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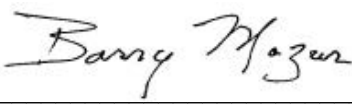
Perverse mod p Sheaves on Affine Flag Varieties

presented by **Robert Cass**

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Date: April 22, 2021

Perverse mod p Sheaves on Affine Flag Varieties

A dissertation presented

by

Robert Cass

to

The Department of Mathematics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

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Dissertation Advisor: Mark Kisin

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Perverse mod p Sheaves on Affine Flag Varieties

Abstract

We prove a mod p analogue of the geometric Satake equivalence and we give a geometric construction of central elements in affine mod p Hecke algebras. To accomplish these goals we apply techniques from F -singularities to the theory of perverse mod p sheaves on schemes in characteristic p . Along the way we prove new results about the F -singularities of affine Schubert varieties and equal characteristic analogues of local models of Shimura varieties.

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Chapter 1

Introduction

1.1 Motivation

The geometric Satake equivalence [MV07] is a fundamental result in the geometric Langlands program which provides a geometric version of the unramified local Langlands correspondence. Bezrukavnikov [Bez16] builds on the geometric Satake equivalence and provides a geometric version of the tamely ramified local Langlands correspondence. A key input in Bezrukavnikov's work is Gaitsgory's geometrization [Gai01] of an isomorphism between the spherical Hecke algebra and the Iwahori Hecke algebra. In this thesis we construct a mod p analogue of the geometric Satake equivalence as well as Gaitsgory's construction.

To be more precise, let E be a local field of characteristic $p > 0$ with ring of integers $\mathcal{O}_E$ and residue field $\mathbb{F}_q$ where q is a power of p . Let G be a split connected reductive group defined over E , and let $T \subset B \subset G$ be a maximal torus and a Borel subgroup. Let $X_*(T)$ be the group of cocharacters of G and let $X_*(T)^+$ be the monoid of dominant cocharacters. For a compact open subgroup $H \subset G(E)$ let

$$\mathcal{H}_H := \{f: G(E) \rightarrow \mathbb{F}_p : f \text{ has compact support and is } H \text{ bi-invariant}\}.$$

The multiplication on $\mathcal{H}_H$ is by convolution of functions. The mod p Hecke algebras $\mathcal{H}_H$ are important in the study of admissible mod p representations of $G(E)$.

As G is split we may let $K = G(\mathcal{O}_E)$. Then Herzig [Her11b] (and Henniart-Vignéras [HV15] for more general coefficients) constructed a mod p Satake isomorphism

$$\mathcal{S}: \mathcal{H}_K \xrightarrow{\sim} \mathbb{F}_p[X_*(T)^+].$$

We note that the mod p Satake isomorphism is related to the usual Satake isomorphism by an integral Satake isomorphism constructed by Zhu in [Zhu20].

In the first part of this thesis we construct a symmetric monoidal category of perverse $\mathbb{F}_p$ -sheaves on the affine Grassmannian of G which gives a geometrization of the inverse of $\mathcal{S}$. This can be viewed as a mod p version of the geometric Satake equivalence. The first part of this thesis is based on our article [Cas19].

Now let $I \subset K$ be the Iwahori subgroup determined by B and let $Z(\mathcal{H}_I)$ be the center of $\mathcal{H}_I$. If $\mathbf{1}_K \in \mathcal{H}_I$ is the function which is 1 on K and 0 elsewhere then by work of Vignéras [Vig05] and Ollivier [Oll14] there is an isomorphism of $\mathbb{F}_p$ -algebras

$$\mathcal{C}: Z(\mathcal{H}_I) \xrightarrow{\sim} \mathcal{H}_K, \quad f \mapsto f * \mathbf{1}_K.$$

The second part of this thesis is based on our article [Cas20], in which we construct a functor on perverse $\mathbb{F}_p$ -sheaves which geometrizes the inverse of $\mathcal{C}$. This allows us to give geometric proofs of certain combinatorial identities in Iwahori mod p Hecke algebras.

This thesis is part of an ongoing project aimed at providing a categorification of the representation theory of affine mod p Hecke algebras. In future joint work with C. Pépin and T. Schmidt we aim to construct a mod p version of [Bez16]. In Bezrukavnikov's work, perverse sheaves on an affine flag variety are a categorification of the local automorphic side (i.e., representations of Hecke modules), and coherent sheaves on the Steinberg variety

of the Langlands dual group are a categorification of the local arithmetic side (i.e., tamely ramified representations of a local Galois group).

That such an equivalence is possible in the mod p setting is suggested by our results in this thesis, which concern the automorphic side, and those in [PS20] on the arithmetic side. Ideally, such an equivalence would give a geometric construction of some instances of Grosse-Klönne's functor [GK16] from supersingular mod p Hecke modules to Galois representations.

1.2 The mod p geometric Satake equivalence

Let G be a connected reductive group over an algebraically closed field k . The loop group LG is the functor on k -algebras

$$LG: R \mapsto G(R((t))),$$

and the positive loop group L^+G is the functor

$$L^+G: R \mapsto G(R[[t]]).$$

The affine Grassmannian is the fpqc-quotient

$$\mathrm{Gr} := LG/L^+G.$$

The functor Gr is represented by an ind-projective ind-scheme which is a union of affine Schubert varieties analogous to classical Schubert varieties in flag varieties. Moreover, it has important applications in arithmetic geometry and representation theory. For example, fix a prime ℓ different from the characteristic of k . There is a natural left action of L^+G on Gr , and one can construct the category $P_{L^+G}(\mathrm{Gr}, \overline{\mathbb{Q}}_\ell)$ of L^+G -equivariant perverse $\overline{\mathbb{Q}}_\ell$ -

sheaves on Gr . Let $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$ be the category of representations of the dual group $\hat{G}$ on finite-dimensional $\overline{\mathbb{Q}}_\ell$ -vector spaces. Then one version of the geometric Satake equivalence of Mirković and Vilonen [MV07] states that there is a natural symmetric convolution product $*$ on the category $P_{L+G}(\mathrm{Gr}, \overline{\mathbb{Q}}_\ell)$, and there is an equivalence of symmetric monoidal categories $P_{L+G}(\mathrm{Gr}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$. Here the dual group $\hat{G}$ arises naturally via a Tannakian formalism as the group of automorphisms of the tensor functor $\bigoplus_i R^i\Gamma(-): P_{L+G}(\mathrm{Gr}, \overline{\mathbb{Q}}_\ell) \rightarrow \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$. Moreover, the following diagram commutes up to natural isomorphism.

$$\begin{array}{ccc}
P_{L+G}(\mathrm{Gr}, \overline{\mathbb{Q}}_\ell) & \xrightarrow{\bigoplus_i R^i\Gamma(-)} & \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell} \\
& \searrow \sim & \nearrow \text{Forget} \\
& \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G}) &
\end{array}$$

One can also get an analogous statement using coefficients in the finite field $\mathbb{F}_\ell$. The geometric Satake equivalence can be viewed as both providing a canonical construction of $\hat{G}$ from G , and as a geometrization of the Satake isomorphism. Recently, Vincent Lafforgue [Laf18] has used the geometric Satake equivalence (among other things) to prove the automorphic to Galois direction of the Langlands correspondence for reductive groups over global function fields.

Now suppose k has characteristic $p > 0$. The category of étale $\mathbb{F}_p$ -sheaves on a scheme X over k is markedly different from the category of étale $\mathbb{F}_\ell$ -sheaves or $\overline{\mathbb{Q}}_\ell$ -sheaves for $\ell \neq p$. However, one can still define perverse $\mathbb{F}_p$ -sheaves on X . We will define perverse $\mathbb{F}_p$ -sheaves in Section 2.1, but for now we note that when X is smooth, perverse $\mathbb{F}_p$ -sheaves on X are anti-equivalent to the category of lfgu $\mathcal{O}_{F,X}$ -modules introduced by Emerton-Kisin [EK04b]. These are certain quasi-coherent $\mathcal{O}_X$ -modules M equipped with an $\mathcal{O}_X$ -linear isomorphism $F^*M \rightarrow M$ and can be viewed as analogues of D -modules.

In this thesis we define the category $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ of L^+G -equivariant perverse $\mathbb{F}_p$ -sheaves on Gr . We then construct a convolution product $*$ making $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ into an

abelian symmetric monoidal category, and we prove the following theorem.

Theorem 1.1. *The functor $H := \bigoplus_i R^i\Gamma(-): P_{L+G}(\mathrm{Gr}, \mathbb{F}_p) \rightarrow \mathrm{Vect}_{\mathbb{F}_p}$ is exact, faithful, and can be naturally upgraded to a symmetric monoidal functor. The functor on $\mathbb{F}_p$ -algebras whose R -points are the tensor endomorphisms of $H(-) \otimes_{\mathbb{F}_p} R: P_{L+G}(\mathrm{Gr}, \mathbb{F}_p) \rightarrow \mathrm{Mod}_R$ is represented by an affine monoid scheme M_G over $\mathbb{F}_p$. Finally, there is an equivalence of symmetric monoidal categories $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p) \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{F}_p}(M_G)$ such that the following diagram commutes up to natural isomorphism.*

$$\begin{array}{ccc}
 P_{L+G}(\mathrm{Gr}, \mathbb{F}_p) & \xrightarrow{\bigoplus_i R^i\Gamma(-)} & \mathrm{Vect}_{\mathbb{F}_p} \\
 & \searrow \sim & \nearrow \mathrm{Forget} \\
 & \mathrm{Rep}_{\mathbb{F}_p}(M_G) &
 \end{array}$$

See [DM82] for more information on the Tannakian formalism. The only difference in our situation is that because the category $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ lacks dual objects in general then an endomorphism of H is not necessarily invertible, which explains why we recover a monoid scheme M_G rather than a group scheme.

The main difficulty in proving Theorem 1.1 lies in constructing the commutativity isomorphism for the convolution product, which we handle by methods similar to those in [MV07] involving Beilinson-Drinfeld Grassmannians. The geometry of the Beilinson-Drinfeld Grassmannians allows us to construct a certain deformation from an external tensor product of perverse sheaves to the convolution product, and then we show that the commutativity isomorphism for the tensor product extends to the convolution product.

As in the case of $\overline{\mathbb{Q}}_\ell$ -coefficients, the simple objects in $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ are the IC sheaves supported on L^+G -orbit closures. After fixing a maximal torus and a Borel $T \subset B \subset G$ we may parametrize L^+G -orbits in Gr by the dominant cocharacters $X_*(T)^+$. Here $\mu \in X_*(T)^+$ corresponds to the orbit $\mathrm{Gr}_\mu := L^+G \cdot \mu(t) \subset \mathrm{Gr}$ (see Section 5.2 for more details). Let IC_μ denote the IC sheaf supported on the reduced closure $\mathrm{Gr}_{\leq \mu}$ of Gr_μ . Unlike the case of

$\overline{\mathbb{Q}}_\ell$ -coefficients, the convolution product of simple objects remains simple.

Theorem 1.2. *Let $\mu_1, \mu_2 \in X_*(T)^+$. Then*

$$\mathrm{IC}_{\mu_1} * \mathrm{IC}_{\mu_2} = \mathrm{IC}_{\mu_1 + \mu_2}.$$

Furthermore, for every $\mu \in X_*(T)^+$,

$$\dim_{\mathbb{F}_p} H(\mathrm{IC}_\mu) = 1.$$

It follows that the simple representations of M_G are one-dimensional and hence M_G is pro-solvable (see Proposition 7.15 for a precise statement). Our proofs of Theorems 1.1 and 1.2 rely crucially on the nature of the singularities of the affine Schubert varieties $\mathrm{Gr}_{\leq \mu}$. In particular, we use Frobenius splitting techniques to show that IC_μ is the shifted constant sheaf $\mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu}]$ supported on $\mathrm{Gr}_{\leq \mu}$. This will be explained in Section 1.5 below.

The other key input in the proof of Theorem 1.2 is the fact due to Faltings [Fal03] and Pappas-Rapoport [PR08] that $\mathrm{Gr}_{\leq \mu_1 + \mu_2}$ has rational singularities if $p \nmid |\pi_1(G_{\mathrm{der}})|$. More precisely, by considering the Artin–Schreier sequence, to prove that $\mathrm{IC}_{\mu_1} * \mathrm{IC}_{\mu_2} = \mathrm{IC}_{\mu_1 + \mu_2}$ it suffices to show that the derived pushforward along a convolution morphism preserves the structure sheaf. We then appeal to a general result of Kovács [Kov17] on rational singularities.

1.3 Central perverse $\mathbb{F}_p$ -sheaves on affine flag varieties

We now assume that $p > 2$ and that the derived subgroup G_{der} is almost simple. Let $I \subset L^+G$ be the Iwahori group given by the fiber of B under the projection $L^+G \rightarrow G$. Let $\mathcal{F}\ell := LG/I$ be the Iwahori affine flag variety, and let $\pi: \mathcal{F}\ell \rightarrow \mathrm{Gr}$ be the projection.

Then we can also define the category $P_I(\mathcal{F}\ell, \mathbb{F}_p)$ of I -equivariant perverse $\mathbb{F}_p$ -sheaves

on $\mathcal{F}\ell$. In Section 8.2 we define the convolution product $\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet \in D_c^b(\mathcal{F}\ell, \mathbb{F}_p)$ of two I -equivariant perverse sheaves $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_I(\mathcal{F}\ell, \mathbb{F}_p)$. We note that the convolution product on $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ preserves the full subcategory of semisimple objects $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}} \subset P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ by Theorem 1.2. Our main result in this section is the existence of a functor $\mathcal{Z}: P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}} \rightarrow P_I(\mathcal{F}\ell, \mathbb{F}_p)$ which geometrizes the isomorphism of Vignéras [Vig05] and Ollivier [Oll14] between the unramified mod p Hecke algebra and the center of the Iwahori mod p Hecke algebra.

We will use the following notation. Let $\tilde{W} := N_{G(k((t)))}T(k((t)))/T(k[[t]])$ be the Iwahori-Weyl group of $G(k((t)))$, which parametrizes left I -orbits in $\mathcal{F}\ell$ by the Bruhat decomposition. For $w \in \tilde{W}$ let $\mathcal{F}\ell_w \subset \mathcal{F}\ell$ be the corresponding reduced I -orbit closure. Let $\mathrm{IC}_\mu \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}}$ be the shifted constant sheaf $\mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu}]$ supported on $\mathrm{Gr}_{\leq \mu}$.

Given $\mu \in X_*(T)^+$ let $\mathrm{Adm}(\mu) \subset \tilde{W}$ be the μ -admissible set defined in (8.4). Set

$$\mathcal{A}(\mu) := \bigcup_{w \in \mathrm{Adm}(\mu)} \mathcal{F}\ell_w \subset \mathcal{F}\ell.$$

Let $\mathcal{Z}_{\mathcal{A}(\mu)} \in D_c^b(\mathcal{F}\ell, \mathbb{F}_p)$ be the shifted constant sheaf $\mathbb{F}_p[\dim \mathcal{A}(\mu)]$ supported on $\mathcal{A}(\mu)$.

Theorem 1.3. *There exists an exact functor $\mathcal{Z}: P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}} \rightarrow P_I(\mathcal{F}\ell, \mathbb{F}_p)$ that satisfies:*

(i) *For all $\mu \in X_*(T)^+$ there is a canonical isomorphism*

$$\mathcal{Z}(\mathrm{IC}_\mu) \cong \mathcal{Z}_{\mathcal{A}(\mu)}.$$

(ii) *For all $\mathcal{F}^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}}$ there is a canonical isomorphism*

$$R\pi_!(\mathcal{Z}(\mathcal{F}^\bullet)) \cong \mathcal{F}^\bullet.$$

(iii) For all $\mathcal{F}_1^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}}$ and $\mathcal{F}_2^\bullet \in P_I(\mathcal{F}\ell, \mathbb{F}_p)$ the convolution product

$$\mathcal{Z}(\mathcal{F}_1^\bullet) * \mathcal{F}_2^\bullet$$

is perverse and I -equivariant.

(iv) For all $\mathcal{F}_1^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}}$ and $\mathcal{F}_2^\bullet \in P_I(\mathcal{F}\ell, \mathbb{F}_p)$ there is a canonical isomorphism

$$\mathcal{Z}(\mathcal{F}_1^\bullet) * \mathcal{F}_2^\bullet \cong \mathcal{F}_2^\bullet * \mathcal{Z}(\mathcal{F}_1^\bullet).$$

(v) For all $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}}$ there is a canonical isomorphism

$$\mathcal{Z}(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \cong \mathcal{Z}(\mathcal{F}_1^\bullet) * \mathcal{Z}(\mathcal{F}_2^\bullet).$$

Our method is analogous to the case of $\overline{\mathbb{Q}}_\ell$ -coefficients considered in [Gai01] and [Zhu14], but it involves some new ideas because we use a different notion of the nearby cycles functor which is well-suited for our purposes (see Remark 8.7 for the usual definition). More precisely, the functor $\mathcal{Z}$ is a nearby cycles functor along a degeneration from Gr to $\mathcal{F}\ell$. As the derived pushforward functor for $\mathbb{F}_p$ -sheaves need not preserve constructibility, we instead take an intermediate extension in the definition of $\mathcal{Z}$ in Section 8.2.

As in the proof of Theorem 1.1, we exploit subtle properties of the singularities of affine Schubert varieties. In particular, we use Frobenius splitting techniques to verify that our ad-hoc construction of the nearby cycles functor satisfies the necessary properties. This requires us to prove new results on the F -singularities of certain equal-characteristic analogues of local models of Shimura varieties, which will be explained in Section 1.5.

The relevant facts involving F -singularities are that if X is an integral F -rational variety of dimension d then the shifted constant étale sheaf $\mathbb{F}_p[d] \in D_c^b(X, \mathbb{F}_p)$ is a simple perverse sheaf by Theorem 1.12, and that F -rational singularities are pseudo-rational by a result

of Smith [Smi97]. See Theorem 4.1 for the definition of F -rationality. We will combine these facts with a result of Kovács [Kov17] on pseudo-rational singularities to deduce that our nearby cycles functor commutes with pushforward along birational morphisms between certain F -rational varieties appearing in the proof of Theorem 1.3. We refer the reader to Section 8.2 for more discussion on the properties of $\mathcal{Z}$.

Remark 1.4. The functor $\mathcal{Z}$ in Theorem 1.3 can be defined on the category $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$, but then it is not clear that this functor is exact. This is the reason we restrict to the subcategory $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}}$.

1.4 Applications to mod p Hecke algebras

We now assume that G is a split connected reductive group defined over a local field E of characteristic $p > 0$ and residue field $\mathbb{F}_q$. Let $K = G(\mathcal{O}_E)$ and let t be a uniformizer of E . Then $\mathcal{H}_K$ has a natural basis $\{\mathbb{1}_\mu\}$ indexed by the dominant cocharacters $X_*(T)^+$ where $\mathbb{1}_\mu$ is the characteristic function of the double coset $K\mu(t)K$. Similarly if $I \subset K$ is the Iwahori subgroup determined by B then $\mathcal{H}_I$ has a basis $\{\mathbb{1}_w\}$ indexed by $\tilde{W}$. Let $\mathbb{1}_K \in \mathcal{H}_I$ be the function which is 1 on K and 0 elsewhere.

The mod p Satake isomorphism of Herzig [Her11b] and Henniart-Vignéras [HV15] is an isomorphism

$$\mathcal{S}: \mathcal{H}_K \xrightarrow{\sim} \mathbb{F}_p[X_*(T)^+]$$

where $X_*(T)^+$ is the monoid of dominant cocharacters. We review the definition of $\mathcal{S}$ in Section 10.3.

In Section 10.2 we show that a version of Theorem 1.3 also holds when we view Gr and $\mathcal{F}\ell$ as ind-schemes over the finite field $\mathbb{F}_q$. We denote these ind-schemes over $\mathbb{F}_q$ by $\mathrm{Gr}_{\mathbb{F}_q}$ and $\mathcal{F}\ell_{\mathbb{F}_q}$. By applying the function-sheaf correspondence we can associate to an element $\mathcal{F}^\bullet \in P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)$ a function $\mathrm{Tr}(\mathcal{F}^\bullet) \in \mathcal{H}_K$, and similarly for $P_I(\mathcal{F}\ell_{\mathbb{F}_q}, \mathbb{F}_p)$ and $\mathcal{H}_I$. The

function $\mathrm{Tr}(\mathcal{F}^\bullet)$ depends only on the isomorphism class of $\mathcal{F}^\bullet$ in the Grothendieck group of the relevant category.

We then define a certain Serre subcategory $P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N \subset P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)$ which can be viewed as the mod p analogue of ℓ -adic perverse sheaves whose simple subquotients are pure of weight zero. The simple objects in this category are indexed by $\mu \in X_*(T)^+$ and are denoted by IC_μ^N . The category $P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N$ is a symmetric monoidal category which also admits an equivalence with the category of representations an affine monoid scheme as in Theorem 1.1, but in general the affine monoid scheme depends on the field over which Gr is defined.

In the following theorem, we equip $K_0(P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N)$ with the structure of a ring where the multiplication is defined by convolution of perverse sheaves.

Theorem 1.5. *There is a natural isomorphism of $\mathbb{F}_p$ -algebras*

$$\mathbb{F}_p[X_*(T)^+] \xrightarrow{\sim} K_0(P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N) \otimes \mathbb{F}_p$$

which sends $\mu \in X_*(T)^+$ to the class of $[\mathrm{IC}_\mu^N] \otimes 1$. The composition

$$\mathbb{F}_p[X_*(T)^+] \xrightarrow{\sim} K_0(P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N) \otimes \mathbb{F}_p \xrightarrow{\mathrm{Tr}} \mathcal{H}_K$$

is the inverse of the mod p Satake isomorphism $\mathcal{S}$.

Now assume that $p > 2$ and that G_{der} is absolutely almost simple. Then a version of Theorem 1.3 also holds over the finite field $\mathbb{F}_q$, and the following theorem then follows from the function-sheaf correspondence.

Theorem 1.6. *Let $\mathcal{C}$ be the isomorphism $Z(\mathcal{H}_I) \rightarrow \mathcal{H}_K$ such that $\mathcal{C}(f) = f * \mathbf{1}_K$. Then*

$$\mathcal{C}^{-1} \left(\sum_{\lambda \leq \mu} \mathbf{1}_\lambda \right) = \sum_{w \in \mathrm{Adm}(\mu)} \mathbf{1}_w.$$

For $\mu \in X_*(T)^+$ let t_μ be the element μ regarded as an element of $\tilde{W}$, and let $\Lambda_\mu \subset X_*(T)$ be the orbit of μ by the action of the finite Weyl group $W := N_G(T)/T$. In [Vig05], Vignéras constructed integral Bernstein elements $B(\lambda) \in \mathcal{H}_I$ for $\lambda \in X_*(T)$ and showed that $\{\sum_{\lambda \in \Lambda_\mu} B(\lambda)\}_{\mu \in X_*(T)^+}$ is an $\mathbb{F}_p$ -basis for $Z(\mathcal{H}_I)$. Ollivier [Oll14] showed that these Bernstein elements give rise to an isomorphism of $\mathbb{F}_p$ -algebras

$$\mathcal{B}: \mathbb{F}_p[X_*(T)^+] \rightarrow Z(\mathcal{H}_I), \quad \mu \mapsto \sum_{\lambda \in \Lambda_\mu} B(\lambda).$$

Ollivier also showed that $\mathcal{B}$ is compatible with the mod p Satake isomorphism in the sense that $\mathcal{B} = \mathcal{C}^{-1} \circ \mathcal{S}^{-1}$.

For our last application, we note that by [Oll14, 2.3] the coefficient of $\mathbf{1}_w$ appearing in $\sum_{\lambda \in \Lambda_\mu} B(\lambda) \in \mathcal{H}_I$ is 0 if $w \notin \text{Adm}(\mu)$ and it is 1 if $w \in \text{Adm}(\mu)$ and $\ell(w) = \ell(t_\mu)$, where ℓ is the length function on $\tilde{W}$. Using Theorem 1.6 we can compute the rest of the coefficients (*cf.* [Oll15, 5.2]).

Corollary 1.7. *Let $\mu \in X_*(T)^+$. Then the integral Bernstein elements satisfy*

$$\sum_{\lambda \in \Lambda_\mu} B(\lambda) = \sum_{w \in \text{Adm}(\mu)} \mathbf{1}_w.$$

Remark 1.8. Let $\tilde{I} \subset I$ be the pro- p Sylow subgroup. The integral Bernstein elements are usually defined in the larger Hecke algebra $\mathcal{H}_{\tilde{I}}$. There is a central idempotent $\epsilon_1 \in \mathcal{H}_{\tilde{I}}$ such that $\mathcal{H}_I = \epsilon_1 \mathcal{H}_{\tilde{I}}$ (see [Oll14, 2.14]). The Bernstein elements we are considering in this thesis are the images of the Bernstein elements in [Oll14] after multiplication by ϵ_1 . The integral Bernstein elements in *loc. cit.* also depend on a choice of a sign ($\pm$) and a Weyl chamber, but the central integral Bernstein elements $\sum_{\lambda \in \Lambda_\mu} B(\lambda)$ do not depend on these choices by [Oll14, 3.4].

1.5 F -singularities and perverse $\mathbb{F}_p$ -sheaves

In this section we allow k to be an arbitrary perfect field of characteristic $p > 0$. The study of F -singularities refers to an approach to classifying singularities in positive characteristic by their interaction with the Frobenius morphism. A basic definition in the field is that of a Frobenius splitting of a scheme X over k , which is an $\mathcal{O}_X$ -linear splitting of the p th power map $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X$.

Remarkably, certain constraints on the Frobenius morphism imply properties of schemes which also make sense in characteristic zero. For example, F -rationality, which is a property of rings in characteristic p defined by the simplicity of an action involving the Frobenius on local cohomology (see Theorem 4.1), implies normality and Cohen-Macaulayness. Crucially for us, F -rationality also provides a purely algebraic way of showing that a scheme has pseudo-rational singularities by a theorem of Smith [Smi97].

While F -rationality is the main property from F -singularities which we use to study perverse $\mathbb{F}_p$ -sheaves, we establish that certain schemes are F -rational by first showing that their local rings are strongly F -regular. The property of strong F -regularity was introduced by Hochster-Huneke [HH89] in relation to tight closure theory. A reduced ring in characteristic p is strongly F -regular if for every non-zero divisor there exists a Frobenius splitting compatible with the non-zero divisor in a certain sense. See Section 5.4 for the precise definition and the relations between these notions.

Now let G be a split connected reductive group over k with a maximal torus and Borel subgroup $T \subset B \subset G$. In order to prove Theorems 1.1, 1.2 and 1.3 we employ Frobenius splitting techniques. To state our results in this direction, note that for every parabolic subgroup $P \subset G$ containing B there is an associated affine flag variety $\mathcal{F}\ell_P$ with an action of the Iwahori group I determined by B . If $P = B$ we set $\mathcal{F}\ell_P = \mathcal{F}\ell$. Given w in the Iwahori-Weyl group, let $\mathcal{F}\ell_{P,w} \subset \mathcal{F}\ell_P$ be the reduced I -orbit closure associated to w .

Theorem 1.9. *Suppose k is a perfect field of characteristic $p > 0$, and that $p \nmid |\pi_1(G_{\text{der}})|$.*

Then the affine Schubert varieties $\mathrm{Gr}_{\leq \mu}$ and $\mathcal{F}\ell_{P,w}$ are globally F -regular, and hence also strongly F -regular and F -rational.

We refer the reader to Section 5.4 for the definition of global F -regularity, which is a global version of strong F -regularity defined for projective k -schemes by Smith in [Smi00]. Theorem 1.9 adds to results of Faltings [Fal03] and Pappas-Rapoport [PR08] in positive characteristic. In particular, they showed that affine Schubert varieties are normal, Cohen-Macaulay, and Frobenius split if $p \nmid |\pi_1(G_{\mathrm{der}})|$. While global F -regularity is stronger than these properties, our proof uses the Frobenius splittings constructed by Faltings and Pappas-Rapoport. See Section 5.1 for more discussion on their results. We also prove that Schubert subvarieties of a particular Beilinson-Drinfeld Grassmannian are F -rational in Theorem 7.4.

Now assuming again that k is algebraically closed, we use Theorem 1.9 to prove the following result.

Theorem 1.10. *The simple objects in $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ and $P_I(\mathcal{F}\ell, \mathbb{F}_p)$ are the shifted constant sheaves $\mathrm{IC}_{\mu} := \mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu}]$ and $\mathbb{F}_p[\dim \mathcal{F}\ell_w]$ supported on the orbit closures $\mathrm{Gr}_{\leq \mu}$ and $\mathcal{F}\ell_w$.*

Theorems 1.9 and 1.10 generalize results of Lauritzen, Raben-Pedersen, and Thomsen [LRPT06] in the case of classical Schubert varieties in flag varieties. While our proof of Theorem 1.9 is analogous to that in [LRPT06], we derive Theorem 1.10 as a consequence of the following more general results.

Theorem 1.11. *If X is a Cohen-Macaulay k -scheme of finite type and equidimension d then the complex of étale sheaves $\mathbb{F}_p[d] \in D_c^b(X, \mathbb{F}_p)$ is perverse.*

Theorem 1.12. *If X is an integral F -rational k -scheme of finite type and dimension d then $\mathbb{F}_p[d]$ is simple as a perverse sheaf.*

The property of F -rationality enters through its connection to the Frobenius action on local cohomology. More precisely, we show the vanishing in certain degrees of the étale coho-

mology of the constant sheaf $\mathbb{F}_p$ with support in the special fibers of the strict henselizations of the local rings of X . By the Artin-Schreier sequence, this reduces to the study of the Frobenius action on the local cohomology modules of these strict henselizations. The crucial result is Lemma 4.6, which says that the top local cohomology group of an F -rational local ring of positive dimension contains no nonzero elements fixed by the Frobenius endomorphism.

During the course of proving Theorem 1.3 we will also prove a result about the singularities of equal characteristic analogues of local models of Shimura varieties. Returning to the setup in Section 1.4, let $\mathrm{Gr}_{\mathbb{F}_q}$ be the affine Grassmannian of G viewed as an ind-scheme over $\mathbb{F}_q$. Then for $\mu \in X_*(T)^+$ there is an associated local model $M_\mu \rightarrow \mathrm{Spec}(\mathcal{O}_E)$ such that the generic fiber of M_μ is isomorphic to $\mathrm{Gr}_{\mathbb{F}_q, \leq \mu} \times \mathrm{Spec}(E)$ and the reduced special fiber is isomorphic to $\mathcal{A}(\mu)$ (see Definition 8.1).

Theorem 1.13. *Suppose that $p > 2$ and that G_{der} is absolutely almost simple and simply connected. Then for any $\mu \in X_*(T)^+$, every local ring in M_μ is strongly F -regular, F -rational, and has pseudo-rational singularities.*

In the mixed characteristic case, local models are used to study the étale local structure of integral models of Shimura varieties with parahoric level structure. In the equal characteristic case they are related to moduli spaces of shtukas. We refer the reader to [HR19] for more information on local models, where it is also shown that certain local models are Cohen-Macaulay by using Frobenius splittings of global affine Schubert varieties constructed in [Zhu14].

We prove Theorem 1.13 by combining the same Frobenius splittings in [Zhu14] with Theorem 1.9. In fact, we prove that certain global affine Schubert varieties are strongly F -regular (Theorem 8.14) and then we deduce the local statement in Theorem 1.13. We also show that Schubert subvarieties in related Beilinson-Drinfeld and convolution Grassmannians are strongly F -regular in Theorem 9.6.

1.6 Outline

In Chapters 2 and 3 we prove basic facts about the category $P_c^b(X, \mathbb{F}_p)$ of perverse $\mathbb{F}_p$ -sheaves on a scheme X of finite type over k which are well-known for perverse $\overline{\mathbb{Q}}_\ell$ -sheaves. For example, we show that for an action of a smooth connected group scheme G on X , the G -equivariant perverse sheaves form a full subcategory of $P_c^b(X, \mathbb{F}_p)$ which is stable under taking subquotients. The usual proofs for $\overline{\mathbb{Q}}_\ell$ -sheaves do not always work in our setting, largely because only three of the six functors preserve constructibility. Our main tool is the Riemann-Hilbert correspondence of Emerton-Kisin [EK04b].

In Chapter 4 we prove Theorems 1.11 and 1.12. For the proofs we use the Artin-Schreier sequence to transform the problem into deciding whether certain local cohomology modules vanish or have any nonzero elements fixed by the Frobenius endomorphism.

In Chapter 5 we introduce the affine Grassmannian and prove that affine Schubert varieties are globally F -regular. We first apply a result of Pappas-Rapoport [PR08] to reduce to the case when G is simply connected. After that our proof is analogous to that of Lauritzen, Raben-Pedersen, and Thomsen [LRPT06] in the classical setting.

In Chapter 6 we define the categories $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ and $P_I(\mathcal{F}\ell_{\mathcal{P}}, \mathbb{F}_p)$, and we prove Theorem 1.10. Then we define the convolution product of two perverse sheaves in $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$. The fact that the convolution product preserves perversity follows from the fact that $\mathrm{IC}_{\mu_1} * \mathrm{IC}_{\mu_2} = \mathrm{IC}_{\mu_1 + \mu_2}$ (Theorem 1.2).

In Theorem 6.9 we show that $R\Gamma(\mathrm{IC}_\mu) = \mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu}]$, which completes the proof of Theorem 1.2. By contrast, in the case of $\overline{\mathbb{Q}}_\ell$ -coefficients the IC sheaves generally have nonzero hypercohomology in multiple degrees. Theorem 6.9 is an immediate consequence of the global F -regularity of $\mathrm{Gr}_{\leq \mu}$. We then apply Theorem 6.9 to show that the functor $H : P_{L+G}(\mathrm{Gr}, \mathbb{F}_p) \rightarrow \mathrm{Vect}_{\mathbb{F}_p}$ is exact and faithful. Finally, we investigate extensions between objects in $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ in Section 6.4.

In Chapter 7 we construct a commutativity constraint on $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$, and we show

that the global cohomology functor H is a symmetric monoidal functor. Our proof follows the standard technique of interpreting the convolution product as a fusion product on a Beilinson-Drinfeld Grassmannian, and we again make use of Frobenius splitting techniques (see Lemma 7.8). At this point the same Tannakian formalism as in the case of $\overline{\mathbb{Q}}_\ell$ -coefficients provides us with an affine monoid scheme M_G such that $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p) \cong \mathrm{Rep}_{\mathbb{F}_p}(M_G)$. We conclude Chapter 7 with a few results about M_G .

We then turn to the construction of $\mathcal{Z}$ in Chapter 8, starting with an introduction to the global affine Schubert varieties associated to a non-constant group scheme over a curve. This allows us to define the local models M_μ , as well as the functor $\mathcal{Z}$. We then prove Theorem 1.13 and parts (i) and (ii) of Theorem 1.3.

In order to prove the remaining parts of Theorem 1.3, in Chapter 9 we introduce the Beilinson-Drinfeld and convolution Grassmannians associated to a non-constant group scheme over a curve. After establishing that Schubert subvarieties in these Grassmannians have strongly F -regular singularities in Theorem 9.6, the rest of the proof of Theorem 1.3 follows the method in [Zhu14].

In Chapter 10 we recall the function-sheaf correspondence for $\mathbb{F}_p$ -sheaves. We then prove Theorem 1.5 by using Theorem 1.2 and an explicit formula for the inverse of the mod p Satake isomorphism due to Herzig [Her11a]. Finally, we use Theorem 1.3 to deduce Theorem 1.6 and Corollary 1.7.

Chapter 2

Perverse $\mathbb{F}_p$ -sheaves

2.1 Basic properties of perverse $\mathbb{F}_p$ -sheaves

For Chapters 2 - 4 we fix an algebraically closed field k of characteristic $p > 0$. By a k -scheme we will always mean a separated scheme of finite type over k unless we specify otherwise. In this section we introduce the category of perverse $\mathbb{F}_p$ -sheaves on a k -scheme X , and we prove some basic facts analogous to those for $\overline{\mathbb{Q}}_\ell$ -coefficients as in [BBD82].

We let $D^*(X, \mathbb{F}_p)$ for $* = \emptyset, +, -, b$ denote the corresponding derived category of étale $\mathbb{F}_p$ -sheaves on X . Let $D_c^b(X, \mathbb{F}_p)$ denote the triangulated subcategory of $D^b(X, \mathbb{F}_p)$ consisting of objects having constructible cohomology sheaves. For a morphism $f: X \rightarrow Y$ between k -schemes, we have the three functors

$$\begin{aligned} Rf^* &: D(Y, \mathbb{F}_p) \rightarrow D(X, \mathbb{F}_p) \\ Rf_! &: D^+(X, \mathbb{F}_p) \rightarrow D^+(Y, \mathbb{F}_p) \\ - \otimes_{\mathbb{F}_p}^L -: & D^-(X, \mathbb{F}_p) \times D^-(X, \mathbb{F}_p) \rightarrow D^-(X, \mathbb{F}_p). \end{aligned}$$

Each of these functors restricts to a functor involving $D_c^b(X, \mathbb{F}_p)$ and $D_c^b(Y, \mathbb{F}_p)$. We also

have the three functors

$$\begin{aligned} Rf^! &: D^+(Y, \mathbb{F}_p) \rightarrow D^+(X, \mathbb{F}_p) \\ Rf_* &: D^+(X, \mathbb{F}_p) \rightarrow D^+(Y, \mathbb{F}_p) \\ R\mathcal{H}om_{\mathbb{F}_p}(-, -) &: D(X, \mathbb{F}_p) \times D^+(X, \mathbb{F}_p) \rightarrow D(X, \mathbb{F}_p). \end{aligned}$$

Contrary to the case of $\mathbb{F}_\ell$ -sheaves for $\ell \neq p$, these last three functors do not in general preserve constructibility. In particular, there is no good notion of a dualizing complex, and there is no Verdier duality functor. Additionally, $Rf^!$ does not in general agree with a shift of Rf^* when f is smooth.

Despite these drawbacks, we can still define perverse $\mathbb{F}_p$ -sheaves (for the middle perversity function) in a way similar to perverse $\overline{\mathbb{Q}}_\ell$ -sheaves. Fix a point $x \in X$, and let Y be the reduced closed subscheme with topological space $\overline{\{x\}}$. For a complex $\mathcal{F}^\bullet \in D^+(X, \mathbb{F}_p)$ and $i \in \mathbb{Z}$ we say that $H^i(Ri_x^* \mathcal{F}^\bullet) = 0$ if the following condition holds:

- There exists a dense open subscheme $U \subset Y$ such that $H^i(Ri_U^* \mathcal{F}^\bullet) = 0$, where $i_U: U \rightarrow X$ is the immersion.

We can similarly define what it means to have $H^i(Ri_x^! \mathcal{F}^\bullet) = 0$. Now consider the following two subcategories of $D^+(X, \mathbb{F}_p)$:

$$\begin{aligned} {}^pD^{\leq 0}(X, \mathbb{F}_p) &= \{\mathcal{F}^\bullet \in D^+(X, \mathbb{F}_p) : H^i(Ri_x^* \mathcal{F}^\bullet) = 0 \text{ for all } x \in X \text{ and } i > -\dim \overline{\{x\}}\} \\ {}^pD^{\geq 0}(X, \mathbb{F}_p) &= \{\mathcal{F}^\bullet \in D^+(X, \mathbb{F}_p) : H^i(Ri_x^! \mathcal{F}^\bullet) = 0 \text{ for all } x \in X \text{ and } i < -\dim \overline{\{x\}}\}. \end{aligned}$$

Gabber [Gab04] has shown that these subcategories underlie a t -structure on $D^+(X, \mathbb{F}_p)$. Moreover, he has shown these subcategories also induce a t -structure on $D_c^b(X, \mathbb{F}_p)$. We

denote the category of perverse sheaves in $D^+(X, \mathbb{F}_p)$ by

$$P^+(X, \mathbb{F}_p) := {}^pD^{\leq 0}(X, \mathbb{F}_p) \cap {}^pD^{\geq 0}(X, \mathbb{F}_p).$$

Let $P_c^b(X, \mathbb{F}_p)$ be the full abelian subcategory of perverse sheaves in $D_c^b(X, \mathbb{F}_p)$.

Remark 2.1. For $x \in X$, let $j: \operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}}) \rightarrow X$ be the strict henselization of the local ring of X at x , and let $i: \bar{x} \rightarrow \operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}})$ be the inclusion of the closed point. Then the condition $H^i(Ri_x^* \mathcal{F}^\bullet) = 0$ is equivalent to $H^i(Ri^*(Rj^* \mathcal{F}^\bullet)) = 0$. Similarly, the condition $H^i(Ri_x^! \mathcal{F}^\bullet) = 0$ is equivalent to $H^i(Ri^!(Rj^* \mathcal{F}^\bullet)) = 0$ (see [Gab04]).

Remark 2.2. When X is smooth, Emerton and Kisin have given an alternative description of $P_c^b(X, \mathbb{F}_p)$ as follows. For any k -algebra R let $R[F]$ be the non-commutative polynomial algebra in one variable F such that $Fa = a^p F$ for $a \in R$. Let $\mathcal{O}_{F,X}$ be the sheaf of non-commutative rings whose value on an open affine $\operatorname{Spec}(R)$ is $R[F]$. If $\mathcal{M}$ is an $\mathcal{O}_{F,X}$ -module there is a natural map of $\mathcal{O}_X$ -modules

$$\varphi_M: F^* \mathcal{M} \rightarrow \mathcal{M},$$

where F is the absolute Frobenius endomorphism on X . The $\mathcal{O}_{F,X}$ -module $\mathcal{M}$ is said to be *locally finitely generated unit* (lfgu) if it is quasi-coherent as an $\mathcal{O}_X$ -module, φ_M is an isomorphism and $\mathcal{M}$ is, locally on X , finitely generated over $\mathcal{O}_{F,X}$. Let $D_{\text{lfgu}}^b(\mathcal{O}_{F,X})$ be the bounded derived category of $\mathcal{O}_{F,X}$ -modules whose cohomology sheaves are lfgu $\mathcal{O}_{F,X}$ -modules. When X is smooth the category $D_c^b(X, \mathbb{F}_p)$ is anti-equivalent to the category $D_{\text{lfgu}}^b(\mathcal{O}_{F,X})$ by [EK04b, 11.3]. Under this equivalence, the perverse t -structure on $D_c^b(X, \mathbb{F}_p)$ corresponds to the standard t -structure on $D_{\text{lfgu}}^b(\mathcal{O}_{F,X})$ by [EK04b, 11.5.4]. If $f: Y \rightarrow X$ is a morphism between smooth k -schemes then Emerton and Kisin have also given a description of the functors Rf^* , $Rf_!$, and $- \otimes_{\mathbb{F}_p}^L -$ in terms of functors on quasi-coherent sheaves.

We now establish some basic properties of perverse $\mathbb{F}_p$ -sheaves. To begin, we have the following result of Gabber, which is in fact valid for a larger class of noetherian schemes defined over $\mathbb{F}_p$.

Theorem 2.3 ([Gab04, 12.4]). *Let X be a k -scheme.*

- (i) *Every object of $P_c^b(X, \mathbb{F}_p)$ is artinian and noetherian.*
- (ii) *Every perverse subquotient of an object of $P_c^b(X, \mathbb{F}_p)$ that lies in $P^+(X, \mathbb{F}_p)$ in fact lies in $P_c^b(X, \mathbb{F}_p)$.*

Given a complex $\mathcal{F}^\bullet \in D^+(X, \mathbb{F}_p)$, we let ${}^p H^i(\mathcal{F}^\bullet) \in P^+(X, \mathbb{F}_p)$ denote its i th perverse cohomology sheaf. If $\mathcal{F}^\bullet \in D_c^b(X, \mathbb{F}_p)$, then ${}^p H^i(\mathcal{F}^\bullet) \in P_c^b(X, \mathbb{F}_p)$. For a morphism between k -schemes $f: X \rightarrow Y$, define

$${}^p f_* \mathcal{F}^\bullet := {}^p H^0(Rf_* \mathcal{F}^\bullet) \in P^+(Y, \mathbb{F}_p).$$

By this recipe we get functors

$$\begin{aligned} {}^p f_*, {}^p f_! : P^+(X, \mathbb{F}_p) &\rightarrow P^+(Y, \mathbb{F}_p) \\ {}^p f^*, {}^p f^! : P^+(Y, \mathbb{F}_p) &\rightarrow P^+(X, \mathbb{F}_p). \end{aligned}$$

In general, two of these restrict to functors

$$\begin{aligned} {}^p f_! : P_c^b(X, \mathbb{F}_p) &\rightarrow P_c^b(Y, \mathbb{F}_p) \\ {}^p f^* : P_c^b(Y, \mathbb{F}_p) &\rightarrow P_c^b(X, \mathbb{F}_p). \end{aligned}$$

Let $j: U \rightarrow X$ be an open immersion of k -schemes, and let $i: Z \rightarrow X$ be a complementary closed immersion. In the following lemma, all functors are defined on the categories $D^+(X, \mathbb{F}_p)$, $D^+(U, \mathbb{F}_p)$, $D^+(Z, \mathbb{F}_p)$, or their corresponding hearts for the perverse t -structure.

Lemma 2.4.

(i) Rj^* and Ri_* are t -exact, and the functors ${}^pj^*$ and ${}^pi_*$ are exact.

(ii) $Rj_!$ and Ri^* are right t -exact, i.e.,

$$Rj_!({}^pD^{\leq 0}(U, \mathbb{F}_p)) \subset {}^pD^{\leq 0}(X, \mathbb{F}_p), \quad Ri^*({}^pD^{\leq 0}(X, \mathbb{F}_p)) \subset {}^pD^{\leq 0}(Z, \mathbb{F}_p),$$

and the functors ${}^pj_!$ and ${}^pi^*$ are right exact.

(iii) Rj_* and $Ri^!$ are left t -exact, and the functors ${}^pj_*$ and ${}^pi^!$ are left exact.

(iv) The triples $({}^pj_!, {}^pj^*, {}^pj_*)$ and $({}^pi^*, {}^pi_*, {}^pi^!)$ are adjoint sequences.

(v) We have

$${}^pj^* \circ {}^pi_* = 0, \quad {}^pi^* \circ {}^pj_! = 0, \quad {}^pi^! \circ {}^pj_* = 0.$$

(vi) We have

$${}^pj^* \circ {}^pj_* = \text{id}, \quad {}^pj^* \circ {}^pj_! = \text{id}, \quad {}^pi^* \circ {}^pi_* = \text{id}, \quad {}^pi^! \circ {}^pi_* = \text{id}.$$

(vii) For $\mathcal{F}^\bullet \in P^+(X, \mathbb{F}_p)$ there are exact sequences

$$0 \rightarrow {}^pi_* {}^pH^{-1}(Ri^* \mathcal{F}^\bullet) \rightarrow {}^pj_! {}^pj^* \mathcal{F}^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow {}^pi_* {}^pi^* \mathcal{F}^\bullet \rightarrow 0$$

and

$$0 \rightarrow {}^pi_* {}^pi^! \mathcal{F}^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow {}^pj_* {}^pj^* \mathcal{F}^\bullet \rightarrow {}^pi_* {}^pH^1(Ri^! \mathcal{F}^\bullet) \rightarrow 0.$$

(viii) The functor ${}^pi_*$ is fully faithful. Its essential image consists of perverse sheaves $\mathcal{F}^\bullet \in P^+(X, \mathbb{F}_p)$ such that ${}^pj^* \mathcal{F}^\bullet = 0$.

Proof. The analogous facts are true for $\overline{\mathbb{Q}}_\ell$ -coefficients [BBD82]. The proofs are the same, so we only provide a sketch. Parts (i) - (iii) are straightforward. The proof of parts (iv) - (vi) uses (i) - (iii) and the analogous facts for the corresponding non-truncated functors. For part (vii), apply the other parts to the exact triangles

$$Rj_!Rj^*\mathcal{F}^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow Ri_*Ri^*\mathcal{F}^\bullet$$

and

$$Ri_*Ri^!\mathcal{F}^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow Rj_*Rj^*\mathcal{F}^\bullet.$$

For part (viii) we note that ${}^pi_*$ is fully faithful because of part (i) and the fact that Ri_* is fully faithful. We have already seen that ${}^pj^* \circ {}^pi_* = 0$, so let $\mathcal{F}^\bullet \in P^+(X, \mathbb{F}_p)$ be such that ${}^pj^*\mathcal{F}^\bullet = 0$. Now apply the first exact sequence in part (vii) to see that $\mathcal{F}^\bullet$ is in the essential image of ${}^pi_*$. $\square$

2.2 The intermediate extension

When X is smooth, Emerton-Kisin [EK04a] defined the intermediate extension $j_{!*}\mathcal{F}^\bullet$ of a perverse sheaf $\mathcal{F}^\bullet \in P_c^b(U, \mathbb{F}_p)$ along an immersion $j: U \rightarrow X$, where U is also smooth. The construction uses their theory of unit $\mathcal{O}_{F,X}$ -modules. They also showed that when X is smooth, a perverse sheaf $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ is simple if and only if it is of the form $j_{!*}\mathcal{L}[\dim U]$ where U is smooth, $j: U \rightarrow X$ is an immersion, and $\mathcal{L}$ is a simple local system of étale $\mathbb{F}_p$ -sheaves on U . We now investigate the properties of the intermediate extension functor for general k -schemes.

Let X be a k -scheme and let $j: U \rightarrow X$ be an immersion. Suppose we have a factorization of j into a closed immersion $i: U \rightarrow Z$ followed by an open immersion $h: Z \rightarrow X$. By Lemma 2.4 (i) we have canonical isomorphisms of functors ${}^ph_! \circ {}^pi_* \cong {}^pj_!$ and ${}^ph_* \circ {}^pi_* \cong {}^pj_*$. Thus by Lemma 2.4 (iv) there is a natural map ${}^pj_!\mathcal{F}^\bullet \rightarrow {}^pj_*\mathcal{F}^\bullet$ for $\mathcal{F}^\bullet \in P^+(U, \mathbb{F}_p)$.

Definition 2.5. The *intermediate extension* of $\mathcal{F}^\bullet \in P^+(U, \mathbb{F}_p)$ along the immersion $j: U \rightarrow X$ is

$$j_{!*}\mathcal{F}^\bullet := \text{im}({}^p j_! \mathcal{F}^\bullet \rightarrow {}^p j_* \mathcal{F}^\bullet).$$

This defines a functor

$$j_{!*}: P^+(U, \mathbb{F}_p) \rightarrow P^+(X, \mathbb{F}_p).$$

The following lemma can be proved using the properties established in Lemma 2.4.

Lemma 2.6. *Suppose we have a factorization of $j: U \rightarrow X$ as a composition of immersions $j = j_1 \circ \cdots \circ j_n$ between k -schemes. Then we have a canonical isomorphism of functors*

$$j_{!*} \xrightarrow{\sim} j_{1,*} \circ \cdots \circ j_{n,*}.$$

We note that $j_{!*}\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ if $\mathcal{F}^\bullet \in P_c^b(U, \mathbb{F}_p)$ by Theorem 2.3 (ii). This definition of the intermediate extension agrees with the one in [EK04a] for smooth k -schemes.

If $\mathcal{F}^\bullet \in D(U, \mathbb{F}_p)$ we say that $\mathcal{G}^\bullet \in D(X, \mathbb{F}_p)$ is an *extension* of $\mathcal{F}^\bullet$ if there is an isomorphism $Rj^*\mathcal{G}^\bullet \cong \mathcal{F}^\bullet$. As in the case of $\overline{\mathbb{Q}_\ell}$ -sheaves, the next lemma shows that the intermediate extension can be characterized as a unique minimal extension.

Lemma 2.7. *Let $j: U \rightarrow X$ be an open immersion of k -schemes and let $\mathcal{F}^\bullet \in P^+(U, \mathbb{F}_p)$. Let $i: Z \rightarrow X$ be a complementary closed immersion. The intermediate extension $j_{!*}\mathcal{F}^\bullet$ is characterized by either of the two equivalent conditions:*

- (i) *The perverse sheaf $j_{!*}\mathcal{F}^\bullet$ is the unique perverse extension $\mathcal{G}^\bullet$ of $\mathcal{F}^\bullet$ such that*

$$Ri^*\mathcal{G}^\bullet \in {}^p D^{\leq -1}(Z, \mathbb{F}_p) \quad \text{and} \quad Ri^!\mathcal{G}^\bullet \in {}^p D^{\geq 1}(Z, \mathbb{F}_p).$$

- (ii) *The perverse sheaf $j_{!*}\mathcal{F}^\bullet$ is the unique perverse extension $\mathcal{G}^\bullet$ of $\mathcal{F}^\bullet$ such that $\mathcal{G}^\bullet$ has no nonzero subobject or quotient in the essential image of ${}^p i_*: P^+(Z, \mathbb{F}_p) \rightarrow P^+(X, \mathbb{F}_p)$.*

Proof. We first show that $j_{!*}\mathcal{F}^\bullet$ satisfies the conditions in (i) except for possibly uniqueness. The complex $j_{!*}\mathcal{F}^\bullet$ is an extension of $\mathcal{F}^\bullet$ because ${}^pj^*$ is exact and ${}^pj^* \circ {}^pj_! = {}^pj^* \circ {}^pj_* = \text{id}$. As ${}^pi^*$ is right exact we get a surjection ${}^pi^*({}^pj_!\mathcal{F}^\bullet) \twoheadrightarrow {}^pi^*(j_{!*}\mathcal{F}^\bullet)$. Since ${}^pi^* \circ {}^pj_! = 0$ it follows that ${}^pi^*(j_{!*}\mathcal{F}^\bullet) = 0$. Similarly by applying the left exact functor ${}^pi^!$ to the injection $j_{!*}\mathcal{F}^\bullet \hookrightarrow {}^pj_*\mathcal{F}^\bullet$ we get that ${}^pi^!(j_{!*}\mathcal{F}^\bullet) = 0$. Now because Ri^* is right t -exact and $Ri^!$ is left t -exact we conclude that $Ri^*(j_{!*}\mathcal{F}^\bullet) \in {}^pD^{\leq -1}(Z, \mathbb{F}_p)$ and $Ri^!(j_{!*}\mathcal{F}^\bullet) \in {}^pD^{\geq 1}(Z, \mathbb{F}_p)$.

We now make a general observation. Suppose $\mathcal{G}^\bullet \in P^+(X, \mathbb{F}_p)$ is an extension of $\mathcal{F}^\bullet$. Then from the natural maps ${}^pj_!{}^pj^*\mathcal{G}^\bullet \rightarrow \mathcal{G}^\bullet$ and $\mathcal{G}^\bullet \rightarrow {}^pj_*{}^pj^*\mathcal{G}^\bullet$, we get maps ${}^pj_!\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ and $\mathcal{G}^\bullet \rightarrow {}^pj_*\mathcal{F}^\bullet$ such that the composition is the natural map ${}^pj_!\mathcal{F}^\bullet \rightarrow {}^pj_*\mathcal{F}^\bullet$. Now suppose $\mathcal{G}^\bullet$ satisfies the conditions in (i) except for possibly uniqueness. Then from the above considerations and the exact sequences in Lemma 2.4 (vii) we get maps

$${}^pj_!\mathcal{F}^\bullet \twoheadrightarrow \mathcal{G}^\bullet \hookrightarrow {}^pj_*\mathcal{F}^\bullet$$

such that the composition is the natural map ${}^pj_!\mathcal{F}^\bullet \rightarrow {}^pj_*\mathcal{F}^\bullet$. From this it follows that $\mathcal{G}^\bullet \cong j_{!*}\mathcal{F}^\bullet$.

For (ii) we start with a couple of general observations. For $\mathcal{G}^\bullet \in P^+(X, \mathbb{F}_p)$ we have a surjection $\mathcal{G}^\bullet \twoheadrightarrow {}^{pi_*}{}^{pi^*}\mathcal{G}^\bullet$ by Lemma 2.4 (vii). By adjunction and Lemma 2.4 (viii) it follows that ${}^{pi_*}{}^{pi^*}\mathcal{G}^\bullet$ is the largest quotient of $\mathcal{G}^\bullet$ in the essential image of ${}^{pi_*}: P^+(Z, \mathbb{F}_p) \rightarrow P^+(X, \mathbb{F}_p)$. Similarly ${}^{pi_*}{}^{pi^!}\mathcal{G}^\bullet$ is the largest subobject of $\mathcal{G}^\bullet$ in the essential image of ^{pi_*} . Now the fact that $j_{!*}\mathcal{F}^\bullet$ is uniquely characterized by (ii) follows from (i). $\square$

Lemma 2.8. *Let $j: U \rightarrow X$ be an immersion of k -schemes. Then the intermediate extension functor $j_{!*}: P^+(U, \mathbb{F}_p) \rightarrow P^+(X, \mathbb{F}_p)$ preserves injections and surjections.*

Proof. This follows from the definitions, and the fact that ${}^pj_!$ is right exact and ${}^pj_*$ is left exact. $\square$

Lemma 2.9. *Let U be a k -scheme and let $\mathcal{F}^\bullet \in P^+(U, \mathbb{F}_p)$ be simple. Suppose $j: U \rightarrow X$ is an immersion into a k -scheme X . Then $j_{!*}\mathcal{F}^\bullet$ is simple.*

Proof. It suffices to treat the case of an open immersion and a closed immersion separately. First suppose j is an open immersion, and that we have an exact sequence

$$0 \rightarrow \mathcal{G}^\bullet \rightarrow j_{!*}\mathcal{F}^\bullet \rightarrow \mathcal{H}^\bullet \rightarrow 0$$

in $P^+(X, \mathbb{F}_p)$. Applying the exact functor ${}^pj^*$, we get that ${}^pj^*\mathcal{G}^\bullet = 0$ or ${}^pj^*\mathcal{H}^\bullet = 0$ because ${}^pj^*(j_{!*}\mathcal{F}^\bullet) = \mathcal{F}^\bullet$ is simple. By Lemma 2.4 (viii) and Lemma 2.7, this means that $\mathcal{G}^\bullet = 0$ or $\mathcal{H}^\bullet = 0$.

If j is a closed immersion, let $h: Z \rightarrow X$ be a complementary open immersion. Then $j_{!*}\mathcal{F}^\bullet = {}^pj_*\mathcal{F}^\bullet$, and so ${}^ph^*(j_{!*}\mathcal{F}^\bullet) = 0$ by Lemma 2.4 (v). Now starting with an exact sequence as above, it follows that ${}^ph^*\mathcal{G}^\bullet = 0$ and ${}^ph^*\mathcal{H}^\bullet = 0$. Hence by Lemma 2.4 (vi) and (viii), the given exact sequence comes from applying ${}^pj_*$ to an exact sequence

$$0 \rightarrow \mathcal{G}'^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow \mathcal{H}'^\bullet \rightarrow 0$$

in $P^+(U, \mathbb{F}_p)$. Here we have that $\mathcal{G}'^\bullet = 0$ or $\mathcal{H}'^\bullet = 0$ because $\mathcal{F}^\bullet$ is simple. $\square$

Lemma 2.10. *Let $j: U \rightarrow X$ be an open immersion of k -schemes, and suppose $\mathcal{F}^\bullet \in P^+(X, \mathbb{F}_p)$ is simple. Then if ${}^pj^*\mathcal{F}^\bullet \neq 0$, we have $\mathcal{F}^\bullet = j_{!*}({}^pj^*\mathcal{F}^\bullet)$ and ${}^pj^*\mathcal{F}^\bullet$ is simple.*

Proof. For the first part, since $\mathcal{F}^\bullet$ is an extension of ${}^pj^*\mathcal{F}^\bullet$, then as in the proof of Lemma 2.7 (ii) we have natural maps ${}^pj_!({}^pj^*\mathcal{F}^\bullet) \rightarrow \mathcal{F}^\bullet$ and $\mathcal{F}^\bullet \rightarrow {}^pj_*({}^pj^*\mathcal{F}^\bullet)$ such that the composition is the natural map ${}^pj_!({}^pj^*\mathcal{F}^\bullet) \rightarrow {}^pj_*({}^pj^*\mathcal{F}^\bullet)$. Because $\mathcal{F}^\bullet$ is simple, and each of these two maps restricts to a nonzero map under ${}^pj^*$, it follows that ${}^pj_!({}^pj^*\mathcal{F}^\bullet) \rightarrow \mathcal{F}^\bullet$ is surjective and $\mathcal{F}^\bullet \rightarrow {}^pj_*({}^pj^*\mathcal{F}^\bullet)$ is injective. Thus $\mathcal{F}^\bullet = j_{!*}({}^pj^*\mathcal{F}^\bullet)$. To see that ${}^pj^*\mathcal{F}^\bullet$ is simple, it now suffices to note that $j_{!*}$ preserves injections and surjections, and that ${}^pj^* \circ j_{!*} = \text{id}$. $\square$

Lemma 2.11. *Let $j: U \rightarrow X$ be an immersion of k -schemes, and suppose $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P^+(U, \mathbb{F}_p)$. Then*

$$\mathrm{Hom}(\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet) = \mathrm{Hom}(j_{!*}\mathcal{F}_1^\bullet, j_{!*}\mathcal{F}_2^\bullet).$$

Proof. This follows from the definition of $j_{!*}$ and the adjunctions in Lemma 2.4 (iv). $\square$

Definition 2.12. Let X be a k -scheme and let $\mathcal{F}^\bullet \in D_c^b(X, \mathbb{F}_p)$. The *support* of $\mathcal{F}^\bullet$ is the union of the supports of each of the cohomology sheaves $H^i(\mathcal{F}^\bullet)$.

The support of $\mathcal{F}$ is a constructible subset of X . We now prove an analogue of the classification of simple perverse sheaves in [BBD82, 4.3.1 (ii)].

Theorem 2.13. *Let X be a k -scheme, and let $j: U \rightarrow X$ be an immersion from an irreducible smooth k -scheme U . Suppose $\mathcal{L}$ is a simple local system of étale $\mathbb{F}_p$ -sheaves on U . Then the complex $\mathcal{L}[\dim U] \in D_c^b(U, \mathbb{F}_p)$ is perverse, and $j_{!*}\mathcal{L}[\dim U]$ is simple. Conversely, every simple perverse sheaf in $P_c^b(X, \mathbb{F}_p)$ is of the form $j_{!*}\mathcal{L}[\dim U]$ for $j: U \rightarrow X$ an immersion from an irreducible smooth k -scheme and $\mathcal{L}$ a simple local system of étale $\mathbb{F}_p$ -sheaves on U .*

Proof. It is shown in [EK04a, 4.3.3] that if U is an irreducible smooth k -scheme and $\mathcal{L}$ is a simple local system of étale $\mathbb{F}_p$ -sheaves on U , then $\mathcal{L}[\dim U]$ is a simple perverse sheaf. Hence if $j: U \rightarrow X$ is an immersion then $j_{!*}\mathcal{L}[\dim U]$ is simple by Lemma 2.9. Conversely, suppose $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ is simple. If we let Z be the closure of the support of $\mathcal{F}^\bullet$ (with its reduced induced structure), then we can replace X by Z so we can assume the support of $\mathcal{F}^\bullet$ is dense in X . Since k is perfect, we may choose an open immersion $j: U \rightarrow X$ such that U is a smooth k -scheme and U is contained in the support of $\mathcal{F}^\bullet$. By Lemma 2.10 we can replace X by U and so we can assume X is smooth. Now the fact that $\mathcal{F}^\bullet$ is of the desired form is proved in [EK04a, 4.3.3]. $\square$

2.3 Pullback along a smooth morphism

We now investigate the behavior of perverse sheaves under smooth pullback.

Lemma 2.14. *Let $f: Y \rightarrow X$ be an étale morphism between k -schemes, and let $\mathcal{F}^\bullet \in P^+(X, \mathbb{F}_p)$. Then $Rf^*\mathcal{F}^\bullet \in P^+(Y, \mathbb{F}_p)$.*

Proof. This follows from the definitions and the fact that $Rf^* = Rf^!$. $\square$

Lemma 2.15. *Let $f: Y \rightarrow X$ be a smooth morphism of relative dimension d between k -schemes and let $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$. Then $Rf^*[d]\mathcal{F}^\bullet \in P_c^b(Y, \mathbb{F}_p)$.*

Proof. We know that $Rf^*[d]\mathcal{F}^\bullet \in D_c^b(Y, \mathbb{F}_p)$ so we only need to prove that $Rf^*[d]\mathcal{F}^\bullet$ is perverse. The problem is Zariski local on Y so we may assume that we have a factorization $Y \xrightarrow{g} \mathbb{A}_X^d \xrightarrow{\pi} X$ where g is étale and π is the projection. Thus by Lemma 2.14 it suffices to prove the result for the projection $\pi: \mathbb{A}_X^d \rightarrow X$. Observe that we can further reduce to the case $X = \text{Spec}(A)$. Now let $i: X \rightarrow \mathbb{A}^n$ be a closed immersion for some n , and consider the Cartesian diagram

$$\begin{array}{ccc} \mathbb{A}_X^d & \xrightarrow{i'} & \mathbb{A}^{n+d} \\ \downarrow \pi & & \downarrow \pi' \\ X & \xrightarrow{i} & \mathbb{A}^n. \end{array}$$

The map π' is the projection onto the first n coordinates. Note that by Lemma 2.4, the complex $R\pi^*[d]\mathcal{F}^\bullet$ is perverse if and only if $Ri'_*(R\pi^*[d]\mathcal{F}^\bullet)$ is perverse. Thus, by the proper base change theorem we can reduce to the case of the projection $\pi: \mathbb{A}^{n+d} \rightarrow \mathbb{A}^n$ onto the first n coordinates. In particular, we have reduced to the case in which X and Y are smooth. Now we can appeal to the Riemann-Hilbert correspondence of Emerton-Kisin [EK04b]. The functor $Rf^*[d]: D_c^b(X, \mathbb{F}_p) \rightarrow D_c^b(Y, \mathbb{F}_p)$ corresponds to the functor

$$f^![-d]: D_{\text{lfu}}^b(\mathcal{O}_{F,X}) \rightarrow D_{\text{lfu}}^b(\mathcal{O}_{F,Y}).$$

On the level of quasi-coherent sheaves the functor $f^![-d]$ is the left derived functor of the usual pullback functor for quasi-coherent sheaves. Because f is flat this functor is already exact, and so $f^![-d]$ is t -exact for the standard t -structures on $D_{\text{lfu}}^b(\mathcal{O}_{F,X})$ and $D_{\text{lfu}}^b(\mathcal{O}_{F,Y})$. These t -structures correspond to the perverse t -structures on $D_c^b(X, \mathbb{F}_p)$ and $D_c^b(Y, \mathbb{F}_p)$. $\square$

We now show that intermediate extensions commute with smooth pullbacks. Stabler [Sta19] has given a proof of this result for k -schemes that admit closed immersions into smooth k -schemes. His proof uses the category of Cartier crystals, which we will not use here. In the first part of the proof we reduce to the case of smooth k -schemes. The theorem then follows from Stabler’s result. We also give a direct argument in the smooth case using unit $\mathcal{O}_{F,X}$ -modules in Lemma 2.17.

Theorem 2.16. *Let $f: Y \rightarrow X$ be a smooth morphism of relative dimension d between k -schemes, and let $j: U \rightarrow X$ be an immersion. Consider the Cartesian diagram*

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{j'} & Y \\ \downarrow f' & & \downarrow f \\ U & \xrightarrow{j} & X \end{array}$$

There is an isomorphism of functors $P_c^b(U, \mathbb{F}_p) \rightarrow P_c^b(Y, \mathbb{F}_p)$:

$$Rf^*[d] \circ j_{!*} \xrightarrow{\sim} j'_{!*} \circ Rf'^*[d].$$

Proof. It suffices to treat the case of a closed immersion and open immersion separately. If j is a closed immersion then $j_{!*} = {}^p j_* = Rj_*$ and so the result is a special case of the proper base change theorem. Now suppose j is an open immersion and let $\mathcal{F}^\bullet \in P_c^b(U, \mathbb{F}_p)$. We will show that $Rf^*[d](j_{!*}\mathcal{F}^\bullet) \cong j'_{!*}(Rf'^*[d]\mathcal{F}^\bullet)$. First observe that $Rf^*[d](j_{!*}\mathcal{F}^\bullet)$ is an extension of $Rf'^*[d]\mathcal{F}^\bullet$. Moreover, from the characterization of $j'_{!*}(Rf'^*[d]\mathcal{F}^\bullet)$ in Lemma 2.7 the problem is Zariski local on Y . Thus we may assume we have a factorization $Y \xrightarrow{g} \mathbb{A}_X^d \xrightarrow{\pi} X$

where g is étale and π is the projection. It suffices to treat the cases when f is étale and when f is the projection $\mathbb{A}_X^d \rightarrow X$ separately.

For the case when f is étale, let $i: Z \rightarrow X$ be a closed immersion complementary to j (with the reduced induced structure on Z), and consider the Cartesian diagram

$$\begin{array}{ccc} Y & \xleftarrow{i'} & f^{-1}(Z) \\ \downarrow f & & \downarrow g \\ X & \xleftarrow{i} & Z \end{array}$$

Using the fact that Rg^* is t -exact we verify that

$$Ri'^*(Rf^*(j_{!*}\mathcal{F}^\bullet)) = Rg^*(Ri^*(j_{!*}\mathcal{F}^\bullet)) \in {}^pD^{\leq -1}(f^{-1}(Z), \mathbb{F}_p).$$

Since $Rf^* = Rf^!$ for étale morphisms we can similarly verify that

$$Ri^!(Rf^*(j_{!*}\mathcal{F}^\bullet)) \in {}^pD^{\geq 1}(f^{-1}(Z), \mathbb{F}_p).$$

This takes care of the case when f is étale.

For later use we note that if we only assume f is smooth then by Lemma 2.7 (i) and Lemma 2.15,

$$Ri'^*(Rf^*[d](j_{!*}\mathcal{F}^\bullet)) = Rg^*[d](Ri^*(j_{!*}\mathcal{F}^\bullet)) \in {}^pD^{\leq -1}(f^{-1}(Z), \mathbb{F}_p).$$

Since $Rf^*[d](j_{!*}\mathcal{F}^\bullet)$ is an extension of $Rf'^*[d]\mathcal{F}^\bullet$, then by Lemma 2.4 (vii) there is a surjection

$${}^pj'_!(Rf'^*[d]\mathcal{F}^\bullet) \twoheadrightarrow Rf^*[d](j_{!*}\mathcal{F}^\bullet).$$

As in the proof of Lemma 2.7 (i) there is also a map $Rf^*[d](j_{!*}\mathcal{F}^\bullet) \rightarrow {}^pj'_!(Rf'^*[d]\mathcal{F}^\bullet)$ whose composition with the above map gives the natural map ${}^pj'_!(Rf'^*[d]\mathcal{F}^\bullet) \rightarrow {}^pj'_*(Rf'^*[d]\mathcal{F}^\bullet)$.

Thus, only assuming f is smooth we always have a surjection

$$Rf^*[d](j_{!*}\mathcal{F}^\bullet) \rightarrow j'_{!*}(Rf'^*[d]\mathcal{F}^\bullet). \quad (2.1)$$

For the case when f is the projection $\mathbb{A}_X^d \rightarrow X$, first note that we can assume $d = 1$ by factoring f as a composition of projections of relative dimension 1. Let $h: \text{Spec}(A) \rightarrow X$ be an open affine, and consider the Cartesian diagram

$$\begin{array}{ccc} \text{Spec}(A) \times \mathbb{A}^1 & \longrightarrow & X \times \mathbb{A}^1 \\ \downarrow & & \downarrow f \\ \text{Spec}(A) & \xrightarrow{h} & X \end{array}$$

Note that $Rh^*(j_{!*}\mathcal{F}^\bullet)$ is the intermediate extension of $\mathcal{F}^\bullet$ restricted to $\text{Spec}(A) \cap U$ along the map $\text{Spec}(A) \cap U \rightarrow \text{Spec}(A)$. Thus we can further assume $X = \text{Spec}(A)$ is affine.

Let $\alpha: \text{Spec}(A) \rightarrow \text{Spec}(B)$ be a closed immersion into a smooth affine k -scheme $\text{Spec}(B)$, and let $\pi: \text{Spec}(B) \times \mathbb{A}^1 \rightarrow \text{Spec}(B)$ be the projection. Using proper base change one can verify that $Rf^*[1](j_{!*}\mathcal{F}^\bullet) \cong j'_{!*}(Rf'^*[1]\mathcal{F}^\bullet)$ if and only if $R\pi^*[1](R\alpha_*(j_{!*}\mathcal{F}^\bullet))$ is the intermediate extension of $Rf'^*[1]\mathcal{F}^\bullet$ along the composition

$$f^{-1}(U) \rightarrow \text{Spec}(A) \times \mathbb{A}^1 \rightarrow \text{Spec}(B) \times \mathbb{A}^1.$$

Thus we can replace $\text{Spec}(A)$ by $\text{Spec}(B)$ at the cost of j now being an open immersion followed by a closed immersion. However, we can refactor j as a closed immersion followed by an open immersion. We already proved the result for closed immersions, and the desired result holds for a composition of immersions if it holds for each immersion separately. Thus, we have reduced to the following setup: $X = \text{Spec}(A)$ is affine and smooth, $Y = X \times \mathbb{A}^1$, $f: Y \rightarrow X$ is the projection, and $j: U \rightarrow X$ is an open immersion. As mentioned before we

can now appeal to [Sta19] to get an isomorphism

$$Rf^*[1](j_{!*}\mathcal{F}^\bullet) \xrightarrow{\sim} j'_{!*}(Rf'^*[1]\mathcal{F}^\bullet),$$

or alternatively use Lemma 2.17 below. From the considerations in the proof of Lemma 2.7 (i) these isomorphisms can be chosen so as to give an isomorphism of functors. The point is that an isomorphism as above is uniquely determined by its restriction to $f^{-1}(U)$. $\square$

Lemma 2.17. *Let $X = \operatorname{Spec}(A)$ be affine and smooth, $Y = X \times \mathbb{A}^1$, $f: Y \rightarrow X$ the projection, and $j: U \rightarrow X$ an open immersion. Let $\mathcal{F}^\bullet \in P_c^b(U, \mathbb{F}_p)$. Then there is an isomorphism*

$$Rf^*[1](j_{!*}\mathcal{F}^\bullet) \cong j'_{!*}(Rf'^*[1]\mathcal{F}^\bullet).$$

Proof. We will argue using the Riemann-Hilbert correspondence of Emerton-Kisin. The essential ideas are the same as in [Sta19]. First, we note that for a morphism $g: S \rightarrow T$ between smooth k -schemes the usual pullback functor for quasi-coherent sheaves induces a functor

$$\{\text{lfgu } \mathcal{O}_{F,T} \text{ - modules}\} \xrightarrow{g^*} \{\text{lfgu } \mathcal{O}_{F,S} \text{ - modules}\}.$$

Let M be the lfgu $\mathcal{O}_{F,U}$ -module corresponding to $\mathcal{F}^\bullet$. The intermediate extension $j_{!*}\mathcal{F}^\bullet$ corresponds to the smallest lfgu $\mathcal{O}_{F,X}$ submodule N of j_*M such that $j^*N = M$. Denote the intermediate extension by $j_{!+}M$. The complex $Rf^*[1](j_{!*}\mathcal{F}^\bullet)$ corresponds to

$$f^*(j_{!+}M) = j_{!+}M \otimes_A A[x],$$

where $\mathbb{A}^1 = \operatorname{Spec}(k[x])$. Note that $f^*(j_{!+}M)$ is an extension of f'^*M . By (2.1) there is a surjection

$$Rf^*[1](j_{!*}\mathcal{F}^\bullet) \twoheadrightarrow j'_{!*}(Rf'^*[1]\mathcal{F}^\bullet).$$

As the Riemann-Hilbert correspondence is an anti-equivalence, this corresponds to an injection

$$j'_{!+}(f'^*M) \hookrightarrow f^*(j_{!+}M) = j_{!+}M \otimes_A A[x].$$

It therefore suffices to show that if $V \subset j_{!+}M \otimes_A A[x]$ is a sub lfgu $\mathcal{O}_{F,Y}$ -module that is an extension of f'^*M , then $V = j_{!+}M \otimes_A A[x]$.

To prove this, let $i: X \rightarrow X \times \mathbb{A}^1$ be the inclusion corresponding to the map $A[x] \rightarrow A$ which sends x to 0. Then we have a diagram with Cartesian squares:

$$\begin{array}{ccccc} \mathrm{Spec}(A) & \xrightarrow{i} & \mathrm{Spec}(A[x]) & \xrightarrow{f} & \mathrm{Spec}(A) \\ j \uparrow & & \uparrow j' & & \uparrow j \\ U & \xrightarrow{i'} & f^{-1}(U) & \xrightarrow{f'} & U \end{array}$$

Note that $i^*V = V \cap j_{!+}M$ is an lfgu $\mathcal{O}_{F,X}$ -module which is an extension of M . Because $V \cap j_{!+}M \subset j_{!+}M$ and $j_{!+}M$ is the intermediate extension of M , it follows that

$$V \cap j_{!+}M = j_{!+}M.$$

Now because V is an $A[x]$ -module we must have $V = j_{!+}M \otimes_A A[x]$, as desired. $\square$

We conclude this section by showing that one can glue perverse sheaves in the smooth topology.

Lemma 2.18. *Let $\{\varphi_i: Y_i \rightarrow X\}_{i \in I}$ be a family of étale morphisms from k -schemes Y_i such that $X = \bigcup \varphi_i(Y_i)$. Let $\mathcal{F}^\bullet \in D^+(X, \mathbb{F}_p)$ be such that $R\varphi_i^* \mathcal{F}^\bullet \in P^+(Y_i, \mathbb{F}_p)$ for all i . Then $\mathcal{F}^\bullet \in P^+(X, \mathbb{F}_p)$.*

Proof. This follows from the definitions. $\square$

Lemma 2.19. *Let $\{\varphi_i: Y_i \rightarrow X\}_{i \in I}$ be a finite family of étale morphisms from k -schemes Y_i such that $X = \bigcup \varphi_i(Y_i)$. Then the category $P^b(X, \mathbb{F}_p)$ satisfies descent with respect to*

the cover $\{\varphi_i: Y_i \rightarrow X\}$. The same is true for $P_c^b(X, \mathbb{F}_p)$

Proof. We use [BBD82, 3.2.2, 3.2.4] to do the gluing. Note that the condition in [BBD82] on vanishing Ext terms is satisfied by the definition of a t -structure. Lemma 2.18 guarantees that the result of gluing perverse sheaves is a perverse sheaf. $\square$

Lemma 2.20. *Let $\{\varphi_i: Y_i \rightarrow X\}_{i \in I}$ be a finite family of smooth morphisms of relative dimension d_i from k -schemes Y_i such that $X = \bigcup \varphi_i(Y_i)$. Let $\mathcal{F}^\bullet \in D_c^b(X, \mathbb{F}_p)$ be such that $R\varphi_i^*[d_i]\mathcal{F}^\bullet \in P_c^b(Y_i, \mathbb{F}_p)$ for all i . Then $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$.*

Proof. As the question is local on X , it suffices to consider the case where $I = \{*\}$ and φ factors as

$$Y \xrightarrow{g} \mathbb{A}_X^d \xrightarrow{\pi} X,$$

where g is étale, π is the projection, and $\pi \circ g$ is surjective.

First, we claim that $R\pi^*[d]\mathcal{F}^\bullet \in P_c^b(\mathbb{A}_X^d, \mathbb{F}_p)$. By Lemma 2.18, the restriction of $R\pi^*[d]\mathcal{F}^\bullet$ to $g(Y)$ is perverse. Now the claim follows because $\mathbb{A}_X^d$ is covered by the translates of $g(Y)$ under the automorphisms of $\mathbb{A}_X^d$ which preserve the projection $\mathbb{A}_X^d \rightarrow X$.

Thus, we have reduced to the case when φ is the projection $\pi: \mathbb{A}_X^d \rightarrow X$. We may also assume X is affine, so that there is a closed embedding $i: X \rightarrow \mathbb{A}^n$ for some n . As $\mathcal{F}^\bullet$ is perverse if and only if $Ri_*\mathcal{F}^\bullet$ is perverse, then by the proper base change theorem we can reduce to the case when φ is the projection $\mathbb{A}^{n+d} \rightarrow \mathbb{A}^n$ onto the first n coordinates. In particular, we can assume X is smooth. Now the result follows by the Riemann-Hilbert correspondence of Emerton-Kisin, as in the proof of Lemma 2.15. $\square$

Because smooth morphisms admit sections étale locally, then we can now extend our étale descent result (Lemma 2.19) to the smooth setting. Note that we have to restrict to the constructible bounded derived category because this is the only context in which we have proved that pullback along smooth morphisms preserves perversity (Lemma 2.15).

Corollary 2.21. *Let $\{\varphi_i: Y_i \rightarrow X\}_{i \in I}$ be a finite family of smooth morphisms from k -schemes Y_i such that $X = \bigcup \varphi_i(Y_i)$. Then the category $P_c^b(X, \mathbb{F}_p)$ satisfies descent with respect to the cover $\{\varphi_i\}$.*

Remark 2.22. Later we will be concerned with extensions between perverse sheaves. We record here that for $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in P^+(X, \mathbb{F}_p)$ we have

$$\mathrm{Ext}_{P^+(X, \mathbb{F}_p)}^1(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{D(X, \mathbb{F}_p)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[1]).$$

This is a general fact about t -structures. We do not know if a similar statement holds for higher Ext terms.

Chapter 3

Equivariant perverse $\mathbb{F}_p$ -sheaves

3.1 Basic properties of equivariant perverse $\mathbb{F}_p$ -sheaves

In this section we define equivariant perverse sheaves and establish some of their basic properties. We fix a k -scheme S and an S -group scheme G of relative dimension d such that the map $G \rightarrow S$ is smooth, affine, and has geometrically connected fibers. A G -scheme is an S -scheme of finite type equipped with an action of G . If X is a G -scheme, we let $\rho: G \times_S X \rightarrow X$ be the action map, and $\pi: G \times_S X \rightarrow X$ be the projection.

Definition 3.1. A G -equivariant perverse sheaf on X is a perverse sheaf $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ such that there exists an isomorphism $R\rho^*\mathcal{F}^\bullet \cong R\pi^*\mathcal{F}^\bullet$ in $D_c^b(G \times_S X, \mathbb{F}_p)$.

Lemma 3.2. Let $f: Y \rightarrow X$ be a G -equivariant morphism between G -schemes.

- (i) If $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ is G -equivariant and $Rf^*[n]\mathcal{F}^\bullet \in P_c^b(Y, \mathbb{F}_p)$ for some integer n , then $Rf^*[n]\mathcal{F}^\bullet$ is a G -equivariant perverse sheaf.
- (ii) If $\mathcal{F}^\bullet \in P_c^b(Y, \mathbb{F}_p)$ is G -equivariant and $Rf_![n]\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ for some integer n , then $Rf_![n]\mathcal{F}^\bullet$ is a G -equivariant perverse sheaf.

Proof. We have Cartesian diagrams

$$\begin{array}{ccc} G \times_S Y & \xrightarrow{\rho_Y} & Y \\ \text{id}_G \times f \downarrow & & \downarrow f \\ G \times_S X & \xrightarrow{\rho_X} & X \end{array} \qquad \begin{array}{ccc} G \times_S Y & \xrightarrow{\pi_Y} & Y \\ \text{id}_G \times f \downarrow & & \downarrow f \\ G \times_S X & \xrightarrow{\pi_X} & X \end{array}$$

Now (i) follows from a straightforward diagram chase, and (ii) follows similarly after applying the proper base change theorem. $\square$

Lemma 3.3. *Let $j: U \rightarrow X$ be a G -equivariant immersion of G -schemes, and let $\mathcal{F}^\bullet \in P_c^b(U, \mathbb{F}_p)$ be G -equivariant. Then $j_{!*}\mathcal{F}^\bullet$ is G -equivariant.*

Proof. The proof is the same as Lemma 3.2 (ii), except we use that taking intermediate extensions commutes with smooth pullback (Theorem 2.16) instead of the proper base change theorem. $\square$

Lemma 3.4. *Let X be an S -scheme, and let Y be a smooth S -scheme with geometrically connected fibers of relative dimension d . Then if Y admits an S -point and $\pi: Y \times_S X \rightarrow X$ is the projection, the functor*

$$R\pi^*[d]: P_c^b(X, \mathbb{F}_p) \rightarrow P_c^b(Y \times_S X, \mathbb{F}_p)$$

is fully faithful. This functor also preserves simple objects.

Proof. We first show that $R\pi^*[d]$ is fully faithful when restricted to simple objects. Let $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ be simple. Then by Theorem 2.13 there is a smooth irreducible k -scheme U , an immersion $j: U \rightarrow X$, and a simple local system $\mathcal{L}$ of étale $\mathbb{F}_p$ -sheaves on U such that

$\mathcal{F}^\bullet \cong j_{!*}\mathcal{L}[\dim U]$. We have a Cartesian diagram

$$\begin{array}{ccc} Y \times_S U & \xrightarrow{j'} & Y \times_S X \\ \downarrow \pi' & & \downarrow \pi \\ U & \xrightarrow{j} & X \end{array}$$

By Theorem 2.16 we have an isomorphism

$$R\pi^*[d]\mathcal{F}^\bullet \cong j'_*(R\pi'^*\mathcal{L}[\dim U + d]).$$

We claim that $Y \times_S U$ is connected. To prove this, note that each connected component of $Y \times_S U$ surjects onto a dense open subset of U since the map π' is a smooth cover and U is irreducible. Since the fibers of π' are geometrically connected then $Y \times_S U$ is connected.

Let $\bar{y}$ be a geometric point of $Y \times_S U$ and let $\bar{x} = \pi'(\bar{y})$. Suppose $\mathcal{L}$ corresponds to a simple finite-dimensional representation V of $\pi_{1,\text{ét}}(U, \bar{x})$. Because $Y \times_S U$ is connected then the local system $\pi'^*\mathcal{L}$ corresponds to V viewed as a representation of $\pi_{1,\text{ét}}(Y \times_S U, \bar{y})$ via the induced map $\pi_{1,\text{ét}}(Y \times_S U, \bar{y}) \rightarrow \pi_{1,\text{ét}}(U, \bar{x})$. This map is surjective because π' admits a section. By Lemma 2.11,

$$\text{Hom}(R\pi^*[d]\mathcal{F}^\bullet, R\pi^*[d]\mathcal{F}^\bullet) = \text{Hom}(\pi'^*\mathcal{L}, \pi'^*\mathcal{L}) = \text{Hom}_{\pi_{1,\text{ét}}(Y \times_S U, \bar{y})}(V, V).$$

Similarly, $\text{Hom}(\mathcal{F}^\bullet, \mathcal{F}^\bullet) = \text{Hom}_{\pi_{1,\text{ét}}(U, \bar{x})}(V, V)$. Now because $\pi_{1,\text{ét}}(Y \times_S U, \bar{y}) \rightarrow \pi_{1,\text{ét}}(U, \bar{x})$ is surjective it follows that

$$\text{Hom}(R\pi^*[d]\mathcal{F}^\bullet, R\pi^*[d]\mathcal{F}^\bullet) = \text{Hom}(\mathcal{F}^\bullet, \mathcal{F}^\bullet).$$

Moreover, we see that $\pi'^*\mathcal{L}$ is also a simple local system, so $R\pi^*[d]$ preserves simple objects.

Finally, if $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are non-isomorphic simple objects then

$$\mathrm{Hom}(R\pi^*[d]\mathcal{F}^\bullet, R\pi^*[d]\mathcal{G}^\bullet) = \mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = 0.$$

This follows from the fact that both pullbacks are simple, and the map π admits a section.

Thus $R\pi^*[d]$ is fully faithful when restricted to simple objects.

Now let $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ be simple and let $\mathcal{G}^\bullet \in P_c^b(X, \mathbb{F}_p)$ have length $n > 1$. We will show by induction on n that the map

$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \rightarrow \mathrm{Hom}(R\pi^*[d]\mathcal{F}^\bullet, R\pi^*[d]\mathcal{G}^\bullet)$$

is an isomorphism. Let

$$0 \rightarrow \mathcal{G}_1^\bullet \rightarrow \mathcal{G}^\bullet \rightarrow \mathcal{G}_2^\bullet \rightarrow 0$$

be an exact sequence in $P_c^b(X, \mathbb{F}_p)$ where the outer two terms have length $< n$. Then we get a morphism between exact triangles in the derived category of abelian groups:

$$\begin{array}{ccccc} \mathrm{RHom}(\mathcal{F}^\bullet, \mathcal{G}_1^\bullet) & \longrightarrow & \mathrm{RHom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) & \longrightarrow & \mathrm{RHom}(\mathcal{F}^\bullet, \mathcal{G}_2^\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{RHom}(R\pi^*[d]\mathcal{F}^\bullet, R\pi^*[d]\mathcal{G}_1^\bullet) & \longrightarrow & \mathrm{RHom}(R\pi^*[d]\mathcal{F}^\bullet, R\pi^*[d]\mathcal{G}^\bullet) & \longrightarrow & \mathrm{RHom}(R\pi^*[d]\mathcal{F}^\bullet, R\pi^*[d]\mathcal{G}_2^\bullet) \end{array}$$

These complexes are concentrated in non-negative degrees because we are dealing with perverse sheaves. Moreover, the induced map

$$\mathrm{Ext}^1(\mathcal{F}^\bullet, \mathcal{G}_1^\bullet) \rightarrow \mathrm{Ext}^1(R\pi^*[d]\mathcal{F}^\bullet, R\pi^*[d]\mathcal{G}_1^\bullet)$$

is injective because π admits a section. By induction and the five lemma the map

$$\mathrm{Hom}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \xrightarrow{\sim} \mathrm{Hom}(R\pi^*[d]\mathcal{F}^\bullet, R\pi^*[d]\mathcal{G}^\bullet)$$

is an isomorphism. We can complete the proof by a similar induction on the length of $\mathcal{F}^\bullet$. $\square$

Lemma 3.5. *The essential image of the functor in Lemma 3.4 is an abelian subcategory of $P_c^b(Y \times_S X, \mathbb{F}_p)$ which is stable under taking subquotients of objects.*

Proof. Because $R\pi^*[d]$ is exact and preserves simple objects, then it preserves the length of objects. In fact, applying $R\pi^*[d]$ to a composition series for $\mathcal{F}^\bullet$ gives a composition series for $R\pi^*[d]\mathcal{F}^\bullet$. By the Jordan-Hölder theorem, this implies that every simple subquotient of $R\pi^*[d]\mathcal{F}^\bullet$ is in the essential image of $R\pi^*[d]$.

For the general case, suppose $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ has length $n > 1$. Let $\mathcal{G}^\bullet \subseteq R\pi^*[d]\mathcal{F}^\bullet$ be a subobject of length m . We will prove by induction on the integer $c = n - m$ that $\mathcal{G}^\bullet$ is in the essential image of $R\pi^*[d]$. The case $c = 0$ is clear. If $c > 0$, let $\mathcal{F}'^\bullet$ be a subobject of $R\pi^*[d]\mathcal{F}^\bullet$ of length $m + 1$ containing $\mathcal{G}^\bullet$. By the induction hypothesis, $\mathcal{F}'^\bullet$ is in the essential image of $R\pi^*[d]$. We have an exact sequence

$$0 \rightarrow \mathcal{G}^\bullet \rightarrow \mathcal{F}'^\bullet \rightarrow \mathcal{H}^\bullet \rightarrow 0$$

where $\mathcal{H}^\bullet$ is simple. We have already shown that $\mathcal{H}^\bullet$ is in the essential image of $R\pi^*[d]$. Because $R\pi^*[d]$ is exact and fully faithful, it follows that $\mathcal{G}^\bullet$ is in the essential image of $R\pi^*[d]$. This completes the induction. Now that we know the essential image of $R\pi^*[d]$ is stable under taking subobjects, the fact that it is stable under taking subquotients also follows from the fact that $R\pi^*[d]$ is exact and fully faithful. $\square$

Lemma 3.6. *Let G act on $G \times_S X$ by left translation on the first coordinate, and let $\pi: G \times_S X \rightarrow X$ be the projection. If G has relative dimension d over S then the functor $R\pi^*[d]$ induces an equivalence of categories between $P_c^b(X, \mathbb{F}_p)$ and the full subcategory of $P_c^b(G \times_S X, \mathbb{F}_p)$ consisting of G -equivariant perverse sheaves. If $i: X \rightarrow G \times_S X$ is the inclusion $x \mapsto (1, x)$, then $Ri^*[-d]$ is an inverse to $R\pi^*[d]$.*

Proof. The same proof as in [Let05, 4.2.3] works, except we replace the reference to [BBD82, 4.2.5] by Lemma 3.4. $\square$

Lemma 3.7. *Let $\mathcal{F}^\bullet$ be a G -equivariant perverse sheaf on X . Denote by $m: G \times_S G \rightarrow G$ the multiplication map, $p_2: G \times_S G \rightarrow G$ the projection onto the second factor, and $i: X \rightarrow G \times_S X$ the inclusion $x \mapsto (1, x)$. Then there is a unique isomorphism $\varphi: R\pi^*\mathcal{F}^\bullet \xrightarrow{\sim} R\rho^*\mathcal{F}^\bullet$ such that $Ri^*(\varphi): \mathcal{F}^\bullet \xrightarrow{\sim} \mathcal{F}^\bullet$ is the identity. This isomorphism also satisfies*

$$R(m \times \text{id}_X)^*\varphi = R(\text{id}_G \times \rho)^*(\varphi) \circ R(p_2 \times \text{id}_X)^*(\varphi).$$

Proof. The same proof as in [Let05, 4.2.4] works. $\square$

Definition 3.8. We define the category $P_G(X, \mathbb{F}_p)$ of G -equivariant perverse sheaves as follows. Its objects are G -equivariant perverse sheaves in $P_c^b(X, \mathbb{F}_p)$. To define morphisms, let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be G -equivariant perverse sheaves on X . Let $\varphi: R\pi^*\mathcal{F}^\bullet \rightarrow R\rho^*\mathcal{F}^\bullet$ and $\psi: R\pi^*\mathcal{G}^\bullet \rightarrow R\rho^*\mathcal{G}^\bullet$ be the two unique isomorphisms satisfying the cocycle conditions as in Lemma 3.7. Then morphisms $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ in $P_G(X, \mathbb{F}_p)$ are morphisms $\tau: \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ between perverse sheaves such that the following diagram commutes:

$$\begin{array}{ccc} R\pi^*\mathcal{F}^\bullet & \xrightarrow{R\pi^*(\tau)} & R\pi^*\mathcal{G}^\bullet \\ \downarrow \varphi & & \downarrow \psi \\ R\rho^*\mathcal{F}^\bullet & \xrightarrow{R\rho^*(\tau)} & R\rho^*\mathcal{G}^\bullet \end{array}$$

Proposition 3.9. *The category $P_G(X, \mathbb{F}_p)$ is a full subcategory of $P_c^b(X, \mathbb{F}_p)$.*

Proof. We give the same proof as in [Let05, 4.2.7]. In the notation of Definition 3.8, we need to show that if $\tau: \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ is a morphism between perverse sheaves then

$$R\rho^*(\tau) \circ \varphi = \psi \circ R\pi^*(\tau).$$

Equality certainly holds after applying Ri^* . Thus, by appealing to Lemma 3.6 it suffices to show that each of $R\pi^*[d]\mathcal{F}^\bullet$, $R\pi^*[d]\mathcal{G}^\bullet$, $R\rho^*[d]\mathcal{F}^\bullet$, $R\rho^*[d]\mathcal{G}^\bullet$ is G -equivariant on $G \times X$, where G only acts by left translation on the first factor. The map ρ is G -equivariant, and the map π is G -equivariant when we give X the trivial action of G . Thus the equivariance of these perverse sheaves follows from Lemma 3.2. $\square$

Proposition 3.10. *If $\mathcal{F}^\bullet \in P_G(X, \mathbb{F}_p)$ then any subquotient of $\mathcal{F}^\bullet$ is G -equivariant.*

Proof. We give the same proof as in [Let05, 4.2.13]. If $\mathcal{G}^\bullet$ is a subquotient of $\mathcal{F}^\bullet$, then $R\rho^*[d]\mathcal{G}^\bullet$ is a subquotient of $R\rho^*[d]\mathcal{F}^\bullet$. Because $\mathcal{F}^\bullet$ is G -equivariant, $R\rho^*[d]\mathcal{G}^\bullet$ is a subquotient of $R\pi^*[d]\mathcal{F}^\bullet$. Hence by Lemma 3.5 there is some $\mathcal{H}^\bullet \in P_c^b(X, \mathbb{F}_p)$ such that

$$R\pi^*[d]\mathcal{H}^\bullet \cong R\rho^*[d]\mathcal{G}^\bullet.$$

Apply the functor $Ri^*[-d]$ to this isomorphism, where $i: X \rightarrow G \times_S X$ is the inclusion $x \mapsto (1, x)$. This gives $\mathcal{H}^\bullet \cong \mathcal{G}^\bullet$. Now by the above isomorphism it follows that $\mathcal{G}^\bullet$ is G -equivariant. $\square$

3.2 Equivariant extensions

Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in P_G(X, \mathbb{F}_p)$. Then an extension of $\mathcal{F}^\bullet$ by $\mathcal{G}^\bullet$ is not necessarily G -equivariant. Now let R be an affine S -scheme, and let $f_R: X_R \rightarrow X$ be the base change of $R \rightarrow S$ by $X \rightarrow S$. Then the group $G(R)$ naturally acts on $\text{Ext}_{D_c^b(X_R, \mathbb{F}_p)}^1(Rf_R^*\mathcal{F}^\bullet, Rf_R^*\mathcal{G}^\bullet)$. The next proposition gives a criterion for determining when an extension is equivariant.

Proposition 3.11. *Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in P_G(X, \mathbb{F}_p)$. Then an element $s \in \text{Ext}_{D_c^b(X, \mathbb{F}_p)}^1(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ represents a G -equivariant extension if and only if the element*

$$f_R^*(s) \in \text{Ext}_{D_c^b(X_R, \mathbb{F}_p)}^1(Rf_R^*\mathcal{F}^\bullet, Rf_R^*\mathcal{G}^\bullet)$$

is fixed by $G(R)$ for every affine S -scheme R .

Proof. Suppose s represents the extension

$$0 \rightarrow \mathcal{G}^\bullet \rightarrow \mathcal{H}^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow 0.$$

If $\mathcal{H}^\bullet$ is G -equivariant then it follows from Proposition 3.9 that $f_R^*(s)$ is fixed by $G(R)$ for every R . To prove the converse, consider the case $R = G$ and the element $\varphi \in G(R)$ corresponding to the identity map of G . The fact that φ fixes s implies that we have an isomorphism $R\pi^*\mathcal{H}^\bullet \cong R\rho^*\mathcal{H}^\bullet$. Thus $\mathcal{H}^\bullet$ is G -equivariant. $\square$

3.3 Change of group

Lemma 3.12. *Suppose X has the structure of a G -scheme and an H -scheme, and suppose the action of G on X factors through a morphism $f: G \rightarrow H$. Then there is a natural functor $P_H(X, \mathbb{F}_p) \rightarrow P_G(X, \mathbb{F}_p)$.*

Proof. In view of Proposition 3.9, we only need to show that if $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ is H -equivariant then it is also G -equivariant. We have commutative diagrams

$$\begin{array}{ccc} G \times_S X & & G \times_S X \\ \downarrow f \times \text{id}_X & \searrow \rho_G & \searrow \pi_G \\ H \times_S X & \nearrow \rho_H & \nearrow \pi_H \\ & X & X \end{array}$$

Thus from an isomorphism $R\rho_H^*\mathcal{F}^\bullet \cong R\pi_H^*\mathcal{F}^\bullet$ we get an isomorphism $R\rho_G^*\mathcal{F}^\bullet \cong R\pi_G^*\mathcal{F}^\bullet$. $\square$

Remark 3.13. Consider the following setup (see also [Ric14]). Let G_i be an inverse system of smooth affine S -schemes with geometrically connected fibers, and let $G = \varprojlim G_i$. Let

$X = \varinjlim X_i$ be an ind-scheme, where each X_i is an S -scheme of finite type and the transition maps are closed immersions of S -schemes. Suppose that G acts on each X_i and that furthermore the action of G on each X_i factors through the quotient G_i . We also assume that for $j \geq i$ the immersion $X_i \rightarrow X_j$ is G -equivariant. Define

$$P_c^b(X, \mathbb{F}_p) := \varinjlim P_c^b(X_i, \mathbb{F}_p).$$

Note that if $j \geq i$ then by Lemma 3.2 and Lemma 3.12 there is a natural fully faithful functor

$$P_{G_i}(X_i, \mathbb{F}_p) \rightarrow P_{G_j}(X_j, \mathbb{F}_p).$$

Thus we can define

$$P_G(X, \mathbb{F}_p) := \varinjlim P_{G_i}(X_i, \mathbb{F}_p).$$

Remark 3.14. In our definition of $P_G(X, \mathbb{F}_p)$ in Remark 3.13 we note that it is possible that there exists an object in $P_G(X, \mathbb{F}_p)$ supported on X_i but that is not in the essential image of $P_{G_i}(X_i, \mathbb{F}_p) \rightarrow P_G(X, \mathbb{F}_p)$. This is because there could be a perverse sheaf $\mathcal{F}^\bullet \in P_c^b(X_i, \mathbb{F}_p)$ which is G_j -equivariant but not G_i -equivariant for some $j > i$. Thus, the subcategory of $P_G(X, \mathbb{F}_p)$ consisting of perverse sheaves supported on X_i should be thought of as those perverse sheaves in $P_c^b(X_i, \mathbb{F}_p)$ which are G_j -equivariant for large enough $j \geq i$.

Proposition 3.15. *Suppose in the notation of Lemma 3.12 that f is surjective and the kernel $N := \ker f$ is a smooth, affine S -scheme with geometrically connected fibers. Then the morphism $P_H(X, \mathbb{F}_p) \rightarrow P_G(X, \mathbb{F}_p)$ is an isomorphism.*

Proof. Let $\mathcal{F}^\bullet \in P_G(X, \mathbb{F}_p)$, and

$$\alpha := f \times_S \text{id}_X : G \times_S X \rightarrow H \times_S X.$$

Note that f is smooth because N is smooth and f is an N -torsor. Then α is a smooth cover

of $H \times_S X$. As in the proof of Lemma 3.12 we have $R\alpha^* R\rho_H^* = R\rho_G^*$ and $R\alpha^* R\pi_H^* = R\pi_G^*$. We want to descend the isomorphism $\beta: R\rho_G^* \mathcal{F}^\bullet \cong R\pi_G^* \mathcal{F}^\bullet$ along α to an isomorphism $R\rho_H^* \mathcal{F}^\bullet \cong R\pi_H^* \mathcal{F}^\bullet$. Consider the Cartesian diagram

$$\begin{array}{ccc} G \times_S X & \xleftarrow{p} & N \times_S G \times_S X \\ \downarrow \alpha & & \downarrow m \\ H \times_S X & \xleftarrow{\alpha} & G \times_S X \end{array}$$

Here p is the projection onto $G \times_S X$ and $m(n, g, x) = (ng, x)$. By smooth descent (Corollary 2.21) we only need to check that the following diagram commutes

$$\begin{array}{ccc} Rp^* R\alpha^* R\rho_H^* \mathcal{F}^\bullet & \xrightarrow{Rp^* \beta} & Rp^* R\alpha^* R\pi_H^* \mathcal{F}^\bullet \\ \downarrow \gamma & & \downarrow \delta \\ Rm^* R\alpha^* R\rho_H^* \mathcal{F}^\bullet & \xrightarrow{Rm^* \beta} & Rm^* R\alpha^* R\pi_H^* \mathcal{F}^\bullet \end{array}$$

Here the vertical maps are the natural isomorphisms coming from commutativity of the previous diagram. Let N act on $G \times_S X$ by $n \cdot (g, x) = (ng, nx) = (ng, x)$. The projection $\pi_G: G \times_S X \rightarrow X$ is N -equivariant, so $R\alpha^* R\pi_H^* \mathcal{F}^\bullet = R\pi_G^* \mathcal{F}^\bullet$ is an N -equivariant perverse sheaf on $G \times_S X$ (up to a shift). By Lemma 3.7, the identity

$$\delta = (Rm^* \beta) \circ \gamma \circ (Rp^* \beta)^{-1}$$

holds because after applying Ri^* , both morphisms reduce to the identity map of $R\pi_G^* \mathcal{F}^\bullet$. $\square$

3.4 Classification of simple equivariant perverse $\mathbb{F}_p$ -sheaves

For the rest of this section we assume $S = k$.

Lemma 3.16. *Suppose $S = k$ and that $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ is simple and G -equivariant. Then there exists a G -equivariant immersion $j: U \rightarrow X$ from a smooth irreducible G -scheme U*

and a simple G -equivariant local system $\mathcal{L}$ on U such that $\mathcal{F}^\bullet \cong j_{!*}(\mathcal{L}[\dim U])$.

Proof. As our classification of the simple objects in $P_c^b(X, \mathbb{F}_p)$ in Theorem 2.13 is analogous to the case of $\overline{\mathbb{Q}}_\ell$ -coefficients, the same proof as in [Jan04, 12.20] works, except we replace the reference to 12.14(5) by Lemma 2.10. $\square$

Lemma 3.17. *Suppose $S = k$ and that G acts on X with finitely many orbits. Then if $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ is simple and G -equivariant, there is a unique orbit $j: \mathcal{O} \rightarrow X$ and a unique simple G -equivariant local system $\mathcal{L}$ on $\mathcal{O}$ such that $\mathcal{F}^\bullet \cong j_{!*}(\mathcal{L}[\dim \mathcal{O}])$. Conversely, every such pair $(\mathcal{O}, \mathcal{L})$ determines a unique simple G -equivariant perverse sheaf.*

Proof. Let $\mathcal{F}^\bullet \in P_c^b(X, \mathbb{F}_p)$ be simple and G -equivariant. Then in the notation of Lemma 3.16, since G acts with finitely many orbits there is an open dense orbit $\mathcal{O}$ in U . By Lemma 2.10 we may replace U by $\mathcal{O}$ and $\mathcal{L}$ by its restriction to $\mathcal{O}$. For the uniqueness of $\mathcal{O}$, note that $\mathcal{O}$ is uniquely determined as the orbit which is open in the support of $\mathcal{F}^\bullet$. Then $\mathcal{L}$ is determined by the restriction of $\mathcal{F}^\bullet$ to $\mathcal{O}$. $\square$

Lemma 3.18. *Suppose $S = k$ and that G acts transitively on X with connected stabilizers. Then the only G -equivariant local systems $\mathcal{L}$ on X are constant.*

Proof. We follow the proof in [Jan04, 12.7]. Let $x \in X(k)$ and let $H \subset G$ be the closed reduced subgroup such that $H(k)$ is the stabilizer of x . Then the map $G \rightarrow X$, $g \mapsto gx$ factors as a composition

$$G \xrightarrow{p_1} G/H \xrightarrow{p_2} X$$

where p_1 is an H -torsor and p_2 is a purely inseparable bijective morphism. Each of these maps is G -equivariant, and $p_2^*\mathcal{L}$ and $p_1^*p_2^*\mathcal{L}$ are G -equivariant. By [SGA 1, IX 4.10] the map p_2 induces an isomorphism of fundamental groups, so $\mathcal{L}$ is constant if and only if $p_2^*\mathcal{L}$ is constant. Because G/H is smooth then $p_2^*\mathcal{L}$ and constant sheaves on G/H are perverse (when shifted by $\dim G/H$). We claim that $p_1^*p_2^*\mathcal{L}$ is constant. Assuming this claim for a

moment, let $\mathcal{F}$ be the constant sheaf on G associated to the stalk of $p_1^*p_2^*\mathcal{L}$ at the identity element. Then we have a canonical isomorphism $\mathcal{F} \cong p_1^*p_2^*\mathcal{L}$. Since H is connected one can descend this isomorphism to G/H using the same ideas as in the proof of Proposition 3.15, so $p_2^*\mathcal{L}$ is constant. Thus $\mathcal{L}$ is also constant.

To prove the claim that $p_1^*p_2^*\mathcal{L}$ is constant, we will in fact show that the only G -equivariant local systems $\mathcal{L}$ on G for the usual left action of G are constant sheaves. Consider the map $j: G \rightarrow G \times G$ given by $g \mapsto (g^{-1}, g)$. Then $\rho \circ j$ is the constant map $G \rightarrow G$, $g \mapsto 1$, so $j^*\rho^*\mathcal{L}$ is a constant sheaf. The map $\pi \circ j$ is the identity map id_G . Thus, applying j^* to an isomorphism $\rho^*\mathcal{L} \cong \pi^*\mathcal{L}$ gives that $\mathcal{L}$ is constant. $\square$

Corollary 3.19. *Suppose $S = k$ and that G acts on X with finitely many orbits, each having connected stabilizers. Then the simple G -equivariant perverse sheaves in $P_c^b(X, \mathbb{F}_p)$ are the intermediate extensions of the constant sheaves $\mathbb{F}_p[\dim \mathcal{O}]$ along the orbits $\mathcal{O}$.*

Chapter 4

Perverse $\mathbb{F}_p$ -sheaves on Cohen-Macaulay and F -rational schemes

4.1 F -rational rings

In this chapter we prove Theorems 1.11 and 1.12. To begin, we will take the following theorem of Smith as our definition of F -rationality.

Theorem 4.1. *[Smi97, 2.6] Let $(A, \mathfrak{m})$ be an excellent Cohen-Macaulay local ring of dimension d and characteristic $p > 0$. Then A is F -rational if and only if the local cohomology module $H_{\mathfrak{m}}^d(A)$ is a simple left $A[F]$ -module.*

The original definition of F -rationality is related to tight closure theory (see [HH94]). A k -scheme is F -rational if all of its local rings are F -rational. The idea of the proofs will be to show all of the hypotheses are preserved under passage to strict henselizations. We will then use the Artin-Schreier sequence to show that the necessary cohomology sheaves vanish.

Lemma 4.2. *Let $(A, \mathfrak{m})$ be an excellent local ring of characteristic $p > 0$, and let A^{sh} be a strict henselization of A . Then A^{sh} is an excellent local ring of characteristic $p > 0$. Moreover, if A has one of the following four properties then A^{sh} has the same property:*

- (i) *A is Cohen-Macaulay,*
- (ii) *A is normal,*
- (iii) *$\dim(A) = d$,*
- (iv) *A is F -rational.*

Proof. The ring A^{sh} is noetherian by [EGA IV₄, 18.8.8]. The excellence of A^{sh} is not shown in *loc. cit.*, but this can be found, for example, in [FK88, Ch. 1, §1]. The Cohen-Macaulay property and normality are preserved by [EGA IV₄, 18.8.13]. The dimension is preserved by [Sta21, 06LK].

To see that A^{sh} is F -rational if A is F -rational, we will use the construction of A^{sh} as in [Sta21, 04GP]. Fix a separable closure $k(\mathfrak{m})^{\text{sep}}$ of the residue field $k(\mathfrak{m})$ and consider the collection of triples $(S, \mathfrak{q}, \alpha)$ where $A \rightarrow S$ is étale, $\mathfrak{q}$ is the only prime of S lying over $\mathfrak{m}$, and $\alpha: k(\mathfrak{q}) \rightarrow k(\mathfrak{m})^{\text{sep}}$ is an embedding of $k(\mathfrak{m})$ -algebras. Note that $\mathfrak{q} = \mathfrak{m}S$ (see [Sta21, 04GN]).

Then A^{sh} is identified with the filtered colimit $\text{colim}_{(S, \mathfrak{q}, \alpha)} S$. Note that $A \rightarrow A^{\text{sh}}$ is flat and $\mathfrak{m}A^{\text{sh}}$ is the maximal ideal of A^{sh} . Then

$$H_{\mathfrak{m}A^{\text{sh}}}^d(A^{\text{sh}}) = H_{\mathfrak{m}}^d(A) \otimes_A A^{\text{sh}}.$$

As tensor products commute with colimits,

$$H_{\mathfrak{m}}^d(A) \otimes_A A^{\text{sh}} = \text{colim}_{(S, \mathfrak{q}, \alpha)} H_{\mathfrak{m}}^d(A) \otimes_A S = \text{colim}_{(S, \mathfrak{q}, \alpha)} H_{\mathfrak{q}}^d(S).$$

For each $(S, \mathfrak{q}, \alpha)$ the map $S \rightarrow A^{\text{sh}}$ factors through the localization map $S \rightarrow S_{\mathfrak{q}}$. Hence

the map $H_q^d(S) \rightarrow H_{\mathfrak{m}A^{\text{sh}}}^d(A^{\text{sh}})$ factors through the map $H_q^d(S) \rightarrow H_{qS_q}^d(S_q)$. By [Vél95, Thm. 3.1], the ring S is F -rational. Now we would like to apply [HH94, Thm. 4.2] to conclude S_q is also F -rational, but to do this we need to know that S is a homomorphic image of a Cohen-Macaulay ring. Since A is local and F -rational then it is already Cohen-Macaulay by [HH94, Thm. 6.27]. Thus S is also Cohen-Macaulay, so S_q is F -rational and $H_{qS_q}^d(S_q)$ is a simple $S_q[F]$ -module.

At this point we have shown $H_{\mathfrak{m}A^{\text{sh}}}^d(A^{\text{sh}}) = \text{colim}_{(S,q,\alpha)} H_q^d(S)$ where the image of each $H_q^d(S)$ generates a simple $S_q[F]$ -submodule of $H_{\mathfrak{m}A^{\text{sh}}}^d(A^{\text{sh}})$. It follows that $H_{\mathfrak{m}A^{\text{sh}}}^d(A^{\text{sh}})$ is a simple $A^{\text{sh}}[F]$ -module, so A^{sh} is F -rational. $\square$

4.2 Perverse $\mathbb{F}_p$ -sheaves on Cohen-Macaulay schemes

The following proposition is the main computation from which we will derive the necessary cohomological vanishing results.

Proposition 4.3.

- (i) *Let $(A, \mathfrak{m})$ be a Cohen-Macaulay, noetherian local ring of characteristic $p > 0$ and dimension d . Let $i: k(\mathfrak{m}) \rightarrow \text{Spec}(A)$ be the inclusion of the closed point and let $j: U = \text{Spec}(A) - k(\mathfrak{m}) \rightarrow \text{Spec}(A)$ be the inclusion of the complement. Then*

$$R^m \Gamma(Ri^! \mathbb{F}_p[0]) = 0, \quad m < d.$$

- (ii) *If $\text{Spec}(A)$ is furthermore irreducible and $d > 0$ then*

$$R^d \Gamma(Ri^! \mathbb{F}_p[0]) = \begin{cases} \ker(H_{\text{ét}}^1(\text{Spec}(A), \mathbb{F}_p) \rightarrow H_{\text{ét}}^1(U, \mathbb{F}_p)) & d = 1 \\ \ker(H_{\mathfrak{m}}^d(A) \xrightarrow{F-\text{id}} H_{\mathfrak{m}}^d(A)) & d > 1. \end{cases}$$

Proof. If $d = 0$ the result is clear, so we now assume $d \geq 1$. Note that

$$R^m \Gamma(Ri^! \mathbb{F}_p[0]) = R^m \Gamma(Ri_* Ri^! \mathbb{F}_p[0]).$$

We can compute the latter group by applying $R\Gamma$ to the exact triangle

$$Ri_* Ri^! \mathbb{F}_p[0] \rightarrow \mathbb{F}_p[0] \rightarrow Rj_* \mathbb{F}_p[0] \quad (4.1)$$

in $D_c^b(\mathrm{Spec}(A), \mathbb{F}_p)$. From the Artin–Schreier sequence

$$0 \rightarrow \mathbb{F}_p \rightarrow \mathcal{O}_{\mathrm{Spec}(A)} \xrightarrow{F-\mathrm{id}} \mathcal{O}_{\mathrm{Spec}(A)} \rightarrow 0$$

and the fact that $\mathrm{Spec}(A)$ is affine, we deduce that

$$R^m \Gamma(\mathbb{F}_p[0]) = \begin{cases} \ker(A \xrightarrow{F-\mathrm{id}} A) = \mathbb{F}_p & m = 0 \\ \mathrm{cok}(A \xrightarrow{F-\mathrm{id}} A) & m = 1 \\ 0 & m > 1. \end{cases}$$

There is an exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(A) \rightarrow A \rightarrow H^0(U, \mathcal{O}_U) \rightarrow H_{\mathfrak{m}}^1(A) \rightarrow 0.$$

Moreover, for $m \geq 1$ there are isomorphisms

$$H^m(U, \mathcal{O}_U) \cong H_{\mathfrak{m}}^{m+1}(A).$$

First suppose $d = 1$. The map $R^0 \Gamma(\mathbb{F}_p[0]) \rightarrow R^0 \Gamma(Rj_* \mathbb{F}_p[0])$ is the natural map

$$H_{\acute{e}t}^0(\mathrm{Spec}(A), \mathbb{F}_p) \rightarrow H_{\acute{e}t}^0(U, \mathbb{F}_p).$$

It is possible that U is disconnected, but in any case this map is always injective. By appealing to (4.1), this observation proves (i) in the case $d = 1$. For (ii) we note that if $\text{Spec}(A)$ is irreducible then $R^0\Gamma(\mathbb{F}_p[0]) \rightarrow R^0\Gamma(Rj_*\mathbb{F}_p[0])$ is an isomorphism.

Now assume $d \geq 2$. We will denote the derived global sections functor on U by $R\Gamma_U$. We have $R\Gamma_U = R\Gamma \circ Rj_*$, and

$$R^m\Gamma(Rj_*\mathbb{F}_p[0]) = R^m\Gamma_U(\mathbb{F}_p[0]).$$

Because A is Cohen-Macaulay,

$$H_{\mathfrak{m}}^m(A) = 0, \quad m < d.$$

As $d \geq 2$ it follows that $H^0(U, \mathcal{O}_U) = A$ and $H^m(U, \mathcal{O}_U) = 0$ for $0 < m < d - 1$. Thus $H_{\text{ét}}^0(\text{Spec}(A), \mathbb{F}_p) \rightarrow H_{\text{ét}}^0(U, \mathbb{F}_p)$ is an isomorphism. Additionally,

$$R^m\Gamma_U(\mathbb{F}_p[0]) = 0, \quad 1 < m < d - 1, \quad (4.2)$$

and if $d > 2$,

$$R^{d-1}\Gamma_U(\mathbb{F}_p[0]) = \ker(H^{d-1}(U, \mathcal{O}_U) \xrightarrow{F-\text{id}} H^{d-1}(U, \mathcal{O}_U)). \quad (4.3)$$

Suppose $d = 2$. From the Artin–Schreier sequences on U and $\text{Spec}(A)$ we get a diagram of long exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{\text{ét}}^0(U, \mathbb{F}_p) & \longrightarrow & A & \xrightarrow{F-\text{id}} & A & \longrightarrow & H_{\text{ét}}^1(U, \mathbb{F}_p) & \longrightarrow & H^1(U, \mathcal{O}_U) & \xrightarrow{F-\text{id}} \\ & & \uparrow f & & \uparrow \sim & & \uparrow \sim & & \uparrow g & & \uparrow & \\ 0 & \longrightarrow & H_{\text{ét}}^0(\text{Spec}(A), \mathbb{F}_p) & \longrightarrow & A & \xrightarrow{F-\text{id}} & A & \longrightarrow & H_{\text{ét}}^1(\text{Spec}(A), \mathbb{F}_p) & \longrightarrow & 0 & \longrightarrow \end{array}$$

From this we deduce that the map $R^m\Gamma(\mathbb{F}_p[0]) \rightarrow R^m\Gamma_U(\mathbb{F}_p[0])$ is an isomorphism for $m = 0$ and an injection for $m = 1$ (these correspond to the maps f and g). This implies

$R^m\Gamma(Ri^!\mathbb{F}_p[0]) = 0$ for $m < 2$. Now for (ii) we have an exact sequence

$$0 \rightarrow H_{\text{ét}}^1(\text{Spec}(A), \mathbb{F}_p) \rightarrow H_{\text{ét}}^1(U, \mathbb{F}_p) \rightarrow R^2\Gamma(Ri^!\mathbb{F}_p[0]) \rightarrow 0.$$

Hence by considering the previous diagram and using that $H^1(U, \mathcal{O}_U) \cong H_{\mathfrak{m}}^2(A)$ we get $R^2\Gamma(Ri^!\mathbb{F}_p[0]) = \ker(H_{\mathfrak{m}}^2(A) \xrightarrow{F-\text{id}} H_{\mathfrak{m}}^2(A))$.

Finally, suppose $d > 2$. As $H^1(U, \mathcal{O}_U) = 0$ the maps $R^m\Gamma(\mathbb{F}_p[0]) \rightarrow R^m\Gamma_U(\mathbb{F}_p[0])$ are isomorphisms for $m = 0$ and $m = 1$. Thus $R^m\Gamma(Ri^!\mathbb{F}_p[0]) = 0$ for $m \leq 2$, and $R^m\Gamma(Ri^!\mathbb{F}_p[0]) = R^{m-1}\Gamma_U(\mathbb{F}_p[0])$ for $m > 2$. Now (i) follows from (4.2) and (ii) follows from (4.3). $\square$

Proof of Theorem 1.11. We will check the criteria for perversity as in Remark 2.1. Fix a point $x \in X$ of codimension c , so $\dim\{\bar{x}\} = d - c$. Let $i: \bar{x} \rightarrow \text{Spec}(\mathcal{O}_{X,x}^{\text{sh}})$ be the inclusion of the closed point in a strict henselization of the local ring of X at x . Clearly $H^i(Ri^*\mathbb{F}_p[d]) = 0$ for $i > -d + c$. By Lemma 4.2, $\mathcal{O}_{X,x}^{\text{sh}}$ is a Cohen-Macaulay noetherian local ring of dimension c . Moreover, $H^i(Ri^!\mathbb{F}_p[d]) = R^i\Gamma(Ri^!\mathbb{F}_p[d])$ for all i because $\mathcal{O}_{X,x}^{\text{sh}}$ is strictly henselian. Now it follows from Proposition 4.3 that $H^i(Ri^!\mathbb{F}_p[d]) = 0$ for all $i < -d + c$. $\square$

Remark 4.4. The constant sheaf $\mathbb{F}_p[\dim X]$ is not always perverse for an arbitrary k -scheme X . For example, let X be a union of two planes meeting only at a single point p . Then if $i: \text{Spec}(k) \rightarrow X$ is the inclusion of p , we claim that $H^{-1}(Ri^!\mathbb{F}_p[2]) \neq 0$. To prove this, let $j: U \rightarrow X$ be the inclusion of the complement of p . Then by using the exact triangle (4.1), there is an embedding of $H_{\text{ét}}^0(U, \mathbb{F}_p)/H_{\text{ét}}^0(X, \mathbb{F}_p)$ into $H^{-1}(Ri^!\mathbb{F}_p[2])$. As U is disconnected, this proves the claim.

4.3 Perverse $\mathbb{F}_p$ -sheaves on F -rational schemes

The remainder of this section is devoted to the proof of Theorem 1.12. We will use the following two lemmas to show the vanishing of the cohomology groups in part (ii) of Proposition 4.3.

Lemma 4.5. *Let $(A, \mathfrak{m})$ be a normal local domain of characteristic $p > 0$, and let U be the complement of the closed point. Then*

$$\ker(H_{\text{ét}}^1(\text{Spec}(A), \mathbb{F}_p) \rightarrow H_{\text{ét}}^1(U, \mathbb{F}_p)) = 0.$$

Proof. From the Artin-Schreier sequences on $\text{Spec}(A)$ and U we get a diagram of long exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{ét}}^0(U, \mathbb{F}_p) & \longrightarrow & H^0(U, \mathcal{O}_U) & \xrightarrow{F-\text{id}} & H^0(U, \mathcal{O}_U) \longrightarrow H_{\text{ét}}^1(U, \mathbb{F}_p) \longrightarrow \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & H_{\text{ét}}^0(\text{Spec}(A), \mathbb{F}_p) & \longrightarrow & A & \xrightarrow{F-\text{id}} & A \longrightarrow H_{\text{ét}}^1(\text{Spec}(A), \mathbb{F}_p) \xrightarrow{0} \end{array}$$

If an element in the kernel is represented by $a \in A$ then there exists $x \in H^0(U, \mathcal{O}_U)$ such that $a = x^p - x$. As A is normal then $x \in A$, so the image of a in $H_{\text{ét}}^1(\text{Spec}(A), \mathbb{F}_p)$ is already 0. $\square$

Lemma 4.6. *Let $(A, \mathfrak{m})$ be an excellent F -rational local ring of characteristic $p > 0$ and dimension $d > 0$. Then $\ker(H_{\mathfrak{m}}^d(A) \xrightarrow{F-\text{id}} H_{\mathfrak{m}}^d(A)) = 0$.*

Proof. An element in the kernel generates an $A[F]$ -submodule of $H_{\mathfrak{m}}^d(A)$ which is finitely generated as an A -module. As $H_{\mathfrak{m}}^d(A)$ is a simple $A[F]$ -module and $H_{\mathfrak{m}}^d(A)$ is never finitely generated as an A -module [BS13b, 7.3.3], then the kernel is zero. $\square$

Proof of Theorem 1.12. By [HH94, 4.2], X is normal and Cohen-Macaulay, so $\mathbb{F}_p[d]$ is perverse by Theorem 1.11. By Theorem 2.13 it suffices to show that $\mathbb{F}_p[d]$ is the intermediate

extension of its own restriction to the smooth locus. Let $x \in X$ be a point of codimension c contained in the singular locus. Let $i: \bar{x} \rightarrow \operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}})$ be the inclusion of the closed point in a strict henselization of the local ring of X at x . By appealing to Lemma 2.7, we need to check that $H^{-d+c}(Ri^*\mathbb{F}_p[d]) = 0$ and $H^{-d+c}(Ri^!\mathbb{F}_p[d]) = 0$. We have $H^{-d+c}(Ri^*\mathbb{F}_p[d]) = 0$ because $c > 0$. Since $\mathcal{O}_{X,x}^{\text{sh}}$ is strictly henselian, $H^{-d+c}(Ri^!\mathbb{F}_p[d]) = R^{-d+c}\Gamma(Ri^!\mathbb{F}_p[d])$. By Lemma 4.2, $\mathcal{O}_{X,x}^{\text{sh}}$ is Cohen-Macaulay, normal, F -rational and has dimension c . Moreover, $\operatorname{Spec}(\mathcal{O}_{X,x}^{\text{sh}})$ is irreducible because X is normal, and in particular, geometrically unibranch. The ring $\mathcal{O}_{X,x}^{\text{sh}}$ is also reduced by [EGA IV₄, 18.8.12], so $\mathcal{O}_{X,x}^{\text{sh}}$ is a domain. Thus if $c = 1$ then $H^{-d+1}(Ri^!\mathbb{F}_p[d]) = 0$ by Proposition 4.3 and Lemma 4.5. If $c > 1$ then $H^{-d+c}(Ri^!\mathbb{F}_p[d]) = 0$ by Proposition 4.3 and Lemma 4.6. $\square$

Chapter 5

Global F -regularity of affine Schubert varieties

5.1 Background

There have been multiple proofs of the normality of classical Schubert varieties in flag varieties (see [And85], [RR85], [Ses87]). If the base field k has positive characteristic $p > 0$, one method due to Mehta-Srinivas [MS87] of proving normality uses the fact that classical Schubert varieties are Frobenius split (F -split). A scheme X over k is said to be F -split if the natural p th-power map of $\mathcal{O}_X$ -modules $\mathcal{O}_X \rightarrow F_*\mathcal{O}_X$ admits a splitting. F -splitting also implies $H^i(L, X) = 0$ for all $i > 0$ and L an ample line bundle on X . Mehta-Ramanathan first introduced Frobenius splitting techniques for classical Schubert varieties in [MR85].

By extending the methods of Mehta-Ramanathan to the affine case, Faltings [Fal03] showed affine Schubert varieties are normal, Cohen-Macaulay, F -split, and have rational singularities when G is simple and simply connected. Pappas-Rapoport [PR08] further extended this to the case $p \nmid |\pi_1(G_{\text{der}})|$. F -splitting is one example of a broader family of properties related to the Frobenius endomorphism, such as strong F -regularity and F -

rationality. See Section 5.4 and also [HH89], [HH94] for definitions and consequences of these and other similar notions. While these properties are typically defined locally on varieties, Smith [Smi00] has defined the notion of global F -regularity of a projective variety.

A globally F -regular variety enjoys several favorable properties. For example, such a variety is strongly F -regular [Smi00], and hence it is also reduced, normal, Cohen-Macaulay, F -split and F -rational. Lauritzen, Raben-Pedersen, and Thomsen [LRPT06] showed that classical Schubert varieties are globally F -regular. Their proof is remarkably short, and relies on the fact that Mehta-Ramanathan proved something more powerful than F -splitting: they showed that Demazure varieties are compatibly Frobenius split along certain divisors. In this section we prove that affine Schubert varieties are globally F -regular.

5.2 Affine Schubert varieties

Let k be a perfect field of characteristic $p > 0$. For a smooth affine group scheme G over the power series ring $k[[t]]$ we define the loop group LG as the functor on k -algebras

$$LG: R \mapsto G(R((t))).$$

The positive loop group L^+G is the functor

$$L^+G: R \mapsto G(R[[t]]).$$

For each integer $n > 0$ we also have the n -th jet group

$$L^nG: R \mapsto G(R[t]/t^n).$$

We now specialize to the case where G is a split connected reductive group defined over k (note that G can also be viewed as a constant group scheme over $k[[t]]$). Let $T \subset B \subset G$

be a maximal torus and a Borel subgroup. Let $I \subset L^+G$ be the Iwahori group given by the fiber of B under the projection $L^+G \rightarrow G$. The affine Grassmannian is the fpqc-quotient $\mathrm{Gr} := LG/L^+G$ and the affine flag variety is the fpqc-quotient $\mathcal{F}\ell := LG/I$. Both Gr and $\mathcal{F}\ell$ are represented by ind-projective k -schemes.

The left L^+G -orbits in Gr are indexed by the dominant cocharacters $X_*(T)^+$ and the left I -orbits in $\mathcal{F}\ell$ are indexed by the Iwahori-Weyl group $\tilde{W} := N_{G(k((t)))}T(k((t)))/T(k[[t]])$. Given $\mu \in X_*(T)^+$ let $\mathrm{Gr}_\mu = L^+G \cdot \mu(t)$ be the corresponding reduced orbit. The reduced closure of Gr_μ is denoted $\mathrm{Gr}_{\leq \mu}$, and it is the union of those Gr_λ for $\lambda \leq \mu$. For $w \in \tilde{W}$ we define $C(w)$ to be the corresponding reduced I -orbit and we denote its reduced closure by $\mathcal{F}\ell_w$. The scheme $C(w)$ is isomorphic to $\mathbb{A}_k^{\ell(w)}$, where ℓ is the length function on $\tilde{W}$. If $\lambda \in X_*(T)$ we denote by t_λ the element λ viewed as an element of $\tilde{W}$. Note that $\mu - \lambda$ is a sum of positive coroots with non-negative integer coefficients if and only if $t_\lambda \leq t_\mu$ in the Bruhat order on $\tilde{W}$ by [Zhu14, 9.4], so there is no ambiguity in the choice of order on $X_*(T)^+$.

We can also define partial affine flag varieties as follows. Let $P \supset B$ be a parabolic subgroup, with preimage $\mathcal{P} \subset L^+G$. If $P = B$ then we write I instead of $\mathcal{P}$. The (partial) affine flag variety is the fpqc-quotient $\mathcal{F}\ell_{\mathcal{P}} := LG/\mathcal{P}$.

If $P = B$ then $\mathcal{P} = I$ and we drop the subscript on $\mathcal{F}\ell$. Given $w \in \tilde{W}$, choose a representative of w in $LG(k)$ and let $w' \in \mathcal{F}\ell_{\mathcal{P}}(k)$ be its image in the quotient. The affine Schubert variety $\mathcal{F}\ell_{\mathcal{P},w}$ is the reduced orbit closure

$$\mathcal{F}\ell_{\mathcal{P},w} := \overline{I \cdot w'} \subset \mathcal{F}\ell_{\mathcal{P}}.$$

The affine Schubert varieties $\mathrm{Gr}_{\leq \mu}$ and $\mathcal{F}\ell_{\mathcal{P},w}$ are projective and integral (see [Zhu17, 1.2.2]).

Theorem 5.1 ([Fal03], [PR08]). *If $p \nmid |\pi_1(G_{\mathrm{der}})|$ then the affine Schubert varieties $\mathrm{Gr}_{\leq \mu}$ and $\mathcal{F}\ell_{\mathcal{P},w}$ are normal, Cohen-Macaulay, F -split and have rational singularities.*

Our proof of global F -regularity is analogous to the case of classical Schubert varieties in [LRPT06]. To begin, we use results in [PR08] to reduce to the case when G is simply connected. In this setting the relevant facts about Frobenius splittings of Demazure varieties are known and follow from the original techniques of Mehta-Ramanathan. We also note that it is necessary to pass to the reductions, otherwise some affine Schubert varieties are non-reduced (see [PR08]) and therefore not globally F -regular.

5.3 Demazure varieties

There is a semi-direct product decomposition

$$W = W_{\text{aff}} \rtimes \Omega$$

where W_{aff} is the affine Weyl group associated to the root system for (G, B, T) and Ω is the normalizer of I in W . The choice of I determines a Coxeter group structure on W_{aff} . Given $w \in W_{\text{aff}}$, let $w = s_1 \cdots s_r$ be an arbitrary expression of w as a product of simple reflections. We will use the notation $\tilde{w}$ to denote a pair consisting of $w \in W_{\text{aff}}$ together with an expression for w as a product of simple reflections.

Let $\mathcal{P}_i \subset LG$ be the unique parahoric subgroup containing both I and a fixed representative for s_i such that $\mathcal{P}_i/I \cong \mathbb{P}^1$. The Demazure variety associated to $\tilde{w}$ is

$$D_{\tilde{w}} := \mathcal{P}_1 \times \cdots \times \mathcal{P}_r / I^r.$$

Here I^r acts by the formula $(p_1, \dots, p_r) \cdot (i_1, \dots, i_r) = (p_1 i_1, i_1^{-1} p_2 i_2, \dots, i_{r-1}^{-1} p_r i_r)$. There is a multiplication map

$$\pi_w : D_{\tilde{w}} \rightarrow \mathcal{F}\ell_w, \quad [(p_1, \dots, p_r)] \mapsto [p_1 \cdots p_r].$$

If $v = s_1 \cdots s_{r-1}$ so that $w = v \cdot s_r$ then there is a $\mathbb{P}^1$ -bundle $D_{\tilde{w}} \rightarrow D_{\tilde{v}}$ obtained by forgetting the last coordinate. As $D_{\tilde{w}}$ is an iterated $\mathbb{P}^1$ -bundle it is smooth. If $\tilde{v} \leq \tilde{w}$ is obtained from w by removing one or more of the s_i there is a natural inclusion $D_{\tilde{v}} \rightarrow D_{\tilde{w}}$ (see [Zhu14, 6.4.1]).

5.4 F -singularities

Let R be a k -algebra, and let F be the absolute Frobenius endomorphism on $\text{Spec}(R)$. For every integer $e \geq 0$ there is a map of R -modules $R \rightarrow F_*^e R$ which sends a to a^{p^e} . For any $c \in R$, by applying F_*^e to the map $R \rightarrow R$ given by multiplying by c and precomposing with $R \rightarrow F_*^e R$, we get a map $R \rightarrow F_e^* R$ sending 1 to c .

Definition 5.2. [HH89] A finitely generated k -algebra R is *strongly F -regular* if for every $c \in R$ not contained in a minimal prime of R there is an integer $e > 0$ such that the map

$$R \rightarrow F_e^* R, \quad 1 \mapsto c$$

splits.

More generally, strong F -regularity is defined for noetherian rings R of characteristic $p > 0$ that are F -finite (meaning $F_* R$ is a finite R -module). If R is reduced then R is F -finite if and only if $R^{1/p}$ is a finite R -module. A finitely generated k -algebra is F -finite since k is perfect.

By [HH89, 3.1 (a)], R is strongly F -regular if and only if $R_{\mathfrak{p}}$ is strongly F -regular for every prime ideal $\mathfrak{p}$, so it makes sense to say a locally noetherian scheme is strongly F -regular if all of its local rings are strongly F -regular. The property of F -rationality was defined in Theorem 4.1 for Cohen-Macaulay rings, and is defined for more general rings in [HH94]. We say that a locally noetherian scheme is F -rational if all of its local rings are

F -rational. If R is a homomorphic image of a Cohen-Macaulay ring, then R is F -rational if and only if all of its local rings are F -rational by [HH94, 4.2 (e)].

Let X be a connected projective variety over k . An ample invertible $\mathcal{O}_X$ -module $\mathcal{L}$ determines a section ring

$$S = \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{L}^n).$$

Definition 5.3. [Smi00] The variety X is *globally F -regular* if it admits a section ring which is strongly F -regular.

We have the following chain of implications for projective k -schemes (or more generally projective schemes over an F -finite field):

$$\begin{array}{ccccccc} & & & & \text{[Smi97, 3.1]} & & \\ & & & & & & \text{Pseudo-rational} \\ \text{Globally} & \xrightarrow{\text{[Smi00, 3.10]}} & \text{Strongly} & \xrightarrow{\text{[HH89, 3.1]}} & F\text{-rational} & \xrightarrow{\text{[HH94, 4.2]}} & \text{singularities, normal,} \\ F\text{-regular} & & F\text{-regular} & & & & \text{Cohen-Macaulay.} \end{array}$$

Remark 5.4. We will also use the notion of pseudo-rationality as defined in [Kov17]. Using [Smi97, 1.13] and the flat base change theorem, one can verify the following: If X is a k -scheme of finite type such that every local ring of X is pseudo-rational as defined in [Smi97, 1.8], then X is also pseudo-rational as defined in [Kov17, 1.2].

For our purposes we will need several alternative methods of showing global F -regularity. To describe the first of these methods, let D be an effective Cartier divisor on X defined by a section s . Then there is a map $\mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ that sends 1 to s . By applying F_*^e to this map and precomposing with the natural map $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$, we get a map $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(D)$ of $\mathcal{O}_X$ -modules such that $1 \mapsto s$. The variety X is said to be *Frobenius split along D* if the map $\mathcal{O}_X \rightarrow F_* \mathcal{O}_X(D)$ splits. More generally, X is *stably Frobenius split along D* if $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(D)$ splits for some integer $e > 0$. Note that splitting for one integer e implies

splitting for all integers $e' \geq e$ (see [Smi00, 3.5]).

Lemma 5.5. *Let X be a smooth, connected, projective variety over k . Then X is globally F -regular if and only if X is stably Frobenius split along an ample divisor.*

Proof. This follows from [Smi00, 3.10]. $\square$

Lemma 5.6. *Let $\pi: X \rightarrow Y$ be a morphism between connected projective varieties over k such that $\pi_*\mathcal{O}_X = \mathcal{O}_Y$. Then if X is globally F -regular, Y is also globally F -regular.*

Proof. See [LRPT06, Lemma 1.2]. $\square$

Lemma 5.7. *Let X, Y be globally F -regular varieties. Then $X \times Y$ is globally F -regular.*

Proof. Let A and B be strongly F -regular section rings for X and Y , respectively. These are naturally $\mathbb{N}$ -graded rings. A section ring for the product $X \times Y$ is the Segre product

$$A\#B := \bigoplus_{n \in \mathbb{N}} A_n \otimes B_n.$$

Being a direct summand of the strongly F -regular ring $A \otimes_k B$, the Segre product $A\#B$ is also strongly F -regular.¹ See [Has03, 5.2] for a proof of these claims, which are attributed to Hochster. $\square$

Before starting the proof we need one more notion. Let $Y \subset X$ be a closed subvariety with ideal sheaf $\mathcal{I}$. Then Y is said to be compatibly split in X if there is a splitting $\varphi: F_*\mathcal{O}_X \rightarrow \mathcal{O}_X$ such that $\varphi(F_*\mathcal{I}) \subset \mathcal{I}$. In this case φ induces a splitting of Y . The following lemma relates this property to Frobenius splitting along a divisor.

Lemma 5.8. *Let X be a variety and let $D \subset X$ be an effective Cartier divisor on X defined by a section s . If D is compatibly split in X then X is Frobenius split along the divisor D .*

Proof. X is Frobenius split along $(p-1)D$ by [LRPT06, Lemma 1.1]. This implies X is Frobenius split along D as well (see [Smi00, 3.7]). $\square$

¹We thank Karen Smith for pointing out this proof.

5.5 Proof of global F -regularity

Proposition 5.9. *Let G be a split, simply connected, simple reductive group and let $w \in W_{\text{aff}}$ with reduced expression $w = s_1 \cdots s_r$. Then $D_{\tilde{w}}$ is globally F -regular.*

Proof. Let $v_i = s_1 \cdots s_{i-1} \hat{s}_i s_{i+1} \cdots s_r$. By [PR08, 9.6] there is a single Frobenius splitting of the Demazure variety $D_{\tilde{w}}$ which is compatible with all the closed immersions $D_{\tilde{v}_i} \rightarrow D_{\tilde{w}}$ (see also [Mat88, Ch. 8.18], and [Gör01, 3.20] for the case of SL_n). Strictly speaking, Pappas and Rapoport only make this claim when $\tilde{v}_i$ gives a reduced expression for v_i , so we will say some more to justify our claim.

The issue is that Pappas and Rapoport require reduced expressions in their definition of Demazure varieties. For their proof they cite the original criterion of Mehta-Ramanathan [MR85, Prop. 8], which requires one to show

$$\omega_{D_{\tilde{w}}}^{-1} \cong \sum_{i=1}^r \mathcal{O}(D_{\tilde{v}_i}) \otimes \mathcal{O}(E),$$

where E is an effective divisor whose support does not meet $\cap_i D_{\tilde{v}_i}$. The precise form E takes in our situation can be found in [Zhu14, 6.4.2]. By [MR85, Prop. 8], $D_{\tilde{w}}$ is compatibly Frobenius split with all the closed immersions $D_{\tilde{v}_i} \rightarrow D_{\tilde{w}}$, regardless of whether $\tilde{v}_i$ gives a reduced expression for v_i .

It follows from the definitions that $D_{\tilde{w}}$ is compatibly split along the closed subscheme defined by the product of the ideal sheaves of the $D_{\tilde{v}_i}$. Now by Lemma 5.8, $D_{\tilde{w}}$ is Frobenius split along the divisor $\sum_{i=1}^r D_{\tilde{v}_i}$. Then by [Smi00, 3.7, 3.9], $D_{\tilde{w}}$ is stably Frobenius split along $\sum_{i=1}^r m_i D_{\tilde{v}_i}$ for any integers $m_i > 0$. By Lemma 5.5, we need only show this divisor is ample for some m_i .

In the classical case, Lauritzen-Thomsen [LT04, 6.1] show that $D_{\tilde{v}_r} + \sum_{i=1}^{r-1} m_i D_{\tilde{v}_i}$ is ample for some integers $m_i > 0$. A similar proof works in our setting. The case $r = 1$ is clear because then $D_{\tilde{w}} \cong \mathbb{P}^1$ and $D_{\tilde{v}_1}$ is a point. For $r > 1$ we note that since $\pi: D_{\tilde{w}} \rightarrow D_{\tilde{v}_r}$,

admits a section then for each closed point $b \in D_{\tilde{v}_r}$ there exist isomorphisms $\pi^{-1}(b) \cong \mathbb{P}^1$ and $\mathcal{O}(D_{\tilde{v}_r})|_{\pi^{-1}(b)} \cong \mathcal{O}_{\mathbb{P}^1}(1)$. Using this fact one can show that $V := \pi_*\mathcal{O}(D_{\tilde{v}_r})$ is a vector bundle and $D_{\tilde{w}} \cong \mathbb{P}V$. In particular, $\mathcal{O}(D_{\tilde{v}_r}) \cong \mathcal{O}_{\mathbb{P}V}(1)$ is π -ample, so the claim follows by induction and [Sta21, 0892]. $\square$

Proof of Theorem 1.9. We proceed by a series of reductions.

Reduction to $\mathcal{F}\ell_{\mathcal{P},w}$. The natural projection $\pi: \mathcal{F}\ell \rightarrow \text{Gr}$ is an étale locally trivial G/B -bundle. We claim $\pi^{-1}(\text{Gr}_{\leq \mu})$ is isomorphic to $\mathcal{F}\ell_w$ for some $w \in W$. To prove this, we first observe that $\pi^{-1}(\text{Gr}_{\leq \mu})$ is integral as both $\text{Gr}_{\leq \mu}$ and G/B are integral. It is also clear that $\pi^{-1}(\text{Gr}_{\leq \mu})$ is a union of I -orbits. The I -orbits in $\mathcal{F}\ell$ and their closure relations are determined by the Bruhat order on W_{aff} (see [Gör10, 2.18]). Since $\pi^{-1}(\text{Gr}_{\leq \mu}) \subset \mathcal{F}\ell$ is of finite type it meets only finitely many I -orbits, and being integral and closed it is necessarily the closure of a unique open dense I -orbit. Thus $\pi^{-1}(\text{Gr}_{\leq \mu}) = \mathcal{F}\ell_w$ for some $w \in W$. Since $H^0(G/B, \mathcal{O}_{G/B}) = k$ then by flat base change, $(\pi|_{\mathcal{F}\ell_w})_*(\mathcal{O}_{\mathcal{F}\ell_w}) = \mathcal{O}_{\text{Gr}_{\leq \mu}}$. Thus the global F -regularity of $\text{Gr}_{\leq \mu}$ will follow from that of $\mathcal{F}\ell_w$ by Lemma 5.6.

Reduction to $\mathcal{P} = I$. We apply the same reasoning as above to the P/B -bundle $\mathcal{F}\ell \rightarrow \mathcal{F}\ell_{\mathcal{P}}$.

Reduction to G simply connected. Let $\tilde{G}_{\text{der}}$ be a simply connected cover of G_{der} . By [PR08, 8.e.3, 8.e.4], $\mathcal{F}\ell_w$ is isomorphic to an affine Schubert variety in the affine flag variety for $\tilde{G}_{\text{der}}$. This is where we need $p \nmid |\pi_1(G_{\text{der}})|$.

Reduction to G simply connected and simple. If G simply connected but not simple choose isomorphisms $G \cong \prod_i G_i$ and $B \cong \prod B_i$ where each G_i is simple and simply connected, and the $B_i \subset G_i$ are Borel subgroups. This induces isomorphisms on flag varieties $\mathcal{F}\ell_I \cong \prod_i \mathcal{F}\ell_{I_i}$ and Iwahori-Weyl groups $W \cong \prod_i W_i$. If we write $w = \prod_i w_i$ for $w_i \in W_i$ then $\mathcal{F}\ell_w \cong \prod_i \mathcal{F}\ell_{w_i}$ (see [PR08, 8.e.2]). Now apply Lemma 5.7.

Proof when G is simply connected and simple. Because G is simply connected then

$W = W_{\text{aff}}$. Let $w = s_1 \cdots s_r$ be a reduced expression for w . By Proposition 5.9 the Demazure variety $D_{\tilde{w}}$ is globally F -regular. The map $\pi_w: D_{\tilde{w}} \rightarrow \mathcal{F}\ell_w$ is a rational resolution of singularities by [PR08, 8.4] and [PR08, 9.7(d)], so we conclude by Lemma 5.6. $\square$

Chapter 6

Perverse $\mathbb{F}_p$ -sheaves on the affine Grassmannian

For Chapters 6 - 7 we return to the case k is an algebraically closed field of characteristic $p > 0$, and we fix a connected reductive group G over k . Throughout this section we assume $p \nmid |\pi_1(G_{\text{der}})|$. We will explain how to remove this hypothesis in Remark 7.12. In this section we define the category $P_{L+G}(\text{Gr}, \mathbb{F}_p)$ and prove Theorem 1.2. We also define a fiber functor on $P_{L+G}(\text{Gr}, \mathbb{F}_p)$ and investigate extensions between objects. The basic facts we will use about the geometry of Gr and the associated convolution Grassmannians are well-known (see, for example, [Ric14]), so we will state them without proof. We use the same notation as in Section 5.2.

6.1 Definition of $P_{L+G}(\text{Gr}, \mathbb{F}_p)$

There is a partial order on $X_*(T)^+$ defined by setting $\lambda \leq \mu$ if $\mu - \lambda$ is a sum of positive coroots. For each $\mu \in X_*(T)^+$, there are only finitely many λ such that $\lambda \leq \mu$. There is a

stratification

$$\mathrm{Gr}_{\leq \mu} = \coprod_{\lambda \leq \mu} \mathrm{Gr}_{\lambda}.$$

The orbit Gr_{μ} is open in its closure. Let ρ be half the sum of the positive roots. Then

$$\dim \mathrm{Gr}_{\mu} = 2\langle \rho, \mu \rangle.$$

We note that if $\lambda < \mu$ then $\mathrm{Gr}_{\leq \lambda}$ has codimension at least 2 in $\mathrm{Gr}_{\leq \mu}$. The reduced sub-ind-scheme of Gr is

$$\mathrm{Gr}_{\mathrm{red}} = \lim_{\rightarrow} \mathrm{Gr}_{\leq \mu}.$$

Define

$$D_c^b(\mathrm{Gr}, \mathbb{F}_p) := \lim_{\rightarrow} D_c^b(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p),$$

and

$$P_c^b(\mathrm{Gr}, \mathbb{F}_p) := \lim_{\rightarrow} P_c^b(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p).$$

The setup in Remark 3.13 applies and we can define L^+G -equivariant perverse $\mathbb{F}_p$ -sheaves on Gr . More precisely, for fixed μ , the group scheme L^+G acts on $\mathrm{Gr}_{\leq \mu}$ through the jet group L^nG for n sufficiently large. Suppose $m > n$ and L^+G acts on $\mathrm{Gr}_{\leq \mu}$ through both quotients L^mG and L^nG . Note that there is an exact sequence

$$0 \rightarrow N \rightarrow L^mG \rightarrow L^nG \rightarrow 0$$

where N is a connected, affine, smooth unipotent group. Thus by Proposition 3.15 there is a natural isomorphism $P_{L^nG}(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p) \cong P_{L^mG}(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p)$. Hence we can unambiguously define $P_{L^+G}(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p) := P_{L^nG}(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p)$ where n is an integer such that L^+G acts on

$\mathrm{Gr}_{\leq \mu}$ through the quotient $L^n G$. We can now define

$$P_{L^+G}(\mathrm{Gr}, \mathbb{F}_p) = \varinjlim P_{L^+G}(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p).$$

By similar reasoning we can define the category $P_I(\mathcal{F}\ell_{\mathcal{P}}, \mathbb{F}_p)$.

Proof of Theorem 1.10. By [NP01, 2.3] the stabilizers for the action of L^+G on Gr are connected. Thus by Corollary 3.19, the simple objects in $P_{L^+G}(\mathrm{Gr}, \mathbb{F}_p)$ are the intermediate extensions of constant sheaves along orbits. By the global F -regularity of $\mathrm{Gr}_{\leq \mu}$ we are in a position to apply Theorem 1.12. Thus the simple objects are the constant sheaves $\mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu}]$ supported on $\mathrm{Gr}_{\leq \mu}$. The stabilizers for the action of I on $\mathcal{F}\ell$ are also connected by [Ric16, 2.3], so the same argument works for $\mathcal{F}\ell$. $\square$

Remark 6.1. By the same reasoning as in the proof of Theorem 1.10 it follows that for any $\mathcal{P}$ the constant sheaves supported on I -orbit closures in $\mathcal{F}\ell_{\mathcal{P}}$ are simple. If the stabilizers for the action of I are connected then these are all the simple objects in $P_I(\mathcal{F}\ell_{\mathcal{P}}, \mathbb{F}_p)$.

6.2 Convolution

The convolution diagram is

$$\mathrm{Gr} \times \mathrm{Gr} \xleftarrow{p} LG \times \mathrm{Gr} \xrightarrow{q} LG \times^{L^+G} \mathrm{Gr} \xrightarrow{m} \mathrm{Gr}. \quad (6.1)$$

Here p is the quotient map on the first factor, and q is the quotient by the diagonal action of L^+G given by $g \cdot (g_1, g_2) = (g_1 g^{-1}, g g_2)$. The map m is the multiplication map $m(g_1, g_2) = g_1 g_2$. Note that p and q are L^+G -torsors. For simplicity we ignore the fact that $LG \times \mathrm{Gr}$ is not of ind-finite type and allow ourselves to speak of perverse sheaves on $LG \times \mathrm{Gr}$. The same technical method as in [Ric14, 3.21] works here for overcoming this issue. The idea

is that for any particular perverse sheaves we are working with, we can replace p and q by $L^n G$ -torsors for n sufficiently large.

More generally there is a convolution morphism $LG \times^{L^+G} \dots \times^{L^+G} \mathrm{Gr} \rightarrow \mathrm{Gr}$. To describe the ind-scheme structure on the convolution Grassmannian, let $\mu_i \in X_*(T)^+$ for $i = 1, \dots, n$ be a collection of dominant cocharacters and let μ be their sum. Let $f: LG \rightarrow \mathrm{Gr}$ be the quotient map. Then we define

$$\mathrm{Gr}_{\leq \mu_\bullet} := f^{-1}(\mathrm{Gr}_{\leq \mu_1}) \times^{L^+G} \dots \times^{L^+G} f^{-1}(\mathrm{Gr}_{\leq \mu_{n-1}}) \times^{L^+G} \mathrm{Gr}_{\leq \mu_n},$$

where we always take the reduced subscheme structure. If $n = 2$ we may also denote $\mathrm{Gr}_{\leq \mu_\bullet}$ by $\mathrm{Gr}_{\leq \mu_1} \tilde{\times} \mathrm{Gr}_{\leq \mu_2}$.

The convolution morphism restricts to a map $m: \mathrm{Gr}_{\leq \mu_\bullet} \rightarrow \mathrm{Gr}_{\leq \mu}$ which is proper, birational, and an isomorphism over Gr_μ . Using the construction in Remark 3.13 we can define

$$P_{L^+G}(LG \times^{L^+G} \dots \times^{L^+G} \mathrm{Gr}, \mathbb{F}_p),$$

where L^+G acts on the leftmost factor LG .

Lemma 6.2. *Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in P_{L^+G}(\mathrm{Gr}, \mathbb{F}_p)$. Then there is a unique perverse sheaf*

$$\mathcal{F}^\bullet \tilde{\boxtimes} \mathcal{G}^\bullet \in P_{L^+G}(LG \times^{L^+G} \mathrm{Gr}, \mathbb{F}_p)$$

such that

$$Rp^*(\mathcal{F}^\bullet \tilde{\boxtimes} \mathcal{G}^\bullet) \cong Rq^*(\mathcal{F}^\bullet \tilde{\boxtimes} \mathcal{G}^\bullet).$$

Proof. We claim that $\mathcal{F}^\bullet \tilde{\boxtimes} \mathcal{G}^\bullet$ is perverse. In general we do not know if the box product of two perverse $\mathbb{F}_p$ -sheaves is perverse (the analogous fact is true for $\overline{\mathbb{Q}}_\ell$ -coefficients with the appropriate t -structure [BBD82, 4.2.8]). However, we can prove that $\mathcal{F}^\bullet \tilde{\boxtimes} \mathcal{G}^\bullet$ is perverse in our situation. First, note that the schemes $\mathrm{Gr}_{\leq \mu} \times \mathrm{Gr}_{\leq \lambda}$ for $\mu, \lambda \in X_*(T)^+$ are Cohen-

Macaulay because a product of Cohen-Macaulay schemes is Cohen-Macaulay [Sta21, 045Q]. Now if $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are simple then $\mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}^\bullet$ is perverse because it is a constant sheaf supported on a Cohen-Macaulay scheme (and it is in the correct degree).

Now suppose $\mathcal{F}^\bullet$ is simple and $\mathcal{G}^\bullet$ has length $n > 1$. Because subquotients of equivariant perverse sheaves are equivariant (Proposition 3.10), there is an exact sequence

$$0 \rightarrow \mathcal{G}_1^\bullet \rightarrow \mathcal{G}^\bullet \rightarrow \mathcal{G}_2^\bullet \rightarrow 0$$

in $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ such that $\mathcal{G}_1^\bullet, \mathcal{G}_2^\bullet$ have length $< n$. This gives rise to an exact triangle

$$\mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}_1^\bullet \rightarrow \mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}^\bullet \rightarrow \mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}_2^\bullet$$

in $D_c^b(\mathrm{Gr} \times \mathrm{Gr}, \mathbb{F}_p)$. By induction the outer two terms are perverse, thus so is $\mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}^\bullet$. A similar induction on the length of $\mathcal{F}^\bullet$ completes the proof that $\mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}^\bullet$ is perverse.

Now that we know $\mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}^\bullet$ is perverse then so is $Rp^*(\mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}^\bullet)$ (up to a shift) because p is smooth. Because $\mathcal{G}^\bullet$ is L^+G -equivariant then $Rp^*(\mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}^\bullet)$ is equivariant for the diagonal action of L^+G . This equivariance is precisely the data needed to use Corollary 2.21 and descend $Rp^*(\mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}^\bullet)$ along the quotient map q to a perverse sheaf $\mathcal{F}^\bullet \overset{\sim}{\boxtimes} \mathcal{G}^\bullet \in P_c^b(LG \times^{L^+G} \mathrm{Gr}, \mathbb{F}_p)$. Finally, one can use smooth descent and the fact that $\mathcal{F}^\bullet$ is L^+G -equivariant to show $\mathcal{F}^\bullet \overset{\sim}{\boxtimes} \mathcal{G}^\bullet$ is L^+G -equivariant. $\square$

If $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ we define their *convolution product* to be

$$\mathcal{F}^\bullet * \mathcal{G}^\bullet := Rm_*(\mathcal{F}^\bullet \overset{\sim}{\boxtimes} \mathcal{G}^\bullet) \in D_c^b(\mathrm{Gr}, \mathbb{F}_p).$$

Our next goal is to prove that $\mathcal{F}^\bullet * \mathcal{G}^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$. Once we know that $\mathcal{F}^\bullet * \mathcal{G}^\bullet$ is perverse, the fact that it is equivariant will follow by Lemma 3.2 since m is proper and L^+G -equivariant.

For later use, we also define the n -fold convolution product. Let $\mathcal{F}_i \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ for $i = 1, \dots, n$. By the same reasoning as in Lemma 6.2 we can define their twisted product

$$\tilde{\boxtimes}_i \mathcal{F}_i^\bullet \in P_{L+G}(LG \times^{L+G} \dots \times^{L+G} \mathrm{Gr}, \mathbb{F}_p).$$

Let $m: LG \times^{L+G} \dots \times^{L+G} \mathrm{Gr} \rightarrow \mathrm{Gr}$ be the n -fold convolution map. Then we define

$$*_i \mathcal{F}_i^\bullet := Rm_*(\tilde{\boxtimes}_i \mathcal{F}_i^\bullet) \in D_c^b(\mathrm{Gr}, \mathbb{F}_p).$$

For $\overline{\mathbb{Q}}_\ell$ -coefficients, the perversity of the convolution product can be proved by using the fact that m is a stratified semi-small map (see [MV07]) or the notion of universally locally acyclic complexes (see [Ric14]). We will take a different approach which uses the fact that Gr has rational singularities. Before giving the proof of Theorem 1.2, we illustrate why the case of $\mathbb{F}_p$ -coefficients differs from $\overline{\mathbb{Q}}_\ell$ -coefficients with the following example.

Example 6.3. Let $G = \mathrm{GL}_2$ and use the standard choice of $T \subset B \subset G$ and the identification $X_*(T) \cong \mathbb{Z}^2$. Consider the two minuscule cocharacters $\mu_1 = (1, 0)$ and $\mu_2 = (0, -1)$ with corresponding IC sheaves IC_{μ_1} and IC_{μ_2} . These are both constant sheaves (shifted by 1) supported on schemes that are isomorphic to $\mathbb{P}^1$. Let $\mu = \mu_1 + \mu_2$. Then $\mathrm{Gr}_{\leq \mu}$ is a two-dimensional surface with a unique singular point e_0 . The convolution Grassmannian $\mathrm{Gr}_{\leq \mu_\bullet}$ over $\mathrm{Gr}_{\leq \mu}$ is a resolution of singularities which is an isomorphism away from e_0 and whose fiber over e_0 is isomorphic to $\mathbb{P}^1$.

For both $\overline{\mathbb{Q}}_\ell$ and $\mathbb{F}_p$ -coefficients the convolution $\mathrm{IC}_{\mu_1} * \mathrm{IC}_{\mu_2}$ is given by the derived pushforward of the constant sheaf (shifted by 2) along the convolution map $m: \mathrm{Gr}_{\leq \mu_\bullet} \rightarrow \mathrm{Gr}_{\leq \mu}$. For $\overline{\mathbb{Q}}_\ell$ -coefficients the result is $Rm_*(\overline{\mathbb{Q}}_\ell[2]) \cong \overline{\mathbb{Q}}_\ell[2] \oplus C[0]$ where C is a skyscraper sheaf supported at e_0 . The stalk of the summand $C[0]$ at e_0 is isomorphic to $H_{\mathrm{et}}^2(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)$. When we work with étale $\mathbb{F}_p$ -sheaves, $Rm_*(\mathbb{F}_p[2]) \cong \mathbb{F}_p[2]$ because $H_{\mathrm{et}}^i(\mathbb{P}^1, \mathbb{F}_p) = 0$ for $i > 0$. More generally, we will prove in this section that for any group G the constant sheaf $\mathbb{F}_p[0]$

is preserved under the derived pushforward along any convolution morphism.

Lemma 6.4. *Let $\mu_i \in X_*(T)^+$ for $i = 1, \dots, n$. Then the scheme $\mathrm{Gr}_{\leq \mu_\bullet}$ is projective, integral, normal, and Cohen-Macaulay.*

Proof. There is an isomorphism of ind-schemes $LG \times^{L^+G} \dots \times^{L^+G} \mathrm{Gr} \cong \mathrm{Gr}^n$ (see [Zhu17, 1.2.14]). This implies $\mathrm{Gr}_{\leq \mu_\bullet}$ is projective. For the other properties we proceed by induction on the number n of dominant cocharacters. Let $\lambda_i = \mu_{i+1}$ for $i = 1, \dots, n-1$. Let k be an integer large enough so that L^+G acts on $\mathrm{Gr}_{\leq \lambda_\bullet}$ on the left through the quotient L^kG . Let

$$\mathrm{Gr}_{\leq \mu_1}^k := LG|_{\mathrm{Gr}_{\leq \mu_1}} \times^{L^+G} L^kG.$$

The map $\mathrm{Gr}_{\leq \mu_1}^k \rightarrow \mathrm{Gr}_{\leq \mu_1}$ is a right L^kG -torsor. Thus $\mathrm{Gr}_{\leq \mu_1}^k$ is normal and Cohen-Macaulay because these properties are local in the étale topology. As $\mathrm{Gr}_{\leq \mu_1}$ is also integral and L^kG is connected then $\mathrm{Gr}_{\leq \mu_1}^k$ is integral. By induction $\mathrm{Gr}_{\leq \mu_1}^k \times \mathrm{Gr}_{\leq \lambda_\bullet}$ is normal, integral, and Cohen-Macaulay. Note that $\mathrm{Gr}_{\leq \mu_\bullet}$ is the quotient of $\mathrm{Gr}_{\leq \mu_1}^k \times \mathrm{Gr}_{\leq \lambda_\bullet}$ by the diagonal action of L^kG . Hence $\mathrm{Gr}_{\leq \mu_\bullet}$ is normal and Cohen-Macaulay. It is integral because it is reduced and it is the image of an irreducible scheme. $\square$

Let Y be a Cohen-Macaulay k -scheme. We say that Y has *rational singularities* if there exists a smooth k -scheme X and a proper birational morphism $f: X \rightarrow Y$ such that $\mathcal{O}_Y \rightarrow Rf_*\mathcal{O}_X$ is an isomorphism. Various authors have considered alternative definitions of rational singularities which do not rely on the existence of a resolution of singularities. One such definition is due to Lipman-Teissier [LT81], which they call *pseudo-rational singularities* (see also [Kov17, 1.2]). We will not need the definition here. All we will need is that rational singularities are pseudo-rational ([Kov17, 9.6]).

Proposition 6.5. *Let $m: \mathrm{Gr}_{\leq \mu_\bullet} \rightarrow \mathrm{Gr}_{\leq \mu}$ be the convolution map corresponding to $\mu_i \in$*

$X_*(T)^+$. Let $\mathcal{O}_{\mathrm{Gr}_{\leq \mu_\bullet}}$ be the structure sheaf of $\mathrm{Gr}_{\leq \mu_\bullet}$. Then the natural map

$$\mathcal{O}_{\mathrm{Gr}_{\leq \mu}} \xrightarrow{\sim} Rm_*(\mathcal{O}_{\mathrm{Gr}_{\leq \mu_\bullet}})$$

is an isomorphism.

Proof. Since $\mathrm{Gr}_{\leq \mu_\bullet}$ and $\mathrm{Gr}_{\leq \mu}$ are projective then f is projective. By Theorem 5.1, $\mathrm{Gr}_{\leq \mu}$ is Cohen-Macaulay and has rational singularities. As rational singularities are pseudo-rational then the result follows from [Kov17, 1.4]. $\square$

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Let $\mu = \mu_1 + \mu_2$ and let $m: \mathrm{Gr}_{\leq \mu_\bullet} \rightarrow \mathrm{Gr}_{\leq \mu}$ be the convolution map. By Lemma 6.4 and Theorem 1.11 the constant sheaf $\mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu_\bullet}]$ supported on $\mathrm{Gr}_{\leq \mu_\bullet}$ is perverse. From the definition of $\mathrm{IC}_{\mu_1} * \mathrm{IC}_{\mu_2}$ and the fact that the IC sheaves are constant we see that

$$\mathrm{IC}_{\mu_1} * \mathrm{IC}_{\mu_2} = Rm_*(\mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu_\bullet}]).$$

Thus, to show that $\mathrm{IC}_{\mu_1} * \mathrm{IC}_{\mu_2} = \mathrm{IC}_\mu$ it suffices to show that $Rm_*\mathbb{F}_p[0] \cong \mathbb{F}_p[0]$. This fact follows immediately from Proposition 6.5 by applying Rm_* to the Artin–Schreier sequence on $\mathrm{Gr}_{\leq \mu_\bullet}$. Finally, for general $\mu \in X_*(T)^+$, Theorem 6.9 in the next section implies that $\dim_{\mathbb{F}_p} H(\mathrm{IC}_\mu) = 1$. $\square$

Corollary 6.6. Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$. Then $\mathcal{F}^\bullet * \mathcal{G}^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$.

Proof. We first suppose $\mathcal{F}^\bullet$ is simple and induct on the length n of $\mathcal{G}^\bullet$. Pick an exact sequence

$$0 \rightarrow \mathcal{G}_1^\bullet \rightarrow \mathcal{G}^\bullet \rightarrow \mathcal{G}_2^\bullet \rightarrow 0$$

in $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ such that $\mathcal{G}_1^\bullet, \mathcal{G}_2^\bullet$ have length $< n$. We claim that the sequence

$$0 \rightarrow \mathcal{F}^\bullet \tilde{\boxtimes} \mathcal{G}_1^\bullet \rightarrow \mathcal{F}^\bullet \tilde{\boxtimes} \mathcal{G}^\bullet \rightarrow \mathcal{F}^\bullet \tilde{\boxtimes} \mathcal{G}_2^\bullet \rightarrow 0$$

in $P_{L+G}(LG \times^{L+G} \mathrm{Gr}, \mathbb{F}_p)$ is exact. Indeed, exactness follows because this sequence is constructed by descending an exact sequence of perverse sheaves on $LG \times \mathrm{Gr}$ (again, we are suppressing the ind-finite type issue with $LG \times \mathrm{Gr}$). Now the fact that $Rm_*(\mathcal{F}^\bullet \tilde{\boxtimes} \mathcal{G}^\bullet)$ is perverse follows by induction. A similar induction on the length of $\mathcal{F}^\bullet$ completes the proof that $\mathcal{F}^\bullet * \mathcal{G}^\bullet$ is perverse. Finally, $\mathcal{F}^\bullet * \mathcal{G}^\bullet$ is equivariant by Lemma 3.2 since m is proper. $\square$

From the proof of Corollary 6.6, we see that the convolution product is exact:

Corollary 6.7. *Let $\mathcal{F}^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$. Then the functor*

$$P_{L+G}(\mathrm{Gr}, \mathbb{F}_p) \rightarrow P_{L+G}(\mathrm{Gr}, \mathbb{F}_p), \quad \mathcal{G}^\bullet \mapsto \mathcal{F}^\bullet * \mathcal{G}^\bullet$$

is exact. The same is true when we convolve on the right by $\mathcal{F}^\bullet$.

We conclude this section by giving $(P_{L+G}(\mathrm{Gr}, \mathbb{F}_p), *)$ the structure of a monoidal category.

Theorem 6.8. *There is a natural associativity constraint so that $(P_{L+G}(\mathrm{Gr}, \mathbb{F}_p), *)$ is a monoidal category.*

Proof. This works the same as in the case of $\overline{\mathbb{Q}}_\ell$ -coefficients. First, we note that the argument in the proof of Corollary 6.6 shows that $*_{\mathcal{F}_i} \mathcal{F}_i^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ for any $\mathcal{F}_i^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$. By using the natural isomorphisms

$$(\mathcal{F}_1^\bullet \overset{L}{\boxtimes} \mathcal{F}_2^\bullet) \overset{L}{\boxtimes} \mathcal{F}_3^\bullet \cong \mathcal{F}_1^\bullet \overset{L}{\boxtimes} \mathcal{F}_2^\bullet \overset{L}{\boxtimes} \mathcal{F}_3^\bullet \cong \mathcal{F}_1^\bullet \overset{L}{\boxtimes} (\mathcal{F}_2^\bullet \overset{L}{\boxtimes} \mathcal{F}_3^\bullet)$$

and the proper base change theorem, one can construct isomorphisms

$$(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) * \mathcal{F}_3^\bullet \cong \mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet * \mathcal{F}_3^\bullet \cong \mathcal{F}_1^\bullet * (\mathcal{F}_2^\bullet * \mathcal{F}_3^\bullet).$$

The unit object is the constant sheaf $IC_0 = \mathbb{F}_p[0]$ supported on the point $\mathrm{Gr}_{\leq 0}$. From the definitions it is clear how to construct isomorphisms

$$\mathcal{F}^\bullet * IC_0 \cong \mathcal{F}^\bullet \cong IC_0 * \mathcal{F}^\bullet.$$

The coherence conditions can be checked using the corresponding properties for $-\overset{L}{\boxtimes}-$. $\square$

6.3 Fiber functor

In this section we study the cohomology of objects in $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$. The most important result is:

Theorem 6.9. *Let $\mu \in X_*(T)^+$ be a dominant cocharacter. Then*

$$H_{\mathrm{et}}^0(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p) = \mathbb{F}_p, \quad H_{\mathrm{et}}^i(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p) = 0 \text{ for } i > 0.$$

Proof. As $\mathrm{Gr}_{\leq \mu}$ is projective, $H^0(\mathrm{Gr}_{\leq \mu}, \mathcal{O}_{\mathrm{Gr}_{\leq \mu}}) = k$. Since $\mathrm{Gr}_{\leq \mu}$ is globally F -regular, $H^i(\mathrm{Gr}_{\leq \mu}, \mathcal{O}_{\mathrm{Gr}_{\leq \mu}}) = 0$ for $i > 0$ by [Smi00, 4.3]. The theorem now follows by considering the Artin-Schreier sequence. $\square$

Corollary 6.10. *Suppose $\mathcal{F}^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ is supported on a connected component of Gr having strata whose dimensions have parity $p \in \mathbb{Z}/2$. Then*

$$R^i \Gamma(\mathcal{F}^\bullet) = 0, \quad i \not\equiv p \pmod{2}.$$

Proof. This follows by Theorem 6.9 and induction on the length of $\mathcal{F}^\bullet$. $\square$

Theorem 6.11. *The functor*

$$H := \bigoplus_i R^i \Gamma(-) : P_{L+G}(\mathrm{Gr}, \mathbb{F}_p) \rightarrow \mathrm{Vect}_{\mathbb{F}_p}$$

is exact and faithful. Furthermore

$$\dim_{\mathbb{F}_p} H(\mathcal{F}^\bullet) = \mathrm{length} \mathcal{F}^\bullet.$$

Proof. Exactness follows by Corollary 6.10. The statement $\dim_{\mathbb{F}_p} H(\mathcal{F}^\bullet) = \mathrm{length} \mathcal{F}^\bullet$ holds for $\mathcal{F}^\bullet$ simple by Theorem 6.9. For general $\mathcal{F}^\bullet$ this statement holds by exactness and induction on the length of $\mathcal{F}^\bullet$. Finally, faithfulness follows from exactness and the fact that $H(\mathcal{F}^\bullet) \neq 0$ if $\mathcal{F}^\bullet \neq 0$. $\square$

6.4 Extensions

We now investigate extensions between perverse sheaves in $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$.

Proposition 6.12. *Let $\mu, \lambda \in X_*(T)^+$. Then $\mathrm{Ext}^1(\mathrm{IC}_\mu, \mathrm{IC}_\lambda) = 0$ in $P_c^b(\mathrm{Gr}, \mathbb{F}_p)$ if any one of the following holds:*

- (i) $\mu > \lambda$,
- (ii) $\mu = \lambda$,
- (iii) $\mu \not\leq \lambda$ and $\mu \not\leq \lambda$.

Proof. The same proof as in [Ric14, 4.1] works. We reproduce it here for completeness.

(i): Let $i : \mathrm{Gr}_{\leq \lambda} \rightarrow \mathrm{Gr}_{\leq \mu}$ be the inclusion. Then by adjunction $\mathrm{Ext}^1(\mathrm{IC}_\mu, \mathrm{IC}_\lambda)$ is

$$\mathrm{Hom}_{D_c^b(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p)}(\mathrm{IC}_\mu, Ri_* \mathrm{IC}_\lambda[1]) = \mathrm{Hom}_{D_c^b(\mathrm{Gr}_{\leq \lambda}, \mathbb{F}_p)}(\mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu}], \mathbb{F}_p[\dim \mathrm{Gr}_{\leq \lambda} + 1]).$$

This is zero because $\dim \mathrm{Gr}_{\leq \lambda} + 1 < \dim \mathrm{Gr}_{\leq \mu}$.

(ii): If $\mu = \lambda$ then $\text{Ext}^1(\text{IC}_\mu, \text{IC}_\mu)$ is

$$\text{Hom}_{D_c^b(\text{Gr}_{\leq \mu}, \mathbb{F}_p)}(\mathbb{F}_p[\dim \text{Gr}_{\leq \mu}], \mathbb{F}_p[\dim \text{Gr}_{\leq \mu} + 1]) = H_{\text{ét}}^1(\text{Gr}_{\leq \mu}, \mathbb{F}_p).$$

This group is trivial by Theorem 6.9.

(iii): We can assume that $\text{Gr}_{\leq \mu}$ and $\text{Gr}_{\leq \lambda}$ lie in the same connected component of Gr .

Thus we can pick a dominant cocharacter $\nu > \mu, \lambda$. This gives rise to a Cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{i'_1} & \text{Gr}_{\leq \lambda} \\ \downarrow i'_2 & & \downarrow i_2 \\ \text{Gr}_{\leq \mu} & \xrightarrow{i_1} & \text{Gr}_{\leq \nu}. \end{array}$$

By proper base change and the adjunction between $Ri_{2,*}^*$, $Ri_{2,*}$ the group $\text{Ext}^1(\text{IC}_\mu, \text{IC}_\lambda)$ is

$$\text{Hom}_{D_c^b(\text{Gr}_{\leq \nu}, \mathbb{F}_p)}(Ri_{1,*} \text{IC}_\mu, Ri_{2,*} \text{IC}_\lambda[1]) = \text{Hom}_{D_c^b(\text{Gr}_{\leq \lambda}, \mathbb{F}_p)}(Ri'_{1,*} Ri_2'^* \text{IC}_\mu, \text{IC}_\lambda[1]).$$

By the adjunction between $Ri'_{1,*}$, $Ri_1'^!$ this is the same as

$$\text{Hom}_{D_c^b(Z, \mathbb{F}_p)}(Ri_2'^* \text{IC}_\mu, Ri_1'^! \text{IC}_\lambda[1]).$$

Note that $\dim Z < \dim \text{Gr}_{\leq \mu}$. Let U be the complement of Z in $\text{Gr}_{\leq \mu}$. The restriction of IC_μ to U is nonzero because IC_μ is a constant sheaf on $\text{Gr}_{\leq \mu}$. Thus by Lemma 2.10 the perverse sheaf IC_μ is the intermediate extension of its own restriction to U . Now by Lemma 2.7,

$$Ri_2'^* \text{IC}_\mu \in {}^p D^{\leq -1}(Z, \mathbb{F}_p).$$

By similar reasoning $Ri_1'^! \text{IC}_\lambda[1] \in {}^p D^{\geq 0}(Z, \mathbb{F}_p)$. Hence $\text{Ext}^1(\text{IC}_\mu, \text{IC}_\lambda) = 0$. $\square$

If $G = T$ is a torus $P_{L+T}(\text{Gr}, \mathbb{F}_p)$ is semi-simple because then Gr is a disjoint union

of points. We are not sure when $P_{L+G}(\text{Gr}, \mathbb{F}_p)$ is semi-simple in general, but we can show that $P_{L+G}(\text{Gr}, \mathbb{F}_p)$ is not semi-simple when $G = \text{GL}_2$. To begin we have the following observation, valid for any group G .

Lemma 6.13. *Let $\mu, \lambda \in X_*(T)^+$ be such that $\lambda < \mu$ and $\text{Gr}_{\leq \lambda}$ has codimension c in $\text{Gr}_{\leq \mu}$. Then*

$$\text{Ext}_{D_c^b(\text{Gr}, \mathbb{F}_p)}^1(\text{IC}_\lambda, \text{IC}_\mu) = H_{\text{ét}}^c(\text{Gr}_{\leq \mu} - \text{Gr}_{\leq \lambda}, \mathbb{F}_p).$$

Proof. Let $i: \text{Gr}_{\leq \lambda} \rightarrow \text{Gr}_{\leq \mu}$ and $j: \text{Gr}_{\leq \mu} - \text{Gr}_{\leq \lambda} \rightarrow \text{Gr}_{\leq \mu}$ be the inclusions. By adjunction

$$\text{Ext}^1(\text{IC}_\lambda, \text{IC}_\mu) = R^0\Gamma(Ri^!\mathbb{F}_p[c+1]).$$

As $H^i(\text{Gr}_{\leq \mu}, \mathbb{F}_p) = 0$ for $i > 0$ by Theorem 6.9, then by considering the exact triangle

$$Ri_* Ri^!\mathbb{F}_p[0] \rightarrow \mathbb{F}_p[0] \rightarrow Rj_* \mathbb{F}_p[0]$$

we get

$$R^0\Gamma(Ri^!\mathbb{F}_p[c+1]) = H_{\text{ét}}^c(\text{Gr}_{\leq \mu} - \text{Gr}_{\leq \lambda}, \mathbb{F}_p).$$

□

Now let R be a k -algebra and let $f: \text{Gr}_{\leq \mu} \times \text{Spec}(R) \rightarrow \text{Gr}_{\leq \mu}$ be the projection. As $c > 1$ a similar computation as in the proof of Lemma 6.13 shows

$$\text{Ext}_{D_c^b(\text{Gr}_{\leq \mu} \times \text{Spec}(R), \mathbb{F}_p)}^1(Rf^* \text{IC}_\lambda, Rf^* \text{IC}_\mu) = H_{\text{ét}}^c((\text{Gr}_{\leq \mu} - \text{Gr}_{\leq \lambda}) \times \text{Spec}(R), \mathbb{F}_p). \quad (6.2)$$

Proposition 6.14. *Let $G = \text{GL}_2$ and $\mu = (1, -1)$ as in Example 6.3. Then*

$$\text{Ext}_{P_{L+G}(\text{Gr}, \mathbb{F}_p)}^1(\text{IC}_0, \text{IC}_\mu) \neq 0.$$

Proof. Let $n \geq 0$ be an integer so that L^+G acts on $\mathrm{Gr}_{\leq \mu}$ through the quotient L^nG . By Proposition 3.11 and (6.2), it suffices to exhibit a nonzero class $s \in H_{\text{ét}}^2(\mathrm{Gr}_{\mu}, \mathbb{F}_p)$ whose image in $H_{\text{ét}}^2(\mathrm{Gr}_{\mu} \times \mathrm{Spec}(R), \mathbb{F}_p)$ is fixed by $L^nG(R)$ for every k -algebra R . We first exhibit a subspace of $H^1(\mathrm{Gr}_{\mu}, \mathcal{O}_{\mathrm{Gr}_{\mu}})$ whose image in

$$H^1(\mathrm{Gr}_{\mu} \times \mathrm{Spec}(R), \mathcal{O}_{\mathrm{Gr}_{\mu} \times \mathrm{Spec}(R)}) = H^1(\mathrm{Gr}_{\mu}, \mathcal{O}_{\mathrm{Gr}_{\mu}}) \otimes_k R$$

is fixed pointwise by $L^nG(R)$ for every R . As $H^1(\mathrm{Gr}_{\mu}, \mathcal{O}_{\mathrm{Gr}_{\mu}})$ is an algebraic representation of L^nG , it suffices to exhibit a trivial subrepresentation of $H^1(\mathrm{Gr}_{\mu}, \mathcal{O}_{\mathrm{Gr}_{\mu}})$.

Let $\mu_1 = (1, 0)$ and $\mu_2 = (0, -1)$. Then the convolution map $m: \mathrm{Gr}_{\leq \mu_{\bullet}} \rightarrow \mathrm{Gr}_{\leq \mu}$ is an isomorphism over Gr_{μ} . Let $f: \mathrm{Gr}_{\mu} \rightarrow \mathrm{Gr}_{\mu_1}$ be the restriction of the projection map $\mathrm{Gr}_{\leq \mu_{\bullet}} \rightarrow \mathrm{Gr}_{\mu_1}$ to Gr_{μ} . The space Gr_{μ} is naturally isomorphic to $\underline{\mathrm{Spec}}_{\mathrm{Gr}_{\mu_1}}(\mathrm{Sym} \mathcal{L})$ for an invertible sheaf $\mathcal{L}$ on Gr_{μ_1} via the map f . Thus as vector spaces

$$H^1(\mathrm{Gr}_{\mu}, \mathcal{O}_{\mathrm{Gr}_{\mu}}) \cong \bigoplus_{m \geq 0} H^1(\mathrm{Gr}_{\mu_1}, \mathcal{L}^m).$$

We claim that $H^1(\mathrm{Gr}_{\mu_1}, \mathcal{L})$ is a representation of L^nG . To prove this, let $g \in L^nG(k)$, which induces an automorphism $\varphi_g: \mathrm{Gr}_{\mu} \rightarrow \mathrm{Gr}_{\mu}$. The action of g on $H^1(\mathrm{Gr}_{\mu}, \mathcal{O}_{\mathrm{Gr}_{\mu}})$ is induced by an automorphism of $f_* \mathcal{O}_{\mathrm{Gr}_{\mu}} = \bigoplus_{m \geq 0} \mathcal{L}^m$. There exist abstract isomorphisms $\mathrm{Gr}_{\mu_1} \cong \mathbb{P}^1$ and $\mathcal{L} \cong \mathcal{O}(-2)$ (see [NP01, 7.2]). Thus $\mathrm{Hom}(\mathcal{L}, \mathcal{L}^m) = 0$ for $m \geq 2$. As $H^1(\mathrm{Gr}_{\mu_1}, \mathcal{O}_{\mathrm{Gr}_{\mu_1}}) = 0$, the claim follows.

It also follows that $H^1(\mathrm{Gr}_{\mu_1}, \mathcal{L})$ is one-dimensional over k . To prove it is the trivial representation of L^nG , we first note that the action of L^nG on Gr_{μ} factors through $L^nG^{\mathrm{ad}} = L^n \mathrm{PGL}_2$. This is an extension of PGL_2 by a connected unipotent group, so every one-dimensional representation of $L^n \mathrm{PGL}_2$ is trivial.

Since Gr_{μ} has an open cover by two affines then $H^2(\mathrm{Gr}_{\mu}, \mathcal{O}_{\mathrm{Gr}_{\mu}}) = 0$. Thus by the

Artin-Schreier sequence

$$H_{\text{ét}}^2(\text{Gr}_\mu, \mathbb{F}_p) = \text{cok}(H^1(\text{Gr}_\mu, \mathcal{O}_{\text{Gr}_\mu}) \xrightarrow{F-\text{id}} H^1(\text{Gr}_\mu, \mathcal{O}_{\text{Gr}_\mu})).$$

By construction the image of $H^1(\text{Gr}_{\mu_1}, \mathcal{L}) \subset H^1(\text{Gr}_\mu, \mathcal{O}_{\text{Gr}_\mu})$ in $H_{\text{ét}}^2(\text{Gr}_\mu, \mathbb{F}_p)$ consists of equivariant extensions. It remains to show the image is nonzero.

We will prove the stronger statement that the composition

$$H^1(\text{Gr}_{\mu_1}, \mathcal{L}) \rightarrow H^1(\text{Gr}_\mu, \mathcal{O}_{\text{Gr}_\mu}) \rightarrow H_{\text{ét}}^2(\text{Gr}_\mu, \mathbb{F}_p)$$

is injective. First, note that F acts injectively on $H^1(\text{Gr}_\mu, \mathcal{O}_{\text{Gr}_\mu})$, as can be computed directly from our explicit description of $f_*\mathcal{O}_{\text{Gr}_\mu} = \bigoplus_{m \geq 0} \mathcal{L}^m$. Note also that $H^1(\text{Gr}_\mu, \mathcal{O}_{\text{Gr}_\mu})$ is graded in positive degrees, and F sends elements of degree m to elements of degree mp . It follows that the elements of degree 1 have trivial intersection with the image of $F - \text{id}$. $\square$

Chapter 7

Commutativity constraint

7.1 Beilinson-Drinfeld Grassmannians

In this section we use the Beilinson-Drinfeld Grassmannians to construct a commutativity constraint on $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$. We assume $p \nmid |\pi_1(G_{\mathrm{der}})|$ until Remark 7.12. Fix a smooth curve C over k , and let $I = \{1, \dots, n\}$ be an ordered finite index set. For a k -algebra R , let $C_R := C \times \mathrm{Spec}(R)$. If $x: \mathrm{Spec}(R) \rightarrow C$ is a morphism, let $\Gamma_x \subset C_R$ be the graph of x . Fix a trivial G -torsor $\mathcal{E}_0$ on C . We will also denote by $\mathcal{E}_0$ the base change of $\mathcal{E}_0$ to C_R for any k -algebra R . The Beilinson-Drinfeld Grassmannian Gr_{C^n} over C^n is defined by the functor

$$\mathrm{Gr}_{C^n}: R \mapsto (x, \mathcal{E}, \beta)$$

where

$$\left\{ \begin{array}{l} x = (x_1, \dots, x_n) \in C^n(R), \\ \mathcal{E} \text{ is a } G\text{-torsor on } C_R, \\ \beta: \mathcal{E}|_{C_R \setminus \cup_i \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_0|_{C_R \setminus \cup_i \Gamma_{x_i}}. \end{array} \right.$$

The functor Gr_{C^n} is represented by an ind-scheme of ind-finite type over C^n . The fiber of Gr_{C^n} over $(x_1, \dots, x_n) \in C^I(k)$ is $\prod_{i=1}^m \mathrm{Gr}$ where $\{y_1, \dots, y_m\} = \{x_1, \dots, x_n\}$ with the y_i

distinct (see [Zhu17, 3.1.13]). This relationship between Gr_{C^n} and our definition of Gr is a consequence of a theorem of Beauville and Laszlo [BL95].

For $I = \{1, 2\}$ there is a global version of the convolution diagram (6.1) whose morphisms are defined over C^2 .

$$\mathrm{Gr}_C \times \mathrm{Gr}_C \xleftarrow{p_I} \tilde{\mathcal{L}}G_{C^2} \xrightarrow{q_I} \tilde{\mathrm{Gr}}_{C^2} \xrightarrow{m_I} \mathrm{Gr}_{C^2}. \quad (7.1)$$

Before we give the functor of points of $\tilde{\mathcal{L}}G_{C^2}$, we need some notation. If $x: \mathrm{Spec}(R) \rightarrow C$ is a morphism, let $\hat{\Gamma}_x$ be the formal completion of C_R along Γ_x and let $\hat{\Gamma}_x^\circ = \hat{\Gamma}_x - \Gamma_x$. Then $\tilde{\mathcal{L}}G_{C^2}$ is the functor

$$\tilde{\mathcal{L}}G_{C^2}: R \mapsto (x, \mathcal{E}_i, \beta_i, \sigma)$$

where

$$\left\{ \begin{array}{l} x = (x_1, x_2) \in C^2(R), \\ \mathcal{E}_i \text{ is a } G\text{-torsor on } C_R \text{ for } i = 1, 2 \\ \beta_i: \mathcal{E}_i|_{C_R \setminus \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_0|_{C_R \setminus \Gamma_{x_i}} \text{ for } i = 1, 2 \\ \sigma: \mathcal{E}_0|_{\hat{\Gamma}_{x_2}} \xrightarrow{\sim} \mathcal{E}_1|_{\hat{\Gamma}_{x_2}}. \end{array} \right.$$

The map p_I forgets σ . The convolution Grassmannian $\tilde{\mathrm{Gr}}_{C^2}$ is the functor

$$\tilde{\mathrm{Gr}}_{C^2}: R \mapsto (x, \mathcal{E}_i, \beta_i)$$

where

$$\left\{ \begin{array}{l} x = (x_1, x_2) \in C^2(R), \\ \mathcal{E}_i \text{ is a } G\text{-torsor on } C_R \text{ for } i = 1, 2 \\ \beta_i: \mathcal{E}_i|_{C_R \setminus \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_{i-1}|_{C_R \setminus \Gamma_{x_i}} \text{ for } i = 1, 2. \end{array} \right.$$

The map q_I sends $(x, \mathcal{E}_i, \beta_i, \sigma)$ to $(x, \mathcal{E}'_i, \beta'_i)$ where $\mathcal{E}'_1 = \mathcal{E}_1$ and the torsor $\mathcal{E}'_2$ is obtained by gluing $\mathcal{E}_2|_{C_R \setminus \Gamma_{x_2}}$ to $\mathcal{E}'_1|_{\hat{\Gamma}_{x_2}}$ along $\sigma|_{\hat{\Gamma}_{x_2}^\circ} \circ \beta_2|_{\hat{\Gamma}_{x_2}^\circ}$. The convolution Grassmannian $\tilde{\mathrm{Gr}}_{C^2}$ is an ind-scheme of ind-finite type over C^2 . The ind-scheme $\tilde{\mathcal{L}}G_{C^2}$ is not of ind-finite type, but we

will ignore this issue as we did for LG . See [Ric14, 3.21] for how to overcome this technical issue. The convolution morphism m_I sends $(x, \mathcal{E}_i, \beta_i)$ to $(x, \mathcal{E}_2, \beta_1|_{C_R \setminus \cup_i \Gamma_{x_i}} \circ \beta_2|_{C_R \setminus \cup_i \Gamma_{x_i}})$.

There is also a global version $\mathcal{L}^+G_{C^n}$ of the positive loop group functor, which is represented by an ind-group scheme of ind-finite type over C^n (in [Ric14] this functor is denoted by $\mathcal{L}^+G_I$). The group $\mathcal{L}^+G_{C^2}$ acts on the left on $\tilde{\mathcal{L}}G_{C^2}$, $\tilde{\text{Gr}}_{C^2}$, and Gr_{C^2} . It also acts on $\text{Gr}_C \times \text{Gr}_C$ via a map $\mathcal{L}^+G_{C^2} \rightarrow (\mathcal{L}^+G_C)^2$. The morphisms in the convolution diagram (7.1) are $\mathcal{L}^+G_{C^2}$ -equivariant. We refer the reader to [Ric14] for more details on all of these functors (see also Section 8.1 for the definition of $\mathcal{L}^+G_{C^n}$ in the case $n = 1$).

For the rest of this section we specialize to the case $C = \mathbb{A}^1$, and we set $I = \{1, 2\}$. Then there are natural isomorphisms $\text{Gr}_C = C \times \text{Gr}$ and $\mathcal{L}^+G_C = C \times L^+G$. Let

$$p: \text{Gr}_C \rightarrow \text{Gr}$$

be the projection. The ind-scheme structures on $\tilde{\text{Gr}}_{C^2}$ and Gr_{C^2} are both parametrized by pairs $\mu_1, \mu_2 \in X_*(T)^+$. We denote the corresponding reduced closed subschemes of $\tilde{\text{Gr}}_{C^2}$ and Gr_{C^2} by $\tilde{\text{Gr}}_{C^2}^{\mu_\bullet}$ and $\text{Gr}_{C^2}^{\mu_\bullet}$, respectively. The scheme $\tilde{\text{Gr}}_{C^2}^{\mu_\bullet}$ is obtained by taking the reduced subscheme structure on

$$q_I(p_I^{-1}(p^{-1}(\text{Gr}_{\leq \mu_1}) \times p^{-1}(\text{Gr}_{\leq \mu_2}))) \subset \tilde{\text{Gr}}_{C^2}.$$

The scheme $\text{Gr}_{C^2}^{\mu_\bullet}$ is the image of $\tilde{\text{Gr}}_{C^2}^{\mu_\bullet}$ under m_I with the reduced subscheme structure. The map $m_I: \tilde{\text{Gr}}_{C^2}^{\mu_\bullet} \rightarrow \text{Gr}_{C^2}^{\mu_\bullet}$ is proper and birational. Our discussion in Remark 3.13 applies, and it makes sense to consider the categories

$$P_{\mathcal{L}^+G_{C^2}}(\text{Gr}_{C^2}, \mathbb{F}_p), \quad P_{\mathcal{L}^+G_{C^2}}(\tilde{\text{Gr}}_{C^2}, \mathbb{F}_p).$$

Remark 7.1. Let $U \subset C^2$ be the locus where the coordinates are pairwise distinct, and let

$\Delta: C \rightarrow C^2$ be the diagonal embedding. Then we have a diagram with Cartesian squares

$$\begin{array}{ccccc} \mathrm{Gr}_C & \xrightarrow{i_I} & \mathrm{Gr}_{C^2} & \xleftarrow{j_I} & \mathrm{Gr}_C^2 \big|_U \\ \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{\Delta} & C^2 & \xleftarrow{\quad} & U \end{array}$$

By the methods introduced by Zhu in [Zhu09, 1.2.4], one can show that for $\mu_1, \mu_2 \in X_*(T)^+$ the fiber of $\mathrm{Gr}_{C^2}^{\mu_\bullet} \rightarrow C^2$ over Δ is $C \times \mathrm{Gr}_{\leq \mu_1 + \mu_2}$. In particular this fiber is reduced, and hence the Cohen-Macaulayness of $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ follows from that of $\mathrm{Gr}_{\leq \mu_1}$, $\mathrm{Gr}_{\leq \mu_2}$, and $\mathrm{Gr}_{\leq \mu_1 + \mu_2}$. See also [Zhu17, 3.1.14].

Finally, we observe that the group $\mathbb{G}_a = \mathbb{A}^1$ also acts on $\mathrm{Gr}_C = C \times \mathrm{Gr}$. More generally, the group $\mathbb{G}_a$ acts on $\tilde{\mathcal{L}}_{C^2}$, $\tilde{\mathrm{Gr}}_{C^2}$, and Gr_{C^2} by diagonal translation on the coordinates. The action of $\mathbb{G}_a$ on these objects is relative to the ground field k instead of C^2 , and this action also preserves the schemes $\tilde{\mathrm{Gr}}_{C^2}^{\mu_\bullet}$ and $\mathrm{Gr}_{C^2}^{\mu_\bullet}$. The morphisms p_I , q_I , and m_I in the convolution diagram (7.1) are equivariant for this action of $\mathbb{G}_a$.

7.2 Construction of the commutativity constraint

With the preliminary facts in Section 7.1 we are now ready to start constructing the commutativity constraint.

Lemma 7.2 (cf. [Ric14, 3.27 (i)]). *The functor $Rp^*[1]$ induces a fully faithful functor*

$$Rp^*[1]: P_{L+G}(\mathrm{Gr}, \mathbb{F}_p) \rightarrow P_{L+G_C}(\mathrm{Gr}_C, \mathbb{F}_p)$$

whose essential image consists of those $\mathbb{G}_a$ -equivariant objects in $P_{L+G_C}(\mathrm{Gr}_C, \mathbb{F}_p)$.

Proof. By Lemma 3.6 the functor $Rp^*[1]: P_c^b(\mathrm{Gr}, \mathbb{F}_p) \rightarrow P_c^b(\mathrm{Gr}_C, \mathbb{F}_p)$ is fully faithful and has essential image consisting of $\mathbb{G}_a$ -equivariant objects in $P_c^b(\mathrm{Gr}_C, \mathbb{F}_p)$. To see that the image

of an L^+G -equivariant perverse sheaf is $\mathcal{L}^+G_C$ -equivariant, note that we have a diagram

$$\begin{array}{ccc} \mathcal{L}^+G_C \times_C \mathrm{Gr}_C & \longrightarrow & L^+G \times \mathrm{Gr} \\ \downarrow \rho_C & & \downarrow \rho \\ \mathrm{Gr}_C & \xrightarrow{p} & \mathrm{Gr} \end{array}$$

Here the horizontal maps are the projections and the vertical maps are the action maps. We can get another commutative diagram by replacing the vertical maps with the projections onto Gr_C and Gr . Now it is a simple diagram chase to show that the image of an L^+G -equivariant perverse sheaf is $\mathcal{L}^+G_C$ -equivariant. A similar diagram allows one to show that the inverse to $Rp^*[1]$ as defined in Lemma 3.6 sends $\mathcal{L}^+G_C$ -equivariant perverse sheaves to L^+G -equivariant perverse sheaves. $\square$

We denote the essential image of the functor $Rp^*[1]$ in Lemma 7.2 by $P_{\mathcal{L}^+G_C}(\mathrm{Gr}_C, \mathbb{F}_p)^{\mathbb{G}_a}$.

Lemma 7.3. *The scheme $\tilde{\mathrm{Gr}}_{C^2}^{\mu_\bullet}$ is integral, normal, and Cohen-Macaulay.*

Proof. The $\mathrm{Gr}_{\leq \mu_i}$ satisfy these properties by Theorem 5.1. Thus the scheme $\tilde{\mathrm{Gr}}_{C^2}^{\mu_\bullet}$ is normal and Cohen-Macaulay because these properties are local in the smooth topology (see the global convolution diagram (7.1)). To show that $\tilde{\mathrm{Gr}}_{C^2}^{\mu_\bullet}$ is integral, it suffices to show that $p_I^{-1}(p^{-1}(\mathrm{Gr}_{\leq \mu_1}) \times p^{-1}(\mathrm{Gr}_{\leq \mu_2}))$ is irreducible. This fact follows because $p^{-1}(\mathrm{Gr}_{\leq \mu_1}) \times p^{-1}(\mathrm{Gr}_{\leq \mu_2})$ is irreducible and p_I is a torsor for a smooth group scheme having geometrically connected fibers over C^2 (we again need to make a technical modification so that everything is of finite type). $\square$

Theorem 7.4. *The scheme $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ is integral, F -rational, and has pseudo-rational singularities.*

Proof. $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ is irreducible because it is the image of the irreducible scheme $\tilde{\mathrm{Gr}}_{C^2}^{\mu_\bullet}$. As we are always working with reduced schemes, then $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ is integral. To prove $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ is

F -rational, we first note that it suffices to show the local rings at every closed point are F -rational [HH94, 4.2]. By *loc. cit.*, F -rationality may be proved after passing to the quotient by a regular sequence. The fiber of $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ over a closed point in C^2 is either isomorphic to $\mathrm{Gr}_{\leq \mu_1} \times \mathrm{Gr}_{\leq \mu_2}$ or $\mathrm{Gr}_{\leq \mu_1 + \mu_2}$ (see Remark 7.1). These fibers are globally F -regular by Theorem 1.9 and Lemma 5.7, and in particular, F -rational. The same remark also implies that each of these fibers is cut out locally by a regular element on $\mathrm{Gr}_{C^2}^{\mu_\bullet}$, so this proves $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ is F -rational. Finally, F -rational singularities are pseudo-rational by [Smi97, 3.1]. $\square$

Proposition 7.5. *The global convolution morphism $m_I: \tilde{\mathrm{Gr}}_{C^2}^{\mu_\bullet} \rightarrow \mathrm{Gr}_{C^2}^{\mu_\bullet}$ satisfies*

$$Rm_{I,*}(\mathbb{F}_p[0]) \cong \mathbb{F}_p[0].$$

Proof. By adjunction there is a natural map $\varphi: \mathbb{F}_p[0] \rightarrow Rm_{I,*}(\mathbb{F}_p[0])$. Since m_I is proper then $Rm_{I,*}(\mathbb{F}_p[0])$ has constructible cohomology sheaves. In particular, it suffices to check φ is an isomorphism over closed points in C^2 . The fiber of the convolution morphism m_I over a closed point in the diagonal in C^2 is a convolution morphism for Gr . By Proposition 6.5 and the Artin-Schreier sequence, the convolution morphisms for Gr preserve the constant sheaf. As m_I is an isomorphism over U then we are done. $\square$

We now construct a convolution product on $P_{\mathcal{L}+G_C}(\mathrm{Gr}_C, \mathbb{F}_p)^{\mathbb{G}_a}$. Recall the notation from the global convolution diagram (7.1).

Lemma 7.6. *Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{\mathcal{L}+G_C}(\mathrm{Gr}_C, \mathbb{F}_p)^{\mathbb{G}_a}$. Then there is a unique perverse sheaf*

$$\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet \in P_{\mathcal{L}+G_{C^2}}(\tilde{\mathrm{Gr}}_{C^2}, \mathbb{F}_p)$$

such that

$$p_I^*(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet) \cong q_I^*(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet).$$

The perverse sheaf $\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet$ is $\mathbb{G}_a$ -equivariant.

Proof. The proof is analogous to that of Lemma 6.2, and makes use of Lemmas 7.2 and 7.3. The descent steps and the proof that $\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet$ is $\mathcal{L}^+ G_{C^2}$ -equivariant are the same as in the case of $\overline{\mathbb{Q}}_\ell$ -coefficients (see [Ric14, 3.20]). $\square$

Lemma 7.7. *Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{\mathcal{L}^+ G_C}(\mathrm{Gr}_C, \mathbb{F}_p)^{\mathbb{G}_a}$. Then*

$$Rm_{I,*}(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet) \in P_{\mathcal{L}^+ G_{C^2}}(\mathrm{Gr}_{C^2}, \mathbb{F}_p)^{\mathbb{G}_a}.$$

Proof. We first suppose each $\mathcal{F}_i^\bullet$ is simple and corresponds to the dominant cocharacter μ_i . Then because $\tilde{\mathrm{Gr}}_{C^2}^{\mu_\bullet}$ is Cohen-Macaulay it follows from the definitions that $\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet$ is the constant sheaf supported on $\tilde{\mathrm{Gr}}_{C^2}^{\mu_\bullet}$ (shifted by the dimension). As $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ is integral and Cohen-Macaulay (see Remark 7.1), and $Rm_{I,*}$ preserves the constant sheaf, then we are done in this case.

The fact that $Rm_{I,*}(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet)$ is perverse for general $\mathcal{F}_i^\bullet$ now follows from an induction on the lengths of these objects as in the proof of Corollary 6.6. Finally, the perverse sheaf $Rm_{I,*}(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet)$ is $\mathcal{L}^+ G_{C^2}$ -equivariant and $\mathbb{G}_a$ -equivariant because m_I is $\mathcal{L}^+ G_{C^2}$ -equivariant and $\mathbb{G}_a$ -equivariant. $\square$

The next lemma is the key result which will allow us to construct the commutativity constraint (cf. [MV07, 5.10] and [Ric14, 3.22]). Recall the notation in Remark 7.1.

Lemma 7.8. *Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{\mathcal{L}^+ G_C}(\mathrm{Gr}_C, \mathbb{F}_p)^{\mathbb{G}_a}$. Then*

$$Rm_{I,*}(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet) \cong j_{I,!*}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U).$$

Proof. First consider the case in which the $\mathcal{F}_i^\bullet$ are simple and correspond to dominant cocharacters μ_i . Then by Proposition 7.5, $Rm_{I,*}(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet)$ is the constant sheaf $\mathbb{F}_p$ on $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ shifted by the dimension. Similarly $\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U$ is the constant sheaf $\mathbb{F}_p$ on $\mathrm{Gr}_{C^2}^{\mu_\bullet}|_U$ shifted by the dimension. As $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ is integral and F -rational by Theorem 7.4 then the isomorphism

$Rm_{I,*}(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet) \cong j_{I,!*}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U)$ follows from Theorem 1.12.

The general case now follows by induction on the lengths of the perverse sheaves $\mathcal{F}_i^\bullet$. This induction is similar to the one as in Corollary 6.6 so we only state the new ingredients. First, we observe that $Rm_{I,*}(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet)$ is an extension of $\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U$ because m_I is an isomorphism over U . Next, suppose

$$0 \rightarrow \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet \rightarrow \mathcal{H}^\bullet \rightarrow 0$$

is an exact sequence in $P_c^b(\mathrm{Gr}_{C^2}^{\mu_\bullet}, \mathbb{F}_p)$. Then if $\mathcal{F}^\bullet \cong j_{I,!*}(^p j_I^* \mathcal{F}^\bullet)$ and $\mathcal{H}^\bullet \cong j_{I,!*}(^p j_I^* \mathcal{H}^\bullet)$, we also have $\mathcal{G}^\bullet \cong j_{I,!*}(^p j_I^* \mathcal{G}^\bullet)$. This follows from the characterization of intermediate extensions in Lemma 2.7. $\square$

Proposition 7.9. *Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{\mathcal{L}+G_C}(\mathrm{Gr}_C, \mathbb{F}_p)^{\mathbb{G}^a}$, and let $f_I: \mathrm{Gr}_{C^2} \rightarrow C^2$ be the structure map. Then $Rf_{I,*}(j_{I,!*}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U)) \in D_c^b(C^2, \mathbb{F}_p)$ has constant cohomology sheaves.*

Proof. We first show that the cohomology sheaves of $Rf_{I,*}(j_{I,!*}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U))$ are local systems. To begin, note that $Rf_{I,*}(j_{I,!*}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U)) \in D_c^b(C^2, \mathbb{F}_p)$ because f_I is proper. Next, note that when the $\mathcal{F}_i^\bullet$ are simple and correspond to dominant cocharacters μ_i the complex $j_{I,!*}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U)$ is the constant sheaf $\mathbb{F}_p$ supported on $\mathrm{Gr}_{C^2}^{\mu_\bullet}$ shifted by the dimension. By Remark 7.1 the fibers of $f_I: \mathrm{Gr}_{C^2}^{\mu_\bullet} \rightarrow C^2$ over closed points in C^2 are isomorphic to $\mathrm{Gr}_{\leq \mu_1} \times \mathrm{Gr}_{\leq \mu_2}$ or $\mathrm{Gr}_{\leq \mu_1 + \mu_2}$. Thus, by Theorem 6.9 and the Künneth formula we see that $Rf_{I,*}(j_{I,!*}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U))$ has constant cohomology sheaves when the $\mathcal{F}_i^\bullet$ are simple. Now the claim for general $\mathcal{F}_i^\bullet$ follows by induction on the lengths and the fact that the subcategory of $D_c^b(C^2, \mathbb{F}_p)$ consisting of objects whose cohomology sheaves are locally constant is a triangulated subcategory.

Now let $f^I: \mathrm{Gr}_C^2 \rightarrow C^2$ be the product of the projection maps $\mathrm{Gr}_C \rightarrow C$, and let

$f: \text{Gr}_C \rightarrow C$ be the case in which I is a singleton. Observe that we have an isomorphism

$$Rf_{I,*}(j_{I,!}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U))|_U \cong Rf_*^I(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet)|_U.$$

By the Künneth formula

$$Rf_*^I(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet)|_U \cong Rf_*(\mathcal{F}_1^\bullet) \boxtimes Rf_*(\mathcal{F}_2^\bullet)|_U.$$

We claim that each $Rf_*(\mathcal{F}_i^\bullet)$ has constant cohomology sheaves. To prove this, consider the Cartesian diagram

$$\begin{array}{ccc} \text{Gr}_C & \xrightarrow{p} & \text{Gr} \\ \downarrow f & & \downarrow \\ C & \longrightarrow & \text{Spec}(k) \end{array}$$

By Lemma 7.2 we can write $\mathcal{F}_i^\bullet \cong Rp^*[1]\mathcal{G}_i^\bullet$ for some $\mathcal{G}_i^\bullet \in P_{L+G}(\text{Gr}, \mathbb{F}_p)$. Thus by the proper base change theorem $Rf_*(\mathcal{F}_i^\bullet)$ has constant cohomology sheaves. Hence $Rf_*(\mathcal{F}_1^\bullet) \boxtimes Rf_*(\mathcal{F}_2^\bullet)$ also has constant cohomology sheaves, and by the above so does $Rf_{I,*}(j_{I,!}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U))|_U$.

Let $\bar{x}$ be a geometric point of U . Each $H^m(Rf_{I,*}(j_{I,!}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U)))$ corresponds to a representation of $\pi_{1,\text{ét}}(C^2, \bar{x})$. As C^2 is integral and normal the map $\pi_{1,\text{ét}}(U, \bar{x}) \rightarrow \pi_{1,\text{ét}}(C^2, \bar{x})$ is surjective [Sta21, 0BQI]. Since we already know $H^m(Rf_{I,*}(j_{I,!}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U)))|_U$ is constant, it follows that $H^m(Rf_{I,*}(j_{I,!}(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet|_U)))$ is constant as well. $\square$

Proposition 7.10. *Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{L+G}(\text{Gr}, \mathbb{F}_p)$, and let $f_I: \text{Gr}_{C^2} \rightarrow C^2$ be the structure map. Then there are canonical identifications*

- (i) $H^{m-2}(Rf_{I,*}(j_{I,!}(Rp^*[1]\mathcal{F}_1^\bullet \boxtimes Rp^*[1]\mathcal{F}_2^\bullet|_U)))|_x \cong \bigoplus_{i+j=m} R^i\Gamma(\mathcal{F}_1^\bullet) \otimes R^j\Gamma(\mathcal{F}_2^\bullet)$ for $x \in U(k)$.
- (ii) $H^{m-2}(Rf_{I,*}(j_{I,!}(Rp^*[1]\mathcal{F}_1^\bullet \boxtimes Rp^*[1]\mathcal{F}_2^\bullet|_U)))|_x \cong R^m\Gamma(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet)$ for $x \in C^2(k)$ in the image of the diagonal $\Delta: C(k) \rightarrow C^2(k)$.

Proof. If $x \in U(k)$, note that by the proper base change theorem

$$Rf_{I,*}(j_{I,!}(Rp^*[1]\mathcal{F}_1^\bullet \boxtimes Rp^*[1]\mathcal{F}_2^\bullet|_U))|_x \cong R\Gamma(\mathcal{F}_1^\bullet[1] \boxtimes \mathcal{F}_2^\bullet[1]).$$

By the Künneth formula,

$$R\Gamma(\mathcal{F}_1^\bullet[1] \boxtimes \mathcal{F}_2^\bullet[1]) \cong R\Gamma(\mathcal{F}_1^\bullet[1]) \otimes_{\mathbb{F}_p}^L R\Gamma(\mathcal{F}_2^\bullet[1]).$$

Now by taking cohomology we get the first isomorphism.

Now suppose x is in the diagonal and note that by Lemma 7.8,

$$j_{I,!}(Rp^*[1]\mathcal{F}_1^\bullet \boxtimes Rp^*[1]\mathcal{F}_2^\bullet|_U) \cong Rm_{I,*}(Rp^*[1]\mathcal{F}_1^\bullet \tilde{\boxtimes} Rp^*[1]\mathcal{F}_2^\bullet).$$

There is a diagram with Cartesian squares

$$\begin{array}{ccccc} LG \times^{L+G} \mathrm{Gr} & \xrightarrow{m} & \mathrm{Gr} & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow i & & \downarrow & & \downarrow x \\ \tilde{\mathrm{Gr}}_{C^2} & \xrightarrow{m_I} & \mathrm{Gr}_{C^2} & \xrightarrow{f_I} & C^2 \end{array}$$

Define

$$\mathcal{F}^\bullet := Ri^*[-2](Rp^*[1]\mathcal{F}_1^\bullet \tilde{\boxtimes} Rp^*[1]\mathcal{F}_2^\bullet) \in D_c^b(LG \times^{L+G} \mathrm{Gr}, \mathbb{F}_p).$$

The isomorphism in (ii) will follow from the proper base change theorem if we can produce a canonical isomorphism

$$\mathcal{F}^\bullet \cong \mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet.$$

From the definitions and the previous diagram it follows immediately that

$$Rq^*(\mathcal{F}^\bullet) \cong Rp^*(\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet).$$

Moreover, $\mathcal{F}^\bullet$ is perverse because this property is local in the smooth topology. Thus there is a natural isomorphism $\mathcal{F}^\bullet \cong \mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet$. $\square$

Theorem 7.11. *One can endow the category $(P_{L+G}(\mathrm{Gr}, \mathbb{F}_p), *)$ with the structure of a symmetric monoidal category such that the global cohomology functor $H: (P_{L+G}(\mathrm{Gr}, \mathbb{F}_p), *) \rightarrow (\mathrm{Vect}_{\mathbb{F}_p}, \otimes)$ is a symmetric monoidal functor.*

Proof. Let $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet, \mathcal{F}_3^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$. By Propositions 7.9 and 7.10 there is a canonical isomorphism

$$H(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \cong H(\mathcal{F}_1^\bullet) \otimes H(\mathcal{F}_2^\bullet).$$

There is also a canonical isomorphism

$$H(\mathrm{IC}_0) \cong \mathbb{F}_p.$$

We constructed an associativity constraint on $(P_{L+G}(\mathrm{Gr}, \mathbb{F}_p), *)$ in Theorem 6.8. We claim that H is a monoidal functor, i.e., H sends the associativity constraint on $(P_{L+G}(\mathrm{Gr}, \mathbb{F}_p), *)$ to the usual associativity constraint on $(\mathrm{Vect}_{\mathbb{F}_p}, \otimes)$.

To prove that H is a monoidal functor, we first note that Lemma 7.6 also applies in the case $I = \{1, 2, 3\}$, with essentially the same proof. We can thus form the perverse sheaf

$$\mathcal{F}_{1,2,3}^\bullet := Rp^*[1]\mathcal{F}_1^\bullet \tilde{\boxtimes} Rp^*[1]\mathcal{F}_2^\bullet \tilde{\boxtimes} Rp^*[1]\mathcal{F}_3^\bullet \in P_c^b(\tilde{\mathrm{Gr}}_{C^3}, \mathbb{F}_p).$$

We refer the reader to [Ric14] for a detailed description of the functors $\tilde{\mathrm{Gr}}_{C^3}$, $\tilde{\mathcal{L}}G_{C^3}$, etc.

We claim that $Rf_{I,*}(Rm_{I,*}(\mathcal{F}_{1,2,3}^\bullet)) \in D_c^b(C^3, \mathbb{F}_p)$ has constant cohomology sheaves. Indeed, if the $\mathcal{F}_i^\bullet$ are simple then one can show that $Rm_{I,*}(\mathcal{F}_{1,2,3}^\bullet)$ is a shifted constant sheaf by using the same methods as in the proof of Proposition 7.5. Now the fact that $Rf_{I,*}(Rm_{I,*}(\mathcal{F}_{1,2,3}^\bullet))$ has constant cohomology sheaves follows by the same arguments as in Proposition 7.9 (where we use the functor $Rm_{I,*}$ instead of $j_{I,!}$). Let $U' \subset C^3$ be the

locus where the coordinates are pairwise distinct. Then by arguments analogous to those in Proposition 7.10 we have

- (i) $H^{m-3}(Rf_{I,*}(Rm_{I,*}(\mathcal{F}_{1,2,3}^\bullet)))|_x \cong \bigoplus_{i+j+k=m} R^i\Gamma(\mathcal{F}_1^\bullet) \otimes R^j\Gamma(\mathcal{F}_2^\bullet) \otimes R^k\Gamma(\mathcal{F}_3^\bullet)$ for $x \in U'(k)$.
- (ii) $H^{m-3}(Rf_{I,*}(Rm_{I,*}(\mathcal{F}_{1,2,3}^\bullet)))|_x \cong R^m\Gamma(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet * \mathcal{F}_3^\bullet)$ for $x \in C^3(k)$ in the image of the diagonal $C(k) \rightarrow C^3(k)$.

Now we conclude that H is monoidal by noting that the associativity constraint in Theorem 6.8 arises from the associativity isomorphism

$$(Rp^*[1]\mathcal{F}_1^\bullet \tilde{\boxtimes} Rp^*[1]\mathcal{F}_2^\bullet) \tilde{\boxtimes} Rp^*[1]\mathcal{F}_3^\bullet \cong Rp^*[1]\mathcal{F}_1^\bullet \tilde{\boxtimes} (Rp^*[1]\mathcal{F}_2^\bullet \tilde{\boxtimes} Rp^*[1]\mathcal{F}_3^\bullet)$$

by restriction to the diagonal in C^3 .

For the rest of the proof we let $I = \{1, 2\}$. To construct the commutativity constraint, let $\tau: I \rightarrow I$ be the bijection which swaps coordinates. Then τ induces morphisms of C^2 and Gr_{C^2} by permuting the coordinates. We have a commutative diagram

$$\begin{array}{ccccc} \text{Gr}_C & \xrightarrow{i_I} & \text{Gr}_{C^2} & \xleftarrow{j_I} & \text{Gr}_{C^2}|_U \\ \text{id} \downarrow & & \downarrow \tau & & \downarrow \tau \\ \text{Gr}_C & \xrightarrow{i_I} & \text{Gr}_{C^2} & \xleftarrow{j_I} & \text{Gr}_{C^2}|_U \end{array}$$

Observe that

$$R\tau^*(j_{I,!}*(Rp^*[1]\mathcal{F}_1^\bullet \boxtimes Rp^*[1]\mathcal{F}_2^\bullet)|_U) \cong j_{I,!}*(Rp^*[1]\mathcal{F}_2^\bullet \boxtimes Rp^*[1]\mathcal{F}_1^\bullet)|_U).$$

Combining this with commutativity of the left side of the diagram we get

$$\begin{aligned} Ri_I^*(j_{I,!}*(Rp^*[1]\mathcal{F}_1^\bullet \boxtimes Rp^*[1]\mathcal{F}_2^\bullet|_U)) &\cong Ri_I^*R\tau^*(j_{I,!}*(Rp^*[1]\mathcal{F}_1^\bullet \boxtimes Rp^*[1]\mathcal{F}_2^\bullet|_U)) \\ &\cong Ri_I^*(j_{I,!}*(Rp^*[1]\mathcal{F}_2^\bullet \boxtimes Rp^*[1]\mathcal{F}_1^\bullet|_U)). \end{aligned}$$

By combining this with Proposition 7.10 (ii) we get an isomorphism

$$\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet \cong \mathcal{F}_2^\bullet * \mathcal{F}_1^\bullet.$$

However, we need to modify this commutativity constraint. The problem is that the commutativity constraint on $(P_{L+G}(\text{Gr}, \mathbb{F}_p), *)$ is constructed from the commutativity of the derived tensor product $\otimes^L$. For two chain complexes $\mathcal{C}^\bullet, \mathcal{D}^\bullet$ the usual isomorphism $(\mathcal{C}^\bullet \otimes \mathcal{D}^\bullet)^\bullet \cong (\mathcal{D}^\bullet \otimes \mathcal{C}^\bullet)^\bullet$ sends

$$c_i \otimes d_j \mapsto (-1)^{i+j} d_j \otimes c_i$$

where c_i has degree i and d_j has degree j . Thus, as it stands now the functor H sends the commutativity constraint on $(P_{L+G}(\text{Gr}, \mathbb{F}_p), *)$ to the commutativity constraint on $(\text{Vect}_{\mathbb{F}_p}, \otimes)$ up to some possible sign changes.

We can remedy the situation by the same method as in [MV07], [Ric14]. More specifically, suppose $\mathcal{F}_i^\bullet \in P_{L+G}(\text{Gr}, \mathbb{F}_p)$ for $i \in \{1, 2\}$ is supported on a component of Gr whose L^+G orbits have dimensions of parity $p(i) \in \{1, -1\}$. Then we modify the commutativity isomorphism $\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet \xrightarrow{\sim} \mathcal{F}_2^\bullet * \mathcal{F}_1^\bullet$ by multiplication by $(-1)^{p(1)p(2)}$. Making use of Corollary 6.10 one can check that H sends this modified commutativity constraint to the usual constraint on $(\text{Vect}_{\mathbb{F}_p}, \otimes)$.

To show that $(P_{L+G}(\text{Gr}, \mathbb{F}_p), *)$ is a symmetric monoidal category, we need to verify certain coherence conditions such as the hexagon axiom. By Theorem 6.11, H is faithful, so

these conditions may be checked after applying H . As H sends our constraints to the usual constraints on $(\text{Vect}_{\mathbb{F}_p}, \otimes)$, the coherence conditions are satisfied. Thus $(P_{L+G}(\text{Gr}, \mathbb{F}_p), *)$ is a symmetric monoidal category and H is a symmetric monoidal functor. $\square$

We can now prove Theorem 1.1.

Proof of Theorem 1.1. Combining Theorems 6.11 and 7.11 the functor H is an exact, faithful, $\mathbb{F}_p$ -linear symmetric monoidal functor. Moreover, the identity object $\mathbf{1} \in P_{L+G}(\text{Gr}, \mathbb{F}_p)$ satisfies $\text{End}(\mathbf{1}) = \mathbb{F}_p$ because $\mathbf{1}$ is the constant sheaf $\mathbb{F}_p[0]$ supported on a point. Now we conclude by noting that in their proof of the main result on neutral Tannakian categories (see the paragraph before Remark 2.17 in [DM82]) Deligne and Milne observed that if one removes the rigidity hypothesis then one gets an equivalence with the category of representations of an affine monoid scheme. As in the rigid case, the monoid scheme M_G represents the functor $\underline{\text{End}}^\otimes(H)$ of endomorphisms of the fiber functor. $\square$

Remark 7.12. So far we have proved Theorems 1.1, 1.2, and 1.10 under the assumption $p \nmid |\pi_1(G_{\text{der}})|$. We now explain how to remove this hypothesis.

By [MS82, 3.1] we may let

$$1 \rightarrow N \rightarrow G' \rightarrow G \rightarrow 1$$

be a central extension of split reductive groups over k such that G'_{der} is simply connected and N is a torus. Let $T' \subset B' \subset G'$ be the maximal torus and Borel subgroup given by the preimages of T and B . Denote by $\text{Gr}_{G'}$ and Gr_G the affine Grassmannians for G' and G respectively, and by $\mathcal{F}\ell_{G'}$ and $\mathcal{F}\ell_G$ the affine flag varieties.

Since N is a torus the map $\pi_1(G') \rightarrow \pi_1(G)$ is surjective. Thus by [HV18, 3.1] the map $G' \rightarrow G$ induces a surjection $\varphi: \text{Gr}_{G'} \rightarrow \text{Gr}_G$ such that each connected component of $\text{Gr}_{G'}$ maps onto its image via a universal homeomorphism. This map also sends L^+G' orbits onto L^+G orbits. The analogous facts hold for $\mathcal{F}\ell_{G'} \rightarrow \mathcal{F}\ell_G$.

By the topological invariance of the étale site [Sta21, 04DY] we may therefore deduce Theorem 1.10 for Gr_G and $\mathcal{F}\ell_G$ from the corresponding properties for $\mathrm{Gr}_{G'}$ and $\mathcal{F}\ell_{G'}$. To prove Theorems 1.1 and 1.2, note that $G' \rightarrow G$ also induces a map $LG' \times^{L^+G'} \mathrm{Gr}_{G'} \rightarrow LG \times^{L^+G} \mathrm{Gr}_G$. There is a commutative diagram (see [Zhu17, 1.2.14])

$$\begin{array}{ccc} LG' \times^{L^+G'} \mathrm{Gr}_{G'} & \xrightarrow{\sim} & \mathrm{Gr}_{G'}^2 \\ \downarrow & & \downarrow \varphi \times \varphi \\ LG \times^{L^+G} \mathrm{Gr}_G & \xrightarrow{\sim} & \mathrm{Gr}_G^2 \end{array}$$

Thus the map $LG' \times^{L^+G'} \mathrm{Gr}_{G'} \rightarrow LG \times^{L^+G} \mathrm{Gr}_G$ is also surjective and restricts to a universal homeomorphism on connected components. The analogous facts hold for the n -fold convolution Grassmannians.

Let $C = \mathbb{A}^1$, and let $\mathrm{Gr}_{C^2}^{G'}$ and $\mathrm{Gr}_{C^2}^G$ be the Beilinson-Drinfeld Grassmannians over C^2 for G' and G . The map $G' \rightarrow G$ induces a map $\mathrm{Gr}_{C^2}^{G'} \rightarrow \mathrm{Gr}_{C^2}^G$. By Remark 7.1 there is a diagram (7.2) with Cartesian squares from which we deduce that $\mathrm{Gr}_{C^2}^{G'} \rightarrow \mathrm{Gr}_{C^2}^G$ is surjective and restricts to a universal homeomorphism on connected components. The analogous facts are also true for the global convolution Grassmannians.

$$\begin{array}{ccccc} C \times \mathrm{Gr}_{G'} & \longrightarrow & \mathrm{Gr}_{C^2}^{G'} & \longleftarrow & (C \times \mathrm{Gr}_{G'})^2|_U \\ \downarrow \mathrm{id}_C \times \varphi & & \downarrow & & \downarrow (\mathrm{id}_C \times \varphi)^2|_U \\ C \times \mathrm{Gr}_G & \longrightarrow & \mathrm{Gr}_{C^2}^G & \longleftarrow & (C \times \mathrm{Gr}_G)^2|_U \\ \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{\Delta} & C^2 & \longleftarrow & U \end{array} \tag{7.2}$$

Now we may use the same arguments as in Chapters 6 and 7 to prove Theorems 1.1 and 1.2 in the case $p \mid |\pi_1(G_{\mathrm{der}})|$. The idea is that any time we need to prove a complex is perverse, simple, an intermediate extension, etc., we use the topological invariance of the étale site and appeal to the corresponding fact for G' .

7.3 Properties of the monoid M_G

For the rest of this section we make no assumption about $|\pi_1(G_{\text{der}})|$. We conclude this section with a few results about M_G .

Proposition 7.13. *The affine monoid scheme M_G in Theorem 1.1 is a group scheme if and only if G is a torus.*

Proof. If M_G is a group scheme then for each $\mu \in X_*(T)^+$ there is some $\text{IC}_\mu^* \in P_{L+G}(\text{Gr}, \mathbb{F}_p)$ which is dual to IC_μ . The perverse sheaf IC_μ^* must also be simple, so $\text{IC}_\mu^* = \text{IC}_\lambda$ for some $\lambda \in X_*(T)^+$. We have seen that $\text{IC}_\mu * \text{IC}_\lambda \cong \text{IC}_{\mu+\lambda}$, so we must have $\mu + \lambda = 0$. In particular, $X_*(T)^+$ is a group. This implies $G = T$.

Conversely, if $G = T$ then the corresponding (reduced) affine Grassmannian consists of a disjoint union of points indexed by $X_*(T)$. There are then no non-trivial extensions between objects of $P_{L+T}(\text{Gr}, \mathbb{F}_p)$, and it is straightforward to check that

$$P_{L+T}(\text{Gr}, \mathbb{F}_p) \cong \text{Rep}_{\mathbb{F}_p}(\hat{T}).$$

Here $\hat{T}$ is the torus defined over $\mathbb{F}_p$ with root datum dual to that of T . □

Let $\text{Spec}(\mathbb{F}_p[X_*(T)^+])$ be the affine monoid scheme with multiplication defined by the map

$$\mathbb{F}_p[X_*(T)^+] \rightarrow \mathbb{F}_p[X_*(T)^+] \otimes_{\mathbb{F}_p} \mathbb{F}_p[X_*(T)^+], \quad \mu \mapsto \mu \otimes \mu, \quad \mu \in X_*(T)^+.$$

Proposition 7.14. *There is a morphism of monoids*

$$M_G \rightarrow \text{Spec}(\mathbb{F}_p[X_*(T)^+]).$$

This is an isomorphism if G is a torus.

Proof. Let $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}} \subset P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ be the full subcategory consisting of semi-simple objects. By Theorem 1.2, $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}}$ is a tensor subcategory. The same Tannakian formalism applies to $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}}$, and we get an equivalence

$$P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}} \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{F}_p}(\mathrm{Spec}(\mathbb{F}_p[X_*(T)^+])).$$

The inclusion $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}} \rightarrow P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ then induces the desired morphism of monoid schemes. It is an isomorphism if G is a torus by the proof of Proposition 7.13. $\square$

Proposition 7.15. *There exists an isomorphism*

$$M_G = \varprojlim_i M_i$$

where each M_i is a closed submonoid of a monoid of upper triangular matrices.

Proof. Let V be a finite-dimensional representation of M_G . As every simple object of $\mathrm{Rep}_{\mathbb{F}_p}(M_G)$ is one-dimensional then one can pick a basis of V for which the image of $M_G \rightarrow \mathrm{End}(V)$ consists of upper triangular matrices. Next, by a standard argument identical to that for affine group schemes, M_G is the inverse limit of its finite-type quotients. Now we conclude by noting that, as for affine group schemes, every affine monoid scheme of finite type admits a faithful finite-dimensional representation. $\square$

Chapter 8

Construction of the functor $\mathcal{Z}$

8.1 Global affine Schubert varieties

We continue using the notation introduced in Section 5.2. Let $C = \mathbb{A}_k^1$. We will denote by 0 the origin viewed as a closed point in C . Let $\hat{\mathcal{O}}_0$ be the completed local ring of C at 0 and let $C^\circ = C - 0$. Let $\mathcal{G}$ be a Bruhat-Tits group scheme over C equipped with isomorphisms

$$\mathcal{G}|_{C^\circ} \cong G \times C^\circ, \quad L^+(\mathcal{G}|_{\hat{\mathcal{O}}_0}) \cong I. \quad (8.1)$$

See [Zhu14, §3.2] for more information on the construction of $\mathcal{G}$.

For any smooth group scheme H over C (including $\mathcal{G}$) we let $\mathcal{E}_0$ be H regarded as a trivial H -torsor. For a k -algebra R let $C_R = C \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$. If $x \in C(R)$ let $\Gamma_x \subset C_R$ be the graph of x , i.e., the closed subscheme $\mathrm{Spec}(R) \xrightarrow{(x, \mathrm{id})} C \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$. The global affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ is the functor on k -algebras defined by

$$\mathrm{Gr}_{\mathcal{G}}(R) = \left\{ (x, \mathcal{E}, \beta) \mid x \in C(R), \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on } C_R, \beta: \mathcal{E}|_{C_R - \Gamma_x} \cong \mathcal{E}_0|_{C_R - \Gamma_x} \right\}.$$

In the above definition we really mean the set of such objects up to the equivalence relation

$(x, \mathcal{E}, \beta) \sim (x, \mathcal{E}', \beta')$ if there is an isomorphism $\mathcal{E} \cong \mathcal{E}'$ which respects the trivializations, but we will suppress this detail. The functor $\mathrm{Gr}_{\mathcal{G}}$ is represented by an ind-projective scheme over C by [PZ13, 5.5].

Given $x \in C(R)$ let $\hat{\Gamma}_x$ be the formal completion of C_R along Γ_x and let $\hat{\Gamma}_x^\circ = \hat{\Gamma}_x - \Gamma_x$. The global loop group of $\mathcal{G}$ is the functor

$$\mathcal{LG}(R) = \{(x, \beta) : x \in C(R), \beta \in \mathcal{G}(\hat{\Gamma}_x^\circ)\}.$$

The global positive loop group of $\mathcal{G}$ is the functor

$$\mathcal{L}^+\mathcal{G}(R) = \{(x, \beta) : x \in C(R), \beta \in \mathcal{G}(\hat{\Gamma}_x)\}.$$

As in [Zhu14, 3.1], a lemma of Beauville-Laszlo [BL95] implies that there is a natural bijection

$$\mathrm{Gr}_{\mathcal{G}}(R) \cong \left\{ (x, \mathcal{E}, \beta) \mid x \in C(R), \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on } \hat{\Gamma}_x, \beta : \mathcal{E}|_{\hat{\Gamma}_x^\circ} \cong \mathcal{E}_0|_{\hat{\Gamma}_x^\circ} \right\}.$$

Then $\mathcal{L}^+\mathcal{G}$ acts on $\mathrm{Gr}_{\mathcal{G}}$ by changing the trivialization β , and there is an isomorphism

$$\mathcal{LG}/\mathcal{L}^+\mathcal{G} \cong \mathrm{Gr}_{\mathcal{G}}.$$

Our choice of isomorphisms in (8.1) induces isomorphisms

$$\mathrm{Gr}_{\mathcal{G}}|_{C^\circ} \cong \mathrm{Gr} \times C^\circ, \quad (\mathrm{Gr}_{\mathcal{G}})_0 \cong \mathcal{F}\ell \tag{8.2}$$

and

$$\mathcal{L}^+\mathcal{G}|_{C^\circ} \cong L^+G \times C^\circ, \quad (\mathcal{L}^+\mathcal{G})_0 \cong I. \tag{8.3}$$

Via the isomorphisms (8.2) and (8.3), the action of $\mathcal{L}^+\mathcal{G}$ on $\mathrm{Gr}_{\mathcal{G}}$ is compatible with the

action of L^+G on Gr and the action of I on $\mathcal{F}\ell$.

We define $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ to be the reduced closure of $\mathrm{Gr}_{\leq\mu} \times C^\circ$ in $\mathrm{Gr}_{\mathcal{G}}$. The scheme $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is stable under the action of $\mathcal{L}^+\mathcal{G}$, and our definition agrees with that in [Zhu14, 3.1] because G is split. We can now define the local model M_μ .

Definition 8.1. The local model M_μ is the fiber of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ over the completed local ring at $0 \in C$.

Thus the generic fiber of M_μ is isomorphic to $\mathrm{Gr}_{\leq\mu} \times \mathrm{Spec}(k((t)))$. We now give the definition of the μ -admissible set appearing in Theorem 1.3. Given $\mu \in X_*(T)$ let $\Lambda_\mu = W \cdot \mu \subset X_*(T)$ be the orbit of μ in $X_*(T)$ under the action of the Weyl group $W := N_G(T)/T$. The μ -admissible set is

$$\mathrm{Adm}(\mu) := \{w \in \tilde{W} \mid w \leq t_\lambda \text{ for some } \lambda \in \Lambda_\mu\}. \quad (8.4)$$

The following theorem is due to Zhu in the case G_{der} is absolutely almost simple and simply connected and was extended by Haines-Richarz to the general case.

Theorem 8.2 ([Zhu14, Thm. 3], [HR20, 5.14], [HR19, 2.1]). *If $p \nmid |\pi_1(G_{\mathrm{der}})|$ the fiber $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0$ is reduced. Without any assumptions on p , the reduced fiber satisfies*

$$(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{0,\mathrm{red}} \cong \mathcal{A}(\mu).$$

For each integer $n > 0$ let $\Gamma_{x,n}$ be the n -th nilpotent thickening of Γ_x . The n -th jet group of $\mathcal{G}$ is

$$\mathcal{L}_n^+\mathcal{G}(R) = \{(x, \beta) : x \in C(R), \beta \in \mathcal{G}(\Gamma_{x,n})\}.$$

This functor is represented by a smooth affine group scheme over C . For each $\mu \in X_*(T)^+$ the action of $\mathcal{L}^+\mathcal{G}$ on $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ factors through $\mathcal{L}_n^+\mathcal{G}$ for sufficiently large n depending on μ . If $x \in C^\circ(k)$ then $(\mathcal{L}_n^+\mathcal{G})_x \cong L^n G$, and $(\mathcal{L}_n^+\mathcal{G})_0 \cong L^n(\mathcal{G}|_{\hat{\mathcal{O}}_0})$.

Finally, let $\underline{G}$ be the constant group scheme $G \times C$. By replacing $\mathcal{G}$ with $\underline{G}$ in the above definitions we get the ind-scheme $\mathrm{Gr}_{\underline{G}}$, which is naturally isomorphic to $\mathrm{Gr} \times C$ and was denoted by Gr_C in Section 7.1. There is a natural morphism $\mathcal{G} \rightarrow \underline{G}$ which induces a morphism $\pi_{\mathcal{G}}: \mathrm{Gr}_{\mathcal{G}} \rightarrow \mathrm{Gr}_{\underline{G}}$. By taking the fibers of $\pi_{\mathcal{G}}$ over C° and 0 we get the following diagram with Cartesian squares:

$$\begin{array}{ccccc} \mathrm{Gr}_{\mathcal{G}}|_{C^\circ} & \xrightarrow{j_{\mathcal{G}}} & \mathrm{Gr}_{\mathcal{G}} & \xleftarrow{i_{\mathcal{G}}} & \mathcal{F}\ell \\ \downarrow \sim & & \downarrow \pi_{\mathcal{G}} & & \downarrow \pi \\ \mathrm{Gr}_{\underline{G}}|_{C^\circ} & \xrightarrow{j_{\underline{G}}} & \mathrm{Gr}_{\underline{G}} & \xleftarrow{i_{\underline{G}}} & \mathrm{Gr} \end{array}$$

8.2 The definition of $\mathcal{Z}$

For this section we assume that k is an algebraically closed field of characteristic $p > 0$. Suppose that X is a separated scheme of finite type over C . Let $j: X^\circ \rightarrow X$ be the inclusion of the fiber of X over C° and let $i: Z \rightarrow X$ be the inclusion of the fiber of X over 0. For $\mathcal{F}^\bullet \in P_c^b(X^\circ, \mathbb{F}_p)$ we define the nearby cycles of $\mathcal{F}^\bullet$ by

$$\Psi_X(\mathcal{F}^\bullet) := Ri^*(j_{!*}\mathcal{F}^\bullet)[-1].$$

This defines an additive functor

$$\Psi_X: P_c^b(X^\circ, \mathbb{F}_p) \rightarrow D_c^b(Z, \mathbb{F}_p).$$

Proposition 8.3. *Suppose $f: X' \rightarrow X$ is a smooth, separated morphism over C of relative dimension d and that X' has fibers X'° and Z' over C° and 0, respectively. Then there is a natural isomorphism of functors*

$$R(f|_{Z'})^*[d] \circ \Psi_X \cong \Psi_{X'} \circ R(f|_{X'^\circ})^*[d]: P_c^b(X^\circ, \mathbb{F}_p) \rightarrow D_c^b(Z', \mathbb{F}_p).$$

Proof. This follows from the fact that pullback along a smooth morphism commutes with taking intermediate extensions by Theorem 2.16. $\square$

Remark 8.4. If $f: X' \rightarrow X$ is a proper morphism then the functor $Rf_!$ does not preserve perversity in general. However, if $\mathcal{F}^\bullet \in P_c^b(X'^\circ, \mathbb{F}_p)$ and $R(f|_{X'^\circ})_!(\mathcal{F}^\bullet)$ happens to be perverse, then it makes sense to ask whether $R(f|_{Z'})_!(\Psi_{X'}(\mathcal{F}^\bullet))$ and $\Psi_X(R(f|_{X'^\circ})_!(\mathcal{F}^\bullet))$ are isomorphic. We will see examples (e.g., the proof of Theorem 1.3 (ii)) where the presence of F -rational singularities allows us to prove there is such an isomorphism, but we are unsure about the general case. By [SGA 7 II, Exp. XIII 2.1.7.1], the analogous fact is true for the usual definition of the nearby cycles functor (see Remark 8.7) due to the proper base change theorem.

Remark 8.5. We note that by Lemma 2.7 (ii), $\Psi_X(\mathcal{F}^\bullet) \in {}^pD^{\leq 0}(Z, \mathbb{F}_p)$. By [BBD82, 4.4.2], the same is true for $\mathbb{F}_\ell$ -sheaves with $\ell \neq p$ using the usual definition of the nearby cycles functor. As the usual nearby cycles functor commutes with Verdier duality for $\mathbb{F}_\ell$ -sheaves by [Ill94, 4.2], then it preserves perversity. There is no duality functor for $\mathbb{F}_p$ -sheaves, and we do not know if $\Psi_X(\mathcal{F}^\bullet)$ is always perverse, but it is perverse in all of the examples we have computed.

By the following lemma, we can naturally extend the definition of the nearby cycles functor to the ind-scheme $\mathrm{Gr}_{\mathcal{G}}$.

Lemma 8.6. *Suppose $h: X' \rightarrow X$ is a closed immersion of separated C -schemes of finite type and that X' has fibers X'° and Z' over C° and 0 , respectively. Then there is a natural isomorphism of functors*

$$R(h|_{Z'})_* \circ \Psi_{X'} \cong \Psi_X \circ R(h|_{X'^\circ})_*: P_c^b(X'^\circ, \mathbb{F}_p) \rightarrow D_c^b(Z, \mathbb{F}_p).$$

Proof. By taking the fibers of h over C° and 0 we get the following diagram with Cartesian

squares:

$$\begin{array}{ccccc} X'^{\circ} & \xrightarrow{j'} & X' & \xleftarrow{i'} & Z' \\ \downarrow & & \downarrow h & & \downarrow \\ X^{\circ} & \xrightarrow{j} & X & \xleftarrow{i} & Z \end{array}$$

Because the intermediate extension functor agrees with the usual pushforward functor for closed immersions then by Lemma 2.6 we have

$$Rh_* \circ j'_{!*} \cong j_{!*} \circ R(h|_{X'^{\circ}})_*.$$

Now the lemma follows by applying Ri^* and the proper base change theorem. $\square$

Remark 8.7. Following [SGA 7 II, Exp. XIII §1.3], there is another nearby cycles functor Ψ'_X defined as follows. Replace C with the henselization of its local ring at 0. Let $\bar{\eta}$ be the spectrum of an algebraic closure of the function field of C , and let $\bar{C}$ be the normalization of C in $\bar{\eta}$. Let $X_{\bar{C}} = X \times_C \bar{C}$ and $X_{\bar{\eta}} = X \times_C \bar{\eta}$. Then there are natural morphisms $\bar{j}: X_{\bar{\eta}} \rightarrow X_{\bar{C}}$ and $\bar{i}: Z \rightarrow X_{\bar{C}}$. If $\mathcal{F}^\bullet|_{\bar{\eta}}$ is the restriction of $\mathcal{F}^\bullet \in D_c^b(X^\circ, \mathbb{F}_p)$ to $X_{\bar{\eta}}$, one can define

$$\Psi'_X(\mathcal{F}^\bullet) := Ri^* R\bar{j}_*(\mathcal{F}^\bullet|_{\bar{\eta}}).$$

For $\mathbb{F}_\ell$ -sheaves with $\ell \neq p$, the functor Ψ'_X preserves constructibility by [SGA 4 $\frac{1}{2}$, Exp. 7, Th. 3.2] and commutes with smooth base change as in Proposition 8.3 by [SGA 7 II, Exp. XIII 2.1.7]. The proofs make essential use of the hypothesis $\ell \neq p$, and part of our original motivation for defining Ψ_X was to give short proofs of these two facts for $\mathbb{F}_p$ -sheaves. It would be interesting to compare the functors Ψ_X and Ψ'_X for $\mathbb{F}_p$ -sheaves.

Recall that by Theorem 1.10, the simple objects in $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ and $P_I(\mathcal{F}\ell, \mathbb{F}_p)$ are the shifted constant sheaves

$$\mathrm{IC}_\mu := \mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu}] \in D_c^b(\mathrm{Gr}_{\leq \mu}, \mathbb{F}_p), \quad \mathrm{IC}_w^{\mathcal{F}\ell} := \mathbb{F}_p[\dim \mathcal{F}\ell_w] \in D_c^b(\mathcal{F}\ell_w, \mathbb{F}_p),$$

for $\mu \in X_*(T)^+$, $w \in \tilde{W}$.

Let $\mathcal{F}^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)^{\mathrm{ss}}$. Since C is smooth then $\mathcal{F}^\bullet \boxtimes^L \mathbb{F}_p[1] \in P_c^b(\mathrm{Gr} \times C, \mathbb{F}_p)$ by Lemma 2.15. Via the isomorphism $\mathrm{Gr}_{\mathcal{G}}|_{C^\circ} \cong \mathrm{Gr} \times C^\circ$ we view $\mathcal{F}^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}$ as a perverse sheaf on $\mathrm{Gr}_{\mathcal{G}}|_{C^\circ}$. We set

$$\mathcal{Z}(\mathcal{F}^\bullet) := \Psi_{\mathrm{Gr}_{\mathcal{G}}}(\mathcal{F}^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \in D_c^b(\mathcal{F}\ell, \mathbb{F}_p).$$

In Section 6.2 we defined the convolution product $\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet$ of $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$. We now define the convolution product of two perverse sheaves in $P_I(\mathcal{F}\ell, \mathbb{F}_p)$. Since the situation is analogous to $P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ we will be brief. To begin, we have the convolution diagram

$$\mathcal{F}\ell \times \mathcal{F}\ell \xleftarrow{p} LG \times \mathcal{F}\ell \xrightarrow{q} LG \times^I \mathcal{F}\ell \xrightarrow{m} \mathcal{F}\ell. \quad (8.5)$$

Here p is the quotient map $LG \rightarrow \mathcal{F}\ell$ on the first factor and the identity map on the second factor. The map q is the quotient by the diagonal action of I given by $g \cdot (g_1, g_2) = (g_1 g^{-1}, g g_2)$, and m is the multiplication map. We will also use the notation $\mathcal{F}\ell \tilde{\times} \mathcal{F}\ell$ for $LG \times^I \mathcal{F}\ell$.

Given $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_I(\mathcal{F}\ell, \mathbb{F}_p)$ we claim that there is a unique perverse sheaf $\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet \in P_I(LG \times^I \mathcal{F}\ell, \mathbb{F}_p)$ such that $Rp^*(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet) \cong Rq^*(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet)$. The proof of this is analogous to the case of the affine Grassmannian in Lemma 6.2 so we omit it. We are also suppressing the fact that because $LG \times \mathcal{F}\ell$ is not of ind-finite type, we must replace the I -torsors p and q by torsors for a finite type quotient of I depending on the support of $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$.

The convolution of $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ is

$$\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet = Rm_!(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet) \in D_c^b(\mathcal{F}\ell, \mathbb{F}_p).$$

We may also write Rm_* instead of $Rm_!$ because $\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet$ is supported on a proper scheme. As in the case of $\overline{\mathbb{Q}}_\ell$ -coefficients, $\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet$ is not perverse in general. However, if $\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet$ is perverse then it is also I -equivariant by Lemma 3.2.

Remark 8.8. Using the method in Remark 3.13 we can define the category $P_{\mathcal{L}+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \mathbb{F}_p)$ of $\mathcal{L}^+\mathcal{G}$ -equivariant perverse $\mathbb{F}_p$ -sheaves on $\mathrm{Gr}_{\mathcal{G}}$. By the same reasoning we can define other categories of equivariant perverse sheaves on ind-schemes we will introduce later, such as $P_{\mathcal{L}+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}}, \mathbb{F}_p)$ (see Section 9.1). Since $\mathcal{L}_n^+\mathcal{G} \rightarrow C$ has geometrically connected fibers for every n then $P_{\mathcal{L}+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \mathbb{F}_p)$ is a full subcategory of $P_c^b(\mathrm{Gr}_{\mathcal{G}}, \mathbb{F}_p)$.

8.3 First properties of $\mathcal{Z}$

In this section we prove parts (i) and (ii) of Theorem 1.3. The main ingredient will be the F -rationality of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$. We assume that k is a perfect field of characteristic $p > 2$ until Proposition 8.16, where we require k to be algebraically closed. Throughout this section we also assume that G_{der} is absolutely almost simple and simply connected. We will explain how to remove the simple connectedness hypothesis in Remark 9.9. To begin, we recall the following results.

Theorem 8.9 ([PZ13, 9.1], [HR19, 2.1]). *The schemes $\overline{\mathrm{Gr}}_{\underline{G},\mu}$ and $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ are integral, normal, and Cohen-Macaulay.*

Corollary 8.10. *The fiber $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0$ is Cohen-Macaulay, connected, and equidimensional of dimension equal to that of $\mathrm{Gr}_{\leq\mu}$.*

Proof. Since $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is Cohen-Macaulay and integral then $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0$ is also Cohen-Macaulay by [Sta21, OC6G]. Note that the morphism $\overline{\mathrm{Gr}}_{\mathcal{G},\mu} \rightarrow C$ is flat by [Har77, III 9.7]. Now because the generic fiber of $\overline{\mathrm{Gr}}_{\mathcal{G},\mu} \rightarrow C$ is connected, then so is $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0$ by [EGA IV₃, 15.5.4]. Finally, $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0$ is equidimensional because it is Cohen-Macaulay and connected, and $\dim(\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0 = \dim \mathrm{Gr}_{\leq\mu}$ by [Har77, III 9.6]. $\square$

The following lemmas (8.11, 8.12, 8.13) are well-known to experts, but because we could not find detailed proofs in the literature we provide them here. These three lemmas are also valid if $p = 2$.

Lemma 8.11. *Let A and B be domains that are strongly F -regular k -algebras of finite type. Then if $A \otimes_k B$ is a domain it is strongly F -regular.*

Proof. This is proven in [Has03, 5.2] when A and B are graded rings, but the same proof works in general. Let $R = A \otimes_k B$ and let $a \in A$ and $b \in B$ be such that the localizations A_a and B_b are smooth. Then $A_a \otimes_k B_b$ is smooth and hence strongly F -regular by [HH89, 3.1 (c)], so by [HH89, 3.3 (a)] it suffices to construct a splitting of $R[(a \otimes b)^{1/q}] \subset R^{1/q}$ for some $q = p^e$. Because k is perfect, such a splitting can be constructed from splittings of $A[a^{1/q}] \subset A^{1/q}$ and $B[b^{1/q}] \subset B^{1/q}$, which exist for some common q because A_a and B_b are strongly F -regular. $\square$

Lemma 8.12. *Let R be a domain that is a strongly F -regular k -algebra of finite type and let E be an F -finite field containing k . Then if $R \otimes_k E$ is a domain it is strongly F -regular.*

Proof. Let $R_E = R \otimes_k E$ and let $c \in R$ be such that the localization R_c is a smooth k -algebra. Since $R_c \otimes_k E$ is a smooth E -algebra, it is regular. Hence by [HH89, 3.1 (c)], $R_c \otimes_k E$ is strongly F -regular. Thus, to prove R_E is strongly F -regular, by [HH89, 3.3 (a)] it suffices to show that the inclusion $R_E[c^{1/q} \otimes 1] \subset R_E^{1/q}$ splits for some $q = p^e$. Because k is perfect, then such a splitting can be constructed from splittings of $R[c^{1/q}] \subset R^{1/q}$ and $E \subset E^{1/q}$. A splitting of $R[c^{1/q}] \subset R^{1/q}$ exists for some q because R is strongly F -regular, and a splitting of $E \subset E^{1/q}$ exists because E is F -finite and it is a field. $\square$

Lemma 8.13. *Let $f: Y \rightarrow X$ be a smooth surjective morphism between reduced k -schemes of finite type. Then X is strongly F -regular if and only if Y is strongly F -regular.*

Proof. If Y is strongly F -regular then X is strongly F -regular by [HH89, 3.1 (b)]. Conversely, if X is strongly F -regular then Y is strongly F -regular by [Abe01, 3.6]. In order to apply *loc. cit.* we are using the fact that the local rings of a smooth scheme over a field are regular, and hence they are Gorenstein and F -rational by [HH94, 3.4]. $\square$

We can now prove that the global Schubert varieties are strongly F -regular.

Theorem 8.14. *The schemes $\overline{\text{Gr}}_{\underline{G},\mu}$ and $\overline{\text{Gr}}_{\mathcal{G},\mu}$ are strongly F -regular, F -rational, and have pseudo-rational singularities.*

Proof. By the implications in Section 5.4, it suffices to prove these schemes are strongly F -regular. By Theorem 1.9, $\text{Gr}_{\leq\mu}$ is globally F -regular and hence also strongly F -regular. As C is smooth then it is strongly F -regular by [HH89, 3.1 (c)]. Thus since $\overline{\text{Gr}}_{\underline{G},\mu} \cong \text{Gr}_{\leq\mu} \times C$ then $\overline{\text{Gr}}_{\underline{G},\mu}$ is strongly F -regular by Theorem 8.9 and Lemma 8.11.

Since $\text{Gr}_{\mathcal{G}}|_{C^\circ} \cong \text{Gr}_{\underline{G}}|_{C^\circ}$ then $\text{Gr}_{\mathcal{G}}|_{C^\circ}$ is strongly F -regular. Hence by [HH89, 3.3 (a)], to prove $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is strongly F -regular it suffice to prove $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is Frobenius split along the effective Cartier divisor $(\overline{\text{Gr}}_{\mathcal{G},\mu})_0$. As $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is Frobenius split compatibly with $(\overline{\text{Gr}}_{\mathcal{G},\mu})_0$ by [Zhu14, 6.5] then it is also Frobenius split along $(\overline{\text{Gr}}_{\mathcal{G},\mu})_0$ (see, for example, Lemma 5.8). $\square$

Remark 8.15. We expect that by essentially the same proof, Theorem 8.14 is true more generally for a parahoric subgroup of a tamely ramified reductive group G over $k((t))$ such that $p > 2$ and G_{der} is absolutely almost simple and simply connected. This is because the necessary inputs from [PR08] and [Zhu14] are proved under these more general assumptions.

Before proceeding we prove Theorem 1.13. Recall that in the setup of Theorem 1.13, E is a local field of characteristic $p > 2$ with ring of integers $\mathcal{O}_E$ and residue field $\mathbb{F}_q$. The group G is a split connected reductive group defined over E such that G_{der} is absolutely almost simple and simply connected. We assume that G is the base extension of a split Chevalley group over $\mathbb{Z}$ and that T and B are defined over $\mathbb{F}_q$. Let Gr be the affine Grassmannian of G viewed as a group scheme over $\mathbb{F}_q$. After choosing an isomorphism $\mathbb{F}_q[[t]] \cong \mathcal{O}_E$ we can view M_μ as a projective $\mathcal{O}_E$ -scheme.

Proof of Theorem 1.13. Let $x \in M_\mu$ and let $\mathcal{O}_x$ be the local ring at x . As $\mathcal{O}_E$ is an excellent ring and M_μ is a projective $\mathcal{O}_E$ -scheme then $\mathcal{O}_x$ is excellent. Hence by [Smi97, 3.1] pseudo-rationality will follow if we prove that $\mathcal{O}_x$ is F -rational. The residue field of

$\mathcal{O}_x$ is F -finite because it is finitely generated as a field over the perfect field k . Thus $\mathcal{O}_x$ is F -finite by [Kun76, 2.6]. By [HH89, 3.1 (d)] it suffices to show $\mathcal{O}_x$ is strongly F -regular.

First suppose $x \in M_\mu$ lies in the closed fiber of $M_\mu \rightarrow \operatorname{Spec}(\mathcal{O}_E)$. The completion $\hat{\mathcal{O}}_x$ is excellent and has an F -finite residue field, so it is F -finite by [Kun76, 2.6]. To prove $\mathcal{O}_x$ is strongly F -regular it suffices to prove $\hat{\mathcal{O}}_x$ is strongly F -regular by [HH89, 3.1 (b)]. The ring $\hat{\mathcal{O}}_x$ is isomorphic to the completion of a local ring in $\overline{\operatorname{Gr}}_{\mathcal{G}, \mu}$. Thus, it suffices to take $y \in \overline{\operatorname{Gr}}_{\mathcal{G}, \mu}$ and show that the completion $\hat{\mathcal{O}}_y$ of the local ring $\mathcal{O}_y$ is strongly F -regular. Note that the map $\mathcal{O}_y \rightarrow \hat{\mathcal{O}}_y$ is a flat map between F -finite noetherian local rings. Since $\mathcal{O}_y$ is reduced and excellent then $\hat{\mathcal{O}}_y$ is also reduced. Moreover, the fibers of this map are regular by [EGA IV₂, 7.8.3 (v)]. Thus, since $\mathcal{O}_y$ is strongly F -regular (Theorem 8.14) then $\hat{\mathcal{O}}_y$ is strongly F -regular by [Abe01, 3.6]. This shows that $\mathcal{O}_x$ is strongly F -regular if x lies in the closed fiber of $M_\mu \rightarrow \operatorname{Spec}(\mathcal{O}_E)$.

To complete the proof we need to show that the generic fiber $M_\mu \times_{\operatorname{Spec}(\mathcal{O}_E)} \operatorname{Spec}(E) = \operatorname{Gr}_{\mathbb{F}_q, \leq \mu} \times_{\operatorname{Spec}(\mathbb{F}_q)} \operatorname{Spec}(E)$ is strongly F -regular. The property of strong F -regularity is Zariski local by definition, so it suffices to take an arbitrary open affine $\operatorname{Spec}(R) \subset \operatorname{Gr}_{\mathbb{F}_q, \leq \mu}$ and show that $R \otimes_{\mathbb{F}_q} E$ is strongly F -regular. Thus by Lemma 8.12 we are reduced to showing that $\operatorname{Gr}_{\mathbb{F}_q, \leq \mu}$ is geometrically integral.

We first show that the orbit $\operatorname{Gr}_{\mathbb{F}_q, \mu}$ is geometrically integral. Let $F/\mathbb{F}_q$ be a field extension and let $G_F = G \times_{\operatorname{Spec}(\mathbb{F}_q)} \operatorname{Spec}(F)$. By definition $\operatorname{Gr}_{F, \mu}$ is the scheme-theoretic image of the orbit map $L^n(G_F) \rightarrow \operatorname{Gr}_F$, $g \mapsto g \cdot \mu(t)$ for $n \gg 0$. Since $L^n G$ is geometrically integral and taking scheme theoretic images commutes with flat base change it follows that $\operatorname{Gr}_{\mathbb{F}_q, \mu} \times_{\operatorname{Spec}(\mathbb{F}_q)} \operatorname{Spec}(F) \cong \operatorname{Gr}_{F, \mu}$ is integral.

Now assume F is an algebraic closure of $\mathbb{F}_q$. Because the Cartan decompositions of $G(\mathbb{F}_q((t)))$ and $G(F((t)))$ are the same, the reduced subscheme of $\operatorname{Gr}_{\mathbb{F}_q, \leq \mu} \times_{\operatorname{Spec}(\mathbb{F}_q)} \operatorname{Spec}(F)$ is $\operatorname{Gr}_{F, \leq \mu}$. Thus $\operatorname{Gr}_{\mathbb{F}_q, \leq \mu}$ is geometrically irreducible by [Sta21, 038I]. Finally, since $\mathbb{F}_q$ is perfect then $\operatorname{Gr}_{\mathbb{F}_q, \leq \mu}$ is geometrically reduced and hence also geometrically integral. $\square$

We now begin proving parts (i) and (ii) of Theorem 1.3. For the rest of this section we assume that k is an algebraically closed field of characteristic $p > 2$ and that G is a connected reductive group over k such that G_{der} is almost simple and simply connected.

Proposition 8.16. *The perverse sheaf $j_{\mathcal{G},!}(\text{IC}_{\mu} \boxtimes^L \mathbb{F}_p[1]|_{C^\circ})$ is the shifted constant sheaf $\mathbb{F}_p[\dim \overline{\text{Gr}}_{\mathcal{G},\mu}]$ supported on $\overline{\text{Gr}}_{\mathcal{G},\mu}$.*

Proof. Since $\overline{\text{Gr}}_{\mathcal{G},\mu}$ is integral and F -rational then $\mathbb{F}_p[\dim \overline{\text{Gr}}_{\mathcal{G},\mu}]$ is a simple perverse sheaf by Theorem 1.12. Now the lemma follows by applying Theorem 1.12 on $\overline{\text{Gr}}_{\mathcal{G},\mu}|_{C^\circ}$ and using that $j_{\mathcal{G},!}(\text{IC}_{\mu} \boxtimes^L \mathbb{F}_p[1]|_{C^\circ})$ is simple by Lemma 2.9. $\square$

Proposition 8.17. *The functor $\mathcal{Z}$ is exact and preserves perversity, and $\mathcal{Z}(\text{IC}_{\mu}) \cong \mathcal{Z}_{\mathcal{A}(\mu)}$.*

Proof. The isomorphism $\mathcal{Z}(\text{IC}_{\mu}) \cong \mathcal{Z}_{\mathcal{A}(\mu)}$ follows from Theorem 8.2 and Proposition 8.16, and the fact that $\dim \mathcal{A}(\mu) = \dim \text{Gr}_{\leq \mu}$. As $\mathcal{A}(\mu)$ is Cohen-Macaulay and equidimensional (Corollary 8.10) then $\mathcal{Z}_{\mathcal{A}(\mu)}$ is perverse by Theorem 1.11. Since $P_{L+G}(\text{Gr}, \mathbb{F}_p)^{\text{ss}}$ is semisimple then $\mathcal{Z}$ is also exact and it preserves perversity. $\square$

Proposition 8.18. *$\mathcal{Z}(\mathcal{F}^\bullet)$ is I -equivariant.*

Proof. Note that $\mathcal{F}^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}$ is equivariant for the action of $\mathcal{L}^+\mathcal{G}|_{C^\circ} = L^+G \times C^\circ$ on $\overline{\text{Gr}}_{\mathcal{G},\mu}|_{C^\circ} = \text{Gr}_{\leq \mu} \times C^\circ$. Using the fact that taking intermediate extensions commutes with smooth pullback (Theorem 2.16) one can check that the perverse sheaf $j_{\mathcal{G},!}(\mathcal{F}^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \in P_c^b(\overline{\text{Gr}}_{\mathcal{G},\mu}, \mathbb{F}_p)$ is $\mathcal{L}^+\mathcal{G}$ -equivariant. By taking the fiber over 0 it follows that $\mathcal{Z}(\mathcal{F}^\bullet)$ is I -equivariant. $\square$

This completes the proof of the properties of $\mathcal{Z}$ asserted in the beginning of Theorem 1.3 and also part (i).

Proof of Theorem 1.3 (ii). We first show that $R\pi_!(\mathcal{Z}_{\mathcal{A}(\mu)}) \cong \text{IC}_{\mu}$. Since $\overline{\text{Gr}}_{\mathcal{G},\mu}$ and $\overline{\text{Gr}}_{\underline{\mathcal{G}},\mu}$ are normal, Cohen-Macaulay, and have pseudo-rational singularities then $R\pi_{\mathcal{G},*}(\mathcal{O}_{\overline{\text{Gr}}_{\mathcal{G},\mu}}) \cong$

$\mathcal{O}_{\overline{\text{Gr}_{\underline{G}, \mu}}}$ by [Kov17, 1.8]. Because $\pi_{\mathcal{G},*} = \pi_{\mathcal{G},!}$ as functors on $\mathbb{F}_p$ -sheaves then by applying the Artin-Schreier sequence and Proposition 8.16 it follows that

$$R\pi_{\mathcal{G},!}(j_{\mathcal{G},!}(\text{IC}_{\mu} \overset{L}{\boxtimes} \mathbb{F}_p[1]|_{C^{\circ}})) \cong \text{IC}_{\mu} \overset{L}{\boxtimes} \mathbb{F}_p[1].$$

Now by semisimplicity, for general $\mathcal{F}^{\bullet} \in P_{L+G}(\text{Gr}, \mathbb{F}_p)^{\text{ss}}$ there exists an isomorphism

$$R\pi_{\mathcal{G},!}(j_{\mathcal{G},!}(\mathcal{F}^{\bullet} \overset{L}{\boxtimes} \mathbb{F}_p[1]|_{C^{\circ}})) \cong \mathcal{F}^{\bullet} \overset{L}{\boxtimes} \mathbb{F}_p[1].$$

We can select a canonical isomorphism by requiring that it restricts to the identity map over C° . By restricting this canonical isomorphism to 0 and applying the proper base change theorem we get a canonical isomorphism

$$R\pi_!(\mathcal{Z}(\mathcal{F}^{\bullet})) \cong \mathcal{F}^{\bullet}.$$

□

Chapter 9

Centrality of $\mathcal{Z}$

9.1 Beilinson-Drinfeld and convolution Grassmannians

In this section we establish some F -regularity results that we will use to prove the remaining parts of Theorem 1.3. Let k be an algebraically closed field of characteristic $p > 2$ and let G be a connected reductive group over k such that G_{der} is almost simple. We also assume that G_{der} is simply connected until Remark 9.9. In our proofs we will use the same geometric objects as in [Zhu14]. First, we have the Beilinson-Drinfeld Grassmannian for $\mathcal{G}$ over C defined by the functor

$$\text{Gr}_{\mathcal{G}}^{BD}(R) = \left\{ (x, \mathcal{E}, \beta) : x \in C(R), \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on } C_R, \beta: \mathcal{E}|_{C_R^\circ - \Gamma_x} \cong \mathcal{E}_0|_{C_R^\circ - \Gamma_x} \right\}.$$

This is an ind-proper scheme over C by [Zhu14, 6.2.1]. By arguments similar to those in [Gai01, Prop. 5],

$$\text{Gr}_{\mathcal{G}}^{BD} \big|_{C^\circ} \cong \text{Gr} \times C^\circ \times \mathcal{F}\ell, \quad (\text{Gr}_{\mathcal{G}}^{BD})_0 \cong \mathcal{F}\ell.$$

Let $\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD}$ be the reduced closure of $\text{Gr}_{\leq \mu} \times C^\circ \times \mathcal{F}\ell_w$ in $\text{Gr}_{\mathcal{G}}^{BD}$.

The global convolution Grassmannian for $\mathcal{G}$ is the functor

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}}(R) = \left\{ (x, \mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) : \begin{array}{l} x \in C(R), \mathcal{E}_1, \mathcal{E}_2 \text{ are } \mathcal{G}\text{-torsors on } C_R, \\ \beta_1: \mathcal{E}_1|_{C_R - \Gamma_x} \cong \mathcal{E}_0|_{C_R - \Gamma_x}, \beta_2: \mathcal{E}_2|_{C_R^\circ} \cong \mathcal{E}_1|_{C_R^\circ} \end{array} \right\}.$$

This is an ind-projective scheme over C by [Zhu14, 6.2.3]. There is a map $m_{\mathcal{G}}: \mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}} \rightarrow \mathrm{Gr}_{\mathcal{G}}^{BD}$ which sends $(x, \mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2)$ to $(x, \mathcal{E}_2, \beta_1 \circ \beta_2)$. The map $m_{\mathcal{G}}$ is an isomorphism over C° . By taking the fibers over C° and 0 we get the following diagram with Cartesian squares.

$$\begin{array}{ccccc} \mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}}|_{C^\circ} & \xrightarrow{j^{\mathrm{conv}}} & \mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}} & \xleftarrow{i^{\mathrm{conv}}} & LG \times^I \mathcal{F}\ell \\ \downarrow \sim & & \downarrow m_{\mathcal{G}} & & \downarrow m \\ \mathrm{Gr}_{\mathcal{G}}^{BD}|_{C^\circ} & \xrightarrow{j^{BD}} & \mathrm{Gr}_{\mathcal{G}}^{BD} & \xleftarrow{i^{BD}} & \mathcal{F}\ell \end{array} \quad (9.1)$$

We define $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{\mathrm{conv}}$ to be the reduced closure of $\mathrm{Gr}_{\leq \mu} \times C^\circ \times \mathcal{F}\ell_w$ in $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}}$. In Theorem 9.6 we will show that $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{BD}$ and $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{\mathrm{conv}}$ are strongly F -regular. Before we can prove this, we need to show that $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{BD}$ is Frobenius split compatibly with $(\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, w}^{BD})_0$. Zhu proved such a splitting exists for $w \in X_*(T)^+$ sufficiently dominant [Zhu14, 6.5], and our argument will require only a minor modification of Zhu's argument.

In our proofs we will use the following facts and notation. Recall that for $\lambda \in X_*(T)^+$ we write t_λ for the element λ viewed as an element of $\tilde{W}$. If we write $\tilde{W} = X_*(T) \rtimes W$, then

$$\pi^{-1}(\mathrm{Gr}_{\leq \lambda}) = \mathcal{F}\ell_{t_\lambda^{w_0}}, \quad \text{where } t_\lambda^{w_0} := (\lambda, w_0) \in \tilde{W}.$$

Additionally, for $\mu \in X_*(T)^+$ we have

$$t_\mu \cdot t_\lambda^{w_0} = (\mu, 1) \cdot (\lambda, w_0) = (\mu + \lambda, w_0) = t_{\mu + \lambda}^{w_0}.$$

In this case, if ℓ is the length function on $\tilde{W}$ we have

$$\ell(t_\mu) + \ell(t_\lambda^{w_0}) = \ell(t_{\mu+\lambda}^{w_0}).$$

This can be proved using the formula for ℓ in [Gör10, §1.1] (see also [Zhu14, §9.1]).

Proposition 9.1. *Let $\mu, \lambda \in X_*(T)^+$ and let $w = t_\lambda^{w_0} \in \tilde{W}$. Then $\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD}$ is normal, and the fiber $(\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD})_0$ is reduced and isomorphic to $\mathcal{F}\ell_{t_{\mu+\lambda}^{w_0}}$.*

Proof. Let $\text{Gr}_{\underline{G}}^{BD}$ be the functor

$$\text{Gr}_{\underline{G}}^{BD}(R) = \left\{ (x, \mathcal{E}, \beta) : x \in C(R), \mathcal{E} \text{ is a } \underline{G}\text{-torsor on } C_R, \beta: \mathcal{E}|_{C_R^\circ - \Gamma_x} \cong \mathcal{E}_0|_{C_R^\circ - \Gamma_x} \right\}.$$

Then $\text{Gr}_{\underline{G}}^{BD}$ is isomorphic to $\text{Gr}_{C^2}|_{C \times \{0\}}$ as in Section 7.1. By [Gai01, 3.1.1], $\text{Gr}_{\underline{G}}^{BD}$ is ind-projective, and there are isomorphisms

$$\text{Gr}_{\underline{G}}^{BD}|_{C^\circ} \cong \text{Gr} \times C^\circ \times \text{Gr}, \quad (\text{Gr}_{\underline{G}}^{BD})_0 \cong \text{Gr}.$$

The map $\mathcal{G} \rightarrow \underline{G}$ induces a map $\pi_{BD}: \text{Gr}_{\mathcal{G}}^{BD} \rightarrow \text{Gr}_{\underline{G}}^{BD}$. Over closed points in C° this is the map $\text{id} \times \pi: \text{Gr} \times \mathcal{F}\ell \rightarrow \text{Gr} \times \text{Gr}$ and over 0 this is $\pi: \mathcal{F}\ell \rightarrow \text{Gr}$. Let $\overline{\text{Gr}}_{\underline{G}, \mu, \lambda}^{BD} \subset \text{Gr}_{\underline{G}}^{BD}$ be the reduced closure of $\text{Gr}_{\leq \mu} \times C^\circ \times \text{Gr}_{\leq \lambda}$.

The fiber $(\overline{\text{Gr}}_{\underline{G}, \mu, \lambda}^{BD})_0$ is reduced and isomorphic to $\text{Gr}_{\leq \mu+\lambda}$ by [Zhu09, 1.2.4] (see also [Zhu17, 3.1.14]). As $\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD} \subset \pi_{BD}^{-1}(\overline{\text{Gr}}_{\underline{G}, \mu, \lambda}^{BD})$ then $(\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD})_{0, \text{red}} \subset \mathcal{F}\ell_{t_{\mu+\lambda}^{w_0}}$. Furthermore, $\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD} \rightarrow C$ is flat by [Har77, 9.7], and hence by [Har77, 9.6] the irreducible components of $(\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD})_{0, \text{red}}$ all have dimension equal to $\dim \text{Gr}_{\leq \mu} + \dim \mathcal{F}\ell_{t_\lambda^{w_0}}$. Thus, since $\dim \mathcal{F}\ell_{t_{\mu+\lambda}^{w_0}} = \dim \text{Gr}_{\leq \mu} + \dim \mathcal{F}\ell_{t_\lambda^{w_0}}$, then $(\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD})_{0, \text{red}} = \mathcal{F}\ell_{t_{\mu+\lambda}^{w_0}}$ and $(\pi_{BD}^{-1}(\overline{\text{Gr}}_{\underline{G}, \mu, \lambda}^{BD}))_{\text{red}} = \overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD}$.

As $\mathcal{F}\ell$ is a G/B fibration over Gr , then $(\pi_{BD}^{-1}(\overline{\text{Gr}}_{\underline{G}, \mu, \lambda}^{BD}))_0$ is a G/B fibration over $(\overline{\text{Gr}}_{\underline{G}, \mu, \lambda}^{BD})_0$. Since $(\overline{\text{Gr}}_{\underline{G}, \mu, \lambda}^{BD})_0$ is reduced, then so is $(\pi_{BD}^{-1}(\overline{\text{Gr}}_{\underline{G}, \mu, \lambda}^{BD}))_0$. As $(\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD})_0$ is a closed subscheme of $(\pi_{BD}^{-1}(\overline{\text{Gr}}_{\underline{G}, \mu, \lambda}^{BD}))_0$, and these two schemes have the same reductions, then

$(\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$ is reduced. Finally, since $\mathcal{F}\ell_{t_{\mu+\lambda}^{w_0}}$ is normal and $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{BD}|_{C^\circ} \cong \mathrm{Gr}_{\leq \mu} \times C^\circ \times \mathcal{F}\ell_{t_\lambda^{w_0}}$ is also normal then $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{BD}$ is normal by Hironaka's lemma [EGA IV₂, 5.12.8]. $\square$

Remark 9.2. The functor $\mathrm{Gr}_{\mathcal{G}}^{BD}$ can also be described as the restriction to $C \times \{0\}$ of a quotient of global loop groups over C^2 as in Section 7.1 (see also [Zhu17, §3.1]). The referee has pointed out that it is possible to use this fact to show that π_{BD} is a G/B fibration, and in particular it is smooth. Since normality and reducedness are local in the smooth topology, this leads to an alternative proof of Proposition 9.1.

The following proposition is well-known, but because we could not find a complete proof in the literature we provide one here.

Proposition 9.3. *Let $w, w' \in \tilde{W}$ be such that $\ell(w) + \ell(w') = \ell(w \cdot w')$. Then the convolution morphism $m: LG \times^I \mathcal{F}\ell \rightarrow \mathcal{F}\ell$ maps $C(w) \tilde{\times} C(w')$ isomorphically onto $C(w \cdot w')$.*

Proof. Let W_{aff} be the affine Weyl group of G and let $\Omega \subset \tilde{W}$ be the subgroup of length 0 elements. Then we have a decomposition $\tilde{W} = W_{\mathrm{aff}} \rtimes \Omega$. First suppose that $w, w' \in W_{\mathrm{aff}}$. Let $w = s_1 \cdots s_{\ell(w)}$ and $w' = s'_1 \cdots s'_{\ell(w')}$ be reduced expressions for w and w' as products of simple reflections. For each i let $\mathcal{P}_i$ be the parahoric group scheme corresponding to s_i . Then $L^+(\mathcal{P}_i)/I \cong \mathcal{F}\ell_{s_i}$, and each $\mathcal{F}\ell_{s_i}$ is abstractly isomorphic to $\mathbb{P}^1$. As in Section 5.3 there is an affine Demazure resolution

$$\pi_{w \cdot w'}: \left(\tilde{\times} \mathcal{F}\ell_{s_i} \right) \tilde{\times} \left(\tilde{\times} \mathcal{F}\ell_{s'_j} \right) \rightarrow \mathcal{F}\ell_{w \cdot w'}.$$

By [Fal03, §3], since the product of the s_i and the s'_j gives a reduced expression for $w \cdot w'$, the morphism $\pi_{w \cdot w'}$ induces an isomorphism

$$\left(\tilde{\times} C(s_i) \right) \tilde{\times} \left(\tilde{\times} C(s'_j) \right) \xrightarrow{\sim} C(w \cdot w').$$

As $\pi_{w \cdot w'} = m \circ (\pi_w \tilde{\times} \pi_{w'})$ then by also applying *loc. cit.* to the factors $\tilde{\times} \mathcal{F}\ell_{s_i}$ and $\tilde{\times} \mathcal{F}\ell_{s'_j}$

we see that m maps $C(w) \tilde{\times} C(w')$ isomorphically onto $C(w \cdot w')$.

For the general case, write $w = \tau w_a$ and $w' = w'_a \tau'$ for $w_a, w'_a \in W_{\text{aff}}$ and $\tau, \tau' \in \Omega$. Then $\ell(w) = \ell(w_a)$, $\ell(w') = \ell(w'_a)$ and $\ell(w \cdot w') = \ell(w_a \cdot w'_a)$. Because Ω normalizes I then Ω acts on both $\mathcal{F}\ell$ and $LG \times^I \mathcal{F}\ell$ by right multiplication. Furthermore, any lift $\dot{\tau}^{-1} \in LG(k)$ induces isomorphisms on both of these ind-schemes by left multiplication, and all of these isomorphisms send I -orbits to I -orbits. In particular, there is a commutative diagram as follows:

$$\begin{array}{ccc} C(w) \tilde{\times} C(w') & \xrightarrow[\sim]{\dot{\tau}^{-1}(\cdot) \tau'^{-1}} & C(w_a) \tilde{\times} C(w'_a) \\ \downarrow m & & \downarrow m \\ C(w \cdot w') & \xrightarrow[\sim]{\dot{\tau}^{-1}(\cdot) \tau'^{-1}} & C(w_a \cdot w'_a) \end{array}$$

We have shown the morphism on the right is an isomorphism, thus so is the morphism on the left. $\square$

Proposition 9.4. *For any $\mu \in X_*(T)^+$ and $w \in \tilde{W}$ the scheme $\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD}$ is Frobenius split compatibly with $(\overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD})_0$.*

Proof. If $w = t_\nu$ for $\nu \in X_*(T)^+$ sufficiently dominant then this is [Zhu14, 6.5]. If $\lambda < \nu$ is also dominant then this splitting is compatible with the closed subscheme $\overline{\text{Gr}}_{\mathcal{G}, \mu, t_\lambda}^{BD} \subset \overline{\text{Gr}}_{\mathcal{G}, \mu, t_\nu}^{BD}$ by [Zhu14, 6.8] and [BK05, 1.1.7 (ii)]. A splitting of $\overline{\text{Gr}}_{\mathcal{G}, \mu, t_\nu}^{BD}$ compatible with $\overline{\text{Gr}}_{\mathcal{G}, \mu, t_\lambda}^{BD}$ and $(\overline{\text{Gr}}_{\mathcal{G}, \mu, t_\nu}^{BD})_0$ induces a splitting of $\overline{\text{Gr}}_{\mathcal{G}, \mu, t_\lambda}^{BD}$ compatible with $(\overline{\text{Gr}}_{\mathcal{G}, \mu, t_\lambda}^{BD})_0$. This proves the proposition when $w = t_\lambda$ for any dominant cocharacter λ .

For the general case, note that for every $w' \in \tilde{W}$ there exists $\lambda \in X_*(T)^+$ such that $w' \leq t_\lambda^{w_0}$, so it suffices to show $\overline{\text{Gr}}_{\mathcal{G}, \mu, t_\lambda^{w_0}}^{BD}$ is compatibly Frobenius split with $(\overline{\text{Gr}}_{\mathcal{G}, \mu, t_\lambda^{w_0}}^{BD})_0$ and $\overline{\text{Gr}}_{\mathcal{G}, \mu, w'}^{BD}$ for all $\lambda \in X_*(T)^+$ and $w' \leq t_\lambda^{w_0}$. Henceforth we fix $\lambda \in X_*(T)^+$ and $w = t_\lambda^{w_0} \in \tilde{W}$.

We will proceed by a similar argument as in [Zhu14, 6.7]. More precisely, we will construct an open subscheme $U \subset \overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{\text{conv}}$ such that:

1. The scheme U maps isomorphically onto its image under $m: \overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{\text{conv}} \rightarrow \overline{\text{Gr}}_{\mathcal{G}, \mu, w}^{BD}$.

2. The complement of $m(U)$ in $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}$ has codimension two.

3. The scheme U is Frobenius split.

Since $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}$ is normal, then by [BK05, 1.1.7] the spitting of $m(U)$ extends to a splitting of $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}$. We will complete the proof of the proposition by showing

4. The resulting splitting of $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}$ is compatible with $(\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$ and $\overline{\text{Gr}}_{\mathcal{G},\mu,w'}^{BD}$ for all $w' < w$.

As in [Zhu14], we define $U_1 \subset \overline{\text{Gr}}_{\mathcal{G},\mu,w}^{conv}$ to be the open subscheme which is $\text{Gr}_\mu \times C^\circ \times \mathcal{F}\ell_w$ over C° and $C(t_\mu) \tilde{\times} \mathcal{F}\ell_w$ over 0. We also define $U \subset U_1$ to be the open subscheme which is $\text{Gr}_\mu \times C^\circ \times \mathcal{F}\ell_w$ over C° and $C(t_\mu) \tilde{\times} C(w)$ over 0. Since the lengths of t_μ and w add then $C(t_\mu) \tilde{\times} C(w)$ maps isomorphically onto $C(t_{\mu+\lambda}^{w_0})$ by Proposition 9.3. Thus $(U)_0$ maps isomorphically onto $m(U)_0$. Hence the morphism from U to $m(U)$ is a bijective birational morphism between normal integral k -schemes, so it is an isomorphism by Grothendieck's reformulation of Zariski's main theorem [EGA III₁, 4.4.3].

As $\text{Gr}_{\leq \mu} - \text{Gr}_\mu$ has codimension two in $\text{Gr}_{\leq \mu}$, the complement of $m(U)|_{C^\circ} = \text{Gr}_\mu \times C^\circ \times \mathcal{F}\ell_w$ has codimension two in $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}|_{C^\circ} = \text{Gr}_{\leq \mu} \times C^\circ \times \mathcal{F}\ell_w$. By Proposition 9.1, $(\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD})_0 - m(U)_0 = \mathcal{F}\ell_{t_{\mu+\lambda}^{w_0}} - C(t_{\mu+\lambda}^{w_0})$ has codimension one, so we conclude that U satisfies (2). Finally, (3) and (4) follow from the fact that U_1 is Frobenius split compatibly with $(U_1)_0$ and $U_1 \cap (\text{Gr}_\mu \times C^\circ \times \mathcal{F}\ell_{w'})$ for all $w' \leq w$ by [Zhu14, 6.8]. $\square$

Corollary 9.5. *For any $\mu \in X_*(T)^+$ and $w \in \tilde{W}$ the scheme $(\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$ is reduced, Cohen-Macaulay, Frobenius split, and equidimensional of dimension $\dim \text{Gr}_{\leq \mu} + \dim \mathcal{F}\ell_w$.*

Proof. A Frobenius splitting of $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}$ compatible with $(\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$ induces a Frobenius splitting of $(\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$. Thus $(\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$ is reduced by [BK05, 1.2.1]. Furthermore, as $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}|_{C^\circ} \cong \text{Gr}_{\leq \mu} \times C^\circ \times \mathcal{F}\ell_w$ is Cohen-Macaulay then $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}$ is Cohen-Macaulay by [BS13a, 5.4] (see also [HR19, 5.5]). Thus $(\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$ is also Cohen-Macaulay by [Sta21,

OC6G]. Finally, the morphism $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD} \rightarrow C$ is flat by [Har77, III 9.7], so by [Har77, III 9.6] the fiber $(\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$ is equidimensional of dimension $\dim \text{Gr}_{\leq \mu} + \dim \mathcal{F}\ell_w$. $\square$

Theorem 9.6. *For any $\mu \in X_*(T)^+$ and $w \in \tilde{W}$ the schemes $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}$ and $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{conv}$ are strongly F -regular, F -rational, and have pseudo-rational singularities.*

Proof. As in the proof of Theorem 8.14 it suffices to prove these schemes are strongly F -regular. Since $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD} \big|_{C^\circ} \cong \text{Gr}_{\leq \mu} \times C^\circ \times \mathcal{F}\ell_w$ then $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD} \big|_{C^\circ}$ is strongly F -regular by Lemma 8.11 and Theorem 1.9. Now as in the proof of Theorem 8.14 it follows that $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD}$ is strongly F -regular because it is Frobenius split compatibly with $(\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$.

To prove $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{conv}$ is strongly F -regular, we first note that $\overline{\text{Gr}}_{\mathcal{G},\mu} \times \mathcal{F}\ell_w$ is strongly F -regular by Lemma 8.11. By the isomorphism proceeding [Zhu14, 6.2.3], $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{conv}$ and $\overline{\text{Gr}}_{\mathcal{G},\mu} \times \mathcal{F}\ell_w$ have a common smooth cover. Since the property of strong F -regularity is local in the smooth topology (Lemma 8.13) then $\overline{\text{Gr}}_{\mathcal{G},\mu,w}^{conv}$ is strongly F -regular. $\square$

9.2 Proof of centrality

We continue using the notation introduced in Section 9.1. For $\mu \in X_*(T)^+$ and $w \in \tilde{W}$ let $\mathcal{Z}_{\mu,w}$ be the shifted constant sheaf $\mathbb{F}_p[\dim(\text{Gr}_{\mathcal{G},\mu,w}^{BD})_0]$ supported on $(\text{Gr}_{\mathcal{G},\mu,w}^{BD})_0$. Parts (iii) and (iv) of Theorem 1.3 follow from the following proposition.

Proposition 9.7. *For $\mathcal{F}_1^\bullet \in P_{L+G}(\text{Gr}, \mathbb{F}_p)^{\text{ss}}$ and $\mathcal{F}_2^\bullet \in P_I(\mathcal{F}\ell, \mathbb{F}_p)$ there are natural isomorphisms*

- (i) $\Psi_{\text{Gr}_{\mathcal{G}}^{BD}}(\text{IC}_\mu \boxtimes^L \mathbb{F}_p[1] \big|_{C^\circ} \boxtimes^L \text{IC}_w^{\mathcal{F}\ell}) \cong \mathcal{Z}_{\mu,w}.$
- (ii) $\Psi_{\text{Gr}_{\mathcal{G}}^{BD}}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1] \big|_{C^\circ} \boxtimes^L \mathcal{F}_2^\bullet) \cong \mathcal{Z}(\mathcal{F}_1^\bullet) * \mathcal{F}_2^\bullet.$
- (iii) $\Psi_{\text{Gr}_{\mathcal{G}}^{BD}}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1] \big|_{C^\circ} \boxtimes^L \mathcal{F}_2^\bullet) \cong \mathcal{F}_2^\bullet * \mathcal{Z}(\mathcal{F}_1^\bullet).$

Furthermore, each of these complexes is perverse and I -equivariant.

Proof. Since $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{BD}$ is integral and F -rational then by Theorem 1.12, $j_{!*}^{BD}(\mathrm{IC}_{\mu}^L \boxtimes \mathbb{F}_p[1]|_{C^\circ}^L \boxtimes \mathrm{IC}_w^{\mathcal{F}^\ell})$ is the constant sheaf $\mathbb{F}_p[\dim \overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{BD}]$ supported on $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{BD}$. Hence the isomorphism in (i) follows. As $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{BD})_0$ is Cohen-Macaulay and equidimensional then $\mathcal{Z}_{\mu,w}$ is perverse by Theorem 1.11.

For (ii), suppose that $\mathcal{F}_1^\bullet$ is supported on $\mathrm{Gr}_{\leq \mu}$ and $\mathcal{F}_2^\bullet$ is supported on $\mathcal{F}\ell_w$. Let $I_n := L^n(\mathcal{G}|_{\hat{\mathcal{O}}_0})$. As in [Zhu14, 6.2.3], for some n there is an I_n -torsor $\mathrm{Gr}_{\mathcal{G},0,n}$ over $\mathrm{Gr}_{\mathcal{G}}$ such that

$$\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{conv} \cong \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},0,n} \times^{I_n} \mathcal{F}\ell_w.$$

Let $\varphi_n^{conv} : \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},0,n} \times \mathcal{F}\ell_w \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{conv}$ be the resulting I_n -torsor over $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{conv}$.

By similar reasoning as in Lemma 6.2 we can form the perverse sheaf

$$j_{\mathcal{G},!*}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \tilde{\boxtimes} \mathcal{F}_2^\bullet \in P_{\mathcal{L}+\mathcal{G}}(\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{conv}, \mathbb{F}_p).$$

The key point is that $j_{\mathcal{G},!*}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \tilde{\boxtimes} \mathcal{F}_2^\bullet$ is perverse when $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ are simple because then $j_{\mathcal{G},!*}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \tilde{\boxtimes} \mathcal{F}_2^\bullet$ is a constant sheaf supported on an equidimensional Cohen-Macaulay scheme. Thus $j_{\mathcal{G},!*}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \tilde{\boxtimes} \mathcal{F}_2^\bullet$ is perverse for general $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ by induction on their lengths.

We claim there is an isomorphism

$$Ri^{conv,*}[-1](j_{\mathcal{G},!*}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \tilde{\boxtimes} \mathcal{F}_2^\bullet) \cong \mathcal{Z}(\mathcal{F}_1^\bullet) \tilde{\boxtimes} \mathcal{F}_2^\bullet. \quad (9.2)$$

Let $\varphi_n : \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},0,n} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ be the pullback of $\mathrm{Gr}_{\mathcal{G},0,n} \rightarrow \mathrm{Gr}_{\mathcal{G}}$ along $\overline{\mathrm{Gr}}_{\mathcal{G},\mu} \rightarrow \mathrm{Gr}_{\mathcal{G}}$ and let $\varphi_{0,n} : (\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{0,n} \rightarrow (\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0$ be the fiber of φ_n over 0. By taking the fiber of φ_n^{conv}

over 0 we get the following Cartesian diagram:

$$\begin{array}{ccc}
\overline{\mathrm{Gr}}_{\mathcal{G},\mu} \times_{\mathrm{Gr}_{\mathcal{G}}} \mathrm{Gr}_{\mathcal{G},0,n} \times \mathcal{F}\ell_w & \xleftarrow{i_n^{conv}} & (\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_{0,n} \times \mathcal{F}\ell_w \\
\downarrow \varphi_n^{conv} & & \downarrow \varphi_{0,n}^{conv} \\
\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{conv} & \xleftarrow{i^{conv}} & (\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0 \widetilde{\times} \mathcal{F}\ell_w
\end{array}$$

Then (9.2) follows from the following calculation:

$$\begin{aligned}
& R\varphi_{0,n}^{conv,*}(Ri^{conv,*}[-1](j_{\mathcal{G},!}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \widetilde{\boxtimes} \mathcal{F}_2^\bullet)) \\
& \cong Ri_n^{conv,*}[-1](R\varphi_n^*(j_{\mathcal{G},!}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ})) \boxtimes^L \mathcal{F}_2^\bullet) \\
& \cong R\varphi_{0,n}^*(Ri_{\mathcal{G}}^*[-1](j_{\mathcal{G},!}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}))) \boxtimes^L \mathcal{F}_2^\bullet \cong R\varphi_{0,n}^*(\mathcal{Z}(\mathcal{F}_1^\bullet)) \boxtimes^L \mathcal{F}_2^\bullet.
\end{aligned}$$

To finish the proof of (ii) it suffices to construct a natural isomorphism

$$Rm_*(Ri^{conv,*}[-1](j_{\mathcal{G},!}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \widetilde{\boxtimes} \mathcal{F}_2^\bullet)) \rightarrow \Psi_{\mathrm{Gr}_{\mathcal{G}}^{BD}}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ} \boxtimes^L \mathcal{F}_2^\bullet). \quad (9.3)$$

By the proper base change theorem, to establish an isomorphism as in (9.3) it suffices to construct an isomorphism

$$Rm_{\mathcal{G},*}(j_{\mathcal{G},!}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \widetilde{\boxtimes} \mathcal{F}_2^\bullet) \rightarrow j_{!}^{BD}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ} \boxtimes^L \mathcal{F}_2^\bullet). \quad (9.4)$$

Since $m_{\mathcal{G}}$ is an isomorphism over C° then the left side of (9.4) is naturally an extension of $\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ} \boxtimes^L \mathcal{F}_2^\bullet$, so we just have to show it is the intermediate extension. Indeed, once we know the left side is isomorphic to the intermediate extension, then by Lemma 2.11 there is a unique isomorphism as in (9.4) which restricts to the identity map on $\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ} \boxtimes^L \mathcal{F}_2^\bullet$.

Since the middle term in an exact triangle is an intermediate extension if the outer terms are intermediate extensions (see the proof of Lemma 7.8) then by induction on the lengths of $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ we reduce to the case $\mathcal{F}_1^\bullet = \mathrm{IC}_\mu$ and $\mathcal{F}_2^\bullet = \mathrm{IC}_w^{\mathcal{F}\ell}$. Because all of the

schemes appearing are integral and F -rational, then both $j_{\mathcal{G},!}(\mathrm{IC}_{\mu} \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \boxtimes^L \mathrm{IC}_w^{\mathcal{F}\ell}$ and $j_{!*}^{BD}(\mathrm{IC}_{\mu} \boxtimes^L \mathbb{F}_p[1]|_{C^\circ} \boxtimes^L \mathrm{IC}_w^{\mathcal{F}\ell})$ are constant sheaves.

The map $m: \overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{\mathrm{conv}} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{BD}$ is a birational map between normal, Cohen-Macaulay k -schemes having pseudo-rational singularities. Thus by [Kov17, 1.8], $Rm_*(\mathcal{O}_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{\mathrm{conv}}}) \cong \mathcal{O}_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu,w}^{BD}}$. By applying the Artin-Schreier sequence we complete the proof of (9.4) and (ii). Applying (i), this also shows that $\mathcal{Z}(\mathcal{F}_1^\bullet) * \mathcal{F}_2^\bullet$ is perverse if $\mathcal{F}_1^\bullet = \mathrm{IC}_{\mu}$ and $\mathcal{F}_2^\bullet = \mathrm{IC}_w^{\mathcal{F}\ell}$. Convolution on the left by $\mathcal{Z}(\mathcal{F}_1^\bullet)$ sends short exact sequences in $P_I(\mathcal{F}\ell, \mathbb{F}_p)$ to exact triangles in $D_c^b(\mathcal{F}\ell, \mathbb{F}_p)$ by a proof analogous to the one in Corollary 6.7. Thus, by induction on the lengths of $\mathcal{F}_1^\bullet$ and $\mathcal{F}_2^\bullet$ we conclude that $\mathcal{Z}(\mathcal{F}_1^\bullet) * \mathcal{F}_2^\bullet$ is perverse in general. For equivariance, we note that because $\mathcal{Z}(\mathcal{F}_1^\bullet)$ is I -equivariant then $\mathcal{Z}(\mathcal{F}_1^\bullet) \boxtimes^L \mathcal{F}_2^\bullet$ is I -equivariant for the action of I on the left factor of $LG \times^I \mathcal{F}\ell$. As the map m is I -equivariant then $\mathcal{Z}(\mathcal{F}_1^\bullet) * \mathcal{F}_2^\bullet$ is I -equivariant by Lemma 3.2.

To prove (iii) we use the functor

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}'}(R) = \left\{ (x, \mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) : \begin{array}{l} x \in C(R), \mathcal{E}_1, \mathcal{E}_2 \text{ are } \mathcal{G}\text{-torsors on } C_R, \\ \beta_1: \mathcal{E}_1|_{C_R^\circ} \cong \mathcal{E}_0|_{C_R^\circ}, \beta_2: \mathcal{E}_2|_{C_R - \Gamma_x} \cong \mathcal{E}_1|_{C_R - \Gamma_x} \end{array} \right\}.$$

By [Zhu14, 7.2.6], $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}'}$ is ind-proper over C . There is also a map $m'_{\mathcal{G}}: \mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}'} \rightarrow \mathrm{Gr}_{\mathcal{G}}^{BD}$ which sends $(x, \mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2)$ to $(x, \mathcal{E}_2, \beta_1 \circ \beta_2)$. The map $m'_{\mathcal{G}}$ is an isomorphism over C° and it restricts to the convolution map $m: LG \times^I \mathcal{F}\ell \rightarrow \mathcal{F}\ell$ over 0.

Suppose $\mathcal{F}_1^\bullet$ is supported on $\mathrm{Gr}_{\leq \mu}$ and $\mathcal{F}_2^\bullet$ is supported on $\mathcal{F}\ell_w$. Let n be an integer large enough so that $\mathcal{L}^+ \mathcal{G}$ acts on $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ through the quotient $\mathcal{L}_n^+ \mathcal{G}$. Then as in the proof of [Zhu14, 7.4 (ii)] there is an $\mathcal{L}_n^+ \mathcal{G}$ -torsor $\mathcal{P}_n$ over $\mathcal{F}\ell_w \times C$ such that $\mathcal{P}_n \times^{\mathcal{L}_n^+ \mathcal{G}} \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \subset \overline{\mathrm{Gr}}_{\mathcal{G}}^{\mathrm{conv}'}$ is a closed subscheme with

$$\mathcal{P}_n \times^{\mathcal{L}_n^+ \mathcal{G}} \overline{\mathrm{Gr}}_{\mathcal{G},\mu} \Big|_{C^\circ} \cong \mathcal{F}\ell_w \times C^\circ \times \mathrm{Gr}_{\leq \mu}, \quad (\mathcal{P}_n \times^{\mathcal{L}_n^+ \mathcal{G}} \overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0 \cong \mathcal{F}\ell_w \widetilde{\times} (\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0.$$

The scheme $\mathcal{P}_n \times^{L_n^+ \mathcal{G}} \overline{\text{Gr}}_{\mathcal{G}, \mu}$ is strongly F -regular by Lemma 8.11 and because this property is local in the smooth topology by Lemma 8.13. Let $\varphi_n^{conv'} : \mathcal{P}_n \times_C \overline{\text{Gr}}_{\mathcal{G}, \mu} \rightarrow \mathcal{P}_n \times^{\mathcal{L}_n^+ \mathcal{G}} \overline{\text{Gr}}_{\mathcal{G}, \mu}$ be the resulting $\mathcal{L}_n^+ \mathcal{G}$ -torsor.

Since $j_{\mathcal{G}, !*}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ})$ is $\mathcal{L}_n^+ \mathcal{G}$ -equivariant (see the proof of Proposition 8.18), then by arguments analogous to those in the proof of (ii) we can form the perverse sheaf

$$(\mathcal{F}_2^\bullet \boxtimes^L \mathbb{F}_p[1]) \tilde{\boxtimes} j_{\mathcal{G}, !*}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \in P_c^b(\overline{\text{Gr}}_{\mathcal{G}}^{conv'}, \mathbb{F}_p),$$

which is supported on $\mathcal{P}_n \times^{L_n^+ \mathcal{G}} \overline{\text{Gr}}_{\mathcal{G}, \mu}$. Here we are applying the operation $\tilde{\boxtimes}$ with respect to the fiber product of C -schemes rather than k -schemes. If $\mathcal{F}_1^\bullet = \text{IC}_\mu$ and $\mathcal{F}_2^\bullet = \text{IC}_w^{\mathcal{F}_\ell}$ then $(\text{IC}_w^{\mathcal{F}_\ell} \boxtimes^L \mathbb{F}_p[1]) \tilde{\boxtimes} j_{\mathcal{G}, !*}(\text{IC}_\mu \boxtimes^L \mathbb{F}_p[1]|_{C^\circ})$ is a constant sheaf. Let $\mathcal{F}_{\ell_w, n}$ be the I_n -torsor $(\mathcal{P}_n)_0$ over $\mathcal{F}_{\ell_w}$. By taking the fiber over 0 we get the following Cartesian diagram:

$$\begin{array}{ccc} \mathcal{P}_n \times_C \overline{\text{Gr}}_{\mathcal{G}, \mu} & \xleftarrow{i_n^{conv'}} & \mathcal{F}_{\ell_w, n} \times (\overline{\text{Gr}}_{\mathcal{G}, \mu})_0 \\ \downarrow \varphi_n^{conv'} & & \downarrow \varphi_{0, n}^{conv'} \\ \mathcal{P}_n \times^{\mathcal{L}_n^+ \mathcal{G}} \overline{\text{Gr}}_{\mathcal{G}, \mu} & \xleftarrow{i_n^{conv'}} & \mathcal{F}_{\ell_w} \tilde{\times} (\overline{\text{Gr}}_{\mathcal{G}, \mu})_0 \end{array}$$

Now we can finish the proof by following arguments analogous to those in the proof of (ii) to establish isomorphisms

$$Ri^{conv', *}[-1]((\mathcal{F}_2^\bullet \boxtimes^L \mathbb{F}_p[1]) \tilde{\boxtimes} j_{\mathcal{G}, !*}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ})) \cong \mathcal{F}_2^\bullet \tilde{\boxtimes} \mathcal{Z}(\mathcal{F}_1^\bullet),$$

and

$$Rm_*(Ri^{conv', *}[-1]((\mathcal{F}_2^\bullet \boxtimes^L \mathbb{F}_p[1]) \tilde{\boxtimes} j_{\mathcal{G}, !*}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}))) \cong \Psi_{\text{Gr}_{\mathcal{G}}^{BD}}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ} \boxtimes^L \mathcal{F}_2^\bullet).$$

In this last isomorphism we are applying the natural isomorphism $\mathcal{F}_{\ell_w} \times C^\circ \times \text{Gr}_{\leq \mu} \cong \text{Gr}_{\leq \mu} \times C^\circ \times \mathcal{F}_{\ell_w}$. We leave the details to the reader. $\square$

Before we finish the proof of Theorem 1.3 we introduce the functor $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}''}$ whose R -points are

$$\left\{ (x, \mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) : \begin{array}{l} x \in C(R), \mathcal{E}_1, \mathcal{E}_2 \text{ are } \mathcal{G}\text{-torsors on } C_R \\ \beta_1: \mathcal{E}_1|_{C_R - \Gamma_x} \cong \mathcal{E}_0|_{C_R - \Gamma_x}, \beta_2: \mathcal{E}_2|_{C_R - \Gamma_x} \cong \mathcal{E}_1|_{C_R - \Gamma_x} \end{array} \right\}.$$

There are isomorphisms

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}''} \Big|_{C^\circ} \cong (LG \times^{L^+G} \mathrm{Gr}) \times C^\circ, \quad (\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}''})_0 \cong LG \times^I \mathcal{F}\ell.$$

Moreover, by arguments similar to those proceeding [Zhu14, 6.2.3],

$$\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}''} \cong \mathcal{L}\mathcal{G} \times^{\mathcal{L}^+G} \mathrm{Gr}_{\mathcal{G}} =: \mathrm{Gr}_{\mathcal{G}} \tilde{\times} \mathrm{Gr}_{\mathcal{G}}.$$

Thus $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}''}$ is ind-proper over C . We define $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \lambda}^{\mathrm{conv}''}$ to be the reduced closure of the subscheme $\mathrm{Gr}_{\leq \mu} \tilde{\times} \mathrm{Gr}_{\leq \lambda} \times C^\circ \subset \mathrm{Gr}_{\mathcal{G}}^{\mathrm{conv}''}$.

Proposition 9.8. *The scheme $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \lambda}^{\mathrm{conv}''}$ is Cohen-Macaulay, $(\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \lambda}^{\mathrm{conv}''})_0$ is reduced, and*

$$\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \lambda}^{\mathrm{conv}''} \cong (\overline{\mathrm{Gr}}_{\mathcal{G}, \mu} \tilde{\times} \overline{\mathrm{Gr}}_{\mathcal{G}, \lambda})_{\mathrm{red}}.$$

Proof. Note that $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, \lambda}^{\mathrm{conv}''}$ is a closed subscheme of $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu} \tilde{\times} \overline{\mathrm{Gr}}_{\mathcal{G}, \lambda}$, and these two schemes are isomorphic over C° . Thus the isomorphism in the proposition will follow if we show that $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu} \tilde{\times} \overline{\mathrm{Gr}}_{\mathcal{G}, \lambda}$ is irreducible.

To prove that $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu} \tilde{\times} \overline{\mathrm{Gr}}_{\mathcal{G}, \lambda}$ is irreducible, fix an integer n large enough so that $\mathcal{L}^+G$ acts on $\overline{\mathrm{Gr}}_{\mathcal{G}, \lambda}$ through the quotient $\mathcal{L}_n^+G$. Let $\mathcal{L}\mathcal{G}_{\leq \mu}$ be the preimage of $\overline{\mathrm{Gr}}_{\mathcal{G}, \mu}$ under the quotient map $\mathcal{L}\mathcal{G} \rightarrow \mathcal{L}\mathcal{G}/\mathcal{L}^+G \cong \mathrm{Gr}_{\mathcal{G}}$, and let

$$\overline{\mathrm{Gr}}_{\mathcal{G}, \mu, n} = \mathcal{L}\mathcal{G}_{\leq \mu} \times^{\mathcal{L}^+G} \mathcal{L}_n^+G.$$

Then there is a right $\mathcal{L}_n^+ \mathcal{G}$ -torsor $\varphi_{\mathcal{G},n}: \overline{\text{Gr}}_{\mathcal{G},\mu,n} \rightarrow \overline{\text{Gr}}_{\mathcal{G},\mu}$ such that

$$\overline{\text{Gr}}_{\mathcal{G},\mu} \tilde{\times} \overline{\text{Gr}}_{\mathcal{G},\lambda} \cong \overline{\text{Gr}}_{\mathcal{G},\mu,n} \times^{\mathcal{L}_n^+ \mathcal{G}} \overline{\text{Gr}}_{\mathcal{G},\lambda}.$$

Thus it suffices to show that $\overline{\text{Gr}}_{\mathcal{G},\mu,n} \times_C \overline{\text{Gr}}_{\mathcal{G},\lambda}$ is irreducible.

Because $\mathcal{L}_n^+ \mathcal{G}$ has geometrically irreducible fibers, then to prove that $\overline{\text{Gr}}_{\mathcal{G},\mu,n} \times_C \overline{\text{Gr}}_{\mathcal{G},\lambda}$ is irreducible it suffices to prove that $\overline{\text{Gr}}_{\mathcal{G},\mu} \times_C \overline{\text{Gr}}_{\mathcal{G},\lambda}$ is irreducible. Since $\overline{\text{Gr}}_{\mathcal{G},\mu}$ and $\overline{\text{Gr}}_{\mathcal{G},\lambda}$ are flat over C then $\overline{\text{Gr}}_{\mathcal{G},\mu} \times_C \overline{\text{Gr}}_{\mathcal{G},\lambda}$ is flat over C . Thus since $\overline{\text{Gr}}_{\mathcal{G},\mu} \times_C \overline{\text{Gr}}_{\mathcal{G},\lambda}|_{C^\circ} = \text{Gr}_{\leq \mu} \times \text{Gr}_{\leq \lambda} \times C^\circ$ is irreducible it follows that $\overline{\text{Gr}}_{\mathcal{G},\mu} \times_C \overline{\text{Gr}}_{\mathcal{G},\lambda}$ is irreducible. Putting this all together, we have shown that $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{\text{conv}''} \cong (\overline{\text{Gr}}_{\mathcal{G},\mu} \tilde{\times} \overline{\text{Gr}}_{\mathcal{G},\lambda})_{\text{red}}$.

By Corollary 8.10, the fibers $(\overline{\text{Gr}}_{\mathcal{G},\mu})_0$ and $(\overline{\text{Gr}}_{\mathcal{G},\lambda})_0$ are reduced and Cohen-Macaulay. Hence the product $(\overline{\text{Gr}}_{\mathcal{G},\mu})_0 \times (\overline{\text{Gr}}_{\mathcal{G},\lambda})_0$ is also reduced and Cohen-Macaulay. As these two properties are local in the smooth topology, then $(\overline{\text{Gr}}_{\mathcal{G},\mu})_0 \tilde{\times} (\overline{\text{Gr}}_{\mathcal{G},\lambda})_0$ is reduced and Cohen-Macaulay. Thus by the isomorphism $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{\text{conv}''} \cong (\overline{\text{Gr}}_{\mathcal{G},\mu} \tilde{\times} \overline{\text{Gr}}_{\mathcal{G},\lambda})_{\text{red}}$ it follows that $(\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{\text{conv}''})_0 \cong (\overline{\text{Gr}}_{\mathcal{G},\mu})_0 \tilde{\times} (\overline{\text{Gr}}_{\mathcal{G},\lambda})_0$ is reduced and Cohen-Macaulay. As $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{\text{conv}''}|_{C^\circ}$ is also Cohen-Macaulay then $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{\text{conv}''}$ is Cohen-Macaulay. $\square$

Proof of Theorem 1.3 (v). There is a morphism $m''_{\mathcal{G}}: \text{Gr}_{\mathcal{G}}^{\text{conv}''} \rightarrow \text{Gr}_{\mathcal{G}}$ which sends the element $(x, \mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2)$ to $(x, \mathcal{E}_2, \beta_1 \circ \beta_2)$. Over points in $C^\circ(k)$ the morphism $m''_{\mathcal{G}}$ is the convolution morphism $LG \times^{L^+G} \text{Gr} \rightarrow \text{Gr}$ and over 0 the morphism $m''_{\mathcal{G}}$ is $m: LG \times^I \mathcal{F}\ell \rightarrow \mathcal{F}\ell$. By taking the fibers of $m''_{\mathcal{G}}$ over C° and 0 we get the following diagram with Cartesian squares:

$$\begin{array}{ccccc} (LG \times^{L^+G} \text{Gr}) \times C^\circ & \xrightarrow{j^{\text{conv}''}} & \text{Gr}_{\mathcal{G}}^{\text{conv}''} & \xleftarrow{i^{\text{conv}''}} & LG \times^I \mathcal{F}\ell \\ \downarrow & & \downarrow m''_{\mathcal{G}} & & \downarrow m \\ \text{Gr} \times C^\circ & \xrightarrow{j_{\mathcal{G}}} & \text{Gr}_{\mathcal{G}} & \xleftarrow{i_{\mathcal{G}}} & \mathcal{F}\ell \end{array} \quad (9.5)$$

Because the schemes $\overline{\text{Gr}}_{\mathcal{G},\mu,\lambda}^{\text{conv}''} = (\overline{\text{Gr}}_{\mathcal{G},\mu} \tilde{\times} \overline{\text{Gr}}_{\mathcal{G},\lambda})_{\text{red}}$ for $\mu, \lambda \in X_*(T)^+$ are Cohen-Macaulay,

then by similar reasoning as in Lemma 6.2 we can form the perverse sheaf

$$\mathcal{F}_{1,2}^\bullet := j_{\mathcal{G},!}(\mathcal{F}_1^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \tilde{\boxtimes} j_{\mathcal{G},!}(\mathcal{F}_2^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}) \in P_c^b(\mathrm{Gr}_{\mathcal{G}}^{conv''}, \mathbb{F}_p).$$

To complete the proof it suffices to construct natural isomorphisms

1. $Ri^{conv'',*}[-1](\mathcal{F}_{1,2}^\bullet) \cong \mathcal{Z}(\mathcal{F}_1^\bullet) \tilde{\boxtimes} \mathcal{Z}(\mathcal{F}_2^\bullet),$
2. $Rm_*(Ri^{conv'',*}[-1](\mathcal{F}_{1,2}^\bullet)) \cong \mathcal{Z}(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet).$

Suppose $\mathcal{F}_1^\bullet$ is supported on $\mathrm{Gr}_{\leq \mu}$ and $\mathcal{F}_2^\bullet$ is supported on $\mathrm{Gr}_{\leq \lambda}$. As in the proof of Proposition 9.8, fix an integer n large enough so that $\mathcal{L}^+ \mathcal{G}$ acts on $\overline{\mathrm{Gr}}_{\mathcal{G},\lambda}$ through the quotient $\mathcal{L}_n^+ \mathcal{G}$. Let $\varphi_{\mathcal{G},n}: \overline{\mathrm{Gr}}_{\mathcal{G},\mu,n} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ be the right $\mathcal{L}_n^+ \mathcal{G}$ -torsor such that

$$\overline{\mathrm{Gr}}_{\mathcal{G},\mu} \tilde{\times} \overline{\mathrm{Gr}}_{\mathcal{G},\lambda} \cong \overline{\mathrm{Gr}}_{\mathcal{G},\mu,n} \times^{\mathcal{L}_n^+ \mathcal{G}} \overline{\mathrm{Gr}}_{\mathcal{G},\lambda}.$$

Let $\varphi_n^{conv''}: (\overline{\mathrm{Gr}}_{\mathcal{G},\mu,n} \times_C \overline{\mathrm{Gr}}_{\mathcal{G},\lambda})_{\mathrm{red}} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu,\lambda}^{conv''}$ be the resulting $\mathcal{L}_n^+ \mathcal{G}$ -torsor over $\overline{\mathrm{Gr}}_{\mathcal{G},\mu,\lambda}^{conv''}$. Over points in C° the map $\varphi_{\mathcal{G},n}: \overline{\mathrm{Gr}}_{\mathcal{G},\mu,n} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ restricts to an $L^n G$ -torsor $p_n: \mathrm{Gr}_{\leq \mu,n} \rightarrow \mathrm{Gr}_{\leq \mu}$. By taking the fibers of $\varphi_n^{conv''}$ over C° and 0 we get a diagram with Cartesian squares:

$$\begin{array}{ccccc} \mathrm{Gr}_{\leq \mu,n} \times \mathrm{Gr}_{\leq \lambda} \times C^\circ & \xrightarrow{j_n^{conv''}} & (\overline{\mathrm{Gr}}_{\mathcal{G},\mu,n} \times_C \overline{\mathrm{Gr}}_{\mathcal{G},\lambda})_{\mathrm{red}} & \xleftarrow{i_n^{conv''}} & (\overline{\mathrm{Gr}}_{\mathcal{G},\mu,n})_0 \times (\overline{\mathrm{Gr}}_{\mathcal{G},\lambda})_0 \\ \downarrow & & \downarrow \varphi_n^{conv''} & & \downarrow \varphi_{0,n}^{conv''} \\ \mathrm{Gr}_{\leq \mu} \tilde{\times} \mathrm{Gr}_{\leq \lambda} \times C^\circ & \xrightarrow{j^{conv''}} & \overline{\mathrm{Gr}}_{\mathcal{G},\mu,\lambda}^{conv''} & \xleftarrow{i^{conv''}} & (\overline{\mathrm{Gr}}_{\mathcal{G},\mu})_0 \tilde{\times} (\overline{\mathrm{Gr}}_{\mathcal{G},\lambda})_0 \end{array}$$

An isomorphism as in (1) can be constructed in a manner similar to (9.2) by pulling everything back to $(\overline{\mathrm{Gr}}_{\mathcal{G},\mu,n})_0 \times (\overline{\mathrm{Gr}}_{\mathcal{G},\lambda})_0$. We leave the details to the reader. By using the left side of the above diagram one can also prove that

$$Rj^{conv'',*}(\mathcal{F}_{1,2}^\bullet) \cong \mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}.$$

By the proper base change theorem, to prove (2) it suffices to construct a natural isomorphism

$$(2') \quad Rm''_{\mathcal{G},*}(\mathcal{F}_{1,2}^\bullet) \cong j_{\mathcal{G},!}((\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}).$$

By the description of $m''_{\mathcal{G}}$ over C° there is a natural isomorphism

$$R(m''_{\mathcal{G}}|_{C^\circ})_*(Rj^{conv'',*}(\mathcal{F}_{1,2}^\bullet)) \cong (\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}.$$

Thus the left side of (2') is naturally an extension of $(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) \boxtimes^L \mathbb{F}_p[1]|_{C^\circ}$, so we just need to show it is isomorphic to the intermediate extension. By semisimplicity we reduce to the case $\mathcal{F}_i^\bullet = \mathrm{IC}_{\mu_i}$ for $\mu_i \in X_*(T)^+$. Note that $m''_{\mathcal{G}}$ maps $\overline{\mathrm{Gr}}_{\mathcal{G},\mu_1,\mu_2}^{conv''}$ onto $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ where $\mu = \mu_1 + \mu_2$. Moreover, $\mathrm{IC}_{\mu_1} * \mathrm{IC}_{\mu_2} \cong \mathrm{IC}_\mu$ by Theorem 1.2. Thus, $\mathcal{F}_{1,2}^\bullet$ and the right side of (2') are shifted constant sheaves.

Because the convolution morphism $\mathrm{Gr}_{\leq \mu_1} \widetilde{\times} \mathrm{Gr}_{\leq \mu_2} \rightarrow \mathrm{Gr}_{\leq \mu}$ is birational then the morphism $m''_{\mathcal{G}}: \overline{\mathrm{Gr}}_{\mathcal{G},\mu_1,\mu_2}^{conv''} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is also birational. We claim that $m''_{\mathcal{G}}: \overline{\mathrm{Gr}}_{\mathcal{G},\mu_1,\mu_2}^{conv''} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is projective. To prove this, note that $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is projective over C by [PZ13, 5.5]. As $\mathrm{Gr}_{\mathcal{G}}^{conv''} \cong \mathrm{Gr}_{\mathcal{G}} \widetilde{\times} \mathrm{Gr}_{\mathcal{G}} \cong \mathrm{Gr}_{\mathcal{G}} \times_C \mathrm{Gr}_{\mathcal{G}}$ (see, for example, [Zhu17, 1.2.14]), then $\overline{\mathrm{Gr}}_{\mathcal{G},\mu_1,\mu_2}^{conv''}$ is also projective over C . Thus $m''_{\mathcal{G}}: \overline{\mathrm{Gr}}_{\mathcal{G},\mu_1,\mu_2}^{conv''} \rightarrow \overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ is projective. Therefore, since $\overline{\mathrm{Gr}}_{\mathcal{G},\mu}$ has pseudo-rational singularities and $\overline{\mathrm{Gr}}_{\mathcal{G},\mu_1,\mu_2}^{conv''}$ is Cohen-Macaulay then $Rm''_{\mathcal{G},*}(\mathcal{O}_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu_1,\mu_2}^{conv''}}) \cong \mathcal{O}_{\overline{\mathrm{Gr}}_{\mathcal{G},\mu}}$ by [Kov17, 1.4]. Now (2') follows by applying the Artin-Schreier sequence. $\square$

Remark 9.9. We have proved Theorem 1.3 under the hypothesis G_{der} is simply connected and almost simple. We now explain how to remove the simple connectedness hypothesis using the same technique as in Remark 7.12. The same idea is also used in [Zhu14, 3.3].

Let G be a connected reductive group over k such that G_{der} is almost simple. By [MS82, 3.1] there exists a central extension

$$1 \rightarrow N \rightarrow G' \rightarrow G \rightarrow 1$$

such that G'_{der} is simply connected and N is a connected torus. Since G_{der} is almost simple then so is G'_{der} . Let $T' \subset B' \subset G'$ be the maximal torus and Borel subgroup given by the preimages of T and B , and let $I' \subset L^+G'$ be the preimage of B' under the projection $L^+G' \rightarrow G'$. Let Gr_G and $\text{Gr}_{G'}$ be the affine Grassmannians for G and G' , respectively. Similarly, let $\mathcal{F}\ell_G$ and $\mathcal{F}\ell_{G'}$ be the Iwahori affine flag varieties.

Because N is connected the map $X_*(T') \rightarrow X_*(T)$ is surjective. Hence the maps $\text{Gr}_{G'} \rightarrow \text{Gr}_G$ and $\mathcal{F}\ell_{G'} \rightarrow \mathcal{F}\ell_G$ are surjective. There is also an induced surjection $X_*(T')^+ \rightarrow X_*(T)^+$ on dominant cocharacters and on Iwahori-Weyl groups $\tilde{W}' \rightarrow \tilde{W}$. Each connected component of $\text{Gr}_{G'}$ maps onto its image in Gr_G via a universal homeomorphism by [HV18, 3.1]. The same is true of the map $\mathcal{F}\ell_{G'} \rightarrow \mathcal{F}\ell_G$.

There exists a Bruhat-Tits group scheme $\mathcal{G}'$ over C satisfying the conditions (8.2) and equipped with a natural map $\mathcal{G}' \rightarrow \mathcal{G}$. This induces a map $\text{Gr}_{G'} \rightarrow \text{Gr}_G$. By restricting this map to C° and 0 one sees that $\text{Gr}_{G'} \rightarrow \text{Gr}_G$ is surjective and maps each connected component of $\text{Gr}_{G'}$ into its image via a universal homeomorphism. Similarly, there are maps $\text{Gr}_{G'}^{BD} \rightarrow \text{Gr}_G^{BD}$, $\text{Gr}_{G'}^{\text{conv}} \rightarrow \text{Gr}_G^{\text{conv}}$, etc., which are surjective and restrict to universal homeomorphisms on connected components.

All of the diagrams of C -schemes we used in Chapters 8 and 9 to prove Theorem 1.3 for G' intertwine with the corresponding diagrams for G . For example, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Gr}_{G'} & \longrightarrow & \text{Gr}_G \\ \downarrow \pi_{G'} & & \downarrow \pi_G \\ \text{Gr}_{\underline{G}'} & \longrightarrow & \text{Gr}_{\underline{G}} \end{array}$$

Thus, by the topological invariance of the étale site [Sta21, 04DY], our arguments in Chapters 8 and 9 can be used to simultaneously prove Theorem 1.3 for both G' and G .

Chapter 10

Applications

10.1 The function-sheaf correspondence

Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$ and let $\mathcal{F}_0^\bullet \in D_c^b(X_0, \mathbb{F}_p)$. Fix an embedding of $\mathbb{F}_q$ into an algebraic closure $\overline{\mathbb{F}}_q$ and let $F^* \in \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ be the inverse of the map which sends $\alpha \mapsto \alpha^q$. Let $X = X_0 \times_{\text{Spec}(\mathbb{F}_q)} \text{Spec}(\overline{\mathbb{F}}_q)$ and let $\mathcal{F}^\bullet$ be the pullback of $\mathcal{F}_0^\bullet$ to X . For $x \in X_0(\mathbb{F}_q)$ let $\mathcal{F}_x^\bullet$ be the pullback of $\mathcal{F}^\bullet$ along the composition $\text{Spec}(\overline{\mathbb{F}}_q) \rightarrow \text{Spec}(\mathbb{F}_q) \xrightarrow{x} X_0$. Then each $H^i(\mathcal{F}_x^\bullet)$ is a representation of $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ and it makes sense to take the trace $\text{Tr}(F^*, H^i(\mathcal{F}_x^\bullet))$.

We form a function $\text{Tr}(\mathcal{F}_0^\bullet): X_0(\mathbb{F}_q) \rightarrow \mathbb{F}_p$ by setting

$$\text{Tr}(\mathcal{F}_0^\bullet)(x) = \sum_i (-1)^i \text{Tr}(F^*, H^i(\mathcal{F}_x^\bullet)).$$

See also [SGA 4 $\frac{1}{2}$, Ch. 2, §1] for more information on the construction of the function $\text{Tr}(\mathcal{F}_0^\bullet)$. As in the case of $\overline{\mathbb{Q}}_\ell$ -coefficients, we have:

Theorem 10.1. *Let X_0 and Y_0 be separated schemes of finite type over $\mathbb{F}_q$.*

- (i) *Let $\mathcal{F}_0^\bullet \in D_c^b(X_0, \mathbb{F}_p)$ and $\mathcal{G}_0^\bullet \in D_c^b(Y_0, \mathbb{F}_p)$. If $x \in (X_0 \times_{\text{Spec}(\mathbb{F}_q)} Y_0)(\mathbb{F}_q)$ has images*

$p_1(x) \in X_0(\mathbb{F}_q)$ and $p_2(x) \in Y_0(\mathbb{F}_q)$ then

$$\mathrm{Tr}(\mathcal{F}_0^\bullet \boxtimes^L \mathcal{G}_0^\bullet)(x) = \mathrm{Tr}(\mathcal{F}_0^\bullet)(p_1(x)) \mathrm{Tr}(\mathcal{G}_0^\bullet)(p_2(x)).$$

(ii) Let $f: Y_0 \rightarrow X_0$ be a morphism. If $\mathcal{F}_0^\bullet \in D_c^b(X_0, \mathbb{F}_p)$ and $y \in Y_0(\mathbb{F}_q)$ then

$$\mathrm{Tr}(Rf^*(\mathcal{F}_0^\bullet))(y) = \mathrm{Tr}(\mathcal{F}_0^\bullet)(f(y)).$$

(iii) If $\mathcal{F}_0^\bullet \in D_c^b(X_0, \mathbb{F}_p)$ then

$$\sum_{x \in X_0(\mathbb{F}_q)} \mathrm{Tr}(\mathcal{F}_0^\bullet)(x) = \sum_i (-1)^i \mathrm{Tr}(F^*, R^i \Gamma_c(X, \mathcal{F}^\bullet)).$$

Proof. Parts (i) and (ii) are immediate from the definitions, and part (iii) is [SGA 4 $\frac{1}{2}$, 4.1]. $\square$

10.2 Perverse $\mathbb{F}_p$ -sheaves over finite fields

For the rest of Chapter 10 we assume G is a split connected reductive group defined over $\mathbb{F}_q$. Let $\mathrm{Gr}_{\mathbb{F}_q}$ and $\mathcal{F}\ell_{\mathbb{F}_q}$ denote the affine Grassmannian and affine flag variety viewed as ind-schemes over $\mathbb{F}_q$. While we restricted to the case of an algebraically closed ground field before, our constructions also work over an arbitrary perfect field of characteristic $p > 0$. In particular, we can construct the categories $P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)$ and $P_I(\mathcal{F}\ell_{\mathbb{F}_q}, \mathbb{F}_p)$. The main difference when working over $\mathbb{F}_q$ is that there are more simple objects. In particular, Lemma 3.18 needs to be revised in this setting, as there are non-trivial simple étale local systems on $\mathrm{Spec}(\mathbb{F}_q)$.

As in the case of $\overline{\mathbb{Q}}_\ell$ -coefficients in [Zhu17, 5.6], we will restrict ourselves to a certain subcategory of $P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)$ consisting of normalized perverse sheaves. More precisely, let

$\mathcal{L}$ be the étale local system on $\mathrm{Spec}(\mathbb{F}_q)$ corresponding to the representation $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \rightarrow \mathrm{GL}_1(\mathbb{F}_p)$ which sends F^* to -1 . If X_0 is a scheme over $\mathbb{F}_q$ then we can also view $\mathcal{L}$ as a local system on X_0 by pulling back along $X_0 \rightarrow \mathrm{Spec}(\mathbb{F}_q)$.

Let $\mu \in X_*(T)^+$ be such that $\dim \mathrm{Gr}_{\leq \mu}$ has parity $p(\mu) \in \{0, 1\}$. Because Theorem 1.12 also holds when k is perfect (with the same proof), then IC_μ is isomorphic to the shifted constant sheaf $\mathbb{F}_p[\dim \mathrm{Gr}_{\leq \mu}]$ supported on $\mathrm{Gr}_{\mathbb{F}_q, \leq \mu}$. We define the normalized IC complex

$$\mathrm{IC}_\mu^N := \mathrm{IC}_\mu \overset{L}{\otimes} \mathcal{L}^{\otimes p(\mu)} \in P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p).$$

Let $P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N \subset P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)$ be the Serre subcategory consisting of perverse sheaves whose simple subquotients are all the form IC_μ^N for $\mu \in X_*(T)^+$.

We claim the subcategory $P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N$ is monoidal. To prove this, we first note that the identity $\mathrm{IC}_{\mu_1} * \mathrm{IC}_{\mu_2} = \mathrm{IC}_{\mu_1 + \mu_2}$ in Theorem 1.2 also holds over $\mathbb{F}_q$. Indeed, this identity is derived from the result of Kovács [Kov17, 1.4] which is independent of the ground field. From this fact and the projection formula [Sta21, 0B54] it follows that $\mathrm{IC}_{\mu_1}^N * \mathrm{IC}_{\mu_2}^N = \mathrm{IC}_{\mu_1 + \mu_2}^N$.

Using the same arguments as in Chapter 7 one can show that $P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N$ is a symmetric monoidal category. For $\mathcal{F}^\bullet \in P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)$ let $\mathcal{F}_{\mathrm{Gr}}^\bullet \in P_{L+G}(\mathrm{Gr}, \mathbb{F}_p)$ be its pullback to Gr . Then one can also show that

$$P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N \rightarrow \mathrm{Vect}_{\mathbb{F}_p}, \quad \mathcal{F}^\bullet \mapsto \bigoplus_i R^i \Gamma(\mathcal{F}_{\mathrm{Gr}}^\bullet)$$

is an exact, faithful, tensor functor.

For $\mu \in X_*(T)^+$ let

$$\mathcal{Z}_{\mathcal{A}(\mu)}^N := \mathcal{Z}_{\mathcal{A}(\mu)} \overset{L}{\otimes} \mathcal{L}^{\otimes p(\mu)} \in P_I(\mathcal{F}\ell_{\mathbb{F}_q}, \mathbb{F}_p).$$

Let $P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^{N, \mathrm{ss}} \subset P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N$ be the tensor subcategory consisting of semisim-

ple objects. If $p > 2$ and G_{der} is absolutely almost simple then the arguments in Chapters 8 and 9 also work over $\mathbb{F}_q$ and give rise to a functor

$$\mathcal{Z}: P_{L+G}(\text{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^{N, \text{ss}} \rightarrow P_I(\mathcal{F}\ell_{\mathbb{F}_q}, \mathbb{F}_p)$$

which satisfies all parts of Theorem 1.3 and such that $\mathcal{Z}(\text{IC}_{\mu}^N) \cong \mathcal{Z}_{\mathcal{A}(\mu)}^N$.

Remark 10.2. The functor $P_{L+G}(\text{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N \rightarrow P_{L+G}(\text{Gr}, \mathbb{F}_p)$ induced by pullback is faithful and identifies the simple objects in these two categories, but it is not an equivalence of categories in general. The issue is that due to the failure of smooth base change, the group of extensions between two objects depends on the ground field (see the proof of Proposition 6.14).

10.3 Proofs of applications to Hecke algebras

In this section we prove Theorem 1.6 and Corollary 1.7. We fix an isomorphism $\mathbb{F}_q[[t]] \cong \mathcal{O}_E$. Recall that if $H \subset G(E)$ is a compact open subgroup then the multiplication on $\mathcal{H}_H$ is defined by

$$(f * g)(x) = \sum_{y \in G(E)/H} f(xy)g(y^{-1}).$$

Fix a maximal torus and a Borel subgroup $T \subset B \subset G$, and let U be the unipotent radical of B . Let $\mathcal{H}_T$ be the Hecke algebra of T with respect to $T(\mathcal{O}_E)$. Following Herzig [Her11b], we define the map

$$\mathcal{S}': \mathcal{H}_K \rightarrow \mathcal{H}_T$$

by

$$\mathcal{S}'(f)(g) = \sum_{u \in U(F)/U(\mathcal{O})} f(gu).$$

As T is commutative, there is a natural isomorphism $\mathbb{F}_p[X_*(T)] \cong \mathcal{H}_T$, where the element

$\mu \in X_*(T)$ corresponds to the function which is 1 on the coset $\mu(t)T(\mathcal{O})$ and 0 everywhere else. In the case where E is a p -adic field and the coefficients are taken in an algebraically closed field of characteristic p (instead of $\mathbb{F}_p$), Herzig [Her11b] showed that $\mathcal{S}'$ is injective with image spanned by the anti-dominant cocharacters $X_*(T)_-$. Henniart and Vignéras [HV15] extended this result to an arbitrary non-Archimedean local field E and coefficients taken in any field of characteristic p .

Multiplication by the longest element of the Weyl group w_0 induces an isomorphism $w_0: \mathbb{F}_p[X_*(T)_-] \xrightarrow{\sim} \mathbb{F}_p[X_*(T)^+]$. The resulting composition

$$\mathcal{S}: \mathcal{H}_K \xrightarrow{\mathcal{S}'} \mathbb{F}_p[X_*(T)_-] \xrightarrow{w_0} \mathbb{F}_p[X_*(T)^+]$$

is what we will call the mod p Satake isomorphism. In fact, there is a similar mod p Satake isomorphism for any weight (a representation of the finite group $G(\mathbb{F}_q)$), but here we are only considering the case where the weight is trivial.

We now verify that the function-sheaf correspondence respects the convolution of perverse sheaves and functions.

Lemma 10.3. *If $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)$ then*

$$\mathrm{Tr}(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) = \mathrm{Tr}(\mathcal{F}_1^\bullet) * \mathrm{Tr}(\mathcal{F}_2^\bullet) \in \mathcal{H}_K.$$

Similarly, if $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_I(\mathcal{F}\ell_{\mathbb{F}_q}, \mathbb{F}_p)$ then

$$\mathrm{Tr}(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet) = \mathrm{Tr}(\mathcal{F}_1^\bullet) * \mathrm{Tr}(\mathcal{F}_2^\bullet) \in \mathcal{H}_I.$$

Proof. We will only consider the case $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet \in P_I(\mathcal{F}\ell_{\mathbb{F}_q}, \mathbb{F}_p)$; the other case can be handled by similar methods. The proof is essentially the same as that of [Zhu17, 5.6.1] in the case of $\overline{\mathbb{Q}_\ell}$ -coefficients, but because this lemma is important for our applications we will reproduce

the details. Let $m: LG \times^I \mathcal{F}\ell \rightarrow \mathcal{F}\ell$ be the multiplication map and let $x \in \mathcal{F}\ell_{\mathbb{F}_q}(\mathbb{F}_q) = G(E)/I$. There is a natural identification $m^{-1}(x)(\mathbb{F}_q) = \{(xy, y^{-1}) : y \in G(E)/I\}$. By Theorem 10.1 (iii) and the proper base change theorem,

$$\mathrm{Tr}(\mathcal{F}_1^\bullet * \mathcal{F}_2^\bullet)(x) = \sum_{y \in G(E)/I} \mathrm{Tr}(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet)(xy, y^{-1}).$$

Now viewing xy and y^{-1} as elements of $G(E)/I$ and using Theorem 10.1 (i) and (ii) we have

$$\mathrm{Tr}(\mathcal{F}_1^\bullet \tilde{\boxtimes} \mathcal{F}_2^\bullet)(xy, y^{-1}) = \mathrm{Tr}(\mathcal{F}_1^\bullet)(xy) \mathrm{Tr}(\mathcal{F}_2^\bullet)(y^{-1}).$$

□

Lemma 10.4. *Let $\mathcal{F}^\bullet \in P_I(\mathcal{F}\ell_{\mathbb{F}_q}, \mathbb{F}_p)$. Then*

$$\mathrm{Tr}(R\pi_!(\mathcal{F}^\bullet)) = \mathrm{Tr}(\mathcal{F}^\bullet) * \mathbb{1}_K \in \mathcal{H}_K.$$

Proof. If $x \in \mathrm{Gr}_{\mathbb{F}_q}(\mathbb{F}_q)$ then $\pi^{-1}(x)(\mathbb{F}_q) = \{xy : y \in K/I\}$. We claim that

$$\mathrm{Tr}(R\pi_!(\mathcal{F}^\bullet))(x) = \sum_{y \in K/I} \mathrm{Tr}(\mathcal{F}^\bullet)(xy) = \sum_{y \in G(E)/I} \mathrm{Tr}(\mathcal{F}^\bullet)(xy) \mathbb{1}_K(y^{-1}) = (\mathrm{Tr}(\mathcal{F}^\bullet) * \mathbb{1}_K)(x).$$

The first equality follows from Theorem 10.1 (iii) and the proper base change theorem. The other two equalities follow from the definitions. □

Since $\mathrm{IC}_{\mu_1}^N * \mathrm{IC}_{\mu_2}^N = \mathrm{IC}_{\mu_1 + \mu_2}^N$ there is a natural isomorphism of $\mathbb{F}_p$ -algebras

$$f: \mathbb{F}_p[X_*(T)^+] \xrightarrow{\sim} K_0(P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N) \otimes \mathbb{F}_p, \quad \mu \mapsto [\mathrm{IC}_\mu^N].$$

The next proposition shows that this isomorphism is compatible with the mod p Satake isomorphism, and it implies Theorem 1.5.

Proposition 10.5. *The composition*

$$\mathbb{F}_p[X_*(T)^+] \xrightarrow{f} K_0(P_{L+G}(\mathrm{Gr}_{\mathbb{F}_q}, \mathbb{F}_p)^N) \otimes \mathbb{F}_p \xrightarrow{\mathrm{Tr}} \mathcal{H}_K$$

is the inverse of the mod p Satake isomorphism $\mathcal{S}$.

Proof. When G_{der} is simply connected, Herzig [Her11a, 5.1] has computed

$$\mathcal{S}^{-1} \left(\sum_{\lambda \leq \mu} \mathbb{1}_\lambda \right) = \mu \in \mathbb{F}_p[X_*(T)^+].$$

While Herzig works with anti-dominant cocharacters, the formula above can be obtained from Herzig's formula by multiplying by the longest element of the Weyl group. Herzig only uses the hypothesis that G_{der} is simply connected to reduce to the case where the weight (a representation of $G(\mathbb{F}_q)$) is trivial. As the weight is trivial in our situation, the above formula is valid for any G . Since IC_μ is a constant sheaf supported on $\mathrm{Gr}_{\mathbb{F}_q, \leq \mu}$ then the lemma now follows from our choice of normalized IC complexes IC_μ^N . $\square$

Proof of Theorem 1.6 and Corollary 1.7. Theorem 1.6 follows immediately from Theorem 1.3 and Lemmas 10.3, 10.4. By [Oll14] the central integral Bernstein elements are uniquely determined by the identity $\mathcal{B} = \mathcal{C}^{-1} \circ \mathcal{S}^{-1}$, so Corollary 1.7 follows from Theorem 1.6 and Proposition 10.5. $\square$

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