and $\widetilde{g}$ is the mate, which lies in $\mathcal{X}^b$ by the strong inert-induction assumption; indeed, existence and uniqueness of the factorization is simply the universal property for induction.
- Condition (ML-a) follows simply by noting that all active arrows in $\mathrm{Tot}^V \mathrm{Span}_{b,v}(\mathcal{X})$ have domain lying in $\mathrm{Span}_{b,v}\left(\mathcal{X}^V\right)$, so we acquire an equivalence on active arrow categories.
- Condition (ML-b) is obvious by construction. □

In the case that our adequate $\mathcal{T}$-quadruples come from self-indexing, we have a similar statement.

Proposition I.4.6. *Suppose $\left(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f; \mathcal{X}_0\right)$ admits inert-induction and active-induction, and moreover satisfies the property that each $\mathcal{X}^U$ has a terminal object such that the induced functor $\mathrm{Ind}^V_U \mathcal{X}^U \to \mathcal{X}^V_{/\mathrm{Ind}^V_{U^*}}$ is an equivalence. Then, there exists a retract $s\colon \mathrm{Tot}^V \mathrm{Span}_{b,f}(\mathcal{X}) \to \mathrm{Span}_{b,f}(\mathcal{X}^V)$ of ι which is a Morita equivalence; moreover, both patterns are soundly extendable under the assumptions of Conditions (i) to (iii) for s, in which case s is a strong Morita equivalence.*

Proof. Taking successive inductions yields an object $V \in \Gamma^{\mathcal{T}} \mathcal{X}$ for each V such that $\left(\Gamma^{\mathcal{T}} \mathcal{X}\right)_{/V} \simeq \mathcal{X}^V$ and such that induction is postcomposition, so that s can be chosen to be the unslicing functor. The assumption that $\mathcal{X}$ admits *active* induction guarantees that s is a map of algebraic patterns, which is automatically a

retract of ι in AlgPatt. Now, the strong Segal condition asks that $s\colon \mathrm{Tot}^V \mathcal{X}^{\otimes}_{0,/(X,U)} \to \mathcal{X}^{\otimes,V}_{0,/\mathrm{Ind}_U^V X}$ is final, i.e. $\left(\mathrm{Tot}^V \mathcal{X}^{\otimes}_{0,/(U,X)}\right) \times_{\mathcal{X}^{\otimes,V}_{0,/\mathrm{Ind}_U^V X}} \mathcal{X}^{\otimes,V}_{0,E//\mathrm{Ind}_U^V X}$ is weakly contractible. This may be viewed as the choices of f and lifts as the dashed arrow

$$\begin{array}{ccc} (F,W) & & \mathrm{Ind}_W^V F \dashleftarrow E \\ \downarrow f & \rightsquigarrow & \downarrow \swarrow \\ (X,U) & & \mathrm{Ind}_U^W X \end{array}$$

Of course, the map $E \to \mathrm{Ind}_U^W X$ exhibits an equivalence $E \simeq \mathrm{Ind}_U^W E'$, and it follows by unwinding definitions that $(E',U) \to (X,U)$ and the lift witnessing the universal property of induction yields an initial object in this ∞-category, giving the strong Segal condition.

The conditions of Corollary I.3.31 follow similarly to Proposition I.4.5. □

I.4.1.2 Concentration in certain isotropy We now study reparameterization over a full subcategory.

Construction I.4.7. Given $\mathcal{F} \subset \mathcal{T}$ a full subcategory, and $\mathfrak{B}$ a $\mathcal{T}$-pattern, we define the *modified totalization pattern* $\mathrm{Tot}^{\mathcal{T},\mathcal{F}-\mathrm{el}} \mathfrak{B}$ to have the same active and inert morphisms as $\mathrm{Tot}^{\mathcal{T}} \mathfrak{B}$ but only elementary objects living in fibers over $\mathcal{F}$; in particular, this is connected by a pattern map

$$\iota\colon \mathrm{Tot}^{\mathcal{F}} \mathfrak{B} \to \mathrm{Tot}^{\mathcal{T},\mathcal{F}-\mathrm{el}} \mathfrak{B}. \qquad \triangleleft$$

Corollary I.4.8. *Suppose $\mathcal{F} \subset \mathcal{T}$ is a full subcategory and $\left(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f; \mathcal{X}_0\right)$ an adequate $\mathcal{T}$-quadruple strongly admitting inert inductions such that $\mathcal{T}$ has a terminal object which is contained in $\mathcal{F}$. Then, the associated pattern map $\iota\colon \mathrm{Tot}^{\mathcal{F}} \mathrm{Span}_{b,f}(\mathcal{X}) \to \mathrm{Tot}^{\mathcal{T},\mathcal{F}-\mathrm{el}} \mathrm{Span}_{b,f}(\mathcal{X})$ is a Morita equivalence; moreover, both patterns are soundly extendable under the assumptions of Conditions (i) to (iii), in which case ι is a strong Morita equivalence.*

Proof. Once more this is just Corollary I.3.31.

- The condition that ι is a strong Segal morphism follows by cancellation of finality under two applications of the argument in Proposition I.4.5.
- Condition (ML-a) follows from the fact that active arrows are fiberwise, and
- Condition (ML-b) is obvious by construction. □

I.4.1.3 Redundant inert morphisms We now address change of inert morphisms.

Proposition I.4.9. *Fix $(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f; \mathcal{X}_0)$ is an adequate $\mathcal{T}$-quadruple and $\mathcal{X}^{b'} \hookrightarrow \mathcal{X}^b$ a wide $\mathcal{T}$-subcategory s.t.*

(a) $\mathcal{X}_0^{b'} = \mathcal{X}_0^b$, and

(b) $\mathcal{X}^{b'}_{0,/X} \hookrightarrow \mathcal{X}^b_{0,/X}$ is final.

Then, the associated pattern map $\iota\colon \mathsf{Span}_{b',f}(\mathcal{X}) \to \mathsf{Span}_{b,f}(\mathcal{X})$ *is a Morita equivalence. Moreover, both patterns are soundly extendable under the assumptions of Conditions (i) to (iii), in which case* ι *is a strong Morita equivalence.*

Proof. We again verify the conditions of Corollary I.3.31 for ι:

- Condition (b) is equivalent to the condition that ι is strong Segal,
- Condition (ML-a) is automatic, and
- Condition (ML-b) is Condition (a). □

For instance, we acquire a lift of a proposition of Harpaz concerning m-commutative monoids.

Corollary I.4.10. *The inclusion* $\mathsf{Span}_{m-1}(\mathcal{S}_m) \hookrightarrow \mathsf{Span}(\mathcal{S}_m)$ *of Example I.2.18 is a strong Morita equivalence, and both patterns soundly extendable; in particular, pullback yields a natural equivalence*

$$\mathsf{Fun}^{\mathcal{K}_m}(\mathsf{Span}(\mathcal{S}_m), -) \simeq \mathsf{CMon}_m(-)$$

Proof. Condition (a) is obvious and Conditions (b) and (ii) are verified in Corollary I.2.19. □

If the reader ever needs a theory of higher algebra whose corresponding theory of "symmetric monoidal categories" is m-commutative monoid ∞-categories, this shows that the two obvious definitions for such a theory agree. The reader should not interpret it as an interesting proof of the equivalence on Segal objects, as we have essentially used all of Harpaz's proof of this fact as input.

I.4.1.4 Restriction to spans over a full subcategory Lastly, we handle restriction to a full subcategory.

Proposition I.4.11. *Fix* $\left(\mathcal{X}, \mathcal{X}^b\mathcal{X}^f; \mathcal{X}_0\right)$ *an adequate* $\mathcal{T}$*-quadruple and* $\mathcal{X}_0 \subset \mathcal{Y} \subset \mathcal{X}$ *a full* $\mathcal{T}$*-subcategory s.t.*

(a) given a $\mathcal{X}^{f,V}$*-morphism* $X \to Y$ *with* $Y \in \mathcal{Y}^V$*, we have* $X \in \mathcal{Y}^V$*, and*

(b) $\mathcal{X}_0 \subset \mathcal{Y}$*.*

Then, writing $\mathcal{Y}^b = \mathcal{X}^b \cap \mathcal{Y}$ *and similar for* $\mathcal{Y}^f$ *and* $\mathcal{Y}_0 = \mathcal{X}_0$*, the adequate full* $\mathcal{T}$*-subquadruple inclusion* $\left(\mathcal{Y}, \mathcal{Y}^b, \mathcal{Y}^f; \mathcal{Y}_0\right) \subset \left(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f; \mathcal{X}_0\right)$ *induces a Morita equivalence*

$$\iota\colon \mathsf{Span}_{b,f}(\mathcal{Y}; \mathcal{Y}_0) \hookrightarrow \mathsf{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0).$$

Moreover, if $\mathsf{Span}_{b,f}(\mathcal{X}; \mathcal{X}_0)$ *is sound and one of the two patterns is extendable, then both are soundly extendable and* ι *is a strong Morita equivalence.*

Proof. This is once again Corollary I.3.31 and Condition (ii).

- The condition that $\mathcal{Y}_0 = \mathcal{X}_0$ and $\mathcal{Y} \subset \mathcal{X}$ is fully faithful implies that $\mathcal{Y}^b_{0,/Y} \hookrightarrow \mathcal{X}^b_{0,/\iota Y}$ is an equivalence, and hence ι is iso-Segal.

- Condition (ML-b) is equivalent to the condition that $\mathcal{X}_0 = \mathcal{Y}_0$, i.e. Condition (b)
- $\iota\colon \left(\mathcal{Y}^f_{/Y}\right)^{\simeq} \to \left(\mathcal{X}^f_{/\iota Y}\right)^{\simeq}$ fully faithful since $\mathcal{Y} \subset \mathcal{X}$ is fully faithful, and its essential surjectivity is equivalent to Condition (a). □

I.4.2 Weak $\mathcal{N}_\infty$-patterns Before specializing to the effective Burnside case, we point out that we may carry out the program for (weak) $\mathcal{N}_\infty$-patterns on the level of generality of span patterns such that $\mathcal{X}^b = \mathcal{X}$.

I.4.2.1 The classification of weak $\mathcal{N}_\infty$-patterns . Fix $\mathcal{X}$ an ∞-category.

Definition I.4.12. Given $I \subset \mathcal{X}$ a wide subcategory closed under base change and $\mathcal{X}_0 \subset \mathcal{X}$ a full subcategory. An $(I, \mathcal{X}_0)$-*weak indexing category* is a subcategory $J \subset I$ such that:

(a) $J \subset \mathcal{X}$ is closed under base change, and

(b) for all $X \in \mathcal{X}$, the following square is cartesian:

$$\begin{array}{ccc} J \times_I I_{/X} & \longrightarrow & \lim\limits_{\substack{I \\ E \to X \\ X \in \mathcal{X}_0}} J \times_I I_{/E} \\ \downarrow & \lrcorner & \downarrow \\ I_{/X} & \longrightarrow & \lim\limits_{\substack{I \\ E \to X \\ E \in \mathcal{X}_0}} I_{/E} \end{array} \tag{12}$$

We write $\mathsf{wIndexCat}_{(I,\mathcal{X}_0)} \subset \mathsf{Sub}(I)$ for the poset of $(I, \mathcal{X}_0)$-weak indexing categories and inclusions. ◁

Remark I.4.13. In Eq. (12), both vertical arrows are full subcategory inclusions; on the lefthand side this is the inclusion of the full subcategory of arrows $D \to X$ which lie in J (and arbitrary I-maps over X between them, and on the righthand side diagrams of J maps $D_E \to E$ which are compatible with base change along I-maps.

Pullback-stability of full subcategory inclusions reduces Eq. (12) to the following condition:

(b') Given an I-map $f\colon D \to X$ such that each base-changed map $g^*D \to E$ along an I-map $E \to X$ with $E \in \mathcal{X}_0$ lies in J, the map f itself lies in J. ◁

Remark I.4.14. We may view Eq. (12) as reminiscent of an I-relative version of the definition of weakly extensive span pairs of [CHLL24b, Def. 2.2.1]. In particular, if I satisfies the property that $I_{/X} \simeq \lim_{E \to C} I_{/X}$ always (i.e. $\mathsf{Span}_I(\mathcal{X}; \mathcal{X}_0)$ is soundly extendable) then Eq. (12) is further equivalent to the condition that $J \times_I I_{/X} \simeq \lim_{E \to X} J \times_I I_{/E}$. When I satisfies this property, we call it *soundly extendable*.

To make this analogy with weakly extensive span pairs precise, we need $\mathcal{X}$ to be freely generated under finite coproducts, i.e. it must be of the form $\mathbb{F}_{\mathcal{T}}$; in this case if $I \subset \mathbb{F}_{\mathcal{T}}$ is already weakly extensive, it follows by unwinding definitions and setting $\mathcal{X}_0 = \mathcal{T} \subset \mathbb{F}_{\mathcal{T}}$ that $(I, \mathcal{T})$-weak indexing categories which contain all objects of $\mathbb{F}_{\mathcal{T}}$ are precisely weakly extensive subcategories of $\mathbb{F}_{\mathcal{T}}$ contained within I.

As noted in Remark I.1.86, this additionally agrees with the poset of one-color weak indexing categories; by the usual argument, this implies whenever I is a $\mathcal{T}$-weak indexing category, that $\mathsf{wIndexCat}_I \simeq \mathsf{wIndexCat}_{\mathcal{T}_{/I}}$. In particular, choosing the terminal $\mathcal{T}$-weak indexing category $\mathfrak{o}(\mathcal{T})$, we pass the sanity check $\mathsf{wIndexCat}_{\mathfrak{o}(\mathcal{T})} \simeq \mathsf{wIndexCat}_{\mathcal{T}}$. ◁

As with its namesake [BH15], these classify subterminal fibrous patterns.

Theorem I.4.15. *Under the morphism* $\mathsf{Sub}_{\mathsf{Cat}}(I) \to \mathsf{Cat}^{\mathsf{int-cocart}}_{/\mathsf{Span}_I(\mathcal{X},\mathcal{X}_0)}$, *$J$ maps to a fibrous* $\mathsf{Span}_I(\mathcal{X})$*-pattern if and only if J is an $(I,\mathcal{X}_0)$-weak indexing category; moreover, the corresponding functor of $(\infty,2)$-categories*

$$\Lambda_{(-)^\infty} : \mathsf{wIndexCat}_{(I,\mathcal{X}_0)} \to \mathsf{FBRS}(\mathsf{Span}_I(\mathcal{C}))$$

is fully faithful with essential image spanned by the subterminal objects.

Proof. Note that the non-full subcategory inclusion

$$\mathsf{Fbrs}(\mathsf{Span}_I(\mathcal{X})) \subset \mathsf{Cat}_{/\mathsf{Span}_I(\mathcal{X})}$$

is compatible with limits, and in particular, it preserves and reflects monomorphisms and the terminal object. In particular, the subterminal objects of $\mathsf{Fbrs}(\mathsf{Span}_I(\mathcal{X}))$ are precisely the (-1)-truncated objects of $\mathsf{Cat}_{/\mathsf{Span}_I(\mathcal{X})}$ which are fibrous patterns, i.e. the fibrous patterns $\mathfrak{O} \to \mathsf{Span}_I(\mathcal{C})$ which are also subcategory inclusions.

Let $\mathfrak{c}(\mathfrak{X}) \subset \mathcal{C}$ be the full subcategory corresponding with the objects in the essential image of $\pi_{\mathfrak{O}}$. By restriction to $\mathsf{Span}_{I\cap\mathfrak{c}(\mathfrak{O})}(\mathfrak{c}(\mathfrak{O}))$, it suffices to reduce to the case of monic essential surjections $\mathfrak{O} \to \mathsf{Span}_I(\mathcal{X})$, i.e. *wide* subcategory inclusions (henceforth called *one color* weak indexing categories).

Now, in a general ∞-category $\mathcal{D}$ with factorization system $\mathcal{D}^L, \mathcal{D}^R \subset \mathcal{D}$, intersecting with $\mathcal{D}^L$ and $\mathcal{D}^R$ yields a full sub-poset embedding

$$\mathsf{Sub}_{\mathsf{wide}}(\mathcal{D}) \hookrightarrow \mathsf{Sub}_{\mathsf{wide}}(\mathcal{D}^L) \times \mathsf{Sub}_{\mathsf{wide}}(\mathcal{D}^R)$$

with image those pairs of wide subcategories which are closed under taking alternating compositions of the left- and right-handed parts and then factorization. Now, restricting to wide subcategories of $\mathcal{D}$ containing $\mathcal{D}^L$, this condition simplifies: instead, it says that, for all R-maps in the wide subcategory f and L-maps g, the R-part of $f \circ g$ lies in the wide subcategory.

In our case, the result reduces to the following claims.

Claim i $\mathfrak{O}$ lies in $\mathsf{Cat}^{\mathsf{int-cocart}}_{/\mathsf{Span}_I(\mathcal{C})}$ if and only if it contains all backwards morphisms, i.e. it consists of a wide subcategory $\mathfrak{a}(\mathfrak{O}) \subset I$ closed under base change in $\mathcal{C}$;

Claim ii There exists a factorization $\mathcal{O} \to \mathcal{P} \to \mathsf{Span}_I(\mathcal{C})$ if and only if $\mathfrak{a}(\mathcal{O}) \subset \mathfrak{a}(\mathcal{P})$; and

Claim iii $\mathcal{O}$ is a fibrous $\mathsf{Span}_I(\mathcal{C})$-pattern if and only if $\mathfrak{a}(\mathcal{O})$ is an I-weak indexing system.

Claim i is obvious by virtue of the fact that *all* lifts of backwards maps in this setting are cocartesian; Claim ii is immediate. For Claim iii, it suffices to note that Eq. (12) is precisely the square stipulating that

$$\mathcal{O} \times_{\mathsf{Span}_I(\mathcal{X})} \mathscr{A}_{\mathsf{Span}_I(\mathcal{X})} \simeq J \times_I I_{/\bullet} \longrightarrow I_{/\bullet} \simeq \mathscr{A}_{\mathsf{Span}_I(\mathcal{X})}$$

is a relative Segal $\mathsf{Span}_I(\mathcal{X})$-$\infty$-category, which is precisely fibrousness by Remark I.2.21. □

I.4.2.2 Trivial fibrous patterns We now handle an important special case of this.

Proposition I.4.16. *Suppose $\mathcal{X}$ admits inductions and $\mathcal{X}_0, I \subset \mathcal{X}$ is soundly extendable such that $\mathcal{X}$ has one $\mathcal{T}$-object. Then, there is an initial one-color $\left(\mathcal{X}^{f,V}, (\mathcal{T}_{/V})^{\mathrm{op}}\right)$- weak indexing category and a factorization*

$$\begin{array}{ccc} \mathsf{Patt}^{cc}(\mathsf{Tot}\,\underline{V}) & \xrightarrow{\iota} & \mathsf{Tot}^V \mathsf{Span}_f(\mathcal{X}; \mathcal{X}_0) \\ {\scriptstyle M}\downarrow & & \uparrow{\scriptstyle M'} \\ \mathsf{Span}_{\mathrm{triv}}\left(\mathcal{X}^V; \mathsf{Tot}\,\underline{V}\right) & \xrightarrow[N]{} & \mathsf{Span}_f\left(\mathcal{X}^V; \mathsf{Tot}\,\underline{V}\right) \end{array} \tag{13}$$

such that M and M' are strong Morita equivalences and N is a weak $\mathcal{N}_\infty$-pattern inclusion.

Proof. The fact that the map M' is a strong Morita equivalence with soundly extendable domain is Proposition I.4.5. Now, a one-color $\left(\mathcal{X}^{f,V}, \mathsf{Tot}\,\underline{V}\right)$-weak indexing category is the data of a wide subcategory $I \subset \mathcal{X}^{f,V}$ such that

$$I_{/X} \simeq \lim_{E \to X} I_{/E};$$

since this condition holds for the terminal wide subcategory, it also holds for the core by conservativity of equivalences, and we may define $I^{\mathrm{triv}} := \mathcal{X}^{V,\simeq}$, yielding N (with soundly extendable domain by Proposition I.2.27). This is obviously initial among one-color weak indexing categories, so we're left with constructing a strong Morita equivalence M making the diagram commute.

By compatibility of M' and N with elementary morphisms, the morphism M classifying the elementary subcategory of $\mathsf{Span}_{\mathrm{triv}}\left(\mathcal{X}^V; \mathsf{Tot}\,\underline{V}\right)$ makes the diagram commute. We're left with verifying that M is a strong Morita equivalence. Now, it is presented by a subquadruple inclusion in the sense of Proposition I.4.11 along the full subcategory inclusion $\underline{V}^{\mathrm{op}} \hookrightarrow \mathcal{X}^V$, so we're tasked with verifying that, given an $I^{\mathrm{triv},V}$-morphism $X \to Y$ with $Y \in \mathcal{T}^{\mathrm{op}}$, $X \in \mathcal{T}^{\mathrm{op}}$; this is obvious since all such morphisms are equivalences. □

We internalize parameterized higher category theory within parameterized higher algebra.

Corollary I.4.17. *In the situation of Proposition I.4.16, writing* $U = \iota^* \colon \underline{\mathrm{FBRS}}\big(\mathrm{Span}_f(\mathcal{X};\mathcal{X}_0)\big) \longrightarrow \underline{\mathrm{CAT}}_{\mathcal{T}}$*, there exists a fully faithful* $\mathcal{T}$*-*$(\infty,2)$*-left adjoint*

$$\underline{\mathrm{CAT}}_{\mathcal{T}} \underset{U}{\overset{\mathrm{triv}(-)^{\otimes}}{\rightleftarrows}} \underline{\mathrm{FBRS}}\big(\mathrm{Span}_f(\mathcal{X};\mathcal{X}_0)\big) \qquad (\perp)$$

whose V*-essential image consists of the fibrous* $\mathrm{Span}_{\mathrm{triv}}\big(\mathcal{X}^V;\underline{V}^{\mathrm{op}}\big)$*-patterns.*

Proof. Since cores are restriction-stable, we may define a $\mathcal{T}$-subpattern $\mathrm{Span}_{\mathrm{triv}}(\mathcal{X};\mathcal{X}_0) \subset \mathrm{Span}_f(\mathcal{X};\mathcal{X}_0)$ whose active V-morphisms are only equivalences; note that, on V-fibers, this presents the functor N of Eq. (13). We acquire a diagram

$$\underline{\mathrm{CAT}}_{\mathcal{T}} \xleftarrow[U_{\mathrm{triv}}]{} \underline{\mathrm{FBRS}}\big(\mathrm{Span}_{\mathrm{triv}}(\mathcal{X};\mathcal{X}_0)\big) \underset{N^*}{\overset{N_!}{\rightleftarrows}} \underline{\mathrm{FBRS}}\big(\mathrm{Span}_f(\mathcal{X};\mathcal{X}_0)\big); \qquad (\perp)$$

under which $U \simeq U_{\mathrm{triv}} \circ N^*$. Thus it suffices to prove that U_{triv} is an equivalence; but this is detected fiberwise, and fiberwise it follows from Proposition I.4.16. □

I.4.3 Applications to the patterns for parameterized operads

I.4.3.1 Construction of the diagram Let $\underline{\mathcal{S}}_{\mathcal{T}}^{\mathrm{big}}$ denote the self-indexing $\mathbb{F}_{\mathcal{T}}$-∞-category of $\mathcal{S}_{\mathcal{T}}$; that is,

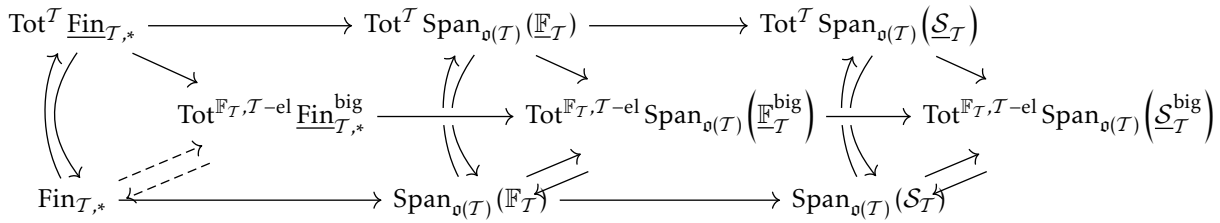
$$\Big(\underline{\mathcal{S}}_{\mathcal{T}}^{\mathrm{big}}\Big)^{S} \simeq \mathcal{S}_{\mathcal{T}/S},$$

with pullback functoriality. Defining $\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{big}}$ similarly, we acquire adequate $\mathcal{T}$-subquadruple inclusions

$$\Big(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{big}}, \text{s.i.}, \underline{\mathrm{Fin}}_{\mathcal{T}}^{\mathrm{big}}; *_{\mathcal{T}}\Big) \hookrightarrow \Big(\underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{big}}, \underline{\mathbb{F}}_{\mathcal{T}}^{\mathrm{big}}, \underline{\mathrm{Fin}}_{\mathcal{T}}^{\mathrm{big}}; *_{\mathcal{T}}\Big) \hookrightarrow \Big(\underline{\mathcal{S}}_{\mathcal{T}}^{\mathrm{big}}, \underline{\mathcal{S}}_{\mathcal{T}}^{\mathrm{big}}, \underline{\mathcal{S}}_{\mathfrak{o}(\mathcal{T})}^{\mathrm{big}}; *_{\mathcal{T}}\Big)$$

Now, the inclusion $\mathcal{T}_{/V} \hookrightarrow \mathbb{F}_{\mathcal{T},/V}$ satisfies the condition of Corollary I.4.8, yielding a diagram of patterns

(14)



We will prove that the solid arrows are strong Morita equivalences.

Warning I.4.18. The dashed arrows are emphatically not Morita equivalences; indeed applying two-out-of-three one is a morita equivalence if and only if the other is a morita equivalence, and as recognized in [BHS22,

Rmk. 5.2.17] in the case $\mathcal{T} = \mathcal{O}_G$, one should not *expect* the vertical one to be a Morita equivalence when $\mathcal{T}$ is not a space, as the vertical arrow fails to be an $\mathbf{el}$-equivalence in that case. ◁

I.4.3.2 Providing the equivalences We begin with enough soundness to propagate everywhere.

Lemma I.4.19. *Every algebraic pattern in the middle and righthand column of Eq. (14) is sound.*

This indeed follows immediately from a much more general claim.

Lemma I.4.20. *If $(\mathcal{X}, \mathcal{X}^b, \mathcal{X}^f; \mathcal{X}_0)$ is an adequate $\mathcal{T}$-quadruple such that $\mathcal{X}_b = \mathcal{X}$, then $\mathrm{Span}_f(\mathcal{X}; \mathcal{X}_0)$ is a sound $\mathcal{T}$-pattern.*

Proof. Letting $\mathrm{Tot}\,\mathcal{X}^\vee$ be the cartesian unstraightening of the cocartesian straightening of $\mathrm{Tot}\,\mathcal{X}$ (i.e. its *dual cartesian fibration*), note that $\mathrm{Tot}\,\mathrm{Span}_f(\mathcal{X}; \mathcal{X}_0)$ occurs as a span pattern

$$\mathrm{Tot}\,\mathrm{Span}_f(\mathcal{X}; \mathcal{X}_0) \simeq \mathrm{Span}_{all, tdeg \cap f}(\mathrm{Tot}\,\mathcal{X}; \mathrm{Tot}\,\mathcal{X}_0),$$

where all backwards morphisms are included and the forward morphisms are morphisms lying in the fibers over $\mathcal{T}$ and which are identified with $\mathcal{X}^{f,V}$-morphisms for some V. Soundness follows immediately from Proposition I.2.16. □

We will propagate extendability from one well-chosen example.

Proposition I.4.21. *$\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$ is a soundly extendable algebraic pattern.*

To prove this, we will need a more approachable sufficient set of generalized Segal conditions.

Lemma I.4.22 ([BHS22, Obs 5.2.9])**.** *Fix $S \in \mathbb{F}_{\mathcal{T}}$. Then, under the equivalence*

$$\left(\left(\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})\right)^{\mathbf{el}}_S\right)^{\mathrm{op}} \simeq \mathcal{T}^{\mathrm{op}} \times_{\mathrm{Fin}_{\mathcal{T}}} \mathrm{Fin}_{\mathcal{T}, /S}$$

the full subcategory of $\left(\mathcal{T} \times_{\mathcal{T}} \mathbb{F}^{\mathcal{P}}_{\mathcal{T}, /S}\right)^{\mathrm{op}}$ consisting of morphisms $f : T \to S$ such that f itself is the inclusion of an orbit is an initial subcategory equivalent to the set $\mathrm{Orb}(S)$.

Proof. The inclusion of the indicated full subcategory has a right adjoint given by taking images, so it is initial. □

Proof of Proposition I.4.21. It is sound by Lemma I.4.19. To see that it is extendable, it is equivalent to prove that $\mathscr{A}_{\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})}$ is a Segal $\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$-$\infty$-category, i.e. for every $S \in \mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$, the associated

functor φ of

$$\begin{array}{ccccc}
\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})^{\mathrm{act}}_{/S} & \xrightarrow{\sim} & \mathrm{Fin}_S & \xrightarrow{\sim} & \prod_{V\in \mathrm{Orb}(S)} \mathrm{Fin}_V \\
\downarrow & & \downarrow & \nearrow_{\varphi} & \\
\lim_{V\in \mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}_{S/}} \mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})^{\mathrm{act}}_{/V} & \xrightarrow{\sim} & \lim_{V\in \mathcal{T}\times_{\mathbb{F}_{\mathcal{T}}}\mathbb{F}_{\mathcal{T},/S}} \mathrm{Fin}_V & &
\end{array}$$

is an equivalence. In fact, it is an equivalence by Corollary I.4.35. □

Now, the following lemma simply follows by unwinding definitions.

Lemma I.4.23.

(1) The middle and righthand vertical arrows of Eq. (14) satisfy the conditions of Proposition I.4.5.

(2) The middle and righthand downward diagonal arrows of Eq. (14) satisfy the conditions of Corollary I.4.8 for the inclusion $\mathcal{T}_{/V} \hookrightarrow \mathbb{F}_{\mathcal{T},/V}$.

(3) The top left horizontal arrow of Eq. (14) satisfies the conditions of Proposition I.4.9.

(4) Every righthand horizontal arrow of Eq. (14) satisfies the conditions of Proposition I.4.11.

Finally, we're able to conclude.

Proof of Theorem I.4.1. After Lemma I.4.19, applying Lemma I.4.23 repeatedly together with Corollary I.3.31 shows that $\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\underline{\mathbb{F}}_{\mathcal{T}})$ is soundly extendable, then $\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}\left(\underline{\mathbb{F}}^{\mathrm{big}}_{\mathcal{T}}\right)$, then $\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}\left(\underline{\mathcal{S}}^{\mathrm{big}}_{\mathcal{T}}\right)$.

Now, all solid arrows of Eq. (14) are generated under two-out-of-three by arrows shown to satisfy the conditions of Corollary I.3.31 via Lemma I.4.23 under two-out-of-three; moreover, the top left horizontal arrow is iso-Segal, so its domain is soundly extendable, and as a consequence, every pattern connected by solid arrows is soundly extendable. Applying two-out-of-three to strong Segal equivalences then yields the claim. □

I.4.4 The algebraic pattern for generalized parameterized operads We now define a disintegrable $\mathcal{T}$-pattern describing a parameterized version of Lurie's *generalized ∞-operads* [HA, § 2.3].

Definition I.4.24. The $\mathcal{T}$-algebraic pattern $\underline{\mathbb{F}}^{\natural}_{I,*}$ has the same active and inert V-morphisms, but elementary V-objects $\{\varnothing_V, *_V\}$. ◁

The following proposition is immediate.

Proposition I.4.25. *$\underline{\mathrm{Fin}}^{\natural}_{\mathcal{T},*}$ is disintegrable and $\underline{\mathrm{Fin}}^{\flat}_{\mathcal{T},*} = \underline{\mathbb{F}}_{\mathcal{T},*}$.*

Given $\mathcal{O}^{\otimes} \in \mathrm{Fbrs}(\underline{\mathrm{Fin}}_{\mathcal{T},*})$, define $\mathrm{Tot}^{\natural}\mathcal{O}$ and $\mathrm{Tot}^{\flat}\mathcal{O}$ for its corresponding prefibrous $\mathcal{T}$-pattern structures over $\underline{\mathrm{Fin}}^{\natural}_{\mathcal{T},*}$ and $\underline{\mathrm{Fin}}_{\mathcal{T},*}$, so that $\mathrm{Tot}^{\flat}\mathcal{O} \simeq \mathrm{Tot}_{\mathcal{T}}\mathcal{O}$. We moreover acquire the following from Example I.3.46.

Corollary I.4.26. *$\mathrm{Tot}^{\natural}\mathcal{O}$ is disintegrable and $\left(\mathrm{Tot}^{\natural}\mathcal{O}\right)^{\flat} \simeq \mathrm{Tot}^{\flat}\mathcal{O}$.*

From here, we will be able to derive results for $\mathrm{Tot}^{\flat}\mathcal{O}$ by studying $\mathrm{Tot}^{\natural}\mathcal{O}$, which is independently useful.

I.4.4.1 Segal conditions for generalized $\mathcal{T}$-operads In particular, we will describe a sufficient set of fibrous generalized Segal conditions for it.

Example I.4.27. Given $V \in \mathcal{T}$, a pushout square of orbit-finite V-subspaces

$$\sigma: \quad \begin{array}{ccc} Q & \longrightarrow & R \\ \downarrow & \ulcorner & \downarrow \\ S & \longrightarrow & R \sqcup_Q S \end{array}$$

and an $R \sqcup_Q S$-color X, we acquire an essentially unique square diagram $\overline{p}_{R\sqcup_Q S,X}\colon \Delta^1 \times \Delta^1 \to \mathrm{Tot}^{\natural}\mathcal{O}$ lying over σ with beginning point X and all arrows cocartesian over the associated inert morphisms. In the case $\mathcal{O}$ has one color, we simply write $\overline{p}_{R\sqcup_Q S}$. ◁

Proposition I.4.28. *Fix $\mathcal{O}^{\otimes} \in \mathrm{Fbrs}\left(\underline{\mathrm{Fin}}^{\flat}_{\mathcal{T},*}\right)$.*

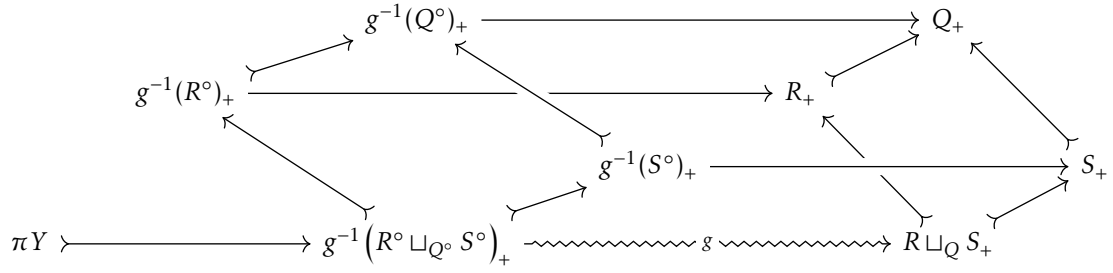
(1) $\left\{\overline{p}_{R\sqcup S,X}\right\}$ is a sufficient set of fibrous generalized $\mathrm{Tot}^{\natural}\mathcal{O}$-Segal conditions.

(2) $\left\{\overline{p}_{R\sqcup_Q S,X}\right\}$ is a sufficient set of fibrous generalized $\mathrm{Tot}^{\natural}\mathcal{O}$-Segal conditions.

Now, to remove pesky ω terms, we note the following lemma, which may be thought of by algebraic pattern theorists as a form of *soundness* for these conditions.

Lemma I.4.29. *The set $\left\{\overline{p}_{R\sqcup_Q S,X}\right\}$ is closed under $\omega_!(-)$.*

Proof. Fix an active arrow $\omega\colon Y \to X$. The entire inert-active factorization diagram for ω and $p_{R\sqcup_Q S,X}$ looks as follows on arity



The middle square is $\omega_!\overline{p}_{R\sqcup_S Q}$, and noting that $g^{-1}(Q) = g^{-1}(R) \cap g^{-1}(S)$, it is equivalent to $\overline{p}_{g^{-1}(R)\sqcup_{g^{-1}(Q)}g^{-1}(S)}$. In particular, we find that $\omega_!\overline{p}_{R\sqcup_Q S,X}$ is a cocartesian lift of some $\overline{p}_{R'\sqcup_{Q'}S'}$, which is what we wanted to show. □

This powers the following observation, which reduces us to the first case.

Observation I.4.30. In general, pullback pasting applied to the diagram

$$\begin{array}{ccccc} F(X_R) & \longleftarrow & F\left(X_Q\right) & \longleftarrow & F(X_\varnothing) \\ \downarrow & & \downarrow & & \downarrow \\ F\left(X_{S\sqcup_Q R}\right) & \longleftarrow & F(X_S) & \longleftarrow & F\left(X_{S-Q}\right) \end{array}$$

expresses a natural equivalence of coassembly maps

$$\begin{array}{ccc} F\left(X_{R\sqcup_\varnothing (S-Q)}\right) & \xrightarrow{c_{R\sqcup(S-Q),X}} & F(X_R)\times_{F(X_\varnothing)} F\left(X_{S-Q}\right) \\ \wr\downarrow & & \wr\downarrow \\ F\left(X_{R\sqcup_Q S}\right) \xrightarrow{c_{R\sqcup_Q S,X}} F(X_R)\times_{F(X_Q)} F(X_S) & \xrightarrow{\mathrm{id}\times_{\mathrm{id}} c_{Q\sqcup(S-Q),X}} & F(X_R)\times_{F(X_Q)} F\left(X_Q\right)\times_{F(X_\varnothing)} F\left(X_{S-Q}\right) \end{array}$$

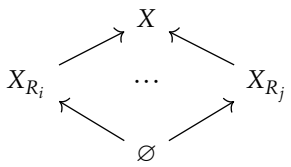
By $\omega_!$-closure, this covers *all* of the Segal conditions considered in the second part of Lemma I.2.48 and so by Lemma I.2.48 and the fact that the conditions in (1) are a *subset* of the conditions in (2), it suffices to prove (2) of Proposition I.4.28. ◁

Now, to characterize $\{p_{R\sqcup S}\}$, we will find it helpful to place it in a more general setting.

Construction I.4.31. Given a finite set I, an I-indexed tuple of finite V-sets (R_i), and a $\sqcup_i R_i$-multicolor X, we define the subdiagram

$$K^{\triangleleft}_{\sqcup_i R_i,X} \subset \mathrm{Tot}^{\natural}\mathcal{O}^{\mathrm{int}}_{/X}$$

to consist of the inert maps lying over either the identity, the inclusion $R_i \hookrightarrow R$ for some i, or the inclusion $\varnothing_H \hookrightarrow R$; that is, it is the $I^{\triangleleft}$-indexed diagram of inert maps



Let $\overline{p}_{\sqcup_i R_i,X}\colon K^{\triangleleft}_{\sqcup_i R_i,X} \to \mathrm{Tot}^{\natural}\mathcal{O}^{\mathrm{int}}_{/X}$ be the composite functor, so that $\overline{p}_{R\sqcup S,X}$ is as previously defined. ◁

Observation I.4.32. By the same argument as Lemma I.4.22, the inclusion $K_{\sqcup_i U_i,X} \hookrightarrow \mathrm{Tot}^{\natural}\mathcal{O}^{\mathrm{el}}_{/\sqcup_i U_i}$ is intial; by Remark I.2.50, this implies that $\left\{\overline{p}_{\sqcup_i U_i}\right\}$ is a sufficient set of generalized fibrous Segal conditions. ◁

Lemma I.4.33. *Each $p_{\sqcup_i R_i,X}$ is a generalized $\mathrm{Tot}^{\natural}\underline{\mathrm{Fin}}_{\mathcal{T},*}$-Segal condition.*

Proof. We induct on the weight $n = \sum_i \left(\left|(R_i)_G\right| - 1\right)$, with base case $n = 0$ covered by Observation I.4.32, so assume this is true for weight $< n$ for some $n > 0$, and assume that $\sum_{i=0}^k R_i$ has weight $n > 0$; since $n > 0$, some R_i has at least 2 orbits, so assume WLOG that's R_0 and choose a nontrivial decomposition $R_0 = S \sqcup P$.

We may apply Lemma I.2.48: pullback pasting yields an equivalence of coassembly arrows

$$\begin{array}{ccc} F\left(\coprod_{i=0}^{k} R_i\right) & \longrightarrow & F(R_0)\times_{F(\varnothing_H)}\cdots\times_{F(\varnothing_H)} F(R_k) \\ \wr\downarrow & & \wr\downarrow \\ F\left(S\sqcup P\sqcup\coprod_{i=1}^{k} R_i\right) & & F(S)\times_{F(\varnothing_H)} F(P)\times_{F(\varnothing_H)} F(R_1)\times_{F(\varnothing_H)}\cdots\times_{F(\varnothing_H)} F(R_k) \\ & \searrow \quad \nearrow & \\ & F(S)\times_{F(\varnothing_H)} F\left(P\sqcup\coprod_{i=1}^{k} R_i\right) & \end{array} \tag{15}$$

Each of the bottom arrows are limits of coassembly arrows for generalized fibrous Segal conditions by the inductive hypothesis, so the top arrow is as well. □

Finally, we're ready to conclude.

Proof of Proposition I.4.28. As observed in Observation I.4.30, it suffices to prove (2), i.e. we may assume $Q = \varnothing$. Lemma I.4.33 proves that this is a set of generalized fibrous Segal conditions, so we just need to prove it's sufficient. We will see this by running the argument of Lemma I.4.33 backwards.

To do so, we introduce auxiliary terminology; say that a set of generalized fibrous $\mathfrak{C}$-Segal conditions $\left\{\overline{p}_\alpha\right\}$ *generates* $\left\{q_\beta\right\}$ if finitely many applications of the first part of Lemma I.2.48 connect $\left\{\overline{p}_\alpha\right\}$ to $\left\{\overline{q}_\beta\right\}$.

Of course, statement (4) of Lemma I.2.48 shows that it suffices to prove that $\left\{p_{\sqcup_{i\in I} R_i, X} \,|\, |I|\leq n\right\}$ is generated by $\{p_{R\sqcup S,X}\}$ for all n, as the former set is sufficient by Observation I.4.32. We prove this inductively on n, with base case $n=2$ tautologically true and $n-1$ trivially true. Indeed, Eq. (15) directly generates an arbitrary $p_{\sqcup_{i\in I} R_i, X}$ with $i\in I$ by cases $p_{\sqcup_{i'\in I'} R'_{i'}, X'}$ with $|I'|<n$, completing the proof. □

I.4.4.2 Generalized segal conditions for $\mathcal{T}$-operads Propositions I.3.50 and I.4.28 imply the following.

Corollary I.4.34. *Fix $\mathcal{O}^{\otimes}\in \mathrm{Fbrs}\left(\underline{\mathrm{Fin}}^{\flat}_{\mathcal{T},*}\right)$. The following are sufficient sets of generalized fibrous $\mathrm{Tot}\,\mathcal{O}$-Segal conditions.*

(1) The diagrams $X_R\hookrightarrow X\hookleftarrow X_S$ for all $V\in\mathcal{T}$, $R,S\in\mathbb{F}_{\mathcal{T}}$, and $X\in\mathcal{O}_{R\sqcup S}$.

(2) The diagrams $X_{V_1},\dots,X_{V_n}\hookrightarrow X$ for all $V_i\in\mathcal{T}$ and $X\in\mathcal{O}_{\sqcup_i V_i}$.

We may additionally prove the following.

Corollary I.4.35. *Fix $\mathcal{C}$ a $\mathcal{T}$-∞-category and $\mathcal{D}$ an ∞-category.*

(1) There is an equality $\underline{\mathrm{Seg}}_{\underline{\mathbb{F}}_{I,*}}(\mathcal{C})\simeq \underline{\mathrm{Fun}}^{I-\oplus}_{\mathcal{T}}\left(\underline{\mathbb{F}}_{I,*},\mathcal{C}\right)$ of subcategories of $\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\underline{\mathbb{F}}_{I,*},\mathcal{C}\right)$.

(2) There is an equality $\mathrm{Seg}_{\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})}(\mathcal{D})\simeq \mathrm{Fun}^{\times}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}),\mathcal{D})$.

Proof. The second claim follows by Proposition I.4.21 under Theorem I.4.1, under the usual reduction of product-preservation to binary and nullary product-preservation and semiadditivity of $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$.

For the first claim, note analogously to Proposition I.4.21 that the inclusion

$$S \hookrightarrow \underline{\mathbb{F}}^{\flat,\mathrm{el}}_{I,*/S_+}$$

is initial, so these together form a sufficient set of *parameterized* generalized Segal conditions for $\underline{\mathbb{F}}^{\flat}_{I,*}$; the corresponding assembly map demands an equivalence

$$F(S_+) \xrightarrow{\sim} \prod_U^S F(U_+) \tag{16}$$

for all $S_+ \in \underline{\mathbb{F}}_{I,*}$. Noting that $S_+ \simeq \bigvee_U^S U_+$ is an I-indexed coproduct, this is certainly *implied* by I-semiadditivity; that it implies I-semiadditivity follows from a simple inductive argument using associativity of indexed coproducts (i.e. that $g_! f_! \simeq g f_!$ for maps of orbit-finite $\mathcal{T}$-spaces $T \xrightarrow{f} S \xrightarrow{g} R$). □

Proof of Corollary I.4.2. Combine Corollaries I.4.2 and I.4.35. □

I.4.4.3 Operadic coefficient systems We now note the following result.

Proposition I.4.36. *The map* $\varphi\colon \mathrm{Patt}^{cc}(\mathcal{T}^{\mathrm{op}}) \times \mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$ *factors essentially uniquely through a strong Morita equivalence*

$$\mathrm{Patt}^{cc}(\mathcal{T}^{\mathrm{op}}) \times \mathrm{Span}(\mathbb{F}) \xrightarrow{\mathrm{SME}} \mathrm{Span}_{I^\infty}(\mathbb{F}_{\mathcal{T}}),$$

where I^∞ *is the minimal indexing category.*

Proof. We already know that the codomain is soundly extendable by Theorem I.4.1, and it is easy to see that φ^{el} and $\left(\varphi^{\mathrm{act}}_{/O}\right)^{\simeq}$ are equivalences. Following Corollary I.3.31, we're left with verifying that φ is a strong Segal morphism, i.e. it induces an initial functor

$$\mathrm{Patt}^{cc}(\mathcal{T}^{\mathrm{op}})^{\mathrm{el}}_{/V} \times \mathrm{Span}(\mathbb{F})^{\mathrm{el}}_{/n} \longrightarrow \mathrm{Span}_{I^\infty}(\mathbb{F}_{\mathcal{T}})^{\mathrm{el}}_{/n\cdot V};$$

this becomes obvious after unwinding the definitions. □

Corollary I.4.37. *The forgetful functor*

$$\mathrm{CMon}_{I^\infty}\left(\mathrm{Coeff}^{\mathcal{T}}(\mathcal{D})\right) \longrightarrow \mathrm{Coeff}^{\mathcal{T}}\mathrm{CMon}(\mathcal{D})$$

is an equivalence.

Proof. Combine Corollary I.4.35 and Proposition I.4.36. □

I.4.4.4 Inflated I-operads From this, when I is an indexing category, we have a *fixed point* procedure

$$(-)^V : \underline{\mathrm{FBRS}}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})) \longrightarrow \underline{\mathrm{FBRS}}(\mathrm{Span}_{I^\infty}(\mathbb{F}_{\mathcal{T}})) \simeq \underline{\mathrm{COFR}^{\mathcal{T}}}(\mathrm{OP}) \xrightarrow{(-)^V} \mathrm{OP}.$$

In particular, taking a limit, this admits a fully faithful $\mathcal{T}$-$(\infty,2)$-left adjoint, by the following immediate corollary.

Corollary I.4.38. *Taking constant values yields exists a fully faithful $(\infty,2)$-left adjoint*

$$\mathrm{OP} \underset{\Gamma^{\mathcal{T}}}{\overset{\mathrm{Infl}_e^{\mathcal{T}}}{\rightleftarrows}} \mathrm{COFR}^{\mathcal{T}}(\mathrm{OP}) \simeq \mathrm{OP}_{I^\infty} \underset{\mathrm{Bor}}{\overset{E}{\rightleftarrows}} \mathrm{OP}_I$$

In particular, this yields a natural equivalence

$$\mathrm{Alg}_{\mathrm{Infl}_e^{\mathcal{T}}\mathcal{O}}(\mathcal{P}) \simeq \mathrm{Alg}_{\mathcal{O}}\left(\Gamma^{\mathcal{T}}\mathcal{P}\right).$$

Here, we've written $E \dashv \mathrm{Bor}$ for the push-pull adjunction along the monomorphism $\mathrm{Span}_{I^\infty}(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$. When $\mathcal{O}^\otimes$ is an operad, we will often write $\mathcal{O}^\otimes \coloneqq \mathrm{Infl}_e^{\mathcal{T}}\mathcal{O}^\otimes$ when it is not likely to cause confusion. This will allow us to import tools from higher algebra into our setting seamlessly.

I.4.4.5 The Boardman-Vogt tensor product We may now prove the following.

Proposition I.4.39. *The $\mathcal{T}$-algebraic pattern map*

$$\wedge \coloneqq \mathrm{Span}_{s.i.}(\times) \colon \underline{\mathrm{Fin}}_{\mathcal{T},*} \times \underline{\mathrm{Fin}}_{\mathcal{T},*} \longrightarrow \underline{\mathrm{Fin}}_{\mathcal{T},*}$$

is a Segal morphism.

Proof. We're tasked with proving that, given a $\mathfrak{o}(\mathcal{T})$-semiadditive $\mathcal{T}$-functor $F \colon \underline{\mathrm{Fin}}_{\mathcal{T},*} \to \mathcal{C}$, $\wedge^* F$ is $\mathfrak{o}(\mathcal{T})$-semiadditive separately in each variable. This follows by distributivity of products and $\mathfrak{o}(\mathcal{T})$-coproducts

in each $\mathbf{Fin}_V$; indeed, we have

$$\begin{aligned}
\wedge^* F\left(Y_+, \bigvee_U^S X_{U,+},\right) &\simeq F\left(Y \times \sqcup_U^S X_U\right)_+ \\
&\simeq F\left(\coprod_U^S \left(\mathrm{Res}_U^V Y \times X_U\right)\right)_+ \\
&\simeq F\left(\bigvee_U^S \left(\mathrm{Res}_U^V Y_+ \wedge X_{U,+}\right)\right) \\
&\simeq \prod_U^S F\left(\mathrm{Res}_U^V Y_+ \wedge X_{U,+}\right) \\
&\simeq \prod_U^S \wedge^* F\left(\mathrm{Res}_U^V Y_+, X_{U,+}\right). \qquad \square
\end{aligned}$$

Remark I.4.40. We've used crucially here that the $\mathcal{T}$-functor $(-)_+ \colon \underline{\mathbf{Fin}}^{\times}_{\mathcal{T}} \longrightarrow \underline{\mathbf{Fin}}^{\wedge}_{\mathcal{T},*}$ is symmetric monoidal. After extending this to $\underline{\mathcal{S}}_{\mathcal{T}} \to \underline{\mathcal{S}}_{\mathcal{T},*}$, we will show in Corollary II.3.69 that this uniquely characterizes the smash symmetric monoidal structure on $\mathcal{T}$-spaces, under which $\underline{\mathbf{Fin}}^{\wedge}_{\mathcal{T},*}$ will turn out to be a symmetric monoidal subcategory. ◁

This endows $\underline{\mathbb{F}}_{\mathcal{T},*}$ with the structure of a magmatic $\mathcal{T}$-algebraic pattern, yielding a Boardman-Vogt tensor product on fibrous patterns in the setting of any of the patterns of Theorem I.4.1.

Chapter II

Parameterized higher algebra

Throughout this section, we will carry out much of the program of *higher algebra*, e.g. as incarnated in [HA], in the context of *I-operads and I-symmetric monoidal ∞-categories*; that is, we study objects in $\underline{HA}(\mathfrak{B})$, for $\mathfrak{B}$ the I-admissible effective Burnside pattern, or any other elements of its strong Morita class via Theorem I.4.1.

Special to this strong Morita class, we will define an underlying *I-symmetric sequence* functor on one colored I-operads, as well as a highly-colored variant, valued in appropriately indexed diagrams of spaces; we note in Proposition II.1.29 that I-operad maps are equivalences if and only if they are essentially surjective on colors and induce an equivalence on appropriately colored I-symmetric sequences. That is, we recognize I-operad equivalences as those maps which are *fully faithful and essentially surjective*. As a result, we acquire a homotopy theory of I-operads which may be reduced conservatively to a homotopy theory of diagrams of spaces.

Using this, in the almost-unital case, we are able to construct I-symmetric monoidal categories and I-operads of immense categorical importance, called *cartesian and cocartesian structures*. For the former, we find that the strong Morita class of $\mathcal{T}$-for I-higher algebra *admits a cartesian structure for* $\mathcal{C}$ in the sense of Section I.3.2 whenever $\mathcal{C}$ admits I-indexed products, satisfying a strong rigidity result as those I-symmetric monoidal ∞-categories whose I-tensor products are I-products and a strong conservativity result that taking algebras in $\underline{\mathcal{S}}_{\mathcal{T}}^{\times}$ detects fully faithful I-operad maps.

As a consequence, we will enter a regime where questions about $\mathcal{O}$-algebras in arbitrary I-operads (or I-symmetric monoidal ∞-categories) may be expressed as writing a map $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ to some other I-operad and detecting that it is an equivalence, which is *equivalent* to φ being a Morita equivalence in the sense of Section I.3.2, and hence amenable to a Morita localization theory lifting Lurie-Harpaz-Barkan-Haugseng-Steinebrunner's approximation theory.

For the latter universal structure, we find an adjunction $U \dashv (-)^{\sqcup}$ of $\mathcal{T}$-$(\infty,2)$-categories in the unital case; a version of the *assembly* regime of [HA, § 2.3] follows. Moreover, we acquire a canonical lifts $\mathcal{C}^{\sqcup} \in \mathrm{Cat}_I^{\otimes}$

whenever $\mathcal{C}$ admits I-indexed *co*products, essentially uniquely so that the I-tensor products of $\mathcal{C}^{\sqcup}$ are I-coproducts. Consequentially, we find $\mathcal{N}_{I\infty}^{\otimes}$ intimately intertwined with I-semiadditivity.

Of course, the pattern $\mathsf{Span}_{\mathfrak{o}(\mathcal{T})}(\mathcal{S}_{\mathcal{T}})$ is magmatic via Proposition I.4.39, yielding an $\mathbb{A}_2$-structure on $\mathsf{Op}_{\mathcal{T}}$ via Boardman-Vogt tensor products as in Section I.3.7; it was shown in [Ste25b] that $\mathsf{CMon}_{\mathcal{T}}(\mathcal{C})$ is an $\mathbb{A}_2$-monoidal localization of $\mathsf{Fun}\big(\mathsf{Span}_{\mathfrak{o}(\mathcal{T})}(\mathcal{S}_{\mathcal{T}})\big)$ as well, and Env an $\mathbb{A}_2$-monoidal subcategory inclusion. Following [BS24a], we saw in [Ste25b] that $\underline{\mathsf{CMon}}_{\mathcal{T}}(\mathcal{S}) \in \mathrm{Pr}^L_{\mathcal{T}}$ is *idempotent* as an $\mathbb{E}_0$-algebra in the sense of [HA], and in particular, it admits an essentially unique distributive symmetric monoidal structure with $\mathcal{T}$-unit the free algebra on $*_{\mathcal{T}}$; we eventually find that the terminal $\mathcal{T}$-operad $\mathsf{Comm}^{\otimes}_{\mathcal{T}}$ is $\overset{\mathrm{BV}}{\otimes}$-idempotent, yielding a canonical presentably symmetric monoidal structure on $\mathsf{Cat}^{\otimes}_{\mathcal{T},/\mathscr{A}_{\mathcal{T}}}$ under which $\mathsf{Op}_{\mathcal{T}}$ is a symmetric monoidal full $\mathcal{T}$-subcategory. As a consequence, $\underline{\mathrm{Op}}_{\mathcal{T}}, \underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T}}$ come equipped with extremely rigid *compatible* presentably symmetric monoidal structures.

We take the perspective that the bulk of higher algebra outside of the theory of fibrous patterns may be studied from these perspectives: the cartesian regime and Boardman-Vogt tensor products create a rich landscape of universal mapping-out properties for I-operads which are detectable via tractable simple higher category theory.

This is made into a relatively complete theory when it is augmented by a theory of *distributivity and presentability*, which we study in Section II.3. The highlight of this is a normed version of the *mode theory* regime of [HA]; for instance, this allows us to prove existence and uniqueness of a well-behaved G-symmetric monoidal structure on G-spectra (with *finite-index* HHR norms), as well as its various localizations.

II.1 First characterizations

We begin by reviewing various corollaries of parameterized pattern structures and the work of Sections I.3 and I.4 in Section II.1.1. We follow this up by developing the *underlying $\mathcal{T}$-symmetric sequence* in Section II.1.2, showing one-color $\mathcal{T}$-operads to be monadic over $\mathcal{T}$-symmetric sequences. Absent coherence, we understand the monad explicitly; consequentially, in Section II.1.3 we review a *nerve functor* extending that of Bonventre and Nardin-Shah [Bon19; NS22] for explicitly constructing topological models of $\mathcal{T}$-operads. Last, in Section II.1.4 we review the Boardman-Vogt tensor product in this setting.

II.1.1 Parameterized operads and symmetric monoidal ∞-categories

II.1.1.1 Basic definitions We make the following definitions.

Definition II.1.1. The $\mathcal{T}$-$(\infty,2)$-category of $\mathcal{T}$-operads is

$$\underline{\mathrm{OP}}_{\mathcal{T}} \coloneqq \underline{\mathrm{FBRS}}\big(\mathsf{Span}_{\mathfrak{o}(\mathcal{T})}(\underline{\mathbb{F}}_{\mathcal{T}})\big).$$

More generally, when $I \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is a weak indexing category, the ∞-*category of* I-*operads* is

$$\underline{\mathrm{OP}}_I \coloneqq \underline{\mathrm{FBRS}}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}})).$$

In particular, we acquire a pair of forgetful ($\mathcal{T}$-)functors

$$\mathrm{Tot}\colon \mathrm{OP}_I \simeq \mathrm{FBRS}(\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}})) \longrightarrow \mathrm{CAT}_{/\mathrm{Span}_I(\underline{\mathbb{F}}_{\mathcal{T}})}$$

$$\mathrm{Tot}_{\mathcal{T}}\colon \underline{\mathrm{OP}}_I \simeq \underline{\mathrm{FBRS}}\left(\underline{\mathbb{F}}_{I,*}\right) \longrightarrow\!\!\!\!\!\!\rightarrow \underline{\mathrm{CAT}}_{\mathcal{T},/\underline{\mathbb{F}}_{I,*}}$$

Given $\mathcal{O}^{\otimes}$ an I-operad and $\mathcal{P},\mathcal{Q} \to \mathcal{O}^{\otimes}$ I-operad maps, we define the $\mathcal{O}$-*relative algebra* ($\mathcal{T}$-)∞-*category* as the associated mapping $\mathcal{T}$-∞-category in $\underline{\mathrm{Op}}_{\mathcal{T},/\mathcal{O}^{\otimes}}$, i.e.

$$\begin{aligned}
\underline{\mathrm{Alg}}_{\mathcal{P}/\mathcal{O}}(\mathcal{Q}) &\simeq \underline{\mathrm{Fun}}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes}, \mathrm{Tot}_{\mathcal{T}}\mathcal{P}^{\otimes}) \\
\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(\mathcal{Q}) &\simeq \Gamma^{\mathcal{T}}\underline{\mathrm{Alg}}_{\mathcal{P}/\mathcal{O}}(\mathcal{Q}) \\
&\simeq \mathrm{Fun}^{\mathrm{int-cocart}}_{/\mathrm{Tot}\,\mathcal{O}^{\otimes}}(\mathrm{Tot}\,\mathcal{P}^{\otimes}, \mathrm{Tot}\,\mathcal{P}^{\otimes})
\end{aligned}$$

If $\mathcal{O}^{\otimes}$ is an I-operad, then the $\mathcal{T}$-∞-*category of small* $\mathcal{O}$-*monoidal* ∞-*categories is*

$$\underline{\mathrm{CAT}}^{\otimes}_{\mathcal{O}} \coloneqq \underline{\mathrm{SEG}}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}}\left(\underline{\mathrm{CAT}}_{\mathcal{T}}\right),$$

so that $\mathrm{CAT}^{\otimes}_{\mathcal{O}} \coloneqq \Gamma^{\mathcal{T}}\underline{\mathrm{CAT}}^{\otimes}_{\mathcal{P}} \simeq \mathrm{SEG}_{\mathrm{Tot}\,\mathcal{O}}(\mathrm{CAT})$. If $\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}$ are $\mathcal{O}$-monoidal ∞-categories, then the ∞-*category of* $\mathcal{O}$-*monoidal functors from* $\mathcal{C}^{\otimes}$ *to* $\mathcal{D}^{\otimes}$ is the functor $\mathcal{T}$-∞-category in $\underline{\mathrm{CAT}}^{\otimes}_{\mathcal{O}}$, i.e.

$$\underline{\mathrm{Fun}}^{\otimes}_{\mathcal{O}}(\mathcal{C},\mathcal{D}) \simeq \underline{\mathrm{Fun}}^{\mathrm{cocart}}_{/\mathcal{O}^{\otimes}}\left(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}\right).$$

A *lax* $\mathcal{O}$-*symmetric monoidal functor* is a functor of I-operads $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $\mathcal{O}^{\otimes}$. ◁

Note that a lax I-symmetric monoidal functor is an I-symmetric monoidal functor if and only if it is a morphism in $\mathrm{Cat}^{\otimes}_I$, i.e. if and only if it preserves cocartesian lifts for arbitrary maps in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$.

We can recover the case of I-operads from the universal case:

Proposition II.1.2. *The pattern map* $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$ *is a fibrous pattern, and moreover outlines a fully faithful* $\mathcal{T}$-$(\infty,2)$-*functor*

$$\mathcal{N}^{\otimes}_{(-)\infty}\colon \underline{\mathrm{wIndexCat}}_{\mathcal{T}} \hookrightarrow \underline{\mathrm{OP}}_{\mathcal{T}}$$

with essential image the sub-terminal objects. In particular, we acquire an equality of colocalizing full $\mathcal{T}$-$(\infty,2)$-*subcategories*

$$\underline{\mathrm{OP}}_I \simeq \underline{\mathrm{OP}}_{\mathcal{T},/\mathcal{N}^{\otimes}_{I\infty}} \hookrightarrow \underline{\mathrm{OP}}_{\mathcal{T}}.$$

If $\mathcal{O}^\otimes \simeq \mathcal{N}_{I\infty}^\otimes$ arises from Proposition II.1.2, we say that $\mathcal{O}^\otimes$ is a *weak* $\mathcal{N}_\infty$ $\mathcal{T}$*-operad* (or simply a weak $\mathcal{N}_\infty$-operad), and if I is an indexing category, then we say that $\mathcal{N}_{I\infty}^\otimes$ is an $\mathcal{N}_\infty$*-operad*; in either case, we write

$$\underline{\mathrm{CAlg}}_I(\mathcal{C}) \coloneqq \underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}}(\mathcal{C})$$

for the $\mathcal{T}$-∞-category of I*-commutative algebras in* $\mathcal{C}$.

Proof of Proposition II.1.2. Since $\mathbb{F}_{\mathfrak{o}(\mathcal{T})} \subset \mathbb{F}_{\mathcal{T}}$ is extensive by Remark I.1.86, Remark I.4.14 identifies $(\mathbb{F}_{\mathfrak{o}(\mathcal{T})}, \mathcal{T}^{\mathrm{op}})$-weak indexing systems and $\mathcal{T}$-weak indexing systems. Theorem I.4.15 then shows fully faithfulness of $\mathcal{N}_{I\infty}^\otimes$ and computes the image; the remaining claim follows by unwinding definitions. □

We may also quickly *internalize* a naive notion of parameterized operads.

Proposition II.1.3. *There exists an equivalence of adjoint pairs of* $\mathcal{T}$*-*$(\infty,2)$*-categories*

$$\begin{array}{ccc} \underline{\mathrm{OP}}_{I^\infty} & \underset{\iota}{\overset{\mathrm{Env}}{\rightleftarrows}} & \underline{\mathrm{CAT}}_{I^\infty}^\otimes \\ \downarrow\sim & & \downarrow\sim \\ \underline{\mathrm{COEFF}}^{\mathcal{T}}(\mathrm{OP}) & \underset{\underline{\mathrm{COEFF}}^{\mathcal{T}}\iota}{\overset{\underline{\mathrm{COEFF}}^{\mathcal{T}}\mathrm{Env}}{\rightleftarrows}} & \underline{\mathrm{COEFF}}^{\mathcal{T}}\left(\mathrm{CAT}^\otimes\right) \end{array}$$

Proof. This is a straightforward combination of Proposition I.4.36 with Proposition I.3.36. □

Remark II.1.4. After Proposition II.1.3, one sees that it is specifically I^∞-operads that are described when one specializes Martini-Wolf's internal higher algebra [MW22, § 2.5] to presheaf toposes; consequentially, our work is strictly more general, and we may not cite their higher algebra directly to characterize ours. ◁

II.1.1.2 The structure of I*-operads* We now explicitly describe I-operads.

Construction II.1.5. Given $\pi_{\mathcal{O}} \colon \mathrm{Tot}\,\mathcal{O}^\otimes \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ a functor and $S \in \mathbb{F}_{\mathcal{T}}$ a finite $\mathcal{T}$-set, we define

$$\mathcal{O}_S \coloneqq \pi_{\mathcal{O}}^{-1}(S).$$

In the case that $\pi_{\mathcal{O}}$ is a prefibrous pattern (i.e. a I*-preoperad*), inert arrows endow on $(\mathcal{O}_V)_{V\in\mathcal{T}}$ the structure of a $\mathcal{T}$-∞-category $U\mathcal{O}$, formally given by the pullback

$$\begin{array}{ccc} \mathrm{Tot}\,U\mathcal{O} & \longrightarrow & \mathcal{O}^\otimes \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{T}^{\mathrm{op}} & \hookrightarrow & \mathrm{Span}(\mathbb{F}_{\mathcal{T}}) \end{array}$$

We call this the *underlying* $\mathcal{T}$*-*∞*-category of* $\mathcal{O}^\otimes$, and refer to it as $\mathcal{O}$ when this won't cause confusion. ◁

The following proposition is an immediate corollary to Corollary I.4.34.

Proposition II.1.6. *A functor $\pi_{\mathcal{O}}\colon \mathrm{Tot}_{\mathcal{T}} \mathcal{O}^{\otimes} \to \mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$ is an I-operad if and only if the following hold:*

(a) $\mathcal{O}^{\otimes}$ has π-cocartesian lifts for backwards maps in $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})$;

(b) (Segal condition for colors) for every $S \in \mathbb{F}_{\mathcal{T}}$, cocartesian transport along the π-cocartesian lifts lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ together induce an equivalence

$$\mathcal{O}_S \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}_U;$$

(c) (Segal condition for multimorphisms) for every map of orbits $T \to S$ in I and pair of objects $(\mathbf{C}, \mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_U$, postcomposition with the π-cocartesian lifts $\mathbf{D} \to D_U$ lying over the inclusions $(S \leftarrow U = U \mid U \in \mathrm{Orb}(S))$ induces an equivalence

$$\mathrm{Map}^{T \to S}_{\mathcal{O}^{\otimes}}(\mathbf{C}, \mathbf{D}) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}^{T \leftarrow T_U \to U}_{\mathcal{O}^{\otimes}}(\mathbf{C}, D_U).$$

where $T_U \coloneqq T \times_S U$.

Furthermore, if $\pi_{\mathcal{O}}$ is a cocartesian fibration, then it is an I-operad if and only if its unstraightening $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Cat}$ is an I-symmetric monoidal ∞-category.

Remark II.1.7. Cocartesian lifts over backwards maps furnish an equivalence

$$\mathrm{Map}^{T \leftarrow T_U \to U}_{\mathcal{O}^{\otimes}}(\mathbf{C}, D_U) \simeq \mathrm{Map}^{T_U \to U}_{\mathcal{O}^{\otimes}}(\mathbf{C}_{T_U}, D_U),$$

where $\mathbf{C}_{T_U} \in \mathcal{O}_{T_U}$ is the T_U-tuple of colors underlying $\mathbf{C}$. Hence in the presence of Conditions (a) and (b), Condition (c) may equivalently stipulate that the map

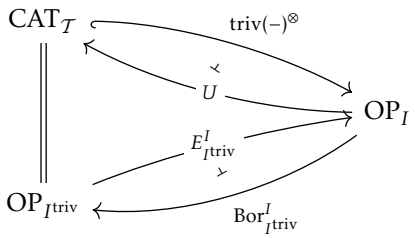
$$\mathrm{Map}^{T \to S}_{\mathcal{O}^{\otimes}}(\mathbf{C}, \mathbf{D}) \to \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}^{T_U \to U}_{\mathcal{O}^{\otimes}}(\mathbf{C}_{T_U}, D_U)$$

is an equivalence. We will generally prefer this version, as the data of a $\mathcal{T}$-operad is most naturally viewed as living over the *active* (i.e. forward) maps. ◂

Remark II.1.8. Practitioners of [HA, Def 2.1.10] should note that, by Remark I.2.21, we may weaken Condition (b) to assert only that cocartesian transport induces a π_0-surjection $\mathcal{O}_S \to \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}_U$. ◂

We also point out an immediate corollary of Corollary I.4.17.

Corollary II.1.9. *If I has one color, then there exists an equality of colocalizing $\mathcal{T}$-$(\infty,2)$-subcategories*



In particular, there exists a natural equivalence

$$\underline{\mathrm{Alg}}_{\mathrm{triv}(\mathcal{C})}(\mathcal{P}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, U\mathcal{P}).$$

As a corollary, U is $\mathcal{T}$-continuous; we thus are somewhat justified in fixing a $\mathcal{T}$-∞ and studying $\mathcal{T}$-operads with those colors. Proposition II.1.6, grants access to the *structure spaces* of such $\mathcal{T}$-operads.

Notation II.1.10. Following [BP22] We will refer to tuples $(V \in \mathcal{T}, S \in \mathbb{F}_V, (\mathbf{C}; D) \in \mathcal{O}_S \times \mathcal{O}_V)$ as *$\mathcal{O}$-profiles*, and we will often abusively refer to the profile $(V, S, (\mathbf{C}; D))$ as $(\mathbf{C}; D)$. Additionally, if $\psi\colon T \to S$ is a map of finite V-sets, we will write $T_U \coloneqq T \times_U S$; we refer to a *composable datum over ψ* as the data of a $\mathcal{O}$-profile $(V, S, (\mathbf{C}; D))$ together with an S-tuple of profiles $((U, T_U, (\mathbf{B}_U, C_U)))_{U \in \mathrm{Orb}(S)}$; in this case, $(\mathbf{B}_U)_{U \in \mathrm{Orb}(S)}$ assemble into a T-tuple $\mathbf{B}$, and we refer to $(V, T, (\mathbf{B}; D))$ as the *composite $\mathcal{O}$-profile*. ◁

Construction II.1.11. Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad. Given an $\mathcal{O}$-profile $(V, S, (\mathbf{C}; D))$, we write

$$\mathcal{O}(\mathbf{C}; D) \coloneqq \mathrm{Map}^{\mathrm{Ind}^{\mathcal{T}}_V S \to V}_{\mathrm{Tot}\,\mathcal{O}}(\mathbf{C}, D).$$

We will sometimes refer to $\mathrm{Mul}^{\psi}_{\mathcal{O}}(\mathbf{C}; \mathbf{D})$ for the same construction in the case that $\pi\mathbf{D}$ is not an orbit. Similarly, given $S \in \mathbb{F}_V$, we write

$$\mathcal{O}(S) \coloneqq \coprod_{(\mathbf{C}, D) \in \mathcal{O}_S \times \mathcal{O}_V} \mathcal{O}(\mathbf{C}; D);$$

we refer to this is the *space of S-ary operations in $\mathcal{O}$*. ◁

This simplifies in a particular setting:

Definition II.1.12. A $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ is:

- *at most one-colored* if $\mathcal{O}_V \in \{\varnothing, *\}$ for all $V \in \mathcal{T}$, i.e. $\mathcal{O}(*_V) \in \{\varnothing, *\}$ for all $V \in \mathcal{T}$,
- *at least one-colored* if $\mathcal{O}_V \neq \varnothing$ for all $V \in \mathcal{T}$, i.e. $\mathcal{O}(*_V) \neq \varnothing$ for all $V \in \mathcal{T}$, and
- *one-colored* if $\mathcal{O}^{\otimes}$ is at least one-colored and at-most one colored.

We denote the associated full subcategories by $\mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}} \subset \mathrm{Op}^{\geq \mathrm{oc}}_{\mathcal{T}}, \mathrm{Op}^{\leq \mathrm{oc}}_{\mathcal{T}} \subset \mathrm{Op}_{\mathcal{T}}$. ◁

One-color $\mathcal{T}$-operads admit a simpler description: they are functors $\mathrm{Tot}\,\mathcal{O}^{\otimes} \to \mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$ with:

(a') cocartesian lifts of backwards maps,

(b') contractible fibers, and

(c') for whom cocartesian transport induces an equivalence

$$\mathrm{Map}_{\mathcal{O}}^{T\to S}(iT, iS) \simeq \prod_{U\in \mathrm{Orb}(S)} \mathcal{O}(T_U),$$

where $\pi^{-1}(T) = \{iT\}$ and $T_U = T \times_S U \to U$, considered as a U-set.

For most applications, algebras over one-color $\mathcal{T}$-operads suffice. For now, we describe the general setting.

Construction II.1.13. Given $\mathcal{O}^\otimes \in \mathrm{Op}_{\mathcal{T}}$ and $(S, V, (\mathbf{C}; D))$ an $\mathcal{O}$-profile, for any $T \leftarrow \mathrm{Ind}_V^{\mathcal{T}} S$, we find

$$\mathcal{O}(\mathbf{C}; D) \simeq \mathrm{Map}_{\pi_{\mathcal{O}}}^{T\leftarrow \mathrm{Ind}_V^{\mathcal{T}} S\to V}(\mathbf{C}; D)$$

due to the existence of cocartesian lifts for inert morphisms. Given a map $U \to V$ in $\mathcal{T}$ and a finite V-set $S \in \mathrm{Fin}_V$, postcomposition with the cocartesian lift of the backwards map $V \leftarrow U = U$ yields a restriction map

$$\begin{array}{ccc} \mathcal{O}(\mathbf{C}; D) & \xrightarrow{\quad\mathrm{Res}_U^V\quad} & \mathcal{O}\left(\mathrm{Res}_U^V \mathbf{C};\, \mathrm{Res}_U^V D\right) \\ \wr & & \wr \\ \mathrm{Map}_{\pi_{\mathcal{O}}}^{\mathrm{Ind}_V^{\mathcal{T}} S\to V}(\mathbf{C}, D) & \longrightarrow & \mathrm{Map}_{\pi_{\mathcal{O}}}^{\mathrm{Ind}_V^{\mathcal{T}} S\leftarrow \mathrm{Ind}_V^{\mathcal{T}} S\times_V U\to U}\left(\mathrm{Res}_U^V \mathbf{C},\, \mathrm{Res}_U^V D\right) \end{array} \tag{17}$$

where the right hand side corresponds with the profile $\left(U, \mathrm{Res}_U^V S, \left(\mathrm{Res}_U^V \mathbf{C};\, \mathrm{Res}_U^V D\right)\right)$ induced by restriction.

Moreover, given a composable datum $\left((\mathbf{C}; D), (\mathbf{B}_U; C_U)_{U\in \mathrm{Orb}(S)}\right)$ lying over a map of V-sets $\varphi_{TS} \colon T \to S$, writing φ_{TV} for the structure map of T, composition in $\mathcal{O}^\otimes$ restricts to a map

$$\begin{array}{ccc} \mathcal{O}(\mathbf{C}; D) \times \prod_{U\in \mathrm{Orb}(S)} \mathcal{O}(\mathbf{B}_U; C_U) & \xrightarrow{\quad\gamma\quad} & \mathcal{O}(\mathbf{B}; D) \\ \wr & & \wr \\ \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(\mathbf{C}, D) \times \mathrm{Map}_{\pi_{\mathcal{O}}}^{\varphi_{TS}}(\mathbf{B}, \mathbf{C}) & \xrightarrow{\quad\circ\quad} & \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{TV}}(\mathbf{B}, D) \end{array} \tag{18}$$

Lastly, define the *color preserving automorphism group* to be the subgroup $\mathrm{Aut}_V(\mathbf{C}) \subset \mathrm{Aut}_V(S)$ consisting of automorphisms σ such that $C_U \simeq C_{\sigma U}$ for all $U \in \mathrm{Orb}(S)$. Note that $\pi_{\mathcal{O}}$-cocartesian lifts of $\mathrm{Aut}_V(\mathbf{C})$ with domain $\mathbf{C}$ are endomorphisms of $\mathbf{C}$; cocartesian transport then yields an action

$$\rho_S : \mathrm{Aut}_V(\mathbf{C}) \times \mathcal{O}(\mathbf{C}; D) \longrightarrow \mathcal{O}(\mathbf{C}; D). \tag{19}$$

We refer to Res_U^V as *restriction*, γ as the *composition*, and ρ_S as Σ-*action*. ◁

Example II.1.14. Let I be a weak indexing category. Recall the example $\mathcal{N}_{I\infty}^{\otimes} = (\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}}))$ of Proposition II.1.2. It follows by definition that $U\mathcal{N}_{I\infty}^{\otimes} \simeq *_{c(I)}$; that is, $\mathcal{N}_{I\infty}$ always has at most one color, and it has one color if and only if I has one color. Moreover, we have

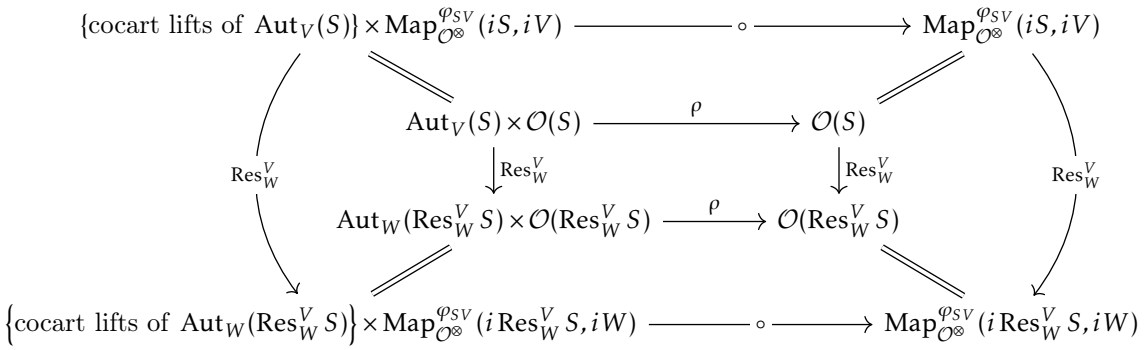
$$\mathcal{N}_{I\infty}(S) \simeq \begin{cases} * & S \in \mathbb{F}_I^V; \\ \varnothing & S \notin \mathbb{F}_I^V. \end{cases}$$

Each of the maps $\mathrm{Res}_U^V, \gamma, \rho_S$ are uniquely determined by their domain and codomain. ◁

For the following observation, we restrict to the one-color case simply for notational clarity; the general case follows identically.

Observation II.1.15. The structures of Eqs. (17) to (19) are compatible in the following ways:

(1) The restriction maps are Borel $\mathrm{Aut}_V(S)$-equivariant, i.e. the following commutes:

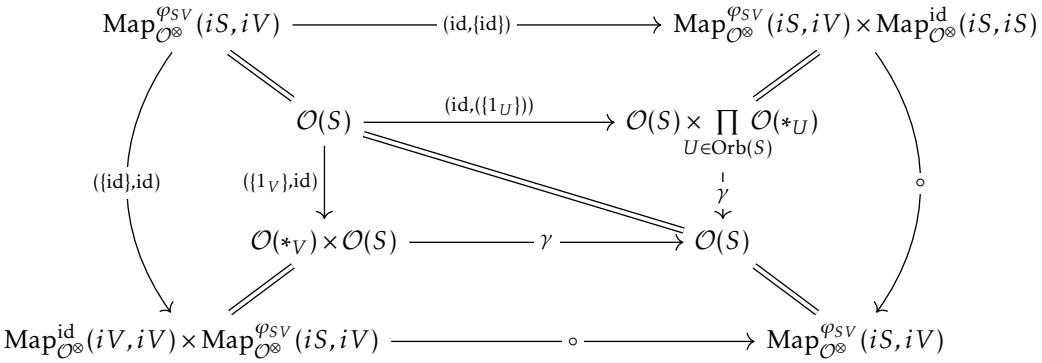


Here we write $\mathcal{O}_S = \{iS\}$.

(2) The composition maps are Borel $\mathrm{Aut}_V(S) \times \prod_{U \in \mathrm{Orb}S} \mathrm{Aut}_U(T_U)$-equivariant in an analogous way.

(3) The identity map on $*_V$ yields an element $1_V \in \mathcal{O}(*_V)$ which is taken to 1_U by Res_U^V.

(4) The composition maps are unital, i.e. the following commutes.



(5) The map γ is compatible with restriction, i.e. given a composable pair of morphisms

$$\mathrm{Ind}_V^{\mathcal{T}} T \xrightarrow{\varphi_{TS}} \mathrm{Ind}_V^{\mathcal{T}} S \xrightarrow{\varphi_{SV}} V \qquad \left(\text{with } \varphi_{TV} \text{ the composite}\right)$$

and $W \to V$ a map in $\mathcal{T}$, the following diagram commutes, where $T \coloneqq \coprod_U^S T_U$:

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(iS,iV) \times \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{TS}}(iT,iS) & \xrightarrow{\quad\circ\quad} & & & \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{TV}}(iT,iV) \\
\| & & & & \| \\
\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U) & & \xrightarrow{\quad\gamma\quad} & & \mathcal{O}(T) \\
\downarrow{\scriptstyle \mathrm{Res}_V^W} & & & & \downarrow{\scriptstyle \mathrm{Res}_V^W} \\
\mathcal{O}\left(\mathrm{Res}_W^V S\right) \times \prod_{U' \in \mathrm{Orb}(\mathrm{Res}_W^V S)} \mathcal{O}(T_{U'}) & & \xrightarrow{\quad\gamma\quad} & & \mathcal{O}\left(\mathrm{Res}_W^V T\right) \\
\| & & & & \| \\
\mathrm{Map}_{\mathcal{O}^\otimes}^{\mathrm{Res}_W^V \varphi_{SV}}\left(i\,\mathrm{Res}_W^V S, iW\right) \times \mathrm{Map}_{\mathcal{O}^\otimes}^{\mathrm{Res}_W^V \varphi_{TS}}(i\,\mathrm{Res}_W^V T, i\,\mathrm{Res}_W^V S) & & \xrightarrow{\quad\circ\quad} & & \mathrm{Map}_{\mathcal{O}^\otimes}^{\mathrm{Res}_W^V \varphi_{TV}}(i\,\mathrm{Res}_W^V T, iW)
\end{array}$$

(with Res_W^V the vertical arcs on the left and right)

(6) The composition maps are associative, i.e. given a collection of maps and composites

$$\mathrm{Ind}_V^{\mathcal{T}} R \xrightarrow{\varphi_{RT}} \mathrm{Ind}_V^{\mathcal{T}} T \xrightarrow{\varphi_{TS}} \mathrm{Ind}_V^{\mathcal{T}} S \xrightarrow{\varphi_{SV}} V, \qquad \varphi_{RV} = \varphi_{SV}\varphi_{TS}\varphi_{RT},\ \varphi_{TV} = \varphi_{SV}\varphi_{TS},\ \varphi_{RS} = \varphi_{TS}\varphi_{RT}$$

the following commutes, where $R \coloneqq \coprod_U^S \coprod_W^{T_U} R_W$:

$$\begin{array}{ccccc}
\mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(iS,iV) \times \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{TS}}(iT,iS) \times \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{RT}}(iR,iT) & \xrightarrow{\quad\circ\quad} & & & \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{TV}}(iT,iV) \times \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{RT}}(iR,iT) \\
\| & & & & \| \\
\left(\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U)\right) \times \prod_{\substack{U \in \mathrm{Orb}(S) \\ W \in \mathrm{Orb}(T_U)}} \mathcal{O}(R_W) & & \xrightarrow{\quad\gamma\quad} & & \mathcal{O}(T) \times \prod_{W \in \mathrm{Orb}(T)} \mathcal{O}(R_W) \\
\| & & & & \downarrow{\scriptstyle \gamma} \\
\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} \left(\mathcal{O}(T_U) \times \prod_{W \in \mathrm{Orb}(T_U)} \mathcal{O}(R_W)\right) & & & & \mathcal{O}\left(\coprod_W^T R_W\right) \\
\downarrow{\scriptstyle \gamma} & & & & \| \\
\mathcal{O}(S) \times \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}\left(\coprod_W^{T_U} R_W\right) & & \xrightarrow{\quad\gamma\quad} & & \mathcal{O}(R) \\
\| & & & & \| \\
\mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{SV}}(iS,iV) \times \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{RS}}(iR,iS) & & \xrightarrow{\quad\circ\quad} & & \mathrm{Map}_{\mathcal{O}^\otimes}^{\varphi_{RV}}(iR,iV)
\end{array}$$

(with $\circ$ the vertical arcs on the left and right)

This is a version of *monoids in $\mathcal{T}$-symmetric sequences in* $\mathrm{ho}\mathcal{S}$. ◁

II.1.1.3 I-symmetric monoidal ∞-categories We also explicitly describe objects of $\mathrm{Cat}_I^{\otimes}$.

Notation II.1.16. Fix some $S \in \mathbb{F}_I^V$ and some I-symmetric monoidal ∞-category $\mathcal{C}$. Associated with the structure map $\varphi\colon \mathrm{Ind}_V^{\mathcal{T}} S \to V$ we have functors

$$\bigotimes_U^S\colon \mathcal{C}^S \to \mathcal{C}^V, \qquad \Delta^S\colon \mathcal{C}^V \to \mathcal{C}^S$$

called the *S-indexed tensor product and S-indexed diagonal*. More generally, given $f\colon T \to S$, we write $f_{\otimes}\colon \mathcal{C}^T \to \mathcal{C}^S$ for the covariant functoriality of $\mathcal{C}^{\otimes}$ on f, and call this the *f-indexed tensor functor*.

We refer to the composite $(-)^{\otimes S}\colon \mathcal{C}_V \xrightarrow{\Delta^S} \mathcal{C}^S \xrightarrow{\otimes_U^S} \mathcal{C}^V$ as the *S-indexed tensor power*. In the case $\mathrm{Ind}_V^{\mathcal{T}} S = W$ is an orbit (i.e. S is a *transitive V-set*), we write

$$N_W^V \coloneqq \bigotimes_U^W\colon \mathcal{C}^W \to \mathcal{C}^V.$$

In general, we will use the inset notation $-\otimes-$ for $\otimes_U^{2\cdot *_V}$, and when $\varnothing_V \in \underline{\mathbb{F}}_I$, we will refer to the $\varnothing_V$-ary operation $* \to \mathcal{C}_V$ as the *V-unit* and denote its essential image as 1_V. ◁

Observation II.1.17. Suppose S, $|\mathrm{Orb}(S)|\cdot *_V$, and all orbits of S are I-admissible V-sets. Then, the following path lies in I

$$\mathrm{Ind}_V^{\mathcal{T}} S \xrightarrow{\mathrm{Ind}_H^G \coprod_{U\in\mathrm{Orb}(S)} \left(U \xrightarrow{!} V\right)} |\mathrm{Orb}(S)|\cdot V \xrightarrow{\quad\nabla\quad} V,$$

In algebra, this yields the commutative diagram

$$\begin{array}{ccc} \mathcal{C}^S & \xrightarrow{\bigotimes_U^S} & \mathcal{C}^V \\ & {\scriptstyle (N_U^V -)}\searrow \quad \nearrow{\scriptstyle \otimes} & \\ & \mathcal{C}^{V,\times \mathrm{Orb}(S)} & \end{array}$$

i.e. $\bigotimes_U^S X_U \simeq \bigotimes_{U\in\mathrm{Orb}(S)} N_U^V X_U$. Thus, when I is an indexing category, the indexed tensor products in an I-symmetric monoidal ∞-category is are determined by their binary tensor products and norms. ◁

The functoriality in categories of correspondences gives a version of the *double coset formula*

$$\mathrm{Res}_W^V N_U^V Z \simeq \bigotimes_X^{U\times_V W} \mathrm{Res}_X^U Z$$

for all cospans $U \to V \leftarrow W$ in $\mathcal{T}$ such that $U \to W$ is in I. Moreover, Res_V^W and N_V^W preserve applicable trivially indexed tensor products.

A construction we will use quite liberally is the *fixed points*.

Construction II.1.18. Right Kan extensions preserve product preserving functors; applying this to the *orbits* functor $F_{\mathcal{T}} : \mathbb{F}_{\mathcal{T}} \to \mathbb{F}$ yields a functor

$$\Gamma \coloneqq \mathsf{Span}(F_{\mathcal{T}})_* : \mathsf{Fun}^{\times}(\mathsf{Span}(\mathbb{F}_{\mathcal{T}}), \mathcal{C}) \to \mathsf{Fun}^{\times}(\mathsf{Span}(\mathbb{F}), \mathcal{C}).$$

In particular, Γ is right adjoint to $\mathsf{Infl}_e^{\mathcal{T}} \coloneqq \mathsf{Span}(F_{\mathcal{T}})^*$. When $\mathcal{C} = \mathsf{Cat}$, the counit of this adjunction is a natural $\mathcal{T}$-symmetric monoidal functor.

$$\mathsf{Infl}_e^{\mathcal{T}} \Gamma \mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}$$

We refer to the (symmetric monoidal) V-value of this as the *symmetric monoidal V-evaluation*

$$\mathrm{ev}_V : \Gamma \mathcal{C}^{\otimes} \to \mathcal{C}_V^{\otimes}. \qquad \triangleleft$$

II.1.1.4 Envelopes For convenience, we spell out an immediate corollary to Corollary I.3.12.

Corollary II.1.19. *If $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a map of $\mathcal{T}$-operads, then the top horizontal composte of the following is an $\mathcal{O}$-monoidal ∞-category:*

$$\begin{array}{ccccc} \mathrm{Env}_{\mathcal{O}} \mathcal{P}^{\otimes} & \longrightarrow & \mathrm{Ar}^{\mathrm{act}}(\mathcal{O}^{\otimes}) & \xrightarrow{\mathrm{ev}_1} & \mathcal{O}^{\otimes} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{ev}_0} & & \\ \mathcal{P}^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes} & & \end{array}$$

This underlies a $\mathcal{T}$-$(\infty, 2)$-functor

$$\mathrm{Env}_{\mathcal{O}} : \underline{\mathrm{OP}}_{\mathcal{T}, /\mathcal{O}^{\otimes}} \to \underline{\mathrm{CAT}}_{\mathcal{O}}^{\otimes}$$

which is left adjoint to the inclusion of $\mathcal{O}$-monoidal ∞-categories into $\mathcal{T}$-operads over $\mathcal{O}^{\otimes}$, and the induced $\mathcal{T}$-$(\infty, 2)$-functor

$$\mathrm{Env}_{\mathcal{O}}^{/\mathscr{A}_{\mathcal{O}}} : \underline{\mathrm{OP}}_{\mathcal{T}, /\mathcal{O}^{\otimes}} \to \underline{\mathrm{CAT}}_{\mathcal{O}, /\mathscr{A}_{\mathcal{O}}}^{\otimes}$$

is fully faithful, with image spanned by equifibrations; in particular, we find

$$\underline{\mathrm{Alg}}_{\mathcal{P}/\mathcal{O}}(\mathcal{C}) \simeq \underline{\mathrm{Fun}}_{\mathcal{O}}^{\otimes}\left(\mathrm{Env}_{\mathcal{O}} \mathcal{P}^{\otimes}, \mathcal{C}\right).$$

To be specific about equifibrations, we prove the following.

Proposition II.1.20. *Suppose I is an indexing category. Then, an I-symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is equifibered if and only if the following diagrams are cartesian, where $U \to V$ is an orbit map lying in I:*

$$\begin{array}{ccc} \mathcal{C}^V \times \mathcal{C}^V & \xrightarrow{\otimes} & \mathcal{C} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \mathcal{D}^V \times \mathcal{D}^V & \xrightarrow{\otimes} & \mathcal{D} \end{array} \qquad \begin{array}{ccc} \mathcal{C}^U & \xrightarrow{N_U^V} & \mathcal{C}^V \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ \mathcal{D}^U & \xrightarrow{N_U^V} & \mathcal{D}^V \end{array}$$

PROOF. It suffices to note that $\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})^{\mathrm{act}} = I$ is generated under composition and disjoint union by binary fold maps, nullary fold maps, and orbit maps, and that the equifibration square for nullary fold maps is always satisfied. □

We will simply write $\mathrm{Env}_I(-) \coloneqq \mathrm{Env}_{\mathcal{N}_{I^\infty}}(-)$ and $\mathrm{Env}(-) \coloneqq \mathrm{Env}_{\mathrm{Comm}_{\mathcal{T}}}(-)$.

Example II.1.21. Let I be a weak indexing category. Then, unwinding definitions, we find that

$$\mathrm{Env}_I \mathcal{N}_{I^\infty}^{\otimes} \simeq \underline{\mathbb{F}}_I^{I-\sqcup},$$

is a structure on $\underline{\mathbb{F}}_I$ whose I-indexed tensor products are I-coproducts. ◁

We record a convenient property of $\mathrm{Env}_I(-)$ here, which follows by unwinding definitions.

Lemma II.1.22 ([HA, Rmk 2.4.4.3])**.** *If $\mathcal{O}^{\otimes} \in \mathrm{Op}_I$ and $\psi\colon T \to S$ is a map of V-sets, then there is an equivalence*

$$\begin{aligned}\mathrm{Ar}\left(\mathrm{Env}_I^V(\mathcal{O})\right)^{\simeq} \times_{\mathrm{Ar}\left(\mathbb{F}_I^V\right)^{\simeq}} \{\psi\} &\simeq \coprod_{(\mathbf{C},\mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_S} \mathrm{Map}^{\psi}_{\mathcal{O}^{\otimes} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})}(\mathbf{C},\mathbf{D}) \\ &\simeq \coprod_{(\mathbf{C},\mathbf{D}) \in \mathcal{O}_T \times \mathcal{O}_S} \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(\mathbf{C}_U; D_U)\end{aligned}$$

In particular, if $\mathcal{O}^{\otimes}$ has one color, then

$$\mathrm{Map}^{\psi}_{\mathrm{Env}_I(\mathcal{O})_V \to \mathbb{F}_{I,V}}(iT, iS) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T_U).$$

II.1.1.5 Preoperads Now, we've defined I-operads in a Morita-invariant way, but we will often want to resolve them by prefibrous patterns, which are *not* Morita-invariant. We correspondingly define the $(\infty,2)$-category of *I-preoperads*

$$\mathrm{POP}_I \coloneqq \mathrm{PFBRS}(\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}}) \simeq \mathrm{CAT}^{\mathrm{int-cocart}}_{/\mathrm{Span}_I(\mathbb{F}_{\mathcal{T}})},$$

and the $\mathcal{T}$-$(\infty,2)$-category of *Γ-I-preoperads*

$$\underline{\mathrm{PreOp}}_I^{\Gamma} \coloneqq \underline{\mathrm{PFBRS}}(\underline{\mathbb{F}}_{I,*}) \simeq \underline{\mathrm{CAT}}^{\mathrm{int-cocart}}_{\mathcal{T},/\underline{\mathbb{F}}_{I,*}}.$$

These admit Op_I and $\underline{\mathrm{Op}}_I$ as localizing ($\mathcal{T}$-)subcategories; we write the localization as $L_{\mathrm{Op}_{\mathcal{T}}}$.

II.1.2 The underlying $\mathcal{T}$-symmetric sequence We now study a forgetful functor to *symmetric sequences*. We begin in Section II.1.2.1 by defining the multi-colored variant of $\mathcal{T}$-symmetric sequences, then move on in

Section II.1.2.2 to construct the underlying $\mathfrak{C}$-symmetric sequence, verifying that it is monadic in the one-color setting. We reframe this somewhat in Section II.1.2.3 to introduce the *n-ary $B_{\mathcal{T}}\Sigma_n$-space* construction.

II.1.2.1 $\mathfrak{C}$-symmetric sequences We define the *symmetric $\mathcal{T}$-space category* $\underline{\Sigma}_{\mathcal{T}} \coloneqq \underline{\mathsf{Fin}}^{\simeq}_{\mathcal{T}}$.

Definition II.1.23. Let $\mathcal{D}$ be a $\mathcal{T}$-∞-category. The *∞-category of $\mathcal{T}$-symmetric sequences in $\mathcal{D}$* is $\mathsf{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \mathcal{D})$. In the case $\mathcal{C} = \underline{\mathcal{S}}_{\mathcal{T}}$, we refer to $\mathsf{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathsf{Fun}(\mathsf{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ simply as the *∞-category of $\mathcal{T}$-symmetric sequences*.

More generally, if $\mathfrak{C}$ is a $\mathcal{T}$-coefficient system of sets, we set the notation $\underline{\Sigma}_{\mathfrak{C}} \coloneqq \mathsf{Tot}_{\mathcal{T}}\,\mathsf{triv}(\mathfrak{C})$; we refer to $\mathsf{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathfrak{C}}, \mathcal{D})$ and $\mathsf{Fun}(\mathsf{Tot}\,\underline{\Sigma}_{\mathfrak{C}}, \mathcal{S})$ as the ∞-categories of $\mathfrak{C}$-symmetric sequences in $\mathcal{D}$ and $\mathfrak{C}$-symmetric sequences, respectively. ◁

Observation II.1.24. For any adequate $\mathcal{T}$-triple $(\mathcal{X}, \mathcal{X}_b, \mathcal{X}_f)$, the inclusion

$$\mathcal{X} \hookrightarrow \mathsf{Span}_{b,f}(\mathcal{X})$$

induces an equivalence on cores; in particular, we acquire an equivalence

$$\underline{\mathsf{Fin}}^{\simeq}_{\mathcal{T}} \simeq \underline{\mathsf{Fin}}^{\simeq}_{\mathcal{T},*} \simeq \underline{\Sigma}_{\mathcal{T}}.$$

In particular, unwinding definitions, we find that

$$\Sigma_V \coloneqq \underline{\Sigma}^V_{\mathcal{T}} \simeq \mathsf{Fin}^{\simeq}_V \simeq \coprod_{S \in \mathsf{Fin}_V} B\,\mathsf{Aut}_V S$$

and the restriction map $\Sigma_V \to \Sigma_W$ is induced by the forgetful maps $B\,\mathsf{Aut}_V S \to B\,\mathsf{Aut}_W S$. ◁

Remark II.1.25. By unwinding definitions, one finds that $\underline{\Sigma}_{\mathcal{T}} \simeq \underline{\Sigma}_{*_{\mathcal{T}}}$. ◁

We may similarly explicitly understand $\underline{\Sigma}_{\mathfrak{C}}$.

Observation II.1.26. A map of coefficient systems $\mathfrak{C} \to \mathfrak{D}$ induces a map of $\mathcal{T}$-operads $\mathsf{triv}(\mathfrak{C})^{\otimes} \to \mathsf{triv}(\mathfrak{D})^{\otimes}$, i.e. a cocartesian fibration of $\mathcal{T}$-∞-categories $\underline{\Sigma}_{\mathfrak{C}} \to \underline{\Sigma}_{\mathfrak{D}}$. Applying this in the case $\mathfrak{D} = *_{\mathcal{T}}$, we acquire a canonical natural cocartesian fibration of $\mathcal{T}$-∞-categories

$$\underline{\Sigma}_{\mathfrak{C}} \to \underline{\Sigma}_{\mathcal{T}};$$

taking V-values yields a natural cocartesian fibration

$$\Sigma_{\mathfrak{C},V} \to \Sigma_V$$

whose straightening has discrete values

$$S \mapsto \mathfrak{C}^S \times \mathfrak{C}^V$$

with functoriality along S-automorphisms given by permuting the factors of the $\mathfrak{C}^S$ part. In particular, the classical Grothendieck construction describes Σ_V as having;

- Objects: profiles $(V, S \in \mathbb{F}_V, (\mathbf{C}; D) \in \mathfrak{C}^S \times \mathfrak{C}^V)$ with orbit V
- Morphisms: V-equivariant automorphisms of S whose action fixes $\mathbf{C}$, i.e. color-preserving S-automorphisms.

In short, we find that

$$\Sigma^V_{\mathfrak{C}} = \coprod_{\substack{S \in \mathbb{F}_V \\ (\mathbf{C};D) \in \mathfrak{C}^S \times \mathfrak{C}^V}} B\mathrm{Aut}_V(\mathbf{C}).$$

Moreover, unwinding definitions, we find that the restriction functor $\mathrm{Res}^V_W \colon \Sigma^V_{\mathfrak{C}} \to \Sigma^W_{\mathfrak{C}}$ is induced from the forgetful maps $B\mathrm{Aut}_V(\mathbf{C}) \to B\mathrm{Aut}_W(\mathbf{C})$. ◁

II.1.2.2 The underlying $\mathfrak{C}$-symmetric sequence We will restrict to the following setting.

Definition II.1.27. Given $\mathfrak{C} \colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Set}$ a coefficient system of sets, we define full subcategory of *$\mathfrak{C}$-colored $\mathcal{T}$-operads* as the pullback

$$\begin{array}{ccc} \mathrm{Op}^{\mathfrak{C}}_{\mathcal{T}} & \hookrightarrow & \mathrm{Op}_{\mathcal{T}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_0 U} \\ \{\mathfrak{C}\} & \hookrightarrow & \mathrm{Coeff}^{\mathcal{T}}(\mathrm{Set}) \end{array} \tag{20}$$

◁

For instance, $\mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}} \subset \mathrm{Op}^{*}_{\mathcal{T}}$. The following construction recovers an *underlying $\mathcal{T}$-symmetric sequence* on one-color $\mathcal{T}$-operads.

Construction II.1.28. Given $\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathfrak{C}}_{\mathcal{T}}$, there is a structure map

$$\mathrm{Env}_{\mathcal{O}}\mathrm{triv}(\mathfrak{C}) \simeq \mathrm{triv}(\mathfrak{C})^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}^{\otimes}) \to \mathrm{triv}(\mathfrak{C})^{\otimes},$$

where $\mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}^{\otimes}) \subset \mathrm{Ar}^{\mathrm{act}}(\mathcal{O}^{\otimes})$ is the full subcategory of active arrows whose codomain is elementary; this is an inert-cocartesian fibration by pullback-stability of inert-cocartesian fibrations [BHS22, Obs. 2.1.7]. The *underlying $\mathfrak{C}$-symmetric sequence of $\mathcal{O}^{\otimes}$* is the unstraightening

$$\mathcal{O}^{\otimes}_{\mathrm{sseq}} := \mathrm{Un}_{\mathrm{triv}(\mathfrak{C})}\mathrm{Env}_{\mathcal{O}}\mathrm{triv}(\mathfrak{C}) \in \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}}, \mathrm{Cat}).$$

Unwinding definitions, we find that there exists a cartesian square

$$\begin{array}{ccccc} \mathcal{O}(\mathbf{C};D) & \longrightarrow & \mathrm{Env}_{\mathcal{O}}\mathrm{triv}(\mathfrak{C}) & = & \mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}} \times_{\mathcal{O}^{\otimes}} \mathrm{Ar}^{\mathrm{act},/\mathrm{el}}(\mathcal{O}) \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \{(\mathbf{C};D)\} & \hookrightarrow & \mathrm{triv}(\mathfrak{C})^{\otimes} & = & \mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}} \end{array}$$

so that $\mathcal{O}^{\otimes}_{\mathrm{sseq}}$ is indeed an $\mathfrak{C}$-symmetric sequence. The associated functor is denoted

$$\mathrm{sseq}\colon \mathrm{Op}^{\mathfrak{C}}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}}, \mathcal{S}). \qquad \triangleleft$$

We will often use the following to reduce questions about $\mathcal{T}$-operads to $\mathcal{T}$-symmetric sequences.

Proposition II.1.29. *A $\mathcal{T}$-operad map $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is an equivalence if and only if it satisfies the following:*

(a) φ induces surjective maps $\pi_0\mathcal{O}_V \to \pi_0\mathcal{P}_V$ for all $V \in \mathcal{T}$, and

(b) for all $V \in \mathcal{T}$, all $S \in \mathbb{F}_V$, all $\mathbf{C} \in \mathcal{O}_S$, and all $D \in \mathcal{O}_V$, the map φ induces equivalences $\varphi : \mathcal{O}(\mathbf{C}; D) \xrightarrow{\sim} \mathcal{P}(\varphi\mathbf{C}; \varphi D)$.

In particular, the underlying $\mathfrak{C}$-symmetric sequence is a conservative functor

$$\mathrm{sseq}\colon \mathrm{Op}^{\mathfrak{C}}_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}}, \mathcal{S})$$

In particular, specializing to $\mathfrak{C} = *_{\mathcal{T}}$, we find out that $\{\mathcal{O}(S) \mid S \in \underline{\mathbb{F}}_{\mathcal{T}}\}$ is jointly conservative on one-color $\mathcal{T}$-operads.

Proof of Proposition II.1.29. The second statement follows immediately from the first, since morphisms of $\mathfrak{C}$-colored $\mathcal{T}$-operads are π_0-isomorphisms by assumption. Moreover, conditions (a) and (b) are certainly necessary, as an inverse to φ induces inverse maps to both of the maps in question.

Fixing φ satisfying (a) and (b), we are left with prove that φ is an equivalence of $\mathcal{T}$-operads. Since $\mathrm{Tot}\colon \mathrm{Op}_{\mathcal{T}} \to \mathrm{Cat}_{/\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}}$ is a subcategory inclusion, it is conservative, so it suffices to prove that $\mathrm{Tot}\,\varphi$ is an equivalence of ∞-categories.

By the Segal condition for colors, we have an equivalence of arrows

$$\begin{array}{ccc} \pi_0\mathcal{O}_S & \simeq & \prod_{V \in \mathrm{Orb}(S)} \pi_0\mathcal{O}_V \\ \downarrow{\scriptstyle \varphi_S} & & \downarrow{\scriptstyle \prod \varphi_V} \\ \pi_0\mathcal{P}_S & \simeq & \prod_{V \in \mathrm{Orb}(S)} \pi_0\mathcal{P}_V \end{array}$$

Since $\pi_0\mathcal{O} \simeq \coprod_S \pi_0\mathcal{O}_S$, (a) implies that φ is essentially surjective. Now, given $\mathbf{C}, \mathbf{D} \in \mathrm{Tot}\,\mathcal{O}$, we have a diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Tot}\,\mathcal{O}}(\mathbf{C}, \mathbf{D}) & \xleftarrow{\quad\varphi(\mathbf{C};\mathbf{D})\quad} & \mathrm{Map}_{\mathrm{Tot}\,\mathcal{P}}(\varphi\mathbf{C}, \varphi\mathbf{D}); \\ & \searrow \quad \swarrow & \\ & \mathrm{Map}_{\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}}(\mathbb{F}_{\mathcal{T}}) & \end{array}$$

to prove that $\mathrm{Tot}\,\varphi$ is fully faithful and conclude, it suffices to prove that the horizontal arrrow is an equivalence of spaces, or equivalently, it induces an equivalence on fibers over every span $f : \pi\mathbf{C} \leftarrow S \to \pi D$; the Segal

condition for colors provides an equivalence of arrows

$$\begin{array}{ccccc} \mathrm{Map}^f_{\mathcal{O}}(\mathbf{C};\mathbf{D}) & \simeq & \prod_{V\in \mathrm{Orb}(\pi(D))} \mathrm{Map}^{f_V}_{\mathcal{O}}(\mathbf{C}_{f_V^{-1}};D_V) & \simeq & \prod_V \mathcal{O}(\mathbf{C}_{f^{-1}V};D_V) \\ \downarrow{\scriptstyle \varphi^f(\mathbf{C};\mathbf{D})} & & \downarrow{\scriptstyle \prod \varphi^{f_V}\left(\mathbf{C}_{f_V^{-1};D_V}\right)} & & \downarrow{\scriptstyle \prod \varphi\left(\mathbf{C}_{f^{-1}V};D_V\right)} \\ \mathrm{Map}^f_{\mathcal{P}}(\varphi\mathbf{C};\varphi\mathbf{D}) & \simeq & \prod_{V\in \mathrm{Orb}(S)} \mathrm{Map}^{f'}_{\mathcal{P}}(\varphi\mathbf{C}_{f^{-1}V},\varphi D_V) & \simeq & \prod_V \mathcal{P}(\varphi\mathbf{C}_{f^{-1}V};\varphi D_V). \end{array}$$

the right arrow is an equivalence by (b), so the leftmost arrow is an equivalence, completing the proof. □

The author learned the $U_\circ$ portion of the following argument from Thomas Blom.

Corollary II.1.30. *The functor* $\mathrm{sseq}_{\mathcal{T}} : \mathrm{Op}^*_{\mathcal{T}} \to \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ *is monadic and preserves sifted colimits.*

Proof. By Corollary I.3.10, $\mathrm{Op}_{\mathcal{T}}$ is presentable; moreover, in view of Corollary II.1.9 in the case $\mathfrak{C} = *$, the bottom right horn of Eq. (20) lies in Pr^R, so by [HTT, Thm. 5.5.3.18] $\mathrm{Op}^*_{\mathcal{T}}$ is presentable. By [HTT, Prop. 5.5.3.6] $\mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathcal{S})$ is presentable as well, so by Barr-Beck [HA, Thm. 4.7.3.5] and the adjoint functor theorem [HTT, Cor. 5.5.2.9], it suffices to prove that sseq is conservative and preesrves limits and sifted colimits.

Conservativity is Proposition II.1.29, and (co)limits in functor categories are computed pointwise by [HTT, Prop. 5.1.2.2], so it suffices to prove that $\mathcal{O} \mapsto \mathcal{O}(S)$ preserves limits and sifted colimits. We separate this into manageable chunks via the following diagram:

$$\begin{array}{ccccccc} \mathrm{Op}^*_{\mathcal{T}} & & \xrightarrow{\mathcal{O}\mapsto\mathcal{O}(S)} & & & & \mathcal{S} \\ \downarrow{\scriptstyle U_{\mathrm{Seg}}} & & & & & & \uparrow{\scriptstyle \mathrm{ev}_{\mathrm{Ind}^{\mathcal{T}}_V S,V}} \\ \mathrm{Cat}^{\mathrm{Int-cocart,core-iso}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})} & \xrightarrow{U_{\mathrm{cocart}}} & \mathrm{Cat}^{\mathrm{core-iso}}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})} & \xrightarrow{\mathrm{Res}^{\mathcal{T}}_V} & \mathrm{Cat}^{\mathrm{core-iso}}_{/\mathrm{Span}(\mathbb{F}_V)} & \xrightarrow{U_\circ} & \mathrm{Fun}\left((\mathrm{Span}(\mathbb{F}_V)^{\simeq})^{\times 2}, \mathcal{S}\right) \end{array}$$

π and $\mathrm{ev}_{\mathrm{Ind}^{\mathcal{T}}_V S,V}$ preserve (co)limits since they are evaluation of functor categories [HTT, Prop. 5.1.2.2]. U_{Cocart} preserves limits and sifted colimits by [BHS22, Cor. 2.1.5]. U_{Seg} preserves limits and sifted colimits, as each commute with finite products. $\mathrm{Res}^{\mathcal{T}}_V$ preserves (co)limits, as it is a left and right adjoint.

By [Hau20, Prop 3.12], $U_\circ$ is equivalent to the forgetful functor

$$\mathrm{Alg}(\mathcal{S}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq},\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq}}) \to \mathcal{S}_{/\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq},\mathrm{Span}(\mathbb{F}_{\mathcal{T}})^{\simeq}},$$

where $\mathcal{S}^{\otimes}_{/Y,Y}$ is a monoidal structure on $\mathcal{S}_{/Y,Y} \simeq \mathcal{S}_{Y\times Y} \simeq \mathrm{Fun}(Y \times Y, \mathcal{S})$. This functor preserves limits and sifted colimits by [HA, Prop. 3.2.3.1], completing the argument. □

In particular, this constructs a left adjoint

$$\mathrm{Fr}\colon \mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}}) = \mathrm{Fun}(\mathrm{Tot}\,\underline{\Sigma}_{\mathcal{T}}, \mathcal{S}) \to \mathrm{Op}^*_{\mathcal{T}}$$

to sseq.

II.1.2.3 Other persiectives on $\mathcal{T}$-symmetric sequences

Remark II.1.31. Let $\mathcal{O}_{G\times\Sigma_n,\Gamma_n} \subset \mathcal{O}_{G\times\Sigma_n}$ be the full subcategory spanned by $[G\times\Sigma_n/\Gamma_S]$ for $\phi_S : H \to \Sigma_n$ a map with associated graph subgroup $\Gamma_S = \{(h,\phi_S(h)) \mid h \in H\} \subset H\times\Sigma_n \subset G\times\Sigma_n$. This possesses an evident forgetful functor $\mathcal{O}^{\mathrm{op}}_{G\times\Sigma_n/\Gamma_S} \to \mathcal{O}^{\mathrm{op}}_G$ taking $[G\times\Sigma_n/\Gamma_S] \to [G/H]$; in [NS22, Ex. 4.3.7], this was shown to be a cocartesian fibration factoring through an equivalence

$$\coprod_{n\in\mathcal{R}} \mathcal{O}^{\mathrm{op}}_{G\times\Sigma_n/\Gamma_S} \simeq \mathrm{Tot}\,\underline{\Sigma}_G \to \mathcal{O}^{\mathrm{op}}_G$$

taking $[G\times\Sigma_n/\Gamma_S] \mapsto (H,S)$, and hence taking the G-space presented by $\mathcal{O}_{G\times\Sigma_n/\Gamma_S}$ equivalently onto the summand $B_G\Sigma_n \subset \underline{\Sigma}_G$, the classifying G-space for equivariant principal Σ_n-bundles.

More generally, we say that an atomic orbital ∞-categories $\mathcal{T}$ is *EI* if the inclusions $\mathsf{Aut}(V) \hookrightarrow \mathsf{End}(V)$ are equivalences. If $\mathcal{T}$ is EI and admits a weakly initial object $e \in \mathcal{T}$, we may define the $\mathcal{T}$-subspace $B_{\mathcal{T}}\Sigma_n \subset \underline{\Sigma}_{\mathcal{T}}$ as corresponding with the $\mathcal{T}$-sets S whose restriction $\mathrm{Res}^{\mathcal{T}}_e S$ is a set with n elements; then, we acquire a splitting

$$\underline{\Sigma}_{\mathcal{T}} \simeq \coprod_{n\in\mathcal{R}} B_{\mathcal{T}}\Sigma_n.$$

Under the above equivalence, given $\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}}$, we define the *$n$-ary $B_{\mathcal{T}}\Sigma_n$-space*

$$\underline{\mathcal{O}}(n)\colon B_{\mathcal{T}}\Sigma_n \subset \underline{\Sigma}_{\mathcal{T}} \xrightarrow{\mathrm{sseq}\,\mathcal{O}} \underline{\Sigma}_{\mathcal{T}}.$$

For instance, if $\mathcal{F} \subset \mathcal{O}_G$ is a family of subgroups, then $\mathcal{F}$ is EI with a weakly initial object, and $B_{\mathcal{F}}\Sigma_n$ is the classifying $\mathcal{F}$-space for $\mathcal{F}$-genuine G-equivariant principal Σ_n-bundles. In the case $\mathcal{T} = \mathcal{O}_G$, $\underline{\mathcal{O}}(n)$ is characterized by its Γ_S-fixed points $\underline{\mathcal{O}}(n)^{\Gamma_S} \simeq \mathcal{O}(S)$, with restriction functors along $\Gamma_{\mathrm{Res}^H_K S} \subset \Gamma_S$ corresponding with restriction map $\mathcal{O}(S) \to \mathcal{O}(\mathrm{Res}^H_K S)$. ◁

We will see in Remark II.1.37 that this intertwines with a nerve functor from operad objects in topological G-spaces. We will also need the following notation.

Notation II.1.32. Given an orbit $V \in \mathcal{T}$, and a finite V-set $S \in \mathbb{F}_V$, we may define a natural "S-ary" V-space in $\mathcal{T}$-symmetric sequences

$$\underline{(-)}(S)\colon \mathrm{Op}^{\mathrm{oc}}_{\mathcal{T}} \to \mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathcal{T}}, \underline{\mathcal{S}}_{\mathcal{T}}) \to \mathrm{Fun}_V(\underline{\Sigma}_V, \underline{\mathcal{S}}_V) \xrightarrow{\mathrm{ev}_S} \mathcal{S}_V.$$

More generally, we may define the analogous V-space for an $\mathcal{O}$-profile:

$$\underline{(-)}(\mathbf{C};D)\colon \mathrm{Op}^{\mathfrak{C}}_{\mathcal{T}} \to \mathrm{Fun}_{\mathcal{T}}(\underline{\Sigma}_{\mathfrak{C}}, \underline{\mathcal{S}}_{\mathcal{T}}) \to \mathrm{Fun}_V(\underline{\Sigma}_{\mathfrak{C},V}, \underline{\mathcal{S}}_V) \xrightarrow{\mathrm{ev}_{(\mathbf{C};D)}} \mathcal{S}_V.$$ ◁

II.1.3 The genuine operadic nerve In this subsection, we assume that $\mathcal{T}$ is an atomic orbital 1-category.

II.1.3.1 The 1-categorical nerve Recall the $\mathcal{T}$-space $\underline{\Sigma}_{\mathfrak{C}}$ of Definition II.1.23. [Bon19] introduced a specialization of the following.

Definition II.1.33. A $\mathfrak{C}$-colored genuine $\mathcal{T}$-operad in a 1-category $\mathcal{V}$ with finite products is the data of:

(1) a $\mathfrak{C}$-symmetric sequence $\mathcal{O}(-;-)\colon \mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}} \to \mathcal{V}$,

(2) for all $V \in \mathcal{T}$ and $C \in \mathfrak{C}_V$, a distinguished "identity" element $1_C \in \mathcal{O}(C;C)$, and

(3) for all composable data $\big((\mathbf{C};D),(\mathbf{B}_U;C_U)_{U\in \mathrm{Orb}(S)}\big)$ lying over a map $T \to S$, a "composition" map

$$\gamma\colon \mathcal{O}(\mathbf{C};D) \times \prod_{U\in\mathrm{Orb}(S)} \mathcal{O}(\mathbf{B}_U;C_U) \to \mathcal{O}(\mathbf{B};D)$$

subject to the following compatibilities:

(a) γ is Borel $\mathbf{Aut}_V(S)\times\prod_{U\in\mathrm{Orb}(S)}\mathbf{Aut}_U(T_U)$-equivariant,

(b) (restriction-stability of the identity) for all $U \to V$ and $C \in \mathfrak{C}_V$, the restriction map

$$\mathrm{Res}^V_U\colon \mathcal{O}(C;C) \to \mathcal{O}\big(\mathrm{Res}^V_U C;\mathrm{Res}^V_U C\big)$$

sends 1_C to $1_{\mathrm{Res}^V_U C}$;

(c) (restriction-stability of composition) for all $U \to V$, the following commutes

$$\begin{array}{ccc}
\mathcal{O}(\mathbf{C};D) \times \prod\limits_{U\in\mathrm{Orb}(S)} \mathcal{O}(\mathbf{B}_U;C_U) & \xrightarrow{\quad\gamma\quad} & \mathcal{O}(\mathbf{B};D) \\
\downarrow{\scriptstyle \mathrm{Res}^W_V} & & \downarrow{\scriptstyle \mathrm{Res}^W_V} \\
\mathcal{O}\big(\mathrm{Res}^V_W \mathbf{C};\mathrm{Res}^V_W D\big) \times \prod\limits_{U\in\mathrm{Orb}(S)} \mathcal{O}\big(\mathrm{Res}^V_W \mathbf{B}_U;C_U\big) & \xrightarrow{\quad\gamma\quad} & \mathcal{O}\big(\mathrm{Res}^V_W \mathbf{B};D\big)
\end{array}$$

(d) (unitality) for all $S \in \mathbb{F}_V$, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{O}(\mathbf{C};D) & \xrightarrow{(\mathrm{id},(\{1_U\}))} & \mathcal{O}(\mathbf{C};D) \times \prod\limits_{U\in\mathrm{Orb}(S)} \mathcal{O}(C_U;C_U) \\
\downarrow{\scriptstyle (1_V,\mathrm{id})} & \searrow & \downarrow{\scriptstyle \gamma} \\
\mathcal{O}(D;D)\times\mathcal{O}(\mathbf{C};D) & \xrightarrow{\quad\gamma\quad} & \mathcal{O}(\mathbf{C};D)
\end{array}$$

(e) (associativity) For all $S \in \mathbb{F}_V$, $(T_U) \in \mathbb{F}_S$ writing $T \coloneqq \coprod_U^S T_U$, and $(R_W) \in \mathbb{F}_T$ writing $R \coloneqq \coprod_W^T R_W$, the following diagram commutes

$$\begin{array}{ccc}
\left(\mathcal{O}(\mathbf{C};D) \times \prod_{U\in \mathrm{Orb}(S_U)} \mathcal{O}(\mathbf{B}_U;C_U)\right) \times \prod_{\substack{U\in\mathrm{Orb}(S)\\ W\in\mathrm{Orb}(T_U)}} \mathcal{O}(\mathbf{A}_W;B_W) & \xrightarrow{\gamma} & \mathcal{O}(\mathbf{B};D) \times \prod_{W\in\mathrm{Orb}(T)} \mathcal{O}(\mathbf{A}_W;B_W) \\
\| & & \\
\mathcal{O}(\mathbf{C};D) \times \prod_{U\in\mathrm{Orb}(S)} \left(\mathcal{O}(\mathbf{B}_U;C_U) \times \prod_{W\in\mathrm{Orb}(T_U)} \mathcal{O}(\mathbf{A}_W;B_W)\right) & & \downarrow \gamma \\
\gamma\downarrow & & \\
\mathcal{O}(\mathbf{C};D) \times \prod_{U\in\mathrm{Orb}(S)} \mathcal{O}(\mathbf{A}_U;C_U) & \xrightarrow{\qquad\gamma\qquad} & \mathcal{O}(\mathbf{A};D)
\end{array}$$

A morphism of $\mathfrak{C}$-colored discrete $\mathcal{T}$-operads in $\mathcal{V}$ is a map of $\mathfrak{C}$-symmetric sequences in $\mathcal{V}$ preserving each 1_C and intertwining γ; we refer to the resulting 1-category as $g\mathrm{Op}_{\mathcal{T}}^{\mathfrak{C}}(\mathcal{V})$. ◁

Construction II.1.34. Given a map of coefficient systems $f\colon \mathfrak{C} \to \mathfrak{C}'$, there is an induced map of $\mathcal{T}$-operads $\mathrm{triv}(\mathfrak{C})^{\otimes} \to \mathrm{triv}(\mathfrak{C}')^{\otimes}$ yielding a $\mathcal{T}$-functor $f\colon \underline{\Sigma}_{\mathfrak{C}} \to \underline{\Sigma}_{\mathfrak{C}'}$, and hence a precomposition functor

$$f^*\colon \mathrm{Fun}\left(\mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}'}, \mathcal{V}\right) \to \mathrm{Fun}\left(\mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}}, \mathcal{V}\right).$$

These are the cocartesian transport functors of a cocartesian fibration, which we call $\mathrm{SSeq}_{\mathcal{T}}^{\bullet}$. A *morphism of colored discrete operads* $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a morphism of their underlying objects of $\mathrm{SSeq}_{\mathcal{T}}^{\bullet}$ which is compatible with identities and composition. We refer to the resulting 1-category as $g\mathrm{Op}_{\mathcal{T}}(\mathcal{V})$. ◁

We write $s\mathrm{Op}_{\mathcal{T}}^{\mathfrak{C}} \coloneqq g\mathrm{Op}_{\mathcal{T}}^{\mathfrak{C}}(s\mathrm{Set})$ and $s\mathrm{Op}_{\mathcal{T}} \coloneqq g\mathrm{Op}_{\mathcal{T}}(s\mathrm{Set})$. In [BP21], a model structure was given to $s\mathrm{Op}_G^{\mathrm{oc}}$; this was later shown to be Quillen equivalent to several other model categorical variations on G-operads (e.g. [BP20, Tab 1]). Complementarily, [Bon19] constructed a *genuine operadic nerve* functor of 1-categories

$$N^{\otimes}\colon s\mathrm{Op}_G \to s\mathrm{Set}^{+}_{/(\mathrm{Tot}\,\underline{\mathbb{F}}_{G,*},\mathrm{Ne})}$$

whose restriction $g\mathrm{Op}_G(\mathrm{Kan})$ lands in fibrant objects in Nardin-Shah's model structure [NS22, § 2.6], and hence presents G-operads.

Moreover, $g\mathrm{Op}_{\mathcal{T}}(\mathrm{Kan})$ agrees with the *fibrant simplicial colored $\mathcal{T}$-operads* of [NS22, Def 2.5.4]; Nardin-Shah [NS22] construct an analogous nerve functor

$$N^{\otimes}\colon g\mathrm{Op}_{\mathcal{T}}(\mathrm{Kan}) \to \mathrm{sSet}^{+}_{\mathrm{CatPatt}(\mathrm{Tot}\,\underline{\mathrm{Fin}}_{\mathcal{T}})}$$

whose specialization to $\mathcal{T} = \mathcal{O}_G$ agrees with Bonventre's nerve.

These nerve functors can be understood as taking $\mathcal{O} \in g\mathrm{Op}_T^{\mathfrak{C}}(\mathbf{Kan})$ to the $\mathbf{Kan}$-enriched category over $\mathrm{Tot}\,\underline{\mathrm{Fin}}_{T,*}$ with $\mathrm{Ob}(\mathcal{O}_S) = \mathfrak{C}_S$ and with mapping spaces

$$\mathrm{Map}_{\mathcal{O}^\otimes}(\mathbf{C},\mathbf{D}) := \coprod_{S \leftarrow \pi_{\mathcal{O}}\mathbf{C} \to \pi_{\mathcal{O}}\mathbf{D}} \prod_{U \in \mathrm{Orb}(\pi_{\mathcal{O}}(\mathbf{D}))} \mathcal{O}(\mathbf{C}_U; D_U),$$

with evident composition and mapping down to $\mathrm{Map}_{\mathrm{Tot}\,\underline{\mathrm{Fin}}_{T,*}}(\pi_{\mathcal{O}}\mathbf{C}, \pi_{\mathcal{O}}\mathbf{D})$ via the evident forgetful map.

II.1.3.2 The conservative ∞-categorical lift $N^\otimes$ has homotopical properties.

Proposition II.1.35. *$N^\otimes$ preserves and reflects weak equivalences between one-color genuine G-operads in Kan complexes.*

Proof. By [BP21, Thm II, Prop 4.31], the functor $U : s\mathrm{Op}_G^{\mathrm{oc}} \to \mathbf{Fun}(\underline{\Sigma}_G, \mathbf{sSet})$ is monadic and $s\mathrm{Op}_G^{\mathrm{oc}}$ admits the transferred model structure from the projective model structure on $\mathbf{Fun}\left(\mathrm{Tot}\,\underline{\Sigma}_G, \mathbf{sSet}_{\mathrm{Quillen}}\right)$; in particular, U preserves and reflects weak equivalences.

It is not hard to see that $\mathbf{sseq}$ is right-derived from a functor

$$s\mathbf{sseq} : \mathbf{sSet}^{+,\mathrm{oc}}_{/(\mathbb{F}_T, Ne)} \to \mathbf{Fun}\left(\mathrm{Tot}\,\underline{\Sigma}_G, \mathbf{sSet}_{\mathrm{Quillen}}\right)_{\mathrm{Proj}}$$

setting $\mathcal{O}_{\mathbf{sseq}}(S) := \pi_{\mathcal{O}}^{-1}(\mathrm{Ind}_H^G S \to G/H)$; by Proposition II.1.29 $\mathbf{sseq}$ is conservative, so $s\mathbf{sseq}$ preserves and reflects weak equivalences between fibrant objects. Hence it suffices unwind definitions and note that the following diagram commutes

$$\begin{array}{ccc} s\mathrm{Op}_G^{\mathrm{oc}} & \xrightarrow{N^\otimes} & \mathbf{sSet}^{+,\mathrm{oc}}_{/(\mathbb{F}_G, Ne)} \\ & \searrow^{U} & \downarrow{\scriptstyle s\mathbf{sseq}} \\ & & \mathbf{Fun}(\mathrm{Tot}\,\underline{\Sigma}_G, \mathbf{sSet}) \end{array}$$

□

In fact, the one-color assumption was unnecessary. We say that a map of genuine simplicial G-operads $\varphi\colon \mathcal{O} \to \mathcal{P}$ is a *weak equivalence* if it is an isomorphism on coefficient systems and for all profiles $(\mathbf{C}; D)$, the map $\mathcal{O}(\mathbf{C}; D) \to \mathcal{P}(\varphi\mathbf{C}; \varphi D)$ is a weak equivalence. These weak equivalences satisfy two-out-of-three (in fact, two-out-of-six) by the same property for isomorphisms and for weak equivalences of simplicial sets.

Proposition II.1.36. *$N^\otimes$ preserves and reflects weak equivalences between arbitrary genine G-operads in Kan complexes.*

Proof. It is not hard to see that $N^\otimes$ preserves and reflects the property of *inducing isomorphism on coefficient systems of colors*, so we may fix a coefficient system of sets of colors $\mathfrak{C}$ and verify that

$$N^\otimes_{\mathfrak{C}}\colon g\mathrm{Op}_G^{\mathfrak{C}}(\mathbf{Kan}) \to s\mathbf{Set}^+_{/(\mathbb{F}_{G,*}, NE)}$$

preserves and reflects weak equivalences. Thankfully, we have the same tools as in the one-color case; Proposition II.1.29 constructs a functor $s\,\mathrm{sseq} : \mathrm{sSet}^{+,\mathfrak{C}}_{/(\underline{\mathbb{F}}_{\mathcal{T}},Ne)} \to \mathsf{Fun}\left(\mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}}, \mathrm{sSet}_{\mathrm{Quillen}}\right)_{\mathrm{Proj}}$ which preserves and reflects weak equivalences between fibrant objects, and $N^{\otimes}_{\mathfrak{C}}$ is a functor over $\mathsf{Fun}\left(\mathrm{Tot}\,\underline{\Sigma}_{\mathfrak{C}}, \mathrm{sSet}_{\mathrm{Quillen}}\right)$; by two-out-of-three for weak equivalences, $N^{\otimes}$ preserves and reflects weak equivalences between fibrant objects. □

Remark II.1.37. In [BP21], another nerve functor $\iota_*\colon g\mathrm{Op}^{\mathrm{oc}}_e(\mathrm{sSet})^{BG} \to g\mathrm{Op}^{\mathrm{oc}}_G(\mathrm{sSet})$ was constructed and shown to furnish a Quillen equivalence for a model structure on $g\mathrm{Op}^{\mathrm{oc}}_e(\mathrm{sSet})^{BG}$ whose weak equivalences and fibrations are preserved and reflected by the *graph subgroup* fixed points $\prod_{n\in\mathcal{R}}\left\{\mathcal{O}(n)^{\Gamma} \mid \Gamma \in \mathcal{O}_{G\times\Sigma_n,\Gamma}\right\}$ for the Quillen model structure on sSet. In particular, under the equivalence $\mathrm{Un}_{\mathcal{O}_G}\mathcal{O}_{G\times\Sigma_n,\Gamma} \simeq B_G\Sigma_n$, an object $\mathcal{O} \in g\mathrm{Op}_e(\mathrm{sSet})^{BG}$ has an n-ary $B_G\Sigma_n$ space $\mathcal{O}(n)$. In fact, unwinding definitions using [BP21, Rmk 4.38], we find that there is an equivalence

$$\underline{N^{\otimes}\iota_*\mathcal{O}}(n) \simeq \mathcal{O}(n).$$ ◁

II.1.4 Boardman-Vogt tensor products of parameterized operads

Construction II.1.38. Fix an object $V \in \mathcal{T}$. The *Boardman-Vogt tensor product* $\overset{\mathrm{BV}}{\otimes}$ is the tensor product associated with the magmatic structure on $\underline{\mathrm{Fin}}_{\mathcal{T},*}$ of Proposition I.4.39, and the *pointwise $\mathcal{T}$-operad structure on algebras* is the associated internal hom via Proposition I.3.67. ◁

Proposition I.3.67 immediately implies the following.

Corollary II.1.39. *Fix $\mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ a map of I-operads and $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a map of I-symmetric monoidal ∞-categories. Then, $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})$ is a I-symmetric monoidal ∞-category, and the canonical lax I-symmetric monoidal functors*

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}), \qquad \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{D})$$

are I-symmetric monoidal.

Using this, in the one-color case we view $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$-algebras as *homotopy-coherently interchanging $\mathcal{O}$-algebra and $\mathcal{P}$-algebra structures on a common $\mathcal{T}$-object*. Now, Proposition I.3.68 specializes to the following.

Corollary II.1.40. *The V-symmetric monoidal envelope intertwines with the localized Day convolution structure*

$$\mathrm{Env}\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}\right) \simeq \mathrm{Env}\left(\mathcal{O}^{\otimes}\right) \circledast \mathrm{Env}\left(\mathcal{P}^{\otimes}\right).$$

In particular, [BS24a, Thm E] shows that this property identifies the non-equivariant Boardman-Vogt tensor product, so we acquire the following.

Corollary II.1.41. *When $\mathcal{T} \simeq *$, $\overset{BV}{\otimes}$ is naturally equivalent to the Boardman-Vogt tensor product of [BS24a; HM23; HA].*

Additionally, we may characterize the $\overset{\mathrm{BV}}{\otimes}$-unit.

Proposition II.1.42. *For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_V$, we have $\mathcal{O}^{\otimes} \simeq \mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathrm{triv}_V^{\otimes}$; hence there exists a natural equivalence*

$$\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}_V}(\mathcal{O}) \to \mathcal{O}^{\otimes}.$$

Proof. The first statement implies the second by the usual folklore argument:

$$\begin{aligned}\mathrm{Map}(\mathcal{O}^{\otimes}, \underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}_V}(\mathcal{P})) &\simeq \mathrm{Map}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_V^{\otimes}, \mathcal{P}^{\otimes}\right), \\ &\simeq \mathrm{Map}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}),\end{aligned}$$

so Yoneda's lemma yields a natural equivalence $\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}_V}(\mathcal{P}) \simeq \mathcal{P}^{\otimes}$. The same argument in reverse shows that the second statement implies the first.

By the definitional equivalences

$$\mathrm{triv}_{\mathcal{T}}^{\otimes} \simeq L_{\mathrm{Fbrs}}\left(\iota_! \mathrm{Patt}(*_{\mathcal{T}})\right) \simeq L_{\mathrm{Op}_{\mathcal{T}}}(*_{\mathcal{T}} \to \mathrm{Span}(\mathbb{F}_{\mathcal{T}})),$$

bifunctors $\mathrm{triv}_{\mathcal{T}}^{\otimes} \times \mathcal{O} \to \mathcal{P}$ correspond canonically with functors of $\mathcal{T}$-operads $\mathcal{O} \to \mathcal{P}$; put another way, using the bifunctor presentation of Observation I.3.63, this demonstrates that the forgetful natural transformation

$$\mathrm{Alg}_{\mathcal{O} \otimes \mathrm{triv}_{\mathcal{T}}}(\mathcal{P}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$$

is a natural equivalence for all $\mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$; Yoneda's lemma then demonstrates that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes} \simeq \mathcal{O}^{\otimes}$. □

Using this, we have a sequence of natural equivalences

$$\begin{aligned}U \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) &\simeq \underline{\mathrm{Alg}}_{\mathrm{triv}_V} \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \\ &\simeq \underline{\mathrm{Alg}}_{\mathcal{O} \otimes \mathrm{triv}_V}(\mathcal{P}) \\ &\simeq \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P});\end{aligned}$$

in particular, we've proved the following corollary.

Corollary II.1.43. *There exists a natural equivalence*

$$U \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P}).$$

We've shown in Proposition I.3.68 that Env intertwines $\overset{\mathrm{BV}}{\otimes}$ with $\circledast$, and we've now seen that $\mathsf{triv}_V^{\otimes}$ is the $\overset{\mathrm{BV}}{\otimes}$-unit. In fact, Env intertwines units.

Proposition II.1.44. $\mathrm{Env}_I(\mathrm{triv}_{\mathcal{T}})$ *is the* $\circledast$*-unit in* $\mathrm{CMon}_I(\mathrm{Cat})^{\circledast}$.

Proof. By symmetric monoidality of the Day convolution structure, , when $\mathcal{C}^{\times}$ is cartesian, the free object $\mathrm{Fr}_I(*) \in \mathrm{CMon}_I(\mathcal{C})$ is the $\circledast$-unit, so we may apply Yoneda's lemma to the equivalence.

$$\begin{aligned}
\mathrm{Fun}_I^{\otimes}(\mathrm{Env}_I(\mathrm{triv}_{\mathcal{T}})^{\otimes}, \mathcal{D}^{\otimes}) &\simeq \mathrm{Alg}_{\mathrm{triv}_{\mathcal{T}}}(\mathcal{D}^{\otimes}) && \text{II.1.19}\\
&\simeq \Gamma^{\mathcal{T}}\mathcal{D}\\
&\simeq \mathrm{Fun}_I^{\otimes}(\mathrm{Fr}_I *, \mathcal{D}^{\otimes})\\
&\simeq \mathrm{Fun}_I^{\otimes}\left(1^{\circledast}, \mathcal{D}^{\otimes}\right).
\end{aligned}$$

□

II.2 Basic examples

We now introduce a large number of basic examples of I-symmetric monoidal ∞-categories and I-operads. As in the classical case, a good first step is *cartesian and cocartesian structures*; we introduce such structures in Section II.2.1 and characterize algebras in the cartesian case in terms of Segal $\mathcal{T}$-objects over the associated $\mathcal{T}$-pattern; notably, this yields fairly simple presentations for the free $\mathcal{O}$-algebra monad in cartesian cases without any serious considerations about distributivity, allowing us to prove early that evaluation on $\mathcal{T}$-spaces detects "fully-faithfulness" of $\mathcal{T}$-operad maps.

Moving on from this, we recognize in Section II.2.2 that the cocartesian structure exists for all $\mathcal{T}$-categories $\mathcal{C}$ as a unital I-operad characterized by mapping-in and mapping-out properties, which is I-symmetric monoidal precisely when $\mathcal{C}$ has I-indexed coproducts. By relating I-cocartesianness to I-semiadditivity, we use this to prove that $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{D})$ is *universal* for the property that it is I-cocartesian and lies I-symmetric monoidally over $\mathcal{D}^{\otimes}$; we recover Cnossen-Lenz-Linskens' semiadditive closure expression of $\underline{\mathrm{CMon}}_I$ as a consequence.

Moving on to operad-leaning examples, we quickly move on to lift Lurie's *disintegration and assembly* procedure to $\mathcal{T}$-operads in Section II.2.3. The major upshot is that we may reduce $\overset{\mathrm{BV}}{\otimes}$-computations of $\mathcal{T}$-space-colored unital I-operads to the reduced case. Following this, we take a short detour in Section II.2.4 to describe the I-operad and I-symmetric monoidal incarnations of Sections I.3.4 and I.3.6; notably, in Section II.2.5 we recognize a type of localization of I-operads lifting the Morita $\mathcal{T}$-localizations of Section I.3.6, ultimately strengthening the expression in terms of *locally constant algebras* from the cartesian case to algebras in arbitrary $\mathcal{T}$-operads.

Then, we move on to a concrete source of examples in equivariant homotopy theory; in Section II.2.6 we review a construction from joint work of the author with Juran of a G-symmetric monoidal ∞-category of G-manifolds with arbitrary tangential structure and C^∞-embedding spaces, ultimately exhibiting the full G-symmetric monoidal subcategory spanned by disks as the envelope of a G-operad $\mathbb{E}^\otimes_\Theta$. In the fully framed case, $\mathbb{E}^\otimes_V$, we exhibit a natural $\mathbb{E}_V$-structure on V-fold loop spaces extending a well-known construction in the finite group case and state a recognition principle due to Juran in the finite group case.

Crucially, in Section II.2.7, we use the disintegration and assembly procedure to assemble $\mathbb{E}^\otimes_\Theta$ (and hence $\underline{\mathrm{Disk}}^{\Theta,\sqcup}_G$) as a B_Θ-indexed G-colimit of the fully framed case. As such, we recover $\mathbb{E}_\Theta$-algebras as B_Θ-indexed diagrams of $\mathbb{E}_{\Theta(x)}$-algebras, with nontrivial action of the loop G-spaces in B_Θ jointly on the algebras and operads. This will form the basis of reducing analysis of such objects to the fully framed case.

Indeed, following this, in the case of a finite group G, in Section II.2.8 we establish a concrete topological model for $\mathbb{E}_V$ in the realm of *rectilinear embeddings of little polydisks*; this interpolates between "affine" and linear little cube embeddings in the nonequivariant case, and it is custom-fit as a model to exhibit later additivity phenomenon.

Finally, in Section II.2.9 we extend the notion of *prefactorization algebras* to this setting, in preparation for later work on additivity of equivariant little disk operads. The upshot is that $\mathbb{E}_V$-algebras embed fully faithfully into $\mathcal{U}$-prefactorization algebras as *locally constant algebras*, if $\mathcal{U} \subset \underline{\mathrm{Open}}(V)$ is a reasonably nice open cover; we show that *decomposable little polydisks* form such a cover, encapsulating the main topological phenomenon behind little disk additivity.

II.2.1 Parameterized Cartesian structures and $\mathcal{O}$-monoids For the following two subsections, fix I an almost-unital weak indexing category. Denote by $\underline{\mathrm{CAT}}^{I-\sqcup}_I, \underline{\mathrm{CAT}}^{I-\times}_I \subset \underline{\mathrm{CAT}}_\mathcal{T}$ the locally fully-faithful $\mathcal{T}$-$(\infty,2)$-subcategories with V-objects the V-∞-categories with I-indexed coproducts (resp. products) and V-morphisms the V-functors which preserve I-indexed coproducts (products). We will prove the following.

Theorem II.2.1. *There are fully faithful embeddings $(-)^{I-\sqcup}, (-)^{I-\times}$ making the following commute:*

$$\begin{array}{ccccc} \underline{\mathrm{CAT}}^{I-\sqcup}_I & \xrightarrow{(-)^{I-\sqcup}} & \underline{\mathrm{CAT}}^{\otimes}_I & \xleftarrow{(-)^{I-\times}} & \underline{\mathrm{CAT}}^{I-\times}_I \\ & {\scriptstyle U}\searrow & \downarrow{\scriptstyle U} & \swarrow{\scriptstyle U} & \\ & & \underline{\mathrm{CAT}}_\mathcal{T} & & \end{array}$$

*The image of $(-)^{I-\sqcup}$ is the I-symmetric monoidal ∞-categories whose $v(I)$-units are initial and whose $\otimes$-*Wirthmüller *transformations*

$$W\colon \coprod^S \Longrightarrow \bigotimes^S$$

are equivalences, and $(-)^{I-\times}$ *their fiberwise opposites. Moreover, there exists a natural transformation* $\mathrm{Tot}_{\mathcal{T}}\,\mathcal{C}^{I-\times} \to \mathcal{C}$ *such that postcomposition factors through a natural equivalence*

$$\underline{\mathrm{Alg}}_{\mathcal{O}}\left(\mathcal{C}^{I-\times}\right) \xrightarrow{\sim} \underline{\mathrm{Seg}}_{\mathcal{O}}(\mathcal{C});$$

in particular, we acquire an equivalence

$$\underline{\mathrm{CAlg}}_{I}\left(\mathcal{C}^{I-\times}\right) \simeq \underline{\mathrm{CMon}}_{I}(\mathcal{C}).$$

We refer to I-symmetric monoidal ∞-categories satisfying the above condition for coproducts as *cocartesian*, and for products as *cartesian*. Many similar definitions have been made in the literature. Luckily, they all agree.

Remark II.2.2. [NS22] constructed a pair of structures $\mathcal{C}^{\amalg}, \mathcal{C}^{\prod}$ which, after unwinding definitions, satisfy the conditions of Theorem II.2.1 in the case $I = \mathcal{T}$. In particular, there are unique I-symmetric monoidal equivalences $\mathrm{Bor}_{I}^{\mathcal{T}}\,\mathcal{C}^{\amalg} \simeq \mathcal{C}^{I-\sqcup}$ and $\mathrm{Bor}_{I}^{\mathcal{T}}\,\mathcal{C}^{\prod} \simeq \mathcal{C}^{I-\times}$ lying over the identity whenever $\mathcal{C}$ admits finite indexed (co)products.

Moreover, [CHLL24b] introduced another structure, specifically a *cartesian I-symmetric monoidal structure* on $\underline{\mathrm{Fin}}_{\mathcal{T}}$, and conjectured it to be equivalent to Nardin-Shah's construction; this conjecture was recently verified in [CLR25] by verifying universality of the unfurling construction used in Nardin-Shah's construction. We acquire an independent proof of this result without serious $(\infty,2)$-category theory: Cnossen-Haugseng-Lenz-Linsken's construction satisfies the condition of Theorem II.2.1, so there is a unique pair of I-symmetric monoidal equivalences $\mathrm{Bor}_{I}^{\mathcal{T}}\,\underline{\mathrm{Fin}}_{\mathcal{T}}^{\prod} \simeq \underline{\mathrm{Fin}}_{\mathcal{T}}^{I-\times} \simeq \underline{\mathrm{Fin}}_{\mathcal{T},\times}$ lying over the identity.

Moreover, after drafts of this result were made public, [LLP25, Def. 2.15] constructed another structure, which specializes to a *cocartesian I-symmetric monoidal structure* $\mathcal{C}^{I-\amalg}$; by [LLP25, Lem. 2.16], in the case that $\mathcal{C}$ has I-indexed coproducts, $\mathcal{C}^{I-\amalg}$ satisfies the conditions of Theorem II.2.1, so there is a unique I-symmetric monoidal equivalence $\mathcal{C}^{I-\amalg} \simeq \mathcal{C}^{I-\sqcup}$ lying over the identity. ◁

Remark II.2.3. It follows from Theorem II.2.1 that, given an inclusion $J \subset I$, the following diagram commutes

$$\begin{array}{ccccc}
\underline{\mathrm{CAT}}_{\mathcal{T}}^{I-\sqcup} & \longrightarrow & \underline{\mathrm{CAT}}_{I}^{\otimes} & \longleftarrow & \underline{\mathrm{CAT}}_{\mathcal{T}}^{I-\times} \\
{\scriptstyle \mathrm{Bor}_{J}^{I}}\downarrow & & {\scriptstyle \mathrm{Bor}_{J}^{I}}\downarrow & & {\scriptstyle \mathrm{Bor}_{J}^{I}}\downarrow \\
\underline{\mathrm{CAT}}_{\mathcal{T}}^{J-\sqcup} & \longrightarrow & \underline{\mathrm{CAT}}_{J}^{\otimes} & \longleftarrow & \underline{\mathrm{CAT}}_{\mathcal{T}}^{J-\times}
\end{array}$$

i.e. $\mathrm{Bor}_{J}^{I}(-)^{I-\sqcup} \simeq (-)^{J-\sqcup}$. As a consequence, when it will not cause confusion, we will simply refer to $(-)^{\sqcup}$ and $(-)^{\times}$ and leave the ambient weak indexing category (which is almost always $\mathbb{F}_{\mathcal{T}^{\simeq}}$ or $\mathbb{F}_{\mathfrak{o}(\mathcal{T})}$) implicit. ◁

II.2.1.1 Semiadditivity as an I-symmetric monoidal equivalence We call I-symmetric monoidal ∞-categories of the form $\mathcal{C}^{I-\sqcup}$ *cocartesian*, and $\mathcal{C}^{I-\times}$ *cartesian*. Now, by $\otimes$-Wirthmüller maps, we are referring to the following construction.

Construction II.2.4. Let $\mathcal{C}^{\otimes}$ be an I-symmetric monoidal ∞-category satisfying the property that $1_\bullet \in \Gamma^{\upsilon(I)}\mathcal{C}$ is initial and let $(X_U) \in \mathcal{C}_S$ be an S-tuple for some $S \in \underline{\mathbb{F}}_I$. The $\otimes$-*Wirthmüller map for* (X_U) is the map

$$W_{S,(X_U)} \colon \coprod_U^S X_U \longrightarrow \bigotimes_U^S X_U$$

classified by the summand maps

$$\begin{array}{ccc} X_W & \xrightarrow{W_{S,(X_U),U'}} & \mathrm{Res}_U^V \bigotimes_W^S X_W. \\ \wr & & \wr \\ X_W \otimes \bigotimes_{U'}^{\mathrm{Res}_U^V S-W} 1_{U'} & \xrightarrow{(\mathrm{id};!)} & X_W \otimes \bigotimes_{U'}^{\mathrm{Res}_U^V S-W} \mathrm{Res}_{U'}^{o(U')} X_{o(U')} \end{array}$$

where $o(U') \in \mathsf{Orb}(S)$ is the orbit whose restriction contains U'. 1_U exists and is initial by almost-unitality. Dually, if $\mathcal{C}^{\otimes}$ is an I-symmetric monoidal ∞-category such that $1_\bullet \in \Gamma^{\upsilon(I)}\mathcal{C}$ the $\otimes$-*coWirthmüller map* $W^{co}_{S,(X_U)} \colon \bigotimes_U^S X_U \to \prod_U^S X_U$ is the Wirtmüller map for $(X_U)_S$ in the fiberwise opposite $\mathcal{C}^{\otimes,\mathrm{vop}}$.

Note that, in the case that $\mathcal{C}^{\otimes}$ has all $\otimes$-Wirthmüller maps equivalences, the $\otimes$-coWirthmüller maps are the ordinary Wirthmüller maps; similarly, if the $\otimes$-coWirthmüller maps are equivalences, then the $\otimes$-Wirtmüller maps instead are the ordinary Wirthmüller maps. ◁

This identification of (co)Wirthmüller maps allows us to incarnate the ostensibly *parameterized higher categorical* property of I-semiadditivity within the framework of parameterized higher algebra.

Corollary II.2.5. *An I-symmetric monoidal category is I-semiadditive if and only if it admits I-products and coproducts and there exists an equivalence $\mathcal{C}^{I-\sqcup} \simeq \mathcal{C}^{I-\times}$ lying over the identity.*

Proof. Employing the remarks in Construction II.2.4, note that are assumption is equivalent to either the condition that $\mathcal{C}^{I-\times}$ lies in the essential image of $(-)^{I-\sqcup}$ or that $\mathcal{C}^{I-\sqcup}$ in $(-)^{I-\times}$. Under this assumption, it follows immediately from Theorem II.2.1 that the essential image of $\underline{\mathrm{Fun}}_I^{\otimes}\left(\mathcal{C}^{I-\sqcup},\mathcal{C}^{I-\times}\right) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C},\mathcal{C})$ contains the identity; conversely, if $\mathcal{C}^{I-\times} \simeq \mathcal{C}^{I-\sqcup}$, then $\mathcal{C}^{I-\times}$ satisfies both the $\times$-Wirthmüller and $\times$-coWirthmüller isomorphims, so by the remarks in Construction II.2.4 give the claim. □

II.2.1.2 Detecting equivalences of G-operads by their algebras We postpone the proof of Theorem II.2.1 until the end of the section, instead first focusing on its applications. First, combining it with with Theorem I.4.1 immediately yields the following.

Corollary II.2.6. *Given $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$ and $\mathcal{D}$ an ∞-category with finite products, there exists a natural equivalence*

$$\mathrm{Alg}_{\mathcal{O}}\left(\underline{\mathrm{Coeff}}^{G}(\mathcal{D})^{\times}\right) \simeq \mathrm{Seg}_{\mathrm{Tot}\,\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}}(\mathcal{D}).$$

In particular, we acquire a natural equivalence

$$\mathrm{CAlg}_{I}\left(\underline{\mathrm{Coeff}}^{G}(\mathcal{D})^{\times}\right) \simeq \mathrm{CMon}_{I}(\mathcal{D}).$$

Now, for expositional reasons, we may use this to understand free $\mathcal{O}$-algebra G-spaces far before free $\mathcal{O}$-algebras in other G-symmetric monoidal categories. Explicitly, we find the following.

Theorem II.2.7 ("Equivariant [HM23, Thm 4.1.1]")**.** *A map of $\mathcal{T}$-operads $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is an equivalence if and only if it satisfies the following conditions:*

(a) the $\mathcal{T}$-functor $U\varphi\colon U\mathcal{O} \to U\mathcal{P}$ is essentially surjective, and

(b) the pullback functor $\varphi^{*}\colon \mathrm{Alg}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is an equivalence of ∞-categories.

In fact, we will prove the following stronger result.

Proposition II.2.8. *Suppose a map of $\mathcal{T}$-operads $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ induces a π_0-bijection $U\varphi\colon \mathcal{O} \to \mathcal{P}$ and an equivalence $\mathrm{Alg}_{\mathcal{P}}\left(\mathrm{Coeff}^{\mathcal{T}}\mathcal{C}\right) \to \mathrm{Alg}_{\mathcal{O}}\left(\mathrm{Coeff}^{\mathcal{T}}\mathcal{C}\right)$ for $\mathcal{C}$ a presentable and cartesian closed category. Then, the induced $\mathcal{C}$-map $\mathrm{Fr}_{\mathcal{C}}\mathcal{O}(\mathbf{B};C) \to \mathrm{Fr}_{\mathcal{C}}\mathcal{P}(\varphi\mathbf{B};\varphi C)$ is an equivalence for all $\mathcal{O}$-profiles $(\mathbf{B};C)$.*

Note that Propositions II.1.29 and II.2.8 together immediately prove Theorem II.2.7. Now, we temporarily import the following result of Chu-Haugseng.

Lemma II.2.9 ([CH21, Cor. 8.2,Cor. 8.12])**.** *Suppose $\mathfrak{O}$ is an extendable algebraic pattern satisfying the condition that $\mathfrak{O}^{\mathrm{el}}_{O/}$ admits a zigzag of initial functors to a finite set for all $O \in \mathfrak{O}$ and $\mathcal{C}$ is a presentable and cartesian closed category. Then, the forgetful functor $\mathrm{Seg}_{\mathfrak{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathfrak{O}^{\mathrm{el}},\mathcal{C})$ is monadic, and the associated* free Segal object *endofunctor on $\mathrm{Fun}(\mathfrak{O}^{\mathrm{el}},\mathcal{C})$ has value on F given by*

$$\mathrm{Fr}_{\mathfrak{O}}(F)(O) = \mathrm{colim}_{O' \in (\mathcal{O}^{\mathrm{act}}_{/O})^{\simeq}} \lim_{E \in \mathcal{O}^{\mathrm{el}}_{O'/}} F(E).$$

Proof of Proposition II.2.8. According to Corollary I.4.35 and Theorem I.4.1 we are in the situation of Lemma II.2.9 for the pattern $\mathrm{Tot}\,\mathcal{O}^{\otimes}$. Now, note in this case that the *essentially unique* G-product-preserving and G-colimit preserving functor $\underline{\mathcal{S}}_{G} \longrightarrow \mathcal{C}$ sends

$$\mathrm{Fr}_{\mathcal{O}}(\mathrm{Fr}_{\mathcal{C}}X)^{V}_{D} \simeq \mathrm{Fr}_{\mathrm{Tot}\,\mathcal{O}^{\otimes}}((\mathrm{Fr}_{\mathcal{C}}X))(D,V) \simeq \mathrm{Fr}_{\mathcal{C}}\mathrm{Fr}_{\mathrm{Tot}\,\mathcal{O}^{\otimes}}(X)(D,V) \simeq \mathrm{Fr}_{\mathcal{C}}\mathrm{Fr}_{\mathcal{O}}(X)^{V}_{D}$$

by the formula of Lemma II.2.9. In particular, for each profile $(\mathbf{C};D)$, it suffices to provide a diagram $y_{(\mathbf{C};D)}\colon U\mathcal{O}\longrightarrow \underline{\mathcal{S}}_G$ and a natural splitting

$$\mathrm{Fr}_{\mathcal{O}}\left(y_{(\mathbf{C};D)}\right)_D^V \simeq \mathcal{O}(\mathbf{C};D)\sqcup J_{(\mathbf{C};D),V,\mathcal{O}}$$

for some junk space $J_{(\mathbf{C};D),V}$; applying the monad to $\mathrm{Fr}_{\mathcal{C}}\, y_{(\mathbf{C};D)}$ will then yield an equivalence

$$\mathrm{Fr}_{\mathcal{C}}\,\mathcal{O}(\mathbf{C};D)\sqcup \mathrm{Fr}_{\mathcal{C}}\, J_{(\mathbf{C};D),V,\mathcal{O}} \xrightarrow{\sim} \mathrm{Fr}_{\mathcal{C}}\,\mathcal{P}(\varphi\mathbf{C};D)\sqcup \mathrm{Fr}_{\mathcal{C}}\, J_{(\mathbf{C};D),V,\mathcal{P}}$$

respecting the summand decomposition; in particular, it will yield the desired equivalence.

Now we embark on our goal. Write $\mathfrak{C}\coloneqq \pi_0 U\mathcal{O}\simeq \pi_0 U\mathcal{P}$. Note that we acquire, for each $\mathcal{T}$-diagram $X\colon U\mathcal{O}\to \underline{\mathcal{S}}_{\mathcal{T}}$ and V-equivariant color D, a natural equivalence of spaces

$$\begin{aligned}\mathrm{Fr}_{\mathcal{O}} X^V(D) &\simeq \mathrm{Fr}_{\mathrm{Tot}\,\mathcal{O}^{\otimes}}(X,V)(D)\\ &\simeq \coprod_{(\mathbf{C},D)\in\Sigma_{\mathfrak{C},V}}\left(\mathcal{O}(f\mathbf{C};fD)\times\prod_{U\in\pi\mathbf{C}} X^U(C_U)\right)_{h\,\mathrm{Aut}^V_{\mathfrak{C}}\mathbf{C}}\\ &\simeq \coprod_{(\mathbf{C},D)\in\Sigma_{\mathfrak{C},V}}\left(\mathcal{O}(f\mathbf{C};fD)\times \mathrm{map}_{\mathrm{Fun}_{\mathcal{T}}(\mathfrak{C},\underline{\mathcal{S}}_{\mathcal{T}})}(\mathbf{C},X)^V\right)_{h\,\mathrm{Aut}^V_{\mathfrak{C}}\mathbf{C}}\end{aligned}\tag{21}$$

Now, for a profile $(\mathbf{C};D)$, let $\text{ⱶ}_{\mathcal{C}}(\mathbf{C};D)$ be the somewhat-constant functor

$$U\mathcal{O}\longrightarrow \mathfrak{C}\longrightarrow \underline{\Sigma}_{\mathfrak{C},V}\xrightarrow{\mathrm{map}(-,(\mathbf{C};D))}\underline{\mathcal{S}}_G$$

Indeed, now by definition there is a free $\mathrm{Aut}^V_{\mathfrak{C}}\mathbf{C}$-summand

$$\mathrm{Aut}^V_{\mathfrak{C}}(\mathbf{C})\subset \mathrm{End}^V(\mathbf{C};D)\simeq \text{ⱶ}_{\mathcal{C}}(\mathbf{C};D)^V,$$

and the rest of the proposition follows as argued above. □

II.2.1.3 Pointwise indexed tensor products of $\mathcal{O}$-monoids We will use Theorem II.2.7 to study interchange, for which we need the following.

Proposition II.2.10. *If $\mathcal{C}$ has I-indexed products and $\mathcal{O}^{\otimes}$ is an I-operad, then the pointwise structure $\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}\left(\mathcal{C}^{I-\times}\right)$ is cartesian.*

Now, to prove this, we first need the following lemmas.

Lemma II.2.11. *$U\colon \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{D})\to \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O},U\mathcal{D})$ creates $\mathcal{T}$-limits.*

PROOF. We claim that each of the forgetful functors

$$\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{D}) \to \underline{\mathrm{Fun}}_{\mathcal{T},/\mathrm{Tot}_{\mathcal{T}}\mathcal{O}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O},\mathcal{D}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O},U\mathcal{D})$$

preserves $\mathcal{T}$-limits; indeed the second follows immediately from Proposition I.1.38, and for the latter, it suffices to note that limits are defined pointwise by Proposition I.1.38, and hence take diagrams of inert-cocartesian arrows to inert-cocartesian arrows (see [NS22, Thm. 5.1.3]). □

Lemma II.2.12. *For all $\mathcal{T}$-categories $\mathcal{B},\mathcal{C}$ such that $\mathcal{C}$ has I-indexed products, $\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}(\mathcal{B})}\left(\mathcal{C}^{I\times}\right)$ is cartesian.*

PROOF. Under Theorem II.2.1, simply apply Yoneda's lemma to the natural equivalence

$$\begin{aligned}\mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathrm{triv}(\mathcal{B})}(\mathcal{C}) &\simeq \mathrm{Alg}_{\mathrm{triv}(\mathcal{B})}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C})\\ &\simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{B},\underline{\mathrm{Seg}}_{\mathcal{O}}(\mathcal{C}))\\ &\simeq \mathrm{Seg}_{\mathcal{O}}(\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{B},\mathcal{C}))\\ &\simeq \mathrm{Alg}_{\mathcal{O}}\left(\underline{\mathrm{Alg}}_{\mathrm{triv}(\mathcal{B})}(\mathcal{C})^{I\times}\right).\end{aligned}$$

□

Lemmas II.2.11 and II.2.12 reduce Proposition II.2.10 to part (4) of the following lemma.

Lemma II.2.13. *Fix $F\colon \mathcal{D}^{\otimes}\to\mathcal{C}^{\otimes}$ an I-symmetric monoidal functor.*

(1) If F is I-coproduct preserving then there is a homotopy $FW_{S,(X_U)}\sim W_{S,F(X_U)}$.

(2) If F is I-product preserving then there is a homotopy $FW^{co}_{S,(X_U)}\sim W^{co}_{S,F(X_U)}$.

(3) If F is I-coproduct-preserving and conservative and $\mathcal{C}^{\otimes}$ is cocartesian then $\mathcal{D}^{\otimes}$ is cocartesian.

(4) If F is I-product preserving and conservative and $\mathcal{C}^{\otimes}$ is cartesian then $\mathcal{D}^{\otimes}$ is cartesian.

PROOF. For (1), since F is I-coproduct preserving, we're tasked with constructing a homotopy making the following diagram commute

$$\begin{array}{ccc} F\left(X_U\otimes\bigotimes_W^{\mathrm{Res}^V_U S-U} 1_W\right) & \xrightarrow{(\mathrm{id};!)} & F\left(X_U\otimes\bigotimes_W^{\mathrm{Res}^V_U S-U} X_W\right)\\ \wr & & \wr\\ FX_U\otimes\bigotimes_W^{\mathrm{Res}^V_U S-U} 1_U & \xrightarrow{(\mathrm{id};!)} & FX_U\otimes\bigotimes_W^{\mathrm{Res}^V_U S-U} FX_W\end{array}$$

In fact, there is a contractible space of such choices. (2) follows by applying fiberwise-opposites.

For (3), applying (1) shows that $FW_{S,(X_U)}$ is an equivalence for all $S\in\underline{\mathbb{F}}_I$ and $(X_U)\in\mathcal{D}_S$, so conservativity implies that $W_{S,(X_U)}$ is an equivalence, i.e. $\mathcal{D}^{\otimes}$ is cocartesian. (4) follows similarly from (2). □

II.2.1.4 The trivial $\mathcal{O}$-algebra A toy case for the study of interchange is interchange of $\mathbb{E}_0$-algebras. We now show how to use Theorem II.2.1 to give a relatively elementary proof of the sharp version of [NS22, Thm. 5.2.11], without use of [NS22, Thm. 5.2.10].

Definition II.2.14. We say that an I-operad $\mathcal{O}^\otimes$ is *unital* if $\mathcal{O}(\varnothing_V; C) = *$ for all $C \in \mathcal{O}_V$ with $V \in v(I)$, and *reduced* if also $\mathcal{O}(*_V) = *$ for all $V \in c(I)$. More generally if $\mathcal{F} \subset v(I)$ is a family, we say that $\mathcal{O}^\otimes$ is *$\mathcal{F}$-unital* if $\mathcal{O}(\varnothing_V; C) \simeq *$ for all $C \in \mathcal{O}_V$ and $V \in \mathcal{F}$ and *$\mathcal{F}$-reduced* if also $\mathcal{O}(*_V) = *$ for all $V \in \mathcal{F}$; equivalently, $\mathcal{O}^\otimes$ is $\mathcal{F}$-unital (resp. $\mathcal{F}$-reduced) if and only if $\mathrm{Bor}^I_{I \cap \mathbb{F}_\mathcal{F}} \mathcal{O}^\otimes$ is unital (reduced). ◁

A minimal example is the following.

Example II.2.15. Given the almost-unital weak indexing category $I_{0,\mathcal{F}}$ corresponding with the weak indexing system

$$\mathbb{F}^V_{\mathcal{T},0,\mathcal{F}} \coloneqq \begin{cases} \{\varnothing_V, *_V\} & V \in \mathcal{F} \\ \{*_V\} & \text{otherwise,} \end{cases}$$

we define the $\mathcal{T}$-operad $\mathbb{E}^\otimes_{0,\mathcal{F}} \coloneqq \mathcal{N}^\otimes_{I_{0,\mathcal{F}}}$. ◁

Corollary II.2.16. *Given $\mathcal{O}^\otimes$ a $\mathcal{T}$-operad, the following conditions are equivalent:*

(a) $\mathcal{O}^\otimes$ is $\mathcal{F}$-unital,

(b) the map $\mathcal{O}^\otimes \simeq \mathcal{O}^\otimes \overset{BV}{\otimes} \mathrm{triv}^\otimes_\mathcal{T} \longrightarrow \mathcal{O}^\otimes \overset{BV}{\otimes} \mathbb{E}^\otimes_{0,\mathcal{F}}$ is an equivalence, and

(c) the forgetful $\mathcal{T}$-functor $\underline{\mathrm{Alg}}_{\mathbb{E}_{0,\mathcal{F}}}(\mathcal{O}) \to U\mathcal{O}$ is an equivalence.

(d) for all $\mathcal{T}$-operads $\mathcal{C}^\otimes$ such that $U\mathcal{C}$ admits a terminal $\mathcal{F}$-object, $\underline{\mathrm{Alg}}^\otimes_\mathcal{O}(\mathcal{C})$ is $\mathcal{F}$-unital.

If these are satisfied for $\mathcal{F} = \mathcal{T}$, and moreover $\mathcal{C}^\otimes$ underlies an $\mathcal{O}$-monoidal ∞-category, then the full $\mathcal{T}$-subcategory of V-objects $A \in \underline{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C})^V$ whose unit U-maps $1_U \to A(X)$ are equivalences for all $X \in U\mathcal{O}^U$ and $U \to V$ is contractible.

Finally, if $\mathcal{C}$ underlies an I-symmetric monoidal category and $\mathcal{O}$ has one color, then under the equivalence $\underline{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}_\mathcal{O}(\mathcal{C})$, the trivial V-algebra and unit V-object agree.

We refer to the algebra described in this corollary as *the trivial $\mathcal{O}$-algebra in $\mathcal{C}$*. Before proving it, we note an immediate corollary which will become useful later.

Corollary II.2.17. *$\mathbb{E}^\otimes_{0,\mathcal{F}} \in \mathrm{Op}_\mathcal{T}$ is an $\overset{BV}{\otimes}$-idempotent $\mathbb{E}_0$-algebra under the unique map $\mathrm{triv}^\otimes_\mathcal{T} \to \mathbb{E}^\otimes_{0,\mathcal{F}}$; a $\mathcal{T}$-operad is an $\mathbb{E}^\otimes_{0,\mathcal{F}}$-module if and only if it is $\mathcal{F}$-unital.*

The following lemma follows by unwinding definitions, noting that all V-objects lying over S_+ for S nonempty admit empty spaces of inert arrows to the object lying over $\varnothing_{V,+}$.

Lemma II.2.18. *$\mathcal{O}^{\otimes}$ is $\mathcal{F}$-unital if and only if $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}$ admits initial V-objects for all $V \in \upsilon(\mathcal{F})$; in this case, these V-objects lie in the (contractible) fibers $\mathcal{C}_{\varnothing_V} \subset (\mathrm{Tot}_{\mathcal{T}}\mathcal{C})^V$.*

Motivated by [HA, Rmk. 3.2.1.3], we see that triviality is witnessed by cocartesianness of the corresponding arrows from this initial object.

Lemma II.2.19. *Given an object $X \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$, the following conditions are equivalent:*

(a) X is trivial,

(b) the corresponding section $\mathrm{Tot}_{\mathcal{T}}\mathcal{O} \to \mathrm{Tot}_{\mathcal{T}}\mathcal{C}$ sends the unique V-arrow $u_C \colon 0_V \to C$ to a cocartesian arrow for all $C \in \mathcal{C}^V$, and

(c) the corresponding section $\mathrm{Tot}_{\mathcal{T}}\mathcal{O} \to \mathrm{Tot}_{\mathcal{T}}\mathcal{C}$ is cocartesian.

Proof. Since $0_V \in (\mathrm{Tot}_{\mathcal{T}}\mathcal{O})^V$ is initial, (b) is made equivalent to (c) by noting that, for all V-arrows $f \colon C \to D$, cancellation for cocartesian arrows [kerodon, Tag 01V5] applied to the factorization $X(0) \to X(C) \to X(D)$ of the map $X(0) \to X(D)$ shows that X_f is cocartesian.

Now, the claim is that X_{u_C} is cocartesian if and only if the unit map $X_0 \to X_C$ is an equivalence; this follows by uniqueness of cocartesian arrows. □

The category of such things turns out to be contractible (c.f. [NS22, Prop. 5.2.8]).

Lemma II.2.20. *If $\mathcal{O}^{\otimes}$ is unital and $\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal ∞-category, then $\underline{\mathrm{Fun}}^{\mathrm{cocart}}_{\mathcal{T},/\mathrm{Tot}_{\mathcal{T}}\mathcal{O}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}, \mathrm{Tot}_{\mathcal{T}}\mathcal{C})$ is contractible.*

Proof. As noted in [NS22, Lem. 5.2.7] (which does not use orbitality of $\mathcal{T}$), since $0 \in \mathrm{Tot}_{\mathcal{T}}\mathcal{O}$ is an initial $\mathcal{T}$-object, evaluation at 0 induces an equivalence

$$\underline{\mathrm{Fun}}^{\mathrm{cocart}}_{\mathcal{T},/\mathrm{Tot}_{\mathcal{T}}\mathcal{O}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}, \mathrm{Tot}_{\mathcal{T}}\mathcal{C}) \xrightarrow{\sim} \mathcal{C}_{\underline{0}};$$

the Segal conditions for $\mathcal{O}$-operads confirms that this is contractible. □

Using this, we immediately acquire the following.

Corollary II.2.21. *Suppose $\mathcal{O}^{\otimes}$ is unital. Then, $\underline{\mathrm{Alg}}_{\mathcal{O}}\left(\underline{\mathcal{S}}_{\mathcal{T}}\right) \times_{\mathrm{Fun}_{\mathcal{T}}(U\mathcal{O}, \underline{\mathcal{S}}_{\mathcal{T}})} \{*_{\mathcal{T}}\}$ is contractible.*

Proof of Corollary II.2.16. Applying Theorem II.2.7 and Theorem II.2.1, to exhibit (a) and (b) as equivalent, it suffices to prove that the postcomposition functor

$$\mathrm{Alg}_{\mathbb{E}_0}\underline{\mathrm{Seg}}_{\mathcal{O}_{\underline{V}}}(\mathcal{S}) \simeq \mathrm{Seg}_{\mathcal{O}_{\underline{V}}}(\mathcal{S})_* \longrightarrow \mathrm{Seg}_{\mathcal{O}_{\underline{V}}}(\mathcal{S})$$

is an equivalence for all $V \in \mathcal{F}$, i.e. the righthand ∞-category is pointed. By restriction-stability of our assumptions, we may assume that $V \in \mathcal{T}$ is terminal. Then, we explicitly use the monad description of Eq. (21); note that

$$\mathrm{Fr}_{\mathcal{O}} \varnothing^V(D) \simeq \mathcal{O}(\varnothing; D),$$

by $\times$-absorption and colimit-closure of $\{\varnothing\}$; this is terminal in $\mathcal{S}_V$ for all (V, D) if and only if $\mathcal{O}$ is unital. Since $U\colon \mathrm{Seg}_{\mathcal{O}}(\mathcal{S}) \to \mathrm{Fun}_{\mathcal{T}}\left(U\mathcal{O}, \underline{\mathcal{S}}_{\mathcal{T}}\right)$ is conservative and preserves limits, this is further equivalent to $\mathrm{Fr}_{\mathcal{O}} \varnothing$ being terminal in $\mathrm{Seg}_{\mathcal{O}}(\mathcal{S})$; it is the image of an initial object under a left adjoint, and hence initial, so this is equivalent to pointedness, i.e. (a) $\iff$ (b).

Note that the adjunction $\mathrm{Infl}_e^V \dashv (-)^V$ provides an equivalence

$$\underline{\mathrm{Alg}}_{\mathbb{E}_{0,\mathcal{F}}}(\mathcal{O})^V \simeq \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{O}^V)$$

for all $V \in \mathcal{F}$; in particular, $\mathcal{F}$-unitality is detected fiberwise, so to identify (b) and (c), we are reduced to the nonequivariant unital case. Moreover, applying the arity Borelification adjunction for $I_0 \subset \mathbb{F}$ constructs an equivalence

$$\mathrm{Alg}_{\mathbb{E}_0}(\mathcal{O}) \simeq \mathrm{Alg}_{/\mathbb{E}_0}\left(\mathcal{O} \times_{\mathbb{F}_*} \mathrm{Tot}\, \mathbb{E}_0\right).$$

Now, it is easy to see that the strong Morita class of $\mathbb{E}_0$ is presented by the algebraic pattern structure on Δ^1 with no morphisms inert and $\{0\}$ elementary (and this is claimed as [HA, Ex. 2.1.4.9]), so we are reduced to proving that

$$\mathrm{ev}_0\colon \mathrm{Fun}_{/\Delta^1}\left(\Delta^1, \mathcal{O} \times_{\mathbb{F}_*} \Delta^1\right) \to \mathcal{O} \times_{\mathbb{F}_*} \{1\}$$

is an equivalence if and only if the contractible subcategory $\mathcal{O} \times_{\underline{\mathbb{F}}_*} \{0\} \subset \mathcal{O} \times_{\underline{\mathbb{F}}_*} \Delta^1$ is initial; but using the Segal condition for multimorphisms, this is equivalent to it being initial in $\mathcal{O}$, i.e. $\mathcal{O}$ being unital via Lemma II.2.18, so (b) $\iff$ (c).

Now, after these equivalences, (b) $\iff$ (d) is easy:

$$\begin{aligned}
\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{C}) \text{ is } \mathcal{F}\text{-unital } \forall\, \mathcal{C} &\iff \underline{\mathrm{Alg}}_{\mathbb{E}_{0,\mathcal{F}}} \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\sim} \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \ \forall\, \mathcal{C} && \text{via (c)} \\
&\iff \underline{\mathrm{Alg}}_{\mathbb{E}_{0,\mathcal{F}} \otimes \mathcal{O}}(\mathcal{C}) \xrightarrow{\sim} \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \ \forall\, \mathcal{C} && \\
&\iff \mathcal{O}^{\otimes} \xrightarrow{\sim} \mathbb{E}_{0,\mathcal{F}} \otimes \mathcal{O} && \text{Yoneda's lemma.}
\end{aligned}$$

Contractibility of the space of trivial algebras follows by combining Lemmas II.2.19 and II.2.20. The final statement follows by unwinding definitions. □

II.2.1.5 The I-symmetric monoidal category $\mathcal{C}^{I-\times}$. We now focus on the algebra statements within Theorem II.2.1. Define the (non-full) $\mathcal{T}$-subcategory $\underline{\Gamma}_I^{\times} \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*})$ to have V-objects given by summand inclusions of pointed V-sets $\overline{S}_+ \hookrightarrow S_+$ and morphisms of V-objects given by maps $\alpha\colon S_+ \to T_+$ with the property that $\alpha^{-1}\left(\overline{T}_+\right) \subset \overline{S}_+$.

Lemma II.2.22. *Under the natural equivalence*

$$\{S_+\} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^{\times} \simeq \underline{\mathscr{P}}_{\underline{V}}(S),$$

where $\underline{\mathscr{P}}_{\underline{V}}(S)$ is the V-poset with U-value given by U-subsets of $\mathrm{Res}_U^V S$ ordered under inclusion, a V-arrow $f\colon (\overline{T} \subset T) \to (\overline{S} \subset S)$ is cartesian if and only if $\overline{T} = f^{-1}(\overline{S})$; in particular, $\underline{\Gamma}_I^{\times} \to \underline{\mathbb{F}}_{I,*}$ is a cartesian $\mathcal{T}$-fibration.

As a consequence, we can use this for a clever pairing construction.

Proposition II.2.23. *Given $\mathcal{C}$ a $\mathcal{T}$-∞-category, the exists a cocartesian $\mathcal{T}$-fibration $\widetilde{p}\colon \mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ characterized by a natural equivalence*

$$\underline{\mathrm{Fun}}_{\mathcal{T},/\underline{\mathbb{F}}_{I,*}}(K, \mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I-\times}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^{\times}, \mathcal{C}).$$

Moreover, given a V-arrow $f\colon S_+ \to T_+$ in $\underline{\mathbb{F}}_{I,*}$, under the equivalence

$$\mathrm{Fun}_{V,/\underline{\mathbb{F}}_{I,*}}\left(\underline{\Delta}^1, \mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I-\times}\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(\underline{\Delta}^1 \times_{f,\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_{\mathcal{T}}, \mathcal{C}\right),$$

a V-arrow α from $F\colon \mathscr{P}(S) \to \mathcal{C}_{\underline{V}}$ to $G\colon \mathscr{P}(T) \to \mathcal{C}_{\underline{V}}$ corresponds with a V-natural transformation

$$\alpha\colon F \implies \mathscr{P}(f)^* G,$$

which is cocartesian if and only if, for all $T' \subset \mathrm{Res}_U^V T$, the associated U-map

$$\alpha_{T'}\colon F\left(\alpha^{-1}(T')\right) \longrightarrow G(T')$$

is an equivalence.

Proof. We simply make the definition

$$\mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I-\times} \coloneqq \widetilde{\underline{\mathrm{Fun}}}_{\underline{\mathbb{F}}_{I,*}}\left(\underline{\Gamma}_I^{\times}, \mathcal{C} \times \underline{\mathbb{F}}_{I,*}\right),$$

i.e. writing $p\colon \underline{\Gamma}_I^{\times} \to \underline{\mathbb{F}}_{I,*}$ and $\pi\colon \underline{\mathbb{F}}_{I,*} \to *_{\mathcal{T}}$, we define

$$\mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I-\times} \coloneqq p_* p^* \pi^* \mathcal{C},$$

using cocartesianness (hence exponentiability) of p via Lemma II.2.22. Note moreover that $\mathcal{C} \times \underline{\mathbb{F}}_{I,*} \to \underline{\mathbb{F}}_{I,*}$ is cocartesian, with cocartesian V-arrows those which project to equivalences in $\mathcal{C}$; the mapping property and cocartesianness then follows by combining Proposition I.1.57 with the natural equivalence equivalence

$$\underline{\mathrm{Fun}}_{\mathcal{T}/\mathcal{B}}(-, \pi^*\mathcal{C}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\pi_!-, \mathcal{C}).$$

The description of cocartesian arrows also follows by combining the above descriptions of cartesian arrows of $\underline{\Gamma}^{\times}_{I}$ and cocartesian arrows of $\pi^*\mathcal{C}$ with the description of cocartesian arrows from Proposition I.1.57. □

We're ready to make the correct construction.

Construction II.2.24. We define the full $\mathcal{T}$-subcategory $\mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\times} \subset \mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I-\times}$ spanned by V-objects over S_+ corresponding with V-functors $\mathscr{P}(S) \to \mathcal{C}_{\underline{V}}$ satisfying the property that, for all $U \to V$ and $T \subset \mathrm{Res}^V_U S$, the maps $\mathrm{Res}^V_W F(T) \longrightarrow F(W)$ associated with orbits $W \subset T$ exhibit as $F(T)$ as a T-indexed product

$$F(T) \xrightarrow{\sim} \prod_W^T F(W)$$

in $\mathcal{C}$; since we've quantified over restrictions, this is manifestly restriction stable, and hence a well defined full $\mathcal{T}$-subcategory. ◁

Proposition II.2.25. *$\mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\times} \to \underline{\mathbb{F}}_{I,*}$ is a cocartesian $\mathcal{T}$-fibration; moreover, it is an I-symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits I-indexed products.*

Proof. For the first statement, it suffices to observe that the cocartesian V0arrows described in Proposition II.2.23 lie in $\mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\times}$. For the second, note by unwinding definitions that cocartesian transport induces a fully faithful functor

$$\mathcal{C}_S \to \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U;$$

this is essentially surjective if and only if $\mathcal{C}$ admits I-indexed products, as desired. □

We $\mathcal{C}^{I-\times}$ for the I-symmetric monoidal ∞-category of Proposition II.2.25. For the time being, we only characterize one direction of the essential image.

Lemma II.2.26. *$\mathcal{C}^{I-\times}$ satisfies the condition that W_S is a natural equivalence for all $S \in \underline{\mathbb{F}}_I$.*

Proof. Inert-cocartesian transport to orbits is precisely the arrows $F(T) \to \mathrm{Res}^V_W F(W)$; these altogether classify an equivalence $\bigotimes^T_W F(W) \simeq F(T) \xrightarrow{\sim} \prod^T_W F(W)$ by assumption. □

II.2.1.6 Algebras in $\mathcal{C}^{I-\times}$ We now do painful category theory.

Observation II.2.27. The section $\iota\colon \underline{\mathbb{F}}_{I,*} \to \underline{\Gamma}^\times_I$ induces a natural transformation

$$K \simeq K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\mathbb{F}}_{I,*} \to K \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^\times_I,$$

which induces a natural transformation $\mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I\text{-}\times} \to \mathcal{C}$ under Yoneda's lemma. ◁

We will first prove the following proposition:

Proposition II.2.28. *Given $\mathcal{O}^\otimes$ a Γ-I-preoperad, the composite $\mathcal{T}$-functor*

$$\underline{\mathrm{Alg}}_{\mathcal{O}}\left(\widetilde{\mathcal{C}}^{I\text{-}\times}\right) \subset \underline{\mathrm{Fun}}_{\mathcal{T},/\underline{\mathbb{F}}_{I,*}}\left(\mathrm{Tot}_{\mathcal{T}} \mathcal{O}^\otimes, \mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I\text{-}\times}\right) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Tot}_{\mathcal{T}} \mathcal{O}^\otimes, \mathcal{C}\right)$$

is an equivalence.

To prove this, we first make the following observation:

Observation II.2.29. Given a $\mathcal{T}$-operad, the projection

$$P\colon \mathrm{Tot}_{\mathcal{T}} \mathcal{O}^\otimes \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^\times_I \to \mathrm{Tot}_{\mathcal{T}} \mathcal{O}^\otimes$$

admits a $\mathcal{T}$-left adjoint L sending $X \in \mathcal{O}^\otimes_{S_+}$ to $(X, S \subset S)$; the unit map of this adjunction is evidently an equivalence, so L is additionally fully faithful. ◁

Now, the proposition reduces immediately to the following lemma.

Lemma II.2.30. *The following conditions are equivalent for $\varphi\colon \mathrm{Tot}_{\mathcal{T}} \mathcal{P} \to \mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}$ a map with corresponding map $\varphi'\colon \mathrm{Tot}_{\mathcal{T}} \mathcal{O} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^\times_I \to \mathcal{C}$:*

(a) φ is a map of Γ-I-preoperads.

(b) For all V-morphisms α in $\mathrm{Tot}_{\mathcal{T}} \mathcal{O}^\otimes \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}^\times_I$ whose image in $\mathrm{Tot}_{\mathcal{T}} \mathcal{O}$ is inert, $\varphi'(\alpha)$ is an equivalence.

(c) If $f\colon (\overline{S}_+, \overline{S}, F, X) \to (S_+, \overline{S}, F, X)$ is a cocartesian lift of the corresponding inert V-morphism, then $\varphi(f)$ is an equivalence.

(d) The composite $\mathrm{Tot}_{\mathcal{T}} \mathcal{O}^\otimes \xrightarrow{\varphi} \mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I\text{-}\times} \to \mathcal{C}$ is homotopic to φ'.

(e) The composite $\mathrm{Tot}_{\mathcal{T}} \mathcal{O}^\otimes \xrightarrow{\varphi} \mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I\text{-}\times} \to \mathcal{C}$ is $\mathcal{T}$-right Kan extended from φ' along L.

Proof. Proposition II.2.23 immediately implies that (a) $\iff$ (b) $\iff$ (c) $\iff$ (d). (d) $\iff$ (e) follows from Lemma I.1.40. □

We're ready to characterize algebras.

Proposition II.2.31. *The composite functor*

$$\underline{\mathrm{Alg}}_{\mathcal{O}}\left(\mathcal{C}^{I\text{-}\times}\right) \longrightarrow \underline{\mathrm{Alg}}_{\mathcal{O}}\left(\widetilde{\mathcal{C}}^{I\text{-}\times}\right) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Tot}_{\mathcal{T}} \mathcal{O}^\otimes, \mathcal{C}\right)$$

is fully faithful with image spanned by the Segal $\mathrm{Tot}_{\mathcal{T}} \mathcal{O}$*-spaces.*

PROOF. Since $\mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\times} \hookrightarrow \mathrm{Tot}_{\mathcal{T}} \widetilde{\mathcal{C}}^{I-\times}$ id s monomorphism of $\mathcal{T}$-∞-categories, it is a monomorphism of Γ-I-preoperads, giving fully faithfulness. It suffices to prove that an algebra $\varphi\colon \mathrm{Tot}_{\mathcal{T}} \mathcal{O}^{\otimes} \to \widetilde{\mathcal{C}}^{I-\times}$ factors through $\mathcal{C}^{I-\times}$ if and only if it is a Segal object; but this follows immediately by comparing Construction II.2.24 with the Segal conditions of Corollary I.4.35. □

We will continue putting off the proof of Theorem II.2.1 for a bit longer.

II.2.2 Parameterized cocartesian structures and commutative algebras As in Construction II.2.24, we will construct an extension of $(-)^{I-\sqcup}$ to all input $\mathcal{T}$-categories. In the cartesian setting, this produced *cocartesian fibrations* Γ-I-preoperads, with the Segal conditions encoding existence of I-products; in the cocartesian setting, we will instead construct an I-operad, with cocartesian lifts for active morphisms encoding existence of I-coproducts.

Theorem II.2.32. *The color* $\mathcal{T}$*-*$(\infty,2)$*-functor* $U\colon \underline{\mathrm{OP}}_I^{\mathrm{uni}} \longrightarrow \underline{\mathrm{CAT}}_{\mathcal{T}}$ *admits a fully faithful* $\mathcal{T}$*-right adjoint* $(-)^{I-\sqcup}$ *making the following diagram commute*

$$\begin{array}{ccc} \underline{\mathrm{CAT}}_{\mathcal{T}}^{I-\times} & \longhookrightarrow & \underline{\mathrm{CAT}}_I^{\otimes} \\ \downarrow & & \downarrow \\ \underline{\mathrm{CAT}}_{\mathcal{T}} & \longhookrightarrow & \underline{\mathrm{OP}}_I^{\mathrm{uni}} \end{array}$$

In particular, when $\mathcal{O}^{\otimes}$ *is a unital* I*-operad, the forgetful* $\mathcal{T}$*-functor*

$$\underline{\mathrm{Alg}}_{\mathcal{O}}\left(\mathcal{C}^{I-\sqcup}\right) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O}, \mathcal{C})$$

is an equivalence.

Essential uniqueness of the square is automatic from the fact that all of the arrows involved are locally fully faithful, and hence monomorphisms; moreover, the horizontal arrows are fully faithful, with essential image of the top one quite explicitly described in Theorem II.2.1. As a consequence, our strategy will be to perform another explicit construction $(-)^{I-\sqcup}$, use it to conclude the proof of Theorem II.2.1, and explicitly compute its $\mathcal{T}$-∞-categories of algebras in analogy with the parts of the Theorem II.2.1 we have already developed.

Of course, before we do any of that, we first explore further corollaries.

II.2.2.1 Pointwise tensor products of I*-commutative algebras are coproducts* Theorem II.2.32 shows that every object in a cocartesian I-symmetric monoidal structure bears a canonical I-commutative algebra algebra structure, i.e. $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ is an equivalence. In fact, we may show a converse.

Corollary II.2.33. *The following are equivalent for* $\mathcal{C}^{\otimes} \in \mathrm{Cat}_I^{\otimes}$.

(a) $\mathcal{C}^{\otimes}$ *is cocartesian.*

(b) For all unital I*-operads* $\mathcal{O}^{\otimes}$*, the forgetful functor* $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{T}}(U\mathcal{O}, \mathcal{C})$ *is an equivalence.*

(c) The forgetful $\mathcal{T}$*-functor* $\underline{\mathrm{CAlg}}_I(\mathcal{C}) \to \mathcal{C}$ *is an equivalence.*

(d) There exists an I*-symmetric monoidal* ∞*-category* $\mathcal{D}^{\otimes}$ *and an equivalence* $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{D}) \xrightarrow{\sim} \mathcal{C}^{\otimes}$.

The implications (a) $\implies$ (b), (c) are simply Theorem II.2.32. For the implication (b) $\implies$ (a), note that Corollary II.2.16 implies that $\mathcal{C}^{\otimes}$ is unital; hence Yoneda's lemma applied to $\mathrm{Op}_I^{\mathrm{uni}}$ constructs an I-operad equivalence $\mathcal{C}^{\otimes} \simeq \mathcal{C}^{I-\sqcup}$, which is automatically I-symmetric monoidal. The implication (c) $\implies$ (d) follows by neglect of assumptions. To summarize, we've arrived at the implications

(22)
$$\begin{array}{ccccc} & & (d) & & \\ & \nearrow & & \searrow & \\ (b) & \Longleftrightarrow & (a) & \Longrightarrow & (c) \end{array}$$

Our workhorse lemma for closing the gap is the following.

Lemma II.2.34. *The following are equivalent for* $\mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$*:*

(e) The $\mathcal{T}$*-*∞*-category* $\underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ *is* I*-semiadditive.*

(f) For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_I^{\mathrm{uni}}$*, the forgetful functor*

$$\mathrm{Alg}_{\mathcal{O}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\underline{\mathcal{S}}_{\mathcal{T}}) \longrightarrow \mathrm{Alg}_{\mathrm{triv}(\mathrm{U}\mathcal{O})}\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \mathrm{Fun}_{\mathcal{T}}(U\mathcal{O}, \underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}))$$

is an equivalence.

(g) For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_I^{\mathrm{uni}}$*, the map* $\mathrm{triv}(\mathcal{O})^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}$ *is an equivalence.*

(h) For all $\mathcal{O}^{\otimes} \in \mathrm{Op}_I^{\mathrm{uni}}$ *and* $\mathcal{Q}^{\otimes} \in \mathrm{Op}_I$*, the forgetful* $\mathcal{T}$*-operad map*

$$\underline{\mathrm{Alg}}_{\mathcal{O}\otimes\mathcal{P}}^{\otimes}(\mathcal{Q}) \simeq \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{Q}) \longrightarrow \underline{\mathrm{Alg}}_{\mathrm{triv}(U\mathcal{O})}^{\otimes}\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\mathcal{Q})$$

is an equivalence

Proof. Since Proposition II.2.10 shows that $\mathrm{Bor}_I^{\mathcal{T}}\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\underline{\mathcal{S}}_{\mathcal{T}})$ is cartesian, Corollary II.2.5 identifies the bi-implication (e) $\iff$ (f) with (a) $\iff$ (b) applied to $\mathrm{Bor}_I^{\mathcal{T}}\underline{\mathrm{Alg}}_{\mathcal{P}}^{\otimes}(\underline{\mathcal{S}}_{\mathcal{T}})$. (f) $\implies$ (g) follows from Theorem II.2.7, and the implications (g) $\implies$ (h) $\implies$ (f) are obvious. □

Proof of Corollary II.2.33. After the implications illustrated in Eq. (22), it suffices to show for all $\mathcal{D}^{\otimes} \in \mathrm{Cat}_I^{\otimes}$ that $\underline{\mathrm{CAlg}}_I(\mathcal{D})$ satisfies (b), i.e. (d) $\implies$ (b); by Lemma II.2.34, it suffices to prove that $\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}})$

is I-semiadditive. But in fact, Theorems I.4.1 and II.2.1 and Corollary I.4.35 construct equivalences

$$\underline{\mathrm{CAlg}}_I(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{N}_{I^\infty}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{Seg}}_{\underline{\mathbb{F}}_{I,*}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}^{I-\oplus}\left(\underline{\mathbb{F}}_{I,*}, \underline{\mathcal{S}}_{\mathcal{T}}\right),$$

and the latter is I-semiadditive by Proposition I.1.116. □

In the cartesian setting, we have reproved a result of Cnossen-Lenz-Linskens. To state it, let $\underline{\mathrm{CAT}}_{\mathcal{T}}^{I-\oplus} \subset \underline{\mathrm{CAT}}_{\mathcal{T}}$ be the locally fully faithful $\mathcal{T}$-$(\infty,2)$-subcategory spanned by $\mathcal{T}$-∞-categories with I-indexed products and I-semiadditive $\mathcal{T}$-functors (see [CLL24, Def. 3.18]). The following is a specialization of [CLL24, Thm. B] to the ≤ 0-semiadditive parameterized case.

Corollary II.2.35. *The construction* $\underline{\mathrm{CMon}}_I \colon \underline{\mathrm{CAT}}_{\mathcal{T}}^{I-\times} \longrightarrow \underline{\mathrm{CAT}}_{\mathcal{T}}^{I-\oplus}$ *is right* $\mathcal{T}$*-adjoint to the fully faithful inclusion* $\underline{\mathrm{CAT}}_{\mathcal{T}}^{I-\oplus} \hookrightarrow \underline{\mathrm{CAT}}_{\mathcal{T}}^{I-\times}$.

Proof. We have a counit natural transformation $\underline{\mathrm{CMon}}_I(\mathcal{C}) \longrightarrow \mathcal{C}$ which is an equivalence if and only if $\mathcal{C}$ is I-semiadditive. This construction is *cosmashing*, so this is enough. □

Remark II.2.36. This, for instance, provides an absorption natural equivalence $\underline{\mathrm{CMon}}_I \underline{\mathrm{CMon}}_J(-) \simeq \underline{\mathrm{CMon}}_I(-)$ whenever $J \subset I$; for instance, $\underline{\mathrm{CMon}}_I \underline{\mathrm{CMon}}(-) \simeq \underline{\mathrm{CMon}}_I(-)$ whenever I is an indexing category. We will explore this more in Section III.1, as a form of *Eckmann-Hilton argument*. ◁

II.2.2.2 The construction of $\mathcal{C}^{I-\sqcup}$ Now, we move on to the proof. First, define the full $\mathcal{T}$-subcategory

$$\underline{\Gamma}_I^* := \left\{ U_+ \xrightarrow{s.i.} S_+ \;\middle|\; U \in \mathcal{T}_{/V} \right\} \subset \mathrm{Ar}(\underline{\mathbb{F}}_{I,*});$$

that is, the V-objects of $\underline{\Gamma}_I^*$ are pointed I-admissible V-sets with a distinguished orbit, and the V-morphisms of $\underline{\Gamma}_I^*$ preserve distinguished orbits. This possesses a *target* forgetful $\mathcal{T}$-functor $t \colon \underline{\Gamma}_I^* \to \underline{\mathbb{F}}_{I,*}$.

Proposition II.2.37. *t is exponentiable; hence, given $\mathcal{C}$ a $\mathcal{T}$-∞-category, there exists a $\mathcal{T}$-∞-category* $\mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\sqcup}$ *over* $\mathrm{Tot}\, \underline{\mathbb{F}}_{I,*}$ *satisfying the universal property that there is a natural equivalence*

$$\mathrm{Fun}_{\mathcal{T}/\underline{\mathbb{F}}_{I,*}}(K, \mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\sqcup}) \simeq \mathrm{Fun}_{\mathcal{T}}(\mathcal{D} \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^*, \mathcal{C});$$

Proof. By Corollary I.1.54, given a composable sequence of pointed V-set maps $R_+ \xrightarrow{f} S_+ \xrightarrow{g} T_+$ and orbits $U_R \subset R$ and $U_T \subset T_+$ such that $gf(U_R) = U_T$, we're tasked with proving that the category of orbits $U_S \subset S$ such that $f(U_R) = U_S$ and $g(U_R) = U_T$ is weakly contractible; indeed, it is just $\{U_S\}$, which is contractible. □

A V-object of $\mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\sqcup}$ may be viewed as S_+ a pointed V-set and $\mathbf{C} = (C_W) \in \mathcal{C}_S$ an S-tuple of elements of $\mathcal{C}$, and a V-morphism $f : \mathbf{C} \to \mathbf{D}$ a map of pointed V-sets $S_+ \to T_+$ together with a collection of maps

$$\left\{ f_W : \mathrm{Ind}_W^U C_W \to D_U \mid W \in f^{-1}(U) \right\}$$

for all $U \in \mathrm{Orb}(T)$. Unwinding the universal property for cocartesian arrows, we find the following.[1]

Proposition II.2.38. *A map $f : \mathbf{C} \to \mathbf{D}$ is π-cocartesian if and only if $\{f_W\}$ witness D_U as the indexed coproduct*

$$\coprod_W^{f^{-1}(U)} C_W \xrightarrow{\sim} D_U$$

for all $U \in \mathrm{Orb}(T)$. In particular, f is inert if and only if the following conditions are satisfied:

(a) The projected morphism $\pi(f) : S \to T$ is inert.

(b) The associated map $C_{f^{-1}(U)} \to D_U$ is an equivalence for all $U \in \mathrm{Orb}(T)$.

Hence $\pi : \mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\sqcup} \to \underline{\mathbb{F}}_{I,*}$ presents a Γ-I-preoperad $\mathcal{C}^{I-\sqcup}$.

Corollary II.2.39. *$\mathcal{C}^{I-\sqcup}$ is an I-operad which is an I-symmetric monoidal ∞-category if and only if $\mathcal{C}$ admits I-indexed coproducts.*

Proof. It follows from Proposition II.2.38 that $\mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\sqcup} \to \underline{\mathbb{F}}_{I,*}$ is a cocartesian fibration if and only if $\mathcal{C}$ admits I-indexed coproducts, so it suffices to verify the following conditions:

(b) cocartesian transport yields an equivalence

$$\mathcal{C}_S \simeq \prod_{U \in \mathrm{Orb}(S)} \mathcal{C}_U;$$

(c) cocartesian transport yields an equivalence

$$\mathrm{Map}^{T \to S}_{\mathrm{Tot}\mathcal{O}^{\otimes}}(\mathbf{C}, \mathbf{D}) \simeq \prod_{U \in \mathrm{Orb}(S)} \mathrm{Map}^{T_U \to U}_{\mathrm{Tot}\mathcal{O}^{\otimes}}\left(\mathbf{C}_{\underline{U}}, D\right).$$

In fact, each condition follows from Proposition II.2.38. □

Observation II.2.40. It follows from the above discussion that $U(\mathcal{C}) = \mathcal{C}$. Moreover, it follows from Proposition II.2.38 that $\mathcal{C}^{I-\sqcup}$ is cocartesian when $\mathcal{C}$ has I-indexed coproducts. ◁

[1] It is here that we use almost-unitality for the cocartesian setting; if I was not almost essentially unital, then there would exist some S whose I-admissible orbits do not together cover S, so $\mathcal{C}^{I-\sqcup} \to \mathrm{Tot}\,\underline{\mathbb{F}}_{I,*}$ would not be an inert-cocartesian fibration.

Given $\mathcal{O}^{\otimes}$ a unital I-operad, define a diagram of Cartesian squares in $\mathrm{Cat}_{\mathcal{T}}$.

$$\begin{array}{ccccc} U\mathcal{O} & \xhookrightarrow{\iota} & \mathrm{Tot}_{\mathcal{T}}\mathcal{O}_{\Gamma} & \longrightarrow & \mathrm{Tot}_{\mathcal{T}}\mathcal{O} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ *_{\mathcal{T}} & \hookrightarrow & \underline{\Gamma}^{*}_{I} & \longrightarrow & \underline{\mathbb{F}}_{I,*} \end{array}$$

The V-objects of $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}_{\Gamma}$ consist of triples (S_+, U, X) where $U \in \mathsf{Orb}(S)$ and $X \in \mathcal{O}_S$, and the image of ι is equivalent to the triples where $S \in \mathcal{T}_{/V}$, hence $U = S$.

Further note that cocartesian transport along the inert morphism $U_+ \hookrightarrow S_+$ induces an equivalence

$$\mathsf{Map}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{V}_{\Gamma}}(\iota Y, (S_+, U, X))) \simeq \mathsf{Map}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{V}_{\Gamma}}(\iota Y, (U_+, U, X_U)))$$

for all $Y \in U\mathcal{O}^{V}$.[2] In particular, ι witnesses $U\mathcal{O}$ as a *colocalizing* $\mathcal{T}$-subcategory, with colocalization $\mathcal{T}$-functor

$$R(S_+, U, X) \simeq (U_+, U, X_U).$$

This interacts with Kan extensions via the following lemma.

Lemma II.2.41. *Fix a $\mathcal{T}$-functor $A\colon \mathrm{Tot}_{\mathcal{T}}\mathcal{O}_{\Gamma} \to \mathcal{C}$. Then, the following are equivalent*

(a) The corresponding map $\mathrm{Tot}_{\mathcal{T}}\mathcal{O} \to \mathcal{C}^{I-\sqcup}$ is a map of I-operads.

(b) For all V-morphisms α in $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}_{\Gamma}$ whose image in $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}$ is inert, $A(\alpha)$ is an equivalence in $\mathcal{C}$.

(c) If $f\colon (S_+, U, X) \to (U_+, U, X_U)$ is a cocartesian lift of the corresponding inert V-morphism, then $A(f)$ is an equivalence.

(d) A is $\mathcal{T}$-left Kan extended from $\mathcal{O}$.

Furthermore, every $\mathcal{T}$-functor $F\colon U\mathcal{O} \to \mathcal{C}$ admits a left Kan extension along $U\mathcal{O} \hookrightarrow \mathrm{Tot}_{\mathcal{T}}\mathcal{O}_{\Gamma}$; in particular, the forgetful functor $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_{G}(\mathcal{O}, \mathcal{C})$ is an equivalence.

Proof. (a) $\iff$ (b) follows immediately from Proposition II.2.38. (b) $\iff$ (c) is immediate by definition. (c) $\iff$ (d) and the remaining statement both follow by Lemma I.1.40. □

II.2.2.3 Putting together the proofs The following observation is crucial.

Observation II.2.42. Let $F\colon \mathcal{D}^{\otimes} \to \mathcal{E}^{\otimes}$ be a lax I-symmetric monoidal functor. Then, the universal property for cocartesian arrows constructs, for each active arrow $\mathsf{Ind}^{\mathcal{T}}_{V} S \to V$, an arrow

$$\mu_S\colon \bigotimes_{U}^{S} F(-) \Longrightarrow F\left(\bigotimes_{U}^{S} -\right)$$

[2] This utilizes unitality of $\mathcal{O}^{\otimes}$, as we implicitly use that, for each orbit $U' \in \mathsf{Orb}(S)$ other than U, the space $\mathcal{O}(\varnothing_{U'}; X_{U'})$ is contractible.

such that F is I-symmetric monoidal if and only if μ_S is an equivalence for all $S \in \underline{\mathbb{F}}_I$. In particular, in the case of the lax I-symmetric monoidal functor $F\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{I-\sqcup}$ classified by a functor $G\colon \mathcal{D} \to \mathcal{C}$, the arrow μ has the type

$$\coprod_U^S G(-) \Longrightarrow G\left(\bigotimes_U^S -\right);$$

moreover, unwinding definitions, in the case that $\mathcal{D}^{\otimes}$ is an I-symmetric monoidal structure on $\mathcal{C}$ and, G is the identity, and $1_\bullet \in \Gamma^{\nu(I)}\mathcal{O}$ is initial, this map is precisely the $\otimes$-Wirthmüller map constructed in Construction II.2.4. In particular, we've observed that the identity classifies an I-symmetric monoidal equivalence $\mathcal{C}^{\otimes} \xrightarrow{\sim} \mathcal{C}^{I-\sqcup}$ if and only if $\mathcal{C}^{\otimes}$ is I-cocartesian, i.e. it has I-admissible $\otimes$-Wirthmüller isomorphisms.

Alternatively, if $\mathcal{D}^{\otimes}$ is cocartesian, then the arrow μ is the assembly map for S-indexed coproducts; that is, F is I-symmetric monoidal if and only if G is I-coproduct preserving. ◁

Proof of Theorem II.2.1. Let $\mathbb{C} \subset \underline{\mathrm{CAT}}_I^{\otimes}$ be the full $(\infty,2)$-subcategory spanned by cartesian I-symmetric monoidal ∞-categories, and consider the composite $\mathcal{T}$-$(\infty,2)$-functor

$$\varphi\colon \mathbb{C} \hookrightarrow \underline{\mathrm{CAT}}_I^{\otimes} \longrightarrow \underline{\mathrm{CAT}}_{\mathcal{T}}.$$

It follows by Observation II.2.42 that φ is locally fully faithful, with image the locally fully faithful $\mathcal{T}$-$(\infty,2)$-subcategory $\underline{\mathrm{CAT}}_{\mathcal{T}}^{I-\times}$. The corresponding section $\underline{\mathrm{CAT}}_{\mathcal{T}} \simeq \mathbb{C} \hookrightarrow \underline{\mathrm{CAT}}_I^{\otimes}$ is the desired embedding. Moreover, the expression on algebras simply follows by combining Proposition II.2.31 and Theorem I.4.1. The dual embedding of $\underline{\mathrm{CAT}}_I^{I-\sqcup}$ follows by taking fiberwise opposites. □

Proof of Theorem II.2.32. The algebras statement is a straightforward corollary of the existence of the right $\mathcal{T}$-adjoint, so we prove that and construct the commutativity. Indeed, the construction $(-)^{I-\sqcup} := t_* t^* \pi^*(-)\colon \underline{\mathrm{CAT}}_{\mathcal{T}} \longrightarrow \underline{\mathrm{CAT}}_{\mathcal{T}/\underline{\mathbb{F}}_{I,*}}$ lands in the locally fully faithful subcategory $\underline{\mathrm{OP}}_I^{\mathrm{uni}}$ and makes the requested square commute by Corollary I.1.21 and Observations II.2.40 and II.2.42; moreover, it is fully faithful and left adjoint to the inclusion by Lemma II.2.41. □

II.2.3 Disintegration and assembly of parameterized operads This section concerns various notions of *families* of $\mathcal{T}$-operads, together with a convenient model for them, called *generalized $\mathcal{T}$-operads*.

II.2.3.1 Disintegration for generalized $\mathcal{T}$-operads We will generalize [HA, § 2.3.2] as follows. Recall from Section I.3.3 that we have a $\mathcal{T}$-algebraic pattern $\mathrm{Tot}^{\natural}_{\mathcal{T}} \underline{\mathbb{F}}_{\mathcal{T}}$ with the same underlying category and factorization system as $\mathrm{Tot}\,\underline{\mathbb{F}}_{\mathcal{T},*}$, but elementary objects (S,V) such that $S \in \{\varnothing_V, *_V\}$.

Definition II.2.43. The $(\infty,2)$-category of *generalized $\mathcal{T}$-operads* is

$$\mathrm{Op}_{\mathcal{T}}^{\mathrm{Gen}} := \mathrm{FBRS}(\mathrm{Tot}^{\natural} \underline{\mathbb{F}}^{\mathrm{el}}_{\mathcal{T},*}).$$ ◁

Applying Proposition I.4.28 gives the following concrete description.

Proposition II.2.44. *A functor $\pi_{\mathcal{O}}\colon \mathrm{Tot}^{\natural}\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes} \to \mathrm{Tot}^{\natural}\underline{\mathbb{F}}_{\mathcal{T},*}$ presents a generalized $\mathcal{T}$-operad if and only if the following conditions hold:*

(1) $\mathrm{Tot}^{\natural}\mathrm{Tot}_{\mathcal{T}}\mathcal{O}^{\otimes}$ has inert-cocartesian lifts.

(2) For each orbit $V \in \mathcal{T}$ and finite V-set $S \simeq U_1 \sqcup \cdots \sqcup U_n$, inert-cocartesian transport supplies an equivalence

$$\mathcal{O}_S \xrightarrow{\sim} \mathcal{O}_{U_1} \times_{\mathcal{O}_{\varnothing_V}} \cdots \times_{\mathcal{O}_{\varnothing_V}} \mathcal{O}_{U_n}$$

(3) Inert-cocartesian transport yields an equivalence

$$\mathrm{Mul}^{f}_{\mathcal{O}}(\mathbf{C};\mathbf{D}) \simeq \mathcal{O}\left(\mathbf{C}_{U_1};D_{U_1}\right) \times_{\mathrm{Map}_{\mathrm{sh}\mathcal{O}}(C_{\varnothing_V},D_{\varnothing_V})} \cdots \times_{\mathrm{Map}_{\mathrm{sh}\mathcal{O}}(C_{\varnothing_V},D_{\varnothing_V})} \mathcal{O}\left(\mathbf{C}_{U_n};D_{U_n}\right)$$

Now, $\mathrm{Tot}^{\natural}\mathbb{F}_{\mathcal{T},*}$ is disintegrable by Corollary I.4.26, yielding the following corollary to Proposition I.3.41 and Lemma I.3.45.

Corollary II.2.45. *Fix $\mathcal{O}^{\otimes}$ a $\mathcal{T}$-operad. There exists a fully faithful $\mathcal{T}$-$(\infty,2)$-right adjoint*

$$\underline{\mathrm{OP}}^{\mathrm{Gen}}_{\mathcal{T},/\mathcal{O}^{\otimes}} \underset{\mathrm{const}_{\mathcal{O}}(-)^{\otimes}=(-\times\mathrm{Tot}^{\natural}\mathcal{O})}{\overset{\mathrm{sh}_{\mathcal{O}}}{\rightleftarrows}} \underline{\mathrm{CAT}}_{\mathcal{T}}$$

Moreover, the fibers of $\mathcal{P}^{\otimes} \to \mathrm{const}_{\mathcal{O}}\,\mathrm{sh}(\mathcal{P})^{\otimes}$ are $\mathcal{T}$-operads over $\mathcal{O}^{\otimes}$, assembling to an equivalence

$$\mathrm{Op}^{\mathrm{Gen}}_{\mathcal{T},/\mathcal{O}} \times_{\mathrm{sh},\underline{\mathrm{Cat}}_{\mathcal{T}}} \underline{\mathcal{S}}_{\mathcal{T}} \simeq \int_{X \in \in \underline{\mathcal{S}}_{\mathcal{T}}} \underline{\mathrm{Fun}}_{\mathcal{T}}\left(X, \underline{\mathrm{Op}}_{\mathcal{T},/\mathcal{O}}\right).$$

in the case that $U\mathcal{O} = *_{\mathcal{T}}$. In particular, there exists a factorization of full subcategory inclusions

$$\mathrm{OP}_{\mathcal{T},/\mathcal{O}^{\otimes}} \xrightarrow{i} \mathrm{OP}^{\mathrm{Gen}}_{\mathcal{T},/\mathcal{O}^{\otimes}} \subset \mathrm{CAT}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathrm{Tot}_{\mathcal{T}}\mathcal{O}};$$

yielding an equivalence $\mathrm{OP}_{\mathcal{T},/\mathcal{O}^{\otimes}} \simeq \mathrm{sh}^{-1}_{\mathcal{O}}(*_{\mathcal{T}})$.

II.2.3.2 Reduced endomorphism I-operads We will also disintegrate unital $\mathcal{T}$-operads along maps to cocartesian structures, generally reducing the study of them to reduced $\mathcal{T}$-operads in the $\mathcal{T}$-space colored case. The relevant construction is the following.

Proposition II.2.46. *The inclusion* $\mathrm{Op}_I^{\mathrm{red}} \simeq \mathrm{Op}^{\mathrm{red}}_{I,\mathbb{E}_{0,v(I)/}} \hookrightarrow \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}_{0,v(I)/}}$ *has a right adjoint given by pullback*

$$\begin{array}{ccc} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \eta} \\ \mathcal{N}_{I\infty}^{\otimes} & \xrightarrow[\{X\}]{} & U\mathcal{O}^{\sqcup} \end{array} \tag{23}$$

In the case that $\mathcal{C}^{\otimes}$ *is a unital* I*-symmetric monoidal* ∞*-category and* $X \in \mathcal{C}_V$ *is a* V*-object, mapping in from the free unital* I*-operad* $\mathrm{Fr}_{\mathrm{Op}}(S) \overset{BV}{\otimes} \mathbb{E}_{0,v(I)}$ *on an operation in arity* $S \in \mathbb{F}_{I,V}$ *yields a pullback*

$$\begin{array}{ccc} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})(S) & \longrightarrow & \mathrm{Map}_{\mathcal{C}_V}\left(X^{\otimes S}, X\right) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle W^*_{S,X}} \\ \{\nabla\} & \longrightarrow & \mathrm{Map}_{\mathcal{C}_V}\left(X^{\sqcup S}, X\right) \end{array}$$

i.e. $\mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})(S)$ *is equivalent to the space of lifts as in the following dashed arrow in* $\mathcal{C}^V$

$$\begin{array}{ccc} X^{\sqcup S} & \xrightarrow{\nabla} & X \\ {\scriptstyle W_{S,X}}\downarrow & \nearrow & \downarrow{\scriptstyle !} \\ X^{\otimes S} & \xrightarrow[!]{} & * \end{array}$$

Proof. We will apply the general reduction procedure of [SY19, Prop. 2.1.5] to the *sliced* adjunction

$$U_*\colon \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}^{\otimes}_{0,v(I)/}} \rightleftarrows \mathrm{Cat}_{\mathcal{T},*}\colon \eta^*(-)^{I-\sqcup},$$

whose right adjoint is $(-)^{I-\sqcup}$ together with the precomposed structure map

$$\mathbb{E}^{\otimes}_{0,v(I)} \xrightarrow{\eta} \mathcal{N}^{\otimes}_{I\infty} \simeq *_{\mathcal{T}}^{I-\sqcup} \to \mathcal{C}^{I-\sqcup}.$$

Indeed, $\mathrm{Cat}_{\mathcal{T},*}$ admits an initial object $*_{\mathcal{T}} \simeq U\mathbb{E}_{0,v(I)}$ and $\mathrm{Op}^{\otimes}_{I,\mathbb{E}^{\otimes}_{0,v(I)/}}$ admits all limits, which are preserved by U since it is a right adjoint. Moreover, $\mathbb{E}_{0,v(I)} \in \mathrm{Op}_I^{\mathrm{red}}$ is initial by Corollary II.2.16, there is a unique equivalence $\mathcal{N}^{\otimes}_{I\infty} \simeq *_{\mathcal{T}}^{I-\sqcup}$ by Theorem II.2.32, and $\mathcal{O}^{\otimes} \in \mathrm{Op}^{\mathrm{uni}}_{I,\mathbb{E}_{0,v(I)/}}$ corresponds with a reduced I-operad if and only if $U\mathcal{O}^{\otimes} \in \mathrm{Cat}_{\mathcal{T},*}$ is initial, so the first claim follows by [SY19, Prop. 2.1.5].

To acquire the second pullback square, one need only note the natural equivalences

$$\begin{aligned} \mathrm{Map}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Fr}_{\mathrm{Op}}(S) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}, \mathcal{C}^{\otimes}\right) &\simeq \mathrm{Map}_{\mathcal{C}_V}\left(X^{\otimes S}, X\right), \\ \mathrm{Map}_{\mathrm{Op}_{\mathcal{T}}}\left(\mathrm{Fr}_{\mathrm{Op}}(S) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_{0,v(I)}, \mathcal{N}^{\otimes}_{I\infty}\right) &\simeq * \end{aligned}$$

The result follows by noting that η induces $W^*_{S,X}$ and the bottom arrow includes the fold map ∇. □

[SY19, Prop. 4.2.8] introduced a result on connectivity of such spaces of lifts, immediately yielding the following corollary, which we will need later.

Corollary II.2.47. *If $X \in \mathcal{C}^V$ is a $(k+\ell+2)$-truncated object and the Wirthmüller map $W_{S,X}\colon X^{\sqcup S} \to X^{\otimes S}$ is ℓ-connected, then the space $\mathrm{End}_X^{I,\mathrm{red}}(\mathcal{C})(S)$ is k-truncated.*

Given a reduced I-operad $\mathcal{P}^\otimes$ and a V-object $X \in \mathcal{O}_V$, applying $\mathcal{P}$-algebras to Eq. (23) yields a pullback

$$\begin{array}{ccc} \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O}) & \longrightarrow & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})^V \\ \downarrow & \lrcorner & \downarrow U \\ \{X\} & \hookrightarrow & U\mathcal{O}^V \end{array} \tag{24}$$

In the case that $U\mathcal{O}$ is a $\mathcal{T}$-space, U is automatically a cocartesian fibration; thus the entire $\mathcal{T}$-category $\underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})$ is recoverable from its algebras with fixed underlying V-object. We'll exploit this to assemble $\mathcal{O}^\otimes$ from its reduced endomorphism operads soon; before then, we see that we still recover quite a lot of information in the general case.

Proposition II.2.48. *Suppose $\mathcal{P}^\otimes \to \mathcal{Q}^\otimes$ is a morphism of I-operads inducing an equivalence of spaces*

$$\varphi_X^{*,\simeq}\colon \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{Q}} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})^\simeq \longrightarrow \mathrm{Alg}_{\mathrm{Res}^{\mathcal{T}}_V \mathcal{P}} \mathrm{End}_X^{I,\mathrm{red}}(\mathcal{O})^\simeq$$

for all $V \in \mathcal{T}$ and $X \in U\mathcal{O}_V$. Then, the induced map of $\mathcal{T}$-spaces

$$\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{O})^\simeq \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})^\simeq$$

is an equivalence; in particular, passing to $\mathcal{T}$-fixed points, the induced map of spaces

$$\mathrm{Alg}_{\mathcal{Q}}(\mathcal{O})^\simeq \to \mathrm{Alg}_{\mathcal{P}}(\mathcal{O})^\simeq$$

is an equivalence.

Proof. Taking cores of Eq. (24), we find that that $\varphi_X^{*,\simeq}$ is the induced map on the homotopy fiber over X of the following map of $\mathcal{T}$-spaces over $U\mathcal{O}$:

$$\begin{array}{ccc} \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{O})^\simeq & \xrightarrow{\varphi^{*,\simeq}} & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{O})^\simeq \\ & \searrow \quad \swarrow & \\ & U\mathcal{O} & \end{array}$$

$\varphi^{*,\simeq}$ is an equivalence if and only if its V-fixed points are an equivalence for all $V \in \mathcal{T}$, and the homotopy fibers of $\varphi^{*,\simeq,V}$ are contractible by the above argument, so $\varphi^{*,\simeq,V}$ is an equivalence for all V. Hence $\varphi^{*,\simeq}$ is an equivalence, proving the proposition. □

II.2.3.3 Assembly for unital $\mathcal{T}$-operads We will conclude the following.

Corollary II.2.49. *Taking reduced endomorphism operads yields an equivalence*

$$\underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{uni}} \times_{U,\underline{\mathrm{Cat}}_{\mathcal{T}}} \underline{\mathcal{S}}_{\mathcal{T}} \simeq \int_{X \in \underline{\mathcal{S}}_{\mathcal{T}}} \underline{\mathrm{Fun}}_{\mathcal{T}}\left(X, \underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{red}}\right);$$

the restricted inclusion $\underline{\mathrm{Fun}}_{\mathcal{T}}\left(X, \underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{red}}\right) \hookrightarrow \underline{\mathrm{Op}}_{\mathcal{T}}^{\mathrm{uni}}$ *takes an X-indexed $\mathcal{T}$-colimit, i.e. witnesses an equivalence*

$$\mathcal{O}^{\otimes} \simeq \underline{\mathrm{colim}}_{x \in U\mathcal{O}} \mathrm{End}_x^{\mathrm{red}}(\mathcal{O})$$

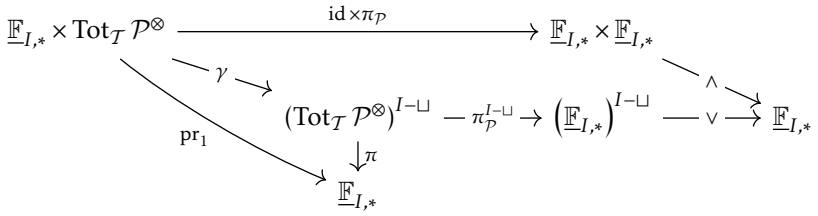
whenever $\mathcal{O}^{\otimes}$ is $\mathcal{T}$-space colored.

In particular, this identifies generalized $\mathcal{T}$-space shaped $\mathcal{T}$-operads with reduced fibers equivalently as unital $\mathcal{T}$-operads. This will be powered by disintegrating along maps to cocartesian structures, for which we need the following.

Construction II.2.50. Fix $\mathcal{C}$ a $\mathcal{T}$-∞-category. Then, pullback along the projection $(-) \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^* \to (-)$ determines a natural transformation

$$\mathrm{Fun}_{\mathcal{T},/\underline{\mathbb{F}}_{I,*}}\left(-, \mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\sqcup}\right) \simeq \mathrm{Fun}_{\mathcal{T}}\left(- \times_{\underline{\mathbb{F}}_{I,*}} \underline{\Gamma}_I^*, \mathcal{C}\right) \longleftarrow \mathrm{Fun}_{\mathcal{T}}(-, \mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{T},/\underline{\mathbb{F}}_{I,*}}\left(-, \mathcal{C} \times \underline{\mathbb{F}}_{I,*}\right),$$

which corresponds with a $\mathcal{T}$-functor $\gamma \colon \mathcal{C} \times \underline{\mathbb{F}}_{I,*} \to \mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{I-\sqcup}$ under Yoneda's lemma. Note that $\gamma(C, S) \simeq \Delta^S C$. Moreover, this is compatible with Γ-I-preoperadic structure in the case $\mathcal{C} = \mathrm{Tot}_{\mathcal{T}} \mathcal{P}^{\otimes}$:



so in particular, γ becomes a *bifunctor* under the alternative structure map ρ. ◂

We will eventually use γ to describe wreath products. First, disintegration.

Lemma II.2.51. *When X is a $\mathcal{T}$-space, the Γ-I-preoperad map $\gamma \colon \mathrm{Patt}(X) \times \underline{\mathbb{F}}_{I,*} \to \mathrm{Tot}_{\mathcal{T}} X^{I-\sqcup}$ is a strong Morita equivalence; in particular, it is an L_{Op_I} localization map.*

Proof. We must verify the conditions of Corollary I.3.31. By unwinding definitions we find that $\mathrm{Tot}\,\gamma$ is an iso-segal morphism, and in particular it is a strong Segal morphism; Condition (ML-b) is obvious.

For Condition (ML-a), note that the V-elements of $\left(X^{I-\sqcup}\right)^{\mathrm{act}}_{/\gamma(x,S)}$ correspond with maps $f \colon T \to S$ in $\mathbb{F}_V$ together with elements $(y_U)_T \in X^T$ with distinguished paths $y_U \sim x$ within X^U; in particular, by

contracting paths, we may construct a deformation retract of $\left(\left(\mathrm{Tot}_{\mathcal{T}} X^{I-\sqcup}\right)^{\mathrm{act}}_{/\gamma(x,S)}\right)^{\simeq}$ onto the subspace $\mathbb{F}^{V,\simeq}_{I,/S} \subset \left(\left(\mathrm{Tot}_I X^{I-\sqcup}\right)^{\mathrm{act}}_{/\gamma(x,S)}\right)^{\simeq}$ of identity paths.

Similarly, we may perform a deformation retract of $\left(\left(X \times \underline{\mathbb{F}}_{i,*}\right)^{\mathrm{act}}_{/(x,S)}\right)^{\simeq}$ onto the summand $\mathbb{F}^{\simeq}_{I,/S} \subset \left(\left(X \times \underline{\mathbb{F}}_{i,*}\right)^{\mathrm{act}}_{/(x,S)}\right)^{\simeq}$ of identity paths. It follows by unwinding definitions that these are taken isomorphically onto each other; alternatively, one may note that the induced endomorphism of $\mathbb{F}^{\simeq}_{I,/S} \simeq \prod_{U \in \mathrm{Orb}(S)} \coprod_{T \in \mathbb{F}_{I,U}} B\,\mathsf{Aut}_U(T)$ is a product of coproducts of maps classified by torsor maps $\mathsf{Aut}_U(T) \to \mathsf{Aut}_U(T)$, which are automatically isomorphisms, implying that our map of 1-truncated spaces is an isomorphism on π_0 and on π_1 at all basepoints. □

Warning II.2.52. A closely related analog of Lemma II.2.51 is claimed in [HA, Rmk. 2.4.3.6] in the case $\mathcal{T} = *$ *without the assumption that* $\mathcal{C}$ *is a space*; as pointed out in [KK24, Rmk 2.3] Lurie's claim (and hence proof) is incorrect in general, but the claim was verified in *loc. cit.* when $\mathcal{C}$ is a space. ◁

As a consequence, we may define the *assembly* $\mathcal{T}$*-functor*

$$\alpha = L_{\mathrm{Op}_I}\gamma_! \colon \underline{\mathsf{Fun}}_{\mathcal{T}}\left(X, \underline{\mathrm{Op}}^{\mathrm{uni}}_I\right) \simeq \underline{\mathsf{Fibrs}}^{\mathrm{uni}}\left(\mathrm{Patt}(X) \times \underline{\mathbb{F}}_{I,\mathcal{T}}\right) \longrightarrow \underline{\mathrm{Op}}^{\mathrm{uni}}_I.$$

This is not such a strange construction:

Lemma II.2.53. *α is naturally equivalent to the $\mathcal{T}$-colimit $\mathcal{T}$-functor.*

Proof. $L_{\mathrm{Op}_I}\pi_{X^{I-\sqcup}!}\gamma_! \dashv \gamma^*\pi^*_{X^{I-\sqcup}}$, and the latter is equivalent to $\Delta \colon \mathrm{Op}_I \to \mathsf{Fun}_{\mathcal{T}}(X, \underline{\mathrm{Op}}_I)$. □

Following this, the author suspects there may be some more general disintegration and assembly picture centered on unstraightening/*lax* colimits which will allow X to have non-invertible morphisms. We won't discuss this further, instead concluding here.

Proof of Corollary II.2.49. As in the proof of Proposition I.3.50, we must verify the conditions of Lemma I.3.49 for the $U \leftrightarrows (-)^{I-\sqcup}$ adjunction of Theorem II.2.32. That is, we must verify given a cartesian square

$$\begin{array}{ccc} \mathcal{P}^{\otimes} & \longrightarrow & \mathcal{O}^{\otimes} \\ \downarrow & \lrcorner & \downarrow \\ X^{I-\sqcup} & \longrightarrow & U\mathcal{O}^{I-\sqcup} \end{array}$$

that $\mathcal{P}^{\otimes} \to X^{I-\sqcup}$ classifies an equivalence $UP \xrightarrow{\sim} X$; this follows by the fact that U itself is a right adjoint by Corollary II.1.9.

In particular, this yields an equivalence

$$\underline{\mathrm{Op}}^{\mathrm{uni}}_I \times_{U,\underline{\mathrm{Cat}}_{\mathcal{T}}} \underline{\mathcal{S}}_{\mathcal{T}} \simeq \int_{X \in \underline{\mathcal{S}}_{\mathcal{T}}} U^{-1}(X).$$

The result then follows by noting that colimits provide a natural equivalence

$$\underline{\mathrm{Fun}}_{\mathcal{T}}\left(X, \underline{\mathrm{Op}}_I^{\mathrm{red}}\right) \xrightarrow{\sim} U^{-1}(X)$$

by Lemmas II.2.51 and II.2.53. □

II.2.4 Parameterized symmetric monoidal subcategories, localization, slices The following is an immediate consequence of Theorem I.3.51, and allows us to construct many examples.

Proposition II.2.54 (Suboperads)**.** *Let $\mathcal{P}^{\otimes}$ be a $\mathcal{T}$-operad and $\iota_U \colon U\mathcal{O} \hookrightarrow U\mathcal{P}$ a $\mathcal{T}$-subcategory.*

(1) The space of monomorphisms $\iota_U \colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ such that $U\iota = \iota_U$ is contractible, and every monomorphism of $\mathcal{T}$-operads arises in such a way.

(2) In this case, for all $\mathcal{T}$-operads $\mathcal{Q}^{\otimes}$, the associated $\mathcal{T}$-subcategory inclusion

$$\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{O}) \hookrightarrow \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{P})$$

has image with V-objects the maps $\varphi \colon \mathfrak{Q}_{\underline{V}}^{\otimes} \to \mathfrak{P}_{\underline{V}}^{\otimes}$ whose

(a) color V-functor $U\mathcal{Q}_{\underline{V}} \to U\mathcal{P}_{\underline{V}}$ factoring through $\mathcal{O}_{\underline{V}}$, and

(b) structure space maps $\mathcal{Q}(\mathbf{C}; D) \to \mathcal{O}(\varphi\mathbf{C}; \varphi D)$ which factor through $\mathcal{P}(\varphi\mathbf{C}; \varphi D)$,

and morphisms those whose color $\mathcal{T}$-natural transformation lies in $U\mathcal{O}_{\underline{V}}$.

(3) If $\mathcal{P}^{\otimes}$ is an I-symmetric monoidal ∞-category, then ι_U lifts to a monomorphism of I-symmetric monoidal ∞-categories if and only if $U\mathcal{O} \subset U\mathcal{P}$ is closed under I-indexed tensor products; every monomorphism of I-symmetric monoidal ∞-categories arises in this way.

(4) If ι is I-symmetric monoidal, then the subcategory

$$\underline{\mathrm{Fun}}_I^{\otimes}\left(\mathcal{C}^{\otimes}, \mathcal{O}^{\otimes}\right) \hookrightarrow \underline{\mathrm{Fun}}_I^{\otimes}\left(\mathcal{C}^{\otimes}, \mathcal{P}^{\otimes}\right)$$

consists of the objects and morphisms which lie in the associated subcategory of algebra maps; moreover, Condition (2)b is automatic in this case.

In the situation of Statement (3), we say that $\mathcal{O}^{\otimes}$ is an *I-symmetric monoidal subcategory.* Now, uniqueness of this structure gives an invariant, called the *multiplicativity* of ι.

Proposition II.2.55. *Write $\mathfrak{m}(\iota_U) = \left\{f \colon T \to S \mid f_{\otimes}\mathcal{O}^T \subset \mathcal{O}^S\right\}$. Then, $\mathfrak{m}(\iota_U)$ is a one-color weak indexing category such that $\mathcal{O}^{\otimes}$ is an I-symmetric monoidal subcategory if and only if $I \subset \mathfrak{m}(\iota_U)$.*

Proof. So long as $\mathfrak{m}(\iota_U)$ is a weak indexing category, its universality is true by fiat. Compositionality of $f_{\otimes}$ shows that $\mathfrak{m}(\iota_U) \subset \mathbb{F}_{\mathfrak{o}(\mathcal{T})}$ is indeed a wide subcategory. The Segal condition is obvious by unwinding

definitions. For pullback-stability, by Observation I.1.87, it suffices to prove that, in the case S is orbital, $\mathfrak{m}(\iota_U)$ is closed under base change along orbit maps $W \to V$; this follows by noting that the resulting functor is $\left(\mathrm{Res}^V_W f\right)_\otimes$. □

In the case that $\mathrm{Tot}_{\mathcal{T}} \mathcal{O}^\otimes \hookrightarrow \mathrm{Tot}_{\mathcal{T}} \mathcal{P}^\otimes$ is full, we call $\mathcal{O}^\otimes$ a *full I-suboperad* (or *full I-symmetric monoidal subcategory*).

Another important source of examples will be *localizations*, per a corollary to Theorem I.3.58:

Proposition II.2.56. *Given $\iota\colon \mathcal{O}^\otimes \hookrightarrow \mathcal{P}^\otimes$ a full I-suboperad, the following are equivalent:*

(a) $U\iota$ admits a left adjoint UL such that, for all S-tuples of UL-equivalences $(f_U \mid U \in \mathrm{Orb}(S))$, the map $\bigotimes^S_U f_U$ is a UL-equivalence.

(b) $\mathcal{O}^\otimes$ admits an I-symmetric monoidal structure and L lifts to an I-symmetric monoidal functor whose underlying $\mathcal{T}$-operad map is left adjoint to ι.

In this case, the image of the full subcategory inclusions

$$L^*\colon \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \hookrightarrow \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C});$$

$$L^*\colon \underline{\mathrm{Fun}}^\otimes_I(\mathcal{O},\mathcal{C}) \hookrightarrow \underline{\mathrm{Fun}}^\otimes_I(\mathcal{P},\mathcal{C})$$

consists of $\mathcal{P}$-algebras (resp. I-symmetric monoidal functors) whose color $\mathcal{T}$-functors factor through $U\mathcal{O}$.

We likewise acquire an invariant measuring mutliplicativity of localizations.

Proposition II.2.57. *Let $L\colon \mathcal{D} \to \mathcal{C}$ be a $\mathcal{T}$-localization, and let $\mathfrak{m}(L) \subset \mathbb{F}_{\mathcal{T}}$ denote the collection of maps $f\colon T \to S$ such that $f_\otimes$ preserves L-equivalences. Then, $\mathfrak{m}(L)$ is a one-color weak indexing category.*

Proof. Once again, compositionality and the Segal condition follow by an obvious argument, and restriction-stability follows by noting that $f_\otimes$ restricts to $\left(\mathrm{Res}^V_W f_\otimes\right)$ and restriction preserves L-equivalences. □

For example, we may describe the interaction between coinduction and connectivity this way.

Example II.2.58. Following Lewis [Lew92, Def. 1.1], we define a *$\mathcal{T}$-dimension function* as a function $\nu\colon \pi_0\mathcal{T} \to \overline{\mathbb{Z}}$, where $\overline{\mathbb{Z}} = \{\pm\infty\} \cup \mathbb{Z}$ is the extended integers.

These admit a poset structure under pointwise $\leq$. Additionally, given ν a V-dimension function and $f\colon U \to V$ an orbit map we may define

$$\mathrm{Res}^V_U \nu\colon \pi_0\mathcal{T}_{/U} \xrightarrow{f_!} \mathcal{T}_{/V} \xrightarrow{\nu} \overline{\mathbb{Z}};$$

this respects $\leq$. Given two dimension functions $\nu \leq \nu'$, we define the *$\mathcal{T}$-spaces concentrated in* $[\nu, \nu']$ as

$$\underline{\mathcal{S}}^V_{\mathcal{T},[\nu,\nu']} = \left\{ X \in \mathcal{S}_V \mid \forall U \to V, X^U \in \mathcal{S}_{[\nu^U, \nu'^U]} \right\};$$

that is ν and ν' specify fiberwise connectivity and truncatedness requirements. In the case $\nu' = \infty$ we write $\underline{\mathcal{S}}_{\mathcal{T},\geq\nu}$, and when $\nu = -\infty$ we write $\mathcal{S}_{\mathcal{T},\leq\nu'}$. ◁

Given ν a dimension function, we define the wide subcategory

$$\mathfrak{Rm}(\nu,\nu') = \left\{ U \to V \;\middle|\; \forall V' \to V, \text{ orbits } U' \in \mathsf{Orb}(U \times_V V'), \quad \nu^{U'} \leq \nu^{V'} \quad \text{and} \quad \nu'^{V'} \leq \nu'^{U'} \right\};$$

we define $\mathfrak{m}(\nu) = \mathbb{F}_{\mathfrak{Rm}(\nu)}$. For convenience, we prove the following in the equivariant case, though it applies more generally.

Proposition II.2.59. *Fix ν, ν' a pair of dimension functions for the ∞-category $\mathcal{T} = \mathcal{O}_G$. Then,*

$$\mathfrak{m}\left(\underline{\mathcal{S}}_{G,[\nu,\nu']} \hookrightarrow \underline{\mathcal{S}}^\times_G\right) = \mathfrak{m}(\nu,\nu').$$

Proof. For (1), we define the *connectiveness dimension function* $\mathsf{connt}(\bullet)\colon \mathcal{T}_{/V} \times \mathcal{S}_V \to \overline{\mathbb{Z}}$ by the value

$$\mathsf{connt}^U(X) = \inf\left\{ n \mid X^U \in \mathcal{S}_{\geq n} \right\};$$

the truncatedness dimension function trunc is defined similarly. We make two crucial observations:

Observation i. $X \in \mathcal{S}_{[\nu,\nu']}$ if and only if $\mathsf{connt}(X) \geq \nu$ and $\mathsf{trunc}(X) \leq \nu'$, and

Observation ii. the double coset formula $\left(\mathrm{CoInd}^V_U X\right)^{V'} \simeq \prod_{U' \in \mathsf{Orb}(U \times_V V')} X^{U'}$ (e.g. from $\mathcal{T}$-symmetric monoidality of the cartesian structure) expresses

$$\mathsf{connt}^{V'}\mathrm{CoInd}^V_U(X) \simeq \min_{U' \in \mathsf{Orb}(U \times_V V')} \mathsf{connt}^{U'}(X);$$

$$\mathsf{trunc}^{V'}\mathrm{CoInd}^V_U(X) \simeq \max_{U' \in \mathsf{Orb}(U \times_V V')} \mathsf{trunc}^{U'}(X).$$

Combining these two observations immediately shows that $\mathfrak{m}\left(\underline{\mathcal{S}}_{\mathcal{T},[\nu,\nu']} \hookrightarrow \underline{\mathcal{S}}^\times_{\mathcal{T}}\right) \subset \mathfrak{m}(\nu,\nu')$; for the opposite inclusion, given some $U \to V \leftarrow V'$ and orbit $U' \in \mathsf{Orb}(U \times_V V')$, such that $\nu^{U'} > \nu^{V'}$, Elmendorf's theorem allows us to construct a pathological U-space with fixed points

$$X^H := \begin{cases} * & (H) \subsetneq (U') \\ K\left(\mathbb{Z}, \nu^{U'}\right) & (H) = (U') \\ \varnothing & \text{otherwise,} \end{cases}$$

with trivial Weyl-group action on $K\left(\mathbb{Z}, \nu^{U'}\right)$. By the above observations, we have $X \in \mathcal{S}_{U,[\nu,\nu']}$, but $\mathrm{CoInd}_U^V X \notin \mathcal{S}_{V,[\nu,\nu']}$. □

For instance, in the case that G is a compact Lie group, we define the V-connected locus

$$\underline{\mathcal{S}}^H_{G,\geq V} \coloneqq \left\{X \in \mathcal{S}_H \mid \forall W \subset \mathrm{Res}^H_K V, \;\; \pi_0 \mathrm{Map}^K(S^W, X) \simeq *\right\}.$$

It's a result of Lewis-Waner that this is connectivity for a dimension function, so we may conclude.

Corollary II.2.60. *$\underline{\mathcal{S}}^\times_{G,\geq V} \subset \underline{\mathcal{S}}^\times_G$ is a full G-symmetric monoidal subcategory.*

Proof. Let $V(\bullet)\colon \pi_0\mathcal{O}_G \to \overline{\mathbb{Z}}$ be the dimension function setting $V(K) = \dim V^K$. Lewis (extending an unpublished result of Waner) shows that $\mathcal{S}_{G,\geq V} \simeq \mathcal{S}_{G,\geq V(\bullet)}$ [Lew92, Lem. 1.2], so this follows immediately from Proposition II.2.59. □

This follows a more general pattern: often G-symmetric monoidal ∞-categories will fall into *multiplicative recollement situations* expressing them as glued together from a nonequivariant object for each conjugacy class, and we will be able to use this to lift symmetric monoidal full subcategories or localizations from the nonequivariant situation to the equivariant one; the multiplicaitivity of the resulting situation will often be describable in terms of compatibility of the various fiberwise localizations.

Before moving on, we point out the relationship with weak indexing categories, which is immediate from definitions.

Proposition II.2.61. *If $I \subset \mathbb{F}_\mathcal{T}$ is a pullback-stable wide subcategory, then I is a weak indexing category if and only if $\mathfrak{m}\left(\underline{\mathbb{F}}_I \hookrightarrow \underline{\mathcal{S}}^\sqcup_\mathcal{T}\right) = I$; moreover, $\underline{\mathrm{Set}}^\sqcup_\mathcal{T} \subset \underline{\mathcal{S}}^\sqcup_\mathcal{T}$ is a $\mathcal{T}$-symmetric monoidal localizing subcategory admitting a factorization $\underline{\mathbb{F}}^\sqcup_\mathcal{T} \hookrightarrow \underline{\mathrm{Set}}^\sqcup_\mathcal{T} \hookrightarrow \underline{\mathcal{S}}^\sqcup_\mathcal{T}$.*

II.2.5 Reflective localizations and Morita localizations of parameterized operads

II.2.5.1 Localizations of $\mathcal{T}$-operads We make the following construction.

Construction II.2.62. Given $\mathcal{O}^\otimes$ a $\mathcal{T}$-operad and $UL\colon U\mathcal{O} \to U\mathcal{P}$ a $\mathcal{T}$-localization, we define the associated localization

$$\begin{array}{ccc} \mathrm{triv}(\mathcal{O})^\otimes & \longrightarrow & \mathrm{triv}(\mathcal{P})^\otimes \\ \downarrow & & \downarrow \\ \mathcal{O}^\otimes & \longrightarrow & \mathcal{P}^\otimes \end{array}$$

so that $\mathcal{P}^\otimes$ satisfies the mapping property

$$\underline{\mathrm{Alg}}_\mathcal{P}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}_\mathcal{O}(\mathcal{C}) \times_{\underline{\mathrm{Fun}}_\mathcal{T}(U\mathcal{O}, U\mathcal{C}),\, L^*} \underline{\mathrm{Fun}}_\mathcal{T}(U\mathcal{P}, U\mathcal{C}).$$ ◁

The following proposition follows immediately by comparing universal properties.

Proposition II.2.63. *If $\mathcal{O}^\otimes$ is I-symmetric monoidal and UL compatible with the I-symmetric monoidal structure, then $\mathcal{P}^\otimes$ is the I-symmetric monoidal localization constructed in Proposition II.2.56.*

More importantly, we note the following.

Proposition II.2.64. *Suppose $\varphi\colon \mathcal{O}^\otimes \to \widetilde{\mathcal{P}}^\otimes$ is a $\mathcal{T}$-operad map lying over a $\mathcal{T}$-localization $U\mathcal{P} \to U\mathcal{O}$. Then, the following conditions are equivalent:*

(a) $\mathrm{Tot}_{\mathcal{T}}\,\varphi$ is a Morita $\mathcal{T}$-localization,

(b) $\mathrm{Tot}\,\varphi$ is a Morita localization, and

(c) The associated map $\widetilde{\mathcal{P}}^\otimes \to \mathcal{P}^\otimes$ is an equivalence.

In particular, φ satisfies the universal property

$$\underline{\mathrm{Alg}}_{\widetilde{\mathcal{P}}}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O},U\mathcal{C}),\, L^*} \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{P}, U\mathcal{C}).$$

Proof. Theorem II.2.7 shows that Condition (c) may be verified on pullback between algebras in $\mathcal{T}$-spaces, which is equivalent to it inducing an equivalence on Segal spaces on Tot or on Segal $\mathcal{T}$-spaces by Theorems I.4.1 and II.2.1. Comparing universal properties shows this is equivalent to Conditions (a) and (b). □

That is, algebras over Morita localizations are *locally constant algebras*. Following the classical results of factorization algebras, this will be quite useful to us, so we now study how to recognize such localizations.

II.2.5.2 Recognizing Γ-Morita localizations of $\mathcal{T}$-preoperads We need a variation of [HA, Cor. 2.3.3.16]:

Proposition II.2.65. *Suppose $\alpha\colon \mathcal{P}^\otimes \to \mathcal{E}^\otimes$ is a map of $\mathcal{T}$-preoperads such that $\mathcal{E}^\otimes$ is a one-color $\mathcal{T}$-operad. Then, Condition (ML-a) of Corollary I.3.31 for the associated Γ-$\mathcal{T}$-preoperad map is equivalent to the condition that, for all $P \in \mathrm{Tot}\,\mathcal{P}^\otimes$ and active $\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$-maps $f\colon D \to \pi_{\mathcal{P}}P$, the induced map of spaces*

$$\widetilde{\alpha}^{\mathrm{act},f}_{/P}\colon B\mathcal{P}^{\mathrm{act},f}_{/P} \longrightarrow \mathrm{Map}^{f}_{\mathrm{Tot}\,\mathcal{E}}(D;\pi_{\mathcal{P}}P).$$

is an equivalence, where $\mathcal{P}^{\mathrm{act},f}_{/P} := \mathcal{P}^{\mathrm{act}}_{/P} \times_{\mathbb{F}_{\mathfrak{o}(\mathcal{T}),/\pi_{\mathcal{P}}P}} \{f\}$. If moreover, $\mathcal{P}^\otimes$ is a $\mathcal{T}$-operad, then it suffices to verify this condition when $\pi_{\mathcal{P}}P$ is an orbit.

Proof. Fixing some $\mu \in \mathrm{Mul}^{f}_{\mathcal{E}}(D;\pi_{\mathcal{P}}P)$, draw the diagram

$$\begin{array}{ccccc}
F & \longrightarrow & \mathcal{P}^{\mathrm{act},f}_{/P} & \longrightarrow & \mathcal{P}^{\mathrm{act}}_{/P} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\{\tilde{f}\} & \longrightarrow & \mathrm{Map}^{f}_{\mathrm{Tot}\,\mathcal{E}}(D;\pi_{\mathcal{P}}P) & \longrightarrow & \mathcal{E}^{\mathrm{act}}_{/\alpha P} \\
 & & \downarrow & \lrcorner & \downarrow \\
 & & \{f\} & \longrightarrow & \mathrm{Fin}_{\mathcal{T},/\pi P}
\end{array}$$

We're tasked with verifying that F is weakly contractible. But we may apply the adjunction between B and $\mathcal{S} \hookrightarrow \mathsf{Cat}$ to construct a fiber square

$$\begin{array}{ccc} BF & \longrightarrow & B\mathcal{P}^{\mathrm{act},f}_{/\mathcal{P}} \\ \downarrow & \lrcorner & \downarrow \\ \left\{\tilde{f}\right\} & \longrightarrow & \mathcal{E}^{\mathrm{act},f}_{/\alpha P} \end{array}$$

In particular, weak contractibility of F for all $\left\{\widetilde{f}\right\}$ is equivalent to contractibility of the homotopy fibers of $\widetilde{\alpha}^{\mathrm{act},f}_{/P}$ for all f, which is equivalent to $\widetilde{\alpha}^{\mathrm{act},f}_{/P}$ being an equivalence for all (f,P), giving the main claim.

For remaining claim, note that pullback along inert morphisms constructs a commutative diagram:

$$\begin{array}{ccc} \prod_{U \in \mathrm{Orb}(\pi_{\mathcal{P}} P)} \mathcal{P}^{\mathrm{act},f_U}_{/P_u} & \longleftarrow & \mathcal{P}^{\mathrm{act},f}_{/P} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ \prod_{U \in \mathrm{Orb}(\pi_{\mathcal{P}} P)} \mathcal{E}(D_U) & \longleftarrow & \mathrm{Mul}^f_{\mathcal{E}}(D; \pi_{\mathcal{P}} P) \end{array}$$

The Segal condition for multioperations implies that the bottom horizontal arrow is an equivalence, so to conclude, it suffices to prove that the top horizontal arrow is an equivalence; fully faithfulness and essential surjectivity both follow by unwinding the Segal conditions for $\mathcal{P}^{\otimes}$. □

II.2.6 ***G*-manifolds and equivariant little disk algebras** Fix G a compact Lie group. Generalizing the nonequivariant picture, a regime of of central importance in G-equivariant homotopy theory is that of *little disk G-operads, G-symmetric monoidal ∞-categories of manifolds, and equivariant factorization homology.*

II.2.6.1 G-manifolds The author developed some rudiments of the parameterized higher algebra of G-manifolds in ongoing joint work with Juran, but is obstructed from using text from that project here, as it appears in Juran's thesis [Jur25b]; as such, we will omit some proofs in this subsubsection.

Construction II.2.66. Define the simplicial topological category $\mathbf{Mfld}_{\bullet,G} \colon \Delta^{\mathrm{op}} \to \mathsf{Cat}_{\mathrm{Top}}$ so that $\mathbf{Mfld}_{n,G}$ has

- objects the sequences of topological G-space maps $M \to S_0 \to \cdots \to S_n$ such that M has a distinguished smooth G-manifold structure and each S_i is an orbit-finite G-space, and
- morphisms the natural transformations

$$\begin{array}{ccccccc} M & \longrightarrow & S_0 & \longrightarrow & \cdots & \longrightarrow & S_n \\ {\scriptstyle f_M}\downarrow & & {\scriptstyle f_0}\downarrow & & & & {\scriptstyle f_n}\downarrow \\ N & \longrightarrow & T_0 & \longrightarrow & \cdots & \longrightarrow & T_n \end{array}$$

  such that f_M is a smooth G-equivariant map and each f_i is a G-equivariant map such that:
  - Each map $S_i \to S_{i+1} \times_{T_{i+1}} T_i$ is an equivalence, and
  - the map $M \to N \times_{T_0} S_0$ is an embedding,

topologized as a subspace of $C^\infty(M,N) \times \prod_{i=0}^n \mathrm{Map}_{\mathrm{Top}_G}(S_i, T_i)$;

the simplicial structure is exactly as in the nerve of Top_G. The simplicial topological category $\mathbf{Span}_{\bullet,G}$ is defined similarly, but without M and N (or equivalently, $M = S_0$ and $N = T_0$). ◁

Proposition II.2.67. *Given G a compact Lie group, the evident natural transformation $p\colon \mathbf{Mfld}_\bullet^G \Longrightarrow \mathbf{Span}_\bullet^G$ satisfies the following:*

(i) The homotopy-coherent nerve of $\mathbf{Span}_\bullet^G$ is a factorization double ∞-category in the sense of [Jur25c] which, under the correspondence of [Jur25c, Thm. A] classifies $\mathrm{Span}_{\mathrm{f.i.}}(\mathbb{F}_G)$ with its canonical factorization system.

(ii) p acts on mapping spaces as Serre fibrations.

(iii) The homotopy-coherent nerve of p is a (cocart, right)-fibration in the sense of [Jur25c], and the corresponding op-gray fibration $\mathrm{Tot}^\vee \underline{\mathrm{Mfld}}_G^{\sqcup} \to \mathrm{Span}_{\mathrm{f.i.}}(\mathbb{F}_G)$ is a cocartesian fibration which straightens to a G-symmetric monoidal ∞-category.

(iv) $\pi_0 \mathrm{Mfld}_H$ is in bijection with dimensional manifolds with distinguished smooth H-action, with mapping spaces the topological subspaces

$$\mathrm{Map}_{\mathrm{Mfld}_H}(M,N) \simeq \mathrm{Emb}^H(M,N) \subset C^\infty(M,N),$$

consisting of equivariant smooth embeddings under the (strong) C^∞-topology.

(v) The binary tensor product on $\mathrm{Mfld}_H^{\sqcup}$ takes disjoint unions; the norm takes

$$N_K^H M \simeq M \times_K H.$$

Construction II.2.68. Let $\underline{\mathrm{Mfld}}_G^{n,\sqcup} \subset \underline{\mathrm{Mfld}}_G^{\sqcup}$ be the full G-symmetric monoidal subcategory spanned by n-dimensional G-manifolds. Let $\underline{\mathrm{Emb}}_G \to \underline{\mathrm{Mfld}}_G$ be the G-functor classified by the topological nonfull topological subcategories $\mathbf{Emb}_\bullet^G \subset \mathbf{Mfld}_\bullet^G$ whose objects are diffeomorphic to $D(V) \to \cdots$ for some V and whose morphisms are only codimension-0 embeddings; that is, there is a canonical equivalence $\underline{\mathrm{Emb}}_G \simeq \coprod_n \underline{\mathrm{Emb}}_G^n$ compatible with the subcategories of $\underline{\mathrm{Mfld}}_G^{n,\sqcup}$.

The *tangent classifier G-functor* is the restricted Yoneda embedding

$$\tau\colon \underline{\mathrm{Mfld}}_G^n \xrightarrow{\curlywedge} \underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Mfld}}_G^n, \underline{\mathcal{S}}_G\right) \xrightarrow{\iota^*} \underline{\mathrm{Fun}}_G\left(\underline{\mathrm{Emb}}_G^n, \underline{\mathcal{S}}_G\right) \simeq \underline{\mathcal{S}}_{G,/\underline{\mathrm{Emb}}_G^n}. \qquad ◁$$

Now, we will eventually prove the following, but first postpone it.

Proposition II.2.69. *There exists an equivalence $\underline{\mathrm{Emb}}_G \simeq B_G O(n)$ and the composite functor $U \circ \tau\colon \underline{\mathrm{Mfld}}_G^n \to \underline{\mathcal{S}}_{G,/B_G O(n)} \simeq \underline{\mathcal{S}}_G$ takes the underlying G-space.*

Corollary II.2.70. *τ admits a contractible space of lifts to G-symmetric monoidal functors $\underline{\mathrm{Mfld}}_G^{\sqcup} \to \underline{\mathcal{S}}^{\sqcup}_{G,/B_G O(n)}$*

Proof. Theorem II.2.32 shows that the space of lax G-symmetric monoidal structures on τ is contractible, with lax structure map the assembly map

$$\begin{array}{ccc} \coprod_K^S \mathrm{Re}(M_K) & \longrightarrow & \mathrm{Re}\left(\sqcup_{[H/K]\subset S} H \times_K M_K\right) \\ & \searrow \quad \swarrow & \\ & \underline{\mathrm{Emb}}_G^n & \end{array}$$

where Re is the underlying G-space functor. That this is an equivalence follows by unwinding the fact that disjoint unions in $\mathcal{S}_G$ are coproducts and $M \mapsto H \times_K M$ is left adjoint to restriction. □

Following Horev [Hor19], we may define G-symmetric monoidal ∞-categories of G-manifolds with distinguished tangential structure.

Construction II.2.71. Given $\Theta\colon B_\Theta \to B_G O(n)$, we define the G-symmetric monoidal ∞-category of Θ-*framed G-manifolds* as the pullback

$$\begin{array}{ccc} \underline{\mathrm{Mfld}}_G^{\Theta,\sqcup} & \longrightarrow & \underline{\mathrm{Mfld}}_G^{n,\sqcup} \\ \downarrow & \lrcorner & \downarrow \\ \underline{\mathcal{S}}^{\sqcup}_{G,/B_\Theta} & \xrightarrow[\Theta_!]{} & \underline{\mathcal{S}}^{\sqcup}_{G,/B_G O(n)} \end{array}$$

where $\Theta_!$ is given its essentially unique G-symmetric monoidal structure via Theorem II.2.1 in view of the fact that it is a G-left adjoint, and hence preserves G-coproducts.

Informally, a G-object of $\underline{\mathrm{Mfld}}_G^{\Theta}$ is a Θ-*framed G-manifold*, i.e. a pair of M a G-manifold and $T^\Theta M$ a lift of its equivariant tangent bundle

$$\begin{array}{ccc} & & B_\Theta \\ & \overset{T^\Theta M}{\nearrow} & \downarrow \Theta \\ M & \xrightarrow[TM]{} & B_G O(n) \end{array}$$

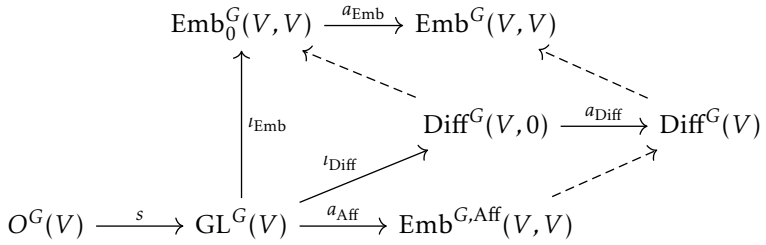
with mapping spaces the C^∞-embedding spaces together with lifts of their tangent factorization

$$\begin{array}{ccc} \mathrm{Emb}^{\Theta,G}(M,N) & \longrightarrow & \mathrm{Emb}^{G}(M,N) \\ \downarrow & \lrcorner & \downarrow d \\ \mathrm{Map}^G_{/B_\Theta}(M,N) & \xrightarrow[\mathrm{Res}^G_H \Theta_!]{} & \mathrm{Map}^G_{/B_G O(n)}(M,N) \end{array} \tag{25}$$

with the expected restriction. In the case of the map $\{V\} \hookrightarrow B_G O(n)$ classifying a representation V, we write $\underline{\mathrm{Mfld}}_G^{V-\mathfrak{r},\sqcup}$, and refer to these as *V-framed G-manifolds.* ◁

II.2.6.2 Self-embeddings Let $\mathrm{Diff}^G(V,0)$ denote the topological group of smooth origin-preserving diffeomorphisms of V under the strong C^∞-topology and $\mathrm{Emb}_0^G(V,V)$ the self-embedding space under the strong C^∞-topology. Let $O^G(V) \subset O(\dim V)$ and $\mathrm{GL}^G(V) \subset \mathrm{GL}(\dim V)$ be the topological subgroups of equivariant

linear isometries and isomorphisms, respectively. Note that, by the same argument as Remark II.2.94, these agree with the strong C^∞ subspace topologies, and in particular, we acquire a diagram of topological submonoid inclusions

(26)


Proposition II.2.72. *All of the maps in Eq. (26) are equivalences.*

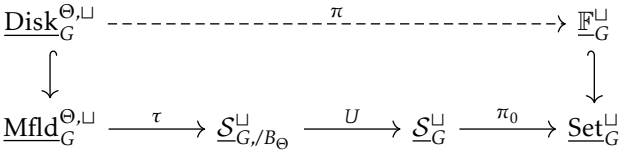
Proof. It suffices to prove that the solid arrows are equivalences, after which the dashed arrows become equivalences by two out of three. s is an equivalence by the Gram-Schmidt procedure. a_{Emb} admits a deformation retract by $h_t(\iota) = \iota - t \cdot \iota(0)$; this restricts to deformation retracts of a_{Diff} and a_{Aff}. To finish, we simply note that the deformation retract constructed in [Kup17, Thm. 2.4] restricts to a deformation retract for ι_{Emb} and ι_{Diff}. □

II.2.6.3 Little disk G-operads For now, we will value this in part for its relationship with *little disks*.

Construction II.2.73. We define $\underline{\mathrm{Disk}}_G^{\Theta,\sqcup} \subset \underline{\mathrm{Mfld}}_G^{\Theta,\sqcup}$ to be the G-symmetric monoidal subcategory generated by Θ-framed G-manifolds whose underlying G-manifolds are representation disks. ◁

As in [Hor19] this ought to occur as the envelope of a Θ-framed *little disks G-operad*. We can recognize this directly via the equifibered perspective.

Proposition II.2.74. *There exists an essentially unique factorization π making the following commute*



Moreover, π is an equifibration of G-symmetric monoidal ∞-categories.

Proof. In light of Propositions II.2.54 and II.2.61, the factorization exists up to contractible ambiguity if π_0 of every Θ-framed H-disk is a finite H-manifold; this follows by definition.

Now, to prove the equifibration condition, by Proposition II.1.20, we must verify the following conditions:

$$\begin{array}{ccc} \mathrm{Disk}_G^\Theta \times \mathrm{Disk}_G^\Theta & \xrightarrow{\sqcup} & \mathrm{Disk}_G^\Theta \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ \mathbb{F}_G \times \mathbb{F}_G & \xrightarrow{\sqcup} & \mathbb{F}_G \end{array} \qquad \begin{array}{ccc} \mathrm{Disk}_K^\Theta & \xrightarrow{(-)\times_K H} & \mathrm{Disk}_H^\Theta \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ \mathbb{F}_K & \xrightarrow{\mathrm{Ind}_K^H} & \mathbb{F}_H \end{array}$$

This is a straightforward verification we leave to the reader, highlighting the fact that H-manifolds with a map to $2 \cdot *_H$ are canonically H-equivariantly diffeomorphic to the coproduct of their fibers, and over H/K canonically equivalent to $H \times_K F$, where F is the fiber over the identity coset eK. □

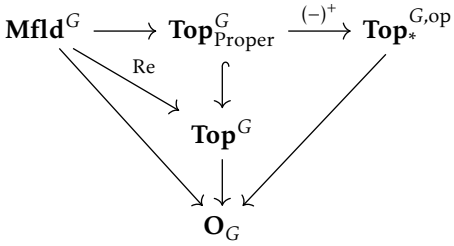
We let $\mathbb{E}_\Theta^\otimes$ be the G-operad classified by this equifibration under Theorem I.2.32; that is, it is the full G-suboperad spanned by Θ-framed G-manifolds of the form $D(V)$.

II.2.6.4 The structure on loop spaces $\mathbb{E}_V$ is perhaps most important because it describes the structure on *V-fold loop spaces*. This is for a simple reason: V-fold loops is corepresentable by S^V, which is the one-point compactification of the *manifold* $D(V)$, and hence acquires a coalgebra structure over the proper self-maps of $D(V)$, which restricts to an $\mathbb{E}_V$-coalgebra structure as embeddings are proper. We bear this out now.

Construction II.2.75. Let $\mathbf{O}_G$ be the topological orbit category of G. Define the pullback topological functor

$$\begin{array}{ccc} \mathbf{Mfld}^G & \longrightarrow & \mathbf{Mfld}_0^G \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{O}_G & \longrightarrow & \mathbf{Span}_0^G \end{array}$$

so that the homotopy-coherent nerve of $\mathbf{Mfld}^G$ straightens to the G-∞-category $\underline{\mathrm{Mfld}}_G$. We may extend the lefthand side to a diagram



where $\mathrm{Top}^G \to \mathbf{O}_G$ is the target functor from the topological category of arrows of topological G-spaces with transitive codomain (which presents $\underline{\mathcal{S}}_G$), $\mathbf{Top}_{\mathrm{proper}}^G$ is the wide topological subcategory of proper maps, and $\mathbf{Top}_*^{G,\mathrm{op}}$ the opposite topological category of pointed G-spaces decorated with a morphism from the complement of the basepoint to a transitive G-space, and $(-)^+$ denotes one-point (i.e. Alexandroff) compactification with its proper-functoriality. After homotopy-coherent nerves and unstraightening, this presents a G-functor

$$(-)^+ \colon \underline{\mathrm{Mfld}}_G^{\mathrm{op}} \to \underline{\mathcal{S}}_{G,*}. \qquad \triangleleft$$

Proposition II.2.76. *The space of G-symmetric monoidal lifts $\underline{\mathrm{Mfld}}_G^{\sqcup,\mathrm{op}} \to \underline{\mathcal{S}}_{G,*}^{\sqcup}$ of $(-)^+$ is contractible, where the domain has the fiberwise opposite G-symmetric monoidal structure.*

Proof. In sight of Theorem II.2.1, we must verify that assembly

$$\bigvee_{[H/K]\subset S} H \times_K M^+ \to \left(\sqcup_{[H/K]\subset S} H \times_K M\right)^+,$$

is an equivalence where the lefthand fiber product is taken in pointed G-spaces. Indeed, this follows from the basic observation that $(-)^+$ takes disjoint unions to wedge products. □

As a consequence, we may restrict this to a G-symmetric monoidal functor

$$S^V : \underline{\mathrm{Disk}}_G^{V-\mathrm{Fr},\sqcup,\mathrm{op}} \hookrightarrow \underline{\mathrm{Mfld}}_G^{V-\mathrm{Fr},\sqcup,\mathrm{op}} \xrightarrow{U} \underline{\mathrm{Mfld}}_G^{\sqcup,\mathrm{op}} \xrightarrow{(-)^+} \underline{\mathcal{S}}_{G,*}^{\sqcup};$$

we call this the *conventional $\mathbb{E}_V$-coalgebra structure on S^V*.

Remark II.2.77. This lifts to a coalgebra structure over the full suboperad $\mathbb{E}^{\otimes}_{BO(V)}$ spanned by the G-object of $B_G O(n)$ classifying the representation V, but we will not need this, so we do not consider it further. ◁

Construction II.2.78. Given a G-∞-category $\mathcal{C}$ with finite G-products and coproducts and $X \in \mathcal{C}^G$, the associated representable G-functor

$$\underline{\mathrm{Map}}_{\mathcal{C}}(-,X) : \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{S}}_G$$

is, by definition, G-continuous; in particular, it lifts canonically to a G-symmetric monoidal functor

$$\underline{\mathrm{Map}}_{\mathcal{C}} : (-,X)\left(\mathcal{C}^{\sqcup}\right)^{\mathrm{op}} \to \underline{\mathcal{S}}_G^{\times};$$

that is, it transforms cocartesian coalgebras to cartesian algebras. We refer to the associated composite

$$\Omega^V X : \underline{\mathrm{Disk}}_G^{V-\mathrm{Fr},\sqcup} \xrightarrow{S^V} \left(\underline{\mathcal{S}}_G^{\sqcup}\right)^{\mathrm{op}} \xrightarrow{\underline{\mathrm{Map}}_{\underline{\mathcal{S}}_{G,*}^{\sqcup}}(-,X)} \underline{\mathcal{S}}_{G,*}^{\times} \xrightarrow{U} \underline{\mathcal{S}}_G^{\times},$$

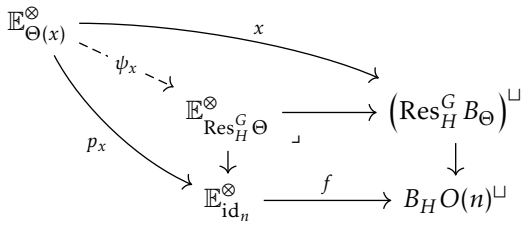
and its corresponding element $\Omega^V X \in \mathrm{Mon}_{\mathbb{E}_V}\left(\underline{\mathcal{S}}_G\right)$ as the *conventional $\mathbb{E}_V$-structure on loop spaces.* ◁

II.2.7 Assembly to equivariant little disk algebras Note that the full $\mathcal{T}$-suboperad $\underline{\mathcal{S}}_{G,/B}^{\sqcup}$ spanned by transitive G-sets is equivalent to $B^{\sqcup}$; indeed cocartesianness is closed under passing to full $\mathcal{T}$-suboperads per Proposition II.2.54 and Theorem II.2.32, and the statement on underlying $\mathcal{T}$-categories follows by unwinding definitions. In particular, we acquire a canonical pullback diagram

$$\begin{array}{ccc} \mathbb{E}^{\otimes}_{\Theta} & \longrightarrow & B^{\sqcup}_{\Theta} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{E}^{\otimes}_{\mathrm{id}_n} & \xrightarrow{f} & B_G O(n)^{\sqcup} \end{array}$$

where f classifies $U\mathbb{E}^{\otimes}_{\mathrm{id}_n} \simeq \underline{\mathrm{Emb}}^G_n \simeq B_G O(n)$.

In particular, given an orbit inclusion $[x]\colon [G/H] \hookrightarrow B_\Theta$, we have an H-equivariant diagram



In particular, this yields a reduction

$$\psi_x^{\mathrm{red}}\colon \mathbb{E}^{\otimes}_{\Theta(x)} \to \mathrm{End}_x^{\mathrm{red}}\, \mathbb{E}^{\otimes}_{\Theta}.$$

In this subsection, we will prove the following.

Theorem II.2.79. *The map $\psi_x^{\mathrm{red}}\colon \mathbb{E}^{\otimes}_{\Theta(x)} \xrightarrow{\sim} \mathrm{End}_x^{\mathrm{red}}\, \mathbb{E}^{\otimes}_{\Theta}$ is an equivalence.*

The disintegration and assembly procedure will immediately yield the following.

Corollary II.2.80. *Pullback along the maps ψ_x yields a natural G-limit presentation*

$$\underline{\mathrm{Alg}}_{\mathbb{E}_\Theta}(\mathcal{C}) \simeq \underline{\lim}_{x \in B_\Theta} \underline{\mathrm{Alg}}_{\mathbb{E}_{\Theta(x)}}(\mathcal{C}).$$

This underlies an equivalence of G-symmetric monoidal ∞-categories

$$\underline{\mathrm{Disk}}^{\Theta,\sqcup}_G \simeq \underline{\mathrm{colim}}_{x \in B_\Theta} \underline{\mathrm{Disk}}^{\Theta(x)-\mathrm{Fr},\sqcup}_G.$$

II.2.7.1 Germs of embeddings It is a classical fact that, up to homotopy, the space germs of open embeddings $k \cdot \mathbb{R}^n \to M$ differs from the configuration space of k points in M by $O(n)^k$. We will equivariantize this, for which we need the following definition.

Definition II.2.81. An *S-indexed multidisk* is a smooth G-manifold D admitting a sequence of maps

$$D \hookrightarrow E \twoheadrightarrow S,$$

where $E \twoheadrightarrow S$ is an equivariant vector bundle and $D \subset E$ is the open unit disk. Given $[G/H] \subset S$ an orbit, we write $D_{[G/H]}$ for the corresponding path orbit lying over $[G/H]$, and we write D_H for the corresponding H-representation over the identity coset, so that

$$D \simeq \coprod_{[G/H] \subset S} D_{[G/H]} \simeq \coprod_{[G/H] \subset S} G \times_H D_H. \qquad \triangleleft$$

These are the local type about a configuration. Indeed, writing $e_{0,\epsilon}\colon D^V \hookrightarrow D^V$ for the "shrink by a factor of ϵ" self-embedding, we will find the following.

Proposition II.2.82. *Given a G-manifold M, the point-set sequential colimit*

$$\mathrm{Germ}^G_D(M) \coloneqq \mathrm{colim}\left(\mathrm{Emb}^G(D,M) \xrightarrow{e^*_{0,\frac{1}{2}}} \mathrm{Emb}^G(D,M) \xrightarrow{e^*_{0,\frac{1}{2}}} \cdots\right) \longrightarrow \mathrm{Conf}^G_S(M)$$

admits the structure of a fiber bundle over its image with fiber $\prod_{[H/K]\subset S} O^K(D_K)$.

We prove local triviality by an equivariant lift of the usual argument.

Lemma II.2.83 (Isotopy extension)**.** *If N is compact, then every continuous $\mathrm{Diff}^G(M)$-equivariant map $X \to \mathrm{Emb}^G(N,M)$ is a fiber bundle over its image.*

Proof. By [Pal60, Thm. A], we must check that all $\iota \in \mathrm{Emb}^G(N,M)$ admit a neighborhood $\iota \in U$ and map $\chi\colon U \to \mathrm{Diff}^G(M)$ such that $\iota' = \chi(\iota')\cdot\iota$. The proof of [Lim64] straightforwardly shows this, by choosing equivariant tubular neighborhoods using [Bre72].[3] □

To see this as a form of isotopy extension, consider the case of $\iota\colon N \hookrightarrow N'$ a compact invariant submanifold, in which case we find that $\iota^*\colon \mathrm{Emb}^G(N',M) \to \mathrm{Emb}^G(N,M)$ is a fiber bundle onto its image; since $C^\infty(N,M)$ is metrizable when N is compact, its subspaces are paracompact, and hence ι^* is a Serre fibration, and one may extend isotopies using the associated lifting property.

Now, to identify germs with embeddings, we note the following.

Lemma II.2.84 (Isototopy to sub-disk)**.** *Given any $\epsilon \in (0,1]$, the precomposition embedding of topological spaces*

$$e^*_{0,\epsilon}\colon \mathrm{Emb}^G(D,M) \longrightarrow \mathrm{Emb}^G(D,M)$$

admits a deformation retract; in particular, it is an acyclic Serre cofibration.

Proof. It suffices to construct an equivariant isotopy $e_{0,\varepsilon} \sim \mathrm{id}_{D^V}$; we choose $h_{t,\varepsilon} = e_{0,t+(1-t)\cdot\varepsilon}$. □

Proof of Proposition II.2.82. Choosing $N = S$ in Lemma II.2.83 shows that $\mathfrak{c}$ is a fiber bundle over its image and Lemma II.2.83 shows that it is the $\mathrm{Germ}^G_D(M)$ is the *homotopy colimit* in $\mathcal{S}_{/\mathrm{Conf}^G_S(V)}$, and that $\mathrm{Germ}^G_D(M) \to \mathrm{Emb}^G(D,M)$ is an equivalence.

We're left with computing the point-set fiber F_c over a configuration c, showing that the derivative map

$$d\colon F_c \longrightarrow \prod_{[H/K]\subset S} O^K(D_K).$$

[3] The author will explain this in more explicit detail in forthcoming work with Branko Juran.

admits an inverse given by precomposing a fixed embedding with well-chosen elements of $\prod_{[H/K]\subset S} \mathrm{GL}^K(D_K)$. This argument is routine, so we omit it. □

Remark II.2.85. Let $\mathrm{Conf}^G_D(M) \subset \mathrm{Conf}^G_S(M)$ be the image of $\mathrm{Germ}^G_D(M)$ under $\mathfrak{c}$. We claim three things:

(i) $\mathrm{Conf}^G_D(M) \subset \mathrm{Conf}^G_S(M)$ is a disjoint union of path-components.

(ii) A configuration $c\colon S \hookrightarrow M$ lies in $\mathrm{Conf}^G_D(M)$ if and only if, for each point $p \in S$ with stabilizer H, there exists an equivalence of H-representations $T_{c(p)}M \simeq D_H$.

(iii) In the case that M admits a V-framing (such as $M = P^V$), we have

$$\mathrm{Conf}^G_D(M) = \begin{cases} \mathrm{Conf}^G_S(M) & D \simeq S \cdot D^V, \\ \varnothing & \text{otherwise.} \end{cases}$$

Claim (iii) follows straightforwardly from Claim (ii), and Claim (ii) follows by noting that the derivative of an embedding $S \cdot D(V) \hookrightarrow M$ at p is such an equivalence, and conversely by taking equivariant tubular neighborhoods using [Bre72].

For Claim (i), note that we may express configurations spaces homeomorphically as products

$$\begin{aligned} \mathrm{Conf}^G_S(M) &\simeq \prod_{(H)\subset G} \mathrm{Conf}^G_{S_{(H)}}(M) \\ &\simeq \prod_{(H)\subset G} \mathrm{Conf}^G_{S_{(H)}}(M_{(H)}), \end{aligned}$$

where $M_{(H)} \subset M$ denotes the topological subspace of points x such that $(\mathsf{stab}(x)) = (H)$, compatible with our distinguished subspaces. As such, it suffices to prove this in the case that S and M have fixed isotropy (H).

Now, in this case, note that the representation T_xM is *locally constant* in x; Claim (ii) then implies that all paths in $\mathrm{Conf}^G_S(M)$ beginning in $\mathrm{Conf}^G_D(M)$ must end in $\mathrm{Conf}^G_D(M)$, which is what we wanted to show. ◁

Before moving on, we acquire a useful technicality about embedding spaces.

Corollary II.2.86. *The following diagram consists of homotopy-cartesian squares:*

$$\begin{array}{ccccc} \mathrm{Emb}^G(D,M) & \simeq & \mathrm{Germ}^G_D(M) & \xrightarrow{\mathfrak{c}} & \mathrm{Conf}^G_D(M) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{[G/H]\subset S} \mathrm{Emb}^H(D_H,M) & \simeq & \prod_{[G/H]\subset S} \mathrm{Germ}^H_{D_H}(M) & \xrightarrow{\mathfrak{c}} & \prod_{[G/H]\subset S} M^H \end{array}$$

Proof. By [HTT, Lem. 4.4.2.1], the lefthand square is homotopy-cartesian, so it suffices to prove the same for the righthand square. Since M^H and $\mathrm{Conf}^G_D(M)$ are smooth manifolds, they are paracompact, and so each fiber bundle $\mathfrak{c}$ is a Serre fibration. Hence it suffices to prove that this is a point-set cartesian square;

equivalently, it suffices to prove that it induces a homeomorphism on point-set fibers. Indeed, each of the vertical maps are embeddings, and the bottom horizontal map takes image homeomorphically onto image. □

II.2.7.2 Germs with tangential structure Given D a Θ-framed S-indexed multidisk, we have a map

$$\mathfrak{c}_M \colon \mathrm{Emb}^{H,\Theta}(D,M) \longrightarrow \mathrm{Emb}^{H}(D,M) \longrightarrow \mathrm{Conf}^{H}_{D}(M).$$

We may bootstrap a generalization of a result of Horev to compute its homotopy fibers.

Proposition II.2.87. *The homotopy fibers of $\mathfrak{c}_M$ are computed by*

$$\mathfrak{c}_M^{-1}(\iota) \simeq \prod_{G/H \subset S} \Omega_{D_{G/H}} B^H_\Theta.$$

In particular, any Θ-framed embedding $f \colon M \hookrightarrow N$ induces an equivalence $\mathfrak{c}_M^{-1}(\iota) \xrightarrow{\sim} \mathfrak{c}_N^{-1}(f\iota)$.

Proof of Proposition II.2.87. Consider the diagram

$$\begin{array}{ccccc}
\prod_{G/H \in S} \Omega_{\Theta\iota(G/H)} B^H_\Theta & \longrightarrow & \mathrm{Emb}^{\Theta,G}_M(S,M) & \longrightarrow & \prod_{G/H \subset S} \mathrm{Emb}^{\Theta,G}_M(G/H,M) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\{\iota\} & \longrightarrow & \mathrm{Conf}^{G}_{D}(M) & \longrightarrow & \prod_{G/H \subset S} M^H
\end{array}$$

The righthand square is cartesian by Corollary II.2.86; by pullback-pasting, it suffices to prove that the outer diagram is cartesian. By commuting limits past limits, we may thus assume that $S \simeq [G/H]$; then, the claim results from an identical argument to [Hor19, Prop. 3.8.9] and [Hor19, Ex. 3.8.10], simply assuming that G is a compact Lie group instead of a finite group. □

In particular, when B_Θ is contractible, we get a strong corollary.

Corollary II.2.88. *If M is a V-framed G-manifold, then the center map*

$$\mathfrak{c} \colon \mathrm{Emb}^{V-\mathrm{Fr},H}\left(S \cdot D^V, M\right) \longrightarrow \mathrm{Conf}^{H}_{S}(M)$$

is an equivalence.

This allows us to perform the main step in proving Theorem II.2.79.

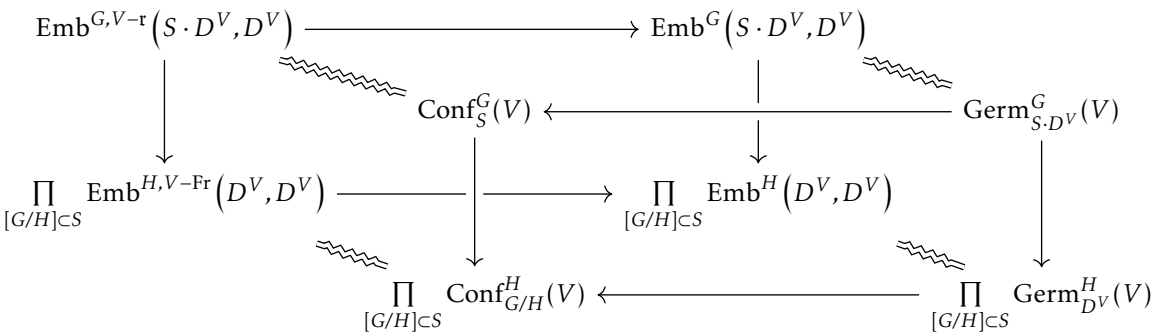
Corollary II.2.89. *The following diagram consists of homotopy-cartesian squares:*

$$\begin{array}{ccccc}
\mathrm{Emb}^{G,V-\mathrm{Fr}}\left(S\cdot D^V, D^V\right) & \longrightarrow & \prod_{[G/H]\subset S} \mathrm{Emb}^{H,V-\mathrm{Fr}}\left(S\cdot D^V, D^V\right) & \xrightarrow{t} & * \\
\downarrow{\scriptstyle \iota} & \lrcorner & \downarrow{\scriptstyle \iota} & \lrcorner & \downarrow \\
\mathrm{Emb}^{G}\left(S\cdot D^V, D^V\right) & \longrightarrow & \prod_{[G/H]\subset S} \mathrm{Emb}^{H}\left(D^V, D^V\right) & \xrightarrow{t} & \prod_{[G/H]\subset S} O^H(V)
\end{array}$$

Proof. Both arrows marked t are equivalences by Corollary II.2.88 and Proposition II.2.72, so the righthand square is homotopy-cartesian by pullback pasting; another application of pullback pasting shows that it suffices to prove that the lefthand square is homotopy-cartesian. Now, note that the sequence

$$\mathrm{Emb}^{G,V-\mathrm{r}}\left(S\cdot D^V, D^V\right) \longrightarrow \mathrm{Emb}^{G}\left(S\cdot D^V, D^V\right) \longrightarrow \mathrm{Germ}^G_{S\cdot D^V}(V) \longrightarrow \mathrm{Conf}^G_S(V)$$

is natural in S and has composite and middle arrow an equivalence; hence it has a coassembly natural transformation



whose back face is exactly the square in question. Here, the squiggly double headless arrows are equivalences by Proposition II.2.82, so the left and right faces are homotopy-cartesian by [HTT, Lem. 4.4.2.1].

Again, pullback pasting shows that it suffices to prove that the front face is homotopy-cartesian; that is, our square admits a retract (as an arrow in the arrow category), so we may check that the retraction square is homotopy-cartesian. But this is Corollary II.2.86. □

We're ready to conclude.

Proof of Theorem II.2.79. Passing to restrictions, we may assume $H = G$. ψ^{red}_x is the coassembly map for the diagram

$$\begin{array}{ccccc}
\mathbb{E}^{\otimes}_{\Theta(x)} & \longrightarrow & \mathbb{E}^{\otimes}_{\Theta} & \longrightarrow & \mathbb{E}^{\otimes}_{\mathrm{id}_n} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Comm}^{\otimes}_G & \longrightarrow & B^{\sqcup}_{\Theta} & \longrightarrow & B_G O(n)^{\sqcup}
\end{array}$$

so by pullback pasting, it suffices to prove that the outer square is cartesian, i.e. we may also assume $\Theta = \mathrm{id}_n$. Now, ψ is a map of one-color G-operads so it suffices to prove that it is an equivalence on S-ary structure

spaces for each $S \in \mathbb{F}_H$; this is the coassembly map for the outer square of the following diagram.

$$\begin{array}{ccccc} \mathrm{Emb}^{G,V-\mathrm{Fr}}\left(S\cdot P^V, P^V\right) & \longrightarrow & \prod_{[G/H]\subset S} \mathrm{Emb}^{H,V-\mathrm{Fr}}\left(S\cdot P^V, P^V\right) & \simeq & * \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Emb}^{G}\left(S\cdot P^V, P^V\right) & \longrightarrow & \prod_{[G/H]\subset S} \mathrm{Emb}^{G}\left(P^V, P^V\right) & = & \prod_{[G/H]\subset S} \mathrm{Emb}^{G}\left(P^V, P^V\right) \end{array}$$

The righthand square is (homotopy)-cartesian as its vertical arrows are equivalences and the left by Corollary II.2.89, so the claim follows by pullback pasting. □

II.2.8 The linear model for equivariant little disks When G is a *finite* group, we have a preferred topological model for G-operads; we will construct more traditional topological models for $\mathbb{E}_V^{\otimes}$ in this case, using *rectlinear little disk embeddings.*

II.2.8.1 Rectilinear little V-polydisks Our geometry admits the following as its local types.

Definition II.2.90. An *orthogonal G-polyrepresentation* $V_\bullet = (V_1, \ldots, V_n)$ is a finite list of finite-dimensional orthogonal representations of G. The *polydisk in* $V_\bullet$ is the open sub-G-manifold

$$P^{V_\bullet} := \prod_{i=1}^n D^{V_i} \quad \subset \quad \prod_{i=1}^n V_i \simeq \bigoplus_{i=1}^n V_i.$$

When it will not cause confusion, we will write $V := \bigoplus_{i=1}^n V_i$, omit the bullet, and write $P^V := P^{V_\bullet}$. ◁

Note that the inclusion $P^V \subset V$ is a G-isotopy equivalence; indeed, this follows by taking products of Lemma II.2.84. Crucially, little V-polydisks will provide a *homotopy-equivalent* theory to little V-disks. For additivity, we need the following.

Construction II.2.91. Given two polyrepresentations $V_\bullet$ and $W_\bullet$, we write $V_\bullet \oplus W_\bullet$ for the orthogonal G-polyrepresentation given by the concatenation $(V_1, \ldots, V_n, W_1, \ldots, W_m)$. Note that

$$P^{V\oplus W} = P^V \times P^W \quad \subset \quad V \times W \simeq V \oplus W.$$ ◁

We henceforth fix $(V_\bullet, W_\bullet)$ a pair of orthogonal G-polyrepresentations.

Construction II.2.92. Given $H \subset G$ a closed subgroup and $S \in \mathrm{Fin}_H$ a finite H-set, we let

$$S \cdot P^V := \coprod_{[H/K]\subset S} H \times_K P^{\mathrm{Res}^G_K V}$$

denote the *S-indexed copower* of P^V; we refer to H-manifolds of this form as *$V_\bullet$-multi-polydisks.* ◁

II.2.8.2 Rectilinear embedding spaces The embeddings for *little disks* will generally be the following.

Definition II.2.93. We refer to embeddings $D^{V_i} \hookrightarrow D^{V_i}$ of the form $v \mapsto \lambda v + v_0$ for some $\lambda \in \mathbb{R}_{>0}$ and $v_0 \in D^V$ as *equidiameter embeddings* and more generally, we refer to embeddings $P^V \hookrightarrow P^V$ given by products of equidiameter embeddings as *rectilinear embeddings.*

Given a map of fintie G-sets $f\colon T \to S$, we refer to a smooth map H-equivariant map $\iota\colon T \cdot P^V \to S \cdot P^V$ as an *f-indexed little V-polydisk multiembedding* if

(a) the induced map $\pi_0\iota\colon \pi_0\left(T \cdot P^V\right) = T \to S = \pi_0\left(S \cdot P^W\right)$ is f,

(b) ι is an embedding of H-manifolds, and

(c) for each orbit $[H/K] \subset T$, the mate $P^{\mathrm{Res}^G_K V} \to P^{\mathrm{Res}^G_K W}$ to the restricted map $\iota_{[H/K]}$ of

$$\begin{array}{ccc} H \times_K P^{\mathrm{Res}^G_K V} & \xrightarrow{\iota_{[H/K]}} & P^{\mathrm{Res}^G_H V} \\ \downarrow & & \downarrow \\ T \cdot P^V & \xrightarrow{\iota} & S \cdot P^V \end{array}$$

is a rectilinear embedding.

The topological subspaces satisfying Conditions (a) to (c), Conditions (a) and (b), and Condition (a) are written

$$\mathrm{Emb}_f^{H,\mathrm{Rect}}\left(T \cdot P^V, S \cdot P^V\right) \subset \mathrm{Emb}_f^{H}\left(T \cdot P^V, S \cdot P^V\right) \subset C_f^{\infty,H}\left(T \cdot P^V, S \cdot P^V\right) \subset C^\infty\left(T \cdot P^V, S \cdot P^V\right),$$

the latter having the (strong) C^∞-topology. In the case $S = *_H$ is the terminal H-set, we simply write

$$\mathrm{Emb}^{H,\mathrm{Rect}}\left(T \cdot P^V, P^V\right) \coloneqq \mathrm{Emb}_!^{H,\mathrm{Rect}}\left(T \cdot P^V, *_H \cdot P^V\right),$$

and we write $\mathrm{Emb}^{H}\left(T \cdot P^V, P^V\right)$ and $C^{\infty,H}\left(T \cdot P^V, P^V\right)$ similarly. ◁

Remark II.2.94. Tracking diameters and centers under the diffeomorphism $V \simeq P^V$ yields an injection

$$(\mathfrak{c}, \mathfrak{t})\colon \mathrm{Emb}_f^{H,\mathrm{Rect}}\left(T \cdot P^V, S \cdot P^V\right) \hookrightarrow \left(\mathbb{R}^{|V|} \times \mathbb{R}^n\right)^{|T|}.$$

Moreover, the codomain admits a structure as a real sub-vector space

$$(\mathbb{R}^n \times \mathbb{R})^{|T|} \subset C^\infty(T \cdot V, S \cdot V) \simeq C^\infty\left(T \cdot P^V, S \cdot P^V\right),$$

and its subspace topology endows it with the structure of a finite dimensional Hausdorff real topological vector space, under which $(\mathfrak{c}, \mathfrak{t})$ is a topological subspace embedding. But a Hausdorff topological structure exists on a finite dimensional real vector space uniquely up to linear homeomorphism; hence $(\mathfrak{c}, \mathfrak{t})$ is also a topological subspace embedding for the (homemorphic) standard topological structure on the codomain. ◁

Warning II.2.95. In the case of the one-term polyrepresentation $V = V_1$, these are called *affine* in [BH15; Bon19; GM17; Hil22; Hor19]; to avoid confusion with the more common definition of *affine transformations*, we will avoid this term. ◁

Remark II.2.96. In the case of an n-step polyrepresentation of the trivial group, $\mathbb{R}^{\oplus n} = \mathbb{R} \oplus \cdots \oplus \mathbb{R}$, this recovers May's open little disk embeddings [May72, Def. 4.1]. In the nonequivariant case of a one-term polyrepresentation $\mathbb{R}^n = \mathbb{R}^n$, these are *equidiameter embeddings* in the sense of [May72, Thm. 4.8]. ◁

Now, first off, note that these behave well with respect to restriction.

Observation II.2.97. Given $K \subset H$ a subgroup inclusion, we acquire a *restriction* factorizations

$$\begin{array}{ccc}
\mathrm{Emb}_f^{H,\mathrm{Rect}}\left(T\cdot P^V, S\cdot P^V\right) & \xrightarrow{\mathrm{Res}_K^H} & \mathrm{Emb}^{K,\mathrm{Rect}}_{\mathrm{Res}_K^H f}\left(T\cdot P^V, S\cdot P^V\right) \\
\downarrow & \searrow & \downarrow \\
C_f^{\infty,H}\left(T\cdot P^V, S\cdot P^V\right) & \xrightarrow[\mathrm{Res}_K^H]{} & C^{\infty,K}_{\mathrm{Res}_K^H f}\left(T\cdot P^V, S\cdot P^V\right)
\end{array} \longrightarrow C^\infty\left(T\cdot P^V, S\cdot P^V\right).$$

We refer to the associated operation as *restriction*. ◁

Additionally, we observe that rectilinear embeddings satisfy a Segal condition.

Observation II.2.98. By restricting and pulling back along the inclusion of the identity coset summand

$$*_K = [K/eKe^{-1}] \subset \coprod_{KhK \in [K\backslash H/K]} [K/hKh^{-1}] \simeq \mathrm{Res}_K^H [H/K].$$

we acquire a chain of homeomorphisms

$$\begin{aligned}
C_f^{\infty,H}\left(T\cdot P^V, S\cdot P^V\right) &\simeq \prod_{[H/K]\subset S} C^{\infty,H}_{\mathrm{Ind}_K^H T_K \to [H/K]}\left(\mathrm{Ind}_K^H T_K \cdot P^V, [H/K]\cdot P^V\right) \\
&\simeq \prod_{[H/K]\subset S} C^{\infty,K}\left(T_K\cdot P^V, P^V\right),
\end{aligned}$$

where we $\mathrm{Ind}_K^H T_K = T \times_S [H/K]$. Moreover, note that the composite homeomorphism preserves and reflects Conditions (b) and (c) of Definition II.2.93, restricting to another homeomorphism

$$\mathrm{Emb}_f^{H,\mathrm{Rect}}\left(T\cdot P^V, S\cdot P^V\right) \simeq \prod_{[H/K]\subset S} \mathrm{Emb}^{K,\mathrm{Rect}}\left(T_K\cdot P^V, P^V\right).$$ ◁

II.2.8.3 Composition An essential structure on rectilinear embedding spaces is *composition*.

Lemma II.2.99. *Composition of continuous maps restricts uniquely to operations*

$$\gamma\colon \mathrm{Emb}_g^{H,\mathrm{Rect}}\left(S\cdot P^V, R\cdot P^V\right) \times \mathrm{Emb}_f^{H,\mathrm{Rect}}\left(T\cdot P^V, S\cdot P^V\right) \longrightarrow \mathrm{Emb}_{g\circ f}^{H,\mathrm{Rect}}\left(T\cdot P^V, R\cdot P^V\right),$$

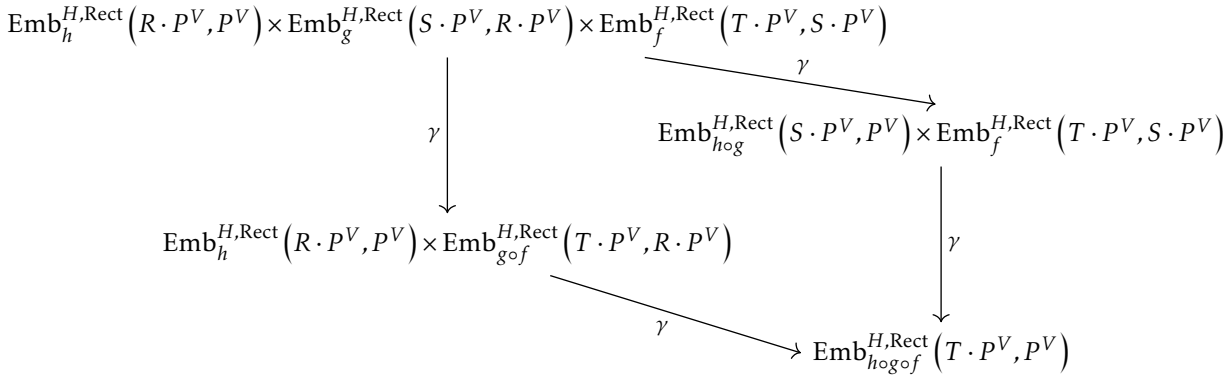
compatible with inclusions of structures. Moreover, γ is compatible with restriction, i.e. the following diagram commutes.

$$\begin{array}{ccc} \mathrm{Emb}_g^{H,\mathrm{Rect}}\left(S\cdot P^V, R\cdot P^V\right)\times \mathrm{Emb}_f^{H,\mathrm{Rect}}\left(T\cdot P^V, S\cdot P^V\right) & \xrightarrow{\gamma} & \mathrm{Emb}_{g\circ f}^{H,\mathrm{Rect}}\left(T\cdot P^V, R\cdot P^V\right). \\ \downarrow \mathrm{Res}_K^H & & \downarrow \mathrm{Res}_K^H \\ \mathrm{Emb}_g^{K,\mathrm{Rect}}\left(S\cdot P^V, R\cdot P^V\right)\times \mathrm{Emb}_f^{K,\mathrm{Rect}}\left(T\cdot P^V, S\cdot P^V\right) & \xrightarrow{\gamma} & \mathrm{Emb}_{g\circ f}^{K,\mathrm{Rect}}\left(T\cdot P^V, R\cdot P^V\right). \end{array}$$

Proof. Since π_0 is functorial, we're tasked with verifying that Conditions (b) and (c) are closed under composition; unwinding definitions, this follows by taking products of the fact that equidiameter embeddings are closed under composition. Restriction compatibility follows from restriction compatibility of composition of smooth equivariant maps. □

We will use this as a G-operadic composition map, so we must observe associativity.

Observation II.2.100. Given V an orthogonal G-representation, the following diagram commutes.



Indeed, this admits a natural monomorphism of topological spaces to the same square for C^∞ mapping spaces, which commutes by the fact that smooth manifolds form a topological category, so the property follows by unique factorization through monomorphisms. ◁

II.2.8.4 External products We now define a staple of the *additivity phenomenon.*

Construction II.2.101. Given X, X', Y, Y' a quadruple of connected H-manifolds and composable morphisms $T \xrightarrow{f} S \xrightarrow{g} R$ we define the *external product map*

$$\varphi\colon C_g^{\infty,H}(S\cdot X, R\cdot Y)\times C_f^{\infty,H}(T\cdot X', S\cdot Y') \to C_{g\circ f}^{\infty,H}(T\cdot(X\times X'), R\cdot(Y\times Y'))$$

elementwise by the composition

$$\begin{array}{ccccc} T\cdot(X\times X') & & \xrightarrow{\qquad\qquad\varphi(\tilde{g};\tilde{f})\qquad\qquad} & & R\cdot(Y\times Y') \\ \wr\downarrow & & & & \wr\downarrow \\ X\times(T\cdot X') & \xrightarrow{(\mathrm{id},\tilde{g})} & X\times(S\cdot Y') \simeq (S\cdot X)\times Y' & \xrightarrow{(\tilde{f},\mathrm{id})} & (R\cdot Y)\times Y' \end{array} \tag{27}$$

Now, note that this restricts uniquely to a map

$$\begin{array}{ccc}
\mathrm{Emb}_g^{H,\mathrm{Rect}}\big(S\cdot P^V, R\cdot P^V\big)\times \mathrm{Emb}_f^{H,\mathrm{Rect}}\big(T\cdot P^W, S\cdot P^W\big) & \xrightarrow{\varphi} & \mathrm{Emb}_{g\circ f}^{H,\mathrm{Rect}}\big(T\cdot P^{V\oplus W}, R\cdot P^{V\oplus W}\big) \\
\downarrow & & \downarrow \\
C_g^{\infty,H}\big(S\cdot P^V, R\cdot P^V\big)\times C_f^{\infty,H}\big(T\cdot P^W, S\cdot P^W\big) & \xrightarrow{\varphi} & C_{g\circ f}^{\infty,H}\big(T\cdot P^{V\oplus W}, R\cdot P^{V\oplus W}\big)
\end{array}$$

We refer to φ as the *external product map*. ◂

We see that φ behaves well with respect to restriction and composition.

Observation II.2.102. Eq. (27) is manifestly restriction-stable, i.e. the following diagram commutes

$$\begin{array}{ccc}
\mathrm{Emb}_g^{H,\mathrm{Rect}}\big(S\cdot P^V, R\cdot P^V\big)\times \mathrm{Emb}_f^{H,\mathrm{Rect}}\big(T\cdot P^W, S\cdot P^W\big) & \xrightarrow{\varphi} & \mathrm{Emb}_{g\circ f}^{H,\mathrm{Aff}}\big(T\cdot P^{V\oplus W}, R\cdot P^{V\oplus W}\big). \\
\downarrow{\scriptstyle \mathrm{Res}_K^H} & & \downarrow{\scriptstyle \mathrm{Res}_K^H} \\
\mathrm{Emb}_{\mathrm{Res}_K^H g}^{K,\mathrm{Rect}}\big(S\cdot P^V, R\cdot P^V\big)\times \mathrm{Emb}_{\mathrm{Res}_K^H f}^{K,\mathrm{Rect}}\big(T\cdot P^W, S\cdot P^W\big) & \xrightarrow[\varphi]{} & \mathrm{Emb}_{\mathrm{Res}_K^H g\circ f}^{K,\mathrm{Rect}}\big(T\cdot P^{V\oplus W}, R\cdot P^{V\oplus W}\big).
\end{array}$$

Moreover, it is compatible with composition, i.e. the top horizontal composite

$$\begin{array}{ccccccc}
P\cdot P^V & \xrightarrow{\varphi(\tilde h;\mathrm{id})} & T\cdot P^{V\oplus W} & \xrightarrow{\qquad\qquad\varphi(\tilde g;\tilde f)\qquad\qquad} & R\cdot P^{V\oplus W} & \xrightarrow{\varphi(\mathrm{id};\tilde j)} & Q\cdot P^{V\oplus W} \\
\downarrow\!\downarrow & & \downarrow\!\downarrow & & \downarrow\!\downarrow & & \downarrow\!\downarrow \\
P^V\times P\cdot P^W & \xrightarrow{(\mathrm{id},\tilde h)} & P^V\times T\cdot P^W & \xrightarrow{(\mathrm{id},\tilde g)} & P^V\times S\cdot P^W & & \\
 & & & & \downarrow\!\downarrow & & \\
 & & & & S\cdot P^V\times P^W & \xrightarrow{(\tilde f,\mathrm{id})} & R\cdot P^V\times P^W \xrightarrow{(\tilde j,\mathrm{id})} Q\cdot P^V\times P^W
\end{array}$$

clearly agrees with $\varphi\big(\tilde g\circ\tilde h;\tilde j\circ\tilde f\big)$. Here, for visual clarity, we've denoted equivalences as squiggly headless double arrows. Moreover, Eq. (27) is associative, in the sense that we may define an element $\varphi(f;g;h)$ making following diagram commute

$$\begin{array}{ccccccc}
T\cdot P^{U\oplus V\oplus W} & & \xrightarrow{\qquad\qquad\varphi(f;g;h)\qquad\qquad} & & & & P\cdot P^{U\oplus V\oplus W} \\
\downarrow\!\downarrow & & & & & & \downarrow\!\downarrow \\
P^U\times T\cdot P^{V\oplus W} & & \xrightarrow{\qquad(\mathrm{id},\varphi(g;h))\qquad} & & P^U\times R\cdot P^{V\oplus W} & & \\
\downarrow\!\downarrow & & & & \downarrow\!\downarrow & & \\
 & & P^U\times S\cdot P^V\times P^W & \xrightarrow[(\mathrm{id},g,\mathrm{id}))]{} & P^U\times R\cdot P^V\times P^W & & \\
 & & \downarrow\!\downarrow & & \downarrow\!\downarrow & & \\
P^U\times P^V\times T\cdot P^W & \xrightarrow{(\mathrm{id},\mathrm{id},h)} & P^U\times P^V\times S\cdot P^W & & R\cdot P^U\times P^V\times P^W & \xrightarrow[(f,\mathrm{id},\mathrm{id})]{} & P\cdot P^U\times P^V\cdot P^W \\
 & & \downarrow\!\downarrow & & & & \downarrow\!\downarrow \\
 & & S\cdot P^{V\oplus W}\times P^W & & \xrightarrow[(\varphi(f;g),\mathrm{id})]{\qquad\qquad} & & P\cdot P^{V\oplus U}\times P^W
\end{array}$$

We moreover see that "products with the identity self-embedding" yields a map

$$\iota\colon \mathrm{Emb}_f^{H,\mathrm{Rect}}\big(S\cdot P^V, R\cdot P^V\big)\longrightarrow \mathrm{Emb}_f^{H,\mathrm{Rect}}\big(S\cdot P^{U\oplus V}, R\cdot P^{U\oplus V}\big),$$

which makes the following diagram commute

$$\begin{array}{ccc} \mathrm{Emb}_g^{H,\mathrm{Rect}}\big(S\cdot P^V, R\cdot P^V\big)\times \mathrm{Emb}_f^{H,\mathrm{Rect}}\big(T\cdot P^W, S\cdot P^W\big) & \xrightarrow{\varphi} & \mathrm{Emb}_{g\circ f}^{H,\mathrm{Rect}}\big(T\cdot P^{V\oplus W}, R\cdot P^{V\oplus W}\big) \\ \downarrow{\scriptstyle (\iota,\mathrm{id})} & & \downarrow{\scriptstyle \iota} \\ \mathrm{Emb}_g^{H,\mathrm{Rect}}\big(S\cdot P^{U\oplus V}, R\cdot P^{U\oplus V}\big)\times \mathrm{Emb}_f^{H,\mathrm{Rect}}\big(T\cdot P^W, S\cdot P^W\big) & \xrightarrow{\varphi} & \mathrm{Emb}_{g\circ f}^{H,\mathrm{Rect}}\big(T\cdot P^{U\oplus V\oplus W}, R\cdot P^{U\oplus V\oplus W}\big) \end{array}$$

Moreover, ι intertwines with restriction and composition in the obvious way. ◁

II.2.8.5 The definition and the comparison We may now define our linear model.

Construction II.2.103. We define the topological G-symmetric sequence

$$\mathbb{E}^t_{V_\bullet}(S) \coloneqq \mathrm{Emb}^{H,\mathrm{Rect}}\big(S\cdot P^{V_\bullet}, P^{V_\bullet}\big).$$

with $\mathrm{Aut}_H S$ acting by precomposition and restriction as in Observation II.2.97; note that restriction is Borel $\mathrm{Aut}_H S$-equivariant by Lemma II.2.99. ◁

Note that the Segal condition of Observation II.2.98 characterizes $\mathbb{E}^t_V$-multioperations:

$$\mathrm{Mul}^f_{\mathbb{E}^t_{V_\bullet}}(T;S) \simeq \mathrm{Emb}_f^{H,\mathrm{Rect}}\big(T\cdot P^V, S\cdot P^V\big).$$

In particular, the composition map γ yields a composition operation

$$\gamma\colon \mathrm{Mul}^g_{\mathbb{E}^t_{V_\bullet}}(S;R)\times \mathrm{Mul}^f_{\mathbb{E}^t_{V_\bullet}}(T;S) \longrightarrow \mathrm{Mul}^{g\circ f}_{\mathbb{E}^t_{V_\bullet}}(T;R).$$

Proposition II.2.104. *$(\mathbb{E}^t_{V_\bullet}, (\mathrm{id}_{P^V}), \gamma)$ is a genuine topological G-operad.*

Proof. We're tasked with verifying Conditions (a) to (e). To start, note that Conditions (b) and (d) (i.e. unitality and restriction-stability of units) are obvious. Moreover, Condition (e) is simply Observation II.2.100 and Condition (c) is verified in the second statement of Lemma II.2.99. Lastly, equivariance of γ follows from the same property for composition of equivariant maps. □

Using this, we define $\mathbb{E}^\otimes_{V_\bullet} \coloneqq N^\otimes \mathbb{E}^t_{V_\bullet}$. Now, let $\mathbf{Emb}^n_G \subset \mathbf{Mfld}^n_G$ be the topological subcategory spanned by disks and with only after restriction. We have the following.

Proposition II.2.105. *There exists an embedding of topological genuine G-operads $\mathbb{E}^t_{V_\bullet} \hookrightarrow \mathbf{Emb}^\sqcup_n$ whose homotopy-coherent nerve factors through an equivalence*

$$\begin{array}{ccc} \mathbb{E}^\otimes_{V_\bullet} & \xrightarrow{\ \sim\ } & \mathbb{E}^\otimes_V \\ & \searrow \quad \swarrow & \\ & \underline{\mathrm{Emb}}^\sqcup_n & \end{array}$$

PROOF. Unwinding all of the definitions, we are tasked with constructing a weak equivalence of topological spaces d, equivariant and compatibly with restriction and composition, making the diagram commute

$$\begin{array}{ccc} \mathrm{Emb}_f^{H,\mathrm{Rect}}\left(T\cdot P^V, S\cdot P^V\right) & \xrightarrow{\quad d \quad} & \mathrm{Emb}_f^{H,V-\mathfrak{r}}\left(T\cdot P^V, S\cdot P^V\right) \\ & \searrow \quad \mathrm{Emb}^H\left(T\cdot P^V, S\cdot P^V\right) \quad \swarrow & \\ & \downarrow & \\ & \prod_{[H/K]\subset S}\prod_{[K/J]\subset T_K}\mathrm{Emb}^K\left([K/J]\cdot P^V, P^V\right) & \end{array}$$

To construct the map is easy: just take derivatives. To verify that it is an equivalence, we may factor though $\mathfrak{r}$ to construct a triangle

$$\begin{array}{ccc} \mathrm{Emb}_f^{H,\mathrm{Rect}}\left(T\cdot P^V, S\cdot P^V\right) & \xrightarrow{\quad d \quad} & \mathrm{Emb}_f^{H,V-\mathfrak{r}}\left(T\cdot P^V, S\cdot P^V\right) \\ & \searrow \quad \swarrow & \\ & \prod_{[H/K]\subset S}\mathrm{Conf}_{T_K}^H\left(P^V\right) & \end{array}$$

The righthand diagonal is an equivalence by Corollary II.2.88, so by two-out-of-three, it suffices to show that the lefthand diagonal is a weak equivalence. Indeed, the usual argument (e.g. as in [May72, Thm. 4.8]) shows this. We will generalize this later on, so we don't elaborate more for now. □

II.2.8.6 Juran's recognition principle From this we may state a recognition principle in the finite group case, for which we need the following.

Observation II.2.106. A linear isometric embedding $V \hookrightarrow W$ induces a map $\mathbb{E}_V^\otimes \to \mathbb{E}_W^\otimes$ by taking products with a complementary subrepresentation; this yields an H-operad map $\iota_H\colon \mathbb{E}^\otimes_{\dim V^H} \to \mathrm{Res}_H^G \mathbb{E}_V^\otimes \simeq \mathbb{E}^\otimes_{\mathrm{Res}_H^G V}$. Since taking fixed points is symmetric monoidal under the cartesian structure, we acquire a lift

$$\begin{array}{ccccccc} \mathrm{Alg}_{\mathbb{E}_V}\left(\underline{\mathcal{S}}_G\right) & \xrightarrow{\mathrm{Res}_H^G} & \mathrm{Alg}_{\mathbb{E}_{\mathrm{Res}_H^G V}}\left(\underline{\mathcal{S}}_H\right) & \xrightarrow{\iota_H^*} & \mathrm{Alg}_{\mathbb{E}_{\dim V^H}}\left(\mathcal{S}_H\right) & \xrightarrow{(-)^H} & \mathrm{Alg}_{\mathbb{E}_{\dim V^H}}\left(\mathcal{S}\right) \\ \downarrow U & & & & & & \downarrow U \\ \mathcal{S}_G & & & \xrightarrow{\qquad (-)^H \qquad} & & & \mathcal{S} \end{array}$$

We refer to the top composition simply as $(-)^H$. ◁

We say that an $\mathbb{E}_V$-G-space X is *grouplike* if, for all $H \subset G$ such that $\dim V^H \geq 1$, the $\mathbb{E}_{\dim V^H}$-space X^H is grouplike. Juran proved the following.

Proposition II.2.107 ([Jur25a, Thm. A])**.** *Fix G a finite group. Then, there exists a factorization*

$$\begin{array}{ccc} & & \underline{\mathrm{Alg}}_{\mathbb{E}_V}\left(\underline{\mathcal{S}}_G\right) \\ & \overset{\varphi}{\nearrow} & \downarrow U \\ \underline{\mathcal{S}}_{G,*} & \xrightarrow[\Omega^V]{} \underline{\mathcal{S}}_{G,*} \simeq & \underline{\mathrm{Alg}}_{\mathbb{E}_0}\left(\underline{\mathcal{S}}_G\right) \end{array}$$

Such that the restriction of φ to $\underline{\mathcal{S}}_{G,*,\geq V}$ is fully faithful, with essential image the grouplike $\mathbb{E}_V$-G-spaces.

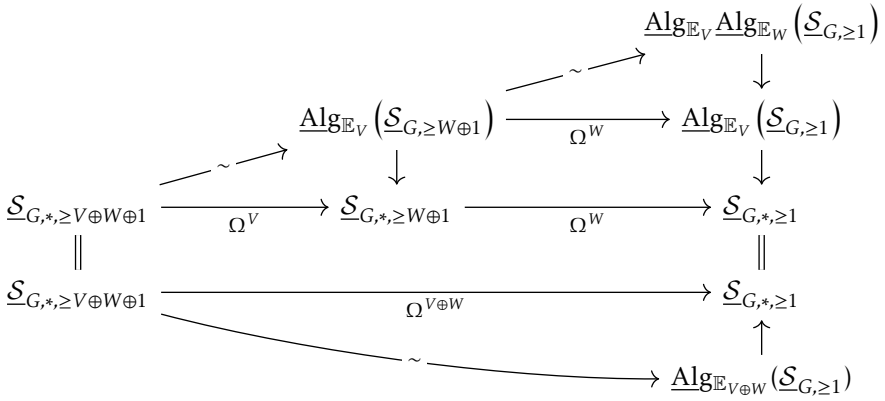
This easily entails quite a powerful corollary.

Corollary II.2.108. *If G is a finite group, then there exists an equivalence of G-∞-categories over $\underline{\mathcal{S}}_{G,\geq 1}$*

$$\underline{\mathrm{Alg}}_{\mathbb{E}_{V\oplus W}}\left(\underline{\mathcal{S}}_{G,\geq 1}\right). \xrightarrow{\sim} \underline{\mathrm{Alg}}_{\mathbb{E}_V}\underline{\mathrm{Alg}}_{\mathbb{E}_W}\left(\underline{\mathcal{S}}_{G,\geq 1}\right)$$

Proof. First note that Proposition II.2.107 implies this in the case $W = 0$, i.e. the grouplike condition is not necessary when restricted to G-connected G-spaces; this is because each fixed point is connected, and hence all $\mathbb{E}_n$-structures on a fixed point space are automatically grouplike.

To conclude, we claim that the following diagram commutes, and $\sim$-marked arrows are equivalences



Indeed each the bottom and lefthand triangle diagram are simply Proposition II.2.107 and the righthand one follows by noting that each of the arrows in Proposition II.2.107 attain an essentially unique G-symmetric monoidal structure under the *cartesian* structure by Theorem II.2.1, which agrees with the relevant pointwise structure on algebras by Proposition II.2.10. The lower rectangular diagram is the obvious equivalence $\Omega^W \circ \Omega^V \simeq \Omega^{W\oplus V}$ coming from the equivalence $S^V \wedge S^W \simeq S^{V\oplus W}$. The upper right square diagram is G-continuity of U from Lemma II.2.11, under the observation that Ω^W is a constant S^W-indexed limit. The result follows by composing the inverse of the bottom equivalence with the top equivalences. □

II.2.9 Equivariant prefactorization algebras We now give a further presentation of $\mathbb{E}_V$-algebras as *locally constant prefactorization algebras*.

II.2.9.1 Configuration spaces and Weiss covers For now, fix X a topological space with G-action. We will study configurations in X in the following sense.

Definition II.2.109. Given S a finite H-set, we write $\mathrm{Conf}_S^H(X) \subset \mathrm{Map}(S,X)$ for the topological subspace of H-equivariant embeddings of S into X under the compact-open topology.[4] We let $C_n(X)$ be its Σ_n-orbits. ◁

These satisfy a type of *codescent*, for which we need the following notions.

Definition II.2.110. Let $\mathrm{Open}(X) \subset \mathscr{P}(X)$ denote the poset of open subspaces of X under inclusion. The poset of *H-invariant open subspaces of X* is the fixed points

$$\mathrm{Open}^H(X) \coloneqq \mathrm{Open}(X)^H,$$

i.e. the collection of open subspaces $U \subset X$ such that $hu \in U$ for all $u \in U$ and $h \in H$. These naturally assemble into a G-poset $\underline{\mathrm{Open}}(X)$ such that $\underline{\mathrm{Open}}(X)^H \simeq \mathrm{Open}^H(X)$. An *open G-precover of X* is a G-subposet $\mathcal{U} \subset \underline{\mathrm{Open}}(X)$.

We additionally make the following definitions.

- A *G-invariant open cover of X* is a subposet $\mathcal{U} \subset \mathrm{Open}^G(X) \subset \mathrm{Open}(X)$ which is an open cover. An *open G-cover of X* is a G-subposet $\mathcal{U} \subset \underline{\mathrm{Open}}(X)$ such that $\mathcal{U}^H \subset \mathrm{Open}^H(X)$ is an H-invariant open cover for all $H \subset G$.
- A subposet $\mathcal{U} \subset \mathrm{Open}(X)$ is a *Weiss cover* if, for all finite configurations $S \subset X$, there exists a single open $U \in \mathcal{U}$ such that $S \subset U$; G-invariant Weiss covers and G-Weiss covers are defined similarly.
- A subposet $\mathcal{U} \subset \mathrm{Open}(X)$ is a *complete Weiss cover* if, for all finite subsets $\{U_i \mid i \in I\} \subset \mathcal{U}$,

  $$\mathcal{U}_{\bigcap_{i\in I} U_i} \coloneqq \mathrm{Open}\left(\bigcap_{i\in I} U_i\right) \cap \mathcal{U} \subset \mathrm{Open}\left(\bigcap_{i\in I} U_i\right)$$

  is a Weiss cover. G-invariant complete Weiss covers and complete G-Weiss covers are defined similarly.
- A subposet $\mathcal{U} \subset \mathrm{Open}(X)$ is a *complete separating cover* if, for all finite subsets $\{U_i \mid i \in I\} \subset \mathcal{U}$, and all configurations $c\colon [n] \hookrightarrow \bigcap_{i\in I} U_i$ the poset of opens $\mathcal{U}_{\bigcap_{i\in I} U_i}$ contains a neighborhood $\mathrm{im}(c) \subset U$ such that the induced map $[n] \to \pi_0 U$ is a bijection. Complete separating G-invariant covers and complete separating G-covers are defined similarly. ◁

Importantly, these yield a sort of hyperdescent for configuration spaces.

Proposition II.2.111. *Given a complete G-Weiss cover $\mathcal{U} \subset \underline{\mathrm{Open}}(X)$ and finite H-set S, the assembly map*

$$\mathrm{colim}_{U\in\mathcal{U}} \mathrm{Conf}_S^{\mathrm{stab}(U)}(U) \longrightarrow \mathrm{Conf}_S^H(X)$$

is an equivalence.

[4] Given a smooth structure on X, it follows by triviality of tangent spaces on a 0-manifold that this agrees with the strong C^∞-topology. We will implicitly use this later.

PROOF. It follows by unwinding definitions that $\left(\mathrm{Conf}_S^G(U)\right) \subset \mathrm{Open}^G\left(\mathrm{Conf}_S^G(M)\right)$ is a complete cover in the sense of [DI04], so [DI04] yields the claim. □

Construction II.2.112. Given a configuration $c\colon S \to P^V$ and a separating neighborhood $c \subset U$, we let

$$\mathrm{Conf}_{S,U}^H(V) \subset \mathrm{Conf}_S^H(V)$$

be the (contractible) subspace of configurations $c'\colon S \to U \to P^V$ subordinate to U such that, for all $[G/H] \subset S$, the points $c([G/H])$ and $c'([G/H])$ lie in the same component of U.

This is compatible with inclusions, and thus defines a strict diagram

$$\mathcal{U}_S^H(V)^\triangleright \to \mathrm{Top}$$

with cone point $\mathrm{Conf}_S^H(V)$ for any V-disk G-precover $\mathcal{U}$. In particular, taking coherent nerves, the universal property for colimits then yields a map

$$\alpha_{\mathcal{U},\mathrm{Conf}}\colon B\mathcal{U}_S^H \simeq \mathrm{colim}_{U\in\mathcal{U}} \mathrm{Conf}_{S,U}^H(V) \longrightarrow \mathrm{Conf}_S^H(V). \qquad \triangleleft$$

We also have the following.

Lemma II.2.113. *If $\mathcal{U}$ is a complete separating G-cover, then $\alpha_{\mathcal{U},\mathrm{Conf}}$ is an equivalence.*

PROOF. Note that $\left\{\mathrm{Conf}_{S,U}^H(V)\right\} \subset \mathrm{Open}(\mathrm{Conf}_S^H(V))$ is a complete cover and apply [DI04]. □

II.2.9.2 V-polydisk G-precovers and (de)composition Our G-precovers will generally come as follows.

Construction II.2.114. Let $\underline{\mathrm{RPDisk}}_S(V) \subset \underline{\mathrm{Open}}\left(P^V\right)$ denote the images of little polydisk embeddings $S \cdot P^V \hookrightarrow P^V$, and define the G-precover

$$\underline{\mathrm{RPDisk}}(V) \coloneqq \bigcup_{S\in\underline{\mathbb{F}}_G} \underline{\mathrm{RPDisk}}_S(V).$$

We refer to sub-G-precovers of $\underline{\mathrm{RPDisk}}(V)$ as *V-polydisk G-precovers.* $\triangleleft$

The following proposition is our entire reason for considering polydisks.

Proposition II.2.115 (Factorization decomposability)**.** *φ underlies an embedding of posets*

$$\varphi_{\underline{\mathrm{RPDisk}}}\colon \underline{\mathrm{RPDisk}}(V) \times_{\pi_0,\mathrm{Ob}\,\mathbb{F}_G} \coprod_{T\in\mathrm{Ob}\,\mathbb{F}_G} \underline{\mathrm{RPDisk}}(W)^T \longrightarrow \underline{\mathrm{RPDisk}}(V\oplus W);$$

that is, factorizations of decomposable rectilinear little $V\oplus W$-polydisk embeddings factor if and only if their decomposed V- and W-polydisk embeddings factor.

PROOF. A rectilinear embedding $\iota\colon S \cdot P^V \to P^V$, by definition, consists of an equivariant configuration $\mathfrak{c}_\iota\colon S \hookrightarrow P^V$ together with a function $\mathsf{t}_\iota\colon S \times n_V \to \mathbb{R}_{>0}$ subject to an injectivity condition. Given a point $p \in S$, let ι_p denote the restricted $\mathsf{stab}(p)$-equivariant little disk embedding $P^V \to P^V$.

We may explicitly describe factorization of another embedding $\iota'\colon T \cdot P^V \hookrightarrow P^V$ through $\iota\colon S \cdot P^V \to P^V$ via a map $f\colon T \to S$ on a per-representation basis: ι' factors through ι via f if and only if, for all $i \in [n_V]$ and $p \in T$,

(a) $\pi_{V_i}\mathfrak{c}_\iota(f(p)) \in \mathsf{im}\left(\pi_{V_i}\iota_p\right)$, and
(b) $d_{V_i}\left(\pi_{V_i}\mathfrak{c}_\iota(f(p)),\ \pi_{V_i}\mathfrak{c}_{\iota'}(p)\right) + \mathsf{t}_{\iota'}(p,i) \le \mathsf{t}_\iota(f(p),i)$.

That is, the center of ι lies within the f-corresponding components of the image of ι' *after projection to each* V_i, and the diameters are small enough to land within each projected image of ι'.

Now, suppose we have a pair of decomposed arity data $g\colon T \to S, g'\colon T' \to S'$ in $\mathbb{F}_H$ together with decomposed little disk embeddings $\iota_V, \iota_W, \iota'_V, \iota'_W$. We are tasked with proving that $\varphi\left(\iota'_V, \iota'_W\right)$ factors through $\varphi(\iota_V, \iota_W)$ via some f if and only if there exist some (unique) extension diagram

$$\begin{array}{ccc} T' & \xrightarrow{g'} & S' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ T & \xrightarrow{g} & S \end{array}$$

such that ι'_W factors through ι_W via h, and for each $[H/K] \subset S'$, $\iota'_{V,f(K)}$ through $\iota_{V,K}$ via f_K.

Suppose such a factorization of $\varphi\left(\iota'_V; \iota'_W\right)$ through $\varphi(\iota_V; \iota_W)$ via f exists. Define the map h taking $p \in S'$ to the unique $h(p) \in S$ such that $\pi_W(\mathfrak{c}_{\iota_W}(p)) \in \mathsf{im}\pi_W(\iota_{w,h(p)})$. Note that Condition (a) applied to the summand $V_i \subset V \oplus W$ is identical to the same condition for $V_i \subset V$ applied to factorization of ι'_V through ι_V via f, and similarly Condition (b) for $V_i \subset V \oplus W$ agrees with the same condition for $V_i \subset V$. Similar logic identifies these conditions for $W_i \subset V \oplus W$ with those for $W_i \subset W$, so $\varphi_{\underline{\mathrm{RPDisk}}}$ is full. Faithfulness follows by reversing this logic. □

Remark II.2.116. We chose polydisk models precisely so that Conditions (a) and (b) in the proof of Proposition II.2.115 were measured independently for each of the summands V and W. This does not hold for disk models. ◁

We refer to the G-poset essential image of $\mathcal{U}_V \times_{\pi_0 \mathrm{Ob}\mathbb{F}_G} \coprod_{T \in \mathrm{Ob}\mathbb{F}_G} \mathcal{U}_W^T$ under $\varphi_{\underline{\mathrm{RPDisk}}}$ as $\mathcal{U}_V \mid \mathcal{U}_W$; in particular, we will write

$$\underline{\mathrm{RPDisk}}(V \mid W) := \underline{\mathrm{RPDisk}}(V) \mid \underline{\mathrm{RPDisk}}(W).$$

For the purpose of *deformation*, we also note a weak convexity property of $\underline{\mathrm{RPDisk}}(V \mid W)$.

Proposition II.2.117. *Suppose $\iota, \iota' \colon S \cdot P^{V\oplus W} \hookrightarrow P^{V\oplus W}$ are decomposable rectilinear little disk embeddings with the same center. Then, for all $t \in [0,1]$, the linear combination*

$$h_t(\iota,\iota')(v) = t\cdot \iota(v) + (1-t)\iota'(v)$$

is a rectilinear decomposable little disk embedding.

This follows immediately from the following lemma, which itself is obvious.

Lemma II.2.118. *An S-indexed general little $V\oplus W$-polydisk embedding ι is decomposable precisely when the following conditions hold for all distinct points $p \neq p' \in S$:*

(a) If $\pi_V \mathfrak{c}_\iota(p) = \pi_V \mathfrak{c}_\iota(p')$, then we have $\mathfrak{t}\pi_V \iota_p = \mathfrak{t}\pi_V \iota_{p'}$, and

(b) if $\pi_V \mathfrak{c}_\iota(p) \neq \pi_V \mathfrak{c}_\iota(p')$, then for all $i \in [n_V]$, we have

$$d_{V_i}\left(\pi_{V_i}\mathfrak{c}_\iota(p),\ \pi_{V_i}\mathfrak{c}_\iota(p')\right) \geq \mathfrak{t}_\iota(i,p)\pi_{V_i} + \mathfrak{t}_{\iota'}(i,p')$$

Intuitively, this says that *V-cocentric disks have identical V-data* and *V-noncocentric disks have disjoint projections to V*, each part of which has the desired convexity property.

II.2.9.3 Factorization and topology We now somewhat systematize the strategy of [May72, Thm. 4.8].

Definition II.2.119. Given a configuration $i \colon [n] \hookrightarrow V^e$ and a sufficiently small number $\varepsilon \in \mathbb{R}_{>0}$, we define the *equidiameter embedding*

$$e_{i,\epsilon}(v) \coloneqq \iota + \epsilon \cdot v.$$

This assembles to an $\mathsf{Aut}^H(S)$-equivariant continuous map

$$e \colon \mathrm{Conf}^H_S(V) \times \mathbb{R}_{>0} \longrightarrow C^{\infty,H}\left(S\cdot P^V, V\right)$$

landing on maps which are componentwise-rectilinear, but which may not be embeddings or land within P^V. Given a V-disk G-precover $\mathcal{U}$, an *equidiameter datum for $\mathcal{U}$* is, for each n, a one parameter family of continuous maps,

$$\varepsilon_{(-)}(-) \colon (0,1) \to \mathrm{Map}\left(C_n(X)_G, \mathbb{R}_{>0}\right),$$

(with righthand side having the compact-open topology) such that

(1) (infinitesimal values) $\lim_{s\to 0}\varepsilon_s = 0$, and

(2) (admissibility) for all $s \in (0,1)$, all $t \in [0,1]$, and all $\mathcal{U}$-admissible ι, the linearly interpolated map

$$h_{s,t}\iota(v) \coloneqq t\cdot e_{c(\iota),\varepsilon_s(c(\iota))} + (1-t)\iota(v)$$

is a $\mathcal{U}$-admissible embedding.[5]

If $\mathcal{U}$ admits such ε for all n, we say that it is an *equidiameterizable V-disk G-cover*. ◁

Observation II.2.120. Equidiameter G-covers are automatically complete separating G-covers. ◁

Now, we may set up a comparisons between embedding spaces, factorization, and configuration spaces.

Construction II.2.121. Given an S-indexed little V-polydisk embedding $\iota\colon S\cdot P^V \hookrightarrow P^V$ with image U, there is a natural inclusion of *ι-subordinate embeddings* and *ι-subordinate configurations* making the following commute

$$\begin{array}{ccc} \mathrm{Emb}^{H,\mathrm{Rect}}_{\mathrm{id}_S}\left(S\cdot P^V, U\right) & \longrightarrow & \mathrm{Emb}^{H,\mathrm{Rect}}\left(S\cdot P^V, P^V\right) \\ \downarrow & & \downarrow \\ \mathrm{Conf}^H_{S,U}(V) & \longrightarrow & \mathrm{Conf}^H_S(V) \end{array}$$

$\mathrm{Conf}^H_{S,U}(V)$ is contractible and the above is natural under inclusion, so this yields a diagram

$$\begin{array}{ccccc} & & \mathrm{colim}_{U\in \mathrm{RPDisk}_S(V)} \mathrm{Emb}^{H,\mathrm{Rect}}_{\mathrm{id}_S}\left(S\cdot P^V, U\right) & \xrightarrow{\alpha_V} & \mathrm{Emb}^{H,\mathrm{Rect}}\left(S\cdot P^V, P^V\right) \\ & & \mathfrak{c}\downarrow & & \mathfrak{c}\downarrow \\ B\mathrm{RPDisk}_S(V) & \xrightarrow{\sim} & \mathrm{colim}_{U\in \mathrm{RPDisk}_S(V)} \mathrm{Conf}^H_{S,U}(V) & \xrightarrow{\alpha_{\mathrm{Conf}}} & \mathrm{Conf}^H_S(V) \end{array} \tag{28}$$

and similar for $\mathcal{U}$. Here, α_{Conf} is the map of Lemma II.2.113 in the case $\mathcal{U} = \underline{\mathrm{RPDisk}}(V)$. ◁

We find that, in good cases, these comparison maps are actually equivalences.

Proposition II.2.122 (c.f. [May72, Thm. 4.8])**.** *Every map in Eq. (28) is an equivalence. Moreover, if $\mathcal{U}$ is an equidiameterizable V-disk G-cover, then taking centers yields a commutative diagram of weak equivalences*

$$\begin{array}{ccccc} B\mathcal{U}^H_S & \xrightarrow{\alpha_{\mathcal{U}}} & \mathrm{Emb}^{H,\mathrm{Rect},\mathcal{U}}\left(S\cdot D(V), D(V)\right) & \xrightarrow{\mathfrak{c}} & \mathrm{Conf}^H_S(V) \\ \downarrow & & \wr\downarrow & & \| \\ B\mathrm{RPDisk}^H_S(V) & \xrightarrow{\alpha_V} & \mathrm{Emb}^{H,\mathrm{Rect}}\left(S\cdot D(V), D(V)\right) & \xrightarrow{\mathfrak{c}} & \mathrm{Conf}^H_S(V), \end{array}$$

Now, for the first statement to follow from the second, it suffices to prove the following.

Lemma II.2.123. $\underline{\mathrm{RPDisk}}(V)$ *is equidiameterizable.*

Proof. Take the equidiameter datum $\varepsilon_s := s\cdot \varepsilon^{\max}$, defined as follows.

$$\varepsilon^{\max}\left\{x_p \mid p\in S\right\} := \min\left\{\min_{p\neq q}\frac{d(x_p,x_q)}{2},\ \min_{\substack{i\leq n_V\\ p\in S}}\left(1-\left|\pi_{V_i}x_p\right|\right)\right\}$$

This is clearly continuous, as it's a polynomial, so to verify that this is an equidiameter datum, we're tasked with verifying that $h_t(\iota, \lambda\cdot\varepsilon^{\max})$ is an embedding landing in P^V; as it's rectilinear, if it is injective then it is

[5] Here, we've implicitly lifted ε to a $G\times\Sigma_n$-equivariant map $\mathrm{Conf}_{|S|}(V)\to\mathbb{R}_{>0}$, then restricted to $\mathrm{Conf}^H_S(V)$.

an immersion, so it suffices to check that it's injective and lands in P^V. But these are the defining properties of $\varepsilon^{\max}$. □

Proof of Proposition II.2.122. Choose an equidiameter datum ε for $\mathcal{U}$. We begin with each map $\mathfrak{c}$. The (essentially unique) lifted $\mathrm{Aut}^H(S)$-equivariant map s of

$$\begin{array}{ccc} \mathrm{Conf}_S^H(V) & \xrightarrow{\ s\ } & \mathrm{Emb}^{H,\mathrm{Rect},\mathcal{U}}\left(S\cdot P^V, P^V\right) \\ {\scriptstyle (\mathrm{id},\varepsilon)}\downarrow & & \downarrow \\ \mathrm{Conf}_S^H(V)\times\mathbb{R}_{>0} & \xrightarrow{\ e\ } & C^\infty(S\cdot P, V) \end{array}$$

is a section of $\mathfrak{c}$, and h_t is a deformation retract of $\mathrm{Conf}_S^H(V)$ onto $\mathrm{im}(s)$, so $\mathfrak{c}$ is an equivalence.

This and Observation II.2.98 construct a composite equivalence

$$\mathrm{Emb}_{\mathrm{id}_S}^{H,\mathrm{Rect}}\left(S\cdot P^V, S\cdot P^V\right) \simeq \prod_{[H/K]\subset S} \mathrm{Conf}_{*_K}^K(V) \simeq \prod_{[H/K]\subset S} V^K \simeq *.$$

In particular, the map $\mathrm{Emb}_{\mathrm{id}_S}^{H,\mathrm{Rect}}\left(S\cdot P^V, U\right) \to \mathrm{Conf}_{S,U}^H(V)$ is an equivalence, which allows us to construct the commutative diagram and note that $\mathfrak{c}\circ\alpha_{\mathcal{U}}$ is homotopic to $\alpha_{\mathcal{U},\mathrm{Conf}}$; Lemma II.2.113 shows that this is an equivalence, so $\alpha_{\mathcal{U}}$ is as well. The remaining equivalences follow from Lemma II.2.123 and two-out-of-three. □

Now, our main example to which we'll apply Proposition II.2.122 is the following.

Proposition II.2.124. $\underline{\mathrm{RPDisk}}(V \mid W)$ *is equidiameterizable; hence we have an equivalence*

$$\alpha\colon B\underline{\mathrm{RPDisk}}_S^H(V\mid W) \xrightarrow{\ \sim\ } \mathrm{Emb}^{H,\mathrm{Rect}}\left(S\cdot P^{V\oplus W}, P^{V\oplus W}\right).$$

By Proposition II.2.117 it suffices to prove the following and use the datum $\varepsilon_s = s\cdot\varepsilon^{V|W}$.

Lemma II.2.125. *Define the map* $\varepsilon^{V|W}\colon C_n(V)\to\mathbb{R}_{>0}$ *by*

$$\varepsilon^{V|W}\{x_1,\dots,x_n\} \coloneqq \min\left\{\epsilon^{\max}\{x_1,\dots,x_n\}, \min_{\pi_V(x_i)\neq\pi_V(x_j)} \frac{d(\pi_V(x_i),\pi_V(x_j))}{2}\right\}.$$

Then, the equidiameter embedding $e_{\varepsilon,c}$ *is decomposable if and only if* $\varepsilon \le \varepsilon^{V|W}(c)$.

Proof. This is a straightforward application of Lemma II.2.118. □

We'd like to discuss change of decomposition, for which we need a variant.

Variant II.2.126. Given $\mathcal{U}$ a V-polydisk G-precover satisfying the property that $\mathcal{U}$-admissible embeddings are closed under composition, we acquire a topological suboperad $\mathbb{E}^t_{V_\bullet,\mathcal{U}}\subset\mathbb{E}^t_{V_\bullet}$ described on structure spaces by

$$\mathrm{Emb}_f^{H,\mathrm{Rect},\ \mathcal{U}}\left(T\cdot P^V, S\cdot P^V\right) \subset \mathrm{Emb}_f^{H,\mathrm{Rect}}\left(T\cdot P^V, S\cdot P^V\right).$$ ◂

Proposition II.2.127. *If $\mathcal{U}$ is closed under composition and equidiameterizable, then the inclusion $\mathbb{E}^t_{V,\mathcal{U}} \to \mathbb{E}^t_V$ is a weak equivalence of genuine topological G-operads.*

PROOF. This follows immediately from Proposition II.2.122. □

Using this, we are free to choose convenient decompositions, arriving at the same G-operad regardless.

Corollary II.2.128. *If V and V' are polyrepresentations with the same direct sum, then there exists a zigzag of equivalences connecting $\mathbb{E}^t_V$ and $\mathbb{E}^t_{V'}$.*

PROOF. After Proposition II.2.127, it suffices to construct equidiameterizable V- and V'-polydisk G-covers $\mathcal{U}$ and $\mathcal{U}'$ and an equivalence $\mathbb{E}^t_{V,\mathcal{U}} \xrightarrow{\sim} \mathbb{E}^t_{V',\mathcal{U}'}$. Indeed, we let $\mathcal{U}$ be the equidiamter V-polydisk embeddings whose corresponding equidiameter V'-polydisk map (with the same diameter and center) is an embedding, and we define $\mathcal{U}'$ to be the corresponding V'-polydisk embeddings.

It is obvious that this correspondence yields an equivalence $\mathbb{E}^t_{V,\mathcal{U}} \xrightarrow{\sim} \mathbb{E}^t_{V',\mathcal{U}'}$, and $\mathcal{U}$ and $\mathcal{U}'$ are tautologically equidiameterizable. □

The reader is now free to assume $V_\bullet$ and $W_\bullet$ are one-step polyrepresentations.

II.2.9.4 Prefactorization algebras and self-similarity prefactorization algebras Let $\underline{\mathrm{Open}}_*(X) \subset \underline{\mathrm{Open}}(X)$ denote the G-subposet of connected open subsets.

Construction II.2.129. Given a topological space X with G-action, we define the *factorization* $\underline{\pi}_0\underline{\mathrm{Open}}_*(X)$*-symmetric sequence*

$$\underline{\mathrm{Open}}(X)((Y_K)_S; Z) = \begin{cases} * & \exists \text{ disjoint factorization} \coprod_K^S Y_K \hookrightarrow Z \hookrightarrow X; \\ \varnothing & \text{otherwise.} \end{cases}$$

By "disjoint factorization," we mean that $\coprod_K^S Y_K \hookrightarrow Z$ itself is an embedding. Note that the factorization is unique, if it exists. ◁

Proposition II.2.130. *There exists an essentially unique $\underline{\pi}_0\underline{\mathrm{Open}}$-colored G-0-operad $\underline{\mathrm{Open}}(X)^\otimes$ with $\underline{\pi}_0\underline{\mathrm{Open}}$-symmetric sequence $\underline{\mathrm{Open}}(X)(-;-)$.*

PROOF. Since $\underline{\mathrm{Open}}(X)(-;-)$ has 0-truncated structure spaces, there is at most one choice of maps 1_C and of γ, and if they exist, the associativity, unitality, equivariance, and restriction-stability diagrams all commute.

For existence of 1_C, it suffices to note that $\underline{\mathrm{Open}}(U;U)$ is nonempty. For γ, we're tasked with verifying that, whenever $\underline{\mathrm{Open}}(X)(U';U'') \times \prod_{[H/K] \subset \underline{\pi}_0 U'} \underline{\mathrm{Open}}(X)\left(U_K; U'_K\right)$ is nonempty, so is $\underline{\mathrm{Open}}(X)(U;U'')$ where

$U = \bigcup_K U_K$. Each arrow in $U \to U' \to U''$ is injective, so the composite is injective, which is what we wanted to show. □

More generally, given a G-precover $\mathcal{U}$, we define the G-subpreoperad $\mathcal{U}^\otimes \subset \underline{\mathrm{Open}}(X)^\otimes$ to have objects $\underline{\pi}_0\mathcal{U}_* \hookrightarrow \underline{\pi}_0\underline{\mathrm{Open}}_*(X)$ and an operation of signature (U, U') when such an operation exists in $\underline{\mathrm{Open}}(X)$ and the composite embedding $U \subset X$ is $\mathcal{U}$-admissible. This encodes a *genuine equivariant* version of prefactorization algebras.

We would like to use this to exercise fine control over different versions of $\mathbb{E}_V$, as *locally constant prefactorization algebras,* but to do so, we this must be realized as *all* spaces of active arrows in our corepresenting object. To do so, we introduce a variant.

Construction II.2.131. Given a V-polydisk G-precover $\mathcal{U} \subset \underline{\mathrm{RPDisk}}(V)$, we define the G-subpreoperad

$$\mathbb{P}^\otimes_{\mathcal{U}} \subset \underline{\mathrm{RPDisk}}(V)^\otimes$$

with colors the G-subposet $\mathcal{U}_* \subset \underline{\mathrm{RPDisk}}_*(V)$ and operations

$$\mathbb{P}_{\mathcal{U}}(U'; U) \simeq \begin{cases} * & U' \in \imath^*\mathcal{U} \subset \underline{\mathrm{Open}}(U) \subset \underline{\mathrm{Open}}(V), \\ \varnothing & \text{otherwise;} \end{cases}$$

That is, the operations with target U correspond with tuples of operations with terminal target in $\mathrm{Res}^G_{\mathrm{stab}(U_i)}\mathcal{U}^\otimes$ under the diffeomorphism $U_i \simeq \mathrm{Res}^G_{\mathrm{stab}(V)}$ for each path orbit $U_i \subset V$. ◁

Indeed, the following proposition is immediate and establishes our desired property.

Proposition II.2.132. *Given $\mathcal{U} \subset \mathcal{U}'$ a related pair of V-polydisk G-precovers, the inclusion $\mathcal{U} \subset \mathcal{U}'$ underlies an essentially-unique G-preoperad map $\mathbb{P}^\otimes_{\mathcal{U}} \to \mathbb{P}^\otimes_{\mathcal{U}'}$; given $U \subset V$ an $\mathcal{U}$-admissible embedded little V-polydisk, the induced map*

$$\mathcal{U}_{/P^V} \simeq \mathbb{P}^{\mathrm{act}}_{\mathcal{U},/U} \to \mathbb{P}^{\mathrm{act}}_{\mathcal{U}',/U} \simeq \mathcal{U}'_{/P^V}$$

is the inclusion $\mathcal{U} \subset \mathcal{U}'$.

Remark II.2.133. If a V-polydisk G-precover $\mathcal{U} \subset \underline{\mathrm{RPDisk}}(V)$ satisfies the property that $U' \in i^*\mathcal{U}$ if and only if $U' \in \mathcal{U}$, then $\mathbb{P}^\otimes_{\mathcal{U}} = \mathcal{U}^\otimes$ as G-suboperads of $\underline{\mathrm{Open}}(X)$, and in particular, $\mathbb{P}^\otimes_{\mathcal{U}}$ is a G-operad. ◁

Example II.2.134. The G-cover $\mathcal{U} = \underline{\mathrm{RPDisk}}(V)$ satisfies the property called for in Remark II.2.133; we define the G-operad $\mathbb{P}^\otimes_V := \mathbb{P}^\otimes_{\underline{\mathrm{RPDisk}}(V)} \simeq \underline{\mathrm{RPDisk}}(V)^\otimes$. In the case $G = e$, these are called $\mathbb{D}_n$ in [Har, Def. 5.2.3]. We also write $\mathbb{P}^\otimes_{V|W} := \mathbb{P}^\otimes_{\underline{\mathrm{RPDisk}}(V|W)}$. ◁

II.2.9.5 Locally constant prefactorization algebras We now need to define a map $\mathbb{P}^\otimes_V \to \mathbb{E}^\otimes_V$.

Variant II.2.135. We start by defining a "fattened" version of $\mathbb{E}_V^t$. Consider the evident inclusion $\underline{\mathrm{RPDisk}}_*(V) \hookrightarrow \mathbf{Mfld}^G$. Let $\mathcal{C}^t \subset \mathbf{Mfld}^G$ be the full topological G-suboperad spanned by its essential image. We define the wide topological G-suboperad $\widetilde{\mathbb{E}}_V^t \subset \mathcal{C}^t$ to have operations

$$\widetilde{\mathbb{E}}_V^t(U'; U) \simeq \mathrm{Emb}^{H,\mathrm{Rect}}(U', U),$$

where here we write $\mathrm{Emb}^{H,\mathrm{Rect}}(U', U)$ for those embeddings which become rectilinear under the pre-postcomposition homeomorphism $\mathrm{Emb}^H(U', U) \simeq \mathrm{Emb}^H\left(S \cdot P^V, P^V\right)$. Of course, we acquire a topological full G-suboperad embedding $\phi\colon \mathbb{E}_V^t \to \widetilde{\mathbb{E}}_V^t$ by including in P^V. ◂

Proposition II.2.136. *The map $\phi\colon \mathbb{E}_V^t \to \widetilde{\mathbb{E}}_V^t$ presents an equivalence of G-operads.*

Proof. First note that $U\widetilde{\mathbb{E}}_V^t$ is a connected G-space; moreover, its loop space is $\mathrm{Emb}^{H,\mathrm{Rect}}\left(P^V, P^V\right) \simeq *$, so $\widetilde{\mathbb{E}}_V^t$ has one color. Hence it suffices to prove that ϕ induces equivalences on operation spaces, i.e. the map

$$\mathrm{Emb}^{H,\mathrm{Rect}}(U', U) \to \mathrm{Emb}^{H,\mathrm{Rect}}\left(S \cdot P^V, P^V\right)$$

is an equivalence; this follows from the fact that $S \cdot P^V \to U'$ and $U \to P^V$ are isotopy equivalences. □

Construction II.2.137. Given a rectilinear little V-polydisk embedding $\iota\colon U \to U'$, the tautological point $\iota \in \mathrm{Emb}^{H,\mathrm{Rect}}(U, U')$ is compatible with G-operadic composition in $\widetilde{\mathbb{E}}_V^t$, and so it constructs a wide topological G-suboperad

$$\widetilde{\alpha}\colon \mathbb{P}_V^t \subset \widetilde{\mathbb{E}}_V^t;$$

such that $N^{\otimes}\mathbb{P}_V^t \simeq \mathbb{P}_V^{\otimes}$. We refer to the G-operad map under Proposition II.2.136 as $\alpha_{\mathcal{U}}\colon \mathbb{P}_{\mathcal{U}}^{\otimes} \to \mathbb{P}_V^{\otimes} \to \mathbb{E}_V^{\otimes}$. ◂

Remark II.2.138. It is unclear to the author how to interpret the construction called for in [Har, Eqn. (5.8)] directly; to define a topological operad map $\mathbb{P}_V^t \to \mathbb{E}_V^t$ lying over the unique map $f\colon \mathrm{Disk}_*(V) \to *$ would require choices of points $\chi_U \in \mathrm{Emb}^{H,\mathrm{Rect}}(S \cdot P^V, P^V)$ for each S-indexed U, which must be strictly compatible with composition and *strictly agree* along rectilinear isotopy equivalences $U' \hookrightarrow U$. Instead, our construction appears closely related to Lurie's map ψ used in [HA, Rmk. 5.4.5.8]. ◂

Our central observation about $\alpha_{\mathcal{U}}$ is the following.

Observation II.2.139. Suppose $P^V \in \mathcal{U}$. Under Proposition II.2.132, α_U induces on active arrow categories the functor $\mathcal{U}_S^H \to \mathrm{Emb}^{H,\mathrm{Rect}}\left(S \cdot P^V, U\right)$ whose restriction to $U' \hookrightarrow U$ is equivalent to the inclusion of

U'-subordinate $\mathcal{U}$-admissible embeddings:

$$\begin{array}{ccccc} \mathbb{P}^{\mathrm{act}}_{\mathcal{U},S//U} & \simeq & \mathcal{U}_S & \longleftarrow & \{U' \hookrightarrow U\} \\ \downarrow & & \downarrow & & \downarrow \sim \\ \widetilde{\mathbb{E}}^{\mathrm{act}}_{V,S//\alpha(U)} & \simeq & \mathrm{Emb}^{H,\mathrm{Rect}}\big(S \cdot P^V, U\big) & \xleftarrow{\iota} & \mathrm{Emb}^{H,\mathrm{Rect},\mathcal{U}}_{\mathrm{id}_S}(S \cdot P^V, U') \end{array}$$

In particular, if $\mathcal{U}$ is equidiameterizable, then under Proposition II.2.122 we find that $\alpha_{\mathcal{U}}$ induces an equivalence

$$\alpha : B\mathbb{P}^{\mathrm{act}}_{\mathcal{U},S//U} \xrightarrow{\sim} B\widetilde{\mathbb{E}}^{\mathrm{act}}_{V,S//\alpha(U)}. \qquad \triangleleft$$

We have already observed the difficult step of the following proposition.

Proposition II.2.140. *Given an equidiameterizable V-polydisk G-cover $\mathcal{U} \subset \underline{\mathrm{RPDisk}}(V)$ such that $P^V \in \mathcal{U}^G$,*

$$\alpha_U : \mathbb{P}^{\otimes}_{\mathcal{U}} \to \mathbb{E}^{\otimes}_V$$

is a Morita localization. In particular, it yields natural equivalences

$$\mathrm{Alg}_{\mathbb{E}_V}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{lc}}_{\mathbb{P}_V}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{lc}}_{\mathbb{P}_{\mathcal{U}}}(\mathcal{C}).$$

Proof. The fact that $P^V \in \mathcal{U}$ implies that $U\mathbb{P}_{\mathcal{U}}$ has a terminal G-object, giving Condition (ML-b). Moreover, applying Proposition II.2.65 for $\mathcal{C} = *_G$, we find that Observation II.2.139 implies Condition (ML-a). □

Corollary II.2.141. *The maps $\mathbb{P}^{\otimes}_{V|W} \to \mathbb{E}^{\otimes}_{V\oplus W}$ and $\mathbb{P}^{\otimes}_{V\oplus W} \to \mathbb{E}^{\otimes}_{V\oplus W}$ are Morita localizations.*

Proof. These each satisfy the conditions of Proposition II.2.140 by Proposition II.2.124. □

II.3 Distributivity and its consequences

We now delve into a topic which is decidedly more complicated in the equivariant setting than the nonequivariant setting: *distributivity*. In Section II.3.1, we develop this condition into a definition of a $\mathcal{T}$-suboperad of $\mathrm{Pr}^{L,\otimes}_{\mathcal{T}} \subset \widehat{\mathrm{Cat}}^{\times}_{\mathcal{T}}$, which is $\mathcal{T}$-symmetric monoidal, yielding a parameterized version of the *Lurie tensor product* and an associated norm; this extends work of Nardin [Nar17], filling in some gaps along the way. In Section II.3.2, we show this theory of distributivity to be equivalent to a special case of that of Elmanto-Haugseng and Lenz-Linskens-Pützstück [EH23; LLP25]; then, in Section II.3.2 we use this to extract a construction of $\mathcal{T}$-operadic left Kan extension from Lenz-Linskens-Pützstück's work, ultimately showing algebras to be monadic over objects in the distributive setting.

Following this, we return to the hunt for examples: in Section II.3.4, we develop a parameterized Day convolution extending that of Nardin-Shah [NS22] and fully develop its functoriality. Then, in Section II.3.5,

we study idempotent I-commutative algebras and the corresponding theory of *I-modes*, i.e. idempotent I-commutative monoids in $\mathrm{Pr}^{L,\otimes}_{\mathcal{T}}$; for instance, we use Day convolution to exhibit pointed spaces as a $\mathcal{T}$-mode.

Following this, in Section II.3.6 we extend Robalo-Cnossen's *formal inversion* to constructions of I-modes, including G-spectra as a G-mode, for G a compact Lie group. In Section II.3.7 we conclude by describing sharp version of a result of Hill on chromatic localizations in our setting.

II.3.1 The distributivity $\mathcal{T}$-symmetric monoidal ∞-category Fix $\mathcal{M} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ a full $\mathcal{T}$-subcategory, i.e. a restriction-stable class of equivalence classes of V-categories for all $V \in \mathcal{T}$. We will refer to the S-tuples $\mathrm{Cat}_S = \underline{\mathrm{Cat}}^S_{\mathcal{T}}$ as *S-categories* and $\mathcal{M}^S$ the *S-categories in $\mathcal{M}$*.

Definition II.3.1. Let $f\colon T \to S$ be an $\mathcal{R}$-admissable map of orbit-finite $\mathcal{T}$-spaces. Let $\mathcal{C}$ be a T-category, and let $\mathcal{D}$ be an S-category. An *f-multifunctor* is an S-functor $F\colon f_*\mathcal{C} \to \mathcal{D}$. We write $\mathsf{Fun}_f(\mathcal{C},\mathcal{D}) = \mathsf{Fun}_S(f_*\mathcal{C},\mathcal{D})$.

Given an T-colimit diagram $\overline{m}\colon \mathcal{K}^{\triangleright} \to \mathcal{C}$, we say that F *distributes* $\overline{m}$ if the S-functor

$$(f_*\mathcal{K})^{\triangleright} \xrightarrow{\mathsf{can}} f_*(\mathcal{K}^{\triangleright}) \xrightarrow{f_*\overline{m}} f_*\mathcal{C} \xrightarrow{F} \mathcal{D}$$

is an S-colimit diagram, where $\mathsf{can}\colon (f_*\mathcal{K})^{\triangleright} \to f_*(\mathcal{K}^{\triangleright})$ is adjunct to the lifted counit map

$$f^*(f_*\mathcal{K})^{\triangleright} \simeq (f^*f_*\mathcal{K})^{\triangleright} \xrightarrow{\epsilon^{\triangleright}} \mathcal{K}^{\triangleright}$$

for $f^* \dashv f_*$.

We say that an f-multifunctor F is *weakly $\mathcal{M}$-distributive* if it distributes T-colimit diagrams $\overline{m}\colon \mathcal{K}^{\triangleright} \to \mathcal{C}$ whenever $\mathcal{K} \in \mathcal{M}^T$. Moreover, we say that F is (strongly) *$\mathcal{M}$-distributive* if for all maps of orbit-finite T-spaces $S' \to S$ and pullback diagrams in $\underline{\mathcal{S}}_{\mathcal{T}}$

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ \downarrow{\scriptstyle f'} & \lrcorner & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

the induced f'-multifunctor

$$g^*F\colon f'_*(g')^*\mathcal{C} \simeq g^*f_*\mathcal{C} \longrightarrow g^*\mathcal{D}$$

is weakly $\mathcal{M}$-distributive. We write $\mathsf{Fun}^{\mathcal{M}-\delta}_f(\mathcal{C},\mathcal{D}) \subset \mathsf{Fun}_f(\mathcal{C},\mathcal{D})$ for the full subcategory of $\mathcal{M}$-distributive f-multifunctors.

We say that an $\mathcal{R}$-symmetric monodial $\mathcal{T}$-∞-category $\mathcal{C}$ is *$\mathcal{M}$-distributive* if, for each map $\mathcal{R}$-admissible map $S \to V$ with $V \in \mathcal{T}$, the indexed tensor V-functor $\underline{f}_{\underline{\otimes}}\colon f_*\mathcal{C}^{\underline{T}} \to \mathcal{C}^{\underline{V}}$ is $\mathcal{M}$-distributive as an f-multifunctor.[6] If $\mathcal{M} = \underline{\mathrm{Cat}}_{\mathcal{T}}$, we simply say that $\mathcal{C}^{\otimes}$ is *distributive.* ◁

[6] Here, the f-indexed tensor multifunctor is given by the functoriality along f of the corresponding $\underline{\mathbb{F}}_{I,*}$-Segal $\mathcal{T}$-∞-category.

Remark II.3.2. Assume $\mathcal{C}$ and $\mathcal{D}$ both admit colimits for $\mathcal{M}$-indexed diagrams. We can describe weak $\mathcal{M}$-distributivity more informally: the fact that the f-multifunctor $F\colon f_*\mathcal{C} \to \mathcal{D}$ distributes them is equivalent to the stipulation that the assembly map

$$\beta_{m,f}\colon \operatorname{colim}\left(f_*\mathcal{K} \to f_*\mathcal{C} \xrightarrow{F} \mathcal{D}\right) \longrightarrow F\left(\operatorname{colim}(\mathcal{K} \to \mathcal{C})\right)$$

for parameterized colimits is an equivalence for any functor $m\colon \mathcal{K} \to \mathcal{C}$ with $\mathcal{K} \in \mathcal{M}$. Now, by construction, this map is the value at m of the Beck-Chevalley transformation for horizontal left-adjointability of the following square of ∞-categories

$$\begin{array}{ccc}
\mathcal{D}(f_*\mathcal{K}) & \xleftarrow{\Delta^{f_*\mathcal{K}}} & \mathcal{D}(S) \\
{\scriptstyle F}\uparrow \quad \nwarrow^{F} & & \uparrow{\scriptstyle F} \\
\mathcal{C}(f^*f_*\mathcal{K}) \simeq f_*\mathcal{C}(f_*\mathcal{K}) & & \\
{\scriptstyle \epsilon^*}\uparrow \quad \nearrow_{f_*} & & \\
\mathcal{C}(\mathcal{K}) & \xleftarrow[\Delta^{\mathcal{K}}]{} & \mathcal{C}(T)
\end{array}$$

Here, we have written $\mathcal{C}(\mathcal{E}) \coloneqq \mathsf{Fun}_S(\mathcal{E},\mathcal{C})$ whenever $\mathbb{E}$ and $\mathcal{C}$ are S-categories. We will eventually use this to compare our notion with that appearing in [EH23]. ◁

The purpose of this section is to prove the following.

Theorem II.3.3. *There exists an essentially $\mathcal{T}$-suboperad $\widetilde{\underline{\mathrm{Cat}}}{}^{\delta}_{\mathcal{T}} \subset \widetilde{\underline{\mathrm{Cat}}}{}^{\times}_{\mathcal{T}}$ (the latter denoting the cartesian $\mathcal{T}$-symmetric monoidal structure) whose V-colors are the non-massive V-∞-categories admitting small colimits and whose multi-operation spaces over an active map $f\colon T \to S$ are*

$$\mathrm{Map}^{f}_{\widetilde{\underline{\mathrm{Cat}}}{}^{\delta}_{\mathcal{T}}}(\mathcal{C},\mathcal{D}) \simeq \mathsf{Fun}^{\mathcal{M}-\delta}_{f}(\mathcal{C},\mathcal{D})^{\simeq};$$

moreover, $\widetilde{\underline{\mathrm{Cat}}}{}^{\delta}_{\mathcal{T}}$ is a massive $\mathcal{T}$-symmetric monoidal ∞-category, and the full $\mathcal{T}$-suboperad $\underline{\mathrm{Pr}}{}^{L,\otimes}_{\mathcal{T}} \subset \widetilde{\underline{\mathrm{Cat}}}{}^{\delta}_{\mathcal{T}}$ is $\mathcal{T}$-symmetric monoidal.

Now, Theorem I.3.51 applied to $\mathsf{Span}_I(\mathbb{F}_{\mathcal{T}})$ immediately describes the corresponding algebras.

Corollary II.3.4. *The forgetful functor*

$$\mathrm{Alg}_{\mathcal{O}}\left(\widetilde{\underline{\mathrm{Cat}}}{}^{\mathcal{M}-\delta}_{\mathcal{T}}\right) \longrightarrow \mathrm{Alg}_{\mathcal{O}}\left(\widetilde{\underline{\mathrm{Cat}}}{}^{\times}_{\mathcal{T}}\right) \simeq \widetilde{\underline{\mathrm{Cat}}}{}^{\otimes}_{\mathcal{O}}$$

is a subcategory inclusion, with image having objects the $\mathcal{M}$-distributive $\mathcal{O}$-monoidal ∞-categories and morphisms the $\mathcal{O}$-monodial functors whose underlying natural transformations of diagrams $U\mathcal{O} \to \underline{\mathrm{Cat}}_{\mathcal{T}}$ are pointwise $\mathcal{M}$-cocontinuous.

II.3.1.1 Composition-stability We begin on the proof, first verifying that we have defined a $\mathcal{T}$-suboperad.

Construction II.3.5. Given $T \xrightarrow{f} S \xrightarrow{g} R$ a composable sequence of $\mathfrak{o}(\mathcal{T})$-admissible maps of orbit-finite $\mathcal{T}$-spaces, and $\mathcal{C},\mathcal{D},\mathcal{E}$ a T, S, and R-category, we define the *composition* operation

$$\begin{array}{ccc}
\mathrm{Fun}_g(\mathcal{D},\mathcal{E}) \times \mathrm{Fun}_f(\mathcal{C},\mathcal{D}) & \xrightarrow{\quad\circ\quad} & \mathrm{Fun}_{gf}(\mathcal{C},\mathcal{E}) \\
\| & & \| \\
\mathrm{Fun}_R(g_*\mathcal{D},\mathcal{E}) \times \mathrm{Fun}_S(f_*\mathcal{C},\mathcal{D}) & & \mathrm{Fun}_R(gf_*\mathcal{C},\mathcal{E}) \\
& \searrow^{\mathrm{id}\times g_*} \quad \nearrow_{\circ} & \\
& \mathrm{Fun}_R(g_*\mathcal{D},\mathcal{E}) \times \mathrm{Fun}_R(gf_*\mathcal{C},g_*\mathcal{D}) &
\end{array}$$

Unwinding definitions, this is the operadic composition operation in $\widetilde{\underline{\mathrm{Cat}}}^{\times}_{\mathcal{T}}$. ◁

Proposition II.3.6. *Composition restricts to a functor of the full subcategories of $\mathcal{M}$-distributive multifunctors:*

$$\begin{array}{ccc}
\mathrm{Fun}_g^{\mathcal{M}-\delta}(\mathcal{D},\mathcal{E}) \times \mathrm{Fun}_f^{\mathcal{M}-\delta}(\mathcal{C},\mathcal{D}) & \dashrightarrow^{\circ} & \mathrm{Fun}_{gf}^{\mathcal{M}-\delta}(\mathcal{C},\mathcal{E}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}_g(\mathcal{D},\mathcal{E}) \times \mathrm{Fun}_f(\mathcal{C},\mathcal{D}) & \xrightarrow{\quad\circ\quad} & \mathrm{Fun}_{gf}(\mathcal{C},\mathcal{E})
\end{array}$$

Proof. We show this for *weakly* $\mathcal{M}$-distributive multifunctors; composition is compatible with restriction, so this implies the desired statement.

Let $F\colon f_*\mathcal{C} \to \mathcal{D}$ and $G\colon g_*\mathcal{D} \to \mathcal{E}$ be weakly distributive parameterized functors. Suppose we start with a T-colimit diagram $\overline{m}\colon \mathcal{K}^{\triangleright} \to \mathcal{C}$, and let

$$\overline{n}\colon (f_*\mathcal{K})^{\triangleright} \xrightarrow{\mathrm{can}} f_*(\mathcal{K}^{\triangleright}) \xrightarrow{f_*\overline{m}} f_*\mathcal{C} \xrightarrow{F} \mathcal{D}$$

be the composite functor. This is an S-colimit by weak distributivity of F. Consider the following diagram:

$$\begin{array}{ccccccccc}
(gf_*\mathcal{K})^{\triangleright} & & \xrightarrow{\quad\mathrm{can}\quad} & & gf_*(\mathcal{K}^{\triangleright}) & \xrightarrow{gf_*\overline{m}} & gf_*\mathcal{C} & \xrightarrow{\quad GF\quad} & \mathcal{E} \\
\| & & & & \wr\uparrow & & \wr\uparrow & & \| \\
 & & g_*(f_*\mathcal{K})^{\triangleright} & \xrightarrow{g_*\mathrm{can}} & g_*f_*(\mathcal{K}^{\triangleright}) & \xrightarrow{g_*f_*\overline{m}} & g_*f_*\mathcal{C} & \xrightarrow{g_*F} & g_*\mathcal{D} \\
 & & \| & & & & & & \| \\
(gf_*\mathcal{K})^{\triangleright} & \xrightarrow{\mathrm{can}} & g_*(f_*\mathcal{K})^{\triangleright} & & \xrightarrow{\qquad g_*\overline{n}\qquad} & & & & g_*\mathcal{D} \xrightarrow{G} \mathcal{E}
\end{array}$$

The lower diagram commutes by definition of $\overline{n}$. The upper diagram obviously commutes. The right diagram commutes by the definition of composition. To see that the left diagram commutes, note that each composite is mate to a map

$$gf^*(gf_*\mathcal{K})^{\triangleright} \simeq (gf^*gf_*\mathcal{K})^{\triangleright} \longrightarrow \mathcal{K}^{\triangleright};$$

one is gotten by applying $\triangleright$ to the $gf^* \dashv gf_*$-counit, and other to the composite of the $f^* \dashv f_*$-counit and the $g^* \dashv g_*$-counit. Commutativity follows by composing adjunction data.

Weak distributivity of G then implies that the bottom horizontal composite is a colimit diagram, implying that the top horizontal composite is a colimit diagram, yielding weak distributivity of GF. □

From here, we are able to make a definition.

Proposition II.3.7. *There exists an essentially unique* $\mathrm{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$*-relative subcategory*

$$\mathrm{Tot}\,\underline{\widetilde{\mathrm{Cat}}}^{\delta}_{\mathcal{T}} \hookrightarrow \mathrm{Tot}\,\underline{\widetilde{\mathrm{Cat}}}^{\times}_{\mathcal{T}}$$

with objects over S *the cocomplete* S*-categories and with relative morphism spaces*

$$\mathrm{Map}^{R\xleftarrow{g}T\xrightarrow{f}S}_{\underline{\widetilde{\mathrm{Cat}}}^{\delta}_{\mathcal{T}}}(\mathcal{C},\mathcal{D}) = \mathrm{Fun}^{\delta}_{f}(g^{*}\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}_{f}(g^{*}\mathcal{C},\mathcal{D}) \simeq \mathrm{Map}^{R\xleftarrow{g}T\xrightarrow{f}S}_{\underline{\widetilde{\mathrm{Cat}}}^{\delta}_{\mathcal{T}}}(\mathcal{C},\mathcal{D});$$

moreover, this is an inert-cocartesian fibration.

Proof. We will apply Propositions I.2.72 and I.2.73. Now, a connected component F of $\mathrm{Map}_{\underline{\widetilde{\mathrm{Cat}}}^{\times}_{\mathcal{T}}}(\mathcal{C},\mathcal{D})$ may be identified with a span $R \xleftarrow{g} T \xrightarrow{f} S$ together with a functor $f_{*}g^{*}\mathcal{C} \to \mathcal{D}$, i.e. an f-multifunctor $F^{\mathrm{act}}\colon g^{*}\mathcal{C} \to \mathcal{D}$. We define the summand

$$\mathrm{Map}_{\underline{\widetilde{\mathrm{Cat}}}^{\delta}_{\mathcal{T}}}(\mathcal{C},\mathcal{D}) \coloneqq \left\{F \;\middle|\; F^{\mathrm{act}} \text{ is a distributive } \pi F^{\mathrm{act}}\text{-multifunctor}\right\} \subset \mathrm{Map}_{\underline{\widetilde{\mathrm{Cat}}}^{\times}_{\mathcal{T}}}(\mathcal{C},\mathcal{D});$$

this satisfies the above formula by fiat. To verify that it is a relative wide subcategory, it suffices to note that it contains all equivalences by assumption and is closed under composition by Proposition II.3.6. The fact that it contains all inert arrows with domain in $\mathrm{Tot}\,\underline{\widetilde{\mathrm{Cat}}}^{\delta}_{\mathcal{T}}$ reduces to observing that equivalences are distributives. □

II.3.1.2 Constructing norms The difficult remaining statement is the $\mathcal{T}$-symmetric monoidality statement, so we begin constructing locally-cocartesian arrows.

Proposition II.3.8. $\mathrm{Tot}\,\underline{\widetilde{\mathrm{Cat}}}^{\delta}_{\mathcal{T}}$ *has locally cocartesian lifts* $c_{f,\mathcal{C}}\colon \mathcal{C} \to f_{\otimes}\mathcal{C}$ *for active morphisms satisfying*

$$g_{\otimes}f_{\otimes}\mathcal{C} \simeq gf_{\otimes}\mathcal{C}.$$

Before we dive into this, we need the following auxiliary lemma:

Lemma II.3.9. *Let* $\mathcal{D} \to \mathcal{C}$ *be a left fibration of* T*-categories, straightening to a functor* $\mathcal{D} \to \underline{\mathcal{S}}_{T}$ *and let* $f\colon T \to S$ *a map of orbit-finite* $\mathcal{T}$*-spaces. Then* $f_{*}\mathcal{D} \to f_{*}\mathcal{C}$ *straightens to* $f_{*}\mathcal{D} \to f_{*}\underline{\mathcal{S}}_{T} \to \underline{\mathcal{S}}_{S}$ *where the last morphism is the limit functor* $f_{*}f^{*}\underline{\mathcal{S}}_{S} \to \underline{\mathcal{S}}_{S}$.

Proof. It is enough to verify the claim for the universal right fibration $(\underline{\mathcal{S}}_{T})_{T/} \to \underline{\mathcal{S}}_{T}$. Applying f_{*} yields the right fibration $f_{*}(\underline{\mathcal{S}}_{T})_{T/} \simeq (f_{*}\underline{\mathcal{S}}_{T})_{f_{*}T/} \to f_{*}\underline{\mathcal{S}}_{S}$. This straightens to the functor taking maps out of the terminal object in $f_{*}\underline{\mathcal{S}}_{S}$ which indeed represents taking the limit. □

Now, we will extend Proposition I.1.42 to indexed products of presheaf $\mathcal{T}$-categories.

Proposition II.3.10 (c.f. [Nar17, Prop. 3.19]). *Let $f\colon T \to S$ be an $\mathfrak{v}(\mathcal{T})$-admissible map of orbit-finite $\mathcal{T}$-spaces. Given $\mathcal{A}$ a T-∞-category, $\mathcal{R}$ a collection of colimit cocones in $\mathcal{A}$, and $\mathcal{D}$ a cocomplete S-∞-category, precomposition with $f_* \mathbin{⫰}_{\mathcal{A},\mathcal{R}}$ yields an equivalence $\underline{\mathrm{Fun}}^{\delta}_{f}(\mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A}),\mathcal{D}) \simeq \underline{\mathrm{Fun}}^{L-\mathcal{R}}_{f}(\mathcal{A},\mathcal{D})$.*

Proof. We elaborate Nardin's proof. The inverse functor will be given by $\mathcal{T}$-left Kan extension along $f_* \mathbin{⫰}_{\mathcal{A}}$. $\mathbin{⫰}_{\mathcal{A},\mathcal{R}}$ is fully faithful by Proposition I.1.42 and f_* preserves fully-faithfulness, so

$$\left(f_* \mathbin{⫰}_{\mathcal{A},\mathcal{R}}\right)_! : \underline{\mathrm{Fun}}^{L-\mathcal{R}}_{f}(\mathcal{A},\mathcal{D}) \longrightarrow \underline{\mathrm{Fun}}_{f}(\mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A}),\mathcal{D})$$

is fully faithful. We need to identify its image precisely with the distributive f-multifunctors.

The forward direction. To verify that an arbitrary distributive f-functor $F\colon f_*\mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A}) \to \mathcal{D}$ is $\mathcal{T}$-left Kan extended along $\mathbin{⫰}_{\mathcal{A},\mathcal{R}}$, it is equivalent to verify that the counit S-natural transformation

$$\varepsilon\colon (f_* \mathbin{⫰}_{\mathcal{A},\mathcal{R}})_!(f_* \mathbin{⫰}_{\mathcal{A},\mathcal{R}})^* F \longrightarrow F$$

is an equivalence. This in turn can be checked after pulling back along an each orbit $V \hookrightarrow S$; as everything in sight commutes with restriction, we may further reduce to the case that $S = V$. Now, that ε can also be checked pointwise, i.e. we must check for each V-object $X \in (f_*\mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A}))^V$ that the counit is an equivalence

$$\varepsilon_X\colon \left(f_* \mathbin{⫰}_{A,\mathcal{R}}\right)_! \left(f_* \mathbin{⫰}_{A,\mathcal{R}}\right)^* F(X) \to F(X).$$

Recall that the T-colimit of $\mathcal{A}_{/X} \to \mathcal{A} \to \mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A})$ is $\mathbin{⫰}_{A,\mathcal{R}}(X)$, so that we have a colimit cocone

$$\overline{m}\colon \mathrm{colim}\,(f_*\mathcal{A}_{/X})^{\triangleright} \longrightarrow \mathrm{Psh}_{V,\mathcal{R}}(f_*\mathcal{A}).$$

Consider the following commutative diagram of V-categories

$$\begin{array}{ccccc} \left((f_*\mathcal{A})_{/X}\right)^{\triangleright} & \longrightarrow & f_*\mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A}) & \xrightarrow{F} & \mathcal{D} \\ \downarrow\simeq & & \uparrow & & \\ (f_*(\mathcal{A}_{/X}))^{\triangleright} & \xrightarrow{\mathrm{can}} & f_*((\mathcal{A}_{/X})^{\triangleright}) & & \end{array}$$

It follows from the commutativity of the lower diagram and distributivity of F that the upper horizontal composite is a colimit cocone. Next, unwinding definitions, this expresses that $F(X)$ is the colimit of

$$(f_*\mathcal{A})_{/X} \longrightarrow f_*\mathcal{A} \xrightarrow{f_* \mathbin{⫰}_{\mathcal{A},\mathcal{R}}} f_*\mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A}) \xrightarrow{F} \mathcal{D}$$

In turn this is precisely the pointwise formula for $\mathcal{T}$-left Kan extension of $(f_* \mathbin{⫰}_{\mathcal{A},\mathcal{R}})^* F$ along $f_* \mathbin{⫰}_{\mathcal{A},\mathcal{R}}$ at X.

The backward direction. We are left to show that any f-multifunctor $\mathrm{Psh}_S(\mathcal{A}) \to \mathcal{D}$ which is S-left Kan extended along $f_* \yoneda_{\mathcal{A}}$ is automatically distributive. We start in the case that $\mathcal{D} \simeq \mathrm{Psh}_S(f_*\mathcal{A})$ and the S-functor we are extending is the Yoneda embedding $\yoneda_{f_*\mathcal{A}, f_*\mathcal{R}} \colon f_*\mathcal{A} \longrightarrow \mathrm{Psh}_{S, f_*\mathcal{R}}(f_*\mathcal{A})$.

Now, restriction along $D \to f_*\mathcal{A}^{\mathrm{op}}$ for D some orbit-finite S-space gives a family of S-colimit preserving S-functors out of $\mathrm{Psh}_{S, f_*\mathcal{R}}(f_*\mathcal{A})$ which jointly detects colimits, so the claim in this case reduces to showing that the composition with all those functors is distributive. Verifying the claim also for the restriction, we may further reduce to $D = S$, i.e. we want to show that for any $A \colon S \to f_*\mathcal{A}^{\mathrm{op}}$, the composite

$$f_*\mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A}) \longrightarrow \mathrm{Psh}_{S, f_*\mathcal{R}}(f_*\mathcal{A}) \xrightarrow{\mathrm{ev}_A} \underline{\mathcal{S}}_S$$

is distributive. As we already verified that any distributive f-multifunctor is Kan extended along $\yoneda_{\mathcal{A},\mathcal{R}}$, it suffices to provide a distributive f-multifunctor which has the same restriction along $f_* \yoneda_{\mathcal{A},\mathcal{R}}$; thinking of A as T-object in $\mathcal{A}^{\mathrm{op}}$, such a distributive f-multifunctor is

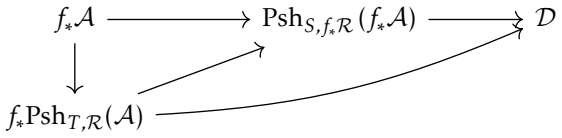
$$F \colon f_*\mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A}) \xrightarrow{f_*(\mathrm{ev}_A)} f_*\underline{\mathcal{S}}_T \xrightarrow{f_*} \underline{\mathcal{S}}_S.$$

First of all, this f-multifunctor is distributive by Proposition II.3.6, as ev_A preserves colimits and the limit f-multifunctor is distributive by Proposition II.3.19. Second, $F|_{f_*\mathcal{A}}$ is given by

$$f_*\mathcal{A} \xrightarrow{f_*(\underline{\mathrm{Map}}_{\mathcal{A}}(A,-))} f_*\underline{\mathcal{S}}_T \longrightarrow \underline{\mathcal{S}}_S\,.$$

We are left to argue that this agrees with with $\underline{\mathrm{Map}}_{f_*\mathcal{A}}(A,-)$ where we instead regard A as an S-object in $f_*\mathcal{A}$. But this follows from Lemma II.3.9 by taking the left fibration $\mathcal{A}_{A/} \to \mathcal{A}$.

This finishes the argument for Kan extending $\yoneda_{f_*\mathcal{A},\mathcal{R}}$. The general case follows: any $f_*\mathcal{R}$-cocontinuous S-functor $f_*\mathcal{A} \to \mathcal{D}$ factors, via S-left Kan extension, through the Yoneda embedding $f_*\mathcal{A} \to \mathrm{Psh}_{S, f_*\mathcal{R}}(f_*\mathcal{A})$ via a cocontinuous S-functor $\mathrm{Psh}_{S,\mathcal{R}}(f_*) \to \mathcal{D}$ by the parameterized Yoneda lemma Proposition I.1.41. Post-composition with this preserves Kan extension, so we have a commutative diagram



where the two diagonal functors are the left Kan extensions. The lower right triangle express the left Kan extension $f_*\mathrm{Psh}_{T,\mathcal{R}}(\mathcal{A}) \to \mathcal{D}$ as a composition of a distributive f-multifunctor and an S-left adjoint, so it is f-distributive by Proposition II.3.6. □

Construction II.3.11. Applying Proposition II.3.10 to $\mathcal{D} = \mathrm{Psh}_S(f_*\mathcal{C})$ identifies the modified Yoneda embedding $\yoneda_{\mathcal{A},f_*C_\mathcal{C}} : f_*\mathcal{A} \to \mathrm{Psh}_{S,f_*C_\mathcal{C}}(\mathcal{C})$ with a distributive f-multifunctor,

$$c_{f,\mathcal{C}} : \mathcal{C} \longrightarrow \mathrm{Psh}_{S,f_*C_\mathcal{C}}(f_*\mathcal{C})$$

where $C_\mathcal{C}$ consists of the colimit diagrams in $\mathcal{C}$. Correspondingly, we write $f_\otimes\mathcal{C} \coloneqq \mathrm{Psh}_{S,f_*C_\mathcal{C}}(f_*\mathcal{C})$. ◁

Proof of Proposition II.3.8. In sight of [kerodon, Tag 01U8], local cocartesianness is equivalent to asking that pullback $c_{f,\mathcal{C}}$ implements a natural equivalence

$$\underline{\mathrm{Fun}}^\delta_f(\mathrm{Psh}_T(f_*\mathcal{C}),\mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}^L_f(\mathcal{C},\mathcal{D});$$

this is exactly Proposition II.3.10. We're left with compositionality, for which we apply the sequence of natural equivalences

$$\begin{aligned}
\mathrm{Fun}^L_R(g_\otimes f_\otimes\mathcal{C},\mathcal{D}) &\simeq \mathrm{Fun}^\delta_g(g_*f_\otimes\mathcal{C},\mathcal{D}) \\
&\simeq \mathrm{Fun}^\delta_g\left(g_*\mathrm{Psh}_{S,f_*C_\mathcal{C}}(f_*\mathcal{C}),\mathcal{D}\right) \\
&\simeq \mathrm{Fun}^\delta_{gf}(g_*f_*\mathcal{C},\mathcal{D}) \\
&\simeq \mathrm{Fun}^\delta_{gf}(gf_*\mathcal{C},\mathcal{D}) \\
&\simeq \mathrm{Fun}^L_R(gf_\otimes\mathcal{C},\mathcal{D}),
\end{aligned}$$

where each equivalence except for the penultimate one is Proposition II.3.10. □

Before moving on, we note a useful corollary.

Corollary II.3.12. *There exists a natural equivalence $f_\otimes\mathrm{Psh}_T(\mathcal{C}) \simeq \mathrm{Psh}_S(f_*\mathcal{C})$; in particular, $(\varnothing \to S)_\otimes{}^* \simeq \underline{\mathcal{S}}_\mathcal{T}$.*

Proof. Apply Yoneda's lemma to the equivalence

$$\begin{aligned}
\underline{\mathrm{Fun}}^L_S(f_\otimes\mathrm{Psh}_T(\mathcal{C}),\mathcal{D}) &\simeq \underline{\mathrm{Fun}}^\delta_f(\mathrm{Psh}_T(\mathcal{C}),\mathcal{D}) \\
&\simeq \underline{\mathrm{Fun}}_f(\mathcal{C},\mathcal{D}) \\
&\simeq \underline{\mathrm{Fun}}_S(f_*\mathcal{C},\mathcal{D}) \\
&\simeq \underline{\mathrm{Fun}}^L_S(\mathrm{Psh}_S(f_*\mathcal{C}),\mathcal{D}).
\end{aligned}$$

□

II.3.1.3 Norms and Bousfield localizations The last thing we will need to do is prove that $f_\otimes\mathcal{C}$ is presentably whenever $\mathcal{C}$ is. To do so, we will show how the Bousfield localizations constructed in Proposition I.1.44 (i.e. [Hil24, Thm. 6.3.7]) interact with norms.

Proposition II.3.13. *Suppose $\mathcal{C} \to f_\otimes \mathcal{C}$ is an active locally-cocartesian arrow, $\mathcal{B} \subset \mathcal{C}$ is a small set of arrows in $\mathcal{C}$, and $\mathcal{C}_0 \subset \mathcal{C}$ is a small full subcategory generating $\mathcal{C}$ under $\mathcal{T}$-colimits. Then, the corresponding active arrow $L_{\mathcal{B}}\mathcal{C} \to L_{f_\otimes(\mathcal{B} \cup \mathcal{C}_0^{\simeq})} f_\otimes \mathcal{C}$ is locally cocartesian.*

Lemma II.3.14. *Let $\mathcal{C}, \mathcal{D} \in \mathrm{Pr}^L_T$ and $\mathcal{C}_0 \subseteq \mathcal{C}$ a small subcategory which generates $\mathcal{C}$ under T-colimits. Let $\mathcal{B}$ be a small set of morphisms in $\mathcal{C}$. Then the evident inclusion yields an equivalence*

$$\underline{\mathrm{Fun}}^{\delta}_{f, f_*(\mathcal{B} \cup \mathcal{C}^{\simeq})}(\mathcal{C}, \mathcal{D}) \subset \underline{\mathrm{Fun}}^{\delta}_{f, f_*\left(\mathcal{B} \cup \mathcal{C}_0^{\simeq}\right)}(\mathcal{C}, \mathcal{D})$$

Proof. Fix some $F \in \mathrm{Fun}_{f, f_*(\mathcal{B} \cup \mathcal{C}_0^{p\simeq})}(\mathcal{C}, \mathcal{D})$. We need to show that it also inverts morphisms in $f_*(\mathcal{B} \cup \mathcal{C}^{\simeq})$. The collection of arrows inverted by F is closed under composition with equivalences, so we may equivalently prove that it inverts $f_*\left(\mathcal{B} \cup \mathcal{C}^{\mathrm{id}}\right)$; moreover, the collection of morphisms inverted by F is closed under colimits by Proposition I.1.46, so this reduces to checking that it inverts $f_*\left(\mathcal{B} \cup \mathcal{C}_0^{\mathrm{id}}\right)$, as desired. □

Lemma II.3.15. *If $\mathcal{B} \subset \mathcal{C}$ is a $\mathcal{T}$-set of arrows, then there exists an equivalence $L_{f_*\mathcal{B}} f_* \mathcal{C} \simeq f_* L_{\mathcal{B}} \mathcal{C}$ under $f_* \mathcal{C}$.*

Proof. f_* is an $(\infty, 2)$-functor and preserves fully faithfulness, so each side is a localization functor; that is, it suffices to characterize the morphisms $\overline{\mathcal{B}}'$ inverted by $f_* L$ as the strong saturation of $f_* \mathcal{B}$. Now, note that $f_* \overline{\mathcal{B}} \subset \overline{\mathcal{B}}'$, the lefthand side has the same strong saturation as $\overline{\mathcal{B}}$, and the latter is strongly saturated, so it will suffice to prove that $\overline{\mathcal{B}} \subset f_* \overline{\mathcal{B}}'$.

But every morphism in $f_* \mathcal{C}$ is expressed as an f-indexed product of morphisms in $\mathcal{C}$; this morphism lies in $\overline{\mathcal{B}}'$ if and only if its factors lie in $\overline{\mathcal{B}}$, giving the claim. □

The following lemma is a straightforward rephrasing of this, in combination with Proposition I.1.44.

Lemma II.3.16. *Let $\mathcal{B}$ be a class of morphsims in a presentable T-∞-category $\mathcal{C}$ with strong saturation $\overline{\mathcal{B}}$ and let $\mathcal{D}$ be a presentable S-∞-category. Then, pullback along $f_* L$ induces an equivalence*

$$(f_* L)^* : \underline{\mathrm{Fun}}^{\delta}_f(L_{\mathcal{B}} \mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \underline{\mathrm{Fun}}^{\delta}_{f, f_* \mathcal{B}}(\mathcal{C}, \mathcal{D}).$$

Proof of Proposition II.3.13. This now follows by the computation

$$\begin{aligned}
\underline{\mathrm{Fun}}^L\left(L_{f_\otimes(\mathcal{B} \cup \mathcal{C}_0^{\simeq})} f_\otimes \mathcal{C}, \mathcal{D}\right) &\simeq \underline{\mathrm{Fun}}^L_{f_\otimes(\mathcal{B} \cup \mathcal{C}_0^{\simeq})}(f_\otimes \mathcal{C}, \mathcal{D}) && \text{I.1.44} \\
&\simeq \underline{\mathrm{Fun}}^{\delta}_{f, f_\otimes\left(\mathcal{B} \cup \mathcal{C}_0^{\simeq}\right)}(\mathcal{C}, \mathcal{D}) && \text{assumption} \\
&\simeq \underline{\mathrm{Fun}}^{\delta}_{f, f_\otimes(\mathcal{B} \cup \mathcal{C}^{\simeq})}(\mathcal{C}, \mathcal{D}) && \text{II.3.14} \\
&\simeq \underline{\mathrm{Fun}}^{\delta}_f\left(L_{f_\otimes \mathcal{B} \cup \mathcal{C}^{\simeq}} \mathcal{C}, \mathcal{D}\right) && \text{II.3.16} \\
&\simeq \underline{\mathrm{Fun}}^{\delta}_f\left(L_{f_\otimes \mathcal{B}} \mathcal{C}, \mathcal{D}\right). && \square
\end{aligned}$$

II.3.1.4 Putting together the proof We are almost ready to conclude. First, to save us some trouble, we give an obvious lemma.

Lemma II.3.17. *Fix $\mathcal{D}$ an ∞-category with a factorization system $\mathcal{D}^L, \mathcal{D}^R \subset \mathcal{D}$, and suppose $\pi\colon \mathcal{C} \to \mathcal{D}$ is a functor with cocartesian lifts over $\mathcal{D}^L$. Then, π is a cocartesian fibration if and only if the following conditions hold.*

(a) π has locally cocartesian lifts for morphisms in $\mathcal{D}^R$.

(b) If f, g are composable π-locally cocartesian such that $\pi f \in \mathcal{D}^L$ and $\pi g \in \mathcal{D}^R$, then $f \circ g$ is π-locally cocartesian.

(c) If f, g are composable π-locally cocartesian such that $\pi f, \pi g \in \mathcal{D}^R$ then $f \circ g$ is π-locally cocartesian.

Proof. These conditions are obviously necessary, so suppose they hold. By composing properties (b) and (c), we find that our locally cocartesian lifts of active morphisms remain locally cocartesian after precomposition along *arbitrary* locally cocartesian morphisms, so [kerodon, Tag 01V5] shows that they are cocartesian. Since cocartesian arrows are closed under composition and $\mathcal{D}^L$ and $\mathcal{D}^R$ generate $\mathcal{D}$ under composition, this implies the claim. □

We are finally ready to conclude.

Proof of Theorem II.3.3. Proposition II.3.7 constructs a $\mathsf{Span}_{\mathfrak{o}(T)}(\mathbb{F}_T)$-relative subcategory

$$\mathrm{Tot}\,\widetilde{\underline{\mathrm{Cat}}}_T^{\delta} \subset \mathrm{Tot}\,\widetilde{\underline{\mathrm{Cat}}}_T^{\times};$$

to lift this to a (massive) $\mathcal{T}$-suboperad, by Proposition II.2.54, we simply need to show that $\mathrm{Tot}\,\widetilde{\underline{\mathrm{Cat}}}_T^{\delta}$ is a $\mathcal{T}$-operad; in fact, we need to prove that it's a (massive) $\mathcal{T}$-symmetric monoidal ∞-category, so we just prove this.

First, we must prove that $\mathrm{Tot}\,\widetilde{\underline{\mathrm{Cat}}}_T^{\delta} \to \mathsf{Span}_{\mathfrak{o}(T)}(\mathbb{F}_T)$ is a cocartesian fibration. Now, it is an *inert*-cocartesian fibration by Proposition II.3.7, and it has locally cocartesian lifts of active morphisms by Proposition II.3.8. We must verify conditions (b) and (c) of Lemma II.3.17; composition with inert morphisms is baked into our restriction-stability assumption, and composition with active arrows is Proposition II.3.8. We're left with the Segal condition that

$$\widetilde{\underline{\mathrm{Cat}}}_T^{\delta} \xrightarrow{\sim} \prod_{U \in \mathrm{Orb}(T)} \widetilde{\underline{\mathrm{Cat}}}_U^{\delta};$$

this follows by the pointwise expression for $\mathcal{T}$-limits in indexed products Proposition I.1.57.

Under Proposition II.2.54, we're left with showing that $f_\otimes\mathcal{C}$ is presentable whenever $\mathcal{C}$ is a presentable T-∞-category. But taking a presentation $\mathcal{C} \simeq L_{\mathcal{B}}\mathrm{Psh}_T(\mathcal{A})$ via Proposition I.1.48, we may apply Corollary II.3.12 and Proposition II.3.13 to arrive at an equivalence

$$f_\otimes L_{\mathcal{B}}\mathrm{Psh}_T(\mathcal{A}) \simeq L_{\mathcal{B}\cup \not\perp(\mathcal{A})^\simeq} f_\otimes \mathrm{Psh}_T(\mathcal{A}) \simeq L_{\mathcal{B}\cup \not\perp(\mathcal{A})^\simeq}\mathrm{Psh}_S(\mathcal{A}),$$

which is again presentable by Proposition I.1.48. □

II.3.1.5 Some basic properties of $\widetilde{\underline{\mathrm{Cat}}}^{\delta}_{\mathcal{T}}$ Before continuing, we prove some assorted facts.

Corollary II.3.18. *Suppose $\mathcal{C}$ is an $\mathcal{R}$-symmetric monoidal $\mathcal{T}$-category such that for any orbit $V \in \mathcal{T}$*

(1) for each binary fold map $\nabla\colon V \sqcup V \to V$, the tensor V-bifunctor

$$\otimes\colon \mathrm{Res}^{\mathcal{T}}_V\mathcal{C} \times \mathrm{Res}^{\mathcal{T}}_V\mathcal{C} \longrightarrow \mathrm{Res}^{\mathcal{T}}_V\mathcal{C}$$

is weakly distributive, and

(2) for each $\mathcal{R}$-admissable map $f\colon U \to V$, the norm f-multifunctor

$$N^V_U\colon \mathrm{CoInd}^V_U\mathrm{Res}^{\mathcal{T}}_U\mathcal{C} \longrightarrow \mathrm{Res}^{\mathcal{T}}_V\mathcal{C}$$

is weakly distributive.

Then, $\mathcal{C}$ is distributive.

Proof. Let $f\colon T \to S$ be an $\mathcal{R}$-admissable map of orbit-finite $\mathcal{T}$-spaces. First, note that the restriction of $f_\otimes\colon f_*\mathrm{Res}^{\mathcal{T}}_T\mathcal{C} \to \mathrm{Res}^{\mathcal{T}}_S\mathcal{C}$ along a map $g\colon S' \to S$ with pullback

$$\begin{array}{ccc} T' & \xrightarrow{g'} & T \\ \downarrow{\scriptstyle f'} & \lrcorner & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

is the map $(f')_\otimes\colon \mathrm{Res}^{\mathcal{T}}_{T'}\mathcal{C} \to \mathrm{Res}^{\mathcal{T}}_{S'}\mathcal{C}$, so it suffices to prove that all indexed tensor multifunctors of $\mathcal{C}$ are *weakly* distributive. Using the Segal condition, we can moreover reduce to the case where the target is a single orbit. Indeed, using associativity of tensor products, the proof of Proposition II.3.6 implies that all n-ary tensor multifunctors of $\mathcal{C}$ are weakly distributive; applying the orbit factorization

$$T = \coprod_{U\in\mathrm{Orb}(T)} U \to \coprod_{U\in\mathrm{Orb}(T)} V \to V$$

to decompose $\underline{f}_\otimes$ as a composition of n-ary tensor multifunctors with norm multifunctors, the result follows by another application of Proposition II.3.6. □

This powers an example; the proof of [NS22, Prop. 3.2.5] applies verbatim.

Proposition II.3.19. *Let $\mathcal{C}$ be a cocomplete $\mathcal{T}$-category and $\mathcal{D} \in \mathsf{Cat}$ be a cocomplete cartesian closed category. Then, $\mathsf{Coeff}^{\mathcal{T}}(\mathcal{C})^{\mathcal{R}-\times}$ is distributive.*

Lastly, we have the following.

Proposition II.3.20. *The unit of $\mathrm{Pr}^{L,\otimes}_{\mathcal{T}}$ is $\underline{\mathcal{S}}_{\mathcal{T}}$, and the binary tensor product computed by $\mathcal{C} \otimes \mathcal{D} \simeq \underline{\mathsf{Fun}}^{R}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$.*

Proof. The unit was proved in Corollary II.3.12. The binary case was pointed out under mildly more restrictive circumstances in [Hil24, Prop. 6.7.5], but the proof applies verbatim: apply Yoneda's lemma to the natural equivalence

$$\begin{aligned}
\underline{\mathsf{Fun}}^{L}_{\mathcal{T}}(\mathcal{C} \otimes \mathcal{D}, -) &\simeq \underline{\mathsf{Fun}}^{\delta}_{2^{*_{\mathcal{T}}} \to *_{\mathcal{T}}}(\mathcal{C} \times \mathcal{D}, -) \\
&\simeq \underline{\mathsf{Fun}}^{L}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathsf{Fun}}^{L}_{\mathcal{T}}(\mathcal{D}, -)\right) \\
&\simeq \underline{\mathsf{Fun}}^{R}_{\mathcal{T}}\left(\mathcal{C}^{\mathrm{op}}, \underline{\mathsf{Fun}}^{R,\mathrm{acc}}_{\mathcal{T}}(-, \mathcal{D})\right)^{\mathrm{op}} \\
&\simeq \underline{\mathsf{Fun}}^{R,\mathrm{acc}}_{\mathcal{T}}\left(-, \underline{\mathsf{Fun}}^{R}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})\right)^{\mathrm{op}} \\
&\simeq \underline{\mathsf{Fun}}^{L}_{\mathcal{T}}\left(\underline{\mathsf{Fun}}^{R}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}), -\right),
\end{aligned}$$

where every equivalence after the second is an application of Proposition I.1.49. □

II.3.1.6 Monoidality of the Yoneda embedding We point out that we've essentially proved a generalization of the main result of [NS22, § 6] from our perspective.

Corollary II.3.21. *The Yoneda embedding underlies a lax $\mathcal{T}$-symmetric monoidal left adjoint*

$$\mathsf{Psh}_{\mathcal{T}} : \widetilde{\underline{\mathsf{Cat}}}^{\times}_{\mathcal{T}} \rightleftarrows \widetilde{\underline{\mathsf{Cat}}}^{\delta}_{\mathcal{T}} : \iota;$$

moreover, よ is $\mathcal{T}$-symmetric monoidal. In particular, if $\mathcal{C}$ is $\mathcal{O}$-monoidal, then there exists an essentially unique $\mathcal{O}$-monoidal structure on $よ : \mathcal{C} \to \mathsf{Psh}_{\mathcal{T}}(\mathcal{C})$ such that $\mathsf{Psh}_{\mathcal{T}}(\mathcal{C})$ is distributively $\mathcal{O}$-monoidal and pullback along よ yields an equivalence

$$よ^{*} : \underline{\mathsf{Fun}}^{\otimes,L}_{\mathcal{O}}(\mathsf{Psh}_{\mathcal{T}}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \underline{\mathsf{Fun}}^{\otimes}_{\mathcal{O}}(\mathcal{C}, \mathcal{D}).$$

Proof. Note that よ is, by Proposition I.1.41, a fiberwise $\mathcal{T}$-left adjoint over $\mathsf{Span}_{\mathfrak{v}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$ to ι, with unit the Yoneda embedding, which lies in a fiber; hence it assembles to a $\mathsf{Span}_{\mathfrak{v}(\mathcal{T})}(\mathbb{F}_{\mathcal{T}})$-relative left adjoint to ι, which attains a canonical oplax structure under Proposition I.3.56. Unwinding this oplax structure map, it

has the form of the cocontinuous S-functor

$$\mathrm{Psh}_S(f_*\mathcal{C}) \to f_\otimes \mathrm{Psh}_T(\mathcal{C}),$$

induced under the universal property of S-presheaves by the canonical functor

$$f_*\mathcal{C} \to f_\otimes \mathrm{Psh}_T(\mathcal{C});$$

this is an equivalence by Corollary II.3.12, so $\curlywedge$ is $\mathcal{T}$-symmetric monoidal. The lax $\mathcal{T}$-symmetric monoidal adjunction then follows by Proposition I.3.56, and the resulting universal property follows by applying $\underline{\mathrm{Alg}}_{\mathcal{O}}(-)$ to this lax $\mathcal{T}$-symmetric monoidal adjunction. □

II.3.2 Equivalence with the Elmanto-Haugseng definition of distributivity Following Remark II.3.2, our next goal is to verify that distributivity is equivalent to a notion given the same name in [EH23]:

Construction II.3.22. Given a pair of arrows $A \xrightarrow{m} T \xrightarrow{f} S$ in $\mathcal{S}_{\mathcal{T}}$ such that f is $\mathfrak{o}(\mathcal{T})$-admissible, we acquire an essentially unique distributivity diagram (in the sense of [EH23, Def. 2.4.1]):

$$\Delta_{m,f} \coloneqq \begin{array}{ccc} & R & \xrightarrow{h} Q \\ {\scriptstyle\epsilon}\swarrow & {\scriptstyle\psi'}\downarrow & \lrcorner \quad \downarrow{\scriptstyle\psi = f_* m} \\ A & & \\ {\scriptstyle m}\searrow & T & \xrightarrow{f} S \end{array} \tag{29}$$

exhibiting ψ as the value of the right adjoint to f^* at m under f^*, i.e. inducing an equivalence

$$\mathrm{Map}_{/S}(\varphi, \psi) \longrightarrow \mathrm{Map}_{/T}(f^*\varphi, \psi') \xrightarrow{\epsilon^*} \mathrm{Map}_{/Y}(f^*\varphi, m)$$

for all maps $\varphi\colon P \to S$. Given $\mathcal{C}$ a $\mathcal{T}$-symmetric monoidal ∞-category, we acquire a commutative diagram

$$\Psi_{m,f} \coloneqq \begin{array}{ccc} \mathcal{C}(Q) & \xleftarrow{\psi^*} & \mathcal{C}(S) \\ {\scriptstyle h_\otimes}\uparrow & & \\ \mathcal{C}(R) & & \uparrow{\scriptstyle f_\otimes} \\ {\scriptstyle \epsilon^*}\uparrow & & \\ \mathcal{C}(A) & \xleftarrow[m^*]{} & \mathcal{C}(T) \end{array} \tag{30}$$

If $\mathcal{C}$ admits left adjoints $\psi_! \dashv \psi^*$ and $m_! \dashv m^*$, we acquire a Beck-Chevalley transformation

$$\beta_{m,f}\colon \psi_! h_\otimes \epsilon^* \longrightarrow f_\otimes m_!.$$ ◁

Definition II.3.23 (c.f. [EH23, Def. 2.4.10])**.** Given a collection of $\mathcal{T}$-spaces $\mathcal{M} \subset \underline{\mathcal{S}}_{\mathcal{T}}$ closed under $\mathcal{R}$-indexed products, we say $\mathcal{C} \in \mathrm{Cat}^{\otimes}_{\mathcal{R}}$ is *$\mathcal{M}$-distributive in the sense of Elmanto-Haugseng* if

(a) $\mathcal{C}$ fiberwise admits sifted colimits and its restrictions and indexed tensor functors preserve them;

(b) $\mathcal{C}$ admits $\mathcal{M}$-colimits; and

(c) for every distributivity diagram $\Delta_{m,f}$, the Beck-Chevalley transformation $\beta_{m,f}$ is an equivalence.

If $\mathcal{M} = \underline{\mathcal{S}}_{\mathcal{T}}$, we simply say that $\mathcal{C}$ is *distributive in the sense of Elmanto-Haugseng.* ◁

The goal of this subsection is to prove the following comparison theorem.

Theorem II.3.24. *An $\mathcal{R}$-symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is distributive in the sense of Elmanto-Haugseng if and only if it is distributive.*

This enables us to use [EH23] and descendent works, such as [LLP25]; for instance, [EH23, Thm. 1.4.1] yields a bispan presentation for distributive $\mathcal{R}$-symmetric monoidal ∞-categories.

Remark II.3.25. Condition (b) can be unwound to *existence* of left adjoints to pullback along the maps in $\mathcal{M}$ and the Beck-Chevalley conditions for restrictions thereof; hence it is equivalent to conditions (2) and (3) of [LLP25, Def. 3.12]. In particular, under Theorem I.4.1, our distributive $\mathcal{R}$-symmetric monoidal ∞-categories are identified with distributive $\mathcal{S}_{\mathcal{R}}$-normed $\mathcal{S}_{\mathcal{T}}$-categories in their sense. Hence [LLP25, Thm. 3.31] and [LLP25, Prop. 3.44] provide a miraculous identification of distributive $\mathcal{R}$-symmetric monoidal ∞-categories with $\underline{\mathbb{U}}^{\times}_{\mathcal{S}_{\mathcal{T}}}$-cocomplete $\mathsf{Span}_{\mathcal{R}}(\mathcal{S}_{\mathcal{T}})$-$\infty$-categories as defined in [LLP25, Def. 3.24]. We will later use this to interpret [LLP25, Thm. 3.39] as constructing $\mathcal{R}$-operadic left Kan extensions. ◁

II.3.2.1 Distributivity is compositional in the diagram We must show that distributivity is equivalent to distributivity for (nonparameterized) sifted diagrams and $\mathcal{T}$-spaces. This will be powered by a version of distributivity for $\mathcal{T}$-left Kan extensions:

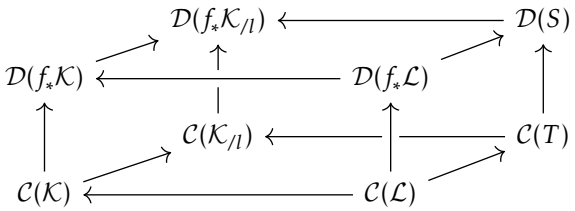
Definition II.3.26. Let $f\colon T \to S$ be an $\mathcal{R}$-admissable map of orbit-finite $\mathcal{T}$-spaces, $\mathcal{C}$ a cocomplete T-category, $\mathcal{D}$ a cocomplete S-category, and $F\colon \mathcal{C} \to \mathcal{D}$ an f-multifunctor. Moreover, let $i\colon \mathcal{K} \to \mathcal{L}$ be a T-functor. We say that F *weakly distributes left Kan extensions along* i if the following diagram is horizontally left-adjointable.

(31)
$$\begin{array}{ccc}
\mathcal{D}(f_*\mathcal{K}) & \xleftarrow{(f_*i)^*} & \mathcal{D}(f_*\mathcal{L}) \\
{\scriptstyle F}\uparrow \quad \nwarrow{\scriptstyle F} & & \\
\mathcal{C}(f^*f_*\mathcal{K}) \simeq f_*\mathcal{C}(f_*\mathcal{K}) & & \uparrow{\scriptstyle F} \\
{\scriptstyle \epsilon^*}\uparrow \quad \nearrow{\scriptstyle f_*} & & \\
\mathcal{C}(\mathcal{K}) & \xleftarrow[i^*]{} & \mathcal{C}(\mathcal{L})
\end{array}$$

In particular, the case $\mathcal{L} = T$ recovers distributivity via Remark II.3.2. ◁

Proposition II.3.27. *In the situation of Definition II.3.26, assume that the T-functor $i\colon \mathcal{K} \to \mathcal{L}$ is such that F is $\mathcal{M}$-distributive for $\mathcal{M} \subset \underline{\mathsf{Cat}}_{\mathcal{T}}$ a subcategory containing $\mathcal{K}_{/l}$ for all $l \in \mathcal{L}$. Then F distributes left Kan extensions along i.*

PROOF. Let $l\colon S \to f_*\mathcal{L}$ be an S-object, i.e. a T-object of $\mathcal{L}$. Consider the lax-commutative cube



where the top and bottom square only commute up to lax natural transformation coming from the projection of the slice onto both factors and the front and back square are the ones from definition of distributivity.

The back square is horizontally left-adjointable by distributivity. The top and bottom square are as well, by the pointwise formula for $\mathcal{T}$-left Kan extension. For the upper square, we additionally use that $f_*(\mathcal{K}_{/l}) \simeq (f_*\mathcal{K})_{/l}$, as f_* commutes with pullbacks and taking arrow categories.

This shows that the natural transformation of functor $\mathcal{C}(\mathcal{K}) \to \mathcal{D}(f_*\mathcal{L})$ is an equivalence when restricted along an arbitrary map $S \to f_*\mathcal{L}$. Applying restriction-stability of everything involved shows that the front square's Beck-Chevalley transformation for horizontal left-adjointability is an equivalence after restriction to all V-objects for all $V \to S$; these are jointly conservative, yielding the claim. □

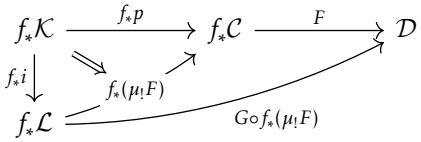
We also point out a simple enough property of this.

Lemma II.3.28. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is an f-multifunctor and $\mathcal{K} \xrightarrow{i} \mathcal{L} \xrightarrow{i'} \mathcal{J}$ a composable pair of T-functors such that F weakly distributes i and i'. Then, F weakly distributes $i' \circ i$.*

PROOF. Compose horizontally left-adjointable diagrams. □

Now, we may re-express this in a manner closer to our original framing.

Corollary II.3.29. *An f-multifunctor $F\colon \mathcal{C} \to \mathcal{D}$ is $\mathcal{M}$-distributive if and only if, for all diagrams*



such that the inner triangle is f_* applied to a T-left Kan extension whose pointwise formulas $\mathcal{K}_{/l} \in \mathcal{M}$ for all $A \in \mathcal{A}$, the outer triangle is an S-left Kan extension.

PROOF. Unwinding definitions, the assembly natural transformation $(f_*i)_!(F \circ f_*p) \implies F \circ f_*(\mu_! F)$ is precisely the Beck-Chevalley transformation for horizontal left adjointability of Eq. (31) applied to $p\colon \mathcal{K} \to \mathcal{C}$; in particular, our condition is equivalent to the conclusion of Proposition II.3.27 always holding, and hence it is implied by $\mathcal{M}$-distributivity. Specializing to $\mathcal{L} = \mathcal{T}$ shows that it implies $\mathcal{M}$-distributivity. □

II.3.2.2 Sifted distributivity To apply this, the correct notion of siftedness is the following.

Definition II.3.30. Given $f\colon T \to S$ a map of orbit-finite $\mathcal{T}$-spaces and $\mathcal{K}$ an S-category, we say that $\mathcal{K}$ is *f-sifted* if the diagonal S-functor

$$\Delta^f : \mathcal{K} \longrightarrow f_* f^* \mathcal{K}$$

is cofinal. More generally, we say that $\mathcal{K}$ is *$\mathcal{R}$-sifted* if it is f-sifted for all $\mathcal{R}$-admissible maps $f\colon T \to S$. ◁

If $\mathcal{R} = \mathcal{T}$ (requiring $\mathcal{T}$ itself atomic orbital), these are the *$\mathcal{T}$-sifted $\mathcal{T}$-categories* of [Sha22, Def. 8.14]. Let $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathcal{R}\text{-sift}} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ be the full $\mathcal{T}$-subcategory of $\mathcal{R}$-sifted categories. Moreover, let $\mathrm{Infl}_e^{\mathcal{T}} \mathrm{Cat}^{\mathrm{sift}} \subset \mathrm{Cat}$ be the category of non-parameterized/constant $\mathcal{T}$-categories with sifted value.

Proposition II.3.31. *Let $f\colon T \to S$ be an $\mathcal{R}$-admissable map of orbit-finite $\mathcal{T}$-spaces and $F\colon \mathcal{C} \to \mathcal{D}$ an f-multifunctor.*

(1) Suppose $\overline{m}\colon \mathcal{K}^{\triangleright} \to f_*\mathcal{C}$ is an S-colimit diagram such that $\mathcal{K}$ is f-sifted. Then, the S-functor underlying F preserves $\overline{m}$ if and only if f distributes the mate diagram $f^*\mathcal{K}^{\triangleright} \to \mathcal{C}$.

(2) F distributes non-parameterized sifted colimits if and only if the S-functor underlying F fixed-pointwise preserves sifted colimits.

(3) Suppose $\mathcal{M}$ contains all non-parameterized sifted categories, $\mathcal{M}' \subset \mathcal{M}$ is a collection of $\mathcal{T}$-spaces which generate $\mathcal{M}$ under non-parameterized sifted colimits, and that F is $\mathcal{M}'$-distributive and preserves (non-parameterized) sifted colimits. Then, F is $\mathcal{M}$-distributive.

Proof. For Statement (1), we closely mirror [Sha22, Prop. 8.19]: indeed, given an S-colimit diagram $\overline{m}\colon \mathcal{K}^{\triangleright} \to f_*\mathcal{C}$, the mate diagram $\overline{p}\colon (f^*\mathcal{K})^{\triangleright} \to \mathcal{C}$ is a T-colimit diagram. Consider the diagram

$$\begin{array}{ccccccc}
\mathcal{K}^{\triangleright} & \xrightarrow{\delta^S} & (f_*f^*\mathcal{K})^{\triangleright} & \xrightarrow{\mathrm{can}} & f_*(f^*\mathcal{K})^{\triangleright} & \xrightarrow{f_*\widetilde{p}} & f_*\mathcal{C} \xrightarrow{F} \mathcal{D} \\
\| & & & & & & \| \quad \| \\
\mathcal{K}^{\triangleright} & & \xrightarrow{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} & & & & f_*\mathcal{C} \xrightarrow{F} \mathcal{D}
\end{array}$$

Siftedness guarantees that the diagonal functor $\eta\colon \mathcal{K} \to f_*f^*\mathcal{K}$ is cofinal, so the top composite is an S-colimit if and only if the composite $(f_*f^*\mathcal{K})^{\triangleright} \to \mathcal{D}$ is an S-colimit; We are left with verifying that the lefthand rectangle commutes, after which we see that the bottom composite is a colimit diagram if and only if $\mathcal{D}$ distributes the designated colimit. But this follows by a routine application of the triangle identity. Statement (2) is a straightforward application of this, as non-parameterized sifted diagrams are $\mathcal{R}$-sifted by the evident extension of [Sha22, Thm. 8.15].

For Statement (3), let $\mathcal{K} \to \mathcal{C}$ be a diagram with $\mathcal{K} \in \mathcal{M}^T$. Then, by assumption, $\mathcal{K}$ is a $\mathcal{T}$-space valued colimit over a sifted diagram $\mathcal{L}$. Using parameterized unstraightening, there is cocartesian T-fibration $\mathcal{K} \to \mathrm{Infl}_e^{\mathcal{T}/T}\mathcal{L}$ such that the slices $\mathrm{Res}_{\mathcal{T}}^{V}\mathcal{K}_{/l}$ are spaces in $\mathcal{M}'$ for all $l \in \mathcal{L} = (\mathrm{Infl}_e^{\mathcal{T}/T}\mathcal{L})^V$. Now it follows from

Proposition II.3.27 that F distributes the left Kan extension along $\mathcal{K} \to \mathrm{Infl}_e^{\mathcal{T}_{/T}} \mathcal{L}$. It also follows from our assumption and Statement (2) that F distributes the left Kan extension along $\mathrm{Infl}_e^{\mathcal{T}_{/T}} \mathcal{L} \to T$. Lemma II.3.28 then implies that F distributes the left Kan extension along $\mathcal{K} \to *$, which is what we had to show. □

We immediately acquire the following corollary.

Corollary II.3.32. *Suppose $\mathcal{C}$ is a cocomplete $\mathcal{R}$-symmetric monoidal $\mathcal{T}$-category.*

(1) $\mathcal{C}$ is $\mathrm{Infl}_e^{\mathcal{T}} \mathrm{Cat}^{\mathrm{sift}}$-distributive if and only if, for each $\mathcal{R}$-admissible map of orbit-finite $\mathcal{T}$-sets $f : T \to S$, f-tensor functor preserves sifted colimits:

$$f_{\otimes} : \mathcal{C}^T \longrightarrow \mathcal{C}^S$$

(2) $\mathcal{C}$ is $\underline{\mathrm{Cat}}_{\mathcal{T}}^{\mathcal{R}-\mathrm{sift}}$-distributive if and only if, for each $\mathcal{R}$-admissible map of orbit finite $\mathcal{T}$-sets $f : T \to S$, the S-functor preserves $\mathcal{R}$-sifted $\mathcal{T}$-colimits:

$$\underline{f}_{\otimes} : f_* \mathcal{C}_{\underline{T}} \to \mathcal{C}_{\underline{S}}$$

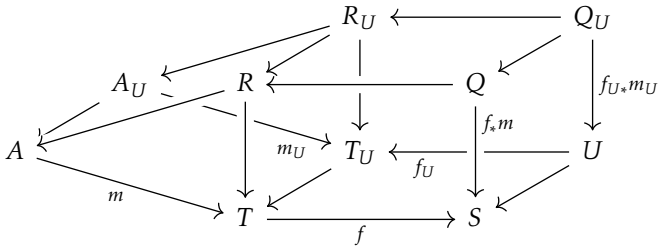
II.3.2.3 Comparing the notions To extend Theorem II.3.24, we actually prove the following.

Proposition II.3.33. *Suppose $\mathcal{C}$ is an $\mathcal{R}$-symmetric monoidal $\mathcal{T}$-category and $\mathcal{M} \subset \underline{\mathcal{S}}_{\mathcal{T}}$ is a collection of G-spaces which is closed under restrictions and indexed products. Then, $\mathcal{C}$ is $\mathcal{M} \cup \mathrm{Infl}_e^{\mathcal{T}} \mathrm{Cat}^{\mathrm{sift}}$-distributive if and only if it is $\mathcal{M}$-distributive in the sense of Elmanto-Haugseng.*

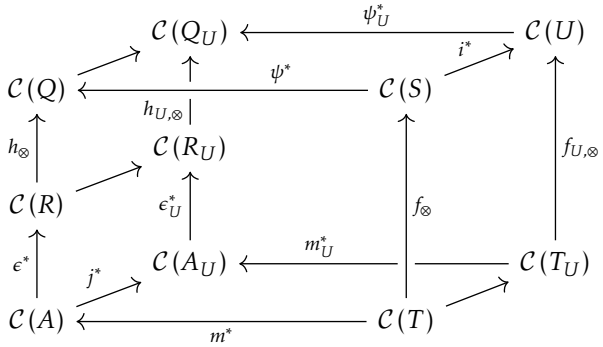
This immediately applies Theorem II.3.24 under Corollary II.3.32 and Proposition II.3.31. Now, in order to start, we simplify Definition II.3.23.(c).

Lemma II.3.34. *Suppose $\mathcal{C}$ satisfies Definition II.3.23.(a) and Definition II.3.23.(b) and $\beta_{m,f}$ is an equivalence whenever S is an orbit. Then, $\mathcal{C}$ is $\mathcal{M}$-distributive in the sense of Elmanto-Haugseng.*

Proof. Given a map from an orbit $U \to S$ and a diagram $A \xrightarrow{m} T \xrightarrow{f} S$ with $m \in \mathcal{M}(T)$ and f $\mathcal{R}$-admissible, we acquire a diagram



where every square is a pullback diagram and the front and back house-shaped faces are distributivity diagrams; that is, we acquire a natural transformation $\Delta_{m_U,f_U} \to \Delta_{m,f}$. This yields a natural transformation $\Psi_{m_U,f_U} \to \Psi_{m,f}$, i.e.



We claim that the two Beck-Chevalley conditions are directly related:

$$\begin{array}{ccc} i^*\psi_! h_\otimes \epsilon^* & \longrightarrow & i^*\psi_! h_\otimes \epsilon^* m^* m_! \\ \wr\downarrow & & \wr\downarrow \\ \psi_{U,!} h_{U,\otimes} \epsilon_U^* j^* & \longrightarrow & \psi_{U,!} h_{U,\otimes} \epsilon_U^* m_U^* m_{U,!} j^* \end{array} \quad \simeq \quad \begin{array}{ccc} i^*\psi_! \psi^* f_\otimes m_! & \longrightarrow & i^* f_\otimes m_! \\ \wr\downarrow & & \wr\downarrow \\ \psi_{U,!} \psi_U^* f_{U,\otimes} m_{U,!} j^* & \longrightarrow & f_{U,\otimes} m_{U,!} j^* \end{array} \tag{32}$$

Here, the top composite is $i^*\beta_{m,f}$ and the bottom is $\beta_{m_U,f_U} j^*$; commutativity follows by commutativity of Equation (32) together with the fact that $\mathcal{C}$ admits $\mathcal{M}$-indexed colimits.

By assumption, each $\beta_{m_U,f_U} j^*$ is an equivalence, implying that $i^*\beta_{m,f}$ is an equivalence for each $U \to S$. The Segal condition that the i assemble to an equivalence

$$\mathcal{C}(S) \xrightarrow{\sim} \lim_{U \to S} \mathcal{C}(U)$$

then implies that $\beta_{m,f}$ is an equivalence, completing the result. □

Proof of Proposition II.3.33. First suppose $\mathcal{C}$ is $\mathcal{M} \cup \mathrm{Infl}_e^{\mathcal{T}} \mathrm{Cat}^{\mathrm{sift}}$-distributive. By assumption, $\mathcal{C}$ satisfies Definition II.3.23.(b) and the cocompleteness part of Definition II.3.23.(a), and Proposition II.3.31.(1) shows that it satisfies the full Definition II.3.23.(a). Moreover, Remark II.3.2 shows that $\beta_{m,f}$ is an equivalence whenever S is an orbit, so $\mathcal{C}$ is $\mathcal{M}$-distributive in the sense of Elmanto-Haugseng by Lemma II.3.34.

Now, suppose $\mathcal{C}$ is $\mathcal{M}$-distributive in the sense of Elmanto-Haugseng. By assumption, $\mathcal{C}$ admits $\mathcal{M} \cup \mathrm{Infl}_e^{\mathcal{T}} \mathrm{Cat}^{\mathrm{sift}}$-colimits, and by Proposition II.3.31.(1), the indexed tensor functors of f distribute $\mathrm{Infl}_e^{\mathcal{T}} \mathrm{Cat}^{\mathrm{sift}}$-colimits. Moreover, Remark II.3.2 shows that the indexed tensor functors of f distributive $\mathcal{M}$-colimits. The result then follows by Proposition II.3.31.(3).

For the remaining statement, set $\mathcal{M} = \underline{\mathcal{S}}_{\mathcal{T}}$. In light of Proposition II.3.31.(1), it suffices to prove that $\underline{\mathcal{S}}_{\mathcal{T}}$ generates $\underline{\mathrm{Cat}}_{\mathcal{T}}$ under sifted colimits. But this follows by [Sha22, Thm. 8.6] by noting that their *corepresentable $\mathcal{T}$-left fibrations* are $\mathcal{T}$-spaces. □

II.3.2.4 Application: transfer ring examples This language makes the following corollary convenient.

Corollary II.3.35. *Fix $\varphi\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a conservative $\mathcal{O}$-monoidal functor which creates $\mathcal{T}$-colimits.*

(1) $\mathcal{C}^{\otimes}$ is distributive if and only if the essential image of φ is distributive.

(2) If $\mathcal{D}^{\otimes}$ is distributive, then so is $\mathcal{C}^{\otimes}$.

Proof. The assumptions guarantee that $\Psi_{m,f}$ of Eq. (30) is left adjointable in $\mathcal{C}$ if and only if the restriction of $\Psi_{m,f}$ to the essential image of $\mathcal{C}$ is left adjointable, giving the first claim. The second claim is clear, as left adjointability of $\Psi_{m,f}$ is closed under restriction along $\mathcal{T}$-cocontinuous fully faithful $\mathcal{O}$-monoidal functors. □

In particular, we immediately acquire the following technique for extending examples.

Corollary II.3.36. *Suppose $\mathcal{D}^{\otimes}$ is a distributive $\mathcal{O}$-monoidal ∞-category.*

(1) If $\mathcal{C}^{\otimes} \subset \mathcal{D}^{\otimes}$ is a $\mathcal{T}$-colimit closed $\mathcal{O}$-monoidal full subcategory, then $\mathcal{C}^{\otimes}$ is distributive.

(2) If $\mathcal{D}^{\otimes} \to \mathcal{E}^{\otimes}$ is an $\mathcal{O}$-monoidal localization, then $\mathcal{E}^{\otimes}$ is distributive; if moreover $\mathcal{D}^{\otimes}$ is presentable and the underlying $\mathcal{T}$-localization $L\colon \mathcal{D} \to \mathcal{E}$ is accessible, then $\mathcal{E}^{\otimes}$ is presentable.

(3) If $p\colon K \to \underline{\mathrm{Alg}}_{/\mathcal{O}}(\mathcal{D})$ is a diagram of $\mathcal{O}$-algebras, then the slice structure on $\mathcal{C}^{\otimes}_{/p}$ is distributive.

Now, we will need the following extension of this in our discussion of formal inversion modes.

Proposition II.3.37 (Sifted generation and distributivity)**.** *Fix $\mathcal{K} \subset \underline{\mathrm{Cat}}_{\mathcal{T}}$ containing all nonequivariant sifted diagrams. Suppose $\otimes\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is an $\mathcal{O}$-monoidal left $\mathcal{T}$-adjoint such that there exists a sifted diagram $\delta\colon K \to \mathcal{D}_{\mathrm{inv}} \times \mathsf{Fun}^{L}_{\mathcal{T}}(\mathcal{C},\mathcal{D})$ whose composite diagram*

$$K \to \mathsf{Fun}_{\mathcal{T}}(\mathcal{C},\mathcal{D}) \times \mathcal{D}_{\mathrm{inv}} \xrightarrow{R^*} \mathsf{Fun}_{\mathcal{T}}(\mathcal{D},\mathcal{D}) \times \mathcal{D}_{\mathrm{inv}} \xrightarrow{\otimes} \mathsf{Fun}_{\mathcal{T}}(\mathcal{D},\mathcal{D})$$

has colimit the identity, where $\mathcal{D}_{\mathrm{inv}} \subset \mathcal{D}$ is the full $\mathcal{T}$-subcategory of $\otimes$-invertible objects. If $\mathcal{C}^{\otimes}$ is $\mathcal{K}$-distributive and $\mathcal{D}^{\otimes}$ is $\mathsf{Infl}^{\mathcal{T}}_{e}\mathsf{Cat}^{\mathrm{sift}}$-distributive, then $\mathcal{D}^{\otimes}$ is $\mathcal{K}$-distributive.

Proof. We must show that, for $f\colon T \to V$ a map of orbit-finite $\mathcal{T}$-spaces with V an orbit,

$$f_{\otimes}\colon f_* f^* \mathcal{D} \longrightarrow \mathcal{D}$$

weakly distributes all K-indexed diagrams for all $K \in \mathcal{K}^T$; by restriction-stability of the assumptions, this will imply $\mathcal{K}$-distributivity.

By assumption, $f_\otimes$ weakly distributes all diagrams factoring through $f^*\varphi$; moreover, as $\underline{f}_\otimes$ is G-symmetric monoidal, for each $X \in \mathcal{D}^V_{\mathrm{inv}}$ we have a commutative diagram

$$\begin{array}{ccc} f_*f^*\mathcal{D} & \xrightarrow{f_\otimes} & \mathcal{D} \\ {\scriptstyle f_*(-\otimes X)}\downarrow & & \downarrow{\scriptstyle -\otimes X^{\otimes f}} \\ f_*f^*\mathcal{D} & \xrightarrow{f_\otimes} & \mathcal{D} \end{array}$$

where the vertical $\mathcal{T}$-functors are equivalences. From this we learn that, if the f-norm weakly distributes a diagram $m\colon K \to f^*\mathcal{D}$, then it weakly distributes $K \xrightarrow{m} f^*\mathcal{D} \xrightarrow{-\otimes X} f^*\mathcal{D}$.

Now, the weak distributivity condition for f-indexed tensors of K-indexed diagrams is equivalent to the Beck-Chevalley condition for horizontal left adjointability of the following square

$$\begin{array}{ccc} \mathrm{Fun}(f_*K, \mathcal{D})^V & \xleftarrow{\Delta^{f_*K}} & \mathcal{D}^V \\ \uparrow{\scriptstyle f_\otimes} & & \\ \mathrm{Fun}(f_*K, f_*f^*\mathcal{D})^V & & \\ \sim\uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle f_\otimes} \\ \mathrm{Fun}(f^*f_*K, f^*\mathcal{D})^T & & \\ \uparrow{\scriptstyle \eta^*} & & \\ \mathrm{Fun}(\mathcal{K}, f^*\mathcal{D})^T & \xleftarrow{\Delta^K} & (f_*f^*\mathcal{D})^V \end{array}$$

i.e. it stipulates that

$$\beta\colon \mathrm{colim}_{f_*K} f_\otimes \varphi\eta^* \longrightarrow \mathrm{colim}_{f_*K} f_\otimes \varphi\eta^*\Delta^K \mathrm{colim}_K \simeq \mathrm{colim}_{f_*K} \Delta^{f_*K} f_\otimes \mathrm{colim}_K \longrightarrow f_\otimes \mathrm{colim}_K\,.$$

is a natural equivalence (see Remark II.3.2). This is a natural transformation between sifted-cocontinuous functors, and in particular, we find that $f_\otimes$ distributes all diagrams $K \to f^*\mathcal{D}$ which can be written as sifted colimits of functors factoring through $X \otimes \varphi$ for some invertible X. But applying δ pointwise always gives such a sifted-colimit expression. □

II.3.3 Parameterized operadic left Kan extension In this section, we use Theorem II.3.24 to apply the *distributed left Kan extension* results of [LLP25] to our setting, as well as describe monoidality.

Theorem II.3.38. *Suppose $\varphi\colon \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ is a morphism of $\mathcal{R}$-operads and $\mathcal{C}$ is a distributive $\mathcal{R}$-symmetric monoidal ∞-category. Then, φ^* preserves $\mathcal{R}$-sifted $\mathcal{T}$-colimits and admits a $\mathcal{T}$-left adjoint*

$$\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \underset{\varphi^*}{\overset{\varphi_!}{\rightleftarrows}} \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}). \qquad (\dashv)$$

if for every $P \in \mathcal{P}_V$, the slice V-category $\mathrm{Env}_{\mathcal{R}}(\mathcal{O})_{\underline{V},/P}$ is $\mathcal{R}$-sifted, then the doctrinal adjunction oplax $\mathcal{R}$-symmetric monoidal structure on $\varphi_!$ is $\mathcal{R}$-symmetric monoidal.

An important and immediate corollary is the following.

Corollary II.3.39. *If $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is a π_0-equivalence of $\mathcal{R}$-operads and $\mathcal{C}^{\otimes}$ is a distributive $\mathcal{R}$-symmetric monoidal ∞-category, then*

$$\varphi^*\colon \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) \longrightarrow \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})$$

is a monadic right $\mathcal{T}$-adjoint; in particular, this is true of $U\colon \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{P}, \mathcal{C})$.

II.3.3.1 Colimits of algebras We start with $\mathcal{T}$-colimits in $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})$, following [NS22, Thm. 5.1.4].

Proposition II.3.40. *Suppose $\mathcal{Q}^{\otimes} \to \mathcal{O}^{\otimes}$ is a morphism of $\mathcal{R}$-operads, $\mathcal{K}$ a collection of $\mathcal{R}$-sifted diagrams, and $\mathcal{A}$ a $\mathcal{K}$-distributive $\mathcal{O}$-symmetric monoidal ∞-category. Then each of the following preserve $\mathcal{K}$-sifted colimits.*

$$\underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{A}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T},/\mathrm{Tot}_{\mathcal{T}}\mathcal{Q}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}, \mathrm{Tot}_{\mathcal{T}}\mathcal{A}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T},/U\mathcal{Q}}(U\mathcal{O}, U\mathcal{A});$$

moreover, in the case $\mathcal{K}$ contains $\mathrm{Infl}_{\ell}^{\mathcal{T}}\mathrm{Cat}^{\mathrm{sift}}$, the $\mathcal{T}$-∞-category $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})$ is cocomplete.

Proof. First note that each of the relative functor $\mathcal{T}$-categories is cocomplete by Proposition I.1.38, and the second arrow preserves all $\mathcal{T}$-colimits by Proposition I.1.38. The first arrow is fully faithful, so it suffices to prove that

Claim i. $\underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{A}) \subset \underline{\mathrm{Fun}}_{\mathcal{T},/\mathrm{Tot}_{\mathcal{T}}\mathcal{Q}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{Q}, \mathrm{Tot}_{\mathcal{T}}\mathcal{A})$ is *closed* under $\mathcal{K}$-sifted colimits, and

Claim ii. $\underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{A})$ admits finitary indexed coproducts.

Claim i follows by applying Proposition II.3.31 to conclude that cocartesian transport $\mathrm{Tot}_{\mathcal{T}}\mathcal{C}^{\otimes}$ preserves $\mathcal{K}$-colimits and applying Corollary I.1.39.

We will not use this proposition to construct operadic left Kan extensions, so we may safely assume they exist; that is, we have already proved that $U\colon \underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{A}) \to \underline{\mathrm{Fun}}_{\mathcal{T},/U\mathcal{O}}(U\mathcal{Q}, U\mathcal{A})$ is a monadic $\mathcal{T}$-right adjoint. As a result, we acquire a natural simplicial resolution

$$\mathrm{colim}\left(\cdots \rightrightarrows T_{\mathcal{O}}^2 X \rightrightarrows T_{\mathcal{O}} X\right) \xrightarrow{\sim} X,$$

where $T_{\mathcal{O}}$ is the monad associated with U; in particular, $\underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{A})$ is generated under nonequivariant sifted colimits by free algebras.

Now, the collection of S-tuples in an arbitrary $\mathcal{T}$-∞-category admitting a coproduct is closed under nonequivariant sifted colimits by commutativity of nonoequivariant sifted colimits with finite indexed coproducts; in particular, it suffices to prove that $\underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{A})$ admits finitary indexed coproducts by free

objects. Indeed, the universal property for free objects guarantees that $\iota_!\left(\coprod_W^S X_W\right) \simeq \coprod_U^S iota_!(X_W)$, where $\iota\colon \mathrm{triv}(U\mathcal{O})^\otimes \to \mathcal{O}^\otimes$ classifies the identity on $U\mathcal{O}$, yielding the claim. □

Now, we can conclude presentability.

Corollary II.3.41. *If $\mathcal{C}$ is presentably $\mathcal{O}$-monoidal, then $\underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C})$ is presentable.*

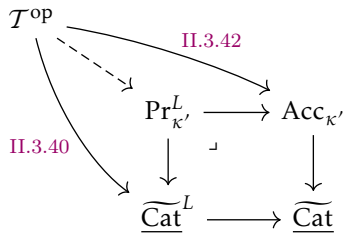
Now, to prove this, following Nardin-Shah following Lurie, we import the following.

Lemma II.3.42 (c.f. [HTT, Prop. 5.4.7.11] and [HTT, Rmk. 5.4.7.13])**.** *Let $\mathcal{C} \to \mathcal{D}$ be a cocartesian $\mathcal{T}$-fibration to a small $\mathcal{T}$-∞-category, such that each fiber $\mathcal{C}_D$ and each covariant transport $\mathcal{T}$-functor $\mathcal{C}_D \to \mathcal{C}_{D'}$ is κ-accessible. Then, given a set of edges $\mathcal{D}^{\mathrm{int}} \subset \mathcal{D}$, the $\mathcal{T}$-∞-category $\underline{\mathrm{Fun}}^{\mathrm{int-cocart}}_{\mathcal{T},/\mathcal{D}}(\mathcal{D},\mathcal{C})$ is κ-accessible.*

Proof of Corollary II.3.41. Fix κ such that $\mathcal{C}$ is κ-presentable, in the sense that the factorization of Proposition I.1.48 factors through Pr^L_κ. Using $\mathcal{R}$-sifted (hence filtered) cocontinuity of cocartesian transport in $\mathcal{C}^\otimes$ via Proposition II.3.31, Lemma II.3.42 applies to $\mathrm{Tot}_{\mathcal{T}}\mathcal{C}^\otimes \to \mathrm{Tot}_{\mathcal{T}}\mathcal{O}^\otimes$; as a consequence, we find that $\underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C})$ is fiberwise κ-accessible.

We must find some κ' such that each functor $\mathrm{Res}^V_U\colon \underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C})^V \to \underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C})^U$ preserves κ'-compact objects; equivalently, CoInd^V_U must commute with κ'-colimits. Fiberwise accessibility allows us to find such a cardinal for each map $U \to V$, so we may take an even larger cardinal to find such a κ' uniformly.

Now, $\mathrm{Acc}_{\kappa'} \subset \widetilde{\underline{\mathrm{Cat}}}$ for the subcategory of κ'-accessible ∞-categories and κ'-compact object preserving κ'-accessible functors and write $\widetilde{\underline{\mathrm{Cat}}}^L \subset \widetilde{\underline{\mathrm{Cat}}}$ for the wide subcategory of cocontinuous functors. It follows by the adjoint functor theorem that $\mathrm{Pr}^L_{\kappa'} \simeq \mathrm{Acc}_{\kappa'} \times_{\widetilde{\underline{\mathrm{Cat}}}} \widetilde{\underline{\mathrm{Cat}}}^L$, so we're provided a factorization



whose diagonal functor classifies $\underline{\mathrm{Alg}}_{\mathcal{Q}/\mathcal{O}}(\mathcal{C})$. Combining this with the cocompleteness of Proposition II.3.40 yields presentability under Proposition I.1.48. □

Warning II.3.43. In the case $\mathcal{O} = \mathrm{Comm}^\otimes_{\mathcal{T}}$, $\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{C})$ is presentable and $\underline{\mathrm{Alg}}^\otimes_{\mathcal{Q}}(\mathcal{C})$ is $\mathcal{T}$-symmetric monoidal, but $\underline{\mathrm{Alg}}^\otimes_{\mathcal{Q}}(\mathcal{C})$ is seldom *presentably $\mathcal{T}$-symmetric monoidal*.

A simple example is when $\mathcal{Q} = \mathrm{Comm}^\otimes_{\mathcal{T}}$ as well; then, $\underline{\mathrm{CAlg}}^\otimes(\mathcal{C})$ is cocartesian. Distributivity for binary tensor products then implies for any object X that $3 \cdot X \simeq (X \sqcup X) \sqcup X \simeq (X \sqcup X) \sqcup (X \sqcup X) \simeq 4 \cdot X$, which is almost never true. ◁

II.3.3.2 Operadic Kan extensions To start, given $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ an $\mathcal{R}$-symmetric monoidal functor, write

$$\mathcal{C}^S_{/d} \coloneqq \mathcal{C}^S \times_{\mathcal{D}^S} \mathcal{D}^S_{/d}.$$

We start by importing the import result of [LLP25], for which we need the following.

Proposition II.3.44. *The following conditions on $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ $\mathcal{R}$-symmetric monoidal are equivalent:*

(a) F is equifibered in the sense of Definition I.2.37,

(b) for all $\mathcal{R}$-admissible maps $f\colon T \to S$ of orbit-finite $\mathcal{T}$-spaces and T-tuples $(X_U) \in \mathcal{C}^T$, the functor $f_{\otimes}$ induces an equivalence $f_{\otimes}\colon f_*\left(\mathcal{C}^U_{/X_U} \mid U \subset T\right) \longrightarrow \mathcal{C}^V_{f_{\otimes}(X_U \mid U \subset T)}$, and

(c) Condition (b) holds when f is either a binary fold map or an orbit map.

Proof. It suffices to prove this *locally*, i.e. that F is equifibered over f if and only if Condition (b) holds for f. Then, Conditions (a) and (b) are made equivalent by unwinding definitions and Conditions (a) and (c) by using the reduction to elementary codomains and composition-closure of Proposition I.2.39 and Corollary I.2.40. □

Remark II.3.45. This is a mirror for non-cartesian structures of *disjunctiveness*; the reader would be justified in remembering this as a notion of disjunctive $\mathcal{R}$-symmetric monoidal functors. ◁

Now, Lenz-Linskens-Pützstück essentially prove the following.

Proposition II.3.46 (Modified and specialized [LLP25, Thm. 3.39])**.** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an $\mathcal{R}$-symmetric monoidal equifibration and $\mathcal{E}$ a distributive $\mathcal{R}$-symmetric monoidal ∞-category. Then F^* admits a left adjoint*

$$\underline{\mathrm{Fun}}^{\otimes}_{\mathcal{R}}(\mathcal{C},\mathcal{E}) \underset{F^*}{\overset{F_!}{\rightleftarrows}} \underline{\mathrm{Fun}}^{\otimes}_{\mathcal{R}}(\mathcal{D},\mathcal{E}) \quad (\dashv)$$

Moreover, the Beck-Chevalley transformation

$$\begin{array}{ccc} \underline{\mathrm{Fun}}^{\otimes}_{\mathcal{R}}(\mathcal{C},\mathcal{E}) & \longrightarrow & \underline{\mathrm{Fun}}^{\otimes}_{\mathcal{R}}(\mathcal{D},\mathcal{E}) \\ \downarrow & \nrightarrow & \downarrow \\ \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{C},\mathcal{E}) & \longrightarrow & \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{D},\mathcal{E}) \end{array}$$

is an equivalence.

Proof. It suffices to verify the assumptions of [LLP25, Lem. 3.40] under the identification with $\mathcal{S}_{\mathcal{R}}$-normed $\mathcal{S}_{\mathcal{T}}$-categories of Remark II.3.25. We've already identified the appropriate cocompleteness assumption on $\mathcal{E}$ over $\mathsf{Span}_{\mathcal{R}}(\mathcal{S}_{\mathcal{T}})$, so we are left with verifying their condition for slice categories.

We may proceed as in the proof of [LLP25, Lem. 3.42], in which case we have $T = E = \mathrm{Span}_{\mathcal{R}}(\mathcal{S}_{\mathcal{T}})$ and $M = \mathcal{S}_{\mathcal{T}}$. They show that the assumption of the previous lemma is fulfilled if the family of functors they denoted by (11) is an equivalence

$$\mathcal{C}(C) \times_{\mathcal{D}(C)} \mathrm{Ar}\mathcal{D}(C) \times_{\mathcal{D}(C)} \{\gamma^* d\} \longrightarrow \mathcal{C}(B) \times_{\mathcal{D}(B)} \mathrm{Ar}\mathcal{D}(B) \times_{\mathcal{D}(B)} \{e^*\gamma^* d\} \tag{33}$$

for all $d \in \mathcal{D}(C)$. Unwinding definitions, this is exactly the map appearing in Condition (b). □

Proof of Theorem II.3.38. By Corollary I.2.40 we know that $\mathrm{Env}_{\mathcal{R}}(\mathcal{O})^{\otimes} \to \mathrm{Env}_{\mathcal{R}}(\mathcal{P})^{\otimes}$ is equifibered, so Proposition II.3.46 proves that the diagram of $\mathcal{T}$-∞-categories

$$\begin{array}{ccc}
\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) & \xleftarrow{\varphi^*} & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) \\
\wr\downarrow & & \wr\downarrow \\
\mathrm{Fun}_J^{\otimes}\left(\mathrm{Env}_J(\mathcal{O})^{\otimes}, \mathrm{Bor}_J^I \mathcal{C}^{\otimes}\right) & \xleftarrow{\varphi^*} & \mathrm{Fun}_J^{\otimes}\left(\mathrm{Env}_J(\mathcal{P})^{\otimes}, \mathrm{Bor}_J^I \mathcal{C}^{\otimes}\right) \\
\downarrow U_{\mathrm{Env}} & & \downarrow U_{\mathrm{Env}} \\
\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Env}_J(\mathcal{O}), \mathcal{C}) & \xleftarrow[\mathrm{Env}_I(\varphi)^*]{} & \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathrm{Env}_J(\mathcal{P}), \mathcal{C})
\end{array}$$

is horizontally left adjointable, providing the desired left $\mathcal{T}$-adjoint.

Now, this proves that φ^* is accessible, but we would like to show that it further commutes with $\mathcal{R}$-sifted colimits. First, we may draw a pair of diagrams

$$\begin{array}{ccc}
\mathcal{P}^{\otimes} & \xleftarrow{\varphi} & \mathcal{O}^{\otimes} \\
{\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota} \\
\mathrm{triv}_{\mathcal{T}}(U\mathcal{P})^{\otimes} & \xleftarrow{U\varphi} & \mathrm{triv}_{\mathcal{T}}(U\mathcal{O})^{\otimes}
\end{array}
\quad \xmapsto{\underline{\mathrm{Alg}}_{(-)}(\mathcal{C})} \quad
\begin{array}{ccc}
\underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) & \xrightarrow{\varphi^*} & \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C}) \\
{\scriptstyle \iota^*}\downarrow & & \downarrow{\scriptstyle \iota} \\
\underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{P}, \mathcal{C}) & \xrightarrow{U\varphi^*} & \underline{\mathrm{Fun}}_{\mathcal{T}}(U\mathcal{O}, \mathcal{C})
\end{array}$$

whose right side is algebras in $\mathcal{C}$ taken along the left side. $\mathcal{R}$-sifted cocontinuity is automatic for the bottom arrow by Proposition I.1.38. Moreover, the vertical arrows are conservative; by cancellation, it suffices to prove that the vertical arrows are $\mathcal{R}$-sifted cocontinuous. This is simply Proposition II.3.40.

All that is left is to consider $\mathcal{R}$-symmetric monoidality. Of course, the oplax structure is supplied by Proposition I.3.56; by the Beck-Chevalley condition, the underlying natural transformation of the oplax structure map, $U_{\mathrm{Env}} o_{\varphi_!, \mathrm{Ind}_V^{\mathcal{T}} S \to V, (X_U)_{U \in \mathrm{Orb}(S)}}(P)$, is given by the map

$$\underline{\mathrm{colim}}_{O \in \mathrm{Env}_J(\mathcal{O})_{\underline{V},/P}} \bigotimes_U^S X_U(O) \longrightarrow \bigotimes_U^S \underline{\mathrm{colim}}_{O \in \mathrm{Env}_J(\mathcal{O})_{\underline{V},/P}} X_U(O)$$

adjunct the unit of the adjunction $\mathrm{Env}_J(\varphi)_! \dashv \mathrm{Env}_J(\varphi)^*$; unwinding definitions, it is the assembly map for the indicated $\mathcal{T}$-colimits in $\mathcal{C}$ along the functor $\otimes^S$. This is an equivalence by the $\mathcal{R}$-sifted assumption and Proposition II.3.31, so conservativity of U implies that $o_{\varphi_!, \mathrm{Ind}_V^{\mathcal{T}} S \to V, (X_U)}$ is a natural equivalence for all $S, (X_U)$, as desired. □

II.3.4 Parameterized Day convolution We lift an important technical theorem of Nardin-Shah (generalizing Lurie, generalizing Glasman) to our setting, as well as functoriality reminiscent of [BS24b, § 3].

Definition II.3.47. We say that a map of $\mathcal{T}$-operads $\varphi\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *exhibits* $\mathcal{C}^{\otimes}$ *as an* $\mathcal{O}$*-promonoidal* ∞*-category* if the associated $\mathcal{T}$-functor

$$\mathrm{Tot}_{\mathcal{T}} \varphi^{\mathrm{act}}\colon \mathrm{Tot}_{\mathcal{T}} \mathcal{C}^{\mathrm{act}} \longrightarrow \mathrm{Tot}_{\mathcal{T}} \mathcal{O}^{\mathrm{act}}$$

is exponentiable; for instance, this holds if $\mathcal{C}^{\otimes}$ is $\mathcal{O}$-monodial. ◁

Theorem II.3.48 (Day convolution)**.** *If* $\varphi\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *is* $\mathcal{O}$*-promonoidal, then pullback admits a right adjoint*

$$\underline{\mathrm{OP}}_{\mathcal{T},/\mathcal{O}^{\otimes}} \underset{\varphi_*}{\overset{\varphi^*}{\rightleftarrows}} \underline{\mathrm{OP}}_{\mathcal{T},/\mathcal{C}^{\otimes}}.$$

Writing $\mathrm{Day}_{\mathcal{O}}(\mathcal{C},-) \coloneqq \varphi_*\varphi^*\colon \mathrm{OP}_{\mathcal{T},/\mathcal{O}^{\otimes}} \to \mathrm{OP}_{\mathcal{T},/\mathcal{O}^{\otimes}}$*, if* $\mathcal{E}^{\otimes}$ *is a distributive (resp presentable)* $\mathcal{O}$*-monoidal* ∞*-category, then so is* $\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E})$*; moreover, when* $\mathcal{O}^{\otimes}$ *has one color, this assembles to a lift*

$$\begin{array}{ccc} \mathrm{Op}^{\mathrm{pro,op}}_{\mathcal{T},/\mathcal{O}} \times \mathrm{Alg}_{\mathcal{O}}\left(\mathrm{Pr}^L_{\mathcal{T}}\right) & \xrightarrow{\mathrm{Day}_{\mathcal{O}}(-,-)} & \mathrm{Alg}_{\mathcal{O}}\left(\mathrm{Pr}^L_{\mathcal{T}}\right) \\ \downarrow U & & \downarrow U \\ \mathrm{Cat}^{\mathrm{op}}_{\mathcal{T}} \times \mathrm{Pr}^L_{\mathcal{T}} & \xrightarrow{\underline{\mathrm{Fun}}_{\mathcal{T}}(-,-)} & \mathrm{Pr}^L_{\mathcal{T}} \end{array}$$

contravariantly natural in $\mathcal{O}$*, characterized by the formula for algebras for* $\psi\colon \mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$*:*

$$\underline{\mathrm{Alg}}_{\mathcal{P}/\mathcal{O}}\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) \simeq \underline{\mathrm{Alg}}_{\psi^*\mathcal{C}/\mathcal{P}}(\psi^*\mathcal{E}).$$

This last equation specifies that $\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E})$ is the $\mathcal{O}$-monoidal structure on $\underline{\mathrm{Fun}}_{\mathcal{T},/U\mathcal{O}}(\mathcal{C},\mathcal{E})$ whose $\mathcal{P}$-algebras are the lax $\mathcal{P}$-monoidal functors; the commutative diagram specifies that it is functorial under $\mathcal{T}$-cocontinuous $\mathcal{O}$-monoidal functors in $\mathcal{E}$ via postcomposition and under arbitrary $\mathcal{T}$-operad maps in $\mathcal{C}$ under $\mathcal{T}$-left Kan extension. The contravariant functoriality in $\mathcal{O}$ specifies that the Day convolution $\mathcal{P}$-monoidal structure underlies the Day convolution $\mathcal{O}$-monoidal structure; in particular, when $\mathcal{E}$ is $\mathcal{T}$-symmetric monoidal, one may simply study the undecorated $\mathrm{Day}(\mathcal{C},\mathcal{E}) \coloneqq \mathrm{Day}_{\mathrm{Comm}_{\mathcal{T}}}(\mathcal{C},\mathcal{E})$.

Remark II.3.49. Exponentiability of φ under promonoidality was recently proved for more general algebraic patterns [BLR26]. For completeness, we will still prove it.

Moreover, this was shown to be *sharp* for fibrous pattern maps over an algebraic patterns $\mathfrak{O}$ satisfying the property that $\mathfrak{O}^{\mathrm{el}}_{O/}$ admits an initial functor from a finite set for every $O \in \mathfrak{O}$ (i.e. it is a *weak* ∞*-operad* in the sense of [Har]) in [BLR26, Cor. 5.23]. We may see that this applies to $\mathrm{Tot}\,\underline{\mathrm{Fin}}_{\mathcal{T},*}$ by unwinding

Corollary I.4.34 (or directly applying [Ste25b, Lem. A.2]);[7] in particular, the first sentence Theorem II.3.48 may be strengthened to say this exponentiability is true *if and only if* $\mathcal{C}$ is $\mathcal{O}$-promonoidal. ◁

Our approach for Theorem II.3.48, influenced greatly by [NS22] (which is turn influenced greatly by [HA]), is the following expression for φ^*.

Observation II.3.50. Consider the span

$$\begin{array}{ccccc} & & \mathrm{Ar}^{\mathrm{int}}(\mathrm{Tot}^\flat \mathcal{O}) \times_{\mathrm{ev}_1, \mathrm{Tot}^\flat \mathcal{O}} \mathrm{Tot}^\flat \mathcal{C} & & \\ & \swarrow^{\mathrm{ev}_0} & & \searrow^{\pi_{\mathcal{C}}} & \\ \mathrm{Tot}^\flat \mathcal{O} & & & & \mathrm{Tot}^\flat \mathcal{C} \end{array} \tag{34}$$

Then, there is a natural equivalence

$$\mathrm{Cat}_{/\mathrm{Tot}^\flat \mathcal{O}} \underset{\varphi^*}{\overset{\pi_{\mathcal{C}!}\,\mathrm{ev}_0^*}{\Downarrow \sim}} \mathrm{Cat}_{/\mathrm{Tot}^\flat \mathcal{C}}$$

In particular, to construct φ_*, it suffices to prove that the Eq. (34) is a pairable span or algebraic patterns. ◁

We will postpone our proof of pairability until the end of the section, where it will follow mostly the same way as the proof in [NS22].

Proposition II.3.51. *In the situation of Theorem II.3.48, Eq. (34) is a pairable span of algebraic patterns.*

Before then, we will go through the rest of the proof, together with a number of examples.

II.3.4.1 Monoidality of Day convolution Our next task is distributivity, which we also prove by noting that Nardin-Shah's proof generalizes.

In general, a V-arrow $\underline{\Delta}^1 \longrightarrow \mathrm{Tot}_V \mathrm{Day}_{\mathcal{O}}(\mathcal{C}, \mathcal{E}) \simeq \varphi_* \varphi^* \mathrm{Tot}_V \mathcal{E}$ corresponds under the universal property of star-pushforward with a V-functor

$$\underline{\Delta}^1 \times_{\mathrm{Tot}_V \mathcal{O}} \mathrm{Ar}^{\mathrm{int}} \mathrm{Tot}_V \mathcal{O} \times_{\mathrm{Tot}_V \mathcal{O}} \mathrm{Tot}_V \mathcal{C} \simeq \varphi_! \varphi^* \underline{\Delta}^1 \to \mathrm{Tot}_V \mathcal{E};$$

its underlying $\mathrm{Tot}_V \mathcal{O}$-arrow corresponds with the postcomposition to $\mathrm{Tot}_V \mathcal{O}$, and its domain (resp. codomain) correspond with its two pullbacks to $\mathrm{Ar}^{\mathrm{int}} \mathrm{Tot}_V \mathcal{O} \times_{\mathrm{Tot}_V \mathcal{O}} \mathrm{Tot}_V \mathcal{C}$.

Notation II.3.52. Given $\mathcal{C}$ an $\mathcal{O}$-operad and $x \in \mathcal{O}_V$, we define the V-∞-category

$$\mathcal{C}_{\underline{x}} := \mathrm{Tot}_V \mathcal{C} \times_{\mathrm{Tot}_V \mathcal{O}} \{x\};$$

[7] To compare $\mathcal{T}$-exponentiability with exponentiability, note that the active arrows in $\mathrm{Tot}\,\mathrm{Tot}_{\mathcal{T}} \mathcal{O}$ lie in fibers; that is, the active wide subcategory is a natural disjoint union of the fiber V-categories of the active wide $\mathcal{T}$-subcategory, so the two exponentiability statements are equivalent.

more generally, if $\mu\colon x \to y$ is an arrow in $\mathcal{O}_V$, we define the V-∞-category

$$\mathcal{C}_{\underline{\mu}} \coloneqq \underline{\Delta}^1 \times_{\mu, \mathrm{Tot}_V \mathcal{O}} \mathrm{Tot}_V \mathcal{C}. \qquad \triangleleft$$

We prove the following mostly the same way as [NS22, Prop. 3.2.2].

Proposition II.3.53. *Let $\varphi\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be an $\mathcal{O}$-promonoidal ∞-category and $\mathcal{E}^\otimes$ an $\mathcal{O}$-monodial ∞-category. Let $\mu\colon x \to y$ be an active arrow in $\mathcal{O}$ of arity f.*

(1) There exists a natural equivalence $U\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) \simeq \widetilde{\underline{\mathrm{Fun}}}_{U\mathcal{O}}(U\mathcal{C}, U\mathcal{E})$ over $U\mathcal{O}$.

(2) An active V-arrow in $\mathrm{Tot}_{\mathcal{T}} \mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E})$ lying over $\mu\colon x \to y$ corresponds with a diagram

$$\begin{array}{ccc} \{x\} \times_{\mathrm{Tot}_V \mathcal{O}} \mathrm{Ar}^{\mathrm{int}} \mathrm{Tot}_V \mathcal{O} \times_{\mathrm{Tot}_V \mathcal{O}} \mathrm{Tot}_V \mathcal{C} & \xrightarrow{F} & \mathrm{Tot}_V \mathcal{E} \\ \downarrow & \overset{\overline{\mu}}{\nearrow} & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ \underline{\Delta}^1 \times_{a, \mathrm{Tot}_V \mathcal{O}} \mathrm{Ar}^{\mathrm{int}} \mathrm{Tot}_V \mathcal{O} \times_{\mathrm{Tot}_V \mathcal{C}} \mathrm{Tot}_V \mathcal{C} & \xrightarrow{\mu} & \mathrm{Tot}_V \mathcal{O} \end{array} \tag{35}$$

$\overline{\mu}$ is locally cocartesian if and only if this is a weak $\pi_{\mathcal{E}}$-relative V-left Kan extension diagram.

(3) $\overline{\mu}$ is locally cocartesian if and only if it restricts to a V-left Kan extension diagram

$$\begin{array}{ccccc} f_*\mathcal{C}_{\underline{x}} & \xrightarrow{(F_U)} & f_*\mathcal{E}_{\underline{x}} & \xrightarrow{\mu_\otimes} & \mathcal{E}_{\underline{y}}; \\ {\scriptstyle \mu_\otimes}\downarrow & & & \nearrow_{\widetilde{\mu}_\otimes F_U} & \\ \mathcal{C}_{\underline{\mu}} & & & & \end{array}$$

in this case, covariant transport is the composite arrow $\mu_\otimes F_U\colon \mathcal{C}_{\underline{y}} \hookrightarrow \mathcal{C}_{\underline{\mu}} \xrightarrow{\widetilde{\mu}_\otimes F_U} \mathcal{E}_{\underline{y}}$.

(4) If $U\mathcal{E} \to U\mathcal{O}$ has cocomplete fibers, then $\pi_{\mathcal{E}}$ is a locally cocartesian $\mathcal{T}$-fibration.

(5) If $\mathcal{E}^\otimes$ is a distributive $\mathcal{O}$-monoidal ∞-category, then so is $\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E})$.

(6) If $\omega\colon \mathcal{E}^\otimes \to \mathcal{D}^\otimes$ is a $\mathcal{T}$-cocontinuous map of distributive $\mathcal{O}$-monoidal ∞-categories, then so is

$$\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\omega)\colon \mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) \longrightarrow \mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{D}).$$

(7) In the case that $\mathcal{E}^\otimes$ is distributive and $\mathcal{C}$ is $\mathcal{O}$-monodial, covariant transport is V-left Kan extension:

$$\begin{array}{ccccc} f_*\mathcal{C}_{\underline{x}} & \xrightarrow{f_*F} & f_*\mathcal{E}_{\underline{x}} & \xrightarrow{\mu_\otimes} & \mathcal{E}_{\underline{y}}; \\ {\scriptstyle \mu_\otimes}\downarrow & & & \nearrow_{\mu_\otimes F} & \\ \mathcal{C}_{\underline{y}} & & & & \end{array}$$

Lemma II.3.54. *Suppose we have a cartesian square of $\mathcal{T}$-functors*

$$\begin{array}{ccc} \mathcal{A}' & \xhookrightarrow{\iota'} & \mathcal{B}' \\ \downarrow{\scriptstyle F'} & \lrcorner & \downarrow{\scriptstyle F} \\ \mathcal{A} & \xhookrightarrow{\iota} & \mathcal{B} \end{array}$$

whose vertical arrows are exponentiable and whose horizontal arrows are fully faithful. Then, the Beck-Chevalley natural transformation

$$\iota^* F_* \Longrightarrow F'_* \iota'^*$$

is a natural equivalence.

Proof. Given $\mathcal{T}$-functors $\mathcal{D} \to \mathcal{A}$ and $\mathcal{D}' \to \mathcal{B}'$, we have natural equivalences

$$\mathsf{Fun}_{\mathcal{T},/\mathcal{A}}(\mathcal{D}, \iota^* F_* \mathcal{D}') \simeq \mathsf{Fun}_{\mathcal{T},/\mathcal{B}'}(F^* \iota_! \mathcal{D}, \mathcal{D}')$$

$$\mathsf{Fun}_{\mathcal{T},/\mathcal{A}}(\mathcal{D}, F'_* \iota'^* \mathcal{D}') \simeq \mathsf{Fun}_{\mathcal{T},/\mathcal{B}'}\left(\iota'_! F'^* \mathcal{D}, \mathcal{D}'\right),$$

under which the Beck-Chevalley transformation corresponds with the Beck-Chevalley transformation

$$\beta \colon \iota'_! F'^* \Longrightarrow F^* \iota_!;$$

we're tasked with proving that β is a natural equivalence. But this is obvious. □

Proof of Proposition II.3.53. For Statement (1), as in the proof of [NS22, Prop. 3.1.9], noting that every inert arrow in $\mathrm{Tot}_{\mathcal{T}} \mathcal{O}$ with elementary domain is an equivalence, we acquire a diagram

$$\begin{array}{ccccccccc} & & U\mathcal{C} & = & U\mathcal{C} & \xrightarrow[U\varphi]{} & U\mathcal{O} & & \\ & \overset{U\varphi}{\nearrow} & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \iota_{U\mathcal{C}}} & & \downarrow{\scriptstyle \iota_{U\mathcal{O}}} & & \\ U\mathcal{O} \times_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}} \mathrm{Ar}^{\mathrm{int}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}) \times_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}} \mathrm{Tot}_{\mathcal{T}}\mathcal{C} & \underset{i}{\rightrightarrows} & \mathrm{Ar}^{\mathrm{int}}(\mathrm{Tot}_{\mathcal{T}}\mathcal{O}) \times_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}} \mathrm{Tot}_{\mathcal{T}}\mathcal{C} & \underset{\pi_{\mathcal{C}}}{\rightrightarrows} & \mathrm{Tot}_{\mathcal{T}}\mathcal{C} & \underset{\varphi}{\rightrightarrows} & \mathrm{Tot}_{\mathcal{T}}\mathcal{O} & \\ \downarrow{\scriptstyle \pi_{U\mathcal{O}}} & & \downarrow{\scriptstyle \mathrm{ev}_0} & & & & & & \\ U\mathcal{O} & & \xrightarrow{\qquad\qquad \iota \qquad\qquad} & & \mathrm{Tot}_{\mathcal{T}}\mathcal{O} & & & & \end{array}$$

and apply the Beck-Chevalley equivalence of Lemma II.3.54 to acquire an equivalence over $U\mathcal{O}$

$$\begin{aligned} U\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) &\simeq \iota^* \mathrm{ev}_{0*} \pi_{\mathcal{C}}^* \varphi^* \mathcal{E}^{\otimes} \\ &\simeq \pi_{U\mathcal{O}*} i^* \pi_{\mathcal{C}}^* \varphi^* \mathcal{E}^{\otimes} \\ &\simeq \pi_{U\mathcal{O}*} \iota_{U\mathcal{C}}^* \varphi^* \mathcal{E}^{\otimes} \\ &\simeq \pi_{U\mathcal{O}*} U\varphi^* \iota_{U\mathcal{O}}^* \mathcal{E}^{\otimes} \\ &\simeq U\varphi_* U\varphi^* U\mathcal{E} \\ &\simeq \underline{\widetilde{\mathrm{Fun}}}_{U\mathcal{O}}(U\mathcal{C}, U\mathcal{E}). \end{aligned}$$

For Statement (2), the correspondence between diagrams of the form of Eq. (35) and V-morphisms $\widetilde{\alpha}\colon F \to F'$ lying over α was described above; in particular, for $\widetilde{\alpha}$ to be locally cocartesian (i.e. initial among all such lifts) is equivalent to $\widetilde{\alpha}$ being initial among all such fillers. [Sha22, Thm. 6.2] establishes this as precisely the universal property of *weak* $\pi_{\mathcal{E}}$-relative V-kan extension diagrams, giving the claim.

For Statement (3), we will show that the two have the same pointwise formulas. Indeed, write $\widetilde{\mathcal{C}}_{\underline{x}}$ and $\widetilde{\mathcal{C}}_{\underline{\mu}}$ for the top and bottom term of the lefthand side, so that the pointwise formula demands a weak $\pi_{\mathcal{E}}$-relative V-colimit diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{C}}^{/\ell}_{\underline{x}} & & \longrightarrow & & \mathrm{Tot}_V \mathcal{E} \\ \downarrow & & & & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ \left(\widetilde{\mathcal{C}}^{/\ell}_{\underline{x}}\right)^{\triangleright} & \longrightarrow & \left(\widetilde{\mathcal{C}}^{/\ell}_{\underline{\mu}}\right)^{\triangleright} & \longrightarrow \widetilde{\mathcal{C}}_{\underline{\mu}} \longrightarrow & \mathrm{Tot}_V \mathcal{O} \end{array}$$

Now, $\pi_{\mathcal{E}}$ is a cocartesian V-fibration by assumption, so [Sha22, Prop. 5.8] gives a pointwise formula for this relative colimit diagram: informally, we may apply cocartesian transport along every morphism pointing to the cocone point to replace this with a diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{C}}^{/\ell}_{\underline{x}} & & \longrightarrow & & \mathrm{Tot}_V \mathcal{E} \\ \downarrow & & & & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ \left(\widetilde{\mathcal{C}}^{/\ell}_{\underline{x}}\right)^{\triangleright} & \longrightarrow & \{y\} & \longrightarrow & \mathrm{Tot}_V \mathcal{O} \end{array}$$

Then, [Sha22, Lem. 5.6] identifies weak relative colimits over constant diagrams as colimits in the fiber, i.e. this is a weak relative colimit diagram if and only if it is refines to a V-colimit diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{C}}^{/\ell}_{\underline{x}} & \longrightarrow & \mathcal{E}_{\underline{y}} & \longrightarrow & \mathrm{Tot}_V \mathcal{E} \\ \downarrow & \nearrow & \downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\ \left(\widetilde{\mathcal{C}}^{/\ell}_{\underline{x}}\right)^{\triangleright} & \longrightarrow & \{y\} & \longrightarrow & \mathrm{Tot}_V \mathcal{O} \end{array}$$

Now, we may employ an observation mentioned in [NS22]: whenever $\ell \in \mathcal{C}_{\underline{\mu}}$, we acquire equivalences:

$$\begin{array}{ccccccc}
\mathcal{C}_{\underline{x}}^{/\ell} & \simeq & \widetilde{\mathcal{C}}_{\underline{x}}^{/\ell} & \longrightarrow & \mathcal{E}_{\underline{y}} & \longrightarrow & \mathrm{Tot}_V \mathcal{E} \\
\downarrow & & \downarrow & \nearrow & \downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{E}}} \\
\left(\mathcal{C}_{\underline{x}}^{/\ell}\right)^{\triangleright} & \simeq & \left(\widetilde{\mathcal{C}}_{\underline{x}}^{/\ell}\right)^{\triangleright} & \longrightarrow & \{y\} & \longrightarrow & \mathrm{Tot}_V \mathcal{O}
\end{array}$$

Moreover, this composite lifting problem is precisely that for the pointwise formula for our distinguished V-left Kan extension, yielding the "only if" direction. To prove the "if" direction, we have to prove that Eq. (35) is a weak $\pi_{\mathcal{E}}$-relative V-left Kan extension diagram whenever it satisfies the pointwise formula for $\ell \in \mathcal{C}_{\underline{\mu}} \subset \widetilde{\mathcal{C}}_{\underline{\mu}}$. To do so, as in [NS22, Prop. 3.2.2], note that a W-object in $\widetilde{\mathcal{C}}_{\underline{\mu}}$ comes in one of three forms:

(i) a W-object of $\widetilde{\mathcal{C}}_{\underline{x}}$,

(ii) a W-object of $\mathcal{C}_{\underline{\mu}}$, or

(iii) a pair of a morphism $y \to p(e)$ and a lift $e \in \mathcal{E}_{p(e)}$ such that $\pi p(e) = \varnothing_W$.

In the first case there is nothing to do; in the third case, $\mathcal{E}_{p(e)}$ is contractible, so the pointwise formula is always satisfied. In the second case, we have already assumed that the pointwise formula is satisfied, giving Statement (3). Then, Statement (4) follows from the pointwise formula for V-left Kan extension.

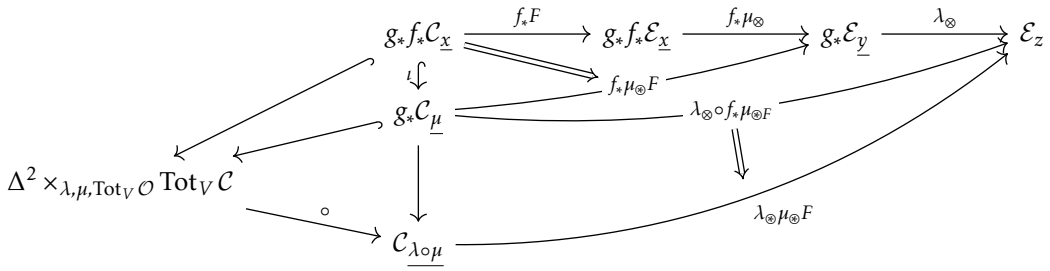
To conclude Statement (7), again following [NS22, Prop 3.2.2], we perform the usual trick for expressing arbitrary Kan extension as Kan extension along full subcategory inclusions: the projection $\mathcal{C}_{\underline{\mu}} \to \underline{\Delta}^1$ is a cocartesian V-fibration whose fibers over $\{0\}$ and $\{1\}$ are $f_* \mathcal{C}_{\underline{x}}$ and $\mathcal{C}_{\underline{y}}$, so [Sha22, Rmk. 2.14] shows that the inclusions *create V-Kan extensions*, in the sense that given a diagram

$$\begin{array}{ccccc}
f_* \mathcal{C}_{\underline{x}} & = & f_* \mathcal{C}_{\underline{x}} & \xrightarrow{G} & \mathcal{D} \\
\downarrow{\scriptstyle \mu_{\otimes}} & & \downarrow{\scriptstyle \iota_x} & \searrow & \\
\mathcal{C}_{\underline{y}} & \xhookrightarrow[\iota_y]{} & \mathcal{C}_{\underline{\mu}} & &
\end{array}$$

the assembly transformation $\mu_{\otimes !} G \implies \iota_{x!} G \circ \iota_y$ is an equivalence. The claim then follows by applying two-out-of-three to identify the assmebly transformations for the two triangle diagrams.

Now, for Statement (5), $\mathcal{O}$-monoidality is equivalent to compositionality of locally cocartesian arrows; applying Statement (3), we're tasked with verifying that the outer triangle in the following diagram is a

V-left Kan extension diagram, where f is the arity of λ:



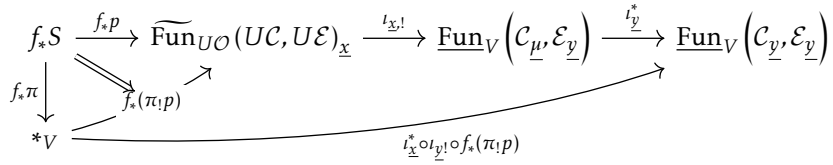
By composing parameterized left Kan extensions, it suffices to prove that coassembly is a natural equivalence

$$\iota_! (\lambda_\otimes \circ f_* \mu_\otimes \circ f_* F) \implies \lambda_\otimes \circ \iota_! (f_* \mu_\otimes \circ f_* F);$$

this is implied by distributivity of $\mathcal{E}^\otimes$ via Corollary II.3.29.

Now, we have an $\mathcal{O}$-monoidal structure on $\mathrm{Day}_{\mathcal{O}}(\mathcal{C}, \mathcal{E})$, and we must argue that it is distributive. Applying Proposition II.3.31, it suffices to prove that it distributes $\mathcal{R}$-sifted $\mathcal{T}$-colimits and $\mathcal{R}$-indexed coproducts separately. $\mathcal{R}$-sifted distributivity is easy; by Proposition II.3.31 it is equivalent to sifted cocontinuity of the tensor multifunctors, and it follows by unwinding definitions that f_* is $\mathcal{R}$-sifted cocontinuous and the relevant pre- and post-composition functors $\mathcal{T}$-cocontinuous.

For $\underline{\mathbb{F}}_{\mathcal{R}}$-distributivity, draw the diagram



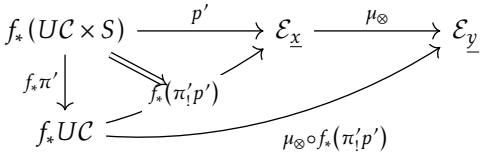
Then, the distributivity arrow is the assembly transformation

$$(f_* p)_! \left(\iota_{\underline{y}}^* \circ \mu_{\otimes,!} \circ f_* p \right) \implies \iota_{\underline{y}}^* \circ \mu_{\otimes,!} \circ f_* (\pi_! p);$$

of course, $\iota_{\underline{y}}^*$ is V-colimit-preserving, so it suffices to prove that the transformation

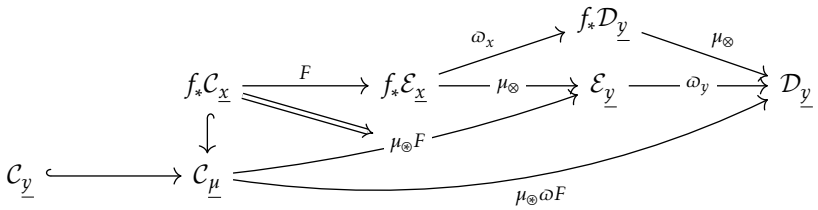
$$(f_* p)_! (\mu_{\otimes,!} \circ f_* p) \implies \mu_{\otimes,!} \circ f_* (\pi_! p)$$

is an equivalence. The universal property for pairing identifies this with the assembly arrow for a triangle



whose inner triangle is f_* applied to an S-left Kan extension diagram. Indeed, this transformation is an equivalence by Corollary II.3.29.

Lastly, for Statement (6), using Statement (4) we find that the lax structure map of ϖ is given by the assembly arrow, restricted to $\mathcal{C}_{\underline{y}}$, of the following diagram



said another way, to verify $\mathcal{O}$-monoidality, it suffices to show that postcomposition with ϖ_y preserves V-left Kan extensions; $\mathcal{T}$-cocontinuity of ϖ suffices. □

II.3.4.2 Functoriality of Day convolution in the base Now, in general given a map $\psi\colon \mathcal{P}^\otimes \to \mathcal{O}^\otimes$ we may expand this into a cartesian square

$$\begin{array}{ccc} \psi^*\mathcal{C}^\otimes & \xrightarrow{\varphi'} & \mathcal{P}^\otimes \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} \\ \mathcal{C}^\otimes & \xrightarrow{\varphi} & \mathcal{O}^\otimes \end{array}$$

Lemma II.3.55. *Given a map $\psi\colon \mathcal{P}^\otimes \to \mathcal{O}^\otimes$ and an $\mathcal{O}$-promonoidal ∞-category $\mathcal{C}$, $\psi^*\mathcal{C}^\otimes$ is a $\mathcal{P}$-promonoidal ∞-category.*

Proof. φ' restricts on active arrows to the pullback of φ^{act}, so it suffices to prove that exponentiable arrows are closed under pullback. But exponentiability is measured fiberwise by Corollary I.1.54, so it reduces to the nonequivariant case, which is [kerodon, Tag 05MQ]. □

Applying right adjoints to pullback on horizontal arrows yields a Beck-Chevalley transformation

$$\mathrm{Day}_{\mathcal{P}}(\psi^*\mathcal{C}, \psi^*-) \simeq \varphi'_*\varphi'^*\psi^*(-) \simeq \varphi'_*\psi'^*\varphi^*(-) \xrightarrow{\beta\varphi^*} \psi^*\varphi_*\varphi^*(-) \simeq \psi^*\mathrm{Day}_{\mathcal{P}}(\mathcal{C}, -). \tag{36}$$

We use this to show naturality of Day convolution in the base.

Proposition II.3.56. *Suppose $\varphi\colon \mathcal{P}^\otimes \to \mathcal{O}^\otimes$ is a U-equivalence and $\mathcal{E}^\otimes$ is a distributive $\mathcal{O}$-monoidal ∞-category. Then the lax $\mathcal{P}$-monoidal functor*

$$\gamma\colon \mathrm{Day}_{\mathcal{P}}(\psi^*\mathcal{C}, \psi^*\mathcal{E}) \longrightarrow \psi^*\mathrm{Day}_{\mathcal{O}}(\mathcal{C}, \mathcal{E})$$

of Eq. (36) is an equivalence of $\mathcal{P}$-monoidal ∞-categories.

PROOF. Applying Statement (3) and unwinding definitions shows that γ is $\mathcal{P}$-monoidal, so it suffices to prove that it is a U-equivalence.

Now, note for a general $\mathcal{O}$-operad $\mathcal{P}^{\otimes}$ that the map $U\mathcal{P} \to U\mathcal{O}$ classifies a $\mathcal{T}$-operad map $\mathsf{triv}(U\mathcal{P})^{\otimes} \to \mathcal{O}^{\otimes}$, and moreover that the following square is cartesian:

$$\begin{array}{ccc} \mathsf{triv}(U\mathcal{P})^{\otimes} & \longrightarrow & \mathcal{P}^{\otimes} \\ \downarrow & \lrcorner & \downarrow \\ \mathsf{triv}(U\mathcal{O})^{\otimes} & \xrightarrow{i_{\mathcal{O}}} & \mathcal{O}^{\otimes} \end{array}$$

That is, the sliced underlying $\mathcal{T}$-category functor $\mathrm{Op}_{\mathcal{T},/\mathcal{O}^{\otimes}} \to \mathrm{Cat}_{\mathcal{T},/U\mathcal{O}}$ is given by $i_{\mathcal{O}}^*$ under the equivalence $\mathrm{Cat}_{\mathcal{T},/U\mathcal{O}} \simeq \mathrm{Op}_{\mathcal{T},/\mathsf{triv}(U\mathcal{O})^{\otimes}}$. Applying compositionality of Beck-Chevalley conditions to the above square yields a diagram

$$\begin{array}{ccccc} \mathrm{Day}_{\mathsf{triv}(\mathcal{P})}(\mathcal{C},\mathcal{E}) & \xrightarrow{\gamma_{i_{\mathcal{P}}}} & i_{\mathcal{P}}^*\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) & \xrightarrow{\sim} & U\mathrm{Day}_{\mathcal{P}}(\mathcal{C},\mathcal{E}) \\ \downarrow{\scriptstyle \gamma_{\mathsf{triv}(U\psi)}} & & \downarrow{\scriptstyle i_{\mathcal{P}}^*\gamma_{\psi}} & & \downarrow{\scriptstyle U\gamma_{\psi}} \\ & & i_{\mathcal{P}}^*\psi^*\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) & \xrightarrow{\sim} & U\psi^*\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) \\ & & \wr & & \\ \mathsf{triv}(U\psi)^*\mathrm{Day}_{\mathsf{triv}(\mathcal{O})}(\mathcal{C},\mathcal{E}) & \xrightarrow[\mathsf{triv}(U\psi)^*\gamma_{i_{\mathcal{O}}}]{} & \mathsf{triv}(U\psi)^*i_{\mathcal{O}}^*\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) & & \end{array}$$

where we have suppressed internal star-pullbacks for readability. Applying two-out-of-three, it then suffices to prove our claim in two cases:

Case i the case that $\mathcal{P}^{\otimes} \simeq \mathsf{triv}(U\mathcal{O})^{\otimes}$, or

Case ii the case that $\mathcal{P}^{\otimes}$ and $\mathcal{O}^{\otimes}$ are trivial.

By our previous analysis, the first case corresponds with the claim that $U\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) \simeq \widetilde{\mathrm{Fun}}_{U\mathcal{O}}(U\mathcal{C},U\mathcal{E})$ of Statement (1). Moreover, under this equivalence, the second case amounts to verifying an equivalence

$$\widetilde{\mathrm{Fun}}_{U\mathcal{P}}(U\mathcal{C}\times_{U\mathcal{O}}U\mathcal{P},U\mathcal{E}\times_{U\mathcal{O}}U\mathcal{P}) \simeq \psi^*\widetilde{\mathrm{Fun}}_{U\mathcal{O}}(U\mathcal{C},U\mathcal{E})$$

for $\mathcal{C},\mathcal{E},\mathcal{P} \in \mathrm{Cat}_{\mathcal{T},/U\mathcal{O}}$; this is Proposition I.1.56. □

II.3.4.3 Algebras and functoriality of Day convolution in the domain This Beck-Chevalley condition allows us to describe algebras in this case.

Corollary II.3.57. *If $\mathcal{P}^{\otimes} \to \mathcal{O}^{\otimes}$ is a U-equivalence and $\mathcal{E}^{\otimes}$ a distributive $\mathcal{O}$-monoidal ∞-category, then*

$$\underline{\mathrm{Alg}}_{\mathcal{P}/\mathcal{O}}\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) \simeq \underline{\mathrm{Alg}}_{\psi^*\mathcal{C}/\mathcal{P}}(\pi^*\mathcal{E}),$$

naturally in $\mathcal{P}$. In particular, we have a commutative diagram of equivalences

$$\begin{array}{ccc} \underline{\mathrm{Alg}}_{/\mathcal{O}}\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) & \xrightarrow{\sim} & \underline{\mathrm{Alg}}_{\mathcal{C}/\mathcal{O}}(\mathcal{E}) \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ U\mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) & \xrightarrow{\sim} & \underline{\mathrm{Fun}}_{\mathcal{T},/\underline{\mathcal{O}}}(U\mathcal{C},U\mathcal{E}) \end{array}$$

Proof. After Proposition II.3.56 it suffices to prove this in the case $\mathcal{P}^{\otimes} \simeq \mathcal{O}^{\otimes}$. But this follows by abstract nonsense, as

$$\begin{aligned} \underline{\mathrm{Alg}}_{/\mathcal{O}}\varphi_*\varphi^*\mathcal{E} &\simeq \underline{\mathrm{Alg}}_{\varphi_!\varphi^*\mathcal{O}/\mathcal{O}}\mathcal{E} \\ &\simeq \underline{\mathrm{Alg}}_{\mathcal{C}/\mathcal{O}}\mathcal{E}. \end{aligned}$$

□

Now having addressed functoriality in two of the three variables for Day convolution, we are left with discussing functoriality in the third, the promonoidal one. This is not too hard; by the previous corollary, to construct a natural lax $\mathcal{O}$-monodial functor

$$\mathrm{Day}_{\mathcal{O}}(\mathcal{C},-) \to \mathrm{Day}_{\mathcal{O}}(\mathcal{B},-),$$

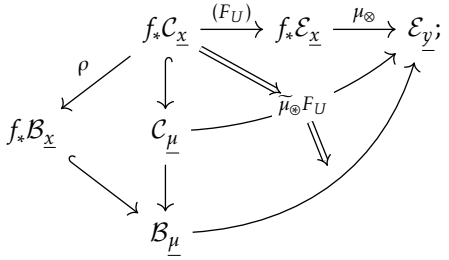
it is equivalent to construct a natural transformation

$$\underline{\mathrm{Alg}}_{\mathcal{C}/\mathcal{O}}(-) \Longrightarrow \underline{\mathrm{Alg}}_{\mathcal{B}/\mathcal{O}}(-);$$

following Theorem II.3.38, we may choose $\mathcal{O}$-operadic left Kan extension.

Proposition II.3.58. *The $\mathcal{O}$-operadic left Kan extension lax $\mathcal{O}$-monoidal functor $F_!\colon \mathrm{Day}_{\mathcal{O}}(\mathcal{C},\mathcal{E}) \to \mathrm{Day}_{\mathcal{O}}(\mathcal{B},\mathcal{E})$ is $\mathcal{O}$-monoidal.*

Proof. Once more, this follows by examining Statement (3); unwinding definitions and using commutativity of the lefthand diagram, the lax structure map for μ is the assembly transformation of outer triangle for the following diagram



Of course, this is an equivalence by transitivity of V-left Kan extensions. □

II.3.4.4 Pairability, and putting the proof together

PROOF OF PROPOSITION II.3.51. As in [NS22], we have a morphism of spans

$$\begin{array}{ccccc} \mathrm{Tot}^{\flat}\mathcal{O} & \xleftarrow{\mathrm{ev}_0} & \mathrm{Ar}^{\mathrm{int}}(\mathrm{Tot}^{\flat}\mathcal{O}) \times_{\mathrm{ev}_1,\mathrm{Tot}^{\flat}\mathcal{O}} \mathrm{Tot}^{\flat}\mathcal{C} & \xrightarrow{\pi_{\mathcal{C}}} & \mathrm{Tot}^{\flat}\mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Tot}^{\mathrm{big}}\mathcal{O} & \xleftarrow{\mathrm{ev}_0} & \mathrm{Ar}^{\mathrm{int}}(\mathrm{Tot}^{\mathrm{big}}\mathcal{O}) \times_{\mathrm{ev}_1,\mathrm{Tot}^{\mathrm{big}}\mathcal{O}} \mathrm{Tot}^{\mathrm{big}}\mathcal{C} & \xrightarrow{\pi_{\mathcal{C}}} & \mathrm{Tot}^{\mathrm{big}}\mathcal{C} \end{array}$$

whose left and right vertical morphisms are Morita equivalences by Theorem I.4.1. In particular, it suffices to prove that the upper span is pairable.

To prove this, we verify the conditions of Proposition I.2.60, mostly by analogous proofs to the paragraphs of the same numbers appearing in [NS22, Prop. 3.1.4]. For completeness, we include those proofs here, improving or simplifying whenever possible.

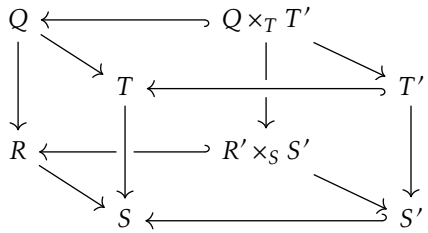
For Condition (p-1), Shah wrote his work on promonoidal Day convolution to be sufficiently general. In particular, we may simply apply [Sha22, Lem. 10.1] for the inert-active factorization system on $\mathrm{Tot}\,\mathrm{Tot}^{\mathrm{big}}\mathcal{O}$.

Likewise for Condition (p-4), part (1) of [Sha22, Prop. 3.5] shows that $\mathrm{ev}_0 \colon \mathrm{Ar}^{\mathrm{int}}\mathrm{Tot}^{\mathrm{big}}\mathcal{O} \to \mathrm{Tot}^{\mathrm{big}}\mathcal{O}$ is a cartesian fibration with cartesian edges those squares who evaluate to active arrows (equivalently, to equivalences) under ev_1.

Now, Condition (p-5') demonstrates why we (and Nardin-Shah) chose *big* parameterization: drawing the diagram

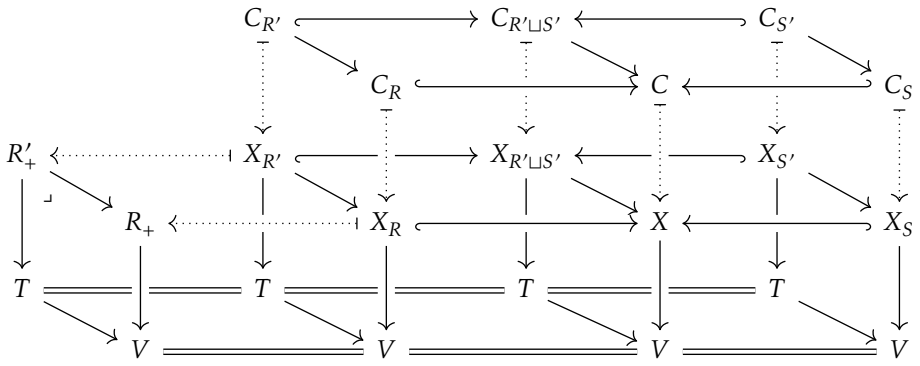
$$\begin{array}{ccc} \mathrm{Ar}^{\mathrm{int}}\mathrm{Tot}^{\mathrm{big}}\mathcal{O} & \longrightarrow & \mathrm{Ar}^{\mathrm{int}}\mathrm{Tot}^{\mathrm{big}}\underline{\mathbb{F}}_{*,\mathcal{T}} \\ \downarrow & & \downarrow \\ \mathrm{Tot}^{\mathrm{big}}\mathcal{O} & \longrightarrow & \mathrm{Tot}^{\mathrm{big}}\underline{\mathbb{F}}_{*,\mathcal{T}} \end{array}$$

and noting that each horizontal arrow has cocartesian lifts for inert morphisms, cancellation for cocartesian morphisms implies that it suffices to verify Condition (p-5') in the case $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes}_{\mathcal{T}}$. Indeed, $\mathrm{Tot}^{\mathrm{big}}\underline{\mathbb{F}}^{\mathrm{int}}_{\mathcal{T},*} \subset \mathrm{Ar}(\mathbb{F}_{\mathcal{T}})^{\mathrm{op}}$ is the wide subcategory whose morphisms are pointwise summand inclusions and $\mathrm{Ar}^{\mathrm{int}}\mathrm{Tot}^{\mathrm{big}}\underline{\mathbb{F}}_{*,\mathcal{T}} \subset \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathbb{F}_{\mathcal{T}})^{\mathrm{op}}$ the full subcategory of squares whose horizontal arrows are summand inclusions; ev_0-cocartesian arrows then correspond with cubes



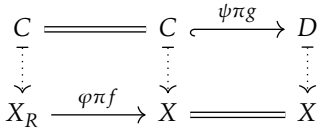
whose top and bottom face are cartesian, which exist in $\mathrm{ArAr}(\mathbb{F}_{\mathcal{T}})$ by atomic orbitality and in $\mathrm{Ar}^{\mathrm{int}}\mathrm{Ar}(\mathbb{F}_{\mathcal{T}})$ by pullback-stability of summand inclusions. Note that $R \times_S S'$ is not in general an orbit, which is why we passed to the $\mathbb{F}_{\mathcal{T}}$-parameterized pattern for $\mathcal{T}$-operads.

For Condition (p-6), we must cleverly choose a sufficient set of generalized $\mathrm{Tot}^{\mathrm{big}}\mathcal{O}$-Segal conditions. We suggest a simpler choice than Nardin-Shah: following Lurie's convention, choosing the sufficient set of generalized Segal conditions $\{p_{R\sqcup S,X}\}$ of Corollary I.4.34 (and using Theorem I.4.1 to transfer them to $\mathrm{Tot}^{\mathrm{big}}\mathcal{O}$), a generic ev_0-cocartesian section of $\varphi_{R\sqcup S,X}$ looks as follows, where each dotted $\mapsto$ arrow denotes cocartesian lifts, each C_- lives in $\mathcal{C}$, X_- in $\mathcal{O}$, and the rest in $\mathbb{F}_{\mathcal{T},*}$ or $\mathbb{F}_{\mathcal{T}}$ as appropriate.



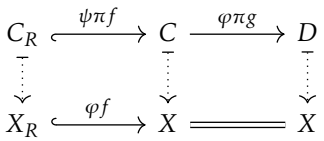
The other diagonal arity quadrilaterals are also pullback diagrams. Now, the summand decomposition of $R' \sqcup S'$ induced by its morphism to $R \sqcup S$ under disjunctiveness of $\mathbb{F}_{\mathcal{T}}$ has inclusion maps the arities of the middle (or top) top horizontal morphisms. In particular, the lower back squares together and outer back squares form the generalized Segal conditions $p_{R'\sqcup S', X_{R'\sqcup S'}}$, and $p_{R'\sqcup S', C_{R'\sqcup S'}}$; the later yields Condition (p-6).

For Condition (p-7), this is easy; we find ourself in the situation



Unwinding definitions, we're tasked with asking that the arrows lying over f are inert if and only if the composite arrows are inert; both are equivalent to $\varphi\pi f$ being inert.

Lastly, Condition (p-8) is similarly easy; we have a diagram



whose hooked arrows are inert, and we're asking that πg is inert if and only if $\pi g \circ \pi f$ is inert; this is just cancellation for inert arrows. □

We've proved every part of the claim, so we may now put them together.

Proof of Theorem II.3.48. Proposition II.3.51 proves pairability of a span underlying a span of $\mathcal{T}$-algebraic patterns; we may apply Proposition I.3.17 to acquire an adjunction of $\mathcal{T}$-$(\infty,2)$-categories, whose left adjoint is φ^* by Observation II.3.50.

The distributivity and presentability statements are Proposition II.3.53, as is the fact that equivalence $U\mathrm{Day}_{\mathcal{O}}(-,-) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(U-,U-)$ in the one color case. The functoriality in the codomain and domain called for are established in Proposition II.3.53 and Proposition II.3.58 respectively; functoriality in the base is Proposition II.3.56, and the desired naturality square follows by applying this along the unique map $\mathrm{triv}^{\otimes}_{\mathcal{T}} \to \mathcal{O}^{\otimes}$. The formula for algebras is Corollary II.3.57. □

II.3.5 Idempotent parameterized commutative algebras and parameterized mode theory

II.3.5.1 Idempotent algebras We may now approach parameterized modes via *idempotence*. Fix, for this section, I a unital weak indexing category.

Definition II.3.59. We say that an $\mathbb{E}_0$-algebra $1 \to Y$ is *idempotent* if the induced map

$$\mathrm{id}\otimes\eta\colon Y \simeq Y\otimes 1 \longrightarrow Y\otimes Y$$

is an equivalence. More generally, we say that an I-commutative algebra X is *idempotent* if, for all I-admissible maps of finite $\mathcal{T}$-spaces $f\colon T\sqcup S \to S$ whose underlying map $S \to S$ is identity, the structure map

$$\mu\colon X^{\otimes T}\otimes X^{\otimes S} \longrightarrow X^{\otimes S}$$

is an equivalence. We write $\mathrm{CAlg}^{\mathrm{idem}}_{I}(\mathcal{C}) \subset \mathrm{CAlg}_{I}(\mathcal{C})$ for the full subcategory of these. ◁

We extend the main beats of idempotent algebras to our setting. For convenience, assume that I is an indexing category; equivalently, replace it with an atomic orbital subcategory $\mathcal{R} \subset \mathcal{T}$.

Proposition II.3.60. *Let $\mathcal{C}$ be an $\mathcal{R}$-symmetric monoidal ∞-category. The composite functor*

$$U\colon \mathrm{CAlg}^{\mathrm{idem}}_{\mathcal{R}}(\mathcal{C}) \hookrightarrow \mathrm{CAlg}_{\mathcal{R}}(\mathcal{C}) \simeq \mathrm{CAlg}_{\mathcal{R}}\underline{\mathrm{Alg}}^{\otimes}_{\mathbb{E}_0}(\mathcal{C}) \longrightarrow \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C})$$

is fully faithful with image those $1 \to Y$ satisfying the following equivalent conditions:

(a) for all $\mathcal{R}$-admissible finite V-sets T, the unit map tensors to an equivalence

$$Y^{\otimes V} \simeq Y^{\otimes V}\otimes 1^{\otimes T} \xrightarrow{\sim} Y^{\otimes T\sqcup V},$$

(b) Y is idempotent and, for all $\mathcal{R}$-admissible finite V-sets T, $Y^{\otimes V}$ admits the structure of a module over the unique idempotent commutative algebra lifting $Y^{\otimes T}$,

(c) Y is idempotent and $-\otimes Y\colon \mathcal{C} \to \underline{\mathrm{Mod}}_{Y}(\mathcal{C})$ is an $\mathcal{R}$-symmetric monoidal $\mathcal{T}$-localization, or

(d) Y is idempotent and admits an $\mathcal{R}$-commutative algebra structure.

Moreover, $\mathrm{CAlg}^{\mathrm{idem},\otimes}_{\mathcal{R}}(\mathcal{C}) \subset \mathrm{CAlg}^{\otimes}_{\mathcal{R}}(\mathcal{C})$ *is a symmetric monoidal full subcategory which is a poset with relation*

$$Y' \leq Y \iff Y \in \mathrm{Mod}_{Y'}(\mathcal{C}). \tag{37}$$

and with tensor product determined by the property

$$\mathrm{Mod}_{Y' \otimes Y}(\mathcal{C}) = \mathrm{Mod}_{Y'}(\mathcal{C}) \cap \mathrm{Mod}_{Y}(\mathcal{C}) \subset \Gamma^{\mathcal{T}}\mathcal{C}. \tag{38}$$

We refer to these as $\mathcal{R}$-*idempotents*. Throughout, by $\underline{\mathrm{Mod}}_Y(\mathcal{C})$ we mean $\underline{\mathrm{Alg}}_{\mathrm{Infl}^{\mathcal{T}}_{\mathcal{C}}\mathrm{LMod}}(\mathcal{C}) \times_{\underline{\mathrm{Alg}}(\mathcal{C})} \{Y\}$, and likewise for $\mathrm{Mod}_Y(\mathcal{C})$, where LMod is the left-modules operad of [HA, § 4.2.1]. We begin as follows.

Lemma II.3.61. *If* $1 \to Y$ *is an idempotent* $\mathbb{E}_0$*-algebra, then so is* $1 \simeq 1^{\otimes T} \to Y^{\otimes T}$.

Proof. This is the image of $1 \to Y$ under the composite symmetric monoidal functor

$$f_{\otimes}\mathcal{C}^V \xrightarrow{\Delta^T} \mathcal{C}^T \xrightarrow{f_{\otimes}} \mathcal{C}^V,$$

and symmetric monoidal functors preserve idempotence of $\mathbb{E}_0$-algebras.[8] □

Lemma II.3.62. *Given* $X, X' \in \mathrm{CAlg}^{\mathrm{idem}}_{\mathcal{R}}(\mathcal{C})$ *whose underlying* $\mathbb{E}_0$*-algebra satisfies (a) through (d), we have*

$$\mathrm{Map}_{\mathrm{CAlg}_{\mathcal{R}}\mathcal{C}}(X, X') \simeq \begin{cases} * & X' \in \mathrm{Mod}_X(\mathcal{C}) \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. The proof of [HA, Prop. 4.8.2.9] generalizes; now, note that the second case is clear by the existence of the functor $\underline{\mathrm{CAlg}}^{\mathrm{idem}}_{\mathcal{R}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_0^{\otimes}}(\mathcal{C})$ and the same property (due to Lurie) for idempotent $\mathbb{E}_0$-algebras, so we're tasked with proving that the mapping space is contractible in the case that $X' \in \mathrm{Mod}_X(\mathcal{C})$; for this, we simply use that the image of the unit under the free Y-module functor $Y \simeq 1 \otimes Y \in \underline{\mathrm{CAlg}}_{\mathcal{R}}(\mathrm{Mod}_Y\mathcal{C})$ is initial, as $1 \in \underline{\mathrm{CAlg}}_{\mathcal{R}}(\mathcal{C})$ is initial by Corollary II.2.16. □

Proof of Proposition II.3.60. We first prove that the above conditions are equivalent. Now, (a) $\iff$ (b) is obvious in light of the characterization of modules over idempotent algebras [HA, Prop. 4.8.2.10].

Assuming (b), suppose $f : A \to b$ is an $- \otimes Y_{\mathcal{U}}$-equivalence; then, under the naturality square

$$\begin{array}{ccc} N^V_U A \otimes N^V_U Y_{\mathcal{U}} & \xrightarrow{N^V_U f \otimes N^V_U Y} & N^V_U B \otimes N^V_U Y_{\mathcal{U}} \\ \wr \uparrow & & \wr \uparrow \\ N^V_U (A \otimes Y) & \xrightarrow{N^V_U (f \otimes Y)} & N^V_U (B \otimes Y) \end{array}$$

[8] To get the symmetric monoidal structure on $f_{\otimes}$, pass through the equivalence $\mathrm{CMon}_{\mathcal{R}}(\mathrm{Cat}) \simeq \mathrm{CMon}_{\mathcal{R}}\mathrm{CMon}(\mathrm{Cat})$.

we find that $N_U^V f$ is a $-\otimes N_U^V Y_{\underline{U}}$-equivalence, and hence an $-\otimes Y_{\underline{V}}$-equivalence under (b), i.e. (b) $\implies$ (c).

Now, (c) $\implies$ (d) follows by the fact that Y is the image of the normed algebra 1 under the $\mathcal{R}$-symmetric monoidal localization $-\otimes Y$ (see Proposition II.2.57). Finally, the equivalence $\underline{\mathrm{CAlg}}_{\mathcal{R}}(\mathcal{C}) \simeq \underline{\mathrm{CAlg}}_{\mathcal{R}}\mathrm{CAlg}\mathcal{C}$ constructs an $\mathbb{E}_\infty$-structure on the T-indexed multiplication map $Y^{\otimes T} \to Y$, yielding a structure of a $Y^{\otimes T}$-algebra (hence module) on Y, i.e. showing that (d) implies (b).

Now, for the essential image, note that the trick of [HA, Prop. 4.8.2.9] lifts to the parameterized world; indeed by the Segal condition for $\mathcal{R}$-symmetric monoidal $\mathcal{C}$-categories we may assume $S = V$ in the defining property for $\underline{\mathrm{CAlg}}^{\mathrm{idem}}_{\mathcal{R}}(\mathcal{C})$, and unitality shows that the $T \sqcup V \to V$-indexed multiplication map is split by the tensored unit map, so one is an equivalence if and only if the other is. In particular, the essential image is precisely those $\mathbb{E}_0^\otimes$-algebras satisfying (a). Moreover, Equations (37) and (38) follows directly from the same statements non-parameterized, which are shown in the proof of [HA, Prop. 4.8.2.9] and in [CSY20, Prop. 5.1.8].

We are left with verifying that our functor is fully faithful. Combining the mapping space formula with Lemma II.3.62 shows that the morphism of mapping spaces is always an endomorphism of $*$ or $\varnothing$; these are all equivalences, as desired. □

Going in the other direction, fixing Y we have a fixed quantity (i.e. element of the poset of atomic orbital subcategories) characterizing how normed Y is as an idempotent:

Corollary II.3.63. *Fix $\mathcal{C}$ an $\mathcal{R}$-symmetric monoidal ∞-category, and $\mathcal{R}', \mathcal{R}'' \subset \mathcal{R}$ atomic orbital subcategories. If $Y \in \mathrm{Alg}_{\mathbb{E}_0}(\mathcal{C})$ is $\mathcal{R}'$-idempotent and $\mathcal{R}''$-idempotent, then Y is $\mathcal{R}' \vee \mathcal{R}''$-idempotent; in particular, there exists a maximal atomic orbital subcategory $\mathfrak{Rm}(Y) \subset \mathcal{R}$ such that Y is $\mathfrak{m}(Y)$-idempotent.*

Proof. By Propositions II.2.57 and II.3.60 it follows that the *unital weak indexing category* $\mathfrak{m}(\mathcal{C} \to \underline{\mathrm{Mod}}_Y(\mathcal{C}))$ is maximal among weak indexing categories I such that Y admits an I-commutative algebra structure. But $\underline{\mathrm{CAlg}}^{\mathrm{idem},\otimes}(\mathcal{C}) \to \underline{\mathrm{Alg}}^\otimes_{\mathbb{E}_0}(\mathcal{C})$ is an $\mathcal{R}$-*symmetric monoidal* full subcategory by Proposition II.3.60 and Lemma II.3.61, and so an idempotent I-commutative algebra interchanges with the underlying $\mathbb{E}_\infty$-structure in an essentially unique way, lifting it to an idempotent $I \cap I^\infty$-structure, so $\mathfrak{m}(\mathcal{C} \to \underline{\mathrm{Mod}}_Y(\mathcal{C}))$ is an indexing category. $\mathfrak{Rm}(Y)$ is the corresponding atomic orbital subcategory under Proposition I.1.97. □

We refer to elements of $\mathrm{CAlg}^{\mathrm{idem}}_{\mathcal{R}}\underline{\mathrm{Pr}}^L_{\mathcal{T}}$ as $\mathcal{R}$-*modes*, or $\mathcal{T}$-*modes* in the case $\mathcal{R} \simeq \mathfrak{o}(\mathcal{T})$. The following crucial corollary follows immediately.

Corollary II.3.64. *If $M \colon \underline{\mathcal{S}}_{\mathcal{T}} \to \mathcal{M}$ witnesses $\mathcal{M}$ as an idempotent $\mathbb{E}_0$-algebra in $\underline{\mathrm{Pr}}^L_{\mathcal{T}}$, then the following conditions are equivalent.*

(a) M witnesses $\mathcal{M}$ as an $\mathcal{R}$-mode,

(b) $\mathcal{M}$ and M jointly extend to distributive $\mathcal{R}$-symmetric monoidal structures, and

(c) the smashing localization on $\underline{\mathrm{Pr}}^L_{\mathcal{T}}$ classified by M is $\mathcal{R}$-symmetric monoidal.

In this situation, the space of distributive $\mathcal{R}$-symmetric monoidal structures on $\mathcal{M}$ with unit classified by M is contractible; otherwise it is empty.

II.3.5.2 Localization of modes We begin by lifting [CSY20, Prop. 5.2.10]. Morally, if $\mathcal{M}$ is a mode, then $\mathcal{M}$-modules are canonically enriched and tensored over $\mathcal{M}$, and $L\mathcal{M}$ becomes a mode measuring which $\mathcal{M}$-modules have L-local mapping objects, as multiplicative as L itself.

Proposition II.3.65. *Suppose $\mathcal{M} \to L\mathcal{M}$ is an $\mathcal{R}$-symmetric monoidal accessible localization of an $\mathcal{R}$-mode. Then, the composite functor $\underline{\mathcal{S}}_{\mathcal{T}} \to L\mathcal{M}$ exhibits $L\mathcal{M}$ as an $\mathcal{R}$-mode; moreover, for $\mathcal{C} \in \underline{\mathrm{Pr}}^L_{\mathcal{T}}$, $\mathcal{C}$ is an $L\mathcal{M}$-module if and only if it is an $\mathcal{M}$-module whose $\mathcal{M}$-enriched Yoneda embedding*

$$\text{ɣ}^{\mathcal{M}}_{\mathcal{C}} \colon \mathcal{C} \simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{M}) \longrightarrow \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{M})$$

factors through the inclusion $\underline{\mathrm{Fun}}(\mathcal{C}^{\mathrm{op}}, L\mathcal{M}) \hookrightarrow \underline{\mathrm{Fun}}(\mathcal{C}^{\mathrm{op}}, \mathcal{M})$.

Proof. Our proof is extremely similar to [CSY20, Prop. 5.2.10]; indeed, since $L\mathcal{M}$ admits a presentably $\mathcal{R}$-symmetric monoidal structure under $\underline{\mathcal{S}}_{\mathcal{T}}$ by Corollary II.3.36, it suffices to prove that $\underline{\mathcal{S}}_{\mathcal{T}} \to L\mathcal{M}$ exhibits it as an idempotent algebra classifying the described property. This follows mutatis mutandis from *loc. cit.*, but we include a minor variation of that proof here for completeness.

Now¸ we're tasked with proving that the tensored unit $\mathcal{T}$-functor

$$\eta' \colon L\mathcal{M} \simeq L\mathcal{M} \otimes \underline{\mathcal{S}}_{\mathcal{T}} \longrightarrow L\mathcal{M} \otimes L\mathcal{M}$$

is an equivalence; since $L\mathcal{M}$ admits a presentably symmetric monoidal structure, it admits a retract in $\mathrm{Pr}^L_{\mathcal{T}}$ given by binary tensor product, and hence a retract in $\mathsf{Cat}_{\mathcal{T}}$. Now, moreover note that η is equivalent to the composition

$$L\mathcal{M} \simeq L\mathcal{M} \otimes \mathcal{M} \xrightarrow{\mathrm{id} \otimes L} L\mathcal{M} \otimes L\mathcal{M};$$

in particular, applying the right adjoint to L yields a section to η' in $\mathsf{Cat}_{\mathcal{T}}$. It suffices to note that every morphism in a 1-category (in this case, $\mathsf{hoCat}_{\mathcal{T}}$) which is a section and retract is an isomorphism.

To conclude, we're tasked with characterizing $L\mathcal{M}$-modules. Now, note that

$$\underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, L\mathcal{M}) \simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{M}) \times_{\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{M})} \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, L\mathcal{M});$$

that is, for a general $\mathcal{M}$-module $\mathcal{C}$,

$$\mathcal{C} \otimes L\mathcal{M} \simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, L\mathcal{M}) \subset \underline{\mathrm{Fun}}^R_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}}, \mathcal{M}) \simeq \mathcal{C}$$

is the full subcategory spanned by $\mathcal{T}$-continuous $\mathcal{M}$-valued presheaves which are pointwise L-local. The result follows. □

Example II.3.66. $\underline{\mathcal{S}}_{\mathcal{T},\leq n}$ is a $\mathcal{T}$-mode; it classifies the property of a presentable $\mathcal{T}$-∞-category that each of its fixed points is an $(n+1)$-category. ◁

II.3.5.3 Pointed $\mathcal{T}$-spaces We have a canonical inclusion

$$\underline{\Delta}^1 \subset \mathrm{Infl}^{\mathcal{T}}_e \mathcal{S} \subset \underline{\mathcal{S}}_{\mathcal{T}};$$

its essential image has V-objects the initial or terminal V-objects.

Lemma II.3.67. *$\underline{\Delta}^1 \subset \underline{\mathcal{S}}_{\mathcal{T}}$ is a $\mathcal{T}$-symmetric monoidal full subcategory*

Proof. It suffices to note that the tuples of initial and terminal V-spaces are closed under arbitrary finite $\mathcal{T}$-products; indeed initial V-objects are *absorptive* since they have empty mapping spaces from all other V-objects, and terminal V-spaces preserve this class as they are the $\mathcal{T}$-symmetric monoidal unit. □

Now, we may lift an easy result of Nardin-Shah.

Proposition II.3.68. *The localization*

$$L\colon \mathrm{Day}\left(\underline{\Delta}^1, \underline{\mathcal{S}}_{\mathcal{T}}\right) \longrightarrow \underline{\mathcal{S}}_{\mathcal{T},*}$$

is $\mathcal{T}$-symmetric monoidal, and the resulting $\mathcal{T}$-symmetric monoidal ∞-category $\mathcal{S}^{\wedge}_*$ is presentable.

Proof of Proposition II.3.68. $\mathcal{T}$-symmetric monoidality of the localization follows by applying the argument as [NS22, Ex. 3.2.8] verbatim. For accessibility, it suffices to note that

$$\underline{\mathcal{S}}_{\mathcal{T},*} \subset \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\underline{\Delta}^1, \underline{\mathcal{S}}_{\mathcal{T}}\right) \times_{\mathrm{ev}_0, \underline{\mathcal{S}}_{\mathcal{T}}} \{*_{\mathcal{T}}\}$$

is closed under weakly contractible $\mathcal{T}$-colimits by $\mathcal{T}$-cocontinuity of ev_0 Proposition I.1.38, and filtered $\mathcal{T}$-colimits are weakly contractible. Presentability then follows by Corollary II.3.36. □

Corollary II.3.69. *$(-)_+\colon \underline{\mathcal{S}}_{\mathcal{T}} \to \underline{\mathcal{S}}_{\mathcal{T},*}$ witnesses pointed $\mathcal{T}$-spaces as a $\mathcal{T}$-mode classifying pointedness.*

Proof. If we know that it exhibits it as a $\mathcal{T}^{\simeq}$-mode classifying pointedness, then by Proposition II.3.68, the associated symmetric monoidal structure extends to a presentable $\mathcal{T}$-symmetric monoidal structure, so the result follows by Corollary II.3.64.

The $\mathcal{T}^{\simeq}$-mode property and its corresponding property is well-known: one simply notes that $\mathcal{C} \simeq \mathcal{C}_*$ if and only if $\mathcal{C}$ is pointed [CLL23, Cor. 4.16], that $\underline{\mathcal{S}}_{\mathcal{T},*}$ is pointed, and that

$$\begin{aligned}\mathcal{C} \otimes \underline{\mathcal{S}}_{\mathcal{T},*} &\simeq \underline{\mathrm{Fun}}^R_{\mathcal{T}}\left(\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Fun}}^{I_0-\oplus}_{\mathcal{T}}\left(\underline{\Delta}^1, \underline{\mathcal{S}}_{\mathcal{T}}\right)\right) \\ &\simeq \underline{\mathrm{Fun}}^{I_0-\oplus}_{\mathcal{T}}\left(\underline{\Delta}^1, \underline{\mathrm{Fun}}^R_{\mathcal{T}}\left(\mathcal{C}^{\mathrm{op}}, \underline{\mathcal{S}}_{\mathcal{T}}\right)\right) \\ &\simeq \underline{\mathrm{Fun}}^{I_0-\oplus}_{\mathcal{T}}\left(\underline{\Delta}^1, \mathcal{C}\right) \\ &\simeq \mathcal{C}_*. \end{aligned}$$

□

II.3.6 Parameterized formal inversion modes Following the theme that we are simply describing *multiplicativity* of constructions well known to experts, we now describe multiplicativity of Cnossen's parameterized lift of Robalo's *formal inversions.* We'll use the following terminology.

Definition II.3.70. A full $\mathcal{T}$-subcategory $\mathcal{B} \subset \mathcal{C}$ is *generated in top-degree* if, for all maps $U \to V$ and $X \in \mathcal{B}^U$, there exists some $Y \in \mathcal{B}^V$ and equivalence $\mathrm{Res}^V_U Y \simeq X$. ◁

Cnossen-Robalo's relevant results are the following [Cno23].

Proposition II.3.71. *Suppose $\mathcal{E} \subset \mathcal{C}$ is a small-generated full $\mathcal{T}$-subcategory of a presentably symmetric monoidal $\mathcal{T}$-∞-category which is generated in top degree. Then, the assignment $\mathcal{C}\left[\mathcal{E}^{-1}\right] \colon V \mapsto \mathcal{C}^V\left[\mathcal{E}^{V,-1}\right]$ constructs a presentably symmetric monoidal $\mathcal{T}$-∞-category under $\mathcal{C}$ satisfying the universal property that precomposition*

$$\underline{\mathrm{Fun}}^{L,\otimes}_{I^\infty}\left(\mathcal{C}\left[\mathcal{E}^{-1}\right], \mathcal{D}\right) \longrightarrow \underline{\mathrm{Fun}}^{L,\otimes}_{I^\infty}(\mathcal{C}, \mathcal{D})$$

with essential image those symmetric monoidal $\mathcal{T}$-left adjoints whose restriction to $\mathcal{E}$ has $\otimes$-invertible values.

Our theorem concerning this is the following; the case $\mathcal{R} = \mathcal{T}^{\simeq}$ is alluded to in [Cno23].

Theorem II.3.72. *Suppose $\mathcal{E} \subset \mathcal{M}$ is a small-generated symmetric monoidal full $\mathcal{T}$-subcategory of an $\mathcal{R}$-mode Then the $\mathcal{E}$-inversion map $\Sigma^\infty_{\mathcal{E}} \colon \mathcal{M} \to \mathcal{M}\left[\mathcal{E}^{-1}\right]$ exhibits the codomain as an $\mathcal{T}^{\simeq}$-mode such that $\mathcal{C}$ is an $\mathcal{M}\left[\mathcal{E}^{-1}\right]$-module if and only if it is an $\mathcal{M}$-module such that, for all $X \in \mathcal{E}^V$, the composite endofunctor*

$$\mathcal{C}^V \simeq \mathcal{C}^V \times \{X\} \hookrightarrow \mathcal{C}^V \times \mathcal{M}^V \xrightarrow{\otimes} \mathcal{C}^V$$

is invertible. Suppose moreover that the following conditions hold for each $X \in \mathcal{E}^V$:

(a) (symmetric) there exists some $n \in \mathbb{N}$ such that $X^{\otimes 3} \otimes X^{\otimes n} \in \mathrm{Fun}(B\Sigma_3, \mathcal{E}^V)$ has trivial action.

(b) (accessible) X is κ-compact for some regular cardinal κ.

Then multiplicativity of the formal inversion agrees with that of the subcategory:

$$\mathfrak{m}\left(\mathcal{M}\left[\mathcal{E}^{-1}\right]\right) \simeq \mathfrak{m}\left(\left\{M \in \mathcal{M} \;\middle|\; \left(\Sigma^\infty_{\mathcal{E}} M\right)^{-1} \in \mathcal{M}\left[\mathcal{E}^{-1}\right]\right\}\right),.$$

i.e. the multiplicativity of those M becoming invertible in $\mathcal{M}\left[\mathcal{E}^{-1}\right]$.

Before proving this, we prove our main corollary to it, using the following definition.

Definition II.3.73. An *$\mathcal{R}$-multiplicative G-universe* is a G-universe V admitting a linear isometric embedding $N_K^H \operatorname{Res}_K^G V \hookrightarrow \operatorname{Res}_H^G V$ for all $\mathcal{R}$-admissible inclusions $K \subset H$. ◁

Corollary II.3.74. *Given G a compact Lie group and V an $\mathcal{R}$-multiplicative G-universe, $\underline{\mathrm{Sp}}_V$ is an $\mathcal{R}$-mode; in particular, $\underline{\mathrm{Sp}}_G$ is a G-mode.*

Proof. By Corollary II.3.69, we already know that $\underline{\mathcal{S}}^\wedge_{G,*}$ is a G-mode; the uniqueness of the Mode structure ensures that the smash product and smash norm are the indexed tensor functors for this structure, so $\mathbf{S}^V = \left\{S^W \mid \exists \text{ embedding } W \hookrightarrow \operatorname{Res}_H^G V\right\} \subset \underline{\mathcal{S}}^\wedge_{G,*}$ is a small $\mathcal{R}$-symmetric monoidal full G-subcategory. Hence Corollary II.3.74 implies that $\underline{\mathcal{S}}_{G,*}\left[\mathbf{S}^{-V}\right]$ is a $\mathcal{T}^\sim$-mode, and this agrees with $\underline{\mathrm{Sp}}_V$ by definition.

Condition (a) is routine, and addressed e.g. in [Hoy17, Lem. 5.21]. All that remains is Condition (b), for which we note that each S^W is compact; indeed, as orbits are obviously compact, every finite G-CW complex is a finite colimit of compact objects, and hence itself compact. □

Remark II.3.75. This is often sharp; to see so, fix V a G-universe. If $W \subset V$ is a finite-dimensional K-subrepresentation, $\underline{\mathrm{Sp}}_G$ is an $\mathcal{R}$-mode, and $K \to H$ lies in $\mathcal{R}$, then $\Sigma^\infty S^{\mathrm{Ind}_K^H W} \simeq N_K^H \Sigma^\infty S^W$ is $\otimes$-invertible since N_K^H admits a symmetric monoidal structure. That is, replacing V with $V' = \bigoplus_{S^{-W} \in \mathrm{Sp}_V} W$ and writing $\mathfrak{m}(V') = \max\{\mathcal{R} \mid V' \text{ is } \mathcal{R}\text{-multiplicative}\}$, we find that

$$\mathfrak{m}(V') = \mathfrak{m}\left(\underline{\mathrm{Sp}}_V\right);$$

in particular, whenever $V = V'$ and $\mathcal{R} = \mathfrak{m}(V')$, Corollary II.3.74 is sharp. ◁

Remark II.3.76. In the case $\mathcal{T} = \mathrm{Pro}(\mathsf{FinGpd})$ is the category of profinite groupoids and $\mathcal{R}$ consists of finitely-presented maps, our construction agrees with that of [BH21, § 9], and in particular demonstrates their normed structure to underly a $\mathrm{Pro}(\mathsf{FinGpd})$-parameterized fp-mode. In particular, restricting to the finite group case, we have demonstrated their $\underline{\mathrm{Sp}}^\otimes_G$ to be a G-mode. This was already done in Nardin's thesis [Nar17] via a different argument, which had notable gaps.

More generally, [GM23, § C] identifies the underlying symmetric monoidal ∞-category $\mathrm{Sp}^\otimes_G$ with the symemtric monoidal ∞-category presented by orthogonal G-spectra, and $\Sigma^\infty \dashv \Omega^\infty$ the derived adjunction of a pointset model. ◁

Before moving on, we construct $\mathbb{E}_V$-structures on Thom G-spectra of V-fold loop maps.

Construction II.3.77. Since $(-)^{\simeq}\colon \underline{\mathrm{Cat}}_{\mathcal{T}} \to \underline{\mathcal{S}}_{\mathcal{T}}$ is a G-right adjoint, it is G-continuous, and hence lifts to a cartesian G-symmetric monoidal functor under Theorem II.2.1; applying a similar argument constructs a G-symmetric monoidal functor $(-)^{\times}\colon \mathrm{Alg}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \mathrm{Alg}^{\mathrm{gp}}(\underline{\mathcal{S}}_{\mathcal{T}})$. We define the G-symmetric monoidal *Picard space* functor $\mathrm{Pic}\colon \underline{\mathrm{Cat}}^{\otimes}_{\mathcal{T},\mathbb{E}_1} \to \underline{\mathcal{S}}_{\mathcal{T}}$ as the pullback G-symmetric monoidal functor

$$\begin{array}{ccc} \mathrm{Pic}(\mathcal{C}) & \longrightarrow & \mathcal{C}^{\simeq} \\ \downarrow & \lrcorner & \downarrow \\ \pi_0^{\times}\mathcal{C}^{\simeq} & \longrightarrow & \pi_0\mathcal{C}^{\simeq} \end{array}$$

In particular, given a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$, $\mathcal{O}\otimes\mathbb{E}_1$-$\mathcal{T}$-space maps $X \to \mathrm{Pic}(\mathcal{C})$ are in correspondence with $\mathcal{O}\otimes\mathbb{E}_1$-functors $X \to \mathcal{C}$ in the case that X has grouplike underling $\mathbb{E}_1$-$\mathcal{T}$-space. We define the *$\mathcal{O}$-algebra Thom object functor* as the object

$$\begin{array}{ccccc} \mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{S}}_G)}(X,\mathrm{Pic}(\mathcal{C})) & \longrightarrow & \mathrm{Alg}_{\mathcal{O}}\mathrm{Day}_{X/\mathcal{O}}(\mathcal{C})^{\simeq} & \xrightarrow[p_!]{} & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathrm{Map}_{\underline{\mathcal{S}}_G}(X,\mathrm{Pic}(\mathcal{C})) & \longrightarrow & \mathrm{Fun}_G(X,\mathcal{C})^{\simeq} & \xrightarrow[\mathrm{colim}]{} & \mathcal{C} \end{array}$$

and in the case of loop maps, using the recognition principle, we acquire a composition

$$\begin{array}{ccccccc} \mathrm{Map}_{\underline{\mathcal{S}}_G}\left(B^V X, B^V \mathrm{Pic}(\mathcal{C})\right) & \xrightarrow{\sim} & \mathrm{Map}_{\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{S}}_G)}(X,\mathrm{Pic}(\mathcal{C})) & \longrightarrow & \mathrm{Alg}_{\mathbb{E}_V}\mathrm{Day}_{X/\mathcal{O}}(\mathcal{C})^{\simeq} & \xrightarrow[p_!]{} & \mathrm{Alg}_{\mathbb{E}_V}(\mathcal{C}) \\ & \searrow^{\Omega^V} & \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ & & \mathrm{Map}_{\underline{\mathcal{S}}_G}(X,\mathrm{Pic}(\mathcal{C})) & \longrightarrow & \mathrm{Fun}_G(X,\mathcal{C})^{\simeq} & \xrightarrow[\mathrm{colim}]{} & \mathcal{C} \end{array}$$

We refer to this as the *conventional $\mathbb{E}_V$-structure on V-loop Thom spectra.* ◁

We will not study this further, but the author believes a strategy akin to that of [QZ25] will be helpful to raise this to $\mathcal{T}$-symmetric functoriality functoriality out of $\underline{\mathcal{S}}_{\mathcal{T},/\mathrm{Pic}(\mathcal{C})}$ and express a universal property relating it to more traditional notions of equivariant Thom spectra.

II.3.6.1 The $\mathcal{T}^{\simeq}$-mode Now, we will prove this via the same trick as always: prove it is a $\mathcal{T}^{\simeq}$-mode in an essentially nonequivariant manner, then construct a distributive $\mathfrak{m}(\mathcal{E})$-symmetric monoidal lift. For the firststep , we give a parameterized analog to Robalo's (nonparameterized) ∞-categorical lift of a result of Tabuada [Rob15, Prop. 2.2]

Proposition II.3.78. *Let $\mathcal{M}^{\otimes}$ be a presentably symmetric monoidal $\mathcal{T}$-∞-category and $\mathcal{E} \subset \mathcal{M}$ a symmetric monoidal full $\mathcal{T}$-subcategory small-generated in top degree.*

(1) *Base-change along* $\Sigma^\infty_{\mathcal{E}}\colon \mathcal{M}^\otimes \to \mathcal{M}\left[\mathcal{E}^{-1}\right]^\otimes$ *implements a fully faithful functor*

$$\Sigma^{\infty,*}_{\mathcal{E}}\colon \mathrm{Mod}_{\mathcal{M}[\mathcal{E}^{-1}]}(\widehat{\mathrm{Cat}}_{\mathcal{T}}) \hookrightarrow \mathrm{Mod}_{\mathcal{M}}(\widehat{\mathrm{Cat}}_{\mathcal{T}})$$

with essential image spanned by those $\mathcal{M}$*-modules* $\mathcal{E}$ *satisfying the condition that, for all* $V \in \mathcal{T}$ *and* $E \in \mathcal{E}^V$*, the action* V*-functor yields an automorphism*

$$M \otimes (-)\colon \mathcal{C}^V \simeq \{E\} \times \mathcal{C}^V \hookrightarrow \mathcal{M}^V \times \mathcal{C}^V \longrightarrow \mathcal{C}^V.$$

(2) *Base-change along* $\Sigma^\infty_{\mathcal{E}}\colon \mathcal{M}^\otimes \to \mathcal{M}\left[\mathcal{E}^{-1}\right]$ *implements a fully faithful functor*

$$\Sigma^{\infty,*}_{\mathcal{E}}\colon \mathrm{Mod}_{\mathcal{M}[\mathcal{E}^{-1}]}(\mathrm{Pr}^L_{\mathcal{T}}) \hookrightarrow \mathrm{Mod}_{\mathcal{M}}(\mathrm{Pr}^L_{\mathcal{T}})$$

with essential image spanned those $\mathcal{M}$*-modules which are* $\mathcal{M}\left[\mathcal{E}^{-1}\right]$*-modules in* $\widehat{\mathrm{Cat}}_{\mathcal{T}}$*.*

Proof. Applying the natural equivalence

$$\mathrm{Alg}_{\mathrm{Infl}^{\mathcal{T}}_{e}\mathcal{O}}\underline{\mathrm{Coeff}}^{\mathcal{T}}(\mathcal{D})^\times \simeq \mathrm{Coeff}^{\mathcal{T}}\mathrm{Alg}_{\mathcal{O}}(\mathcal{D}^\times)$$

to $\mathcal{O}^\otimes \simeq \mathrm{LMod}^\otimes$ and noting that formal inversion is defined fiberwise reduces the first claim immediately to the nonequivariant case, where it becomes [Rob15, Prop. 2.2] after universe expansion. The second claim follows from the fact that $\underline{\mathrm{Pr}}^{L,\otimes}_{\mathcal{T}} \subset \left(\underline{\widehat{\mathrm{Cat}}}_{\mathcal{T}}\right)^\times$ is a $\mathcal{T}$-suboperad under Theorem I.3.51. □

We also need a fairly standard lemma in nonequivariant higher algebra.

Lemma II.3.79. *If* $\mathcal{D}^\otimes$ *is a presentably symmetric monoidal* ∞*-category, then the following conditions on a commutative algebra* $X \in \mathrm{CAlg}(\mathcal{D})$ *are equivalent:*

(a) *The underlying* $\mathbb{E}_0$*-algebra of* X *is idempotent, and*

(b) *The forgetful functor* $\mathrm{Mod}_X(\mathcal{C}) \to \mathcal{C}$ *is fully faithful.*

Proof. The fact that (a) implies (b) is [HA, Prop. 4.8.2.10]. Conversely, $(-)\otimes X\colon \mathcal{C} \to \mathcal{C}$ agrees with the free X-module monad by [HA, Cor. 4.2.4.8] and [HA, Prop. 4.5.1.5], so (b) implies (a) under part (2) of [HA, Prop. 4.8.2.4]. □

We may conclude the following.

Proposition II.3.80. *In Theorem II.3.72,* $\Sigma^\infty_{\mathcal{E}}$ *exhibits* $\mathcal{M}\left[\mathcal{E}^{-1}\right]$ *as a* $\mathcal{T}^{\simeq}$*-mode with the indicated modules.*

Proof. Proposition II.3.78 and Lemma II.3.79 shows that the forgetful functor

$$\mathrm{Mod}_{\mathcal{M}[\mathcal{E}^{-1}]}(\mathrm{Pr}^L_{\mathcal{T}}) \xrightarrow{\Sigma^{\infty,*}_{\mathcal{E}}} \mathrm{Mod}_{\mathcal{M}}(\mathrm{Pr}^L_{\mathcal{T}}) \xrightarrow{U} \mathrm{Pr}^L_{\mathcal{T}}$$

is fully faithful, so $\mathcal{M}\left[\mathcal{E}^{-1}\right] \in \mathrm{Pr}^L_{\mathcal{T}}$ is an idempotent algebra via Lemma II.3.79; Proposition II.3.78 provides the correct formula for its modules. □

II.3.6.2 Norms on formal inversion modes For multiplicativity, we use a trick reminiscent of [LLP25]: there exists a fully faithful embedding

$$\mathrm{CMon}_{\mathcal{R}}\underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{D}^{\times} \simeq \mathrm{Seg}_{\mathrm{Span}_{\mathcal{R}}(\mathbb{F}_{\mathcal{T}})}(\mathcal{D}) \hookrightarrow \mathrm{Coeff}^{\mathrm{Span}_{\mathcal{R}}(\mathbb{F}_{\mathcal{T}})}(\mathcal{D}),$$

so $\mathcal{R}$-commutative monoids in $\mathcal{D}$ may be viewed just as a mildly special kind of parameterized $\mathcal{D}$-object. In particular, under the equivalence $\mathrm{CMon}_{\mathcal{R}}(\mathrm{Cat}^{\mathrm{sift}}) \simeq \mathrm{CMon}_{\mathcal{R}}\mathrm{CMon}(\mathrm{Cat}^{\mathrm{sift}})$ of Remark II.2.36, nonequivariantly sifted-distributive $\mathcal{R}$-symmetric monoidal ∞-categories embed into $\mathrm{Coeff}^{\mathrm{Span}_{\mathcal{R}}(\mathbb{F}_{\mathcal{T}})}\left(\mathrm{CMonCat}^{\mathrm{sift}}\right)$.

Lemma II.3.81. *If $\mathcal{E}^{\otimes} \subset \mathcal{M}^{\otimes}$ is an $\mathcal{R}$-symmetric monoidal full subcategory small-generated in top degree, then the full $\mathrm{Span}_{\mathcal{R}}(\mathbb{F}_{\mathcal{T}})$-subcategory $\widetilde{\mathcal{E}} \subset \widetilde{\mathcal{M}}$ spanned by S-objects $\bigcap_{\substack{f\colon S\to V\\ V\in\mathcal{T}}} f^*\mathcal{E}$ is small-generated in top degree.*

Proof. Suppose $\mathcal{E}_0 \subset \mathcal{E}$ is a small full $\mathcal{T}$-subcategory of generators; it follows by unwinding definitions that $\widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{E}}$ is a small full $\mathrm{Span}_{\mathcal{R}}(\mathbb{F}_{\mathcal{T}})$-subcategory of generators. Generation in top degree is automatic by definition. □

From this, we define the $\mathrm{Span}_{\mathcal{R}}(\mathbb{F}_{\mathcal{T}})$-parameterized formal inversion

$$\widetilde{\Sigma}^{\infty}_{\mathcal{E}}\colon \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}\left[\widetilde{\mathcal{E}}^{-1}\right]$$

This is exactly the $\mathcal{R}$-symmetric monoidal structure we want:

Proposition II.3.82. *$\widetilde{\Sigma}^{\infty}_{\mathcal{E}}\colon \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}\left[\mathcal{E}^{-1}\right]$ underlies an $\mathcal{R}$-symmetric monoidal functor whose corresponding symmetric monoidal $\mathcal{T}$-functor is $\Sigma^{\infty}_{\mathcal{E}}$.*

Proof. Unwinding definitions, the formal inversion is curried from a composition

$$\mathrm{Span}_{\mathcal{R}}(\mathbb{F}_{\mathcal{T}}) \xrightarrow{(\mathcal{E}^{\bullet}\hookrightarrow\mathcal{M}^{\bullet})} \mathrm{Cat}^{\otimes}_{\mathrm{aug}} \xrightarrow{\eta_{\mathcal{L}}} \mathrm{ArCat}^{\otimes},$$

where $\mathrm{Cat}^{\otimes}_{\mathrm{aug}} \simeq \mathrm{Cat}^{\otimes} \times_{\mathrm{Cat}} \mathrm{Cat}_{\mathrm{aug}}$ and $\mathrm{Cat}_{\mathrm{aug}} \subset \mathrm{Ar}(\mathrm{Cat})$ consists of the full subcategories and $\eta_{\mathcal{L}}$ is the unit of the adjunction corresponding with the left adjoint $\mathcal{L}$ to $(-)_{\mathrm{inv}}\colon \mathrm{Cat}^{\otimes} \to \mathrm{Cat}^{\otimes}$ constructed by an evident variation of [Cno23, Lem. 2.42].

To prove that $\widetilde{\Sigma}^{\infty}_{\mathcal{E}}$ is $\mathcal{R}$-symmetric monoidal, it suffices to prove that $\eta_{\mathcal{L}}$ is product preserving, i.e. the left adjoint $\mathcal{L}$ is product-preserving. That is, we must prove that

$$\mathcal{C} \times \mathcal{D}\left[\mathcal{E}_{\mathcal{C}} \times \mathcal{E}_{\mathcal{D}}^{-1}\right] \simeq \mathcal{C}\left[\mathcal{E}_{\mathcal{C}}^{-1}\right] \times \mathcal{D}\left[\mathcal{E}_{\mathcal{D}}^{-1}\right],$$

i.e. the righthand side satisfies the universal property of formal inversion of $\mathcal{E}_{\mathcal{C}} \times \mathcal{E}_{\mathcal{D}}$. Indeed, this follows by unwinding the definitions. It moreover follows by naturality of everything involved that the underlying symmetric monoidal $\mathcal{T}$-functor of our construction is $\Sigma^\infty_{\mathcal{E}}$. □

Crucially, this preserves distributivity in the setting where we care for multiplicativity.

Proposition II.3.83. *If $\mathcal{M}^\otimes$ is $\mathcal{K}$-distributive and $\mathcal{E}$ satisfies Conditions (a) and (b), then the induced structure on $\mathcal{M}\left[\mathcal{E}^{-1}\right]$ is $\mathcal{K}$-distributive.*

Proof. Applying Proposition II.3.37, it suffices to prove that there exists a sifted diagram

$$(\varphi, E)\colon \mathcal{K} \longrightarrow \mathrm{Fun}_{\mathcal{T}}\left(\mathcal{M}\left[\mathcal{E}^{-1}\right], \mathcal{M}\right) \times \mathcal{M}_{\mathrm{inv}}$$

such that $\mathrm{colim}_{k\in\mathcal{K}} \varphi(k) \otimes E(k) \simeq \mathrm{id}_{\mathcal{M}}$; following the discussion in [Hoy17, § 6.1], we may reduce to the case that the subcategory $\mathcal{E} \subset \mathcal{M}$ consists of a single $\mathcal{T}$-object X.

Now, it suffices to construct (φ, E) in the nonequivariant case, naturally in $\mathcal{M}$ and X, so specialize to that case. [Rob15, Cor. 2.22] yields a limit diagram

$$\mathcal{M}\left[X^{-1}\right] \simeq \lim\left(\cdots \xrightarrow{\Omega^X} \mathcal{M} \xrightarrow{\Omega^X} \mathcal{M}\right)$$

by Condition (a) where $\Omega^X \simeq \mathrm{map}_{\mathcal{M}}(X, -)$ is the internal hom from X, and the right adjoint $\Omega^\infty_{\mathcal{E}} \dashv \Sigma^\infty_{\mathcal{E}}$ is the corresponding functor to the last element. This lies in Pr^R by Condition (b) and the adjoint functor theorem. The argument of [HTT, Prop. 5.5.7.6] then naturally expresses $\mathrm{id}_{\mathcal{M}[X^{-1}]}$ as a filtered colimit

$$\mathrm{id}_{\mathcal{M}[X^{-1}]} \simeq \mathrm{colim}_{n\to\infty} \Sigma^{-nX} \Sigma^\infty_X \Omega^\infty_X \Omega^{-nX},$$

yielding the claim. □

Proof of Theorem II.3.72. Proposition II.3.80 proves the first sentence. Write $\mathcal{N}$ for the desired multiplicativity. For the remaining statement, by Proposition II.3.60, it suffices to do the following:

(i) construct an $\mathcal{N}$-symmetric monoidal structure on $\Sigma^\infty_{\mathcal{E}}$ such that $\mathcal{M}\left[\mathcal{E}^{-1}\right]$ is distributive, and

(ii) prove that $\mathfrak{m}\left(\Sigma^\infty_{\mathcal{E}}\right) \subset \mathcal{N}$.

Now, replacing $\mathcal{E}$ with $\left\{E \;\middle|\; \left(\Sigma^\infty_{\mathcal{E}} E\right)^{-1} \in \mathcal{M}\left[\mathcal{E}^{-1}\right]\right\}$, (i) follows immediately from Proposition II.3.83. For (ii), given an orbit map $f\colon U \to V$ lying in $\mathfrak{m}\left(\Sigma^\infty_{\mathcal{E}}\right)$, we acquire a commutative diagram

$$\begin{array}{ccc} \mathcal{M}^U & \xrightarrow{\Sigma^\infty_{\mathcal{E}}} & \mathcal{M}\left[\mathcal{E}^{-1}\right]^U \\ \downarrow{\scriptstyle N^V_U} & & \downarrow{\scriptstyle N^V_U} \\ \mathcal{M}^V & \xrightarrow{\Sigma^\infty_{\mathcal{E}}} & \mathcal{M}\left[\mathcal{E}^{-1}\right]^V \end{array}$$

Symmetric monoidality of N_U^V then implies that whenever $E \in \mathcal{M}^U$ has $\Sigma^\infty_{\mathcal{E}} E$ admitting $\otimes$-inverse F, the V-object $N_U^V E$ admits $\otimes$-inverse $N_U^V F$, i.e. $f \in \mathcal{N}$. □

II.3.7 Digression: the equivariant chromatic localizations Now, we will work out some examples arising in chromatic equivariant homotopy theory, following Hill [Hil17]; we do not claim to any originality. First, a description of multiplicativity of localizations via their quotients.

II.3.7.1 Multiplicativity via kernels Let $\mathrm{Ker}(L) \coloneqq L^{-1}\{0\}$.

Proposition II.3.84. *Suppose $\mathcal{C}$ is a fiberwise-stable distributive $\mathcal{R}$-symmetric monoidal ∞-category subject to the condition that, for all $\mathcal{R}$-maps $U \to V$, there exists a sifted diagram $p\colon \mathcal{K} \to \mathrm{Fun}\left(\mathcal{C}^V, \mathcal{C}^V\right)$, pointwise factoring through $N_U^V \mathcal{C}^U$, such that $\mathrm{colim}\, p = \mathrm{id}$. Then, given $L\colon \mathcal{C} \to L\mathcal{C}$ a symmetric monoidal $\mathcal{T}$-localization, we have $\mathfrak{m}(\mathrm{Ker}(L) \hookrightarrow \mathcal{C}) = \mathfrak{m}(\mathcal{C} \to L\mathcal{C})$.*

Proof. A V-morphism $f\colon A \to B$ is an L-equivalence if and only if $\mathrm{cof}(f) \in \mathrm{Ker}(L)$ by the fact that L is a left adjoint, and hence righ-exact; moreover, every V-object of the kernel arises in such a way, for instance as a cofiber of the null map to itself.

Now, sifted-cocontinuity of $\mathcal{C}$ shows that N_U^V preserves cofibers of *split maps*, such as the null map; if all L-equivalences of f are taken to L-equivalences under N_U^V, then preservation of the L-equivalence $0 \to A$ shows that L preserves the kernel, i.e. $\mathfrak{m}(\mathcal{C} \to L\mathcal{C}) \subset \mathfrak{m}(\mathrm{Ker}(L) \hookrightarrow \mathcal{C})$.

For the other direction, we may simply resolve a map $A \to N_U^V \mathrm{cof}(f)$ by norms of maps $A_k \to \mathrm{cof}(f_k)$, where A_k is L-acyclic, by sifted-cocontinuity of N_U^K and whenever $f \in \mathfrak{m})\mathrm{Ker}(L) \hookrightarrow \mathcal{C})$; this shows that such maps are null, ie. $f \in \mathfrak{m}(\mathcal{C} \to L\mathcal{C})$. □

Corollary II.3.85. *In the situation of Proposition II.3.84, suppose additionally that*

(a) $\mathcal{C} \to L\mathcal{C}$ is a fiberwise-finite localization *in the sense that* $\mathrm{Ker}(L)$ *is fiberwise generated under colimits by its compact objects* $\mathrm{Ker}(L)^\omega$*, and*

(b) $\mathcal{C}^\omega \subset \mathcal{C}$ is an $\mathcal{R}$-symmetric monoidal full subcategory.

Then, $\mathfrak{m}(\mathcal{C} \to L\mathcal{C}) = \mathfrak{m}(\mathrm{Ker}(L)^\omega \hookrightarrow \mathcal{C}^\omega)$.

To prove this, we write down an easy lemma.

Lemma II.3.86. *Suppose $\mathcal{E}, \mathcal{D} \subset \mathcal{C}$ is a pair of symmetric monoidal full $\mathcal{T}$-subcategory inclusions such that*

(a) $\mathcal{C}$ has a distinguished distributive $\mathcal{R}$-symmetric monoidal structure,

(b) $\mathcal{D}$ is closed under sifted colimits,

(c) $\mathcal{E}$ is an $\mathcal{R}$-symmetric monoidal subcategory, and

(d) $\mathcal{D} \cap \mathcal{E}$ generates $\mathcal{D}$ under sifted colimits.

Then, $\mathfrak{m}(\mathcal{D} \hookrightarrow \mathcal{C}) = \mathfrak{m}(\mathcal{D} \cap \mathcal{E} \hookrightarrow \mathcal{C})$.

Proof. The lefthand side is obviously contained within the right by $\mathcal{R}$-symmetric monoidality of $\mathcal{E}$, so assume that $\mathcal{D} \cap \mathcal{E}$ is closed under norms along the $\mathcal{R}$-admissible map $U \to V$. Given an object $D \in \mathcal{D}^U$, write a sifted resolution $D \simeq \mathrm{colim}_{k \in \mathcal{K}} E_k$ such that $E_k \in \mathcal{D}^U \cap \mathcal{E}^U$ for all k; then, distributivity guarantees that $N_U^V D \simeq \mathrm{colim}_{k \in \mathcal{K}} N_U^V E_k$. This is a sifted colimit of objects in $\mathcal{D}^V \cap \mathcal{E}^V$ by assumption, and hence it lies in $\mathcal{D}^V$, as desired. □

Proof of Corollary II.3.85. Compact objects are closed under finite colimits, so our assumption is equivalent to the condition that $\mathsf{Ker}(L)$ is generated fiberwise under *filtered* (hence sifted) colimits by compact objects. We find ourselves in the situation of Lemma II.3.86 for $\mathcal{D} = \mathsf{Ker}(L)$ and $\mathcal{E} = \mathcal{C}^\omega$ and the result follows. □

II.3.7.2 Norms on equivariant chromatic localizations Fix G a compact Lie group. To stay relatively self-contained, we construct geometric fixed points without residual Weyl-group action.

Construction II.3.87. Under Proposition II.3.71, define the *geometric fixed points* as the essentially unique symmetric monoidal left adjoint making the following diagram commute:

$$\begin{array}{ccc} \mathcal{S}_{G,*} & \xrightarrow{\Sigma^\infty} & \mathrm{Sp}_G \\ \downarrow{\scriptstyle F^G} & & \vdots{\scriptstyle \Phi^G} \\ \mathcal{S}_* & \xrightarrow{\Sigma^\infty} & \mathrm{Sp} \end{array}$$

More generally, we define $\Phi^H := \Phi^H \circ \mathrm{Res}^G_H \colon \mathrm{Sp}_G \to \mathrm{Sp}$. ◁

We use this to give an important invariant of chromatic localizations:

Construction II.3.88. Suppose $\mathcal{X} \subset \mathrm{Sp}^\omega_{G,(p)}$ is a p-local thick $\otimes$-ideal. The *vanishing function of* $\mathcal{X}$ is the function $\mathfrak{v}^\bullet(\mathcal{X}) \colon \pi_0 \mathcal{O}_G \to [0, \infty]$ defined by

$$\mathfrak{v}^H(\mathcal{X}) = \sup\left\{ \ell \in \mathbb{N} \;\middle|\; \forall X \in \mathcal{X},\ K(\ell) \otimes X^{\Phi H} \neq 0 \right\}. \qquad \triangleleft$$

Now, Balmer-Sanders recognized that a standard isotropy separation argument shows that every prime tensor ideal must have geometric fixed points concentrated at some conjugacy class $(H) \subset G$; this was extended to compact Lie groups by Barthel-Greenlees-Hausmann [BGH19]. Hence the Hopkins-Smith thick subcategory theorem identifies these with an extended integer denoting minimal index ℓ at which of $K(\ell)$-homology is nonvanishing for the elements of $\mathcal{X}$. In particular, taking prime decompositions shows that $\mathcal{X}$ is recoverable from $\mathfrak{v}(\mathcal{X})$, as

$$\mathcal{X} = \left\{ X \in \mathrm{Sp}^\omega_{G,(p)} \;\middle|\; \forall (H) \subset G,\ K\left(\mathfrak{v}^H(\mathcal{X})\right) \otimes X^{\Phi H} \neq 0 \right\};$$

see [BGH19, Thm. 1.3]. Hill [Hil17] noted that the double coset formula for interaction of norms and geometric fixed points gives a straightforward characterization of norm-closure of $\mathcal{X}$. We may use it to prove the following.

Proposition II.3.89. *The multiplicativity of the finite localization $L^f_{\mathcal{X}}\colon \underline{\mathrm{Sp}}_{G,(p)} \to \underline{\mathrm{Sp}}_{G,\mathcal{X}}$ associated with $\mathcal{X}$ is an indexing category associated with the transfer system*

$$\mathfrak{Rm}\left(L^f_{\mathcal{X}}\right) = \left\{K \hookrightarrow H \;\middle|\; \forall\, K \subset J \subset H,\;\; \mathfrak{v}^J(\mathcal{X}) \leq \mathfrak{v}^{hKh^{-1}\cap J}(\mathcal{X})\right\};$$

in particular, $L^f_{\mathcal{X}}$-local G-stability is an $\mathfrak{Rm}\left(L^f_{\mathcal{X}}\right)$-mode.

We need a few extremely routine lemmas about geometric fixed points to prove this. We will mostly prove them for completeness. Let $E\mathcal{P}$ be the unique G-subspace of $*_G$ characterized by its fixed points

$$E\mathcal{P}^H \simeq \begin{cases} * & H \subsetneq G \\ \varnothing & H = G. \end{cases}$$

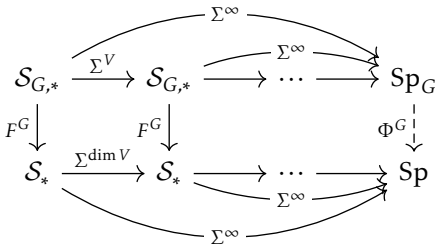
Under Σ^∞_+, this extends to a cofiber sequence of G-spectra $E\mathcal{P}_+ \to \mathbb{S} \to \widetilde{E\mathcal{P}}$.

Lemma II.3.90. *There exists a natural equivalence $\left(\widetilde{E\mathcal{P}} \otimes X\right)^G \simeq X^{\Phi G}$.*

Proof. As in [GM23, § C], note that Sp_G is the colimit in the ∞-category $\widehat{\mathrm{Cat}}^\omega_G$ of compactly-generated (large) ∞-categories and compact object-preserving left adjoints of the diagram

$$\mathcal{S}_G \simeq \mathrm{colim}\left(\mathcal{S}_{G,*} \xrightarrow{\Sigma^V} \mathcal{S}_{G,*}\right),$$

as V ranges across the finite-dimensional subrepresentations of some fixed complete G-universe. In particular, the space of factorizations



whose rightmost vertical arrow is cocontinuous and preserves compact objects is also contractible, and contains Φ^G. Thus it suffices to prove that the cocontinuous and compact object-preserving functor $\left(\widetilde{E\mathcal{P}} \otimes X\right)^G$ makes

this diagram commute; indeed, we have

$$\begin{aligned}\left(\widetilde{E\mathcal{P}} \otimes \Sigma^\infty X\right)^G &\simeq \mathrm{cof}\left(\Sigma^\infty (E\mathcal{P}_+ \wedge X)^G \to (\Sigma^\infty X)^G\right) \\ &\simeq \Sigma^\infty \left(X^G\right),\end{aligned}$$

where the second equivalence is the tom-Dieck splitting. □

This confirms that our geometric fixed points agree with the classical notion, say, from [Sch18, Prop. 3.3.8]. We acquire an immediate corollary, e.g. from [Sch18, Prop. 3.3.10].

Lemma II.3.91. *The functor* $\prod_{H\leq G} \Phi^H \colon \mathrm{Sp}_G \to \prod_{H\leq G} \mathrm{Sp}$ *is conservative.*

More importantly, it confirms that our vanishing function agrees with the type functions of [BGH19]. We will also need the following well-known property of norms.

Lemma II.3.92. *There exists a natural equivalence* $N_K^H \Phi^H \simeq \Phi^K$.

Proof. All of the functors in sight have distinguished symmetric monoidal structures, so this is easy: we first note that the following diagram commutes, and the bottom horizontal composite is determined up to contractible ambiguity as a symmetric monoidal left adjoint by the universal property of formal inversion

$$\begin{array}{ccccc} \underline{\mathcal{S}}_{K,*} & \xrightarrow{\mathrm{CoInd}_K^H} & \underline{\mathcal{S}}_{H_*} & \xrightarrow{F^H} & \mathcal{S}_* \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sp}_K & \xrightarrow[N_K^H]{} & \mathrm{Sp}_H & \xrightarrow[\Phi^H]{} & \mathrm{Sp} \end{array}$$

By composing right adjoints, we find that the top horizontal composite is equivalent to F^K, so the bottom horizontal composite is equivalent to Φ^K, completing the proof. □

Proof of Proposition II.3.89. Since $L^f_{\mathcal{X}}$ is by definition given fiberwise by a fiberwise-finite smashing localization, it is symmetric monoidal, i.e. $\mathfrak{m}\left(L^f_{\mathcal{X}}\right)$ is an indexing category. Note that the $L^f_{\mathcal{X}}$-local objects are generated under sifted colimits by (norms of) local spheres and $\underline{\mathrm{Sp}}^{\otimes}_G$ is distributive, so we are in the situation of Proposition II.3.84 and Corollary II.3.85, yielding an equality $\mathfrak{Rm}\left(L^f_{\mathcal{X}}\right) = \mathfrak{Rm}(\mathcal{X})$; thus we're tasked with proving that $\mathcal{X}$ is closed under N_K^H if and only if the above condition holds. Indeed this follows immediately

by the straightforward computation

$$\begin{aligned} K(\ell)\otimes\Phi^J N_K^H X &\simeq K(\ell)\otimes\Phi^J \operatorname{Res}^J N_K^H X \\ &\simeq K(\ell)\otimes\Phi^J \bigoplus_{KhJ\in[K\backslash H/J]} N^J_{hKh^{-1}\cap J}\operatorname{Res}^H_{hKh^{-1}\cap J} X \\ &\simeq K(\ell)\otimes \bigoplus_{KhJ\in[K\backslash H/J]} \Phi^J N^J_{hKh^{-1}\cap J}\operatorname{Res}^H_{hKh^{-1}\cap J} X \\ &\simeq \bigoplus_{KhJ\in[K\backslash H/J]} K(\ell)\otimes\Phi^J N^J_{hKh^{-1}\cap J}\operatorname{Res}^H_{hKh^{-1}\cap J} X \\ &\simeq \bigoplus_{KhJ\in[K\backslash H/J]} K(\ell)\otimes\Phi^{(hKh^{-1}\cap J)} X: \end{aligned}$$

the first term is contractible if and only if each term in the final direct sum is contractible. □

Applying Propositions II.3.60 and II.3.84 immediately yields the following corollary.

Corollary II.3.93. *The space of $\mathcal{R}$-commutative algebra structures on $L^f_{\mathcal{X}}\mathbb{S}$ with unit $L\colon \mathbb{S}\to L^f_{\mathcal{X}}\mathbb{S}$ is either empty or contractible; it is nonempty if and only if $\mathcal{R}\subset \mathfrak{Rm}\left(L^f_{\mathcal{X}}\right)$.*

II.3.7.3 Chromatic realizability Now, we will briefly speak on the *inverse problem* of characterizing which transfer systems arise in this way. A version of the following input was obtained for finite abelian groups, conditioned on a Log_p*-conjecture* by Balmer-Sanders [BS17, Cor. 10.6]; this conjecture was disproved and corrected to a rank_p-theorem for by Barthel-Hausmann-Naumann-Nikolaus-Noel-Stapleton [BHNNNS19, Cor. 1.4], then extended to all compact Lie groups by Barthel-Greenlees-Hausmann in [BGH19, Thm. 1.3], completing the result.

Proposition II.3.94. $\mathfrak{v}$ *provides an embedding of posets*

$$\mathfrak{v}\colon \mathrm{Thick}-\otimes-\mathrm{Ideal}\left(\mathrm{Sp}^{\omega}_{(p),C_{p^n}}\right)\hookrightarrow [0,\infty]^n_{\leq};$$

the essential image is those sequences $\mathfrak{v}$ such that, for all indices $1\leq j\leq k\leq n$, we have $\mathfrak{v}^j\leq \mathfrak{v}^k+1$.

For instance, we may exhibit that this problem is nontrivial as follows.

Corollary II.3.95. *A C_{p^n}-transfer system is realizable as the multiplicativity of a finite localization of $\mathbb{S}$ if it satisfies two-out-of-three.*

PROOF OF COROLLARY II.3.95. If $\mathcal{R}$ is saturated, we may simply write the increasing recurrence

$$\ell_0 = 0;$$
$$\ell_{i+1} = \begin{cases} \ell_i & C_{p^i} \to C_{p^{i+1}} \in \mathcal{R}; \\ \ell_i + 1 & \text{otherwise.} \end{cases}$$

It follows by unwinding definitions that ℓ is chromatically realizable and that it encodes the same index-p transfers as $\mathcal{R}$; moreover, the transfers correspond with equalities of indices, which satisfy two-out-of-three, so ℓ encodes a saturated transfer system. Two saturated C_{p^n}-transfer systems agree if they agree on index-p inclusions, so we are done. □

This condition is not necessary; for instance, the following C_{p^2}-transfer system (illustrated in Rubin's dot notation [Rub21]) does not satisfy two-out-of-three, but it is realizable via the sequence $(1, 0, 1)$:



We claim that the following C_{p^3}-transfer system is not chromatically realizable



Indeed for the sequence to admit these transfers, it would require $\mathfrak{v}_1 < \mathfrak{v}_2 < \mathfrak{v}_3 \leq \mathfrak{v}_0$ to hold, but such a sequence can not be realizable, as $\mathfrak{v}_0 > \mathfrak{v}_1 + 1$. The author is not sure precisely which transfer systems are chromatically realizable.

Chapter III

Boardman-Vogt tensor product computations

We finish with more computations of Boardman-Vogt tensor products of parameterized and equivariant operads. In particular, in Section III.1, we lift Schlank-Yanovski's ∞-*categorical Eckmann-Hilton argument* [SY19] to arbitrary atomic orbital subcategories, and in Section III.2 we extend Dunn-Lurie to be genuine-equivariant over arbitrary finite groups.

III.1 The equivariant Eckmann-Hilton argument

We say that a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ is *k-connected* if $\mathcal{O}(S)$ is k-connected or empty for all $S \in \underline{\mathrm{Fin}}_{\mathcal{T}}$. In this section, we prove the following analog of the Eckmann-Hilton argument.

Theorem III.1.1 (Equivariant Eckmann-Hilton argument)**.** *If $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are k and ℓ-connected almost-unital $\mathcal{T}$-operads with $\mathfrak{a}\mathcal{O} = \mathfrak{a}\mathcal{P}$, then $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}$ is $(k+\ell+2)$-connected.*

All nonempty $\mathcal{T}$-operads will be (-1)-connected yielding a stabilization result.

Corollary III.1.2 (Equivariant stabilization hypotheses)**.** *If $\mathcal{O}^{\otimes}$ is a nonempty almost-unital $\mathcal{T}$-operad, then $\mathcal{O}^{\otimes(n+1)}$ is $(n-1)$-connected; in particular, for any $\mathcal{T}$-symmetric monoidal n-category $\mathcal{C}^{\otimes}$,*

$$U\colon \underline{\mathrm{CAlg}}^{\otimes}_{A\mathcal{O}}(\mathcal{C}) \xrightarrow{\sim} \overbrace{\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}} \cdots \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}}^{(n+1)\text{-}fold}(\mathcal{C}),$$

where the $(n+1)$-fold tensor product is taken as a colimit (and composition as limit) in the case $n = \infty$.

For instance, this immediately implies a version of the *Baez-Dolan stabilization hypothesis*, in this case stating that $(n+2)$-tuply $\mathbb{E}_V$-monoidal n-categories are equivalent to $\mathbb{E}_{\infty V}$-monoidal n-categories, i.e. $\mathfrak{a}\mathbb{E}_V$-semi-Mackey functors valued in n-categories.

Now, we may remove the assumption $\mathfrak{a}\mathcal{O} = \mathfrak{a}\mathcal{P}$ in Theorem III.1.1, but we will need a more refined notion of connectivity. In general, given a weak indexing category I, we say that $\mathcal{O}^{\otimes}$ is *k-connected at I* if, for

all $T \in \underline{\mathbb{F}}_I$, the structure space $\mathcal{O}(T)$ is k-connected. We define the *connectivity function*

$$\mathsf{conn}_{\mathcal{O}} \colon \mathsf{wIndexCat}_{\mathcal{T}}^{a\mathrm{uni}} \longrightarrow \mathbb{Z} \cup \{\infty\}$$

by the formula $\mathsf{conn}_{\mathcal{O}}(I) \coloneqq \inf\{k \mid \mathcal{O}^{\otimes} \text{ is } k\text{-connected at } I\}$.

Now, $(\mathbb{Z} \cup \{\infty\})^{\mathsf{wIndexCat}_G^{a\mathrm{uni}}}$ forms a *pointwise* commutative monoidal poset, i.e.

$$f \leq g \quad \Longleftrightarrow \quad \forall I,\, f(I) \leq g(I); \qquad\qquad f + g(I) = f(I) + g(I).$$

In this language, we will actually prove the following strengthening of Theorem III.1.1.

Theorem III.1.3. *Given $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ a pair of almost-unital G-operads, we have $\mathsf{conn}_{\mathcal{O}} + \mathsf{conn}_{\mathcal{P}} + 2 \leq \mathsf{conn}_{\mathcal{O}\otimes\mathcal{P}}$.*

The degenerate case of functions concentrated on $\{-2\} \cup \{\infty\}$ demands a tensor product of almost-unital weak $\mathcal{N}_\infty$-operads, so we begin there in Section III.1.1; specializing to the case $\mathcal{T} = \mathcal{O}_G$ and indexing categories, this affirms a conjecture of Blumberg-Hill [BH15, Conj. 6.27].

In Section III.1.2, we define *I-symmetric monoidal d-categories and I-d-operads* in preparation for our theorems; we prove Theorems III.1.1 and III.1.3 in Section III.1.3.

III.1.1 Tensor products of equivariant commutative operads and their consequences

Proposition III.1.4. *The morphism $\mathcal{N}_{I\infty}^{\otimes} \simeq \mathsf{triv}_{\mathcal{T}}^{\otimes} \otimes \mathcal{N}_{I\infty}^{\otimes} \longrightarrow \mathcal{N}_{I\infty}^{\otimes} \overset{BV}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is an equivalence.*

Proof. On algebras in $\mathcal{T}$-spaces, this implements the forgetful $\mathcal{T}$-functor

$$\underline{\mathrm{CMon}}_I \underline{\mathrm{CMon}}_I\left(\underline{\mathcal{S}}_{\mathcal{T}}\right) \longrightarrow \underline{\mathrm{CMon}}_I\left(\underline{\mathcal{S}}_{\mathcal{T}}\right),$$

which is an equivalence by Corollary II.2.35. □

Thus $\mathcal{N}_{I\infty}^{\otimes}$ is an idempotent algebra. We will make heavy use of this in this subsection.

III.1.1.1 Distributivity and coherences of $\overset{BV}{\otimes}$ Since Env is symmetric monoidal, Proposition III.1.4 renders $\underline{\mathbb{F}}_{\mathcal{T}}^{\sqcup} \simeq \mathsf{EnvComm}_{\mathcal{T}}^{\otimes}$ as an idempotent $\mathbb{E}_0$-algebra under the localized Day convolution structure. Using this, we acquire coherences canonically.

Corollary III.1.5. *$\underline{\mathrm{Op}}_{\mathcal{T}} \subset \underline{\mathrm{Cat}}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\sqcup}}^{\otimes}$ is a symmetric monoidal $\mathcal{T}$-colimit closed full $\mathcal{T}$-subcategory with tensor product operation $\overset{BV}{\otimes}$ under the localized Day convolution structure on the righthand side; in particular, the resulting symmetric monoidal $\mathcal{T}$-∞-category $\underline{\mathrm{Op}}_{\mathcal{T}}^{\otimes}$ is presentable.*

Proof. It is a fully faithful $\mathcal{T}$-left adjoint by Corollary II.1.19 which inertwines binary tensor products by Corollary II.1.40, so it is a symmetric monoidal $\mathcal{T}$-colimit preserving full $\mathcal{T}$-subcategory. Moreover, the underlying $\mathcal{T}$-category $\underline{\mathrm{Op}}_{\mathcal{T}}$ is presentable by Corollary I.3.10, so it suffices to prove that it is distributive.

Distributivity is closed under symmetric monoidal $\mathcal{T}$-cocontinuous fully faithful $\mathcal{T}$-functors, localizations, and slicing by Corollary II.3.36, so it suffices to prove that the Day convolution structure on $\underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathrm{Span}_I(\underline{\mathcal{S}}_{\mathcal{T}})^{\wedge}, \underline{\mathrm{Cat}}_{\mathcal{T}}^{\times}\right)$ is distributive; indeed the cartesian structure on coefficient systems on a presentable ∞-category is presentable by Proposition II.3.19, which implies that the Day convolution structure for $\mathcal{T}$-functors into such things is distributive by Theorem II.3.48. □

The following corollary is immediate and extremely useful.

Corollary III.1.6. . *Suppose $\varphi\colon \mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes} \to \mathcal{Q}^{\otimes}$ is a functor of $\mathcal{T}$-operads such that $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}, \mathcal{Q}^{\otimes}$ are each unital and $\mathcal{T}$-space colored. Then φ is an equivalence if and only if the following conditions hold.*

(a) the induced $\mathcal{T}$-space map $U\mathcal{O} \times U\mathcal{P} \to U\mathcal{Q}$ is an equivalence, and

(b) for all $V \in \mathcal{T}$ and $(x, y) \in U\mathcal{O}^V \times U\mathcal{P}^V$, φ restricts to an equivalence

$$\mathrm{End}^{\mathrm{red}}_x \mathcal{O} \overset{BV}{\otimes} \mathrm{End}^{\mathrm{red}}_y \mathcal{P} \longrightarrow \mathrm{End}^{\mathrm{red}}_{\varphi(x,y)} \mathcal{Q}.$$

III.1.1.2 The smashing localization for $\mathcal{N}_{I\infty}^{\otimes}$. $\mathcal{N}_{I\infty}^{\otimes}$ classifies algebraic Wirthmüller isomorphisms:

Theorem III.1.7. *Let I be an almost-unital weak indexing system. Then, an at-least one color $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ possesses an equivalence $\mathcal{P}^{\otimes} \overset{BV}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \simeq \mathcal{P}^{\otimes}$ if and only if $\underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive.*

Proof. Just as in the proof of Lemma II.2.34, note that Corollary II.2.33 implies that the conditions of Lemma II.2.34 are equivalent to the additional condition

(j) The forgetful $\mathcal{T}$-functor $\underline{\mathrm{Alg}}_{\mathcal{N}_{I\infty}\otimes\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \simeq \underline{\mathrm{CAlg}}_I \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}}) \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is an equivalence.

But this is equivalent to the desired equivalence by Theorem II.2.7. □

Thus we may answer a conjecture of Blumberg-Hill [BH15, Conj. 6.27].

Corollary III.1.8. *When I and J are almost-unital, there is an equivalence $\mathcal{N}_{I\infty}^{\otimes} \overset{BV}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes}$.*

Proof. By Lemma II.3.62, $\mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \in \mathrm{Op}_{\mathcal{T}} \simeq \mathrm{Op}_{\mathcal{T},\mathrm{triv}^{\otimes}_{\mathcal{T}}/}$ is an idempotent $\mathbb{E}_0$-algebra classifying the conjunction of the properties which are classified by $\mathcal{N}_{I\infty}^{\otimes}$ and $\mathcal{N}_{J\infty}^{\otimes}$; that is, $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ is fixed by $(-) \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes}$ if and only if $\underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is I-semiadditive and J-semiadditive. Corollary I.1.111, identifies this with the property that $\underline{\mathrm{Alg}}_{\mathcal{O}}(\underline{\mathcal{S}}_{\mathcal{T}})$ is $I \vee J$-semiadditive , i.e. $\mathcal{O}^{\otimes}$ is fixed by $(-) \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\vee J}^{\otimes}$. Thus, we have

$$\mathcal{N}_{I\vee J\infty}^{\otimes} \simeq \mathcal{N}_{I\vee J\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}^{\otimes} \simeq \mathcal{N}_{I\infty}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{J\infty}.$$ □

Following Corollary III.1.8, we may go on to reframe Theorem III.1.7, motivated by the following.

Observation III.1.9. The computation of [HA, § 2.3.1] and resulting theory may be stated simply: the operad $\mathbb{E}_0^{\otimes}$ is an idempotent $\mathbb{E}_0$-algebra in $\mathrm{Op}^{\otimes}$ under the unique map $\mathsf{triv}^{\otimes} \to \mathbb{E}_0^{\otimes}$, and the corresponding smashing localization classifies unitality. In particular, a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ is a $\mathbb{E}_0^{\otimes}$-module with respect to $\overset{\mathrm{BV}}{\otimes}$ if and only if the unit object $1_{\mathcal{C}} \in \mathcal{C}^{\otimes}$ is initial, i.e. for all $X \in \mathcal{C}$, the unique map $X^{\sqcup \varnothing} \to X^{\otimes \varnothing}$ is an equivalence; that is, $(-) \overset{\mathrm{BV}}{\otimes} \mathbb{E}_0^{\otimes}$ classifies I_0-*cocartesianness* in the symmetric monoidal case. ◁

We say that a $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ is I-*cocartesian* if it is $v(I)$-unital and $\mathsf{id}_{U\mathcal{O}}$ classifies an equivalence $\mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}^{\otimes} \xrightarrow{\sim} U\mathcal{O}^{I-\sqcup}$. We have an I-operadic variant of Corollary II.2.33 and Theorem III.1.7.

Proposition III.1.10. *Given I almost-unital, the following are equivalent for $\mathcal{P}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$.*

(a) $\mathcal{P}^{\otimes}$ is I-cocartesian.

(b) For all unital I-operads $\mathcal{O}^{\otimes}$, the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \to \underline{\mathrm{Fun}}_{\mathcal{T}}(I\mathcal{O}, \mathcal{P})$ is an equivalence.

(c) The forgetful $\mathcal{T}$-operad map $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{P}) \to \mathcal{P}^{\otimes}$ is an equivalence.

(d) There exists a $\mathcal{T}$-operad $\mathcal{Q}^{\otimes}$ and an equivalence of $\mathcal{T}$-operads $\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{Q}) \xrightarrow{\sim} \mathcal{P}^{\otimes}$.

(k) The canonical map $\mathcal{P}^{\otimes} \to \mathcal{P}^{\otimes} \overset{BV}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ is an equivalence.

Proof. The proof of the equivalence between Conditions (a) to (d) is identical to Corollary II.2.33, so we omit it. Now, by a standard two-out-of-three argument, $\mathcal{P}^{\otimes}$ is local for the smashing localization associated with $\mathcal{N}_{I\infty}^{\otimes}$ if and only if pullback along the localization map $\mathcal{O}^{\otimes} \to \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}^{\otimes}$ induces an equivalence $\mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{P})^{\simeq} \simeq \mathrm{Alg}_{\mathcal{O}\otimes\mathcal{N}_{I\infty}}(\mathcal{P})^{\simeq} \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\simeq}$ for all $\mathcal{O}^{\otimes} \in \mathrm{Op}_{\mathcal{T}}$, which is equivalent to the condition that $\mathrm{CAlg}_I^{\otimes}(\mathcal{P}) \to \mathcal{P}^{\otimes}$ is an equivalence by Yoneda's lemma. □

Remark III.1.11. Said another way, we understand semiadditivty of Segal $\mathcal{T}$-objects over almost-unital $\mathcal{T}$-operads:

$$I \leq \mathfrak{s}\left(\underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}}\left(\underline{\mathcal{S}}_{\mathcal{T}}\right)\right) \iff \mathrm{Bor}_I^{\mathcal{T}} \mathcal{O}^{\otimes} \simeq (U\mathcal{O})^{I-\sqcup}.$$

In particular, this computes that

$$\mathfrak{s}\left(\underline{\mathrm{CMon}}_I(\underline{\mathcal{S}}_{\mathcal{T}})\right) \simeq \mathfrak{s}\left(\underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{N}_{I\infty}}\left(\underline{\mathcal{S}}_{\mathcal{T}}\right)\right) \simeq I. \qquad ◁$$

Given a related pair of weak indexing categories $I \subset J$, let $\mathrm{Op}_J^{I-\mathrm{cocart}} \subset \mathrm{Op}_J$ be the full subcategory of I-cocartesian J-operads. We find that $\mathrm{Op}_J^{I-\mathrm{cocart}}$ is absorptive under $\overset{\mathrm{BV}}{\otimes}$ and the internal hom.

Corollary III.1.12. *Suppose $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ are at-least-one-color J-operads such that either $\mathcal{O}^{\otimes}$ or $\mathcal{P}^{\otimes}$ are I-cocartesian. Then, $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}$ and $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{P})$ are I-cocartesian; in particular, there exists an equivalence*

$$\mathcal{C}^{I-\sqcup} \overset{BV}{\otimes} \mathcal{D}^{I-\sqcup} \simeq (\mathcal{C} \times \mathcal{D})^{I-\sqcup}.$$

PROOF. I-cocartesianness of $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ follows from Proposition III.1.10. If $\mathcal{P}^{\otimes}$ is I-cocartesian, then Proposition III.1.10 constructs equivalences

$$\mathrm{CAlg}_I \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \simeq \mathrm{Alg}_{\mathcal{O}} \underline{\mathrm{CAlg}}^{\otimes}_I(\mathcal{P}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{P}),$$

so the result follows from another application of Proposition III.1.10. If $\mathcal{O}^{\otimes}$ is I-cocartesian, the result follows from two more applications of Proposition III.1.10. as we acquire equivalences

$$\underline{\mathrm{CAlg}}_I \underline{\mathrm{Alg}}^{\otimes}_{\mathcal{O}}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}_{\mathcal{N}_{I^\infty} \otimes \mathcal{O}}(\mathcal{P}) \simeq \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{P}). \tag{39}$$

□

Finally, we vastly extend the results of [HA, § 2.3.1].

Corollary III.1.13. *Suppose $I \subset J$ is a related pair of almost-unital weak indexing categories. Then, $\mathrm{Op}_J^{I\text{-cocart}} \subset \mathrm{Op}_J$ is a smashing localization and a cosmashing colocalization:*

$$\mathrm{Op}_J^{I\text{-cocart}} \underset{\underline{\mathrm{CAlg}}_I^{\otimes}(-)}{\overset{\mathcal{N}_{I^\infty}^{\otimes} \overset{BV}{\otimes} (-)}{\rightleftarrows}} \mathrm{Op}_J \qquad (\perp,\ \perp)$$

PROOF. Proposition III.1.10 exhibits the top adjunction and Corollary II.2.33 shows that $\underline{\mathrm{CAlg}}_I^{\otimes} \colon \mathrm{Op}_J \to \mathrm{Op}_J$ factors through $\mathrm{Op}_J^{I\text{-cocart}} \subset \mathrm{Op}_J$. Moreover, applying $\Gamma^{\mathcal{T}}(-)^{\simeq}$ to Eq. (39) yields a natural equivalence

$$\mathrm{Map}_{\mathrm{Op}_J}\left(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}\right) \simeq \mathrm{Map}_{\mathrm{Op}_J^{I\text{-cocart}}}\left(\mathcal{O}^{\otimes}, \underline{\mathrm{CAlg}}_I^{\otimes}(\mathcal{P})\right)$$

for all J-operads $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ such that $\mathcal{O}^{\otimes}$ is I-cocartesian, yielding the bottom adjunction. □

III.1.1.3 The mapping-out property for cocartesian structures

Corollary III.1.14. *There exists a triple adjunction of $\mathcal{T}$-$(\infty,2)$-categories*

$$\underline{\mathrm{OP}}_I^{\mathrm{uni}} \underset{\underline{\mathrm{CAlg}}_I(-)}{\overset{\mathrm{triv}(-)^{\otimes} \overset{BV}{\otimes} \mathbb{E}_{0,v(I)}^{\otimes}}{\underset{(-)^{I-\sqcup}}{\overset{U}{\rightrightarrows}}}} \underline{\mathrm{CAT}}_{\mathcal{T}}$$

whose leftmost and third-leftmost adjoint are fully faithful. In particular, there is a natural equivalence

$$\underline{\mathrm{Alg}}_{\mathcal{C}^{I-\sqcup}}(\mathcal{P}) \simeq \underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{C}, \underline{\mathrm{CAlg}}_I(\mathcal{P})).$$

PROOF. The top adjunction follows by composing the localization adjunction $\mathbb{E}^{\mathrm{uni}}_{0,\nu(I)}\otimes(-)\colon \underline{\mathrm{OP}}_I \leftrightarrows \underline{\mathrm{OP}}^{\mathrm{uni}}_I\colon \iota$ of Corollary II.2.17 with the triv-adjunction of Corollary II.1.9. The middle is Theorem II.2.32. The bottom adjunction is equivalent to the indicated natural equivalence, which we construct as follows:

$$\begin{aligned}\underline{\mathrm{Alg}}_{\mathcal{C}^{I-\sqcup}}(\mathcal{P}) &\simeq \underline{\mathrm{Alg}}_{\mathcal{C}^{I-\sqcup}\otimes\mathcal{N}_{I_\infty}}(\mathcal{P}) && \text{III.1.10}\\ &\simeq \underline{\mathrm{Alg}}_{\mathcal{C}^{I-\sqcup}}\underline{\mathrm{CAlg}}^{\otimes}_I(\mathcal{P})\\ &\simeq \underline{\mathrm{Fun}}_{\mathcal{T}}\left(\mathcal{C}, \underline{\mathrm{CAlg}}^{\otimes}_I(\mathcal{P})\right) && \text{II.2.32.}\end{aligned}$$ □

III.1.2 I-symmetric monoidal d-categories, I-d-operads, and arity-support Recall that a space is (-2)-*truncated* if it is empty, (-1)-*truncated* if it is empty or contractible, and for $d \geq 0$, a space X is d-*truncated* if it is a disjoint union of connected spaces $(X_\alpha)_{\alpha\in A}$ such that $\pi_m(X_\alpha) = 0$ for all $m > d$ and $\alpha \in A$.

III.1.2.1 I-symmetric monoidal d-categories Recall that a $(d+1)$-*category* is an ∞-category $\mathcal{C}$ such that the space $\mathrm{Map}(X,Y)$ is d-truncated for all $X, Y \in \mathcal{C}$. We say that an ∞-category is a (-1)-*category* if it is either $*$ or empty. In general, we write $\mathrm{Cat}_d \subset \mathrm{Cat}$ for the full subcategory spanned by the ∞-categories with the property that they are d-categories.

Definition III.1.15. The $\mathcal{T}$-∞-category of small $\mathcal{T}$-d-categories is

$$\underline{\mathrm{Cat}}_{\mathcal{T},d} \coloneqq \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathrm{Cat}_d.$$

If $I \subset \mathbb{F}_{\mathcal{T}}$ is pullback-stable, the $\mathcal{T}$-∞-*category of small I-symmetric monoidal d-categories* is

$$\underline{\mathrm{Cat}}^{\otimes}_{I,d} \coloneqq \underline{\mathrm{CMon}}_I\mathrm{Cat}_d.$$

We write $\mathrm{Cat}_{\mathcal{T},d} \coloneqq \Gamma^{\mathcal{T}}\underline{\mathrm{Cat}}_{\mathcal{T},d}$ and $\mathrm{Cat}^{\otimes}_{I,d} \coloneqq \Gamma^{\mathcal{T}}\underline{\mathrm{Cat}}^{\otimes}_{I,d}$. ◁

By the following lemma, $\underline{\mathrm{Cat}}_{\mathcal{T},d}$ is a $\mathcal{T}$-$(d+1)$-category and $\mathrm{Cat}_{\mathcal{T},d}$ is a $(d+1)$-category.

Lemma III.1.16 ([HTT, Cor 2.3.4.8, Prop 2.3.4.12, Cor 2.3.4.19])**.** Cat_d *is a $(d+1)$-category and the inclusion* $\mathrm{Cat}_d \hookrightarrow \mathrm{Cat}$ *has a right adjoint* $h_d\colon \mathrm{Cat} \to \mathrm{Cat}_d$.

Construction III.1.17. By Lemmas I.1.30 and III.1.16, the functor $\underline{\mathrm{Cat}}_{\mathcal{T},d} \hookrightarrow \underline{\mathrm{Cat}}_{\mathcal{T}}$ is an inclusion of a localizing $\mathcal{T}$-subcategory; let $h_d\colon \underline{\mathrm{Cat}}_{\mathcal{T}} \to \underline{\mathrm{Cat}}_{\mathcal{T},d}$ be the associated $\mathcal{T}$-left adjoint.

The mapping spaces in a product of categories are the product of the mapping spaces; in particular, the inclusion $\mathrm{Cat}_d \hookrightarrow \mathrm{Cat}$ is product-preserving. Hence we acquire a $\mathcal{T}$-adjunction

$$\underline{\mathrm{Cat}}^{\otimes}_I \underset{\iota}{\overset{h_d}{\rightleftarrows}} \underline{\mathrm{Cat}}^{\otimes}_{I,d} \qquad (\dashv)$$

whose right adjoint is fully faithful. We refer to h_d as the *homotopy I-symmetric monoidal d-category.* ◁

The remainder of this subsection will be dedicated to recognition results for $\mathcal{T}$-symmetric monoidal d-categories, which will be useful throughout the remainder of the paper. We first reduce this consideration to that of plain $\mathcal{T}$-∞-categories; the following proposition follows by unwinding definitions and noting that $\mathsf{Cat}_d \hookrightarrow \mathsf{Cat}$ is closed under products.

Proposition III.1.18. *If $I \subset \mathbb{F}_{\mathcal{T}}$ is a one-object weak indexing system, then $\mathcal{C}^{\otimes} \in \mathsf{Cat}_I^{\otimes}$ is a I-symmetric monoidal d-category if and only if its underlying $\mathcal{T}$-∞-category $\mathcal{C}$ is a $\mathcal{T}$-d-category.*

Often in equivariant higher algebra, we will find that our objects come with natural $\mathcal{T}$-functors to $\mathcal{T}$-1-categories, and we'd like to develop a recognition theorem in this case in terms of mapping spaces.

Proposition III.1.19. *A $\mathcal{T}$-∞-category $\mathcal{C}$ is a $\mathcal{T}$-d-category if and only if*

$$\mathsf{Mor}_V(\mathcal{C}) \coloneqq \mathsf{Fun}(\Delta^1, \mathcal{C}_V)^{\simeq}$$

is $(d-2)$-truncated for all $V \in \mathcal{T}$.

Proof. By definition, it suffices to prove this in the case $\mathcal{T} = *$. Fix $f, g \in \mathsf{Mor}_V(\mathcal{C})$. Then, we may present $\mathsf{Map}(f,g)$ as a disjoint union over a, b of homotopies

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle a}\downarrow & \nearrow & \downarrow{\scriptstyle b} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

For fixed a, b, this is either empty or equivalent to the component of the space $\mathsf{Map}(S^1, \mathsf{Map}(W,Z))$ whose underlying map is homotopic to bf. If $\mathcal{C}$ is a d-category, then this is $(d-2)$-truncated; conversely, choosing $a, b = \mathsf{id}$ and $f = g$, if this is $(d-2)$-truncated for all f, then the mapping spaces of $\mathcal{C}_V$ are $(d-1)$-truncated for all V, i.e. $\mathcal{C}$ is a $\mathcal{T}$-d-category. □

Given a $\mathcal{T}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ and a map $\psi\colon \Delta^1 \to \mathcal{C}_V$, define the pullback space

$$\begin{array}{ccc} \mathsf{Mor}_F^{\psi}(\mathcal{C}) & \longrightarrow & \mathsf{Mor}_V(\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ B\mathsf{Aut}_{\psi} & \hookrightarrow & \mathsf{Mor}_V(\mathcal{D}) \end{array}$$

so that $\mathsf{Mor}_F^{\psi}(\mathcal{C})$ is the disjoint union of the connected components of $\mathsf{Mor}_V(\mathcal{C})$ whose image in $\mathsf{Mor}_V(\mathcal{D})$ is equivalent to ψ. We say that F has $(d-1)$-*truncated mapping fibers* if $\mathsf{Mor}_F^{\psi}(\mathcal{C})$ is $(d-2)$-truncated for all $V \in \mathcal{T}$ and $\psi \in \mathsf{Mor}_V(\mathcal{C})$.

Corollary III.1.20. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a $\mathcal{T}$-functor and $\mathcal{D}$ is a $\mathcal{T}$-1-category. Then, the following are equivalent for $d \geq 1$:*

(1) F has $(d-1)$-truncated mapping fibers.

(2) $\mathcal{C}$ is a $\mathcal{T}$-d-category.

Additionally, the following are equivalent.

(1') $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is fully faithful and F has (-1)-truncated mapping fibers.

(2') F includes $\mathcal{C}$ as a (replete) $\mathcal{T}$-subcategory of $\mathcal{D}$.

Proof. After Proposition III.1.19, the only remaining part is the equivalence between (1') and (2'). Note that $B\mathrm{Aut}_{\psi}$ is (-1)-truncated by Proposition III.1.19, so (1') is equivalent to the conditions that $\mathcal{C}$ is a $\mathcal{T}$-1-category and $F_V : \mathcal{C}_V \to \mathcal{D}_V$ is a faithful functor which is fully faithful on cores, i.e. it is a (replete) subcategory inclusion. □

Lastly, we take note of the following useful fact.

Corollary III.1.21. *Suppose $\mathcal{A}_1$ is a $\mathcal{T}$-∞-category, $\mathcal{A}_2$ and $\mathcal{B}$ are $\mathcal{T}$-d-categories, $\mathcal{C}_1^{\otimes}$ is an I-symmetric monoidal ∞-category, and $\mathcal{C}_2^{\otimes}$ and $\mathcal{D}^{\otimes}$ are I-symmetric monoidal d-categories. Fix $\mathcal{T}$-functors $\mathcal{A}_i \to \mathcal{B}$ and I-symmetric monoidal functors $\mathcal{C}_i^{\otimes} \to \mathcal{D}^{\otimes}$. Then, the relative functor ∞-categories*

$$\mathrm{Fun}_{\mathcal{T},/\mathcal{B}}(\mathcal{A}_1, \mathcal{A}_2) \qquad \textit{and} \qquad \mathrm{Fun}^{\otimes}_{\mathcal{T},/\mathcal{D}^{\otimes}}\left(\mathcal{C}_1^{\otimes}, \mathcal{C}_2^{\otimes}\right)$$

are d-categories.

To prove this, we make use of a lemma which follows nearly directly from [HTT, Cor. 2.3.4.19].

Lemma III.1.22. *A $\mathcal{T}$-space X is d-truncated if and only if it's a $\mathcal{T}$-d-category; moreover, we have $h_d X \simeq \tau_{\leq d} X$.*

Proof. The first statement follows by applying [HTT, Cor. 2.3.4.19] fiberwise. Similarly, since each functor is defined fiberwise, it suffices to prove the second statement in the case $\mathcal{T} = *$. Now, [kerodon, Tag 057W] constructs a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_{\mathcal{T},\leq d} & \longrightarrow & \mathcal{S}_{\mathcal{T}} \\ \uparrow & & \uparrow \\ \mathrm{Cat}_{\mathcal{T},\leq d} & \longrightarrow & \mathrm{Cat}_{\mathcal{T}} \end{array}$$

and the claim follows by taking left adjoints to each arrow. □

Proof of Corollary III.1.21. The same ∞-categories arise when $\mathcal{A}_1$ and $\mathcal{C}_1^{\otimes}$ are replaced with $h_d\mathcal{A}_1$ and $h_d\mathcal{C}_1^{\otimes}$, so we may assume without loss of generality that $\mathcal{A}_1$ is a $\mathcal{T}$-d-category and $\mathcal{C}_1^{\otimes}$ an I-symmetric

monoidal d-category. We employ a trick; $\mathsf{Fun}^{\otimes}_{\mathcal{T},/\mathcal{D}^{\otimes}}(\mathcal{C}_1^{\otimes},\mathcal{C}_2^{\otimes})$ is equivalent to the relative functor category for $\mathsf{Span}(\mathbb{F}_G)$-$\infty$-categories, so we may simply prove the claim for the first category without assuming that $\mathcal{T}$ is atomic orbital. Moreover, $\underline{\mathsf{Fun}}_{\mathcal{T},/\mathcal{B}}(\mathcal{A}_1,\mathcal{A}_2) \simeq *_{\mathcal{T}} \times_{\underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{B})} \underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{A}_2)$ is a pullback of $\mathcal{T}$-categories covered by the first case; $\mathcal{T}$-d-categories are closed under pullback, so it suffices to prove that $\underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{A}_2)$ is a $\mathcal{T}$-d-category.

We can verify this by a standard strategy: given a $(d-1)$-connected $\mathcal{T}$-space X, we have

$$\begin{aligned}\underline{\mathsf{Fun}}_{\mathcal{T}}\left(X,\underline{\mathsf{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}\Delta^1,\underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{A}_2)\right)\right) &\simeq \underline{\mathsf{Fun}}_{\mathcal{T}}\left(\mathcal{A}_1,\underline{\mathsf{Fun}}_{\mathcal{T}}\left(X,\underline{\mathsf{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}\Delta^1,\mathcal{A}_2\right)\right)\right)\\ &\simeq \mathsf{Fun}_{\mathcal{T}}\left(\mathcal{A}_1,\underline{\mathsf{Fun}}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}\Delta^1,\mathcal{A}_2\right)\right)\\ &\simeq \mathsf{Fun}_{\mathcal{T}}\left(\mathrm{Infl}_e^{\mathcal{T}}\Delta^1,\underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{A}_2)\right).\end{aligned} \tag{40}$$

For Eq. (40), we've used Lemma III.1.22 and the natural equivalence $\underline{\mathsf{Fun}}_{\mathcal{T}}(*_{\mathcal{T}},\mathcal{C}) \simeq \mathcal{C}$. Now, writing $\underline{\mathsf{Mor}}(\mathcal{C}) \coloneqq \underline{\mathsf{Fun}}_{\mathcal{T}}(\mathrm{Infl}_e^{\mathcal{T}}\Delta^1,\mathcal{C})^{\simeq}$ and taking cores then implies that

$$\mathsf{map}\left(X,\underline{\mathsf{Mor}}\left(\underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{A}_2)\right)\right) \simeq \underline{\mathsf{Mor}}\left(\underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{A}_2)\right),$$

i.e. $\underline{\mathsf{Mor}}\left(\underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{A}_2)\right)$ is a $(d-2)$-truncated $\mathcal{T}$-space. But this implies that $\underline{\mathsf{Fun}}_{\mathcal{T}}(\mathcal{A}_1,\mathcal{A}_2)$ is a $\mathcal{T}$-d-category by Proposition III.1.19. □

III.1.2.2 $\mathcal{T}$-d-operads We may extend this program to $\mathcal{T}$-operads:

Proposition III.1.23. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad and let $d \geq -1$. Then, the following conditions are equivalent:*

(a) $\mathcal{O}(S)$ is d-truncated for all $S \in \mathbb{F}_V$.

(b) The $\mathcal{T}$-functor $\mathrm{Env}\mathcal{O} \to \underline{\mathbb{F}}_{\mathcal{T}}$ has d-truncated mapping fibers.

Proof. Let $\psi\colon T \to S$ be a map of $\mathcal{T}$-sets over V. Then, by Lemma II.1.22, we have a equivalences

$$\begin{aligned}\mathrm{Mor}^{\psi}_{\mathrm{Env}\mathcal{O}\to\underline{\mathbb{F}}_{\mathcal{T}}}(\mathrm{Env}\mathcal{O}) &\simeq \coprod_{\mathbf{C}\in\mathcal{O}_T,\mathbf{D}\in\mathcal{O}_S} \mathrm{Map}^{\psi}_{\mathrm{Env}\mathcal{O}\to\underline{\mathbb{F}}_{\mathcal{T}}}(\mathbf{C},\mathbf{D})\\ &\simeq \coprod_{\mathbf{C}\in\mathcal{O}_T,\mathbf{D}\in\mathcal{O}_S}\prod_{U\in\mathrm{Orb}(S)} \mathrm{Map}^{\psi}_{\mathrm{Env}\mathcal{O}\to\underline{\mathbb{F}}_{\mathcal{T}}}(\mathbf{C}_U,D_U)\\ &\simeq \coprod_{\mathbf{C}\in\mathcal{O}_T,\mathbf{D}\in\mathcal{O}_S}\prod_{U\in\mathrm{Orb}(S)} \mathcal{O}(\mathbf{C}_U;D_U),\end{aligned} \tag{41}$$

natural in $\mathcal{O}^{\otimes}$. First, in the case $d=-1$, note that conditions (a) and (b) both imply that $\mathcal{O}$ has at most one color, so Eq. (41) specializes to

$$\mathrm{Mor}^{\psi}_{\mathrm{Env}\mathcal{O}\to\underline{\mathbb{F}}_{\mathcal{T}}}(\mathrm{Env}\mathcal{O}) \simeq \prod_{U\in\mathrm{Orb}(S)} \mathcal{O}(S).$$

Thus it suffices to note that a product is -1-truncated if and only if its factors are.

Next, in the case $d \geq 0$, note that a coproduct of spaces is d-truncated if and only if its factors are; hence Eq. (41) shows that (b) is equivalent to the condition that $\prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(\mathbf{C}_U; D_U)$ is d-truncated for all $S, \mathbf{C}, \mathbf{D}$. In fact, the equation

$$\mathcal{O}(S) \simeq \coprod_{(\mathbf{C},D) \in \mathcal{O}_S \times \mathcal{O}_V} \mathcal{O}(\mathbf{C}; D)$$

shows that this (b) equivalent to the condition that $\mathcal{O}(S)$ is d-truncated for all $S \in \mathbb{F}_V$, as desired. □

We define the full subcategory of *d-operads* $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ to be spanned by $\mathcal{T}$-operads satisfying the condition that $\mathcal{O}(S)$ is $(d-1)$-truncated for all $S \in \mathbb{F}_V$ as in Proposition III.1.23. The following corollary follows from Proposition III.1.23 and the mapping fiber truncation characterizations of Corollary III.1.20.

Corollary III.1.24. *Let $\mathcal{O}^{\otimes}$ be a $\mathcal{T}$-operad and let $d \geq 1$. The following conditions are equivalent:*

(a) $\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-d-operad, and

(b) $\mathrm{Env}\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-symmetric monoidal d-category.

Furthermore, the following conditions are equivalent:

(a') $\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-0-operad, and

(b') the $\mathcal{T}$-functor $\mathrm{Env}\mathcal{O} \to \underline{\mathbb{F}}_{\mathcal{T}}$ is a $\mathcal{T}$-subcategory inclusion.

In general, these form a well-behaved subcategory.

Corollary III.1.25. *The inclusion $\mathrm{Op}_{\mathcal{T},d} \hookrightarrow \mathrm{Op}_{\mathcal{T}}$ has a left adjoint h_d satisfying*

$$(h_d\mathcal{O})(S) \simeq \tau_{\leq d-1}\mathcal{O}(S).$$

Furthermore, when $d \geq 1$, this fits into the following diagram

$$\begin{array}{ccc} \mathrm{Op}_{\mathcal{T}} & \xrightarrow{h_d} & \mathrm{Op}_{\mathcal{T},d} \\ \downarrow & & \downarrow \\ \mathrm{Cat}^{\otimes}_{\mathcal{T}} & \xrightarrow{h_d} & \mathrm{Cat}^{\otimes}_{\mathcal{T},d} \end{array}$$

When $\mathcal{C}^{\otimes}$ is a $\mathcal{T}$-symmetric monoidal d-category, the map $\mathcal{O}^{\otimes} \to h_d\mathcal{O}^{\otimes}$ induces an equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{h_d\mathcal{O}}(\mathcal{C}).$$

Proof. We separately argue in the case $d \geq 1$ and $d = 0$ that equifibrations are closed under $h_{\mathcal{T},d}$; this will imply that $h_d\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\mathcal{O}^{\otimes}$ corresponds with a $\mathcal{T}$-d-operad $h_d\mathcal{O}^{\otimes}$, which computes the left adjoint to the inclusions $\mathrm{Op}_{\mathcal{T},d} \subset \mathrm{Op}_{\mathcal{T}}$ by fully faithfulness of $\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\mathcal{O}^{\otimes}$. In particular, the counit $\varepsilon\colon \mathcal{O}^{\otimes} \to h_d\mathcal{O}^{\otimes}$ will induce

the counit $\mathrm{Env}\mathcal{O} \to h_d\mathrm{Env}\mathcal{O}$, which by Eq. (41) shows that $\mathcal{O}(S) \to h_d\mathcal{O}(S)$ is the Postnikov $(d-1)$-truncation map.

In the case $d \geq 1$. In this case, since $h_{\mathcal{T},d} : \mathrm{Cat}_{\mathcal{T}}^{\otimes} \to \mathrm{Cat}_{\mathcal{T},d}^{\otimes}$ is applied pointwise, it preserves equifibrations, so $h_{\mathcal{T},d}\mathrm{Env}^{/\underline{\mathbb{F}}_{\mathcal{T}}}\mathcal{O}^{\otimes}$ corresponds with a d-operad $h_{\mathcal{T},d}\mathcal{O}^{\otimes}$.

The case $d = 0$ is similar, except that we are tasked with replacing equifibered $\mathcal{T}$-symmetric monoidal functors with an equifibered subcategory inclusions. In fact, subcategories are precisely (-1)-truncated maps in Cat_1, so we may do this by taking the pointwise (-1)-truncation functor and applying [HA, Prop. 5.5.6.5] to see that the result is equifibered. □

We easily acquire compatibility of h_n with cocartesian structures.

Corollary III.1.26. *Given $\mathcal{C}$ a $\mathcal{T}$-category, there exists an equivalence $h_n\left(\mathcal{C}^{I-\sqcup}\right) \simeq (h_n\mathcal{C})^{I-\sqcup}$.*

Proof. Take left adjoints to all arrows in the following diagram.

$$\begin{array}{ccc} \mathrm{Op}_{\mathcal{T},d} & \xrightarrow{\mathrm{CAlg}_I(-)} & \mathrm{Cat}_{\mathcal{T},d} \\ \downarrow & & \downarrow \\ \mathrm{Op}_{\mathcal{T}} & \xrightarrow{\mathrm{CAlg}_I(-)} & \mathrm{Cat}_{\mathcal{T}} \end{array}$$

□

Corollary III.1.27. *Let $\mathcal{P}^{\otimes}$ be a $\mathcal{T}$-d-operad.*

(1) if $d \geq 1$, then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ is a d-category; hence $\mathrm{Op}_{\mathcal{T},d}$ is a $(d+1)$-category.

(2) if $d = 0$, then $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ is either empty or contractible; hence $\mathrm{Op}_{\mathcal{T},0}$ is a poset.

Proof. In each case, the second statement follows from the first by noting that the mapping spaces in $\mathrm{Op}_{\mathcal{T}}$ are $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\simeq}$. For the first statements, note that

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) \simeq \mathrm{Alg}_{h_d\mathcal{O}}(\mathcal{P}) \simeq \mathrm{Fun}^{\otimes}_{\mathcal{T},/\underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}}(\mathrm{Env}h_d\mathcal{O}^{\otimes}, \mathrm{Env}\mathcal{P}^{\otimes});$$

if $d \geq 1$, then this is a d-category by Corollaries III.1.21 and III.1.24. If $d = 0$, then this category is either empty or contractible since Corollary III.1.24 shows that $\mathrm{Env}\mathcal{P}^{\otimes} \to \underline{\mathbb{F}}_{\mathcal{T}}^{\mathcal{T}-\sqcup}$ is monic. □

Moreover, we may identify the h_d-equivalences exactly as in Theorem II.2.7.

Corollary III.1.28. *Let $\varphi : \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ be a π_0-bijective $\mathcal{T}$-operad map. The following are equivalent for $d \geq 1$.*

(a) $h_d\varphi$ is an equivalence.

(b) For all $\mathcal{C}^{\otimes} \in \mathrm{Cat}_{\mathcal{T},d}^{\otimes}$, the pullback map $\underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{C})$ is an equivalence.

(c) The pullback map $\underline{\mathrm{Mon}}_{\mathcal{P}}\left(\mathcal{S}_{\mathcal{T},\leq(d-1)}\right) \to \underline{\mathrm{Mon}}_{\mathcal{O}}\left(\mathcal{S}_{\mathcal{T},\leq(d-1)}\right)$ is an equivalence.

PROOF. (a) implies (b) since $h_d\mathrm{Env} = \mathrm{Env}h_d$, and (b) implies (c) by neglect of assumptions. Taking coproducts of Proposition II.2.8 shows that (c) implies that $\tau_{\leq(d-1)}\varphi(S)$ is an equivalence for all S, implying (a) via Corollary III.1.25. □

Corollary III.1.29. *$\mathcal{P}^{\otimes}$ is a $\mathcal{T}$-0-operad if and only if it's a sub-terminal object of $\mathrm{Op}_{\mathcal{T}}$.*

PROOF. The mapping space criterion of monomorphisms equates the latter with the condition that

$$\mathrm{Alg}_{h_0\mathcal{O}}(\mathcal{P})^{\simeq} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{P})^{\simeq} \to \mathrm{Alg}_{\mathcal{O}}(\mathrm{Comm}^{\otimes}_{\mathcal{T}})^{\simeq} \simeq *$$

is a monomorphism, i.e. $\mathrm{Alg}_{h_0\mathcal{O}}(\mathcal{P})^{\simeq} \in \{\varnothing, *\}$; this follows from Corollary III.1.27.

On the other hand, Corollary II.1.30 (together with Kan extensions) constructs a *free $\mathcal{T}$-operad on $*_S$* characterized by the property

$$\mathrm{Alg}_{\mathrm{Fr}_S(*)}(\mathcal{O})^{\simeq} \simeq \mathcal{O}(S);$$

thus the mapping space criterion for a subterminal $\mathcal{T}$-operad $\mathcal{O}^{\otimes}$ implies that $\mathcal{O}(S)$ is either empty or contractible for all S, so $\mathcal{O}^{\otimes}$ is a $\mathcal{T}$-0-operad. □

In particular, in sight of Proposition II.1.2, $h_{\leq 0}$ yields a left adjoint to the construction of weak $\mathcal{N}_\infty$-operads. We may combinatorially describe this as follows.

Construction III.1.30. Given $\mathcal{O} \in \mathrm{Op}_{\mathcal{T}}$, the *arity support of $\mathcal{O}$* is the collection of maps $\mathfrak{a}(\mathcal{O}) \subset \mathbb{F}_{\mathcal{T}}$ defined by

$$\mathfrak{a}(\mathcal{O}) := \left\{ \psi : T \to S \;\middle|\; \prod_{U \in \mathrm{Orb}(S)} \mathcal{O}(T \times_U S) \neq \varnothing \right\} \subset \mathbb{F}_{\mathcal{T}} \qquad \triangleleft$$

Corollary III.1.31. *There exists an essentially unique equivalence $h_{\leq 0}\mathcal{O}^{\otimes} \simeq \mathcal{N}^{\otimes}_{\mathfrak{a}(\mathcal{O})_\infty}$; in particular, $\mathfrak{a}$ is a $\mathcal{T}$-$(\infty,2)$-categorical left adjoint*

$$\underline{\mathrm{wIndexCat}}_{\mathcal{T}} \underset{\mathcal{N}^{\otimes}_{(-)\infty}}{\overset{\mathfrak{a}}{\rightleftarrows}} \underline{\mathrm{OP}}_{\mathcal{T}} \qquad (\mathfrak{a} \dashv \mathcal{N}^{\otimes}_{(-)\infty})$$

PROOF. To construct the fiberwise adjunctions, it suffices to note that $\mathfrak{a}(\mathcal{N}_{I\infty}) = I$ and that $\mathfrak{a}(h_{\leq d}\mathcal{O}) = \mathfrak{a}(\mathcal{O})$ and apply Proposition II.1.2. We're left with restriction-stability of $\mathfrak{a}$, which is obvious. □

III.1.2.3 The smashing localization on $\mathcal{T}$-n-operads associated with $\mathcal{N}^{\otimes}_{I\infty}$ Note the following.

Proposition III.1.32. *If $\varphi\colon \mathcal{O}^{\otimes} \to \mathcal{P}^{\otimes}$ is an h_n-equivalence and $\mathcal{Q}^{\otimes}$ is a $\mathcal{T}$-operad, then the induced map*

$$\mathcal{Q}^{\otimes} \overset{BV}{\otimes} \varphi\colon \mathcal{Q}^{\otimes} \overset{BV}{\otimes} \mathcal{O}^{\otimes} \longrightarrow \mathcal{Q}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}$$

is an h_n-equivalence.

PROOF. By Corollary III.1.28, pullback along $\varphi \otimes \mathcal{Q}^{\otimes}$ yields an equivalence

$$\begin{array}{ccc} \mathrm{Mon}_{\mathcal{Q}}\underline{\mathrm{Mon}}_{\mathcal{P}}(\mathcal{S}_{\leq n}) & \longrightarrow & \mathrm{Mon}_{\mathcal{Q}}\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S}_{\leq n}) \\ \wr \downarrow & & \wr \downarrow \\ \mathrm{Mon}_{\mathcal{Q}\otimes\mathcal{P}}(\mathcal{S}_{\leq n}) & \longrightarrow & \mathrm{Mon}_{\mathcal{Q}\otimes\mathcal{O}}(\mathcal{S}_{\leq n}) \end{array}$$

Noting that $\pi_0 U\varphi$ is an equivalence, applying Corollary III.1.28 shows that $\varphi \otimes \mathcal{Q}^{\otimes}$ is an h_n-equivalence. □

In particular, Propositions II.2.56 and III.1.32 construct a symmetric monoidal structure on $\mathrm{Op}_{\mathcal{T},n}$ together with a symmetric monoidal structure on h_n. The tensor product for this structure is $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes}_n \mathcal{P}^{\otimes} \simeq h_n\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}\right)$, and in particular, Corollary III.1.8 shows that $\mathcal{N}_{I\infty}^{\otimes} \in \mathrm{Op}_{\mathcal{T},n}$ is an idempotent algebra. It's easy to identify its smashing localization, and in fact, its h_n-preimages.

Corollary III.1.33. *Suppose $\mathcal{O}^{\otimes}$ is an almost-unital $\mathcal{T}$-operad. Then, the following conditions are equivalent:*

(a') The map $\mathrm{Bor}_I^{\mathcal{T}}\mathcal{O}^{\otimes} \to (h_n U\mathcal{O})^{I-\sqcup}$ is an h_n-equivalence.

(e') For all $\mathcal{T}$-$(n+1)$-operads $\mathcal{P}^{\otimes}$, the $\mathcal{T}$-operad $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{P})$ is cocartesian at I.

(f') For all $A\mathcal{O}$-symmetric monoidal $(n+1)$-categories $\mathcal{C}^{\otimes}$, the $A\mathcal{O}$-symmetric monoidal $(n+1)$-category $\underline{\mathrm{Alg}}_{\mathcal{O}}^{\otimes}(\mathcal{C})$ is cocartesian at I.

(g') The $\mathcal{T}$-$(n+1)$-category $\underline{\mathrm{Mon}}_{\mathcal{O}}(\mathcal{S}_{\leq n})$ is I-semiadditive.

(j') The unit map tensors to an h_n-equivalence

$$h_n(\mathrm{id}\otimes !) : h_n\mathcal{O}^{\otimes} \simeq h_n\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathrm{triv}_{\mathcal{T}}^{\otimes}\right) \xrightarrow{\sim} h_n\left(\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{N}_{I\infty}^{\otimes}\right).$$

PROOF. The implication Condition (a') $\implies$ Condition (e') is Theorem II.2.32 and Corollary III.1.25; the implication Condition (e') $\implies$ Condition (f') is obvious; the implication Condition (f') $\implies$ Condition (g') is Corollary II.2.5. The implication Condition (g') $\implies$ Condition (j') follows straightforwardly from Corollaries II.2.5 and III.1.28 and Theorems II.2.1 and II.2.32. Lastly, the implication Condition (j') $\implies$ Condition (a') follows from the following:

$$\begin{aligned} h_n\mathrm{Bor}_I^{\mathcal{T}}\left(\mathcal{O}^{\otimes}\right) &\simeq h_n\mathrm{Bor}_I^{\mathcal{T}}\left(\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}_{I\infty}\right) && \text{Condition (j')} \\ &\simeq h_n(U\mathcal{O})^{I-\sqcup} && \text{Corollary III.1.12} \\ &\simeq (h_n U\mathcal{O})^{I-\sqcup} && \text{Corollary III.1.26.} \end{aligned}$$ □

III.1.3 Connectivity of parameterized Boardman-Vogt tensor products Along the way to prove Theorems III.1.1 and III.1.3, we will characterise connectivity of $\mathcal{T}$-operads as follows.

Theorem III.1.34. *Let $\mathcal{P}^{\otimes}$ be a $\mathcal{T}$-operad, I an almost-unital weak indexing category, and ℓ a natural number. Then, the following conditions are equivalent:*

(a) *$\mathcal{P}^{\otimes}$ is ℓ-connected at I.*

(b) *For all n-toposes $\mathcal{C}$ (with $n \leq \infty$), I-admissible V-sets $S \in \mathbb{F}_I^V$, and S-tuples of $\mathcal{P}$-monoids $(X_U) \in \prod_{U \in \mathrm{Orb}(S)} \mathrm{Mon}_{\mathcal{P}_{\underline{U}}}(\mathcal{C})$, the S-indexed $\mathcal{P}$-monoid Wirthmüller map*

$$W_{S,(X_K)} \colon \coprod_U^S X_U \longrightarrow \prod_U^S X_U$$

is ℓ-connected.

(c) *For all I-admissible V-sets $S \in \mathbb{F}_I^V$ and S-tuples of $\mathcal{P}$-monoids $(X_U) \in \prod_{U \in \mathrm{Orb}(S)} \mathrm{Mon}_{\mathcal{P}_{\underline{U}}}(\mathcal{S})$, the S-indexed $\mathcal{P}$-monoid space Wirthmüller map*

$$W_{S,(X_U)} \colon \coprod_U^S X_U \longrightarrow \prod_U^S X_U$$

is ℓ-connected.

Remark III.1.1.35. Note the following.

(1) When $G = e$, only the implication (a) $\implies$ (b) is argued in [SY19], but (b) $\implies$ (c) is obvious and, other than a diagram chase, the core idea of our argument that (c) $\implies$ (a) is already present in [SY19].

(2) In the case that $\mathcal{C}$ is an n-topos for some $0 \leq n \leq \infty$, the above definitions are equivalent to *ℓ-truncatedness* and *$(\ell-1)$-connectiveness* in the sense of [HTT, Def. 6.5.1.10] by [SY19, Lem. 4.2.6] and [HTT, Props. 6.5.1.12, 6.5.1.19].

(3) While proving Theorem III.1.34, we will verify that Condition (b) is further equivalent to the condition that the $\mathrm{Coeff}^H\mathcal{C}$-map underlying $W_{S,(X_K)}$ is pointwise ℓ-connected; moreover, Condition (c) is equivalent to the condition that the underlying H-space map is ℓ-connected, i.e. its associated maps on J-fixed point spaces are surjective on path components with ℓ-connected fiber for each $J \subset H$. ◁

III.1.3.1 Connectivity of algebras can be detected in the value topos Fix $\mathcal{C}$ an n-topos for some $n \leq \infty$.

Lemma III.1.1.36. *A map $f \colon C \to D$ in $\mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ is ℓ-connected if and only if, for all $V \in \mathcal{T}^{\mathrm{op}}$, the fixed point map $C^V \to D^V$ is ℓ-connected.*

Proof. Per Remark III.1.1.35, it is equivalent to prove that ℓ-connectiveness of a morphism in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}},\mathcal{C})$ is measured elementwise. Indeed, since (co)limits in $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}},\mathcal{C})$ are computed elementwise, effective epimorphisms and diagonals are as well. The former proves the statement for (-2)-connectiveness, and the latter together with the diagonal presentation of [HTT, Prop. 6.5.1.18] shows that the statement for $(\ell-1)$-connectiveness implies the statement for ℓ-connectiveness, so the lemma follows by induction. □

Proposition III.1.37. *Given a map $f\colon X \to Y$ in $\mathrm{Seg}_{\mathrm{Tot}_{\mathcal{O}}}(\mathcal{C})$, if the underlying map Uf in $\mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ is ℓ-connected, then f is ℓ-connected.*

Proof. In view of [SY19, Lem. 4.4.1], it suffices to verify that the monad $T_{\mathcal{O}}\colon \mathrm{Coeff}^{\mathcal{T}}\mathcal{C} \to \mathrm{Coeff}^{\mathcal{T}}\mathcal{C}$ preserves ℓ-connected morphisms; by Lemma III.1.36, it suffices to verify that whenever each $\mathcal{C}$-diagram $X^V \to Y^V$ is ℓ-connected, each induced map $T_{\mathcal{O}}X^W \to T_{\mathcal{O}}X^W$ is ℓ-connected. But by the monad formula of Eq. (21), it suffices to note that ℓ-connected morphisms in an ∞-topos are closed under cartesian products and colimits [HTT, Cor. 6.5.1.13, Prop. 5.2.8.6]. □

For instance, U preserves the terminal object and is conservative, so it also reflects the property of being terminal; applying Proposition III.1.37 in the case $Y = *$ shows that U reflects n-connectivity of objects.

Remark III.1.38. Since U is a right adjoint, it preserves n-truncatedness and n-truncated objects. ◁

Warning III.1.39. Proposition III.1.37 is delicate for a few reasons.

(1) If $\mathcal{O}$ is not n-connected, then the free $\mathcal{O}$-algebra monad $T_{\mathcal{O}}\colon \mathcal{C}_V \to \mathcal{C}_V$ may itself may fail to preserve n-connected objects; indeed, we have $T_{\mathcal{O}^*V} \simeq \coprod_{S\in\mathbb{F}_V} \mathrm{Fr}_{\mathcal{C}}\mathcal{O}(S)_{h\,\mathrm{Aut}_V S}$, which is often not much more highly connected than the individual spaces $\mathcal{O}(S)_{h\,\mathrm{Aut}_V S}$.

(2) U does not generally *preserve* ℓ-connectivity of objects or morphisms for instance, given an $\ell \geq (k+1)$-connected space X, the equivalence $\Omega^k\colon \mathcal{S}_{*,\geq k+1} \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{S}_{\geq 1})$ of the recognition principle exhibits $\Omega^k X$ as an ℓ-connected $\mathbb{E}_k$-algebra such that $U\Omega^n X$ is only in general $(\ell - k)$-connected.

(3) For a similar reason, U does not usually reflect ℓ-truncatedness of morphisms or objects. ◁

III.1.3.2 The proof of Theorem III.1.34 We now begin to reduce Theorem III.1.34 to the case $n \leq \ell + 1$ with the following.

Lemma III.1.40. *The truncation functor $\tau_{\leq\ell}\colon \mathcal{C} \to \tau_{\leq\ell}\mathcal{C}$ extends to a $\mathcal{T}$-functor*

$$\tau_{\mathcal{O}}\colon \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}}(\mathcal{C}) \to \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}}(\tau_{\leq\ell}\mathcal{C})$$

satisfying $\tau_{\mathcal{O}}W_{S,X} = W_{S,\tau_{\mathcal{O}}X}$. Moreover, the inclusion $\iota\colon \tau_{\leq\ell}\mathcal{C} \to \mathcal{C}$ extends to a fully faithful $\mathcal{T}$-functor

$$\iota_{\mathcal{O}}\colon \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}}(\tau_{\leq\ell}\mathcal{C}) \hookrightarrow \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{O}}(\mathcal{C})$$

such that $\tau_{\mathcal{O}}W_{S,\iota_{\mathcal{O}}X} = W_{S,X}$.

Proof. Since $\tau_{\leq\ell}$ is product-preserving [HTT, Lem. 6.5.1.2], $\tau_{\leq\ell}\colon \underline{\mathrm{Coeff}}^{\mathcal{T}}\mathcal{C} \to \underline{\mathrm{Coeff}}^{\mathcal{T}}\tau_{\leq\ell\mathcal{C}}$ is a $\mathcal{T}$-symmetric monoidal left adjoint for the cartesian structure Theorem II.2.1; everything other than the equalities involving $W_{S,X}$ then follows straightforwardly from the doctrinal adjunction.

In particular, $\tau_{\mathcal{O}}$ is a $\mathcal{T}$-functor which preserves indexed products and coproducts; this implies that $\tau_{\mathcal{O}} W_{S,X} = W_{S,\tau_{\mathcal{O}}X}$. The remaining equality follows from fully faithfulness by noting that

$$\tau_{\mathcal{O}} W_{S,\iota_{\mathcal{O}}X} = W_{S,\tau_{\mathcal{O}}\iota_{\mathcal{O}}X} = W_{S,X}. \qquad \square$$

We say that a map $f\colon X \to Y$ in an n-topos is an *ℓ-equivalence* if it is a $\tau_{\leq \ell}$-equivalence; if f admits a section, this is equivalent to f being ℓ-connected (see [SY19, Prop. 4.3.5] or note that this follows by splitting the long exact sequence in homotopy). In general, it is *implied* by ℓ-connectedness, as $\tau_{\leq \ell}$ preserves ℓ-connectedness and ℓ-connected maps between ℓ-truncated objects are equivalences. We apply this by equivariantizing [SY19, Lem. 5.1.1].

Lemma III.1.41. *If $\mathcal{C}^{I-\times}$ is a Cartesian I-symmetric monoidal ∞-category for $I = \mathfrak{a}\mathcal{O}$ and $S \in \underline{\mathbb{F}}_I$, then the image of the $\mathcal{O}$-algebra Wirthmüller map $W_{X,S}\colon \coprod_U^S X_U \to \prod_U^S X_U$ under $U\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})_V \to \mathcal{C}_V$ admits a section.*

Proof of Lemma III.1.41. Let $i_U\colon Y_U \to \mathrm{Res}_U^V \coprod_{U'}^S Y_{U'}$ be mate to the inclusion $\mathrm{Ind}_U^V Y_U \hookrightarrow \coprod_{U'}^S Y_{U'}$ and fix an operation $\mu \in \mathcal{O}(S)$. We verify commutativity of the following diagram, giving a section $\mu_{\coprod X_U}\sigma_1 f$ for $W_{X,S}$.

$$\begin{array}{ccccc}
\prod_U^S\left(\mathrm{Res}_U^V \coprod_U^V X_U\right) & \xrightarrow[\sim]{\sigma_1} & \left(\coprod_U^S X_U\right)^{\times S} & \xrightarrow{\mu_{\coprod X_U}} & \coprod_U^S X_U \\
\uparrow{\scriptstyle f=(i_U)_{U\in\mathrm{Orb}(S)}} & & \downarrow{\scriptstyle h=\left(W_{\mathrm{Res}_U^V X,\mathrm{Res}_U^V S}\right)_{U\in\mathrm{Orb}(S)}} & & \downarrow{\scriptstyle W_{X,S}} \\
 & & \left(\prod_U^S X_U\right)^{\times S} & \xrightarrow{\mu_{\prod X_U}} & \prod_U^S X_U \\
 & & \sigma_2\downarrow{\scriptstyle \sim} & & \| \\
\prod_U^S X_U & \xrightarrow{g=(\iota_U)_{U\in\mathrm{Orb}(S)}} & \prod_U^S X_U^{\times \mathrm{Res}_U^V S} & \xrightarrow{\left(\mathrm{Res}_U^V \mu_{X_U}\right)_{U\in\mathrm{Orb}(S)}} & \prod_U^S X_U
\end{array}$$

Here, $\mu_X\colon X^S \to X$ is the structure map corresponding with μ for a $\mathrm{Res}_V^{\mathcal{T}}\mathcal{O}$-algebra X.

Note that the top right square is commutative by the fact that $W_{S,X}$ is an $\mathcal{O}$-algebra morphism and the bottom right follows by unwinding definitions. Moreover, $\mu \circ g$ is the external product of a collection of endomorphisms $X_U \xrightarrow{\iota_U} X_U^{\times \mathrm{Res}_U^V S} \xrightarrow{\mu} X_U$, the first of which is the inclusion of a unit on all but one factor:

$$\begin{array}{ccccc}
X_U & \xrightarrow{\iota_U} & X_U^{\times \mathrm{Res}_U^V S} & \xrightarrow{\mathrm{Res}_U^V \mu} & X_U \\
\wr\downarrow & & \wr\downarrow & \nearrow & \\
X_U \times \prod_W^{\mathrm{Res}_U^V S - \{\alpha\}} 1_W & \xrightarrow{(\mathrm{id},\eta)} & X_U \times \prod_W^{\mathrm{Res}_U^V S - \{\alpha\}} X_W & &
\end{array}$$

in particular, $\mu \circ \iota_U$ is homotopic to the identity by unitality, so $\mu \circ g \sim \mathrm{id}$ and the bottom triangle commutes.

To characterize the composite morphism of the left rectangle, we may equivalently characterize the composite map $\pi_U \sigma_2 h \sigma_1 f \colon \prod_U^S X_U \to \mathrm{CoInd}_U^V X_U^{\times \mathrm{Res}_U^V S}$; in fact, under the expression $X_U^{\times \mathrm{Res}_U^V S} \simeq \prod_W^{\mathrm{Res}_U^V S} \mathrm{Res}_W^U X_U$, it suffices to characterize the composite morphism $\prod_U^S X_U \to \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U$ and verify that it is homotopic to the relevant projection of g for each W, U.

In particular, the relevant projection of g is the composite morphism

$$\prod_U^S X_U \twoheadrightarrow \mathrm{CoInd}_U^V X_U \xrightarrow{\delta_{U,W}} \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U$$

where $\delta_{U,W}$ is a Kronecker delta[1]

$$\delta_{U,W} = \begin{cases} \mathrm{id} & U = W; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, note that the projection $\pi_U \sigma_2 h \sigma_1 \colon \prod_U^S X_U \to X_U^{\times \mathrm{Res}_U^V S}$ itself factors as

$$\prod_U^S \left(\mathrm{Res}_U^V \coprod_U^V X_U \right) \twoheadrightarrow \mathrm{CoInd}_U^V X_U \xrightarrow{\widetilde{f_U}} \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U,$$

so we're tasked with verifying that $\widetilde{f_U}$ is homotopic to $\delta_{U,W}$. Indeed, this follows by examining the following diagram:

$$\begin{array}{ccccccc}
\prod_U^S X_U & \xrightarrow{f} & \prod_U^S \left(\mathrm{Res}_U^V \coprod_U^V X_U \right) & \simeq & \left(\coprod_U^S X_U \right)^{\times S} & \xrightarrow{h} & \left(\prod_U^S X_U \right)^{\times S} \simeq \prod_U^S X_U^{\times \mathrm{Res}_U^V S} \\
\downarrow & \searrow & & & & & \downarrow \\
\mathrm{CoInd}_U^V X_U & \xrightarrow{\mathrm{CoInd}_U^V i_U} & \mathrm{CoInd}_U^V \mathrm{Res}_U^V \coprod_U^V X_U & \xrightarrow{\mathrm{CoInd}_U^V W} & \mathrm{CoInd}_U^V \mathrm{Res}_U^V \prod_U^V X_U & \simeq & X_U^{\times \mathrm{Res}_U^V S} \\
 & \searrow^{\delta_{U,W}} & & & \downarrow & & \\
 & & & & \mathrm{CoInd}_W^V \mathrm{Res}_W^U X_U & &
\end{array}$$

□

Proof of Theorem III.1.34. Assume $\mathcal{P}^{\otimes}$ is ℓ-connected at I, i.e. Condition (a). Writing X for an S-tuple in $\underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{I}} \mathcal{P}}(\mathcal{C})$, we chase $W_{S,X}$ around the following commutative diagram:

$$\begin{array}{ccccc}
\underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{I}} \mathcal{P}}(\tau_{\leq \ell} \mathcal{C}) & \xrightarrow{\iota_{\mathcal{P}}} & \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{I}} \mathcal{P}}(\mathcal{C}) & \xrightarrow{\tau_{\mathcal{P}}} & \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{I}} \mathcal{P}}(\tau_{\leq \ell} \mathcal{C}) \\
\downarrow{\scriptstyle U_{\leq \ell}} & & \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U_{\leq \ell}} \\
\underline{\mathrm{Coeff}}^{\mathcal{I}} \tau_{\leq \ell} \mathcal{C} & \xrightarrow{\iota} & \underline{\mathrm{Coeff}}^{\mathcal{I}} \mathcal{C} & \xrightarrow{\tau_{\leq \ell}} & \underline{\mathrm{Coeff}}^{\mathcal{I}} \tau_{\leq \ell} \mathcal{C}
\end{array}$$

[1] Here, 0 is the image under U of the zero map in the (pointed) ∞-category $\mathrm{Mon}_{\mathrm{Res}_U^V \mathcal{O}}(\mathrm{Res}_U^V \mathcal{C})$.

In particular, by Corollary III.1.33 and Lemma III.1.40, $\tau_{\mathcal{P}} W_{S,X} \sim W_{S,\tau_{\mathcal{O}} X}$ is an equivalence, so

$$U_{\leq \ell} \tau_{\mathcal{P}} W_{S,X} \sim \tau_{\leq \ell} U W_{S,X}$$

is an equivalence, i.e. $U W_{S,X}$ is an ℓ-equivalence. In turn, by Lemma III.1.41 this implies that $U W_{S,X}$ is ℓ-connected, so Proposition III.1.37 implies that $W_{S,X}$ is ℓ-connected, i.e. Condition (b).

The implication Condition (b) $\implies$ Condition (c) is immediate, so assume Condition (c), i.e. fix the case $\mathcal{C} \coloneqq \mathcal{S}$ and assume that that $W_{S,X}$ is ℓ-connected for all $X \in \mathrm{Alg}_{\mathcal{P}} \mathcal{S}$ and $S \in \underline{\mathbb{F}}_I$. We may invert the above argument: this time, since $\tau_{\mathcal{P}}$ is a left adjoint, it preserves ℓ-connectedness, so $\tau_{\mathcal{P}} W_{S,X}$ is ℓ-connected. Hence

$$U_{\leq \ell} \tau_{\mathcal{P}} W_{S,X} = \tau_{\leq \ell} U W_{S,X}$$

is an equivalence, i.e. $U W_{S,X}$ is an ℓ-equivalence. We've shown that Condition (c) implies the following.

(d) For all I-admissible V-sets $S \in \mathbb{F}_{I,V}$ and S-tuples of $\mathcal{P}$-monoids $X = (X_K) \in \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{P}}(\mathcal{S})_S$, the underlying V-space map of the S-indexed $\mathcal{P}$-monoid space Wirthmüller map

$$U W_{S,X} \colon U \coprod_K^S X_K \longrightarrow U \prod_K^S X_K \simeq \prod_K^S U X_K$$

is an ℓ-equivalence.

We're left with showing that Condition (d) implies Condition (a); indeed, choosing $X = \iota_{\mathcal{O}} Y$, the map

$$\tau_{\leq \ell} U W_{S,\iota_{\mathcal{O}} Y} = U_{\leq \ell} \tau_{\mathcal{O}} W_{S,\iota_{\mathcal{O}} Y} = U_{\leq \ell} W_{S,\tau_{\mathcal{O}} \iota_{\mathcal{O}} Y} = W_{S,Y}$$

is an equivalence, i.e. $\underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{O}}(\tau_{\leq \ell} \mathcal{C})$ is I-semiadditive. Thus Corollary III.1.28 implies Condition (a). □

III.1.3.3 The proof of Theorems III.1.1 and III.1.3 Note the following.

Proposition III.1.42. *If $\mathcal{P}^{\otimes}$ is ℓ-connected at I, then for all $(k+\ell+2)$-toposes $\mathcal{C}$, the reduced endomorphism I-operad $\mathrm{End}_X\left(\underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{P}}(\mathcal{C})^{I-\times}\right)$ is an I-$(k+1)$-operad.*

Proof. Since $\mathcal{C}$ is a $(k+\ell+2)$-category, X is $(k+\ell+2)$-truncated, and Theorem III.1.34 implies that $W_{X,S}$ is ℓ-connected, so the result follows from Corollary II.2.47. □

We quickly acquire a slightly weakened version of Theorem III.1.3.

Corollary III.1.43. *Suppose $\mathcal{T}$ is an atomic orbital ∞-category, $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ are unital $\mathcal{T}$-operads and I is a unital weak indexing category. If $\mathcal{O}^{\otimes}$ is k-connected at I and $\mathcal{P}^{\otimes}$ is ℓ-connected at I, then $\mathcal{O}^{\otimes} \overset{BV}{\otimes} \mathcal{P}^{\otimes}$ is $(k+\ell+2)$-connected at I.*

PROOF. By Proposition III.1.42, we know that $\mathrm{End}_X^{I,\mathrm{red}}\left(\underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{P}}(\mathcal{C})^{\mathcal{T}^{-\times}}\right) \simeq \mathrm{End}_X^{I,\mathrm{red}}\left(\underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}}\mathcal{P}}(\mathcal{C})^{I^{-\times}}\right)$ is an I-$(k+1)$-operad for $\mathcal{C}$ a $(k+\ell+2)$-topos; by Corollary III.1.33 this shows that $\mathrm{Alg}_{\mathcal{O}}\mathrm{End}_X\left(\underline{\mathrm{Seg}}_{\mathcal{P}}(\mathcal{C})^{\mathcal{T}^{-\times}}\right)$ is I-cocartesian, so Proposition II.2.48 shows that

$$\mathrm{CMon}_I\underline{\mathrm{Alg}}_{\mathcal{O}}\underline{\mathrm{Alg}}^{\otimes}_{\mathcal{P}}(\mathcal{C})^{\simeq} \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{O}}\underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C})^{\simeq}.$$

By Corollary III.1.28, this implies that the map

$$\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \simeq \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathrm{triv}^{\otimes}_{\mathcal{T}} \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes!} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{N}^{\otimes}_{I\infty}$$

is an $h_{k+\ell+2}$-equivalence, so Corollary III.1.33 shows that $\mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes}$ is $(k+\ell+2)$-connected at I. □

PROOF OF THEOREMS III.1.1 AND III.1.3. Restriction assembles to a symmetric monoidal equivalence

$$\mathrm{Op}^{\otimes}_{\mathcal{T}} \simeq \lim_{V\in\mathcal{T}} \mathrm{Op}^{\otimes}_{V}$$

such that, given a morphism $V \to W$ in $\mathcal{T}$ and S a finite V-set, $\mathrm{Res}^{\mathcal{T}}_{V}\mathcal{O}(S) \simeq \mathcal{O}(S)$. In particular, Theorems III.1.1 and III.1.3 may be verified after restriction to each to $V \in \mathcal{T}$, in which case the base ∞-category $\mathcal{T}_{/V}$ has a terminal object.

Moreover, each $\mathcal{O}(S)$ and $\mathcal{P}(S)$ are easily determined by arity support except in the case $V \in \upsilon(\mathcal{O}) = \upsilon(\mathcal{P})$, and arity support is additive in the predicted way by [Ste25c, Prop. 1.44]; thus Theorems III.1.1 and III.1.3 may be verified after restriction to each $V \in \upsilon(\mathcal{O})$, in which case $\mathrm{Res}^{\mathcal{T}}_{V}\mathcal{O}^{\otimes}$ and $\mathrm{Res}^{\mathcal{T}}_{V}\mathcal{P}^{\otimes}$ are unital. This and Corollary III.1.43 together yield Theorem III.1.3, and Theorem III.1.1 follows by setting $I := \mathfrak{a}\mathcal{O}$. □

III.2 Additivity of equivariant little disk operads

For this section, fix G a finite Lie group. Given a pair of G-equivariant smooth tangential structures $\Theta\colon B_\Theta \to B_G O(k)$ and $\Psi\colon B_\Psi \to B_G O(k')$ we write

$$\Theta\oplus\Psi\colon B_\Theta \times B_\Psi \xrightarrow{\Theta\times\Psi} B_G O(k)\times B_G O(k') \xrightarrow{\oplus} B_G O(k+k')$$

for the *direct sum structure*. We will prove the following.

Theorem III.2.1. *The natural G-operad maps $\mathbb{E}^{\otimes}_{\Theta},\mathbb{E}^{\otimes}_{\Psi} \to \mathbb{E}^{\otimes}_{\Theta\oplus\Psi}$ extend to an equivalence*

$$\mathbb{E}^{\otimes}_{\Theta} \overset{BV}{\otimes} \mathbb{E}^{\otimes}_{\Psi} \xrightarrow{\sim} \mathbb{E}^{\otimes}_{\Theta\oplus\Psi}.$$

In particular, the tensor product product of G-symmetric monoidal ∞-categories acts additively on disks:

$$\times\colon \underline{\mathrm{Disk}}^{\Theta}_{G} \circledast \underline{\mathrm{Disk}}^{\Psi}_{G} \xrightarrow{\sim} \underline{\mathrm{Disk}}^{\Theta\oplus\Psi}_{G}.$$

We will use disintegration and assembly to bootstrap from the case that B_Θ and B_Φ are contractible, which is the following.

Theorem III.2.2. *The natural G-operad maps $\mathbb{E}_V^\otimes, \mathbb{E}_W^\otimes \to \mathbb{E}_{V\oplus W}^\otimes$ extend to an equivalence*

$$\mathbb{E}_V^\otimes \overset{BV}{\otimes} \mathbb{E}_W^\otimes \xrightarrow{\sim} \mathbb{E}_{V\oplus W}^\otimes.$$

III.2.1 Wreath products and related bookkeeping We will define a *Wreath product* operation on $\mathcal{T}$-preoperads which is *easy to map out of* and prove the following theorem.

Theorem III.2.3. *Fix algebraic pattern maps α_i, $\varphi_\mathcal{E}$, and $\varphi_\mathcal{P}$ over $\mathsf{Span}(\mathbb{F}_\mathcal{T})$ making the following commute*

$$\begin{array}{ccc} \mathcal{P}_1^\otimes \wr \mathcal{P}_2^\otimes & \xrightarrow{\varphi_\mathcal{P}} & \mathcal{P}_3^\otimes \\ {\scriptstyle \alpha_1 \wr \alpha_2}\downarrow & & \downarrow{\scriptstyle \alpha_3} \\ \mathcal{E}_1^\otimes \wr \mathcal{E}_2^\otimes & \xrightarrow{\varphi_\mathcal{E}} & \mathcal{E}_3^\otimes \end{array}$$

and satisfying the following conditions:

(BK-a) Each α_i is a Morita localization;

(BK-b) $\varphi_\mathcal{P}$ is a Γ-Morita equivalence; and

(BK-c) each $\mathcal{E}_i^\otimes$ is a reduced $\mathcal{T}$-operad.

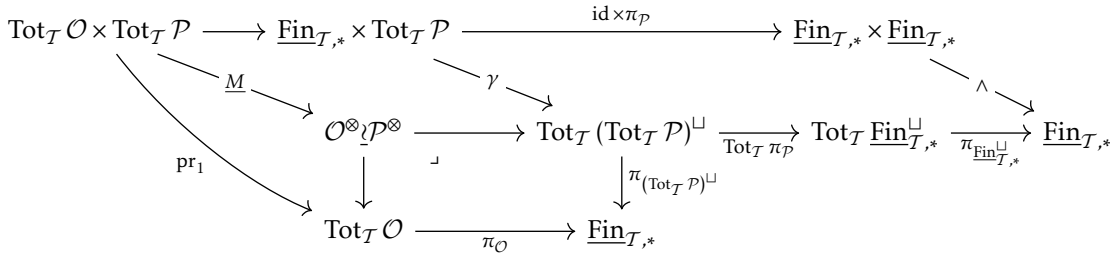
Then, the map $\mathcal{E}_1^\otimes \wr \mathcal{E}_2^\otimes \to \mathcal{E}_3^\otimes$ induces an equivalence

$$\mathcal{E}_1^\otimes \overset{BV}{\otimes} \mathcal{E}_2^\otimes \xrightarrow{\sim} \mathcal{E}_3^\otimes.$$

For this theorem, we define a Γ-*Morita equivalence of $\mathcal{T}$-preoperads* to be a morphism of $\mathcal{T}$-preoperads which presents a Morita equivalence after pullback along the map $s\colon \underline{\mathsf{Fin}}_{\mathcal{T},*} \to \mathsf{Span}_{\mathfrak{o}(\mathcal{T})}(\mathbb{F}_\mathcal{T})$.

III.2.1.1 The definition of the wreath product model We first define a parameterized version of $\wr$.

Construction III.2.4. Following Construction II.2.50, we define the diagram



We refer to $\mathcal{O}^{\otimes} \underline{\wr} \mathcal{P}^{\otimes}$ as the Γ-*wreath product* of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$, and note that M is a bifunctor, and so presents a morphism

$$\underline{M}' \colon \mathrm{Tot}_{\mathcal{T}} \mathcal{O}^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathcal{P}^{\otimes} \longrightarrow \mathcal{O}^{\otimes} \underline{\wr} \mathcal{P}^{\otimes}. \tag{42}$$ ◁

We will eventually prove the following.

Proposition III.2.5. *The map $\underline{M}'$ of Eq. (42) is an $L_{\mathrm{Op}_{\mathcal{T}}}$-equivalence.*

Before then, we make heavy use of the following observation.

Observation III.2.6. $\mathcal{O}^{\otimes} \underline{\wr} \mathcal{P}^{\otimes}$ is a pullback of a cospan of algebraic patterns pulled back from $\mathcal{T}$-operads, and hence $\mathcal{T}$-preoperads; in particular, writing

$$\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} \simeq \mathrm{Tot}\, \mathcal{O} \times_{\mathsf{Span}_{\mathfrak{o}(T)}(\mathbb{F}_T)} \mathrm{Tot}\, (\mathrm{Tot}_{\mathcal{T}} \mathcal{P})^{\sqcup},$$

limit-preservation of the pullback s^* along $s \colon \mathrm{Tot}\, \underline{\mathsf{Fin}}_{G,*} \to \mathsf{Span}(\mathbb{F}_G)$ yields an equivalence

$$\mathcal{O}^{\otimes} \underline{\wr} \mathcal{P}^{\otimes} \simeq s^* \left(\mathcal{O}^{\otimes} \wr \mathcal{O}^{\otimes} \right).$$

We will generally work directly with $\wr$. ◁

Now Proposition III.2.5 indeed implies the following important corollary:

Corollary III.2.7. *If $\mathcal{O}_1^{\otimes} \wr \mathcal{O}_2^{\otimes} \to \mathcal{O}_3^{\otimes}$ is a Γ-Morita equivalence, then it induces an equivalence*

$$\mathcal{O}_1^{\otimes} \overset{BV}{\otimes} \mathcal{O}_2^{\otimes} \xrightarrow{\sim} \mathcal{O}_3^{\otimes}.$$

III.2.1.2 How do we use wreath products? Now, to understand these better, note the following proposition.

Proposition III.2.8. *Let $\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}$ be $\mathcal{T}$-operads. Then, the fiber ∞-categories of $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ are given by*

$$\left(\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes} \right)_T \simeq \coprod_{T \to S} \mathcal{O}_S \times \mathcal{P}_T.$$

Moreover, given a multi-arity $g \colon T' \to T$, the multimorphism spaces of $\mathcal{O}^{\otimes} \wr \mathcal{P}^{\otimes}$ over g are

$$\mathrm{Mul}^{g}_{\mathcal{O} \wr \mathcal{P}}(h' \colon T' \to S', \mathbf{C}', \mathbf{D}';\ h \colon T \to S, \mathbf{C}, \mathbf{D}) \simeq \coprod_{\substack{f \colon S' \to S \\ fh = h'g}} \mathrm{Mul}^{f}_{\mathcal{O}}(\mathbf{C}'; \mathbf{C}) \times \mathrm{Mul}^{g}_{\mathcal{P}}(\mathbf{D}'; \mathbf{D}).$$

The wreath product construction is of a nontrivial combinatorial nature; for simplicity's sake, we pre-macerate these combinatorics, yielding recognition results which will provide our interface for working

with wreath products going forward. First, we describe how to compute wreath products of $\mathcal{T}$-operads with contractible structure spaces.

Proposition III.2.9. *Suppose $\mathcal{T}$ is an atomic orbital 1-category and $\mathcal{O}^t, \mathcal{P}^t, \mathcal{Q}^t$ are genuine topological $\mathcal{T}$-operads such that $\mathcal{Q}^t$ has contractible structure spaces. Then, a relative pattern map $\mathcal{O}^t \wr \mathcal{P}^t \to \mathcal{Q}^t$ is uniquely specified by a map of coefficient systems of sets*

$$\varphi\colon \coprod_{T \to S} \pi_0 \mathcal{O}_S \times \pi_0 \mathcal{P}_T \to \pi_0 \mathcal{Q}_T$$

such that for all data

$$\begin{array}{ccc} T' & \xrightarrow{g} & T \\ \downarrow & & \downarrow \\ S' & \longrightarrow & V \end{array}; \qquad ((\mathbf{C}'; C), (\mathbf{B}'; \mathbf{B})) \in \mathcal{O}_{S'} \times \mathcal{O}_V \times \mathcal{P}_{T'} \times \mathcal{P}_T.$$

such that $\mathcal{O}(\mathbf{C}'; C) \times \mathrm{Mul}^g_{\mathcal{P}}(\mathbf{B}'; \mathbf{B}) \neq \varnothing$, we have $\mathrm{Mul}^h_{\mathcal{Q}}(\mathbf{B}'; \mathbf{B}) \neq \varnothing$.

Moreover, the resulting map satisfies Condition (ML-b) if and only if φ is an equivalence; in this case, φ is a Γ-Morita equivalence if the converse to the above compositionality statement holds.

Proof. First note that φ is a map $\mathrm{Ob}(\mathcal{O}^t \wr \mathcal{P}^t)_T \to \mathcal{Q}_T$; by Proposition III.2.8, to specify a topological functor $\mathcal{O}^t \wr \mathcal{P}^t \to \mathcal{Q}^t$ over $\mathbf{Span}(\mathbb{F}_T)$ it then suffices to define the data on mapping spaces. Indeed, we determine this canonically via the factorization

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{O}^\otimes \wr \mathcal{P}^\otimes}((\mathbf{C}', \mathbf{B}'), (\mathbf{B}', \mathbf{B})) & \longrightarrow & \mathrm{Map}_{\mathcal{Q}^\otimes}(\varphi(\mathbf{C}', \mathbf{B}', \varphi(\mathbf{B}'; \mathbf{B})) \\ \uparrow & & \uparrow \\ \mathrm{Mul}^f_{\mathcal{O}^\otimes \wr \mathcal{P}^\otimes}((\mathbf{C}', \mathbf{B}'); (\mathbf{C}, \mathbf{B})) & \dashrightarrow & \mathrm{Mul}^f_{\mathcal{Q}^\otimes}(\varphi(\mathbf{C}', \mathbf{B}'); \varphi(\mathbf{B}', \mathbf{B})) \end{array}$$

There exists an essentially unique filler as in the dashed arrow since whenever the domain is nonempty, the codomain is contractible by assumption. To verify functoriality, we must verify a that a collection of diagrams among products of multimorphism spaces of $\mathcal{Q}^\otimes$ commute; however, since $\mathcal{Q}^\otimes$ has contractible nonempty structure spaces, all terms in such diagrams are either empty or contractible, so all such diagrams commute.

The remaining statement follows by unwinding definitions to find that our assumption implies *contractible* (not just weakly contractible) fibers on active arrow categories. □

Now, we cover the case of maps to one-color $\mathcal{T}$-operads.

Proposition III.2.10. *Suppose $\mathcal{T}$ is a 1-category and $\mathcal{O}^t, \mathcal{P}^t, \mathcal{Q}^t$ are genuine topological $\mathcal{T}$-operads such that $\mathcal{Q}^t$ has one color. A relative pattern map $\mathcal{O}^t \wr \mathcal{P}^t \to \mathcal{Q}^t$ is uniquely specified by the data of, for each composable arities $\mathrm{Ind}^{\mathcal{T}}_V T \to \mathrm{Ind}^{\mathcal{T}}_V S \to V$ and compatible $\mathcal{P}$ and $\mathcal{O}$ (multi-)profiles $(\mathbf{B}'; \mathbf{B})$ and $(\mathcal{C}'; C)$, a homotopy class*

of continuous map

$$\varphi\colon \mathcal{O}(\mathbf{C}';C)\times \mathrm{Mul}^{g}_{\mathcal{P}}(\mathbf{B}';\mathbf{B}) \to \mathcal{Q}(T)$$

sending $\varphi 1 = 1$ *and such that the following diagrams commute:*

$$\begin{array}{ccc} \mathcal{O}(\mathbf{C}';C)\times \mathrm{Mul}^{g}_{\mathcal{P}}(\mathbf{B}';\mathbf{B}) & \xrightarrow{\quad\varphi\quad} & \mathcal{Q}(T) \\ \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\ \mathcal{O}\left(\mathrm{Res}^{V}_{U}\mathbf{C}';\mathrm{Res}^{V}_{U}C\right)\times \mathrm{Mul}^{\mathrm{Res}^{V}_{U}g}_{\mathcal{P}}\left(\mathrm{Res}^{V}_{U}\mathbf{B}';\mathrm{Res}^{V}_{U}\mathbf{B}\right) & \xrightarrow{\varphi} & \mathcal{Q}\left(\mathrm{Res}^{V}_{U}T\right) \end{array} \tag{43}$$

$$\begin{array}{ccc} \left(\mathcal{O}(\mathbf{C}';\mathbf{C})\times \mathrm{Mul}^{h}_{\mathcal{O}}(\mathbf{C}'';\mathbf{C}')\right)\times\left(\mathrm{Mul}^{g}_{\mathcal{P}}(\mathbf{B}';\mathbf{B})\times \mathrm{Mul}^{f}_{\mathcal{P}}(\mathbf{B}'';\mathbf{B}')\right) & \xrightarrow{(\gamma,\gamma)} & \mathcal{O}(\mathbf{C}'';C)\times \mathrm{Mul}^{f\circ g}_{\mathcal{P}}(\mathbf{B}'';\mathbf{B}) \\ \wr\downarrow & & \\ \left(\mathcal{O}(\mathbf{C}';C)\times \mathrm{Mul}^{f}_{\mathcal{P}}(\mathbf{B}';\mathbf{B})\right)\times\left(\mathrm{Mul}^{h}_{\mathcal{O}}(\mathbf{C}'';\mathbf{C}')\times \mathrm{Mul}^{g}_{\mathcal{P}}(\mathbf{B}'';\mathbf{B}')\right) & & \downarrow{\scriptstyle \varphi} \\ \downarrow{\scriptstyle (\varphi,\varphi)} & & \\ \mathcal{Q}(T)\times \mathrm{Mul}^{f\circ g}_{\mathcal{Q}}(T';T) & \xrightarrow{\quad\gamma\quad} & \mathcal{Q}(T') \end{array} \tag{44}$$

The associated functor acts on active mapping spaces via φ*.*

Proof. Since $\mathcal{Q}^{t} \to \mathbf{Span}(\mathbb{F}_{\mathcal{T}})$ is bijective on objects, there is exactly one map $\mathrm{Ob}\left(\mathcal{O}^{t} \wr \mathcal{P}^{t}\right) \to \mathrm{Ob}\,\mathcal{Q}^{t}$ over $\mathrm{Ob}\,\mathbf{Span}(\mathbb{F}_{\mathcal{T}})$: $(\mathbf{C},\mathbf{B}) \mapsto T'$. To construct such a functor which is compatible with inert arrows, we then must determine the data on mapping spaces, uniquely via the factorization

$$\begin{array}{ccccc} \mathrm{Map}_{\mathcal{O}^{t}\wr\mathcal{P}^{t}}((\mathbf{C}',\mathbf{B}'),(\mathbf{B}',\mathbf{B})) & \hookleftarrow & \mathrm{Mul}^{f}_{\mathcal{O}^{t}\wr\mathcal{P}^{t}}((\mathbf{C}',\mathbf{B}');(\mathbf{C},\mathbf{B})) & \longrightarrow & \mathcal{O}(\mathbf{C}'_{K};C_{K})\times \mathrm{Mul}^{g}_{\mathcal{P}}(\mathbf{B}'_{K};\mathbf{B}_{K}) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Map}_{\mathcal{Q}^{t}}(T',T) & \hookleftarrow & \mathrm{Mul}^{f}_{\mathcal{Q}^{t}}(T',T) & \longrightarrow & \mathcal{Q}(T'_{K}) \end{array}$$

We're left with functoriality; this follows by unwinding definitions. □

III.2.1.3 Wreath products vs tensor products We now compare $\wr$ with $\overset{\mathrm{BV}}{\otimes}$, closely following [Har], via the following construction. For clarity, we will conflate $\mathcal{O}^{\otimes}$ with $\mathrm{Tot}_{\mathcal{T}}\mathcal{O}$, for now.

Construction III.2.11. Define the $\mathcal{T}$-algebraic pattern $\mathcal{P}^{c} \coloneqq \mathrm{Patt}^{cc}(\mathrm{Tot}_{\mathcal{T}}\mathcal{P})$ over $\underline{\mathbb{F}}_{\mathcal{T},*}$. We define the *mixed wreath product* $(\mathcal{O}^{\otimes}\underline{\wr}\mathcal{P}^{\otimes})^{\mathrm{mix}}$ of $\mathcal{O}^{\otimes}$ and $\mathcal{P}^{\otimes}$ as the same $\mathcal{T}$-∞-category over $\mathsf{Span}(\mathbb{F}_{V})$ and inert arrows as $\mathcal{O}^{\otimes}\underline{\wr}\mathcal{P}^{\otimes}$ and the same elementary objects as $\mathcal{O}^{\otimes}\underline{\wr}\mathcal{P}^{c}$, i.e.

$$\left(\mathcal{O}^{\otimes}\underline{\wr}\mathcal{P}^{\otimes}\right)^{\mathrm{el}} = \mathcal{O}^{\mathrm{el}}\times\mathcal{P}^{\mathrm{el}} \subset \mathcal{O}^{\mathrm{el}}\times\mathcal{P}^{\mathrm{int}} = \left(\mathcal{O}^{\otimes}\underline{\wr}\mathcal{P}^{\otimes}\right)^{\mathrm{mix-el}}. \tag{45}$$

◁

There is a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}^{\otimes} \times \mathcal{P}^{c} & & \xrightarrow{\quad t \quad} & & \mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \\ \downarrow{\scriptstyle \underline{M}^{\mathrm{coarse}}} & & & & \downarrow{\scriptstyle \underline{M}} \\ \mathcal{O}^{\otimes}_{\underline{\wr}} \mathcal{P}^{c} & \xrightarrow{\quad c \quad} & (\mathcal{O}^{\otimes}_{\underline{\wr}} \mathcal{P}^{\otimes})^{\mathrm{mix}} & \xrightarrow{\quad m \quad} & \mathcal{O}^{\otimes}_{\underline{\wr}} \mathcal{P}^{\otimes} \end{array}$$

whose solid arrows are maps of patterns lying over $\underline{M}$, and whose dotted arrows lie over the identity. We want to verify that the rightmost vertical arrow is a Morita equivalence, which we will try to deduce from the fact that leftmost vertical arrow is a Morita equivalence, which is the following lemma.

Lemma III.2.12. *$\underline{M}^{\mathrm{coarse}}$ is a Morita equivalence.*

Proof. All active arrows in $\mathcal{P}^{c}$ are equivalences, so including identities induces an equivalence

$$\begin{array}{ccc} (\mathcal{O}^{\otimes} \times \mathcal{P}^{c})^{\mathrm{act}}_{/(\mathbf{C};\mathbf{D})} & \longrightarrow & (\mathcal{O}^{\otimes}_{\underline{\wr}} \mathcal{P}^{c})^{\mathrm{act}}_{/(\mathbf{C};\Delta^{S}\mathbf{D})} \\ \wr \downarrow & & \wr \downarrow \\ \mathcal{O}^{\mathrm{act}}_{/\mathbf{C}} & = & \mathcal{O}^{\mathrm{act}}_{/\mathbf{C}} \end{array}$$

confirming Condition (ML-a). We're left with verifying the strong form of Condition (ML-b), i.e. that

$$\mathcal{O}^{\mathrm{el}} \times \mathrm{Tot}_{\mathcal{T}} \mathcal{P} \simeq \left(\mathcal{O}^{\otimes} \times \mathcal{P}^{c}\right)^{\mathrm{el}} \to \left(\mathcal{O}^{\otimes}_{\underline{\wr}} \mathcal{P}^{c}\right)^{\mathrm{el}}$$

is an equivalence. This follows by unwinding definitions. □

Now, to make comparisons along horizontal arrows, we use a version of a lemma due to Harpaz.

Lemma III.2.13. *Suppose $\mathfrak{O}, \mathfrak{O}'$ are $\mathcal{T}$-algebraic pattern structures on the same $\mathcal{T}$-∞-category such that $\mathfrak{O}^{\mathrm{int}} \subset \mathfrak{O}^{\mathrm{int}'}$ and $\mathfrak{O}^{\mathrm{el}} \subset \mathfrak{O}'^{\mathrm{el}}$.*

(1) Suppose that $\mathfrak{O}^{\mathrm{el}} = \mathfrak{O}^{\mathrm{el}'}$ and they satisfy the additional property that, for every V-path $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $g \circ f$ lies in $\mathfrak{O}^{\mathrm{int}}$, f lies in $\mathfrak{O}'^{\mathrm{int}}$, and Z is elementary, Y is also elementary. Then, $\mathfrak{O}$ and $\mathfrak{O}'$ have the same Segal $\mathcal{T}$-objects.

(2) Suppose instead that $\mathfrak{O}^{\mathrm{int}} = \mathfrak{O}'^{\mathrm{int}}$; Then, $X \colon \mathfrak{O} \to \mathcal{C}$ is a Segal $\mathfrak{O}$-object if and only if it is a Segal $\mathfrak{O}'$-object such that $F|_{\mathfrak{O}'^{\mathrm{el}}}$ is $\mathcal{T}$-right Kan extended from $F|_{\mathfrak{O}^{\mathrm{el}}}$.

Proof. The proof of [Har, Lem. 4.2.21] extends mutatis mutandis, but we reproduce the idea here. For the first, it follows from Joyal's Theorem A [kerodon, Tag 02NY] that this condition implies that $\mathfrak{O}^{\mathrm{el}}_{O/} \subset \mathfrak{O}'^{\mathrm{el}}_{O/}$ is initial, yielding the claim. The second is pasting for $\mathcal{T}$-right Kan extensions. □

We separate out the map c first.

Lemma III.2.14. *c satisfies the condition of part (1) of Lemma III.2.13*

PROOF. We must verify that, for all coarse-inert composite arrows

$$f\colon (\mathbf{C};(\mathbf{D}_U)_S) \xrightarrow{f'} (\mathbf{C}'';(\mathbf{D}''_{U''})_{S''}) \to (\mathbf{C}';(\mathbf{D}'_{U'})_{S'})$$

with f' mixed-inert and $(\mathbf{C}';(\mathbf{D}_{U'})_S)$ mixed-elementary *such that f and f' lie in the initial subcategory of inert morphisms with nonempty domain*, $(\mathbf{C}'';(\mathbf{D}''_{U''})_{S''})$ is coarse-elementary. The assumptions guarantee that $\mathbf{C}'$ lies over a transitive G-set, which implies the same of $\mathbf{C}$ by coarse inertness of f, and $\mathbf{C}''$ by mixed-inertness of f', which is the desired coarse-elementary property. □

PROOF OF PROPOSITION III.2.5. Lemmas III.2.12 to III.2.14 and two-out-of-three construct, for each complete $\mathcal{T}$-category $\mathcal{C}$, a commutative diagram of fully faithful functors

$$\begin{array}{ccccc}
\mathrm{Seg}_{\mathcal{O}}\underline{\mathrm{Seg}}_{\mathcal{P}}(\mathcal{C}) & \simeq & \mathrm{Seg}_{\mathcal{O}\times\mathcal{P}}(\mathcal{C}) & \longleftarrow & \mathrm{Seg}_{\mathcal{O}\underline{\wr}\mathcal{P}}(\mathcal{C}) \\
\downarrow{\scriptstyle \iota_\times} & & \downarrow & & \downarrow{\scriptstyle \iota_{\underline{\wr}}} \\
\mathrm{Seg}_{\mathcal{O}}\underline{\mathrm{Fun}}_{\mathcal{T}}(\mathcal{P},\mathcal{C}) & \simeq & \mathrm{Seg}_{\mathcal{O}\times\mathcal{P}^c}(\mathcal{C}) & \simeq & \mathrm{Seg}_{\mathcal{O}\underline{\wr}\mathcal{P}^c}(\mathcal{C}) \simeq \mathrm{Seg}_{(\mathcal{O}\underline{\wr}\mathcal{P})^{\mathrm{mix}}}(\mathcal{C})
\end{array}$$

It suffices to verify that $\mathrm{im}\iota_{\underline{\wr}} \subset \mathrm{im}\iota_\times$. By Lemma III.2.13, the image of $\underline{M}^{\mathrm{coarse}*}\iota_{\underline{\wr}}$ is those coarse Segal objects $F\colon \mathcal{O}^\otimes \times \mathcal{P}^\otimes \to \mathcal{C}$ such that

(a) $F|_{(\mathcal{O}^\otimes\underline{\wr}\mathcal{P}^\otimes)^{\mathrm{mix-el}}}$ is $\mathcal{T}$-right Kan extended from $F|_{(\mathcal{O}^\otimes\underline{\wr}\mathcal{P}^\otimes)^{\mathrm{el}}}$,

and the image of $\iota_\times$ is those F such that

(b) for all $O \in \mathcal{O}^\otimes$, the $\mathcal{T}$-functor $\{O\}\times\mathcal{P}^\otimes \to \mathcal{C}$ is a Segal object.

We're tasked with verifying that Conditions (a) and (b) are equivalent. Now, in sight of Eq. (45), this compiles down to verifying that a $\mathcal{T}$-functor is $\mathcal{T}$-right Kan extended along a base changed functor $\mathcal{C}\times_{\mathcal{D}}\mathcal{E} \to \mathcal{C}\times_{\mathcal{D}}\mathcal{E}'$ if and only if, for all $C \in \mathcal{C}$, the associated functor is right Kan extended along the functor $\{C\}\times\mathcal{E} \to \{C\}\times\mathcal{E}'$; but this follows from the fact that $\mathcal{T}$-limits in product $\mathcal{T}$-categories are products of $\mathcal{T}$-limits in the factors. □

III.2.1.4 A bookkeeping theorem We now prove Theorem III.2.3. Now, the following lemma follows by compatibility of products with pullbacks and equivalences.

Lemma III.2.15. *In the situation of Theorem III.2.3, the map $\alpha_1\times\alpha_2$ is a weak approximation of $\mathcal{T}$-algebraic patterns.*

PROOF OF THEOREM III.2.3. We may pull back our data to construct the $\mathcal{T}$-algebraic pattern diagram

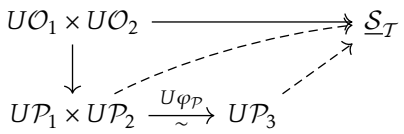
$$\begin{array}{ccccc}
\mathcal{P}_1^\otimes\times\mathcal{P}_2^\otimes & \xrightarrow{\underline{M}} & \mathcal{P}_1^\otimes\underline{\wr}\mathcal{P}_2^\otimes & \xrightarrow{\underline{\varphi}_{\mathcal{P}}} & \mathcal{P}_3^\otimes \\
\downarrow{\scriptstyle \alpha_1\underline{\odot}\alpha_2} & & \downarrow & & \downarrow{\scriptstyle \alpha_3} \\
\mathcal{E}_1^\otimes\times\mathcal{E}_2^\otimes & \xrightarrow{\underline{M}} & \mathcal{E}_1^\otimes\underline{\wr}\mathcal{E}_1^\otimes & \xrightarrow{\underline{\varphi}_{\mathcal{E}}} & \mathcal{E}_3^\otimes
\end{array}$$

and in sight of Corollary III.2.7 and two-out-of-three, we're tasked with proving that $\underline{\varphi_{\mathcal{E}}} \circ \underline{M}$ is a Morita equivalence.

By Lemma III.2.15 we know that α_3 and $\alpha_1 \times \alpha_2$ are Morita $\mathcal{T}$-localizations, by Proposition III.2.5 that each arrow marked $\underline{M}$ is a Morita equivalence, and by Corollary I.3.31 that $\underline{\varphi_{\mathcal{P}}}$ is a Morita equivalence. By two-out-of-three for Morita equivalences, $\underline{\varphi_{\mathcal{P}}} \circ \underline{M}$ is a Morita equivalence, so by Corollary I.3.31, applying $\mathrm{Seg}_{(-)}$ to the outer diagram yields a diagram of fully faithful inclusions

$$\begin{array}{ccc} \mathrm{Seg}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{P}_1} \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{P}_2}(\mathcal{S}) & \simeq & \mathrm{Seg}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{P}_3}(\mathcal{S}) \\ \uparrow & & \uparrow \\ \mathrm{Seg}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{E}_1} \underline{\mathrm{Seg}}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{E}_2}(\mathcal{S}) & \longhookrightarrow & \mathrm{Seg}_{\mathrm{Tot}_{\mathcal{T}} \mathcal{E}_3}(\mathcal{S}); \end{array}$$

It suffices to verify that the images of $(\alpha_1 \times \alpha_2)^*$ and $\varphi_{\mathcal{E}}^* \alpha_3^*$ agree. In fact, by Corollary I.3.31 they correspond with $\mathcal{P}_1 \times \mathcal{P}_2$-monoid $\mathcal{T}$-spaces attaining solutions to the inner and outer lifting problem, respectively:



$U\varphi_{\mathcal{P}}$ is an equivalence by assumption, so the images agree, yielding the result. □

III.2.2 Additivity of little disk G-operads Fix $V_\bullet$ and $W_\bullet$ polyrepresentations and G a finite group.

Following Lurie's proof of nonequivariant additivity with tangential structure, a convenient construction for the additivity map with the universal tangential structure would be as a *restriction* of the canonical map

$$B_G O(k)^{\sqcup} \overset{\mathrm{BV}}{\otimes} B_G O(\ell)^{\sqcup} \simeq (B_G O(k) \times B_G O(\ell))^{\sqcup} \xrightarrow{\oplus} B_G O(k+\ell)^{\sqcup};$$

then, the fact that it disintegrates to the map we study in the linear setting will be automatic. Unfortunately, we run into two large problems:

Problem i. The putative map on structure spaces $\mathrm{Emb}^{H,\mathrm{Rect}}\left(S \cdot P^V, P^V\right) \hookrightarrow \prod_{[H/K] \subset S} \mathrm{Emb}^K(P^V, P^V)$ is not a monomorphism *homotopically*, only a topological subspace inclusion; hence factorizations through the putative associated G-operad map $\mathbb{E}_V^{\otimes} \to B_G O(k)^{\sqcup}$ will only possibly be unique if *models* are chosen on both sides.

Problem ii. We have not topologically modelled the map $\oplus^{\sqcup}$ above, but instead derived it from homotopical formal properties.

To solve these problems, we will topologically model the cocartesian structure. The following construction is enabled by the fact that we've restricted so that $\mathcal{O}_G$ is an atomic orbital 1-category.

Construction III.2.16. $X^\bullet\colon \mathcal{O}_G \to \mathrm{Cat}_{\mathrm{Top}}$ a coefficient system of topological categories and $\mathbf{Tot}X \to \mathcal{O}_G$ the associated topologically enriched unstraightening.

We may define a topological functor $\mathbf{Tot}X^{\sqcup} \to \mathrm{Tot}\,\underline{\mathrm{Fin}}_{G,*}$ as follows:

(1) The objects in the fiber over $S_+ \in \mathrm{Fin}_{H,*}$ are $\prod_{[H/K]\subset S} \pi_0^K X$.
(2) The topological space of morphisms $(x, T_+, H) \to (y, S_+, K)$ lying over a map $(T_+, H) \to (S_+, K)$ classified by a map $f\colon \mathrm{Res}^H_K T_+ \to S_+$ is the space $\prod_{[K/J]\subset S} \prod_{[J/L]\subset f^{-1}([K/J])} \mathrm{Map}^L_X(x_L, y_J)$.
(3) Composition in $\mathbf{Tot}X^{\sqcup}$ is extended by composition of paths under rearranging the above product factorizations.

Note that $\mathbf{Tot}X^{\sqcup} \times_{\mathrm{Tot}\,\underline{\mathrm{Fin}}_{G,*}} \mathcal{O}_G \simeq \mathbf{Tot}X$. This extends to a diagram

$$\begin{array}{ccccccc}
\mathbf{Tot}X^{\sqcup} \underline{\wr} \mathbf{Tot}Y^{\sqcup} & \longrightarrow & (\mathbf{Tot}Y^{\sqcup})^{\sqcup} & \xrightarrow[\mathbf{Tot}\pi_{\mathcal{P}}]{} & \mathrm{Tot}\,\underline{\mathrm{Fin}}^{\sqcup}_{G,*} & \xrightarrow[\pi_{\underline{\mathrm{Fin}}^{\sqcup}_{T,*}}]{} & \mathrm{Tot}\,\underline{\mathrm{Fin}}_{G,*} \\
\downarrow & \lrcorner & \downarrow & & & & \\
\mathbf{Tot}X & \xrightarrow[\pi_{\mathcal{O}}]{} & \mathrm{Tot}\,\underline{\mathrm{Fin}}_{G,*} & & & &
\end{array}$$

Now, given a map of coefficient systems $f\colon X \times Y \to Z$, claim that this may be extended as follows:

$$\begin{array}{ccccccc}
 & & \mathbf{Tot}Z^{\sqcup} & & & & \\
 & \overset{f_{\underline{\wr}}}{\nearrow} & & \searrow & & & \\
\mathbf{Tot}X^{\sqcup} \underline{\wr} \mathbf{Tot}Y^{\sqcup} & \longrightarrow & (\mathbf{Tot}Y^{\sqcup})^{\sqcup} & \xrightarrow[\mathbf{Tot}\pi_{\mathcal{P}}]{} & \mathrm{Tot}\,\underline{\mathrm{Fin}}_{G,*} & \xrightarrow[\pi_{\underline{\mathrm{Fin}}^{\sqcup}_{T,*}}]{} & \mathrm{Tot}\,\underline{\mathrm{Fin}}_{G,*} \\
\downarrow & \lrcorner & \downarrow & & & & \\
\mathbf{Tot}X^{\sqcup} & \xrightarrow[\pi_{\mathcal{O}}]{} & \mathrm{Tot}\,\underline{\mathrm{Fin}}_{G,*} & & & &
\end{array}$$

Indeed, viewing an object of $\mathrm{Tot}\,\underline{\mathrm{Fin}}^{\sqcup}_G$ as a choice of $H \subset G$, $S \in \mathbb{F}_H$, and $T \in \mathbb{F}_{H,/S}$, viewing an object of $\mathbf{Tot}Y^{\sqcup}$ over this as a further choice of points $y_\bullet \in Y^T$, and an object of $\mathbf{Tot}X \underline{\wr} \mathbf{Tot}Z$ over this as a further choice of points $x_\bullet in X^S$, on objects the map $f_{\underline{\wr}}$ takes the object $(x_\bullet, y_\bullet)$ lying over $(H, S, T \to S)$ to the T-object taking an orbit $[H/K] \subset T$ to $f\left(x_{\pi([H/K])}, y_{[H/K]}\right)$; the map on mapping spaces is extended from f in an essentially unique way so that this restricts to f under the evident map $\mathbf{Tot}X \times \mathbf{Tot}Y \to \mathbf{Tot}X \underline{\wr} \mathbf{Tot}Y \times_{\mathrm{Tot}\,\underline{\mathrm{Fin}}_{G,*}} \mathcal{O}_G$ and is compatible with pullback along the evident topological lifts of inert morphisms.

In particular, on homotopy-coherent nerves, this presents a G-preoperad map

$$N^{hc}X^{\sqcup} \underline{\wr} N^{hc}Y^{\sqcup} \to \left(N^{hc}Z\right)^{\sqcup}$$

whose associated G-operad map $N^{hc}(X \times Y)^{\sqcup} \simeq N^{hc}X^{\sqcup} \overset{\mathrm{BV}}{\otimes} N^{hc}Y^{\sqcup} \to N^{hc}Z^{\sqcup}$ restricts to $N^{hc}f$, and hence is classified by $N^{hc}f$ under Theorem II.2.32. ◂

Now, note that there are evident inclusions of topological categories over $\mathcal{O}_G$

$$\mathbb{E}^t_{V_\bullet} \hookrightarrow \mathbb{E}^t_{\mathrm{id}_k} \hookrightarrow \mathbf{Emb}^k_G,$$

compatible with the evident topological preimages of inert morphisms. In particular, we acquire the solid subdiagram diagram of the following, as a diagram of topological functors over $\mathcal{O}_G$:

$$\begin{array}{ccccccccc}
\mathbb{P}^t_{V_\bullet \wr W_\bullet} & \longrightarrow & \widetilde{\mathbb{E}}^t_{V_\bullet \wr W_\bullet} & \longleftarrow & \mathbb{E}^t_{V_\bullet \wr W_\bullet} & \longrightarrow & \mathbb{E}^t_{\mathrm{id}_k \wr \mathrm{id}_\ell} & \longrightarrow & \mathbf{Emb}^k_G \wr \mathbf{Emb}^\ell_G \\
\downarrow{\scriptstyle \varphi_{\mathbb{P}}} & & \downarrow{\scriptstyle \widetilde{\varphi}} & & \downarrow{\scriptstyle \varphi_{\mathbb{E}}} & & \downarrow{\scriptstyle \varphi_{\mathrm{id}}} & & \downarrow{\scriptstyle \oplus} \\
\mathbb{P}^t_{V_\bullet | W_\bullet} & \longrightarrow & \widetilde{\mathbb{E}}^t_{V_\bullet \oplus W_\bullet} & \longleftarrow & \mathbb{E}^t_{V_\bullet \oplus W_\bullet} & \longrightarrow & \mathbb{E}^t_{\mathrm{id}_{k+\ell}} & \longrightarrow & \mathbf{Emb}^{k+\ell}_G
\end{array}$$

and, unwinding definitions, this extends essentially uniquely (by virtue of the fact that topological subcategory inclusions are monomorphisms in $\mathrm{Cat}_{\mathrm{Top}}$) to fill in the dashed arrows, the resulting diagrams also preserving the evident topological preimages of inert morphisms. In particular, taking homotopy-coherent nerves yields a commutative diagram of G-preoperad maps

$$\begin{array}{ccccc}
\mathbb{P}^\otimes_V \wr \mathbb{P}^\otimes_W & \xrightarrow{\alpha_1 \wr \alpha_2} & \mathbb{E}^\otimes_V \wr \mathbb{E}^\otimes_W & \longrightarrow & \mathbb{E}^\otimes_{\mathrm{id}_k} \wr \mathbb{E}^\otimes_{\mathrm{id}_\ell} \\
\downarrow{\scriptstyle \varphi_{\mathbb{P}}} & & \downarrow & & \downarrow \\
\mathbb{P}^\otimes_{V|W} & \xrightarrow{\varphi_{\mathbb{E}}} & \mathbb{E}^\otimes_{V \oplus W} & \xrightarrow{\alpha} & \mathbb{E}^\otimes_{\mathrm{id}_k + \ell}
\end{array} \tag{46}$$

We're ready to conclude equivariant Dunn-Lurie additivity.

Proof of Theorem III.2.2. We are left with verifying that the lefthand square of Eq. (46) satisfies the conditions of Theorem III.2.3. Certainly each (one color) little disk G-operad $\mathbb{E}^\otimes_V$ is reduced, and the fact that each α_i is a Morita localization is Corollary II.2.141. We're left with proving that $\varphi_{\mathbb{P}}$ is a Γ-Morita equivalence, which we verify using Corollary I.3.31.

In general, under Proposition II.2.115, $\varphi_{\mathbb{P}}$ yields an equivalence

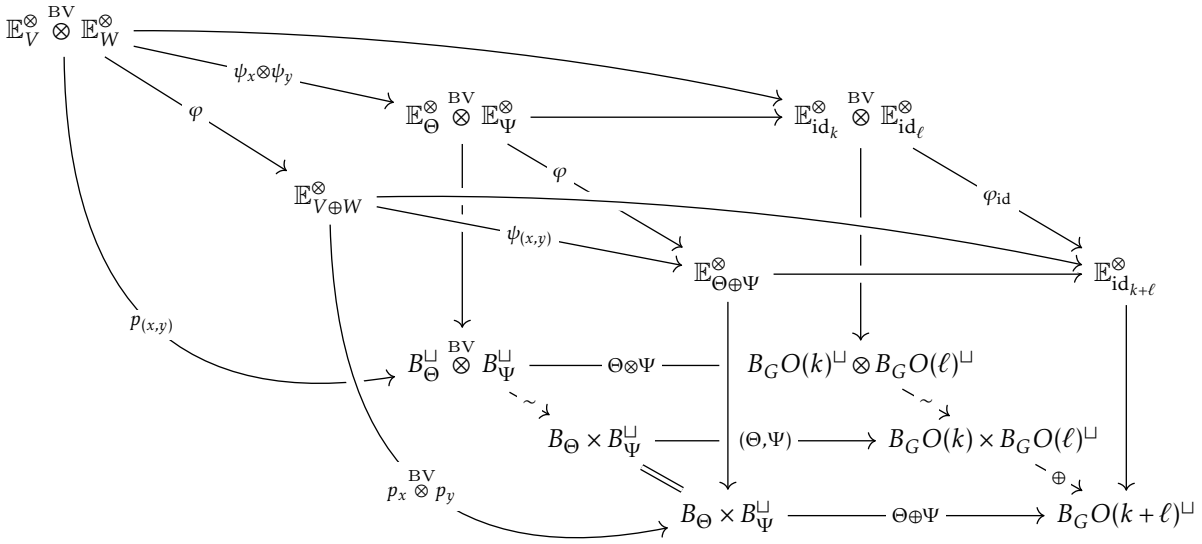
$$U\left(\mathbb{P}^\otimes_{\mathcal{U}_V} \wr \mathbb{P}^\otimes_{\mathcal{U}_W}\right) \simeq \mathcal{U}_V \times_{\mathrm{Ob}\,\mathbb{F}_g} \coprod_T \mathcal{U}^T_W.$$

Under Proposition III.2.9, we're left with checking the dashed equivalence of the following conditions

$$\begin{array}{ccc}
\mathrm{Mul}^g_{\mathbb{P}_{\mathcal{U}_V \wr \mathcal{U}_W}}\left((\mathbf{U}_V;\mathbf{U}_W);\left(\mathbf{U}'_V;U_W\right)\right) = * & \Leftarrow\!=\!=\!=\!\Rightarrow & \mathrm{Mul}^g_{\mathbb{P}_{\mathcal{U}_V}}\left(\mathbf{U}_V;\mathbf{U}'_V\right) \times \mathbb{P}_{\mathcal{U}_W}\left(\mathbf{U}_W;U'_W\right) = * \\
\Updownarrow & & \Updownarrow \\
\varphi(\mathbf{U}_V;\mathbf{U}_W) \in \iota^*(\mathcal{U}_V \,|\, \mathcal{U}_W) & \Longleftrightarrow & \mathbf{U}_V \in \iota^*\mathcal{U}_V \text{ and } \mathbf{U}_W \in \iota^*\mathcal{U}_W
\end{array}$$

Now, the vertical equivalences follow simply by definition and the bottom horizontal by the fact that φ is an equivalence (see Proposition II.2.115), so the desired implication follows by two out of three. □

Now, to approach Theorem III.2.1, we must construct a map which disintegrates to the maps we know to be equivalences via Theorem III.2.2. We may do so via the following diagram:



We're ready to conclude equivariant Dunn additivity with tangential structure.

Proof of Theorem III.2.1. Note that ψ_x, ψ_y and $\psi_{(x,y)}$ reduce to equivalences under Theorem II.2.79, so the upper left quadrilateral shows that $\varphi\colon \mathbb{E}_\Theta^{\otimes} \overset{\mathrm{BV}}{\otimes} \mathbb{E}_\Psi^{\otimes} \to \mathbb{E}_{\Theta\oplus\Psi}^{\otimes}$ disintegrates to an equivalence at each point; Corollary III.1.6 shows that this implies that φ is an equivalence. □

III.2.3 A dream for the future We offer a vision, hypothetical as of yet, for the placement of our additivity theorem within the isotopy theory of G-manifolds, for G potentially a compact Lie group.

Question III.2.17. Given G a compact Lie group, does there exist a categorical Green functor

$$\underline{\mathrm{Mfld}}_G^{\sqcup,\times} \in \mathrm{CAlgCMon}_G(\mathrm{Cat})$$

whose "additive" G-symmetric monoidal structure agrees with $\underline{\mathrm{Mfld}}_G^{\sqcup}$ admits a G-symmetric monoidal full subcategory embedding $\underline{\mathrm{Mfld}}_G^{\Theta,\sqcup} \hookrightarrow \underline{\mathrm{Mfld}}_G^{all,\sqcup}$, whose multiplicative structure takes $(M,N) \mapsto M \times N$, and whose multiplicative binary tensor product induces an equivalence

$$\times\colon \underline{\mathrm{Disk}}_G^{k-\mathrm{r},\sqcup} \circledast \underline{\mathrm{Disk}}_G^{\ell-\mathrm{r},\sqcup} \xrightarrow{\sim} \underline{\mathrm{Disk}}_G^{k+\ell-\mathrm{r},\sqcup}? \tag{47}$$

Moreover, does there exist a morphism of categorical green functors $\pi_0\colon \underline{\mathrm{Mfld}}_G^{\sqcup,\times} \to \underline{\mathrm{Fin}}_G^{\sqcup,\times}$ lifting the equifibration $\underline{\mathrm{Mfld}}_G^{n} \to \underline{\mathrm{Fin}}_G^{\sqcup}$? ◂

The author strongly believes that, once such a structure is constructed, similar techniques to that of Theorem III.2.1 will yield an equivalence

$$\times\colon \underline{\mathrm{Disk}}_G^{\Theta,\sqcup} \circledast \underline{\mathrm{Disk}}_G^{\Psi,\sqcup} \xrightarrow{\sim} \underline{\mathrm{Disk}}_G^{\Theta\oplus\mathrm{Sp},\sqcup},$$

underlying an equivalence $\mathbb{E}_\Theta^\otimes \oplus \mathbb{E}_\Psi^\otimes$. Other than extending Theorem III.2.1 to compact Lie groups, this promises to cohere Dunn additivity with products of manifolds, hopefully for the purpose of matching Fubini's theorem in factorization homology to the additivity phenomenon.

There is also a harder question, which the author does not know how to approach.

Question III.2.18. Given a finite-index closed subgroup inclusion $H \subset G$, is $\underline{\mathrm{CMon}}_G(\underline{\mathcal{S}}_G)$ a G-mode? If so, given an orthogonal H-representation V, does there exist an equivalence $N_H^G \underline{\mathrm{Disk}}_H^{V-\mathfrak{r},\sqcup} \simeq \underline{\mathrm{Disk}}_G^{\mathrm{Ind}_H^G V-\mathfrak{r},\sqcup}$, where the norm is taken with respect to the mode structure? ◁

This would give a *non-geometric* reason that $\mathbb{E}_V$ appears in equivariant homotopy theory whenever V is a permutation representation: all such G-operads would occur as indexed tensor powers of the inflation of the associative operad.

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