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Building a Foundation: Developing a Middle School Number Theory Course to Enhance Mathematical Understanding

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A Thesis in the Field of Mathematics for Teaching

for the Degree of Master of Liberal Arts in Extension Studies

Harvard University

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Abstract

This thesis addresses the lack of emphasis on number theory in middle school mathematics by developing an elementary number theory curriculum. Core concepts, including divisibility, prime numbers, even and odd numbers, and rational numbers, are taught alongside arithmetic skills, culminating in a cryptography unit. The curriculum is designed to strengthen students' number sense and conceptual understanding while building on existing research in K-12 number theory education. By integrating targeted number theory instruction with arithmetic and early algebra, this curriculum aims to enhance mathematical reasoning and provide middle school learners with a more robust foundation for future mathematical study.

Dedication

To my mom, for a lifetime of inspiring my curiosity and guiding me to share what I know; to my dad and grandpa, for their support and nurturing of my love of learning; to my husband, whose patience and unwavering belief in me made this work possible; and to my sister, whose kindness is endless. To all my loved ones, thank you for your support and encouragement.

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Chapter I.

Introduction

Elementary number theory introduces students to core concepts such as divisibility, prime numbers, even and odd numbers, and rational numbers. When these topics are integrated with arithmetic and early algebra in middle school, they play a key role in developing number sense, flexible reasoning, and an understanding of mathematical structure. Despite the National Council of Teachers of Mathematics' (NCTM, 1989) recognition of number theory as a subject that can further develop mathematical concepts and problem-solving, number theory is not typically a subject area of its own in the K-12 curriculum (Campbell, 2002).

Effective implementation of a number theory course requires teachers to first clarify their instructional goals and priorities. By grounding instruction in the broader purposes of number theory, teachers can ensure that the course meaningfully supports students' long-term mathematical understanding and that the course prepares the students for more advanced study. Dedicating structured time to number-theoretic topics allows educators to address common areas of difficulty, strengthen students' number sense, and foster the development of computational fluency, pattern recognition, and algebraic reasoning (Silla et al., 2024).

Curriculum Overview

The four-year curriculum described below is structured to build an understanding of number theory in a systematic, developmental sequence:

- Year 1 focuses on foundational concepts, including number classification, divisibility, factors, multiples, and fractions, as well as patterns in sequences such as Pascal's Triangle and the Fibonacci sequence. Throughout the first year, students develop habits of reasoning, pattern recognition, and reflective thinking, building a strong conceptual base alongside procedural fluency.
- Year 2 builds on the year 1 foundational concepts by introducing remainders, modular reasoning, and the relationships between division, fractions, and decimals. During the second year, students encounter informal cyclic patterns in familiar contexts, such as clocks and calendars, which serve as entry points to modular thinking. The ideas generated from these informal contexts are then formalized and applied to numerical patterns, parity, and sequences, supporting students in developing flexible and abstract reasoning about numbers.
- Year 3 strengthens students' understanding of divisibility, modular arithmetic, and prime factorization. Students study the division algorithm, greatest common factors (GCF), least common multiples (LCM), and factor-based approaches to fraction operations. Connections between sequences, cyclic patterns, and structured reasoning are emphasized. This third-year-course of study is designed to strengthen students' conceptual understanding and to enhance their procedural skills, providing a foundation for symbolic algebra and advanced problem-solving.

- Year 4 applies the number-theoretic foundations developed in previous years to the study of cryptography and digital security. In the fourth year, students begin with classical ciphers and progress to modern symmetric and public-key encryption, investigating how prime numbers, modular arithmetic, and one-way functions form the basis of secure communication. Real-world contexts such as online banking, data privacy, and digital communication are used to illustrate how abstract number theory concepts are applied to tangible problems, integrating conceptual understanding, procedural skill, and ethical considerations.

Structuring the curriculum across four progressive years, culminating in applied contexts such as cryptography, enables students to develop a deep and interconnected understanding of number theory. This structure supports the growth of strong problem-solving and pattern-recognition skills and equips students with the tools needed to be successful with advanced mathematical concepts.

Philosophy

Each student learns differently, and nurturing their unique creativity is essential for developing strong mathematical problem-solving skills. Math teachers can support learners by helping them notice patterns and relationships in numbers, and by facilitating discussions that promote student autonomy. Teachers do not need to have all the answers at all times; being willing to explore students' ideas demonstrates intellectual curiosity and models a growth mindset. A teacher can show genuine engagement by carefully considering students' contributions and responding thoughtfully. Striking a balance between subject-matter expertise and openness to new perspectives enables teachers to

guide learning effectively while fostering an environment where students feel empowered to think critically and creatively.

Balancing teacher expertise with exploring student perspectives is particularly important in a number theory course, where abstract concepts and pattern-based reasoning play a central role. By encouraging exploration, discussion, and multiple approaches to problem-solving, teachers help students build both conceptual understanding and flexible thinking, laying the foundation for success in arithmetic, algebra, and beyond.

Consider, for example, a student who solves the quadratic below by isolating the x in the following steps:

$$4x^2 - 4x - 24 = 0$$

$$4x^2 - 4x = 24$$

$$x^2 - x = 6$$

$$x(x - 1) = 6$$

The student gets to the final step above and concludes that since $6 = 1 \times 6$ or 2×3, x must be 1, 2, 3, or 6. They try these four options and find that $x = 3$ works! Many teachers would stop this process early on, encouraging students to factor or use the quadratic formula as a more effective method. While using the quadratic formula may be more efficient, halting the student's creative process early prevents them from exploring their understanding of number properties.

How could teachers encourage students to consider other solution paths if they try the process above? At what point should a teacher suggest a more efficient route to the correct solutions? When time allows, presenting students with examples where x and x-1 are not positive integers could help students consider why guessing and checking may not always be efficient.

When working with rational equations, there are often multiple correct methods for solving a problem. These methods may involve clearing fractions, cross-multiplying, finding common denominators, or distributing over parentheses, and the particular combination of strategies chosen can influence both the efficiency and simplicity of the paths to the solution. Developing flexibility and creativity in selecting varying approaches helps students solve problems accurately and efficiently while deepening their understanding of the underlying mathematics. When students understand the reasoning behind different strategies, they can choose the approach that best fits the problem rather than relying on rote procedures and potentially overlooking more straightforward options for solving the problem. By guiding students in exploring multiple approaches instead of prescribing a single "correct" method, teachers expand learners' problem-solving abilities and foster critical thinking, reflection, and autonomy.

This curriculum explicitly encourages problem-solving habits of mind by building in metacognitive reflections and continually prompts students to consider why particular methods are used for different calculations. In this way, students are supported in thinking deeply about their choices, connecting concepts, and developing a more flexible and thoughtful approach to mathematics.

Implementation

The number theory curriculum for middle schoolers spans four school years, with roughly 12 classes per year, and is intended for students on a standard math track from fifth through eighth grade. Allocating one month each year should be sufficient to cover the material, though the structure is meant to remain flexible and adaptable to a variety of classroom needs. Many of the practice sets provided include more problems than a class can realistically complete, so teachers should feel free to be selective and choose the exercises that best fit their instructional goals. Adjusting the course pace to meet the needs of the class is both expected and encouraged. If a class is more advanced, moving through the material more quickly can make the overall difficulty more suitable. For example, if students show strong understanding of topics that appear multiple times throughout the curriculum, such as prime factorization, sequences, LCM, and GCF, the pace can be adjusted, as it is perfectly acceptable to move more quickly than the suggested pacing. Conversely, if some topics are too challenging and require additional time, teachers can omit or condense portions of the course. Some optional extensions are suggested throughout the course, but none of them are required. Ultimately, teachers should feel free to adapt both difficulty and pacing to best support students' number sense, problem-solving skills, and understanding of prime numbers.

Class practice problems are intended to be teacher-guided, with students actively reasoning through the mathematics via whole-class discussion, think-pair-share, or partner work. Worksheets, when included, are designed to be completed in small groups unless otherwise noted, and collaborative work is encouraged for multi-step word problems or problems involving more in-depth calculations. Calculators should be used

minimally to support number sense development. Students may check their work with the aid of calculators, but should not rely on the calculators to perform the core reasoning.

Summer Enrichment

The curriculum can also be adapted for a summer math enrichment program, with each year's lessons spanning at least two weeks of daily classes. Instead of 12 classes per year for four years, the curriculum can be taught in 10 classes per summer for about four years, depending on the number of scheduled hours during each summer session. The first three years can be compressed for students who accelerate at a faster pace, allowing more time for the modular arithmetic and cryptography units. The lessons that incorporate review and repetition can be compressed to focus on new material, since summer courses tend to move at a faster pace.

Quizzes, check-ins, and reflections should be adjusted to match the accelerated schedule. Teachers can combine and administer the assessments at the end of the summer sessions or assign them as homework.-Adjusting the curriculum to a condensed timeline should maintain the same sense of progression as the yearlong structure and should adhere to the core ideas of the curriculum.

Chapter II.

Literature Review

The following review of research on elementary number theory in middle school curricula looks at three areas of study: foundational abstract number concepts, rational number operations, and number theory. The foundational numerical concepts section reviews the development of mathematical intuition and the significance of developing number sense for computational skill progression. Next, the rational number operations section looks at the challenges that learners face when working with rational numbers and how number theory can help to resolve some of these challenges. The final section on number theory examines the ways in which elementary number theory topics are incorporated into middle school math education and the consideration of number theory as an advanced course for middle school students. Additionally, the number theory section investigates the interconnectedness of elementary number theoretic concepts and the potential for a middle school number theory curriculum to help students develop a deeper understanding of key mathematical concepts and support mathematical reasoning.

Foundational Abstract Numerical Concepts

Researchers (Dehaene, 1997; Howden, 1989; McIntosh et al., 1997) in the mathematics education, cognitive psychology, and neuroscience spheres have written a wide range of definitions of number sense. For instance, number sense can include the ability to visualize a number line, create mental procedures to aid with computations, decompose numbers and relate them to one another, and adapt to numerical operations and procedures in a flexible manner. Howden (1989) calls number sense a “special feel

for numbers," (p. 6) claiming that students' K-12 education impacts how they view numbers and demonstrate number sense development.

Dehaene (2009) describes mathematical intuition within elementary arithmetic as a sense of numbers which permits a rapid evaluation of a number of objects present in a scene, whether this number is more or less than another number, and how this number is changed by simple operations of addition and subtraction. This mathematical intuition encompasses the ability to gauge quantity and size without computation. Researchers, including Barrow (1992), Bundesen (1990), Miller (1994), and Sperling (1967), show that due to working memory constraints, there are limitations to the number of items that humans can enumerate without counting. Since humans need an understanding of grouping and knowledge of number systems to envision larger quantities, it is crucial to be educated in ways to further develop quantity sense to connect it to numbers and make sense of larger-scale structures and phenomena (Barrow, 1992; Wagner & Davis, 2010).

Number sense is also connected to students' reasoning behind the mathematical operations they use, their selection of comparative approximations when estimating values, and their mastery of operations with rational numbers. Many researchers (Kirkland et al., 2024; Markovits & Sowder, 1994; McIntosh et al., 1997) have shown that stronger number sense has positive outcomes throughout schooling. Number sense enables students to use existing mathematical knowledge in new and creative ways, to recognize the reasonableness of solutions to numerical problems, and to improve mathematics achievement. Kirkland et al. found that mature number sense is a unique predictor of grade-level achievement scores. When the researchers compared two students at the same grade level with similar learning trajectories, the student with higher

number sense had a greater increase in their mathematics content learning over the course of a year. Kirkland et al.'s study, which tested U.S. middle school students' number sense, supported their hypothesis that students' understanding of the relationship between arithmetic operations and their computations reflects their levels of number sense. McIntosh et al.'s (1992) definition of mature number sense includes a comprehension of numbers and operations as well as the ability to apply this knowledge in versatile ways to make mathematical assessments. It is this inclination to apply mathematical knowledge to make strategic judgments that is key to students' success with numbers and operations.

A mental fluidity where the mind can navigate between ideas and their numerical representations shows proficiency with numerical problems. Dehaene (1997) compares the so-called "numerical illiterate" person who haphazardly makes calculations to the "expert calculator" individual who thoughtfully chooses an algorithm for a given problem. When learners are equipped with a true understanding of mathematical concepts and ideas, they are able to fluently choose appropriate approaches for mathematical situations.

An understanding of numbers, then, is not a result of merely memorizing a sequence of steps, algorithms, and procedures. To tackle rich, complex math problems, learners need to gain a practical grasp of what different representations of numbers mean. To give a few examples, 5.3 is the same as 5.30; 200 is ten 20's; and 16/17 is less than 14/13. Without a basis for why these comparisons are true, students resort to drawing from a host of memorized rules which they can easily forget. Reys et al.'s (1989) study, which looked at eight- to fourteen-year-old students in the United States, Australia, Sweden, and Taiwan, demonstrates the challenges encountered by teachers when

assessing their students' responses to estimation problems. The researchers asked questions in unfamiliar ways to encourage strategy generation rather than recall based on predictability. The study asked multiple-choice questions such as, "About how many days have you lived?" "How many different decimal numbers are there between 1.52 and 1.53?" and "How Many Different Fractions Are There Between 2/5 and 3/5?" Interviews with the students showed that even amongst those who chose the correct responses, there were misconceptions about decimals and fractions. The researchers determined that across the different countries, mathematics curricula were heavily weighted toward algorithms and procedures, which did not always lead to computational skill progression. Rather, number sense is the culmination of appropriate activities intended to connect foundational mathematical ideas upon which to build computational fluency. Howden (1989) shows that exploring numerical ideas through activities, when paired with discussion of recorded findings, can be fertile ground for developing number sense in children.

In elementary education, children with strong number sense often break up numbers into their smaller additive parts to find or approximate sums and differences (Geary et al., 2007). When adding and subtracting, decomposition strategies demonstrate a more mature thinking process. Canobi et al. (1998) found that children with advanced conceptual understanding were faster, more accurate, and more flexible in their use of problem-solving strategies than their peers. Cowan et al.'s (2011) study of elementary school students also found that differences in fundamental calculation skills were indicative of future understanding of concepts, even when similar factors were taken into consideration. The more children understand math, the more they may use it on a daily

basis, thus improving their calculation competence (Cowan et al., 2011). Additionally, Sood and Jitendra's (2013) findings suggest that even a brief instructional intervention, for example, four weeks, that focuses on key areas related to the development of number sense can improve learning and retention of number proficiency.

It follows that having a strong grasp of ways to break down numbers into their multiplicative parts (i.e., their factors) would further support the development of a mathematical mind's ability to manipulate and make sense of numbers. Multiplicative reasoning includes counting, combining, and grouping actions that progress from addition to multiplication (Smith & Smith, 2006). Grasping the step-by-step process of multiplication through skip counting or repeated addition could establish a basis for comprehending the formal definitions of different multiplicative methods.

Mack's (1990) study, which examined sixth-grade students' ability to build on informal knowledge to make sense of fraction comparison and arithmetic, found that while students were able to relate symbolic representations of fractions to their intuitive knowledge, rote procedural knowledge interfered with their attempts at constructing meaningful algorithms. For example, when comparing whether a slice of a pizza cut into six slices was larger than a slice of a pizza cut into eight slices, students clearly reasoned that a slice of the pizza cut into fewer slices was larger. When presented with the identical fraction comparison of 1/6 and 1/8, most students responded incorrectly, citing that 8 is larger than 6, so 1/8 must be larger than 1/6. This disconnect between understanding and rote procedure can be minimized with more attention to mathematical intuition and number sense.

Formal educational activities encourage students to make connections and build their mathematical understanding (Kidd et al., 2013; Kipman et al., 2019; van der Ven et al., 2017). As noted by Griffiths (2012), when mathematical intuition is encouraged, students can decompose numbers and develop sensible hypotheses. Griffiths' study on intuiting the fundamental theorem of arithmetic (FTA) shows that high-performing thirteen- and fourteen-year-old students were able to logically construct the conclusions of the FTA without direct instruction.

In Christou and Vamvakoussi's (2021) study on students' multiplication and division skills with natural and non-natural numbers, the researchers found weaknesses in arithmetic between numbers of different types. The challenges that students faced when multiplying and dividing a mixture of natural and non-natural numbers are in line with the high error rates when working with rational numbers in general. It is therefore crucial to better understand rational number operations and to recognize the connections between different types of numbers under these mathematical operations.

Rational Number Operations

Learners have difficulty working with rational, non-whole numbers, when written as fractions in the form a/b, decimals, and percentages. In studies comparing learners' error rates and response times to placing natural numbers, decimals, and fractions in the form a/b on a number line, students' error rates and response times were higher for fractions than they were for natural numbers and decimals (DeWolf et al., 2014; Iuculano & Butterworth, 2011; MacKay et al., 2024). Furthermore, difficulty with fraction arithmetic is well-known to educators. When 1,000 Algebra 1 teachers in the U.S. were surveyed to rate their students' areas of limited knowledge affecting their algebra

learning, “rational numbers and operations involving fractions and decimals” were the second most prevalent responses (Hoffer et al, 2007). The only knowledge limitation rated as more serious was solving word problems.

Lortie-Forgues, Tian, and Siegler (2015) asked 48 U.S. eighth graders in an Algebra 1 course to approximate 12/13 + 7/8, with 1, 2, 19, and 21 as the multiple-choice answer options. Only 27% of the students answered the question correctly. Thirty-six years prior, a similar percentage (only 24%) of 20,000 eighth graders answered this question correctly on the 1978 National Assessment of Educational Progress (NAEP) exam (Carpenter et al., 1980). The 1983 NAEP also showed comparable results for a similar decimal problem. It appears that the many attempts at reform did not help with student performance on the number sense of fractions and decimals.

Fraction arithmetic can be intimidating for many reasons. Due to perceived differences between fractions and decimals as well as fractions and natural numbers, the rules for addition versus multiplication can be confusing. When adding fractions, students find themselves questioning whether adding numerators and denominators is allowed. Similarly, when multiplying fractions, is multiplying numerators and denominators allowed? The interplay between computational algorithms and conceptual meaning may be the reason behind the confusion around fraction arithmetic. Compared to whole numbers and decimals, fractions appear to have a split structure, where the numerator and denominator are considered separate. This split structure may contribute to early misunderstandings and to later flawed use of shortcut strategies in fraction arithmetic (DeWolf et al., 2014).

Some of the issues students have with fractions in the context of word problems may result from the mismatch between the words' mathematical meaning and their everyday meaning. This mismatch, which is typically referred to as lexical ambiguity, is a common source of confusion for math students when solving word problems (Durkin, Crowther & Shire, 1986; Fuchs et al., 2017). Cartwright et al. (2022) found a significant positive correlation between lexically ambiguous math word knowledge and word problem solving, reading comprehension, and vocabulary. In a study on lexical ambiguity, Durkin and Shire (1991) examined seven- to eleven-year-old students' understanding of mathematical words in non-mathematical contexts and vice versa. The researchers found that students tended to erroneously select the nonmathematical meaning for words used in mathematical contexts. For example, students may define the word "make" in the sentence, "two and two make four" as they define the word in the sentence "Mum and John make cake". The students defined the word from a list of response options: eat, add, equals, cook, or no answer. Other words in the study included times, below, near, tables, take, low, above, leaves, and small. The researchers also noted that younger students were more likely to confuse the meanings and select a nonmathematical meaning of a word used in a mathematical context. While unsurprising, this confusion at a younger age exposes students to issues relating to misinterpreting mathematical ideas.

Students can draw from their intuitive, informal understanding of fraction arithmetic when presented with real-world situations using concepts such as partitioning items into discrete groups (Mack, 1990; Viegut et al., 2023). However, this intuition does not always translate to algorithmic procedures. For example, in Mack's (1995) study,

when students were asked questions related to their part-whole interpretation of fractions, some students answered correctly when the concept was stated verbally but not when they were presented with the equivalent fraction problem. Mack (1990) also demonstrated that some students who were able to solve rational number addition and subtraction problems correctly in the context of partitions of a pizza had challenges transferring their knowledge to the equivalent fraction arithmetic problems.

Misconceptions related to symbolic representations for fractions in the form a/b are often tied to knowledge of whole numbers. This confusion between representations of numbers leads many learners to erroneously apply natural number properties to rational number tasks. Some children and adults see "a" and "b" as independent whole numbers, rather than as a ratio expressing its own magnitude. The perception of fractions with disjointed numerator and denominator results in confusion when comparing fraction magnitudes, where the larger fraction (such as 2/3 compared to 5/11) has smaller numerical components (Fazio, DeWolf, & Siegler, 2016; Meert, Grégoire, & Noël, 2010). The relationship between whole number and fraction arithmetic can also be seen in common errors when children independently apply whole number addition and subtraction to numerators and denominators, such as 1/3 + 1/4 = 2/7 (Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2010). Students must be adept at transforming numbers into forms helpful for the problem they are presented with, such as changing denominators or switching fractions to decimals as needed. When Post et al. (1985) tested children's ability to derive meaning from fractions, they found that students struggled to correctly identify which of two fractions of the form 1/n was larger, such as 1/3 and 1/6. Students often encountered such challenges due to a lack of flexibility in transforming numbers

between different forms. Additionally, Silla et al. (2024) established that procedural flexibility with fractions is a crucial predictor of students' knowledge of algebra.

The many factors that can predict students' fraction skills suggest that early education is a crucial time to develop the cognitive knowledge needed for a strong conceptual understanding of fraction skills. Fraction computation skills are uniquely predicted by conceptual knowledge, arithmetic fluency, working memory, and attentive behaviors in the classrooms (Hecht, 1998; Hecht et al., 2003, 2007; Hecht & Vagi, 2010; Mazzocco & Devlin, 2008). For example, first graders' knowledge of whole number magnitudes and arithmetic predicted their knowledge of fraction magnitudes in eighth grade and fraction arithmetic in seventh grade (Bailey et al., 2014). Contrastingly, Braithwaite et al.'s (2017) study, which looked at fraction tasks that did not require computation, suggests that children do not have a strong conceptual understanding of fraction arithmetic.

Whole number and fraction skills are intertwined, so it can be challenging to pinpoint potential causes for trouble with fractions. Although the computation methods typically applied to whole numbers appear different from those applied to fractions, the underlying principles are the same. Ma and Kessel (2022) emphasized the similarities between the definitions of whole numbers and fractions, and their respective mathematical operations. Furthermore, central executive working memory predicts whole number computation skills (Fuchs et al., 2006; Fuchs et al., 2008; Geary et al., 2007; Swanson & Kim, 2007), and the development of fraction skills seems to involve similar cognition.

Despite the similarities with whole number and fractional computations, Seethaler et al.'s (2011) investigation of the cognitive abilities that account for the development of computational skills with rational numbers and whole numbers found that language ability was a unique factor that influenced rational number skills. Additionally, when comparing students with difficulties in only mathematics, only reading, and normally achieving students, students with difficulties in both mathematics and reading demonstrated pervasive issues in more areas of mathematical thinking (Jordan & Hanich, 2000). The connection between mathematics and language highlights the importance of understanding the meaning behind the steps involved in fraction arithmetic.

The first NCTM's (2020) middle school standard, "Knowing and Understanding Meaningful Mathematics," highlights rational numbers as an essential concept for educators' comprehensive conceptual insight and procedural expertise. The positioning of rational numbers as a key focus of the NCTM standards for middle school mathematics, coupled with the dismal results of studies showing a subpar understanding of rational numbers on average, is a significant impetus for change to support student sense-making of rational numbers (Carpenter et al., 1980; Lortie-Forgues et al., 2015).

It follows that understanding the principles behind whole number arithmetic is crucial for a better understanding of fractions. The 2020 NCTM standards for secondary school include number theory, which requires "more sophisticated insights and understandings" (p. 8) of whole numbers and fractions. Within the NCTM secondary school standards, ratios are a subtopic of the field of number theory. As such, studying number theory as a conceptual field may help resolve issues with rational number operations.

Number Theory

Number theory is defined as the study of positive integers, also known as whole numbers (Dunham, 2024). Particularly, elementary number theory uses arithmetic, geometry, and high school algebra to find integer or rational solutions to equations (Terr, 2024). It includes topics such as the divisibility of integers, remainders, prime and composite numbers, and perfect squares. Some early number theory findings include the Euclidean algorithm, which finds the greatest common divisor of two integers; the Sieve of Eratosthenes, which provides a method of finding prime numbers; and Fermat's Last Theorem, which proves that $a! + b! = c!$ only has natural number solutions when $n = 1$ or $n = 2$.

Elementary number theory appears in the Common Core State Standards in Mathematics (CCSSM) early in the sequence of math topics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP-CCSSO], 2010). The foundations of place value appear in the CCSSM as early as kindergarten, continuing to be explicitly mentioned through fifth grade. According to the CCSSM, students in the third grade should be able to view multiplication as finding unknown products and division as finding unknown factors. In the fourth grade, the CCSSM expect students to gain familiarity with factors and multiples and be able to determine whether numbers are prime or composite. Place value understanding is also generalized in the fourth grade, where place value properties should be used to perform arithmetic, including finding whole-number quotients and remainders. In the fifth grade, students continue to develop the math concepts introduced in the earlier grades. The greatest common factor and least common multiple are introduced in the

sixth grade, and knowledge about whole numbers is expanded to include the set of rational numbers at this stage.

These topics introduced in the kindergarten through grade six as listed above relate to number theory and rely on the concept of divisibility (NGACBP-CCSSO, 2010). Divisibility relates to multiplication, division, prime numbers, and place value. It follows that the middle school grades could be an ideal placement for an exploration of divisibility, after its foundations have been introduced in earlier grades, and before the introduction of algebraic applications in later grades, when divisibility is required as foundational knowledge. For example, students in the middle school grades can investigate number systems through projects and activities much like the explorations used by Hall (2007) with K-8 teachers. In the sixth, seventh and eighth grades, activities such as Bhattacharya et al.'s (2016) "Mystery Fractions" can encourage problem-solving using multiple strategies while covering many middle-school mathematics concepts, such as place value, fraction sense, scale factors, and reasoning and proof.

Despite the importance of divisibility in elementary and middle school mathematics, there is room for improvement in how it is taught. When Zazkis and Liljedahl (2004) investigated preservice elementary teachers' understanding of prime numbers, they found that the participants' knowledge of prime numbers did not always result in the use of this knowledge when solving problems. Zazkis and Campbell's (1996) study resulted in similar findings when they investigated preservice teachers' conceptual understanding of divisibility. The researchers designed questions that require significant time and effort to solve procedurally and yet are readily resolvable with the appropriate conceptual understanding. Rather than thinking of divisibility as a relationship between

integers, the majority of participants only regarded divisibility as the result of division. Many participants also misapplied divisibility rules in overgeneralized ways when specific rules were not available. In another study of preservice teachers' conceptual understanding of divisibility, Brown, Thomas, and Tolias (2002) found that teachers had varied degrees of understanding of why the prime-factorization method provides the LCM of a group of numbers. Brown et al. suggested that an understanding of the different methods of calculating the least common multiple and how these methods are connected is critical to teachers' comprehensive grasp of the ideas related to this integral number theory concept.

Interviews with preservice teachers also uncovered difficulties in understanding the multiplicative structure of natural numbers and divisibility (Zazkis & Campbell, 1996). For example, when Zazkis and Campbell asked the participants if any numbers between 12,358 and 12,368 were divisible by 7, preservice teachers chose a variety of approaches. Many teachers used procedural approaches, such as dividing by hand on a calculator; some used conceptual approaches about divisibility, and others used variations between the procedural and conceptual approaches. The non-conceptual approaches suggested teachers' difficulties in understanding whole number division.

One goal for students and teachers should be to reach a depth and breadth of understanding of mathematical concepts. Liping Ma (1999) found that teachers of elementary mathematics with a higher degree of "profound understanding of fundamental mathematics" (PUFM) are more effective than their peers with a lesser degree of PUFM. Ma explained that teachers with PUFM intend to make connections among concepts and procedures, thereby teaching students a unified body of knowledge. To support students

with conceptual understanding of divisibility and the multiplicative structure of natural numbers, teachers must possess this understanding.

While there are extracurricular number theory courses for advanced middle and high school students, typically available online during the school year or through different avenues over the summer, there are limited opportunities for students to engage with these topics in K-8 school curricula. Some programs specifically target students who are looking to prepare for mathematics competitions. As part of the many regional, national, and international mathematics competitions, number theory is commonly on the list of topics competitors are expected to know.

Preparation material for the International Mathematical Olympiad (IMO), such as the IMO Compendium, includes number theory as one of four to six main areas (Djukić et al., 2011). In the compendium, which was written by four IMO medalists, the basic concepts included are algebra, geometry, number theory, and combinatorics. Other competitions, including the American Mathematics Competitions (AMC), MATHCOUNTS, and the Mathematical Olympiads for Elementary and Middle Schools (MOEMS), are popular among advanced math students. The AMC is a large series of national math competitions in the United States that test students on number theory concepts at the eighth (AMC 8), tenth (AMC 10), and twelfth (AMC 12) grade levels (Art of Problem Solving, 2024a, 2024b, 2024c). The MATHCOUNTS competition series for sixth to eighth graders includes various problems about prime numbers, LCM, and GCD (MATHCOUNTS Foundation, 2024). The Mathematical Olympiads for Elementary and Middle Schools, a large elementary and middle school math competition, includes divisibility, LCM, GCF, prime numbers, and many more topics related to integer

properties outlined in the preparation document, “What Every Young Mathlete Should Know” (MOEMS, 2022).

In addition to math competitions, which draw mathematical minds from around the world, many talented youth math programs teach number theory. Elite universities host math camps and courses for high-achieving middle schoolers, such as CyberMath Academy’s Summer Math Camp at Harvard University’s Faculty Club, Math Online Summer Academy for middle schoolers (MOSA), Johns Hopkins’ Center for Talented Youth’s various programs (CTY), and the AwesomeMath Summer Program. Cybermath offers an Advanced Middle School Math course that teaches students to solve mathematical puzzles and cyphers and covers topics that go beyond the Common Core Standards, including number theory and combinatorics (CyberMath, 2022). MOSA, which is run by students and alumni of Stanford University Math Camp (SUMaC), includes number theory as one of its four main topics (MOSA, 2023); Johns Hopkins’ CTY offers a three-week number theory and cryptology course for seventh to eleventh graders (Johns Hopkins CTY, 2024); and AwesomeMath’s three-week course offerings include a number theory course for middle and high schoolers (AwesomeMath, 2024). The value of learning number theory is not lost on programs for advanced students, but it does not seem to translate to school math curricula.

Despite the NCTM’s (1989) recognition of number theory as a subject that can further develop mathematical concepts and problem-solving skills, number theory is not typically a stand-alone subject area in the K-12 curriculum (Campbell, 2002). Children are taught to count objects in the base 10 number system early on (CCSM, 2010). By middle school, the CCSSM expect students to expand that understanding to incorporate

zero, negative numbers, fractions, and decimals. Different types of numbers are introduced, such as even and odd numbers, prime and composite numbers, and squares and cubes. Middle schoolers learn about divisibility, factors, and multiples, a prerequisite for factoring and a key part of algebra and calculus. These topics all relate to prime numbers, the greatest common factor, and the least common multiple, which can be found systematically by looking at prime factorization. By understanding the interconnectedness of these concepts and unifying them in a number theory unit or curriculum, students can further develop their understanding of key mathematical concepts, patterns, and reasoning (Campbell, 2002). As noted by Schmidt and Houang (2012), number theory has the potential to impact students' K-12 education trajectory. Yet, school curricula do not cover the number theory topics of primes and factorization in a way consistent with the CCSSM. There is an opportunity for school systems to engage students with number theory concepts by intentionally developing age-appropriate coursework on these topics and other related material. The following chapter, Chapter III, outlines a number theory curriculum for middle school students, and suggested teaching strategies with examples of lesson excerpts and discussion of assessment and pedagogy are provided in Chapter IV. The final chapter, Chapter V, details the lessons of the curriculum.

Chapter III.

The Number Theory Curriculum

This chapter provides a comprehensive overview of the elementary number theory curriculum developed for middle school students. It summarizes the program's structure for each year, highlighting major themes, skills progression, and the rationales for topic sequencing. By outlining how foundational concepts are introduced and built upon, this chapter illustrates the curriculum's coherence and its alignment with the goal of strengthening students' number sense and mathematical reasoning. Throughout the four years of study, emphasis is placed on developing clear processes, asking questions, and noticing patterns rather than memorizing procedures.

Year 1: Classification of Numbers, Building Blocks, and Patterns

Year 1 of the four-year mathematics curriculum establishes the foundational understanding of numbers and their structures, setting the stage for increasingly sophisticated mathematical reasoning. In the first year of the curriculum, students develop foundational number-theoretic habits by exploring divisibility, factors, and the structures of whole numbers. They investigate patterns in numbers, build intuition about factors and multiples, and practice simplifying fractions. Students encounter prime numbers through tools such as the Sieve of Eratosthenes, and they begin to see prime factorization as a way to understand the composition of numbers. The year concludes with an introduction to two numerical sequences: Pascal's Triangle and the Fibonacci

sequence. These sequences highlight recurring structures in mathematics, such as adding two previous entities to form the next entry in the sequence. Formal terminologies are introduced, while ensuring that students develop flexible reasoning and conceptual understanding alongside procedural skills. Reflection and structured assessments support the development of mathematical habits, including reasoning, questioning, and pattern noticing. By framing Year 1 as the exploration of numerical building blocks, students are prepared to progress toward mastery of factors, multiples, sequence analysis, and ultimately applications in areas such as cryptography, ensuring a coherent and cumulative learning trajectory across the four years.

The Year 1 course of study is designed to address four major themes during the first year of study.

1. Foundations of Number Theory: Understanding the classification of numbers, divisibility, and the building blocks of arithmetic.
2. Factors, Multiples, and Prime Numbers: Exploring how numbers can be broken down into smaller components, leading to prime factorization.
3. Fractions and Division: Developing the concept of fractions as division, simplifying fractions, and connecting fractions to factors and multiples.
4. Pattern Recognition and Sequences: Investigating recurring structures in Pascal's Triangle and the Fibonacci sequence.

Table 1 below outlines the math topics (eight units) and progression of targeted skills, and links lessons 1.1 through 1.12 (Chapter V) to the major-themes.

Table 1. Progression of Skills throughout Year 1.

Unit	Skills developed	Lessons	Major themes
Number classification & divisibility	Identify whole, natural, rational numbers; distinguish odd/even; define divisibility, factors, multiples	Lessons 1.1 and 1.2	Foundations of number theory
Arithmetic operations & formalization	Apply arithmetic operations; describe classifications mathematically	Lesson 1.3	Foundations of number theory
Assessment & reflection	Self-assess understanding of number properties	Lesson 1.4	Foundations of number theory
Fractions	Connect fractions to division; reduce fractions using factors; interpret fraction bar as grouping	Lessons 1.5 and 1.6	Factors, multiples, and prime numbers; fractions and divisions
Prime numbers	Identify prime and composite numbers; understand primes as building blocks; recognize special cases	Lessons 1.7, 1.8, and 1.10	Factors, multiples, and prime numbers
Assessment & reflection	Evaluate understanding of fractions and prime numbers	Lesson 1.9	Factors, multiples, and prime numbers
Sequences & patterns	Define sequences; construct Pascal's Triangle; write Fibonacci sequence	Lesson 1.11	Pattern recognition and sequences
Year wrap-up & assessment	Integrate skills and reflect on learning	Lesson 1.12	Foundations of number theory; factors, multiples, and prime numbers; fractions and divisions; pattern recognition and sequences

The course sequence is designed to scaffold conceptual understanding in a logical progression. Starting with the familiar and concrete, students begin with number classification and divisibility, grounding them in concepts they encounter in everyday arithmetic. As such, students build procedural fluency alongside conceptual

understanding. These lessons on arithmetic operations and fractions allow students to apply their knowledge while developing reasoning skills. Prime numbers and factorization are introduced early to establish the idea that all numbers can be decomposed into primes. Throughout the year, students gain tools for understanding future mathematical structures. The Year 1 curriculum connects abstract concepts to patterns with the introduction of Pascal's Triangle and the Fibonacci sequence at the end of the year, which illustrates the recurring structures in mathematics, reinforcing pattern recognition and intuition. The integration of reflections and assessments encourages students to periodically consolidate their knowledge, reflect on their thinking, and identify areas for growth. Overall, the structure balances conceptual understanding, procedural skills, and mathematical reasoning, ensuring students develop a strong foundation for subsequent years.

The objectives for the eight units with the 12 lessons for Year 1 are as follows: Unit 1 - Number Classification & Divisibility (Lessons 1.1 and 1.2): Unit 1 comprises two lessons, "Introduction to Number Theory and Classification of Numbers" and "Divisibility." Students will identify and classify numbers as whole, natural, or rational; distinguish odd and even numbers; and explain what it means for one number to be divisible by another. Students will define and use the concepts of divisibility, factors, and multiples to analyze and categorize numbers accurately.

Unit 2 - Reinforcing Arithmetic Operations (Lesson 1.3): Unit 2 comprises one lesson, "Arithmetic Operations on Natural Numbers." Students will apply basic arithmetic operations to explore number relationships and will learn to express number

classifications more formally. They will justify their thinking using mathematical vocabulary and demonstrate how operations interact with number sets.

Unit 3 - Assessment & Reflection (Lesson 1.4): Unit 3 is the first Check-Your-Understanding assessment of the year. Students will evaluate their own understanding of number properties through guided reflections and assessment tasks. They will identify areas of mastery and topics that require further practice.

Unit 4 - Fractions (Lessons 1.5 and 1.6): Unit 4 comprises two lessons, “Understanding Fractions” and “More Fractions and Factors.” Students will connect fractions to division and explore the fraction bar as a representation of grouping or comparison. They will use factors to reduce fractions and explain why simplification works. Students will recognize fractions as rational numbers and relate fraction structure to earlier work in Unit 1 on divisibility.

Unit 5 - Prime Numbers (Lessons 1.7, 1.8, and 1.10): Unit 5 comprises three lessons, “Defining Prime Numbers,” “Prime Numbers as Building Blocks,” and “Prime Factorization.” Students will identify prime and composite numbers and describe primes as the fundamental building blocks of whole numbers. They will understand and justify special cases, such as why 1 is neither prime nor composite and why 2 is the only even prime. Students will apply their knowledge of factors to analyze numbers efficiently.

Unit 6 - Assessment & Reflection (Lesson 1.9): Unit 6 is the second Check-Your-Understanding assessment of the year. Students will demonstrate and reflect on their understanding of fractions, prime numbers, and factor relationships. They will revisit their earlier ideas and evaluate how their thinking has grown across the unit.

Unit 7 - Sequences & Patterns (Lesson 1.11): Unit 7 comprises one lesson, "Sequences Involving Integers." Students will define numerical sequences, spot patterns, construct Pascal's Triangle, and write the Fibonacci sequence. They will discuss how patterns emerge from simple rules and connect sequence behavior to earlier number concepts.

Unit 8 - Year Wrap-Up & Assessment (Lesson 1.12): Unit 8 comprises the first year's final, cumulative Check-Your-Understanding assessment. Students will integrate skills from across the unit to analyze, classify, and reason about numbers in new contexts. They will participate in a reflection activity that highlights their growth and prepares them for continued study in more complex mathematical topics.

Year 2: Remainders, Numerical Patterns, and Division

In the second year, students extend their understanding of divisibility by examining the role of remainders in division. Through informal experiences with modular reasoning, such as clock arithmetic, day-of-the-week problems, and other remainder-based contexts, students learn to recognize when a problem is fundamentally about remainders and develop efficient strategies for reasoning about them. Students continue to strengthen their understanding of fractions by exploring multiple ways to interpret division and making connections between division, fractions, and remainders. As they

gain fluency with informal modular reasoning, they apply these ideas to simple number-theoretic questions, such as determining the parity of sums and predicting whether a fraction's decimal representation terminates or repeats. Cyclic patterns in familiar contexts introduce modular thinking, which is then formalized to support flexible problem-solving. Students apply these ideas to numerical patterns, parity, and sequence analysis while strengthening procedural fluency and representational flexibility. This year deepens the foundation laid in Year 1, preparing students to approach advanced number-theoretic concepts and real-world applications, including cryptography, with confidence and conceptual insight.

The Year 2 course of study is designed to address four major themes during the second year of study.

1. Remainders and Modular Reasoning: Recognizing cyclic patterns in numbers and using remainders to solve real-world and abstract problems.
2. Fractions, Decimals, and Division Connections: Linking division, fractions, decimals, mixed numbers, and remainders to strengthen numerical fluency.
3. Prime Factorization Applications: Using prime factorization to predict factors, understand divisibility, and anticipate patterns in decimals and fractions.
4. Numerical Properties and Patterns: Exploring parity, sums of sequences, and patterns in numbers through modular reasoning.

Table 2 below outlines the math topics (10 units) and progression of targeted skills, and links lessons 2.1 through 2.12 (Chapter V) to the major-themes.

Table 2. Progression of Skills throughout Year 2.

Unit	Skills developed	Lessons	Major themes
Review and consolidation	Recall Year 1 concepts: number classification, prime factorization, sequences	Lessons 2.1 and 2.2	Numerical properties and patterns
Remainders in context	Recognize dividends, divisors, quotients, and remainders; identify cyclic/remainder-based problems	Lesson 2.3	Remainders and modular reasoning; numerical properties and patterns
Assessment & reflection	Consolidate understanding of remainders, prime numbers, and divisibility	Lesson 2.4	Remainders and modular reasoning; numerical properties and patterns
Modular arithmetic	Perform modular calculations; reason about cycles and remainders; connect to arithmetic operations	Lessons 2.5 and 2.6	Remainders and modular reasoning
Numerical patterns & parity	Identify patterns in sums of integers; connect parity to mod 2; explore arithmetic sequences	Lesson 2.7	Numerical properties and patterns
Fraction and division connections	Relate improper fractions, mixed numbers, decimals, and remainders; strengthen representational fluency	Lesson 2.8	Fractions, decimals, and division connections
Assessment & reflection	Consolidate modular reasoning, fraction-decimal-remainder connections, and parity concepts	Lesson 2.9	Remainders and modular reasoning; fractions, decimals, and division connections; numerical properties and patterns
Division by powers of 10	Calculate remainders with powers of 10; reinforce base-10 structure; link terminating/repeating decimals to prime factorization	Lesson 2.10	Prime factorization applications
Mixed word problems & procedural fluency	Solve flexible problems involving fractions, decimals, division; create strategy notes; represent results meaningfully	Lesson 2.11	Fractions, decimals, and division connections
Year wrap-up & assessment	Integrate modular reasoning, fractions, decimals, and numerical properties; reflect on learning	Lesson 2.12	Remainders and modular reasoning; fractions, decimals, and division connections; prime factorization applications; numerical properties and patterns

The Year 2 course of study is structured to begin with review and consolidation, reinforcing Year 1 concepts such as number classification and prime factorization to ensure a strong foundation for new material. Students work on finding remainders in familiar contexts, such as clocks and day-of-week problems, to develop intuition about modular thinking. Building on the familiar contexts, modular arithmetic is formally introduced to equip students with systematic ways to handle remainders and to support abstract reasoning. Appropriate modulo notation is also introduced.

As the year progresses, students learn to connect division to fractions and decimals. The connection aims to strengthen representational fluency and to deepen student understanding of division. Ongoing engagement with numerical patterns through the study of parity, sums, sequences, and modular reasoning reinforces connections between arithmetic and number theory. Procedural fluency is developed through focused practice with division by powers of 10 and word problems carefully designed to allow students to flexibly apply concepts across a range of contexts. Reflection and assessment are integrated throughout the course, with “Check Your Understanding” lessons to consolidate learning, surface misconceptions, and encourage students to develop metacognitive awareness of their own mathematical thinking. Overall, Year 2 emphasizes flexible reasoning, abstraction, and representational fluency to prepare students for more advanced topics in number theory, algebra, and problem-solving.

The objectives for the 10 units, consisting of the 12 lessons for Year 2, are as follows:

Unit 1 - Review and Consolidation (Lessons 2.1 and 2.2): Unit 1 comprises two lessons, “Number Theory and Factor Review” and “Prime Factorization and Divisibility.” Students will recall and apply key concepts from the previous year, including number classification, prime factorization, and numerical sequences. They will activate prior knowledge through review tasks and problem sets designed to strengthen foundational understanding. Students will also re-establish vocabulary and reasoning strategies that will be used throughout the year.

Unit 2 - Remainders in Context (Lesson 2.3): Unit 2 comprises one lesson, “Remainder Strategies.” Students will identify and correctly use the terms dividend, divisor, quotient, and remainder. They will solve contextual problems where remainders play an essential role to support students’ recognition of situations in which remainder-based reasoning reveals patterns or cyclic behavior. Students begin to view division as both a computational process and a conceptual exercise.

Unit 3 - Assessment & Reflection (Lesson 2.4): Unit 3 is the first Check-Your-Understanding assessment of the year. Students will evaluate their own understanding of number properties through guided reflections and assessment tasks. They will identify areas of mastery and topics that require further practice.

Unit 4 - Modular Arithmetic (Lessons 2.5 and 2.6): Unit 4 comprises two lessons, “Introduction to Modular Arithmetic” and “More Modular Arithmetic.” Students will perform modular arithmetic calculations and explain how modular arithmetic describes

cycles and repeating patterns. They will connect remainders to arithmetic operations, explore equivalence classes, and use modular reasoning to simplify numerical expressions. This stage lays the foundation for more advanced topics in number theory.

Unit 5 - Numerical Patterns & Parity (Lesson 2.7): Unit 5 comprises one lesson, "Numerical Properties." Students will investigate patterns in sums and differences of integers, with special attention to parity (evenness and oddness). They will connect parity rules to modular arithmetic, particularly mod 2, and use patterns to predict results, justify claims, and analyze arithmetic sequences. Students strengthen their ability to reason about structure within the number system.

Unit 6 - Fraction and Division Connections (Lesson 2.8): Unit 6 comprises one lesson, "Number Formats and Divisibility." Students will deepen fluency with improper fractions, mixed numbers, decimals, and remainders, viewing each as a different representation of division. They will flexibly convert among these forms and explain their relationships. Students will use visual models and symbolic reasoning to build representational fluency across formats.

Unit 7 - Assessment & Reflection (Lesson 2.9): Unit 7 is the second Check-Your-Understanding assessment of the year. Students will demonstrate their understanding of modular reasoning, fraction-to-decimal conversions, remainder connections, and parity. They will reflect on their problem-solving strategies and identify areas that require

additional practice or clarification. This stage emphasizes metacognition and consolidation.

Unit 8 - Division by Powers of 10 (Lesson 2.10): Unit 8 comprises one lesson, “Place Value and Terminating Versus Repeating Decimals.” Students will calculate remainders in problems involving powers of 10 and explore how base-10 structure influences divisibility and remainder patterns. They will connect terminating and repeating decimals to prime factorization, discovering how the structure of numbers determines decimal behavior. This stage strengthens their understanding of place value and number composition.

Unit 9 - Mixed Word Problems & Procedural Fluency (Lesson 2.11): Unit 9 comprises one lesson, “Arithmetic Skill Review and Mixed Word Problems.” Students will solve multi-step problems involving fractions, decimals, division, and remainders. They will create strategy notes, compare solution paths, and justify their reasoning. Students practice representing results in contextually meaningful ways to emphasize clarity and mathematical communication.

Unit 10 - Year Wrap-Up & Assessment (Lesson 2.12): Unit 10 comprises the second year’s final, cumulative Check-Your-Understanding assessment. Students will integrate modular arithmetic, fraction-decimal connections, division concepts, and numerical properties into a comprehensive understanding of number relationships. They will

complete a summative assessment and engage in a reflection activity to highlight growth and prepare for future mathematical study.

Year 3: Divisibility, Factors, and Multiples

In the third year, students build on their earlier work with remainders and factorization by examining the structure underlying the division algorithm and modular arithmetic. They encounter formal terminology such as “n divides m” and “a mod b,” to connect the Year 3 concepts to the informal modular reasoning developed in Year 2. The Year 3 course of study is oriented to deepen students’ understanding of prime factorization and to use the acquired knowledge to determine the greatest common factor (GCF) and least common multiple (LCM) of numbers suitable for hand computation. They also explore the fundamental relationship:

$$LCM(a, b) \times GCF(a, b) = a \times b.$$

Cyclic patterns, sequences, and word problems provide contexts for developing flexible reasoning and pattern recognition. This work strengthens students’ number sense, supports flexible reasoning with fractions, and lays the foundation for early algebraic thinking. The third year’s study emphasizes the interplay between procedural skill, pattern recognition, and conceptual understanding to further consolidate foundational number-theoretic skills and prepare students for more symbolic reasoning, advanced problem-solving, and applications in cryptography in Year 4.

The Year 3 course of study is designed to address five major themes during the first year of study.

1. Formalizing Divisibility and Modular Arithmetic: Moving from informal modular reasoning to formal terminology and structured approaches.

2. Prime Factorization Applied: Using prime factorization to find GCFs, LCMs, and support fraction operations.
3. Patterns in Numbers and Sequences: Recognizing cyclic patterns and sequences to strengthen reasoning and problem-solving.
4. Division Algorithm: Understanding the structure of division and connecting remainders to decimal and fractional representations.
5. Fractions and Number Sense: Applying factorization knowledge to arithmetic with fractions, reinforcing flexibility and conceptual understanding.

Table 3 below outlines the math topics (10 units) and progression of targeted skills, and links lessons 3.1 through 3.12 (Chapter V) to the major-themes.

Table 3. Progression of Skills throughout Year 3.

Unit	Skills developed	Lessons	Major themes
Review and consolidation	Recall Year 2 concepts: number classification, modular arithmetic, prime factorization	Lesson 3.1	Remainders and modular reasoning; prime factorization applications; numerical properties and patterns (from Year 2)
Division algorithm	Understand conceptual steps of division; connect remainder to decimals	Lesson 3.2	Division algorithm
Divisibility rules	Use correct terminology; check divisibility by numbers 2-11	Lesson 3.3	Formalizing divisibility and modular arithmetic
Assessment & reflection	Consolidate understanding of divisibility and division	Lesson 3.4	Formalizing divisibility and modular arithmetic; division algorithm
Sequences and patterns	Recognize and describe patterns; solve cyclic/remainder-based problems	Lessons 3.5 and 3.6	Formalizing divisibility and modular arithmetic; patterns in numbers and sequences
GCF and LCM	Determine GCF and LCM using brute force and prime factorization; understand relationships	Lessons 3.7 and 3.8	Prime factorization applied
Assessment & reflection	Consolidate understanding of modular arithmetic, prime factorization, GCF, LCM	Lesson 3.9	Formalizing divisibility and modular arithmetic; prime factorization applied
Word problems & application	Solve real-world GCF and LCM problems; compare numbers via prime factorization	Lesson 3.10	Prime factorization applied
Fractions and prime factors	Apply factorization to fraction arithmetic; add and subtract fractions	Lesson 3.11	Prime factorization applied; fractions and number sense
Year wrap-up & assessment	Integrate skills and reflect on learning	Lesson 3.12	Formalizing divisibility and modular arithmetic; prime factorization applied; patterns in numbers and sequences

The Year 3 course of study begins with a review, reinforcing Year 2 concepts to ensure fluency in modular reasoning, divisibility, and prime factorization. Formal division structures are then introduced, with the division algorithm and divisibility rules to provide a foundation for systematic reasoning. The goal of the Year 3 study is for students to develop factor-based reasoning by studying GCF and LCM through brute force and prime factorization to support conceptual understanding and procedural fluency. Connections between patterns and reasoning are strengthened through the study of sequences, cyclic patterns, and divisibility to promote flexible thinking and prepare students for future algebraic reasoning. Knowledge is applied through word problems and fraction operations that situate skills in real-world contexts. Reflection and assessment are integrated throughout the course, with regular "Check Your Understanding" lessons that allow students to consolidate learning and monitor their own progress. Overall, Year 3 emphasizes conceptual mastery of divisibility and factorization, building the necessary foundations for symbolic algebra, advanced number theory, and problem-solving in Year 4.

The objectives for the 10 units, consisting of the 12 lessons for Year 3, are as follows:

Unit 1 - Review and Consolidation (Lesson 3.1): Unit 1 comprises one lesson, "Number Theory and Modular Arithmetic Review." Students will recall and apply key Year 2 concepts, including number classification, modular arithmetic, and prime factorization. They will review vocabulary, reinforce prior problem-solving strategies, and reconnect with foundational ideas to prepare for more advanced work in division and number theory.

Unit 2 - Division Algorithm (Lesson 3.2): Unit 2 comprises one lesson, “The Division Algorithm.” Students will explore the conceptual steps of division, including the relationships between the dividend, divisor, quotient, and remainder. They will connect the remainder to decimal representations and understand how division extends beyond whole numbers. This stage emphasizes conceptual understanding alongside procedural fluency.

Unit 3 - Divisibility Rules (Lesson 3.3): Unit 3 comprises one lesson, “Divisibility Rules.” Students will learn and apply divisibility rules for the integers 2 through 11, using correct terminology. They will check whether numbers are divisible, explain their reasoning, and connect these rules to prime factorization and modular arithmetic. This stage strengthens their ability to analyze numbers efficiently.

Unit 4 - Assessment & Reflection (Lesson 3.4): Unit 4 is the first Check-Your-Understanding assessment of the year. Students will consolidate their understanding of division and divisibility by reflecting on strategies, clarifying misconceptions, and self-assessing their readiness for more complex applications.

Unit 5 - Sequences and Patterns (Lessons 3.5 and 3.6): Unit 5 comprises two lessons, “Sequences and Pattern Recognition“ and “Prime Numbers and Pattern-Recognition.” Students will recognize, describe, and extend numerical patterns, including cyclic and

remainder-based patterns. They will solve problems that require pattern recognition and reasoning, using modular arithmetic and other strategies to justify their solutions.

Unit 6 - GCF and LCM (Lessons 3.7 and 3.8): Unit 6 comprises two lessons, “Greatest Common Factor and Prime Factorization” and “Least Common Multiple and Prime Factorization.” Students will determine the greatest common factor (GCF) and least common multiple (LCM) of numbers using brute-force listing and prime factorization. They will explore relationships between GCF, LCM, and divisibility, and use these concepts to compare numbers and simplify calculations.

Unit 7 - Assessment & Reflection (Lesson 3.9): Unit 7 is the second Check-Your-Understanding assessment of the year. Students will consolidate their understanding of modular arithmetic, prime factorization, GCF, and LCM. Reflection tasks will allow them to connect these topics and identify areas needing further practice.

Unit 8 - Word Problems & Application (Lesson 3.10): Unit 8 comprises one lesson, “More GCF and LCM.” Students will solve real-world problems involving GCF and LCM, compare numbers, make predictions, and apply prime factorization to practical contexts. Emphasis will be on reasoning, strategy selection, and clear communication of solutions.

Unit 9 - Fractions and Prime Factors (Lesson 3.11): Unit 9 comprises one lesson, “Fraction Arithmetic and Prime Factors.” Students will apply prime factorization to

fraction arithmetic, particularly in adding and subtracting fractions with unlike denominators. They will see the connection between factors, multiples, and fraction operations to reinforce procedural fluency and conceptual understanding.

Unit 10 - Year Wrap-Up & Assessment (Lesson 3.12): Unit 10 comprises the third year's final, cumulative Check-Your-Understanding assessment. Students will integrate skills from across the year, including divisibility, modular reasoning, GCF/LCM, and fraction operations. They will engage in reflective activities and complete a summative assessment, demonstrating growth and readiness for more advanced mathematical study.

Year 4: Encryption and Ciphers

In the fourth year, students apply the number-theoretic foundations developed over the previous three years to understand how prime numbers underpin modern encryption. They begin by exploring real-world contexts where digital security is essential, such as online communication, banking, and data privacy, and investigate why primes are particularly suited for securing information. Students study classical ciphers, including Caesar and general substitution ciphers, to build intuition about encoding and decoding messages.

As the unit progresses, students explore symmetric (shared-key) encryption and public-key (asymmetric) cryptography, investigating why one-way mathematical structures and large prime numbers are essential for secure communication. Guided activities introduce the principles behind the Diffie-Hellman key exchange and RSA encryption, highlighting how prime factorization and modular arithmetic underlie their security.

The year concludes with a comprehensive review that connects number theory concepts to fractions, modular arithmetic, and classical and modern cryptography. By the end of the curriculum, students can describe, in accessible terms, how asymmetric encryption works, explain its reliance on number theory, and appreciate how abstract mathematical ideas shape real-world digital security.

The Year 4 course of study is designed to address five major themes during the first year of study.

1. Applications of Number Theory: Connecting prime numbers, factors, and modular arithmetic to practical encryption problems.
2. Classical Cryptography: Understanding the mechanics and patterns behind Caesar and general substitution ciphers.
3. Symmetric vs. Asymmetric Encryption: Contrasting shared-key systems with public-key systems and exploring mathematical reasons for their differing security.
4. One-Way Functions and Prime Factorization: Recognizing how one-way mathematical structures and large primes provide the foundation for modern encryption.
5. Integration and Application: Applying number theory concepts to fractions, modular arithmetic, classical ciphers, and modern encryption scenarios.

Table 4 below outlines the math topics (10 units) and progression of targeted skills, and links lessons 4.1 through 4.12 (Chapter V) to the major-themes.

Table 4. Progression of Skills throughout Year 4.

Unit	Skills developed	Lessons	Major themes
Foundational review	Recall divisibility, factors, multiples, and primes	Lesson 4.1	Factors, multiples, and prime numbers (from Year 1)
Introduction to cryptography	Understand basic cryptography concepts, key terms (cipher, plaintext, ciphertext), and the role of primes	Lesson 4.2	Classical cryptography
Classical ciphers	Encode/decode messages using Caesar and other substitution ciphers; recognize patterns in encryption	Lesson 4.3	Classical cryptography
Assessment & reflection	Consolidate introductory cryptography and simple decryption skills	Lesson 4.4	Applications of number theory; classical cryptography
Encryption theory	Compare symmetric vs. asymmetric encryption; understand why one-way functions and primes are essential	Lessons 4.5 and 4.6	Symmetric vs. asymmetric encryption; one-way functions and prime factorization
Applied cryptography	Encrypt and decrypt messages using public/private keys	Lessons 4.7	Applications of number theory
Mathematical constraints and large primes	Understand why factoring large numbers is computationally hard and how this underpins secure encryption	Lesson 4.8	One-way functions and prime factorization
Assessment & reflection	Consolidate understanding of cryptography concepts and their connection to number theory	Lesson 4.9	Applications of number theory
Number theory review	Refresh number theory foundations (divisibility, primes, GCF/LCM, fractions, modular arithmetic) and apply number theory to fractions, modular arithmetic, classical ciphers, and modern encryption	Lessons 4.10 and 4.11	Integration and application
Course wrap-up game	Reflect on learning, consolidate skills, and demonstrate understanding of the 4-year curriculum	Lesson 4.12	Applications of number theory; integration and application

The Year 4 course of study is structured to build on prior foundations, drawing on the number-theoretic tools developed in Years 1, 2, and 3, such as prime factorization, modular arithmetic, fractions, and divisibility, to support an introduction to encryption. Cryptography is introduced gradually, beginning with familiar and tangible examples like the Caesar cipher before progressing to more sophisticated substitution methods and public-key systems. Students compare symmetric and asymmetric encryption to highlight the mathematical reasoning underlying secure communication. A focus on one-way functions and computational constraints, including the role of large primes and modular arithmetic, helps students understand why encryption is secure and why factoring is computationally difficult. Reflection and review are integrated throughout the curriculum, with lessons that consolidate four years of number-theoretic learning, strengthen connections between concepts and applications, and provide opportunities to apply knowledge in authentic contexts.

The objectives for the 10 units, consisting of the 12 lessons for Year 4, are as follows: Unit 1 - Foundational Review (Lesson 4.1): Unit 1 comprises one lesson, “Number Theory and Prime Number Review.” Students will recall and apply key number theory concepts from previous years, including divisibility, factors and multiples, and identifying prime numbers. This stage ensures students are prepared for advanced applications in cryptography and secure communications.

Unit 2 - Introduction to Cryptography (Lesson 4.2): Unit 2 comprises one lesson, “Introduction to Cryptography.” Students will be introduced to the basic concepts of cryptography. They will learn key terms such as cipher, plaintext, and ciphertext, and

explore the role of prime numbers in creating secure systems. This stage emphasizes conceptual understanding and the real-world relevance of number theory.

Unit 3 - Classical Ciphers (Lesson 4.3): Unit 3 comprises one lesson, “Classical Ciphers: Substitution.” Students will encode and decode messages using classical ciphers, including Caesar and other substitution methods. They will identify patterns in encryption and understand how systematic rules can securely transform messages.

Unit 4 - Assessment & Reflection (Lesson 4.4): Unit 4 is the first Check-Your-Understanding assessment of the year. Students will consolidate their understanding of introductory cryptography and demonstrate the ability to encode and decode simple messages. Reflection tasks will encourage self-assessment and clarification of concepts.

Unit 5 - Encryption Theory (Lessons 4.5 and 4.6): Unit 5 comprises two lessons, “Puzzles and Symmetric Encryption” and “Public Key Cryptography: Diffie-Hellman.” Students will compare symmetric and asymmetric encryption systems, exploring the mathematical principles that underpin their security. They will understand the importance of one-way functions and the role of prime numbers in these systems.

Unit 6 - Applied Cryptography (Lessons 4.6 and 4.7): Unit 6 comprises one lesson, “Public Key Cryptography: RSA,” which has some overlap with Unit 5, lesson 4.6, on “Public Key Cryptography: Diffie-Hellman.” Students will apply their understanding to encrypt and decrypt messages using public-key and private-key methods. They will

investigate real-world applications of cryptography, such as ISBN verification, credit card security, HTTPS protocols, and secure email systems.

Unit 7 - Mathematical Constraints and Large Primes (Lesson 4.8): Unit 7 comprises one lesson, “RSA In Practice and Modern-Day Encryption.” Students will explore why factoring very large numbers is computationally difficult and how this property underpins secure encryption. They will connect number theory concepts to the practical security of modern systems.

Unit 8 - Assessment & Reflection (Lesson 4.9): Unit 8 is the second Check-Your-Understanding assessment of the year. Students will consolidate their understanding of cryptography concepts and their connection to number theory. Reflection tasks will highlight key insights and identify areas for further practice or discussion.

Unit 9 - Number Theory Review (Lessons 4.10 and 4.11): Unit 9 comprises two lessons, “Number Theory Concept Review” and “Number Theory Application Review.” Students will recall foundational number theory topics, including divisibility, primes, GCF/LCM, fractions, and modular arithmetic, and apply these concepts to fractions, modular calculations, classical ciphers, and modern encryption techniques. This stage strengthens connections between theoretical and applied mathematics.

Unit 10 - Course Wrap-Up Game (Lesson 4.12): Unit 10 comprises one lesson, “A Final Game,” which is a collaborative game that closes out the four-year curriculum. Students

will integrate the skills and knowledge developed over the four-year curriculum. They will engage in reflective activities, consolidate learning, and demonstrate comprehensive understanding through a summative game.

Chapter IV.

Teaching Strategies

The number theory curriculum for middle school students emphasizes guided inquiry, pattern exploration, and scaffolded problem-solving to develop both conceptual understanding and procedural fluency across four years of study. Students are encouraged to explore mathematical ideas through carefully structured tasks, visual models, and real-world contexts that lead them to identify patterns, relationships, and rules independently. Through guided examples, visual supports, and gradual release of responsibility, supports fade as confidence grows. Collaborative and discussion-based learning is central throughout the curriculum. Routines such as think-pair-share, partner check-ins, small-group roles, and whole-class synthesis discussions allow students to think through problems, explain their reasoning, justify solutions, and solve complex problems. Teachers facilitate learning by circulating the room, asking open-ended questions, and unpacking student strategies to prompt critical and analytical thinking and connect new ideas to prior knowledge.

Year 1 instruction focuses on number classification, divisibility, factors, multiples, fractions, and primes through hands-on exploration, peer discussion, and pattern recognition, with students investigating number sets, constructing sequences such as Pascal's Triangle and the Fibonacci sequence, and reasoning about special cases and structure. As students progress through the four years of the curriculum, instruction incorporates remainders, modular arithmetic, equivalence classes, parity, sequences, GCF

and LCM, division algorithms, and decimal representations, emphasizing cyclic behavior, structure in the base-10 system, and flexible problem-solving. Real-world applications, word problems, and multi-step tasks encourage students to apply ideas in context, communicate solutions clearly, and reason collaboratively by comparing solution strategies and justifying reasoning. In the final year, students engage with number theory in both theoretical and applied contexts, particularly cryptography, where they explore classical and modern encryption methods, test rules, decode patterns, and examine the role of prime numbers in secure communication.

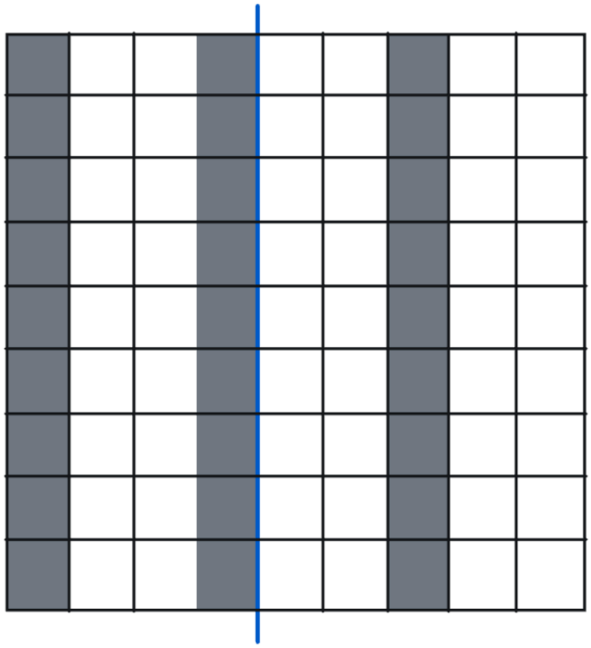
Throughout the curriculum, reflection and assessment activities are embedded through exit tickets, check-your-understanding assessments, self-grading, and reflection. These activities promote metacognition, help students evaluate their understanding, identify areas for growth, consolidate learning, and prepare for continued and more advanced mathematical study.

Lesson Excerpt Examples

Four selected excerpts, one from each year of the study, are provided as models for implementation of the curriculum. Excerpts from Year 1 lesson 1.5, Year 2 lesson 2.5, Year 3 lesson 3.5, and Year 4 lesson 4.6 introduce Understanding Fractions, Introduction to Modular Arithmetic, Sequences and Pattern Recognition, and Public Key Cryptography: Diffie-Hellman, respectively. Pedagogical approaches to the curriculum to maximize students' engagement are described through the four selected scripts. For the complete set of lessons for the four years, see Chapter V.

Lesson 1.5: Below is a sample walkthrough of Lesson 5 from Year 1, illustrating how an instructor might introduce the concepts in the Understanding Fractions lesson and

details possible avenues for interactions with students. Instructors may begin the lesson by presenting the shaded rectangle image, either on a slide or clearly drawn on the board, making sure the shading is visible. Say to the class, “Take a look at this rectangle. What fraction of the rectangle is shaded?”



Students should be given a moment to study the rectangle and think independently. The instructor should also encourage quiet reflection before moving to discussion. Ask students to turn to a partner and share their reasoning, reminding them to explain how they figured out the fraction and to listen to their partner’s idea as well. The instructor can circulate around the room to listen to conversations and gently prompt deeper thinking if needed.

After a few minutes, the instructor should invite students to share their answers with the class, encouraging multiple representations of the fractions. Some students may write the fraction on a whiteboard, while others may explain their thought process verbally. In order to reinforce the importance of logical reasoning, the instructor should

follow up with questions such as, “How did you know that?” or “Can you show your reasoning in another way?”

Once students have shared, the instructor can present the suggested answers, for example: 18/36, 2/4, and 1/2. Instructors can highlight that these are all equivalent fractions and guide a discussion around why they represent the same quantity. “Notice how these fractions look different, but they all show the same part of the rectangle. How can we see that they are the same?” The class may use visuals, multiplication and division, or fraction strips as needed to solidify the idea of equivalence. Instructors can end this segment by asking students to summarize what they learned about identifying fractions and recognizing equivalent fractions, encouraging them to write or say one thing they now understand about fractions that they did not before.

Next, the class should move on to learning that fractions are a division problem waiting to happen. The instructor can introduce the concept by saying something like the following:

> “Now that we’ve practiced identifying fractions and seeing how different fractions can mean the same amount, let’s think about what a fraction really is. Fractions aren’t just parts of shapes; they’re actually closely connected to division.”

The instructor may present a fraction on the board, perhaps $\frac{15}{4}$, and ask, “Who remembers what the top and bottom numbers are called?” After a brief review of numerators and denominators, the discussion can continue,

> "Here's something interesting: 15 divided by 4 is exactly the same as 15/4. And 24 divided by 6 is the same as 24/6. So, every time you see a fraction, you can think, 'I'm going to divide the top number by the bottom number.'

The instructor can then draw the division symbol (÷) on the board and ask students to notice the parts of the symbol, "Take a look at this symbol. Do you see a connection to fractions?" Students should volunteer answers, then the instructor can add or reiterate, "The dot on top is like the numerator (the number we are dividing). The dot on the bottom is like the denominator (the number we are dividing by). And the line in between? That's kind of like the fraction bar!" The discussion should reinforce that fractions are literally division problems, presented in a different format.

The instructor may add, "The line between the numerator and denominator is called a vinculum. Whenever you see it, it's saying, 'Divide the top by the bottom.'" Students should come up with a few fractions and convert each into a division statement. For example: "What about 9/3? What's the division problem?" "How about 7/2?"

Instructors should encourage students to say their answers out loud or write them down. "Next time you see a fraction, remember, it's a division problem waiting to happen!" Once done, the instructor should transition to simplifying fractions, continuing to emphasize exploration and understanding rather than rote memorization of processes.

Lesson 2.5: Below is a sample walkthrough of Lesson 5 from Year 2, illustrating how an instructor might introduce the concept of Modular Arithmetic and detail possible avenues for interactions with students.

Instructors are encouraged to begin the lesson by showing students a regular number line and briefly reviewing how it extends in both directions. The instructor can then introduce the idea of a number line that bends into a circle, using a clock as the primary example, since students are already familiar with the cyclical pattern, that after 12, the numbers start over at 1. Instructors can highlight examples from the lesson, such as 13 o'clock aligning with 1 o'clock and 25 o'clock landing in the same position as 1 o'clock, to emphasize the concept of numbers "wrapping around."

Students should be given a moment to consider why this happens and how it connects to the remainder work they have recently completed. The instructor can explain that after passing 12 hours, one full cycle is completed, and what matters is the leftover amount: the remainder. Students should briefly discuss how this idea relates to the clock problems in the lesson.

Instructors are encouraged to guide students into the first set of problems. Students should use a clock drawing or the concept of remainders to determine the time after a given number of hours. While circulating, the instructor can ask questions such as, "How many full groups of 12 hours did you pass?" or "What is the extra amount after those groups?" The focus should remain on noticing the repeating cycle and identifying the remaining steps after full cycles.

The instructor can then transition to the circular number line with seven positions, explaining that the same wrap-around idea applies, but using 0 through 6 instead of 1 through 12. Instructors should emphasize that the remainder after dividing by 7 indicates where a number lands on the circle. Examples such as 4 plus 5 landing on 2 can be used to show that students can think in terms of stepping around the circle or reducing the total

using remainders. Students should work on the mod 7 addition and multiplication Class Practice Problems, using either the circle or remainder method as appropriate. When reviewing answers, instructors can highlight instances in which a sum or product exceeds 6 and loops back to 0.

For the next set of problems involving larger numbers, such as 16, 27, 45, 63, or 108, instructors are encouraged to guide students in using the remainder method directly. Students should compare approaches, keeping the focus on how the remainder indicates the final position on the modulo 7 circle. Finally, instructors can connect this thinking to the days of the week. Students should use the total number of days and the remainder after dividing by 7 to determine the corresponding day.

Instructors should close the lesson by summarizing the main idea: in any repeating system, such as clocks, circular number lines, or days of the week. The key question is how many steps remain after removing full cycles. That remaining amount is the remainder, which is the central concept in modular arithmetic.

Lesson 3.5: Below is a sample walkthrough of Lesson 5 from Year 3, illustrating how an instructor might introduce Sequences and Pattern Recognition and detail possible avenues for interactions with students. To begin the lesson, the instructor should write the decimal representation of 1/7 on the board: 0.142857142857…, then point out that the digits repeat in a cycle of six: 1, 4, 2, 8, 5, 7. The instructor can then pose the question, “If I asked you for the 333rd digit after the decimal, how would you figure it out?” and allow students a moment to think independently. Afterwards, students should be invited to discuss their strategies with a partner, sharing ideas such as counting the digits,

recognizing cycles, or using division to find remainders. As students share, the instructor can prompt them with questions like, “How can we use the length of the repeating cycle to avoid counting each digit individually?” If the students have not reached a solution yet, the instructor can then explain that because the cycle repeats every 6 digits, the problem becomes: What is 333 mod 6?

The instructor should walk students through the division: $333 \div 6 = 55$ remainder 3. The remainder tells us the position in the repeating sequence that corresponds to the 333rd digit. In this case, the remainder is 3, meaning the 3rd digit in the repeating sequence is 2. The instructor should emphasize that recognizing the repeating pattern and using the remainder makes the problem much easier than counting the digits one by one.

Next, the instructor should guide students through a set of practice problems. The first problem should be written on the board: “If today is Tuesday, what day will it be 359 days from now?” The instructor should encourage students to think of the days of the week as a repeating cycle of length 7. As students work, the instructor should circulate and ask questions such as, “How many full weeks are contained in 359 days?” and “Once full weeks are accounted for, how many extra days remain?” The goal is to lead students to calculate $359 \bmod 7 \equiv 2$; therefore, 359 days from Tuesday will be a Thursday.

For a more complex problem, the instructor should introduce the following:

> “A secret number, when divided by 5, leaves a remainder of 3. When the same number is divided by 6, it leaves a remainder of 2. What might the secret number be?”

The instructor should encourage students to reason step by step: numbers that give a remainder of 3 when divided by 5 end in 3 or 8, and numbers that give a remainder of 2

when divided by 6 are 2 more than a multiple of 6. By comparing these conditions, students should identify 38 and 68 as possible solutions.

The instructor should then present a practical application problem: “I have 34 cupcakes. They are packaged in packages of 6. How many packages will I need?” The instructor should walk through the division: 34 ÷ 6 = 5 remainder 4. In order to package the leftover cupcakes, round up to 6 packages.

The lesson should then transition to cyclic powers. The instructor should write the first few powers of 2 on the board:

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64,\ 2^7 = 128$$

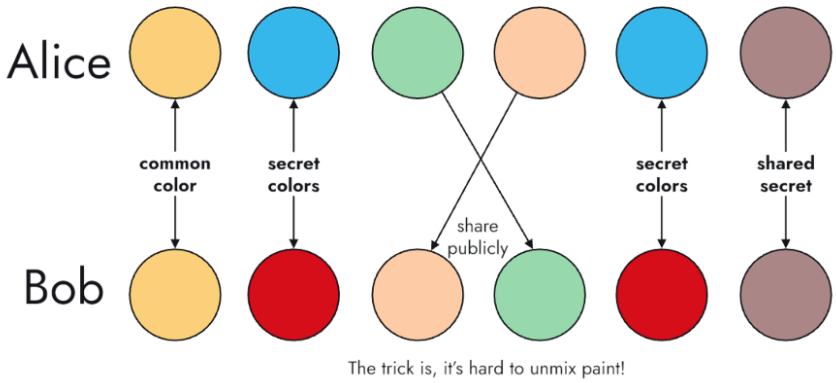
The instructor can then ask, “Can you predict the last digit of the next few powers of 2?” Students should notice the pattern in the units digit: 2, 4, 8, 6. The instructor can encourage students to apply this pattern to other problems, such as calculating the last digit of 3^{13}, 4^{10}, and 7^{43}, guiding them to find the repeating cycles for each and determining the correct last digit based on the remainder when the exponent is divided by the length of the cycle. Finally, the instructor should conclude the lesson by summarizing the main concepts. Recognizing repeating sequences and cycles allows us to predict outcomes without computing every single step. The instructor should emphasize that understanding modular arithmetic, identifying remainders, and recognizing cycles are powerful tools for efficiently solving both mathematical and real-world problems.

Lesson 4.6: Below is a sample walkthrough of Lesson 6 from Year 4, illustrating how an instructor might introduce Public Key Cryptography: Diffie-Hellman and detail possible avenues for interactions with students. Begin the lesson by reminding students that in

Years 2 and 3, they explored modular arithmetic, remainders, and repeating cycles. The instructor should explain that these same ideas appear in real-world applications, particularly in internet security. To frame the lesson, the instructor might say,

> "Almost everything we do online, like sending messages, buying items, and logging in to various accounts, depends on private information staying secret. This requires secret 'key' numbers, but how do two people who have never met agree on a secret number without letting anyone else discover it?"

Students should be given a brief moment to think, and a few should be invited to share ideas, even if they are uncertain. This discussion helps build curiosity around the problem Diffie and Hellman solved.



The instructor should then introduce the color-mixing analogy by displaying the paint-mixing diagram and explaining, "Imagine Alice and Bob want to share a secret color. They can talk out loud, and Eve is listening. How can they agree on a secret mixture without Eve figuring it out?" The instructor should walk carefully through the analogy: there is a public starting color, such as yellow, which everyone can see; Alice and Bob each choose a private color that no one else knows; they mix their private color into the public color and send the mixture to each other; finally, they add their private

color to the mixture they receive. At this point, the instructor should pause and ask questions such as, “What part of this process did Eve see?” and “Why can’t Eve recover Alice’s or Bob’s private color from the mixture?” The instructor should emphasize the key idea that mixing colors is easy, but separating them is extremely difficult. Students should be encouraged to restate this idea in their own words. The instructor can then make the transition explicit by explaining, “In mathematics, we want an operation that behaves the same way: easy to do in one direction, but extremely hard to undo.”

To connect the analogy to numbers, the instructor should write the prime number 23 on the board and remind students that they have previously worked with clock arithmetic. The instructor can explain that today’s clock has 23 positions and that a special number is needed whose powers move around the clock in a way that appears unpredictable. Without going into technical details, the instructor can explain that mathematicians have found that 5 is one such number for the 23-clock. To illustrate, the instructor can demonstrate a single example, showing that $5^7 = 78125 \equiv 17 \bmod 23$ and then ask, “If all you knew was that someone landed on 17 on the 23-clock, how easy would it be to figure out which power of 5 got them there?” Students should recognize that this task is very difficult. The instructor should summarize by explaining that computing $5^n \bmod 23$ is easy in the forward direction, even for very large values of n, but working backward to find n is extremely hard. This one-way property is what makes the method useful for encryption. Students should be reassured that they do not need to understand the full theory behind primitive roots, only the idea that powers of certain numbers spread unpredictably around the circle. The instructor should then walk through the steps of the Diffie-Hellman process slowly, explaining that the color-mixing idea is

now being repeated using numbers instead of paint. The shared public choices should be written on the board: the prime modulus, p=23, and the generator, g=5, with the explanation that everyone, including Eve, knows these values. For Alice's step, the instructor can explain that Alice chooses a private number, such as 4, computes 5^4 mod 23 ≡ 4, and sends the result to Bob. The instructor should pause to ask what information Alice is not sending, guiding students to recognize that her original private number is not revealed. For Bob's step, the instructor should explain that Bob chooses a private number, such as 3, computes 5^3 mod 23 ≡ 10, and sends this value to Alice. The instructor should again pause and ask what Eve knows and what she does not know at this stage, reinforcing the distinction between public and private information.

Next, the instructor should demonstrate how Alice and Bob arrive at the same shared secret. Alice computes 10^4 mod 23 ≡ 18, while Bob computes 4^3 mod 23 ≡ 18. Students should be asked why both end up with the same result despite performing the steps in a different order. The instructor can guide them to see that $(5^3)^4 = (5^4)^3 = 5^{12}$, emphasizing that the order of exponentiation does not matter. Students do not need to compute 5^{12}; they only need to see that exponents can be done in either order. The instructor should help students articulate what Eve is missing: the secret exponents. The key idea to reinforce is that Eve can see the final position on the clock but cannot determine which exponent produced it without trying every possibility, a task that becomes infeasible with large numbers.

Before concluding, the instructor should pause and ask students where modular arithmetic appeared in the lesson, allowing a few students to share their observations. This reflection helps connect the activity back to prior learning. The lesson should

conclude with a clear, student-friendly summary: mathematics can create processes that are easy to perform forward and nearly impossible to reverse; modular arithmetic with prime numbers creates these one-way problems; and public-key cryptography uses this structure to allow strangers to create shared secrets. The instructor should emphasize that students are not expected to master all the technical details, but rather to understand the core ideas: why primes matter, how modular arithmetic creates unpredictable wrap-around behavior, how the paint-mixing analogy maps to numbers, and how two people can securely share information without meeting. Instructors are encouraged to move slowly, pause frequently, and invite students to explain ideas in their own words, as this is where genuine understanding develops.

Assessments and Pedagogy

Assessments across the four-year number theory course are designed to develop both conceptual understanding and procedural fluency while strengthening number sense, structural reasoning, and pattern recognition. Early assessments focus on developing vocabulary for number classification; recognizing relationships among factors, multiples, divisibility, and fractions; and applying knowledge of such relationships to simplification and equivalence. As students progress, assessments emphasize structural patterns in arithmetic, division and remainders, modular reasoning, cycles, sequences, and predictive reasoning. Greatest common factors and least common multiples are assessed using list-based strategies, prime factorization, and Venn diagrams, reinforcing flexible problem-solving approaches. In the final year, assessments shift toward applied contexts, particularly cryptography. Students reason about structure, patterns, and the role of prime numbers, modular arithmetic, and factorization in secure communication through

interpretive prompts, written explanations, and discussion-based tasks. Across all years, assessments connect concrete computation with conceptual interpretation, supporting proportional reasoning, algebraic thinking, and the ability to generalize results.

Throughout the curriculum, students are routinely asked to explain why patterns occur, justify rules and strategies, interpret results in context, and show their reasoning using multiple representations. “Why” questions and prompts requiring explanation and justification help students see coherence across topics, including connections between factors and divisibility, division and remainders, decimal representations, modular arithmetic, sequences, and cyclic behavior. Rather than emphasizing isolated correctness, assessments foreground reasoning, structure, and mathematical communication.

Student-graded assessments are a core component of the course, reinforcing integrity, responsibility, and metacognition. A two-color pencil-and-pen system differentiates student work from self-evaluation and provides transparency in the assessment process. Feedback is framed around engagement, persistence, clarity of reasoning, and evidence of understanding rather than the accuracy of individual responses. Reflection questions follow check-your-understanding assessments, prompting students to identify where their reasoning broke down and offer advice to their future selves to revisit before later assessments.

Check-your-understanding assessments primarily serve a formative purpose, reinforcing core skills and supporting learning rather than simply evaluating it. Exit tickets, in-class problem sets, practice problems, short reasoning checks, and discussion-based assessments appear frequently and are embedded in daily instruction. In-class work provides opportunities for peer discussion around assessment items, allowing students to

compare strategies, debate interpretations, and co-analyze patterns. These conversations deepen understanding, support mathematical discourse, and cultivate precision and argumentation.

Spiral review and cumulative practice are intentionally embedded throughout the entire course. Earlier concepts, such as factors, divisibility, proportional reasoning, and modular thinking, naturally resurface in later work with sequences, cyclic patterns, division structures, and cryptographic applications. Assessments contribute to a coherent developmental trajectory of mathematical reasoning, ensuring continuity across all four years and helping students see that the structure of numbers is both mathematically elegant and genuinely useful in real-world contexts.

Chapter V.

The Lessons

The 48 lessons, arranged in 38 units, cover the four-year Number Theory Curriculum for middle school students. Year 1 has eight units with 12 lessons (lessons 1.1 through 1.12); Year 2 has ten units with 12 lessons (lessons 2.1 through 2.12); Year 3 has ten units with 12 lessons (lessons 3.1 through 3.12); and Year 4 has ten units with 12 lessons (lessons 4.1 through 4.12). The lessons provide varying combinations of introduction and warm-up activities, key ideas, vocabulary, symbolic notations, examples, class practice problems with answer keys, worksheet problems (also with answer keys), "Check Your Understanding" assessments, and closure reflection questions. The Year 1 study focuses on Classification of Numbers, Building Blocks, and Patterns; Year 2 focuses on Remainders, Numerical Patterns, and Division; Year 3 focuses on Divisibility, Factors, and Multiples; and Year 4 culminates in the study of Encryption and Ciphers.

Year 1 Lessons

The following 12 lessons make up the Year 1 course of study.

Lesson 1.1. Introduction to Number Theory and Classification of Numbers

Instructors are encouraged to open the first class with number theory, a topic whose name may be unfamiliar to middle school students, but whose concepts they have likely seen

before. The lesson consists of two parts: Introduction to Number Theory and Classification of Numbers. Definitions and examples are provided along with class practice problems, a worksheet to complete in small groups, and answer keys.

Introduction to Number Theory

What is "number theory" and why does it matter?

Number theory is a branch of mathematics that studies the positive whole numbers: 1, 2, 3, 4, 5, 6, … (also called natural numbers). Number theory helps students study the relationships between different kinds of numbers. If you think of numbers as LEGO blocks, with smaller numbers being combined to create bigger numbers, then number theory is the part of math that studies how those bricks fit together. Number theory supports students' understanding of numbers and the use of numbers in everyday life applications. For example,

1. Number theory helps us understand how numbers work. Just like learning how LEGO pieces snap together helps you build things, learning how numbers fit together helps us solve math problems. For example, number theory helps us understand prime numbers, which are like the basic LEGO blocks of all other numbers.
2. Number theory keeps information safe. Every time you send a text, play an online game, or log in to a website or an application (app), number theory helps keep your information safe. It is used in codes and secret messages (called encryption). Large prime numbers help lock your information so that only you can open it.

3. Number theory helps computers work better. Computers use number patterns to do math quickly. Number theory helps create shortcuts that enable computers to solve problems in seconds instead of hours.
4. Number theory is full of fun puzzles, like figuring out patterns, deciding whether a number is prime, or finding tricks with digits. If you like solving puzzles or brain teasers, you're already thinking like a number theorist.

Classification of Numbers

Class warm-up:

Which number does not belong? Why?

7, 0, 1/2, 14

Potential answers:

- 1/2 is the only non-whole number
- 14 is the only even number
- 0 is the only number that isn't natural
- 7 is the only one that's odd

Definitions and Examples

These are some common classifications of numbers.

- Natural: the counting numbers. Example: 1, 2, 3, 4, …
- Whole: the counting numbers AND zero. Example: 0, 1, 2, 3, …
- Rational: can be written as a fraction (more on this below!). Example: ½, 5.7, 0.3333…, 498
- Even: can be divided by 2 with no remainder. Example: 0, 2, 4, 6, 8, …

- Odd: not even, so it has a remainder when divided by 2. Example: 1, 3, 5, 7, 9, …

Rational numbers are literally ratios, and ratios can always be written as fractions.

- ½ is like the ratio 1:2
- 5.7 can be rewritten as 57/10, or the ratio 57:10
- 0.3333… = 1/3 = 1:3. Repeating decimals can always be written as fractions!
- 498 is a whole number, so it can be rewritten as 498/1, which is the same as 498:1.

Class Practice Problems

As a class, classify each of the following numbers as natural, whole, rational, even, and/or odd:

1. 0
2. 27
3. 1/4
4. 15.5
5. 101
6. 5.25

Answers:

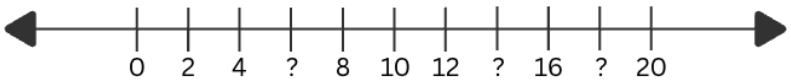
1. 0: ~~natural~~, **whole**, **rational**, **even**, ~~odd~~
2. 27: **natural**, **whole**, **rational**, ~~even~~, **odd**
3. 1/4: ~~natural~~, ~~whole~~, **rational**, ~~even~~, ~~odd~~

4. 15.5: ~~natural~~, ~~whole~~, **rational**, ~~even~~, ~~odd~~
5. 101: **natural**, **whole**, **rational**, ~~even~~, **odd**
6. 5.25: ~~natural~~, ~~whole~~, **rational**, ~~even~~, ~~odd~~

Practice classifying numbers in small groups by completing the Number Line Worksheet below.

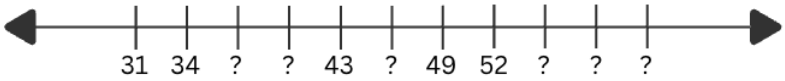
Number Line Worksheet

For each problem below, look at the number line and circle the words that apply to every number shown.

1. 

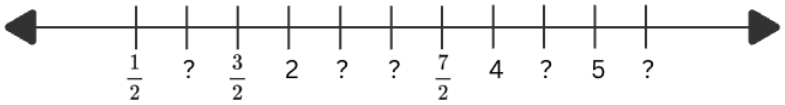
Whole Natural Rational Odd only Even only

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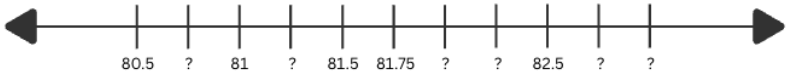
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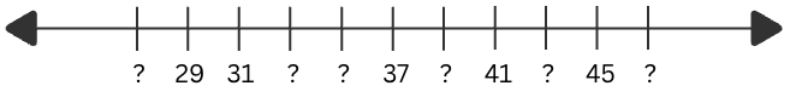
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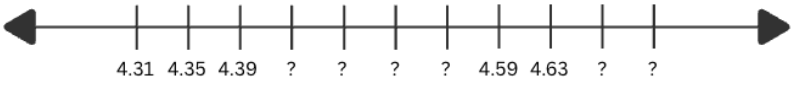
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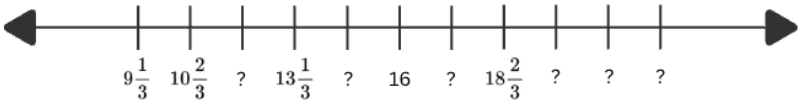
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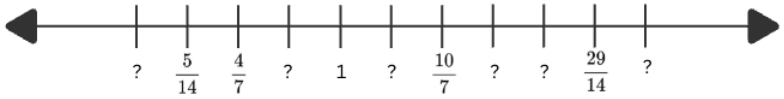
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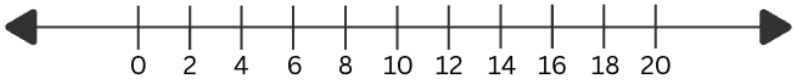
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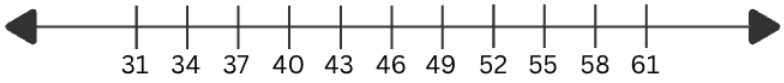
Whole Natural Rational Odd only Even only

Number Line Answer Key

1. 

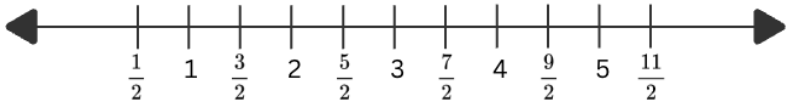
Whole Natural **Rational** Odd only **Even only**

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2. 

Whole **Natural** **Rational** Odd only Even only

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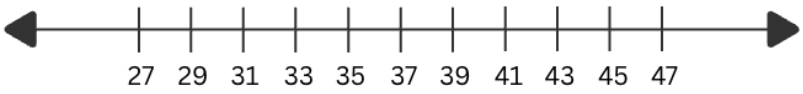
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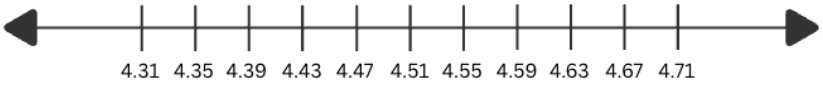
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Whole Natural **Rational** Odd only Even only

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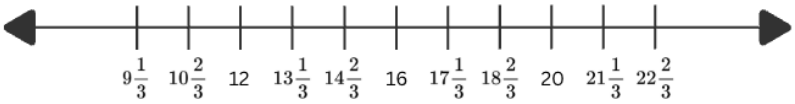
Whole **Natural** **Rational** **Odd only** Even only

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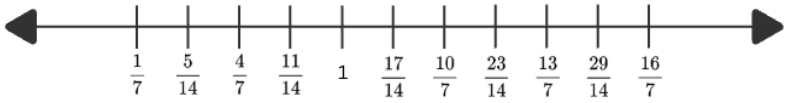
Whole Natural **Rational** Odd only Even only

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7. 

Whole Natural **Rational** Odd only Even only

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8. 

Whole Natural **Rational** Odd only Even only

Lesson 1.2. Divisibility

Instructors can begin with a whole-class discussion of divisibility, using examples to connect to students' prior knowledge of division. Students justify their reasoning and then write and share definitions of divisibility, which are formalized as producing a whole number with no remainder. The concept of factors is introduced next, followed by class practice problems and small-group work on a Factor Factories worksheet.

Defining Divisibility

Discuss as a class what it means for one number to be divisible by another.

Each group of students should come up with answers to the following:

- Is 10 divisible by 5?
- Is 23 divisible by 2?
- Is 4 divisible by 8?

Groups should be able to express how they know, perhaps using long division or drawings that show grouping.

After exploring, each student should write a definition of divisibility: "A number is divisible by another number if _____?" Groups should share their answers with the class.

Common student ideas may include:

- It divides evenly.
- There is no remainder.
- You can make whole groups.
- The answer is a whole number.

- Formally: A number is divisible by another number if the division gives a whole number with no remainder.

Defining Factors

Explore as a class: If 10 is divisible by 5, what is 5 called?

Answer: 5 is a factor of 10.

What do we call numbers that divide others evenly?

Answer: A factor is a number that divides another number evenly.

Class Practice Problems

Decide if each of these numbers is divisible and identify factors.

1. Is 10 divisible by 5?
2. Is 23 divisible by 2?
3. Is 4 divisible by 8?

Answers:

1. Yes, the answer has no remainder, so 5 is a factor of 10.
2. No, the answer has a remainder, so 2 is NOT a factor of 23.
3. No, 8 is bigger than 4, so 8 is NOT a factor of 4.

Factor Factories Worksheet

Everyone can build numbers! You just have to practice "seeing" the pairs of factors that make up a number. Have you ever followed the instructions and built a LEGO? Those instructions show how, brick by brick, you can build a larger product. Factors are like bricks. Together they make up a number. So, remember, factories make products.

Practice some multiplication. Every number has factors and multiples.

1. List 5 multiples of 6.

   6, ____, ____, ____, ____, ____

2. What are some things you notice about the multiples?

3. Now, show a Factor Factory for 6. List the factors of 6.

6	
Number	Factor? (Yes or No)
1	
2	
3	
4	
5	
6	

4. Use dots and draw arrays to represent the factor pairs of 6.

For example, 1×6 can be represented as 1 dot, six times.



Factor Factories Answer Key

1. List 5 multiples of 6.

   6, 12, 18, 24, 30, 36, 42, …

2. What are some things you notice about the multiples?

   They are all even.

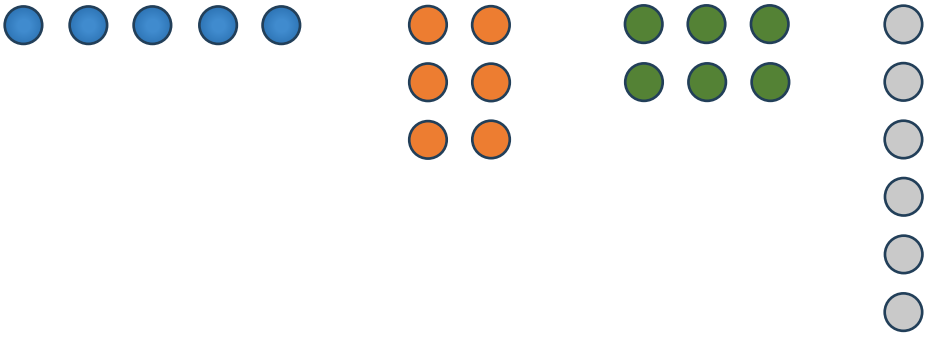
   If they are listed in order, they progressively increase by 6.

   They end in the same cycle of digits: 6, 2, 8, 4, 0

3. Now, show a Factor Factory for 6. List the factors of 6.

6	
Number	Factor?
1	**Yes**
2	**Yes**
3	**Yes**
4	**No**
5	**No**
6	**Yes**

4. Use dots and draw arrays to represent the factor pairs of 6.



Factor Factory Activity

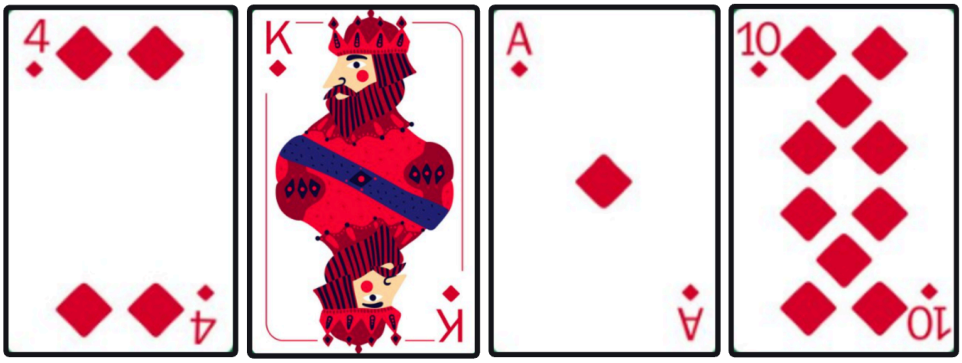
Instructions for Factor Factory Stations

Station 1

Materials needed: Playing cards and graph paper

- Draw 4 cards.
- Add the face value of the cards together. Face cards each have a value of 10.
- Write down that number. This is the sum of your cards' values.
- Find ALL factors of that number by using a Factor Factory.
- Use graph paper to show arrays that can be made for the number.

Example: You draw a 4, a King, an Ace, and a 10. That is 4+10+1+10=25.



The "factor factory" for 25 has 1, 25, and 5, because $25 = 1 \times 25$ and $25 = 5 \times 5$.

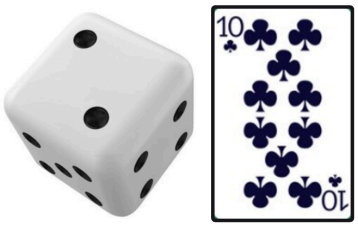
On graph paper, we can show these as 5 by 5 or 1 by 25 arrays.

Station 2

Materials needed: Dice and playing cards

- Roll one dice and draw a card.
- Find the product.
- Use coins or tokens to show arrays for that number.
- Find ALL of the factors of that number in a Factor Factory.

Example: you roll a 2 and draw a 10.



2×10=20, which can be represented as 1×20, 2×10, or 4×5.

Station 3

Materials needed: Hundreds chart, coin, index cards (or other paper)

- Flip a coin onto a Hundreds Chart.
- If it lands on heads, multiply the number by 2. If it lands on tails, multiply by 3.
- Write down the product on one index card (or other paper).
- Create a Factor Factory on the index card to show the factors of your number.
- Play again. Did any numbers show up in multiple factories?

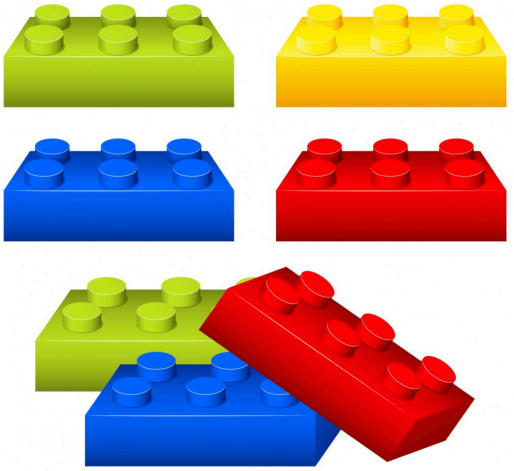


1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Station 4

Materials needed: Dice and Legos or blocks

- Roll 2 dice. If either die shows a 1, reroll that die until both dice show numbers other than 1.
- Find the product of the two numbers you roll.
- Build arrays with Lego blocks. Each block represents 1. For example, 8 is built with 8 blocks: 2 rows of 4 blocks, 4 rows of 2 blocks, 1 row of 8 blocks, etc.
- List your factors based on the factor arrays you build.





Factor Factory Reflection Questions

Please respond to each of the following questions.

1. Which number gave you the most factors? Record it below and list its factors.

2. Did your largest number have the most factors? Explain your answer.

3. What is a factor that every number has?

4. What is a factor that all *even* numbers have?

5. How did building arrays help you show the factors? Explain your reasoning.

6. Can you list all of the factors of 30?

7. Can you find all of the multiples of 30?

Lesson 1.3. Arithmetic Operations on Natural Numbers

Instructors are encouraged to begin the lesson with the Building Natural Numbers worksheet. Students can work in small groups to explore patterns involving even and odd numbers. Once most groups have made progress, the class reconvenes to discuss key observations, such as how multiplying by an even number results in an even number or how adding an even number to any number keeps the number type (i.e., parity) the same.

Instructors then guide students to the Results of Odd and Even Arithmetic activity, where students work in pairs or small groups to explain the outcomes of operations involving odd and even numbers. Through these discussions, students develop the understanding that an odd number can be viewed as one more than an even number, which provides a foundation for explaining the patterns observed in arithmetic operations.

Building Natural Numbers Worksheet

Use the 4 basic math operations (+, -, x, ÷) to create the given number.

1. Make 30 using 3, 4, 5, 6, 10, 11, 41, 120.

_____ X _____ _____ - _____

_____ X _____ _____ ÷ _____

2. Make 54 using 2, 6, 8, 9, 10, 27, 46, 64.

_____ X _____ _____ X _____

_____ + _____ _____ - _____

3. Make 16 using 2, 3, 5, 8, 16, 21, 32, 48.

_____ X _____ _____ - _____

_____- _____ _____ ÷ _____

4. Make 21 using 1, 3, 6, 7, 9, 15, 20, 30.

_____ X _____ _____ + _____

_____ + _____ _____ - _____

5. Make 72 using 2, 6, 8, 9, 12, 60, 78, 144.

_____ + _____ _____ - _____

_____ X _____ _____ ÷ _____

6. Make 50 using 1, 5, 10, 20, 30, 40, 50, 90.

_____ X _____ _____ ÷ _____

_____ + _____ _____ - _____

7. Make 8 using 1, 2, 4, 8, 9, 10, 72, 80.

_____ X _____ _____ ÷ _____

_____ X _____ _____ ÷ _____

8. Make 300 using 4, 15, 20, 120, 180, 400, 700, 1200.

_____ ÷ _____ _____ X _____

_____ + _____ _____ - _____

Building Natural Numbers Answer Key

1. Make 30 using 3, 4, 5, 6, 10, 11, 41, 120.

3 x 10	41 - 11
5 x 6	120 ÷ 4

2. Make 54 using 2, 6, 8, 9, 10, 27, 46, 64.

6 x 9	2 x 27
46 + 8	64 - 10

3. Make 16 using 2, 3, 5, 8, 16, 21, 32, 48.

2 x 8	21 - 5
32 - 16	48 ÷ 3

4. Make 21 using 1, 3, 6, 7, 9, 15, 20, 30.

7 x 3	15 + 6
1 + 20	30 - 9

5. Make 72 using 2, 6, 8, 9, 12, 60, 78, 144.

60 + 12	78 - 6
8 x 9	144 ÷ 2

6. Make 50 using 1, 5, 10, 20, 30, 40, 50, 90.

5 x 10	50 ÷ 1
20 + 30	90 - 40

7. Make 8 using 1, 2, 4, 8, 9, 10, 72, 80.

1 x 8	80 ÷ 10
2 x 4	72 ÷ 9

8. Make 300 using 4, 15, 20, 120, 180, 400, 700, 1200.

1200 ÷ 4	15 x 20
120 + 180	700 - 400

Results of Odd and Even Arithmetic Worksheet

For each math operation below, write one example using the given types of numbers. Then, determine whether the result is even or odd.

1. Even + Even

   Example:

   Even or Odd? _____

2. Even - Even

   Example:

   Even or Odd? _____

3. Odd + Odd

   Example:

   Even or Odd? _____

4. Odd - Odd

   Example:

   Even or Odd? _____

5. Even + Odd

   Example:

   Even or Odd? _____

6. Even - Odd

   Example:

   Even or Odd? _____

7. Odd - Even

   Example:

   Even or Odd? _____

8. Even × Even

   Example:

   Even or Odd? _____

9. Odd × Odd

   Example:

   Even or Odd? _____

10. Even × Odd

    Example:

    Even or Odd? _____

Results of Odd and Even Arithmetic Answer Key

1. Even + Even

   12 + 6 = 18 → Even

2. Even - Even

   20 - 4 = 16 → Even

3. Odd + Odd

   5 + 7 = 12 → Even

4. Odd - Odd

   21 - 3 = 18 → Even

5. Even + Odd (or Odd + Even)

   30 + 5 = 35 → Odd

6. Even - Odd

   20 - 7 = 13 → Odd

7. Odd - Even

   11 - 6 = 5 → Odd

8. Even × Even

   4 × 8 = 32 → Even

9. Odd × Odd

   5 × 11 = 55 → Odd

10. Even × Odd (or Odd × Even)

    20 × 3 = 60 → Even

Lesson 1.4. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

1. Classify the number 0. Which categories does it belong to? Choose all that apply.

   a) Natural

   b) Whole

   c) Even

   d) Rational

2. Is 17 an odd or even number? Explain how you know.

3. Circle all the rational numbers below:

   4 0.75 3 1/5 0.34913723401….

4. Is 24 divisible by 6? Show or explain your reasoning.

5. Is 29 divisible by 3? How do you know?

6. List all factors of 18.

7. List three multiples of 7 (not including 7).

8. Which number is a factor of 30?

   a) 4

   b) 5

   c) 9

   d) 11

9. Which number is NOT a multiple of 8?

   a) 8

   b) 24

   c) 40

   d) 22

10. Fill in the blanks:

    a) A number is divisible by another number if _____.

    b) A factor is _____.

Check Your Understanding Answer Key

1. b) Whole, c) Even, d) Rational. Zero is not natural.
2. 17 is odd, because it is not divisible by 2.
3. Rational numbers: 4, 0.75, -3, 1/5
4. Yes, 24 is divisible by 6. 24 ÷ 6 = 4, an integer with no remainder.
5. Even numbers leave a remainder of 0 when divided by 2 because they are all divisible by 2.
6. Factors of 18: 1, 2, 3, 6, 9, 18
7. Possible correct multiples of 7 (any three): 14, 21, 28, 35, 42, 49, ...
8. b) 5, since 30 ÷ 5 = 6, an integer with no remainder.
9. d) 22. 22 is not a multiple of 8.
10. A number is divisible by another number if the division gives a whole number with no remainder.

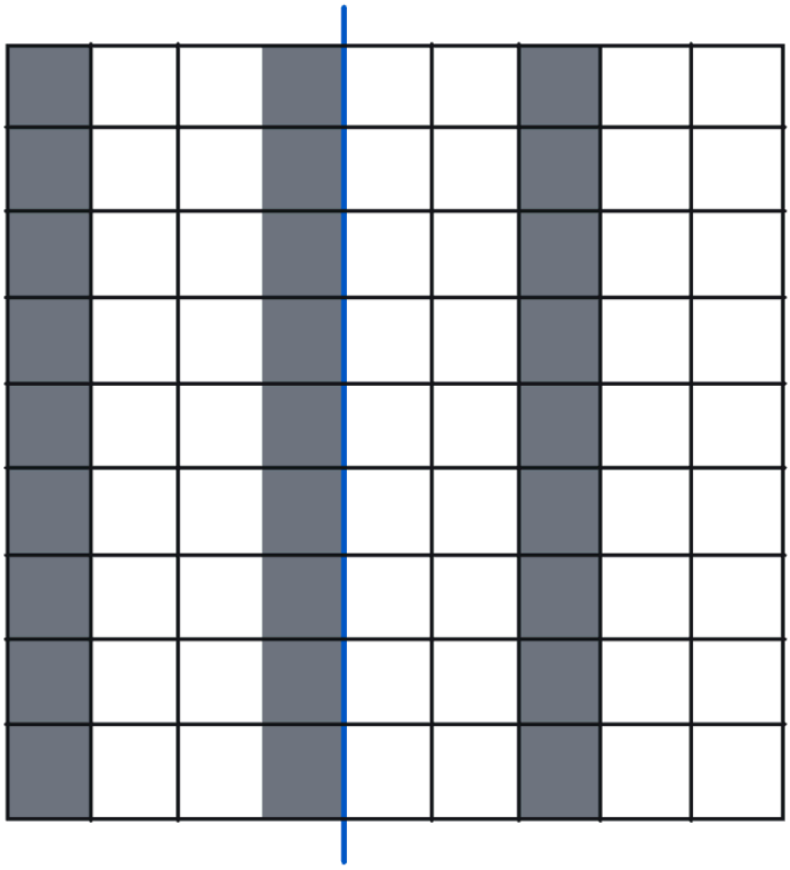
    A factor is a number that divides another number evenly.

Lesson 1.5. Understanding Fractions

Instructors are encouraged to begin the lesson by presenting students with shaded rectangles and asking what fraction of each rectangle is shaded. Students are given time to think independently, then discuss their reasoning with a partner, before sharing and recording their thinking with the class. Through discussion, students conclude that one rectangle shows one-fourth shaded and the other shows one-fifth shaded.

The lesson then connects fractions to division, emphasizing that a fraction represents a division of the numerator by the denominator. Examples are used to illustrate this relationship, along with a discussion of the division symbol and the fraction bar, or vinculum. The lesson continues with work on simplifying fractions using multiple methods.

What fraction of each rectangle is shaded? How do you know?



Give students some time to consider the question: What fraction of each smaller rectangle is shaded?

Once they have had a chance to think about this, have them share their thinking with a partner. After they have talked to a partner, students can show, record, and share their thinking with the class.

Answer: 1/4 of the rectangle on the left, 1/5 of the rectangle on the right.

Fractions and Division

Fractions are like division waiting to happen. The division sign (÷) even looks like a tiny fraction!

For example:

- $\frac{15}{4}$ is the same as 15÷4
- $\frac{24}{6}$ is the same as 24÷6

When you see a fraction, divide the top by the bottom. This is easier and more helpful when the numerator is bigger than the denominator.

Now, let's look at the division sign: ÷. Can you see how the symbol itself relates to fractions? It's kind of like a fraction turned into a tiny symbol!

The dot on top is the numerator (the number you are dividing), and the dot on the bottom is the denominator (the number you are dividing by).

The fraction bar itself also has a special name. That line between the top and bottom numbers in a fraction is called a vinculum. The vinculum says, "Divide the top by the bottom."

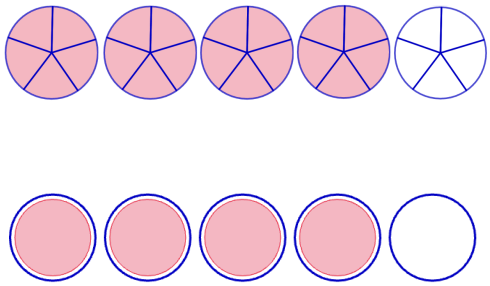
Simplifying Fractions

Simplify the fraction $\frac{20}{25}$.

Method 1: divide the numerator and denominator by a common factor

$\frac{20}{25} = \frac{20 \div 5}{25 \div 5} = \frac{4}{5}$

Visual:



Method 2: rewrite the numerator and denominator as a product of factors

$\frac{20}{25} = \frac{5 \times 4}{5 \times 5} = \frac{5}{5} \times \frac{4}{5} = 1 \times \frac{4}{5} = \frac{4}{5}$

How is this similar to the first method? How is it different?

Answer: both methods divide 5÷5=1. Method 2 shows the math more explicitly.

Class Practice Problems

Now try using both methods to simplify the given fractions.

Method 1: divide the numerator and denominator by a common factor

Method 2: rewrite the numerator and denominator as a product of factors

A) $\frac{8}{12}$

Method 1:

Method 2:

B) $\frac{24}{36}$

Method 1:

Method 2:

Answers:

A) $\frac{8}{12}$

Method 1: $\frac{8}{12} = \frac{8\div4}{12\div4} = \frac{2}{3}$

Method 2: $\frac{8}{12} = \frac{4\times2}{4\times3} = \frac{4}{4} \times \frac{2}{3} = 1 \times \frac{2}{3} = \frac{2}{3}$

B) $\frac{24}{36}$

Method 1: $\frac{24}{36} = \frac{24\div12}{36\div12} = \frac{2}{3}$

Method 2: $\frac{24}{36} = \frac{12\times2}{12\times3} = \frac{12}{12} \times \frac{2}{3} = 1 \times \frac{2}{3} = \frac{2}{3}$

Lesson 1.6. More Fractions and Factors

Instructors are encouraged to begin the lesson with the Mystery Number activity, which includes two problems. Students should work in small groups to explore divisibility, number properties, and arithmetic, applying observations from previous lessons, such as identifying common factors. Following this exploration, students should work independently on the Factor Pair activity to reinforce their understanding of factors and factor relationships.

Mystery Number Class Activity

Mystery #1: Use the clues below to find the mystery fraction:

- Clue 1: The fraction is greater than $\frac{1}{2}$ but less than $\frac{5}{2}$.
- Clue 2: When the numerator is divided by the denominator, the result is a whole number.
- Clue 3: The numerator is twice the denominator or three times the denominator.
- Clue 4: The sum of the numerator and denominator is less than 20.
- Clue 5: The denominator is an even number.
- Clue 6: The numerator is less than 10.

Questions:

- What fraction(s) fit all of these clues?
- Explain how you used all of these clues to find your answers.
- Write one more clue that would make the fraction code unique.

Mystery #2: Use the clues below to find the mystery mixed number:

- Clue 1: The mixed number is greater than 1 but less than 4.
- Clue 2: The fractional part is equivalent to $\frac{1}{2}$, $\frac{2}{4}$, or $\frac{3}{6}$.
- Clue 3: The whole number part is even.
- Clue 4: The sum of the whole number and the numerator of the fraction is less than 8.
- Clue 5: The denominator is an even number.

Questions:

- What fraction(s) fit(s) all of these clues?
- Explain how you used all of these clues to find your answer.

Mystery Number Solutions

Mystery #1:

Step 1: Denominator possibilities

Denominator must be even → 2, 4, 6, 8, …

Step 2: Numerator possibilities (Clue 3)

Case A: Numerator = 2 × denominator → 4, 8, 12, 16, …

Options are $\frac{4}{2}$ or $\frac{8}{4}$. Anything else has a numerator over 10.

Case B: Numerator = 3 × denominator → 6, 12, 18, …

Options are $\frac{6}{2}$, $\frac{12}{4}$, …, but those are all too big since fraction is less than $\frac{5}{2}$.

That means that the fraction is either $\frac{4}{2}$ or $\frac{8}{4}$, so the simplified mystery number is $\frac{2}{1}$.

Mystery #2:

Step 1: Analyze the Fractional Part

Fractional part options (simplest form): $\frac{1}{2}$

Step 2: Whole Number Part

Whole numbers possible between 1 and 3 (even numbers): 2

Step 3: Check the sum constraint

Whole number + numerator is less than 8 → 2 + 1 = 3

Step 4: Denominator is even

The fraction is $\frac{1}{2}$, so the mixed number is $2\frac{1}{2}$.

Factor Pair Worksheet

Find the factor pairs of each of these numbers. For example, the factor pairs of 50 are

$1 \times 50, 2 \times 25, 5 \times 10$

Problem 1: Factors of 12

Factor pairs:

Problem 2: Factors of 18

Factor pairs:

Problem 3: Factors of 24

Factor pairs:

Problem 4: Factors of 30

Factor pairs:

Factor Pair Answer Key

Problem 1: Factors of 12

Factor pairs:

1 × 12

2 × 6

3 × 4

Problem 2: Factors of 18

Factor pairs:

1 × 18

2 × 9

3 × 6

Problem 3: Factors of 24

Factor pairs:

1 × 24

2 × 12

3 × 8

Problem 4: Factors of 30

Factor pairs:

1 × 30

2 × 15

3 × 10

5 × 6

Lesson 1.7. Defining Prime Numbers

Instructors are encouraged to begin the lesson with the Sieve of Eratosthenes activity. Students should work in groups using a number grid from 1-100 to explore and identify prime numbers. After completing the sieve process, groups should respond to reflection questions, which can then be discussed as a whole class. The lesson concludes with the Prime and Composite Numbers worksheet, allowing students to further practice distinguishing between prime and composite numbers.

Sieve of Eratosthenes Activity

Instructions

- Step 1: Start with the number 2 (the first prime).

  Circle or highlight 2 in one color.

  Cross out all multiples of 2 (4, 6, 8, etc.).

- Step 2: Move to the next number that is not crossed out, 3.

  Circle or highlight 3 in a different color.

  Cross out all multiples of 3 (6, 9, 12, etc.).

- Step 3: Repeat the process with the next uncrossed number (5, then 7, etc.)

  Circle/highlight the number and cross out all of its multiples.

- Step 4: Continue until all numbers in the grid are either circled or crossed out.

Sieve of Eratosthenes Reflection Questions

1. Prime numbers are numbers that have only two factors. Where do those come up on the grid created above? Use an example to explain this.

2. Composite numbers are numbers that have more than two factors. Where do those come up on the grid created above? Use an example to explain this.

3. What number didn't fit either definition?

Sieve of Eratosthenes Reflection Answers

1. Prime numbers are circled. For example, the factors of 3 are 1 and 3. Since 3 is not a multiple of anything other than 1 and 3, it did not get crossed off.
2. Composite numbers are crossed off. For example, the factors of 6 are 1, 2, 3, and 6, so it was crossed (multiple times!).
3. The number 1 did not fit either definition because it only has one factor, itself, rather than the two or more required to be considered prime or composite. We started the "sieve" process with the multiples of 2.

Prime and Composite Numbers Worksheet

Complete the table below. List the factors of each number and decide accordingly whether it is prime or composite.

Number	**List of Factors**	**Prime or Composite**
10		
5		
12		
18		
41		
15		
2		
49		
73		
33		
21		

Prime and Composite Numbers Answer Key

Number	List of Factors	Prime or Composite
10	1, 10, 2, 5	Composite
5	1, 5	Prime
12	1, 12, 2, 6, 3, 4	Composite
18	1, 18, 2, 9, 3, 6	Composite
41	1, 41	Prime
15	1, 15, 3, 5	Composite
2	1, 2	Prime
49	1, 49, 7	Composite
73	1, 73	Prime
33	1, 33, 3, 11	Composite
21	1, 21, 3, 7	Composite

Lesson 1.8. Prime Numbers as Building Blocks

This class is dedicated to building numbers from their prime factors. Students will spend the period creating Prime Recipes. Encourage them to look for and discuss any patterns they notice along the way. Please note that the "recipe cards" included below can be printed and folded over.

Prime Recipes Activity

Every whole number (except 1) can be written as a product of prime numbers. These prime recipes show the "ingredients" that build a number.

Instructions

- Look at the number on each card.
- Break the number into its prime ingredients. Write each prime factor on the recipe side.
- Use prime numbers only (2, 3, 5, 7, 11, …) and write them as many times as they are needed to create the number on the card. For example, 20 requires 2, 2, and 5.
- Check your recipe by multiplying your "ingredients" back together.

You may notice patterns as you work:

Some numbers use only one prime (with possible repetition), while others use two or more primes. Larger numbers often have longer recipes, but not always. Different paths of breaking down a number still lead to the same primes. Your goal is to explore how numbers are built, one prime ingredient at a time!

Recipe Cards (1-10)

1	Recipe:
2	Recipe:
3	Recipe:
4	Recipe:
5	Recipe:
6	Recipe:
7	Recipe:
8	Recipe:
9	Recipe:
10	Recipe:

Recipe Cards (11-20)

11	Recipe:
12	Recipe:
13	Recipe:
14	Recipe:
15	Recipe:
16	Recipe:
17	Recipe:
18	Recipe:
19	Recipe:
20	Recipe:

Recipe Cards (21-30)

21	Recipe:
22	Recipe:
23	Recipe:
24	Recipe:
25	Recipe:
26	Recipe:
27	Recipe:
28	Recipe:
29	Recipe:
30	Recipe:

Prime Recipes Answer Key

1: N/A, no prime factorization

2: 2

3: 3

4: 2, 2

5: 5

6: 2, 3

7: 7

8: 2, 2, 2

9: 3, 3

10: 2, 5

11: 11

12: 2, 2, 3

13: 13

14: 2, 7

15: 3, 5

16: 2, 2, 2, 2

17: 17

18: 2, 3, 3

19: 19

20: 2, 2, 5

21: 3, 7

22: 2, 11

23: 23

24: 2, 2, 2, 3

25: 5, 5

26: 2, 13

27: 3, 3, 3

28: 2, 2, 7

29: 29

30: 2, 3, 5

Lesson 1.9. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

1. Is 37 a prime or composite number? Explain how you know.

2. Which of the following numbers are composite: 9, 11, 13, 15? Show your reasoning.

3. Is the number 0 a whole number, a natural number, or both?

4. Circle all the categories that the number 12 belongs to:

   Whole Number • Natural Number • Rational Number • Even Number • Odd Number

5. True or False: Every whole number is also a rational number. Explain.

6. Which fraction is equivalent to $\frac{3}{4}$?

   a) $\frac{6}{8}$

   b) $\frac{9}{16}$

   c) $\frac{4}{3}$

7. Write one fraction that is equivalent to $\frac{2}{5}$?

8. Simplify $\frac{12}{18}$.

9. Which fraction is in simplest form: $\frac{8}{10}$ or $\frac{3}{4}$? Explain why.

10. Simplify the fraction: $\frac{45}{60}$.

Check Your Understanding Answer Key

1. 37 is prime. It has only two factors, 1 and 37.
2. Composite numbers: 9 and 15. 9 = 3×3, 15 = 3×5. 11 and 13 are prime.
3. 0 is a whole number. It is not considered a natural number, since those are the counting numbers (1, 2, 3, 4, …).
4. 12 is a Whole Number, Natural Number, Rational Number, and Even Number. It is not an Odd Number.
5. True. Every whole number can be written as a fraction (for example, 5 = 5/1), so they are all rational.
6. a) 6/8. Multiply the numerator and denominator of 3/4 by 2 (i.e., multiply by 2/2).
7. Possible answers: 4/10, 6/15, 8/20, etc.
8. $\frac{12\div6}{18\div6}=\frac{2}{3}$
9. $\frac{3}{4}$ is in its simplest form. $\frac{8}{10}$ can be simplified to $\frac{4}{5}$.
10. $\frac{45\div15}{60\div15}=\frac{3}{4}$

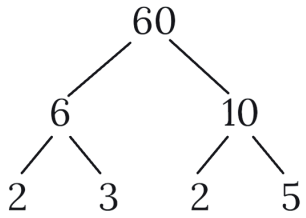
Lesson 1.10. Prime Factorization

Instructors are encouraged to begin the lesson by discussing factoring: separating numbers into factors that may be prime or composite. Students should then explore prime factorization, breaking numbers down into a product of prime numbers. Activities such as factor trees and factor ladders provide structured ways for students to identify prime factors.

Factor Tree Method

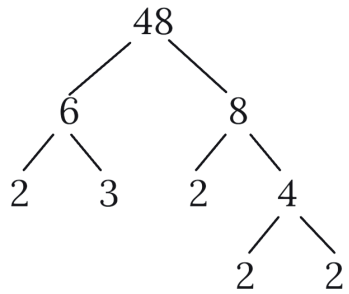
The factor tree is complete when only prime factors appear in the product.

Write the prime factorization of 60.



So, the prime factorization of 60 is 2×2×3×5.

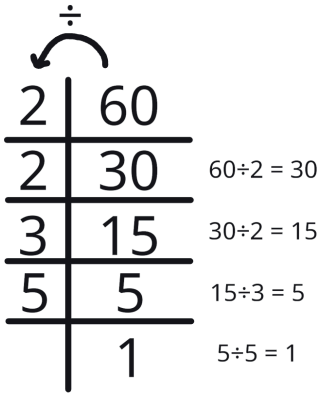
Now, write the prime factorization of 48. Choose any factor pair of 48 to begin the factor tree.

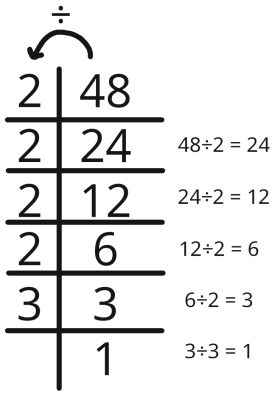


The prime factorization of 48 is 2×2×2×2×3.

Ladder Method

Another way we could do the problems above is by using a ladder instead of a tree.





The numbers in the left column are the prime factors of your starting number!

Class Practice Problems

Write the prime factorization of each number below.

1. 20
2. 88
3. 90
4. 462

Answers:

1. $20 = 2 \times 2 \times 5$
2. $88 = 2 \times 2 \times 2 \times 11$
3. $90 = 2 \times 3 \times 3 \times 5$
4. $462 = 2 \times 3 \times 7 \times 11$

Prime Factorization Worksheet

Write the prime factorization of each given number, then decide if the number is prime.

1. 12 =

   Is the number prime? _____

2. 44 =

   Is the number prime? _____

3. 33 =

   Is the number prime? _____

4. 35 =

   Is the number prime? _____

5. 31 =

   Is the number prime? _____

6. 36 =

   Is the number prime? _____

7. 17 =

   Is the number prime? _____

8. 23 =

   Is the number prime? _____

9. 22 =

   Is the number prime? _____

10. 26 =

    Is the number prime? _____

11. 42 =

    Is the number prime? _____

12. 7 =

    Is the number prime? _____

13. 2 =

Is the number prime? _____

14. 28 =

Is the number prime? _____

15. 9 =

Is the number prime? _____

16. 16 =

Is the number prime? _____

Prime Factorization Answer Key

1. 12 = 2×2×3 (No)
2. 44 = 2×2×11 (No)
3. 33 = 3×11 (No)
4. 35 = 5×7 (No)
5. 31 = 31 (Yes)
6. 36 = 2×2×3×3 (No)
7. 17 = 17 (Yes)
8. 23 = 23 (Yes)
9. 22 = 2×11 (No)
10. 126 = 2×13 (No)
11. 142 = 2×3×7 (No)
12. 17 = 17 (Yes)
13. 2 = 2 (Yes)
14. 28 = 2×2×7 (No)
15. 9 = 3×3 (No)
16. 16 = 2×2×2×2 (No)

Lesson 1.11. Sequences Involving Integers

Instructors are encouraged to begin the lesson by connecting mathematics to real-world pattern recognition. This sets the stage for exploring sequences, a foundational way mathematicians identify patterns in numbers. Students will then investigate sequences, learning to identify terms, rules, and patterns. Later lessons will build on this understanding to explore Pascal's Triangle and Fibonacci Numbers.

What are Sequences?

Many careers that use math focus on finding patterns to make predictions. For example, geologists look for patterns in data to predict earthquakes, and bankers study trends in financial markets to anticipate changes. Professional mathematicians rely on complex tools to do this work, but we will start with a simpler idea: identifying patterns in sequences.

Sequences are sets of numbers that follow a particular pattern or rule to get from number to number. Each number in a pattern is called a term. For example, the rules could involve addition, subtraction, multiplication, or division. The rule should be general enough to help you find the next terms of the sequence.

Class Practice Problems

Find the missing numbers:

1. 3, 6, 9, 12,15, ___, ___
2. 4, 10, 16, 22, 28, ___, ___

3. 3, 4, 7, 8, 11, ___, ___

4. 1, 2, 4, 8, 16, ___, ___

Answers:

1. 3, 6, 9, 12,15, **18**, **21** *(add 3)*

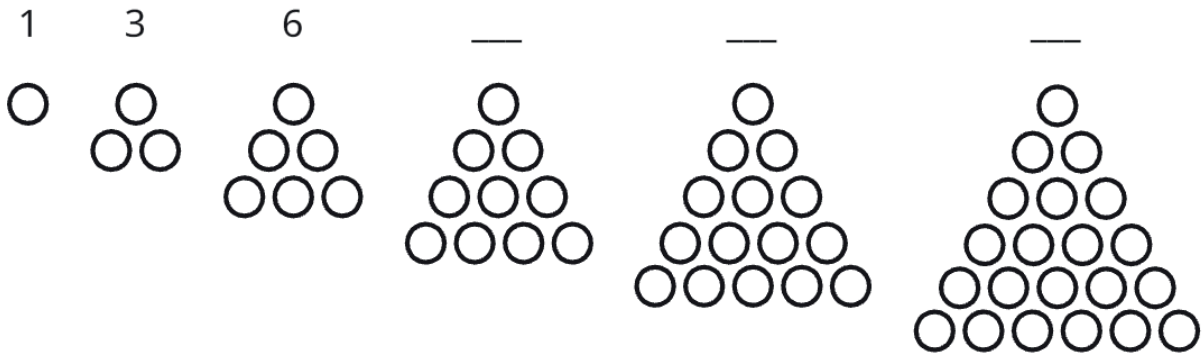
2. 4, 10, 16, 22, 28, **34**, **40** *(add 6)*

3. 3, 4, 7, 8, 11, **12**, **15** *(add 1, then 3, then 1, then 3, ...)*

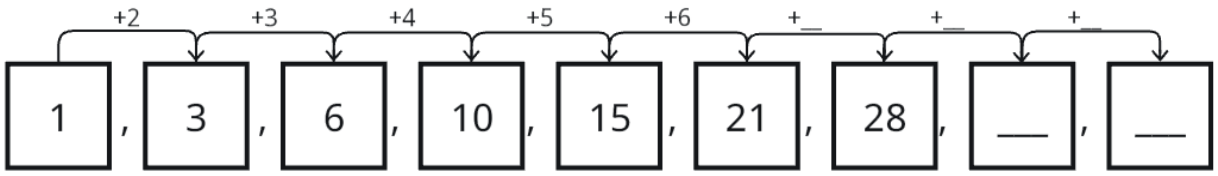
4. 1, 2, 4, 8, 16, **32**, **64** *(multiply by 2)*

Triangle Numbers

Sequences in mathematics do not always have to be numbers. Here is a sequence that consists of geometric shapes:

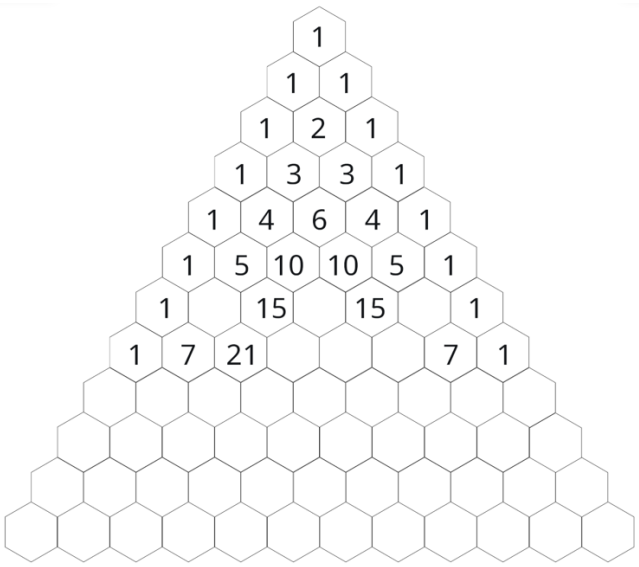


What do you notice about the number of circles that make each triangle?



Pascal's Triangle

For another geometric shape, here is a number pyramid that is created using a simple pattern. What do you notice?



This diagram shows only 5 completed rows, but we could continue forever by adding new rows at the bottom. Notice that the triangle is symmetric, which can help you calculate some of the cells. Try filling in some of the empty spots above!

Answer:

The triangle starts with a single "1" at the top, and every following number is the sum of the two numbers directly above. This triangle is called Pascal's Triangle, after the French mathematician Blaise Pascal. He is famous for studying its patterns and discovering many interesting properties. Even though Pascal explored it in Europe, mathematicians in other parts of the world, such as India, Iran, and China, had known about it many centuries earlier.

Fibonacci Numbers

Class Practice Problem

Find the missing numbers:

1, 1, 2, 3, 5, 8, 13, ___, ___, ___, …

What do you notice about this pattern?

Answer:

Each number from the third term onwards is the sum of the two previous numbers. In other words, you have to add the previous two terms in the sequence to get the next one. The sequence starts with two 1s.

This sequence of numbers is called the Fibonacci Sequence, named after the Italian mathematician Leonardo Fibonacci. Fibonacci was born in 1175, when most people in Europe still used Roman numerals. While traveling with his merchant father in North Africa and the Middle East, he learned the Arabic numeral system and the idea of place value. When he returned to Italy, he introduced these numerals in his book Liber Abaci ("The Book of Calculations"). The system was quickly adopted, and it is the one we still use today.

Fibonacci numbers are all around us in nature, often in ways we don't notice at first. The next time you're outside, take a closer look: count the petals on a flower, trace the spirals of a pinecone or sunflower, or check how many leaves grow along a stem. You might be surprised to find that these numbers often follow the Fibonacci sequence!

Lesson 1.12. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

1. Look at the sequence: 4, 7, 10, 13, ___. What number goes next? Show your reasoning.

2. What are the next two numbers in the Fibonacci sequence 1, 1, 2, 3, 5, ___, ___? Explain your reasoning

3. Use this row of Pascal's triangle to fill in the missing number: 1, 4, ___, 4, 1. What number belongs in the blank? Show your reasoning.

4. Write 36 as a product of its prime factors.

5. Write 60 as a product of prime factors.

6. Simplify the fraction $\frac{14}{30}$. Show your work.

7. Simplify the fraction $\frac{32}{48}$. Show your work.

8. Fill in the blank to make an equivalent fraction: $\frac{5}{6} = \frac{}{18}$. Show your work.

9. Which number is prime? Why are the other numbers not prime?

   a) 21

   b) 19

   c) 25

10. Which number is a multiple of 6? Show your reasoning.

    a) 14

    b) 24

    c) 32

Check Your Understanding Answer Key

1. 16. Pattern: add 3 each time → 4, 7, 10, 13, 16
2. 8, 13. Pattern: sum of the previous two numbers → 1, 1, 2, 3, 5, 8, 13
3. 6. Row: 1, 4, 6, 4, 1. The 6 is the sum of 3+3 from the row above it (1, 3, 3, 1).
4. $36 = 2 \times 2 \times 3 \times 3$
5. $60 = 2 \times 2 \times 3 \times 5$
6. $\frac{14 \div 2}{30 \div 2} = \frac{7}{15}$
7. $\frac{32}{48} = \frac{2}{3}$ (divide by powers of 2 like 2, 4, 8, 16 to reduce)
8. $\frac{5 \times 3}{6 \times 3} = \frac{15}{18}$
9. b) 19 is the only prime. $21 = 3 \times 7$, $25 = 5 \times 5$.
10. b) 24 is the only multiple of 6. $14 \div 6$ is not whole, $24 \div 6 = 4$, and $32 \div 6$ is not whole.

Year 2 Lessons

The following 12 lessons make up the Year 2 course of study.

Lesson 2.1. Number Theory and Factor Review

Instructors are encouraged to begin the class with a brief review of number theory, revisiting what it is and how it is used in familiar mathematical contexts. The lesson then provides a quick refresher on classifying numbers before transitioning into a factor review. Students review and solidify definitions of prime numbers, composite numbers, and common factors through guided examples and practice. The lesson concludes with factor riddles for students to work on in small groups or individually, reinforcing concepts in an engaging way.

Number Theory Review

What is “number theory” and why does it matter?

Students should share their definitions with the class and discuss this first.

Building on those shared ideas, recall the introduction from Year 1, which included the following description of number theory.

Number theory is a branch of mathematics that studies the positive whole numbers: 1, 2, 3, 4, 5, 6, … (also called natural numbers). Number theory helps study the relationships between different kinds of numbers. If you think of numbers as LEGO blocks, with

smaller numbers being combined to create bigger numbers, then number theory is the part of math that studies how those bricks fit together. For example,

1. Number theory helps us understand how numbers work. Just like learning how LEGO pieces snap together helps you build things, learning how numbers fit together helps us solve tricky problems. For example, number theory helps us understand prime numbers, which are like the basic LEGO blocks of all other numbers.
2. Number theory keeps information safe. Every time you send a text, play an online game, or log in to a website or app, number theory helps keep your information safe. It is used in codes and secret messages (called encryption). Large prime numbers help lock your information so that only you can open it.
3. Number theory helps computers work better. Computers use number patterns to do math really quickly. Number theory helps create shortcuts that enable computers to solve problems in seconds instead of hours.
4. Number theory is full of fun puzzles, like figuring out patterns, deciding whether a number is prime, or finding tricks with digits. If you like solving puzzles or brain teasers, you are already thinking like a number theorist.

Definitions and Examples

- Natural: the counting numbers.
    - Example: 1, 2, 3, 4, …
- Whole: the counting numbers AND zero.
    - Example: 0, 1, 2, 3, …
- Rational: can be written as a fraction (more on this below!).
    - Example: ½, 5.7, 0.3333…, 498

- Even: can be divided by 2 with no remainder.
    - Example: 0, 2, 4, 6, 8, …
- Odd: not even, so it has a remainder when divided by 2.
    - Example: 1, 3, 5, 7, 9, …

Work on the Factor Review lesson below for the remainder of class to review primes, composites, and common factors.

Factor Review

Students revisit the concept of factor pairs and apply it to identify the common factors of 90 and 70. Afterwards, they solve a set of number riddles that further reinforce their understanding of prime factorization.

Class Practice Problems

The goal is to identify the common factors of 70 and 90. Here is a step-by-step process for the class to follow.

1. As a class, list the prime numbers between 1 and 10.

   Answer: 2, 3, 5, 7
2. Write the number 90 on the board and let students decide whether it is prime or composite. They should share and explain their thoughts.

   Answer: No, because 90 has factors other than 1 and itself.
3. Work with the class to list the factor pairs of 90.

   Answer: 1×90, 2×45, 3×30, 5×18, 6×15, 9×10
4. Write the number 70 on the board and ask students whether they think 70 and 90 have any common factors. After they have had a minute to discuss their thoughts, students should independently list all the factor pairs of 70.

   Answer: 1×70, 2×35, 5×14, 7×10
5. Finally, identify the factors that 70 and 90 have in common.

   Answer: 1, 2, 5, 10

Factor Riddles Warm-Up

Now, try to figure out this factor riddle as a class.

Clue 1: I am a common factor of 18 and 30.

Students should take some time to list the factor pairs for 18 and 30 independently: 18 = 1×18, 2×9, 3×6 and 30 = 1×30, 2×15, 3×10, 5×6.

Work with students to note the common factors of 18 and 30 at the board (1, 2, 3, and 6).

Write the next two clues on the board and have students use them to identify the mystery number.

Clue 2: I am an even number.

Clue 3: I am not prime.

Answer: 6

Next, students should work on the factor riddles below, either individually or in pairs.

Factor Riddles Worksheet

Instructions:

1. List the factor pairs for each number.
2. Find the factors shared by both numbers.
3. Use the remaining clues to find the answer to the riddle.

Riddle 1

I am a common factor of 21 and 35.

I am an odd number.

When you multiply me by 3, you get a number greater than 10.

Number: _____

Riddle 2

I am a common factor of 35 and 70.

I am also a factor of 105.

I am an odd number.

I am divisible by 7.

Number: _____

Riddle 3

I am a common factor of 48 and 72.

I am an even number greater than 6 that is also divisible by 6.

I am less than 20.

Number: _____

Riddle 4

I am an odd number.

I am a common factor of 77 and 154.

I am greater than 5.

Number: _____

Riddle 5

I am a common factor of 90 and 180.

I am greater than 10 and less than 50.

I am divisible by 5.

I am also divisible by 9.

Number: _____

Riddle 6

I am an odd number.

I am a common factor of 105 and 175.

I am not prime.

Number: _____

Factor Riddles Answer Key

Riddle 1: 7

Riddle 2: 35

Riddle 3: 12

Riddle 4: 11

Riddle 5: 45

Riddle 6: 35

Lesson 2.2. Prime Factorization and Divisibility

This lesson centers on a hands-on group activity in which students build numbers using their prime factors. Working in six small groups, students represent numbers from 2-100 as products of prime numbers, reinforcing prime factorization and divisibility concepts. Each group is assigned a set of numbers and finds the prime factors for each. A completed example is provided to model expectations, and groups that finish their handouts early may assist others. The activity sets the stage for analyzing divisibility by examining shared prime factors.

Building Prime Numbers Activity

Each group will receive a handout with assigned numbers. Work in groups to build the numbers on the handout using their prime factors. Write the prime factorization on the handout. The first number on each handout shows a completed example to model expectations. Consider what the prime factors reveal about a number, including its divisibility. For example, if a number is divisible by 3, its card should have a "3" on it.

Building Prime Numbers Handouts

Group A: Write the prime factors that "make" each of the given numbers. The first one is already completed.

100 **2 × 2 × 5 × 5**	**81**
96	**68**
93	**60**
82	**52**

Group B: Write the prime factors that "make" each of the given numbers. The first one is already completed.

98 **2 × 7 × 7**	**70**
87	**56**
84	**55**
80	**30**

Group C: Write the prime factors that "make" each of the given numbers. The first one is already completed.

94 **2×47**	**65**
90	**58**
76	**57**
69	**20**

Group D: Write the prime factors that "make" each of the given numbers. The first one is already completed.

92 **$2 \times 2 \times 23$**	**74**
91	**63**
85	**46**
78	**15**

Group E: Write the prime factors that "make" each of the given numbers. The first one is already completed.

99 **3 × 3 × 11**	**64**
95	**51**
88	**45**
66	**12**

Group F: Write the prime factors that "make" each of the given numbers. The first one is already completed.

86 **2 × 43**	**62**
77	**54**
75	**42**
72	**10**

Building Prime Numbers Answer Key

Group A

100: $2 \times 2 \times 5 \times 5$

81: $3 \times 3 \times 3 \times 3$

96: $2 \times 2 \times 2 \times 2 \times 2 \times 3$

68: $2 \times 2 \times 17$

93: 3×31

60: $2 \times 2 \times 3 \times 5$

82: 2×41

52: $2 \times 2 \times 13$

Group B

98: $2 \times 7 \times 7$

70: $2 \times 5 \times 7$

87: 3×29

56: $2 \times 2 \times 2 \times 7$

84: $2 \times 2 \times 3 \times 7$

55: 5×11

80: $2 \times 2 \times 2 \times 2 \times 5$

30: $2 \times 3 \times 5$

Group C

94: 2×47

65: 5×13

90: $2 \times 3 \times 3 \times 5$

58: 2×29

76: $2 \times 2 \times 19$

57: 3×19

69: 3×23

20: $2 \times 2 \times 5$

Group D

92: $2 \times 2 \times 23$

74: 2×37

91: 7×13

63: $3 \times 3 \times 7$

85: 5×17

46: 2×23

78: $2 \times 3 \times 13$

15: 3×5

Group E

99: $3 \times 3 \times 11$

64: $2 \times 2 \times 2 \times 2 \times 2 \times 2$

95: 5×19

51: 3×17

88: $2 \times 2 \times 2 \times 11$

45: $3 \times 3 \times 5$

66: $2 \times 3 \times 11$

12: $2 \times 2 \times 3$

Group F

86: 2×43

62: 2×31

77: 7×11

54: $2 \times 3 \times 3 \times 3$

75: $3 \times 5 \times 5$

42: $2 \times 3 \times 7$

72: $2 \times 2 \times 2 \times 3 \times 3$

10: 2×5

Prime Factorization Worksheet

Answer each problem below, making sure to show your reasoning and calculations.

1. The prime factorization of a number N is $N=2\times2\times2\times3\times3$. Decide, for each number below, whether it is a divisor of N.
    a. 3
    b. 5
    c. 16
    d. 8
    e. 18
    f. 6
    g. 20
    h. 27
2. The prime factorization of the number 56 is $2\times2\times2\times7$. Use the prime factorization to find all of the divisors of 56.

3. Extra Challenge: Suppose that a number is divisible by both 14 and 16. What is the smallest number it could be?

Prime Factorization Answer Key

1. a. 3, d. 8, e. 18, f. 6

2. 1, 2, 4, 7, 8, 14, 28, 56

   2, 2×2=4, 2×2×2=8 7, 2×7=14, 2×2×7=28, 2×2×2×7=56

   Also, 1, because it is a factor of all numbers.

3. The least common multiple of 14 and 16 is 112: 2×2×2×2×7.

Lesson 2.3. Remainder Strategies

This lesson introduces students to division problems that result in remainders and emphasizes different ways remainders can be interpreted. Using a mnemonic framework, students review four common strategies for handling remainders. The lesson includes guided practice followed by small-group work to reinforce these ideas.

Remainder Strategies

What happens if we try dividing something by a number that is not one of its factors? By the end of this lesson, students should recognize four different ways to handle remainders.

Once you've got a remainder, think of the mnemonic, "**DIRT**:"

- **D**rop It (ignore the remainder)
- **I**t's Your Answer (use the remainder)
- **R**ound Up (add one more)
- Keep I**T** (along with your quotient)

Class Practice Problems

1. **D**rop It (ignore the remainder)

You go to the pizza store with $7. It costs $2 to buy a slice of pizza. How many slices can you buy?

Answer:

With $7, at $2 per slice, 7÷2 = 3 ½ slices of pizza or 3 remainder $1.

The ½ represents the remainder. Since you cannot buy half of a slice, you drop (or ignore) the remainder because you can only buy 3 slices. It does not matter that you have money left over. If it is not enough to buy another slice, you must ignore it. So, you can buy 3 slices of pizza.

2. **I**t's Your Answer (use the remainder)

The students in Mr. Diegnan's class are forming groups of 4 for a math game. There are 29 students in his class. How many students will be left after groups of 4 are formed?

Answer:

29÷4 = 7 R 1. The 7 represents the number of groups that are formed, while the 1 represents the number of students who will be left after the groups are formed. In order to answer the question, use the remainder. So, there will be 1 student left after groups are formed.

3. Round Up (add one more)

Mrs. Pascarella is buying cupcakes for her daughter's birthday party. The cupcakes come in boxes of 4. There are 29 children attending. How many boxes of cupcakes does Mrs. Pascarella need to buy if she wants to give each child at the party a cupcake?

Answer:

29÷4 = 7 R 1. The 7 represents the need for at least 7 boxes of cupcakes to be ordered, while the 1 represents the number of children who would not get a cupcake if only 7 boxes of cupcakes are ordered. For everyone to get a cupcake, round up to 8 boxes. 8×4 = 32, so 8 boxes will be enough cupcakes.

4. Keep I**T** (along with your quotient)

Ms. Dolan has a long sheet of paper. The paper is 29 inches long, and she must use the entire sheet. She wants to cut the sheet of paper into 4 equal pieces. How long is each piece after it is cut?

Answer:

29÷4 = 7 R 1. The 7 represents the length of each piece: at least 7 inches long. The 1 represents the remaining 1 inch of paper after cutting the 7-inch-long pieces. In order to get 7 equal pieces and use the entire sheet of paper, the leftover 1-inch piece of paper need to also be cut into 4 equal parts, each ¼ inch long. So, each person gets 7 ¼ inches of paper. Here, the quotient and remainder are used.

Remainder Worksheet

For each problem below, show your calculations and list the remainder strategy used.

1. Boat Ride

   At an amusement park, 16 people are waiting in line for a water ride. Each boat holds 6 people. How many boats will be completely filled?

   Which of the remainder strategies did you use? _____

2. Apples

   Miss Eddy is giving out 13 apples to 4 of her students. She wants each student to get the same number of apples. She also wants no apple pieces left over. How many apples should each student get?

   Which of the remainder strategies did you use? _____

3. Trading Cards

   Thirty-three trading cards are to be placed in plastic sheets. Each sheet can display 8 cards. How many plastic sheets are needed to protect all the cards?

   Which of the remainder strategies did you use? _____

4. Field Trip

   Twenty-six students are going on a field trip, and parents have offered to drive. If they can fit 4 students in each car, how many students will be in a partially full car?

   Which of the remainder strategies did you use? _____

Remainder Answer Key

1. Boat Ride

   At an amusement park, 16 people are waiting in line for a water ride. Each boat holds 6 people. How many boats will be completely filled?

   16 people ÷ 6 people per boat = 2 boats, remainder 4 people

   There will be 2 full boats.

   Which of the remainder strategies did you use? **D**rop It

2. Apples

   Miss Eddy is giving out 13 apples to 4 of her students. She wants each student to get the same number of apples. She also wants no apple pieces left over. How many apples should each student get?

   13 apples ÷ 4 students = 3.25 apples per student

   Which of the remainder strategies did you use? Keep I**T**

3. Trading Cards

   Thirty-three trading cards are to be placed in plastic sheets. Each sheet can display 8 cards. How many plastic sheets are needed to protect all the cards?

   33 cards ÷ 8 cards per sheet = 4 sheets, remainder 1 card

   5 sheets are needed to protect all the cards.

   Which of the remainder strategies did you use? **R**ound Up

4. Field Trip

   Twenty-six students are going on a field trip, and parents have offered to drive. If they can fit 4 students in each car, how many students will be in a partially full car?

   26 students ÷ 4 students per car = 6 cars, remainder 2 students

   2 students will be in a partially full car

   Which of the remainder strategies did you use? **I**t's Your Answer

Additional Remainder Practice Worksheet

Please answer the problems below, showing all calculations clearly.

1. Skittles come in packages of 8. Theresa ate 11 Skittles. How many whole boxes did she eat, and how many Skittles does she have left in an open package?

2. Atomic Fireballs come in bags of 6. Janet ate 22 Atomic Fireballs. How many whole bags did she eat, and how many Atomic Fireballs does she have left?

3. Madison sold 30 boxes of Trefoils. How many cases of 4 boxes does Madison need to deliver? How many extra boxes will she have left?

4. Deborah had 19 Sweettarts. If Deborah gave equal numbers of Sweettarts to her 8 friends and she ate what was left, how many Sweettarts did each person eat?

5. It is currently 4:30 p.m. What time will it be in 2022 hours? Hint: What time will it be in 24 hours? What about in 48 hours?

6. Challenge problem: You and a group of 3 friends find a treasure chest filled with gold coins. You find that if you split the coins up into piles of 4, there are 2 coins remaining. If you split the coins into stacks of 9, you find that there is a remainder of 1 coin. You know that there are at least 100 coins. What is the least amount of gold coins possible?

Additional Remainder Practice Answer Key

1. Skittles come in packages of 8. Theresa ate 11 Skittles. How many whole boxes did she eat, and how many Skittles does she have left in an open package?

   11÷8=1 remainder 3

   1 full package eaten and 3 Skittles eaten from a partial package, with 5 left over.

2. Atomic Fireballs come in bags of 6. Janet ate 22 Atomic Fireballs. How many whole bags did she eat, and how many Atomic Fireballs does she have left in an open bag?

   22÷6 = 3 remainder 4

   3 whole bags eaten, 6-4 = 2 Atomic Fireballs left over

3. Madison sold 30 boxes of Trefoils. How many cases of 4 boxes does Madison need to deliver? How many extra boxes will she have left?

   30÷4 = 7 remainder 2

   7 cases of Trefoils, 2 extra boxes left

4. Deborah had 19 Sweettarts. If Deborah gave equal numbers of Sweettarts to her 8 friends and then she ate what was left, how many Sweettarts did each person eat?

   19÷8 = 2 remainder 3

   2 Sweettarts for each friend and 3 Sweettarts for Deborah

5. It is currently 4:30 p.m. What time will it be in 2022 hours? Hint: What time will it be in 24 hours? What about in 48 hours?

   2022 hours ÷ 24 hours per day = 84 days remainder 6 hours.

   We don't care how many days pass, only the remaining hours. Adding 6 hours to 4:30 pm, we get 10:30 p.m.

6. Challenge problem: You and a group of 3 friends find a treasure chest filled with gold coins. You find that if you split the coins up into piles of 4, there are 2 coins remaining. If you split the coins into stacks of 9, you find that there is a remainder of 1 coin. You know that there are at least 100 coins. What is the least amount of gold coins possible?

   The stacks of 4 have a remainder of 2, so we know that the number of coins is 2 more than a multiple of 4. Similarly, it is 1 more than a multiple of 9.

   Since there are at least 100 coins and 99 is divisible by 9, we can start testing from 100 coins (1 more than 99), then add 9 over and over (to check the next number with a remainder of 1 when divided by 9) to see what works.

   100-2=98, which is not divisible by 4.

   109-2=107, which is odd and therefore not divisible by 4.

   118-2=116 which is divisible by 4. So, the answer is 118 coins!

   Check: 118÷4 = 29 piles remainder 2 coins. 118÷9 = 13 piles remainder 1 coin.

   Note: We started with cycles of 9 since that is faster to check than cycles of 4!

Lesson 2.4. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

1. a) 47 ÷ 5 = _____ remainder _____

   b) 92 ÷ 8 = _____ remainder _____

2. A baker has 55 muffins and puts them into boxes of 6 each.

   a) How many full boxes can the baker make? Show your reasoning.

   b) How many muffins will be left over? Show your reasoning.

3. If it is 9 o'clock now, what time will it be in 17 hours? Hint: Think about the remainder when dividing 17 by 12.

4. Circle the prime numbers:

   12 17 23 28 31 40

5. List all the factors of 24: _____

6. List all the multiples of 7 less than 50: _____

7. Identify the common factors of 18 and 24: _____

8. A teacher has 134 stickers and wants to give each of 12 students the same number of stickers.

   a) How many stickers will each student get?

   b) How many stickers will be left over?

   c) If the leftover stickers are saved for a class activity, should you focus on the remainder or the quotient? Explain why.

Check Your Understanding Answer Key

1. a) 47 ÷ 5 = 9 remainder 2

   b) 92 ÷ 8 = 11 remainder 4

2. a) 9 full boxes (55÷6 = 9 remainder 1)

   b) 1 muffin left over (55÷6 = 9 remainder 1)

3. 2 o'clock. 17 ÷ 12 = 1 remainder 5, so 5 extra hours past 9 o'clock → 2 o'clock

4. Primes: 17, 23, 31

5. Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24

6. Multiples of 7 less than 50: 7, 14, 21, 28, 35, 42, 49

7. Common factors of 18 and 24: 1, 2, 3, 6

8. a) 11 stickers per student (134÷12 = 11 remainder 2)

   b) 2 stickers will be left over

   c) The remainder tells us how many stickers do not fit evenly into the groups. If we ignored the remainder, we wouldn't know how many extra stickers are left for the class activity.

Lesson 2.5. Introduction to Modular Arithmetic

This lesson introduces modular arithmetic using a circular number line. Students begin by reviewing 12-hour clock arithmetic through class practice with individual think time, then extend these ideas to a 7-hour clock. Formal modulo language is introduced and applied through guided practice involving addition, multiplication, equivalence, and connections to remainders. The lesson concludes with guided practice using day-of-the-week contexts to reinforce these concepts.

12-Hour Clock Arithmetic

Most people are familiar with a regular number line that stretches endlessly in both directions. But what if we imagined a number line that loops back on itself, forming a circle?

A common example of this is a clock. On a standard 12-hour clock, the numbers run from 1 to 12. However, once we pass 12, we don't continue to 13 in the same way as a straight number line. Instead, the count starts over: 13 o'clock is the same as 1 o'clock, 14 o'clock is the same as 2 o'clock, and this pattern continues forever. In this way, numbers "wrap around" the clock face. For instance, 25 o'clock points to the same position as 1 o'clock (and also the same as 13 o'clock). This circular numbering system is a simple introduction to what mathematicians call modular arithmetic.

For example, if it is 10 o'clock now, what time will it be 5 hours later? 10 o'clock + 5 hours is the same as 10 o'clock + 2 hours + 3 hours = 12 o'clock + 3 hours = 3 o'clock.

How can we end up here if we just add 10+5 = 15? What is "15 o'clock?" What we're really asking is, how far past 12 o'clock (or a multiple of 12) are we? 15 is 3 more than 12, so that brings us to 3 o'clock. Similarly, if we want to know the time 19 hours after 11 o'clock, 11+19 = 30, which is 6 more than 24, bringing us to 6 o'clock.

Class Practice Problems

1. If it is 2 o'clock now, what time will it be 36 hours later?
2. If it is 4 o'clock now, what time will it be 9 hours later?
3. If it is 6 o'clock now, what time will it be 13 hours later?
4. If it is 5 o'clock now, what time will it be 20 hours later?

Answers:

1. 2 o'clock

   We can think of this as 2+36 = 38, which is 2 more than a multiple of 12.

   Another option is to notice that 36 = 12×3, so we just loop around the clock 3 times and land back at 2 o'clock.

2. 1 o'clock

   4+9 = 13, which is 1 more than 12.

3. 7 o'clock

   6+13 = 19, which is 7 more than 12.

4. 1 o'clock

   5+20 = 25, which is 1 more than a multiple of 12.

Other Circular Number Lines

We do not have to limit ourselves to clocks! Imagine a number line from 0 to 7. If we bend this line into a circle so that 0 and 7 meet at the same point, we can use the same "wrap-around" idea. In this system, we treat the overlapping point as 0.

You can use this circular number line to perform addition in a slightly different way. For example, when you calculate something like 3 + 2, instead of simply moving two steps to the right along a straight line, you move two steps clockwise around the circle. Since this circle has 7 positions (including 0), we are looking for how far we are from a multiple of 7. In this system, numbers "wrap around" after reaching 7.

For instance, 4+5 = 9, which is 2 away from 7, so the addition "lands" at 2. In other words, starting at 4 and moving 5 steps clockwise around the 0-7 circle brings you to position 2.

This calculation method is called "modular arithmetic," and what we are doing is calculating modulo 7. We call 7 the modulus. So, the problem we just did can be written as "4+5 ≡ 2 mod 7." The symbol "≡" is used instead of an equal sign, and we read it as "is equivalent to."

Class Practice Problems

Try calculating these sums and products using the circular number line with 7 positions (0 through 6). Remember that after reaching 6, you wrap back to 0.

1. $1 + 4 \equiv ? \bmod 7$
2. $3 + 5 \equiv ? \bmod 7$
3. $2 + 6 \equiv ? \bmod 7$
4. $5 \times 2 \equiv ? \bmod 7$
5. $4 \times 3 \equiv ? \bmod 7$

Answers:

1. $1 + 4 \equiv 5 \bmod 7$ (since 5 is less than 7, it is our final answer)
2. $3 + 5 \equiv 1 \bmod 7$ (since $3 + 5 = 8$ and $8 \bmod 7 \equiv 1$)
3. $2 + 6 \equiv 1 \bmod 7$ ($2 + 6 = 8 \rightarrow 8 \bmod 7 \equiv 1$)
4. $5 \times 2 \equiv 3 \bmod 7$ ($5 + 5 = 10 \rightarrow 10 \bmod 7 \equiv 3$)
5. $4 \times 3 \equiv 5 \bmod 7$ ($4 \times 3 = 12 \rightarrow 12 \bmod 7 \equiv 5$)

We can continue moving around our 7-position circle. For example, to represent the number 16, start at 0 and move 16 steps clockwise. After wrapping around the circle twice, we land on 2. In modular arithmetic, we write this as: $16 \equiv 2 \bmod 7$ (or $16 \bmod 7 \equiv 2$).

Class Practice Problems

Find the position of these numbers on a modulo 7 circle:

1. $27 \equiv ? \bmod 7$
2. $45 \equiv ? \bmod 7$
3. $63 \equiv ? \bmod 7$
4. $108 \equiv ? \bmod 7$
5. $? \equiv 5 \bmod 7$

Hint: Divide by 7 and use the remainder to find the position. For part (e), find a number that would land on 5.

Answers:

1. $27 \equiv 6 \bmod 7$ ($27 \div 7$ leaves remainder 6)
2. $45 \equiv 3 \bmod 7$ ($45 \div 7$ leaves remainder 3)
3. $63 \equiv 0 \bmod 7$ ($63 \div 7$ leaves remainder 0)
4. $108 \equiv 3 \bmod 7$ ($108 \div 7$ leaves remainder 3)
5. Possible answers: 5, 12, 19, … Anything that is 5 more than a multiple of 7 works!

These concepts can also help us work with the days of the week. Using your understanding of arithmetic modulo 7, try solving the following set of problems.

Class Practice Problems

1. If today is Wednesday, what day will it be in 45 days' time?
2. If May 5th is on a Tuesday, what day will July 4th be on?
3. If February 1st is a Monday and it is a leap year, what day will February 1st be on next year?

Hint: Use arithmetic modulo 7 to calculate the number of days and determine the weekday.

Answers:

1. Saturday

   The problem we have to solve is 45 mod 7, which is 3. Move forward 3 days to Saturday.

2. Tuesday

   Count the days from May 5 to July 4: May has 31 days → $31 - 5 = 26$ days in May, and June has 30 days, so we have 26+30=56 days.

   $56 \bmod 7 \equiv 0$ → same day of the week, so July 4 will also be a Tuesday.

3. Wednesday

   Leap years have 366 days, so we calculate $366 \bmod 7 \equiv 2$. Two days from Monday, it will be Wednesday.

Lesson 2.6. More Modular Arithmetic

This class begins with a short, guided review of modular arithmetic and remainders. Students then work in small groups to complete a practice worksheet that reinforces these concepts and builds fluency.

Class Review Problem

Let's say that you plan to leave your home at 5:00 a.m. and it is a 9-hour drive. What time do you expect to arrive?

Answer:

You would expect to arrive at 2:00 p.m., right? But does that mean that 9 + 5 = 2? Isn't 9 + 5 = 14? When we do clock math, 2 and 14 mean the same thing.

We can say that two numbers are "congruent modulo 12" if they are equally far from a multiple of 12. That is, the remainder when you divide them by 12 is the same. This means that the difference between them is 12, 24, 36, or any other multiple of 12.

For example, 2 and 14 are congruent modulo 12 because their difference, twelve, is divisible by 12. This means that on a typical clock, 2 and 14 mean the same thing.

What are some other numbers that are congruent to 2 and 14 modulo 12?

Answer: 14+12=26, 26+12=38, 38+12=50, …

Modular Arithmetic and Remainder Worksheet

Please answer the problems below, showing all calculations clearly.

1. Find two numbers congruent to 5 modulo 12.

2. What time will it be 53 hours after 4 pm?

3. What time will it be 95 hours after 7 am?

4. $19 + 10 \equiv$ _____ (mod 7). Hint: The answer is less than 7.

5. Anne is making cards to send to her grandparents. She always puts 4 stickers on each card. If she has 23 stickers, how many cards can she make?

6. Mrs. Jordan is buying food for a picnic. A tray of sandwiches serves 11 people. How many trays should Mrs. Jordan buy to serve 60 people?

7. The hiking trail to the Mayan ruins is 32 miles long. The Moser family took three days to hike to the ruin. If they travelled the same amount each day, how many miles did they travel in one day?

8. Mr. Barona wants to put students into relay race teams. There needs to be four students on each team. Ms. Morgan's class has 30 students. How many teams can Mr. Barona make?

9. There are 14 children. A van can hold 5 children. How many vans are needed to take all of the children to the zoo?

10. Four children share fourteen cookies evenly. How many cookies will each child get?

11. Find the quotient and remainder when:

    a) 108 is divided by 3

    b) 129 is divided by 7

12. What is the remainder when:

    a) 16×2 is divided by 5?

    b) 12039102394 + 12340984921425 is divided by 5?

13. Sally the swimmer has a quirky way of recovering after her ultra-swim competitions: she takes a special floating nap in her recovery tank. If Sally climbs into her tank at 9 am on Monday and sets the timer for 78 hours, what day and time will she wake up?

14. On Tuesday morning, Lily discovered a new hobby: juggling flaming torches (safely, of course!). She loved it so much that she decided she would practice every day for the next 123 days. What day of the week will it be when Lily can finally take a break from juggling?

Modular Arithmetic and Remainder Answer Key

1. Find two numbers congruent to 5 modulo 12.

   5+12=17; 17+12=29

   Keep adding 12: 17, 29, 41, 53, 65, …

2. What time will it be 53 hours after 4 pm?

   Method 1: 53 mod 24 ≡ 5 (or 53÷24 = 2 days remainder 5 more hours), 4 pm + 5 hours = 9 pm.

   Method 2: 53+4 = 57; 57 mod 24 ≡ 9 → 9 pm.

3. What time will it be 95 hours after 7 am?

   Method 1: 95 mod 24 ≡ 23 (or 95 ÷ 24 = 3 days remainder 23 more hours), 7 am + 23 hours = 6 am.

   Method 2: 95+7 = 102; 102 mod 24 ≡ 6 → 6 am

4. 19 + 10 ≡ _____ (mod 7). Hint: The answer is less than 7.

   19+10=29. 29÷7 = 4 remainder 1, so 19+10 ≡ 1 (mod 7).

5. Anne is making cards to send to her grandparents. She always puts 4 stickers on each card. If she has 23 stickers, how many cards can she make?

   23÷4 = 5 cards, remainder 3 stickers. Drop the remainder since each card needs 4 stickers on it → 5 cards

6. Mrs. Jordan is buying food for a picnic. A tray of sandwiches serves 11 people. How many trays should Mrs. Jordan buy to serve 60 people?

   60÷11 = 5 trays, remainder 5 people. Round up to 6 trays since getting 5 trays would only feed 5×11=55 people → 6 trays

7. The hiking trail to the Mayan ruins is 32 miles long. The Moser family took three days to hike to the ruin. If they travelled the same amount each day, how many miles did they travel in one day?

   32 miles ÷ 3 days = 10 2/3 miles per day.

8. Mr. Barona wants to put students into relay race teams. There needs to be four students on each team. Ms. Morgan's class has 30 students. How many teams can Mr. Barona make?

   30 students ÷ 4 students per team = 7 teams, remainder 2 students. Round up to 8 teams.

9. There are 14 children. A van can hold 5 children. How many vans are needed to take all of the children to the zoo?

   14 children ÷ 5 children per van = 2 vans, remainder 4 children. Round up to 3 vans.

10. Four children share fourteen cookies evenly. How many cookies will each child get?

    14 cookies ÷ 4 children = 3 2/4=3 1/2 cookies per child.

11. Find the quotient and remainder when:

    a) 108 is divided by 3.

    108÷3 = 36, remainder 0

    b) 129 is divided by 7

    129÷7 = 18, remainder 3

12. What is the remainder when:

    a) 16 × 2 is divided by 5?

    Answer: 2

    16×2 = 32, 32÷5 = 6, remainder 2, i.e. 32 mod 5 ≡ 2

    b) 12039102394 + 12340984921425 is divided by 5?

    Answer: 4

    The last digit is 9, so the remainder is 9 - 5 = 4

13. Sally the swimmer has a quirky way of recovering after her ultra-swim competitions: she takes a special floating nap in her recovery tank. If Sally climbs into her tank at 9 am on Monday and sets the timer for 78 hours, what day and time will she wake up?

    Answer: 3:00 PM on Thursday

    Break 78 hours into days and hours:

    1 day = 24 hours

    78 ÷ 24 = 3 days and 6 hours (because 24 × 3 = 72, and 78 − 72 = 6)

    Add 3 days to Monday 9:00 AM:

    Monday 9:00 AM + 3 days = Thursday 9:00 AM

    Add the remaining 6 hours:

    Thursday 9:00 AM + 6 hours = Thursday 3:00 PM

14. On Tuesday morning, Lily discovered a new hobby: juggling flaming torches (safely, of course!). She loved it so much that she decided she would practice every day for the next 123 days. What day of the week will it be when Lily can finally take a break from juggling?

    Answer: Lily can take a break on Saturday.

    Since a week has 7 days, divide 123 by 7 to find how many full weeks and extra days:

    123÷7=17 weeks with a remainder of 4 days

    17 full weeks bring her to the same day of the week (Tuesday)

    Add the 4 extra days to end up on a Saturday.

Lesson 2.7. Numerical Properties

The first part of this lesson explores patterns and sums. Students will examine consecutive integers, consecutive odd integers, and consecutive even integers, and make conjectures about their sums. Next, the lesson connects to modular arithmetic. Students will add different numbers and use modular arithmetic to determine the result's parity.

Class Practice Problems

Think about how we can find these sums without adding each number manually. Look for any patterns you notice and discuss them.

1. 6+7+8 = 21
2. 24+25+26 = 75
3. 79+80+81 = 240

Answers:

Think of "lending" 1 from the biggest number to the smallest one.

1. (6**+1**)+7+(8**-1**) = 7+7+7 = **7×3** = 21
2. (24**+1**)+25+(26**-1**) = 25+25+25 = **25×3** = 75
3. (79**+1**)+80+(81**-1)** = 80+80+80 = **80×3** = 240

Class Practice Problems

If we know the sum of three consecutive numbers, how can we find the numbers?

1. _____ + _____ + _____ = 36
2. _____ + _____ + _____ = 300
3. _____ + _____ + _____ = 510
4. _____ + _____ + _____ = 102

Answers:

1. 36÷3 = 12 → this is the middle number! **11, 12, 13**
2. 300÷3 = 100 → **99, 100, 101**
3. 510÷3 = 170 → **169, 170, 171**
4. 102÷3 = 34 → **33, 34, 35**

Class Practice Problems

Can this be extended to more than three numbers?

1. 4+5+6+7 = ?
2. 10+11+12+13 = ?
3. 23+24+25+26 = ?

Answers:

1. 4+5+6+7 = (middle number) × 4 = 5.5×4 = 22
2. 10+11+12+13 = (middle number) × 4 = 11.5×4 = 46
3. 23+24+25+26 = (middle number) × 4 = 24.5×4 = 98

Class Practice Problems

What about for consecutive odd numbers? And even numbers?

1. 7+9+11 = ?
2. 90+92+94 = ?
3. 10+12+14+16 = ?
4. 15+17+19+20 = ?

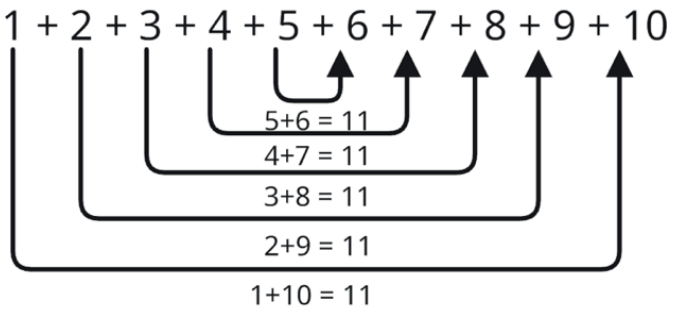
Answers:

1. 7+9+11 = (middle number) × 3 = 9×3 = 27
2. 90+92+94 = (middle number) × 3 = 92×3 = 276
3. 10+12+14+16 = (middle number) × 4 = 13×4 = 52
4. 15+17+19+20 = (middle number) × 4 = 18×4 = 72

We can think of any list of consecutive numbers in terms of the number in the middle of the list!

1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = ?

The number in the middle here is 5.5, which is not a whole number, so it might be easier to think of what each pair of numbers on this list adds up to.



1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 11×5 = 55

Class Practice Problems

Try adding the following numbers:

1. 10 + 11 + 12 + 13 + 14 + 15 = ?
2. 35 + 37 + 39 + 41 + 43 + 45 = ?
3. 1000 + 1001 + 1002 + 1003 + 1004 + 1005 = ?
4. 1 + 2 + 3 + 4 + … + 97 + 98 + 99 + 100 = ?

Answers:

1. 10 + 11 + 12 + 13 + 14 + 15 = **25×3** = 75 (since 10+15 = 11+14 = 12+13 = 25)
2. 37 + 39 + 41 + 43 = **80×2** = 160 **or 40×4** = 160
3. 1000 + 1001 + 1002 + 1003 + 1004 + 1005 = **2005×3** = 6,015
4. 1 + 2 + 3 + 4 + … + 97 + 98 + 99 + 100 = **101×50** = 5,050

Modular Arithmetic and Parity

Class Practice Problems

Compute the following.

1. 3 (mod 2)
2. 8 (mod 2)
3. 100 (mod 2)
4. 25 (mod 2)
5. 444 (mod 2)
6. What do you notice about the value of odd numbers modulo 2? What about even numbers?

Answers:

1. 3 (mod 2) ≡ 1 (mod 2)
2. 8 (mod 2) ≡ 0 (mod 2)
3. 100 (mod 2) ≡ 0 (mod 2)
4. 25 (mod 2) ≡ 1 (mod 2)
5. 444 (mod 2) ≡ 0 (mod 2)
6. We can always think of even numbers as "multiples of 2," so they will never have a remainder when divided by 2. That means that even numbers are always equivalent to 0 modulo 2.

(even number) **≡ 0 (mod 2)**

On the other hand, odd numbers are always 1 away from an even number. For example, 3 is 1 more than 2, and 25 is 1 more than 24. Since odd numbers are 1 more than a multiple of 2, they have a remainder of 1 when divided by 2. That means that odd numbers are always equivalent to 1 modulo 2.

$$(\text{odd number}) \equiv \mathbf{1 \pmod 2}$$

Based on the results of the conclusion above, consider the following sums as a class and try to come up with a conjecture.

Class Practice Problems

Is the sum of these numbers even or odd? Focus on whether the numbers you are adding are odd or even, not necessarily on their actual values.

1. $1 + 9 + 13 + 21$
2. $32 + 15 + 18$
3. $6{,}262 + 7{,}444 + 132 + 2{,}468$
4. $313 + 58{,}575 + 149$

Answers:

1. $1 + 9 + 13 + 21 \equiv 1 + 1 + 1 + 1 \pmod 2 \equiv 4 \equiv 0 \pmod 2$ → even
2. $32 + 15 + 18 \equiv 0 + 1 + 0 \pmod 2 \equiv 1 \pmod 2$ → odd
3. $6{,}262 + 7{,}444 + 132 + 2{,}468 \equiv 0 + 0 + 0 + 0 \pmod 2 \equiv 0 \pmod 2$ → even
4. $313 + 58{,}575 + 149 \equiv 1 + 1 + 1 \pmod 2 \equiv 3 \equiv 1 \pmod 2$ → odd

We can just add all the extra "1s" contributed by odd numbers to see if the result is odd or even!

Numerical Properties Worksheet

Please answer the problems below, showing all calculations clearly.

1. The sum of three consecutive odd numbers is 93. Find the numbers.

   a. 29, 30, 31

   b. 29, 31, 33

   c. 27, 31, 35

   d. 30, 31, 32

2. Find three consecutive integers with a sum of 168. Which of these is the greatest of the three integers?

   a. 59

   b. 57

   c. 19

   d. 53

3. Find three consecutive integers with a sum of 204.

4. Find three consecutive integers with a sum of 99.

5. Is the following sum odd or even? 53 + 28 + 101 + 320

6. Is the following sum odd or even? 300 + 129 + 4321 + 8,765,432

Numerical Properties Answer Key

1. The sum of three consecutive odd numbers is 93. Find the numbers. 93÷3=31

   b. 29, 31, 33

2. Find three consecutive integers with a sum of 168. Which is the greatest of the three?

   168÷3=56

   b. 57

3. Find three consecutive integers with a sum of 204.

   204÷3=68 → **67, 68, 69**

4. Find three consecutive integers with a sum of 99.

   99÷3 = 33 → **32, 33, 34**

5. Is the following sum odd or even? 53 + 28 + 101 + 320

   1 + 0 + 1 + 0 (mod 2) ≡ 2 ≡ 0 (mod 2) → **even**

6. Is the following sum odd or even? 300 + 129 + 4321 + 8,765,432

   0 + 1 + 1 + 0 (mod 2) ≡ 2 ≡ 0 (mod 2) → **even**

Lesson 2.8. Number Formats and Divisibility

This lesson reviews common number formats and divisibility concepts. Instruction begins by ensuring students can convert between mixed numbers and improper fractions and perform long division by hand. Students practice expressing results as mixed numbers, decimals, and quotients with remainders, then work independently on a practice worksheet, checking answers in pairs.

Guided Examples

Consider the following numbers, expressed as improper fractions, mixed numbers, decimals, and quotients with remainders.

$\frac{100}{3} = 33\frac{1}{3} = 33.333\ldots$ is the same as 100÷3 = 33 Remainder 1

$\frac{574}{5} = 114\frac{4}{5} = 114.8$ is the same as 574÷5 = 114 Remainder 4

Class Practice Problem

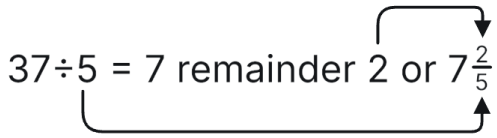
What are some different ways to present the number $33\frac{1}{3}$?

Answer: $33\frac{1}{3} = \frac{100}{3} = 33.333\ldots$

This is the same as the division problem 100÷3.

Remainders as a Fraction Worksheet

When division results in a remainder, we can write that remainder as a fraction of the divisor. For each of the division problems below, find the remainder, then represent the answer as a mixed number and an improper fraction. For example,



1. 22 ÷ 5 = 4 remainder ☐ or 4 ☐/☐ or $\frac{22}{5}$

2. 32 ÷ 10 = 3 remainder ☐ or 3 ☐/☐ or ☐/☐

3. 29 ÷ 5 = 5 remainder ☐ or 5 ☐/☐ or ☐/☐

4. 28 ÷ 3 = 9 remainder ☐ or 9 ☐/☐ or ☐/☐

5. 26 ÷ 4 = 6 remainder ☐ or 6 ☐/☐ or ☐/☐

6. 84 ÷ 9 = 9 remainder ☐ or 9 ☐/☐ or ☐/☐

Remainders as a Fraction Answer Key

1. $22 \div 5 = 4$ remainder 2 or $4\frac{2}{5}$ or $\frac{22}{5}$

2. $32 \div 10 = 3$ remainder 2 or $3\frac{2\div2}{10\div2} = 3\frac{1}{5}$ or $\frac{32\div2}{10\div2} = \frac{16}{5}$

3. $29 \div 5 = 5$ remainder 4 or $5\frac{4}{5}$ or $\frac{29}{5}$

4. $28 \div 3 = 9$ remainder 1 or $9\frac{1}{3}$ or $\frac{28}{3}$

5. $26 \div 4 = 6$ remainder 2 or $6\frac{2\div2}{4\div2} = 6\frac{1}{2}$ or $\frac{26\div2}{4\div2} = \frac{13}{2}$

6. $84 \div 9 = 9$ remainder 3 or $9\frac{3\div3}{9\div3} = 9\frac{1}{3}$ or $\frac{84\div3}{9\div3} = \frac{28}{3}$

Lesson 2.9. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

1. Simplify: 17 ≡ _____ (mod 5). Show your calculations or logic.

2. If *a number* is equivalent to 4 (mod 7), what is *that number + 10* equivalent to (mod 7)?

3. A clock shows that it's 8 o'clock.

   a) After 11 hours, what time will it show?

   b) Explain how this connects to modular arithmetic.

4. Complete the pattern and explain each result using modular arithmetic.

   a) odd + odd = _____ because _____

   b) even × odd = _____ because _____

   c) odd × odd = _____ because _____

5. a) If a number is even, what can you say about its remainder when divided by 2?

   b) How does this relate to modular arithmetic?

6. a) Find the sum of these numbers efficiently:

   3,998 + 3,999 + 4,000 + 4,001 + 4,002 = _____

   b) Find the sum of these numbers efficiently:

   5,047 + 5,049 + 5,051 + 5,053 = _____

7. Write 13 ÷ 4 as:

   a) a mixed number _____

   b) a fraction _____

   c) a decimal _____

8. A baker divides 11 cookies among 2 friends. Write 11 ÷ 2 as a fraction, mixed number, and remainder. Which form best describes how many cookies each person gets?

9. Which division problem does the fraction 7/3 represent? Explain your answer.

10. Why does 10 ÷ 4 give the same result as 5 ÷ 2? Use a visual (like a pie chart) or a short, written explanation. Feel free to be creative!

11. Reflection: When you see a remainder in a problem, how do you decide whether to write it as a remainder, a fraction, or a decimal? Give an example.

Check Your Understanding Answer Key

1. $17 \equiv 2 \pmod 5$; $17 \div 5 = 3$ remainder 2 or $\frac{17}{5} = 3\frac{2}{5}$
2. $(4 + 10) \equiv 14 \equiv 0 \pmod 7$.
3. a) After 11 hours, it will be 7 o'clock.

   b) Clock math is like doing arithmetic mod 12.
4. a) odd + odd = even; $(\text{odd}+\text{odd}) \bmod 2 \equiv 1+1 \equiv 2 \equiv 0 \bmod 2$

   b) even × odd = even; $(\text{even}\times\text{odd}) \bmod 2 \equiv 0\times1 \equiv 0 \bmod 2$

   c) odd × odd = odd; $(\text{odd}\times\text{odd}) \bmod 2 \equiv 1\times1 \equiv 1 \bmod 2$

   An attempt at an explanation with remainders when dividing by 2 (i.e., making a mod 2 connection) is ideal.
5. a) Even numbers leave a remainder of 0 when divided by 2 because they are all divisible by 2.

   b) Even numbers are divisible by 2, so any even number is equivalent to 0 mod 2.
6. a) 4,000×5= 20,000

   b) 5,050×4 = 20,000+200 = 20,200 or 10,100×2 (from the sum of number pairs)
7. 13 ÷ 4 = 3 remainder 1 → a) $3\frac{1}{4}$, b) $\frac{13}{4}$, c) 3.25.
8. 11 ÷ 2 = 5 remainder 1 = 5 ½ = 5.5; a mixed number best describes the answer, 5 ½ cookies.
9. 7 ÷ 3 = 7/3; the fraction bar and the division symbol give the same result.
10. Both simplify to 2½ or 2.5; dividing the numerator and denominator by 2 gives the same quotient. A visual with correct shading is correct!
11. Reflection. Students should reason that context determines the best form: decimals for measurement, remainders for counting, and fractions for sharing.

Lesson 2.10. Place Value and Terminating Versus Repeating Decimals

This lesson begins with guided examples exploring remainders when dividing by powers of 10 and connecting the number of decimal places to the remainder to reinforce place value. Students then work in groups on a worksheet practicing division by powers of 10. The class moves on to investigate why some decimals terminate while others repeat using prime factorization, followed by additional group work on a related worksheet.

Division by Powers of 10

Class Practice Problems

What is the remainder when:

1. 5,329 is divided by 10?
2. 5,329 is divided by 100?
3. 5,329 is divided by 1,000?

Answers:

1. 5,329÷10 has a remainder of 9
2. 5,329÷100 has a remainder of 29
3. 5,329÷1,000 has a remainder of 329

Class Practice Problem

How do the remainders above relate to the original number?

Answer:

The remainder's digits correspond to the number of times we divide by 10.

1. Dividing by 10 → 1-digit remainder → last digit
2. Dividing by 100 = 10*10 → 2-digit remainder → last 2 digits
3. Dividing by 1,000 = 10*10*10 → 3-digit remainder → last 3 digits

Class Practice Problem

How does the remainder relate to the decimal representation of the division?

Answer:

- 5,329÷10 = 532.9
- 5,329÷100 = 53.29
- 5,329÷1,000 = 5.329

The remainder *is* the decimal part of the result.

Class Practice Problems

Divide the following numbers. Express your answer as a number with a remainder and as a decimal.

1. 48÷10 = ?
2. 735÷100 = ?
3. 9,102÷1000 = ?

Answers:

1. 48÷10 = 4 remainder 8 or 4.8
2. 735÷100 = 7 remainder 35 or 7.35
3. 9,102÷1000 = 9 remainder 102 or 9.102

Dividing by Powers of 10: Remainders as Decimals Worksheet

Please answer the problems below, showing all calculations clearly.

Example: $43 \div 10 = 4$ remainder $3 = 4\ \frac{3}{10} = 4.3$

1. $36 \div 10 =$ ☐ remainder ☐ = ☐ ☐/☐ = ☐.☐

2. $52 \div 10 =$ ☐ remainder ☐ = ☐ ☐/☐ = ☐.☐

3. $47 \div 10 =$ ☐ remainder ☐ = ☐ ☐/☐ = ☐.☐

4. $291 \div 100 =$ ☐ remainder ☐ = ☐ ☐/☐ = ☐.☐

5. $3685 \div 1000 =$ ☐ remainder ☐ = ☐ ☐/☐ = ☐.☐

Dividing by Powers of 10: Remainders as Decimals Answer Key

1. $36 \div 10 = 3$ remainder $6 = 3\ \frac{6}{10} = 3.6$

2. $52 \div 10 = 5$ remainder $2 = 5\ \frac{2}{10} = 5.2$

3. $47 \div 10 = 4$ remainder $7 = 4\ \frac{7}{10} = 4.7$

4. $291 \div 100 = 2$ remainder $91 = 2\ \frac{91}{100} = 2.91$

5. $3685 \div 1000 = 3$ remainder $685 = 3\ \frac{685}{1000} = 3.685$

Terminating Versus Repeating Decimals

Why do some decimals terminate while others repeat?

Which of these fractions have decimals that stop? Which ones go on forever?

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$$

Groups should investigate further and discuss their thoughts. They can try more examples as needed. A hint to give if they are stuck is to consider the prime factors of the divisor (the denominator).

Answer:

Only denominators (after simplification) that are made up of 2s and 5s (the factors of 10) result in terminating decimals. Since decimals represent fractions of 10, the decimal part of fractions that have terminating decimals can be written as $\frac{remainder}{10}$, $\frac{remainder}{100}$, $\frac{remainder}{1000}$, etc. For example, 9.34 can be written as 9 $\frac{34}{100}$ and $\frac{1}{2}$ can be written as $\frac{5}{10}$, but $\frac{1}{3}$ cannot be written as $\frac{number}{10}$.

Class Practice Problems

Without dividing, decide if the following fractions will terminate or repeat.

1. $\frac{1}{10}$
2. $\frac{1}{15}$
3. $\frac{7}{20}$

4. $\frac{2}{7}$

5. $\frac{3}{60}$

Answers:

1. $\frac{1}{10}$ terminates since 10=2×5.

2. $\frac{1}{15}$ repeats since 15=3×5, which includes a prime other than 2 and 5.

3. $\frac{7}{20}$ terminates since 20=2×2×5.

4. $\frac{2}{7}$ repeats since 7 is prime, which is a prime other than 2 and 5.

5. $\frac{3}{60}$ terminates since 3/60 = 1/20 and 20=2×2×5.

Decimals: Terminating or Repeating? Worksheet

Please find the prime factors of each denominator, then decide if each of these divisions results in a terminating or repeating decimal.

Number	Prime factors of denominator	Terminating or repeating?
$\frac{1}{2}$		
$\frac{3}{4}$		
$\frac{5}{8}$		
$\frac{7}{10}$		
$\frac{9}{20}$		
$\frac{1}{3}$		
$\frac{2}{7}$		
$\frac{5}{6}$		
$\frac{11}{12}$		
$\frac{7}{15}$		
$\frac{8}{25}$		
$\frac{3}{40}$		
$\frac{1}{11}$		
$\frac{13}{16}$		
$\frac{7}{30}$		

Decimals: Terminating or Repeating? Answer Key

Please find the prime factors of each denominator, then decide if each of these divisions results in a terminating or repeating decimal.

Number	Prime factors of denominator	Terminating or repeating?
$\frac{1}{2}$	2	Terminating
$\frac{3}{4}$	2	Terminating
$\frac{5}{8}$	2	Terminating
$\frac{7}{10}$	2, 5	Terminating
$\frac{9}{20}$	2, 5	Terminating
$\frac{1}{3}$	3	Repeating
$\frac{2}{7}$	7	Repeating
$\frac{5}{6}$	2, 3	Repeating
$\frac{11}{12}$	2, 3	Repeating
$\frac{7}{15}$	3, 5	Repeating
$\frac{8}{25}$	5	Terminating
$\frac{3}{40}$	2, 5	Terminating
$\frac{1}{11}$	11	Repeating
$\frac{13}{16}$	2	Terminating
$\frac{7}{30}$	2, 3, 5	Repeating

Lesson 2.11. Arithmetic Skill Review and Mixed Word Problems

This lesson focuses on consolidating skills with decimals and fractions. Students first create a personal note sheet with examples and key points, making connections between earlier arithmetic skills and recent work on numerical properties, prime numbers, and parity. For the rest of class, students work in pairs on mixed word problems that bring together these concepts.

Arithmetic Skill Review

For each problem below:

- Solve the problem and show your work
- Write notes to yourself to help you remember how to do this type of problem.

Adding and subtracting decimals

Example: 25.32 + 4.8

Notes:

Example: 108.5 - 24.31

Notes:

Multiplying and dividing decimals

Example: 2.45×6

Notes:

Example: $24.9 \div 3$

Notes:

Adding and subtracting fractions

Example: $\frac{1}{7} + \frac{4}{7}$

Notes:

Example: $\frac{7}{4} - \frac{2}{3}$

Notes:

Multiplying and dividing fractions

Example: $\frac{1}{3} \times \frac{5}{7}$

Notes:

Example: $\frac{7}{4} \div \frac{2}{3}$

Notes:

Mixed Word Problems Worksheet

Please answer the problems below, showing all calculations clearly. Make sure to choose a representation of your answer that makes sense for the context of the problem. For example, decimals are useful for measurements and money, remainders often make sense when counting items, and fractions are helpful when sharing.

1. A bank account has $150. Three transactions occur: a withdrawal of $75.25, a deposit of $60.80, and a withdrawal of $25.10. What is the final balance?

2. A hiker begins at -85.5 meters. They ascend 120.25 meters, then descend 45.75 meters. What is their final elevation?

3. The temperature is -2.5°C in the morning. It rises 6.75°C during the day, then drops 8.25°C overnight. What is the temperature the next morning?

4. A gymnast receives scores of 8.5, 9.25, 7.75, and 8.0. What is her average score?

5. A recipe calls for $\frac{2}{3}$ cup of oil. You want to make half the recipe. How much oil do you need?

6. A customer purchases three bracelets priced at $24.95 each, but there is a storewide discount of $18.50 applied at checkout. What is the final cost of the purchase?

7. A tank starts with 100 gallons. It leaks 12.5 gallons overnight and an additional $\frac{3}{4}$ gallon in the morning. What is the remaining volume?

8. Easter eggs are packed in boxes of 24. How many boxes are needed to pack 250 eggs?

9. 355 children are going on a trip to Washington, D.C. A coach bus seats 53 children. How many buses are needed for the children?

10. I have 366 cans of soda. They are packed in cases of 24. How many whole cases do I have?

11. I have made 265 cakes for a party. Boxes hold 25 cakes. How many boxes do I need to hold all the cakes?

12. A ferry to the Isle of Wight holds 230 people. How many trips would the ferry need to take to carry 1,200 people to the island?

Mixed Word Problems Answer Key

1. $110.45

   150-75.25 = 74.75; 74.75+60.80 = 135.55; 135.55-25.10 = 110.45

2. 11 meters below sea level

   -85.5+120.25 = 34.75; 34.75-45.75 = -11

3. -4°C

   -2.5+6.75 = 4.25; 4.25-8.25 = -4

4. 8.375

   8.5+9.25+7.75+8.0 = 33.5; 33.5÷4 = 8.375

5. $\frac{1}{3}$ cup of oil

   Divide $\frac{2}{3}$ by 2: $\frac{2}{3} \div 2 = \frac{2}{3} \div \frac{2}{1} = \frac{2}{3} \times \frac{1}{2} = \frac{2}{6} = \frac{2 \div 2}{6 \div 2} = \frac{1}{3}$

6. $56.35

   3×24.95 = 74.85; 74.85-18.50 = 56.35

7. 86.75 gallons

   100-12.5 = 87.5; ¾=0.75; 87.5-0.75 = 86.75

8. 11 boxes

   250÷24 = 10 boxes, remainder 10 eggs → 11 boxes for all of the eggs

9. 7 buses

   355÷53 = 6 buses with some remainder → 7 buses for everyone

10. 15 whole cases

    366÷24 = 15 with some remainder → 15 whole cases and 1 partially full case

11. 11 boxes

    265÷25 = 10 boxes with some remainder → 11 boxes for all of the cakes

12. 6 trips

    1200÷230 = 5 with some remainder → 6 trips

Lesson 2.12. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

1. Without dividing, decide whether each fraction has a terminating or repeating decimal. Explain briefly.

   a) $\frac{3}{8}$

   b) $\frac{4}{45}$

   c) $\frac{5}{40}$

   d) $\frac{7}{18}$

2. Write 54 ÷ 8 in four forms: a mixed number, a whole number with a remainder, an improper fraction, and a decimal.

3. The number $\frac{5}{2}$ can be written in many forms. List three different ways to represent this number.

4. If the sum of three consecutive integers is 81, what are the integers? Show or explain your reasoning.

5. Find each sum using your knowledge of patterns. Show your calculations.

   a) 14+15+16+17+18 =

   b) 51+53+55+57+59 =

   c) 1+2+3+4+…+48+49+50 =

6. A group of 95 students is going on a museum trip. Each bus holds 24 students. How many buses are needed to transport all students? Explain why your answer makes sense.

7. The biking trail around Lake Orion is 27 miles long. Omar bikes the same distance each day and finishes in 4 days. How many miles does he bike per day?

8. It is 11 p.m. right now. What time will it be 50 hours from now?

9. $9 \times 4 \equiv ? \pmod{11}$

10. I am a factor of both 84 and 126.

    - Clue 1: I am greater than 5.
    - Clue 2: I am even.
    - Clue 3: I am also a multiple of 7.
    - What number am I?

Check Your Understanding Answer Key

1. a) $\frac{3}{8}$ is terminating (denominator = 2×2×2)

   b) $\frac{4}{45}$ is repeating (denominator = 3×3×5, which includes a prime other than 2 and 5)

   c) $\frac{5}{40}$is terminating (simplifies to $\frac{1}{8}$, with a denominator of 2×2×2)

   d) $\frac{7}{18}$ is repeating (denominator has the factor 3, which is not 2 or 5)

2. Mixed number: $6\frac{6}{8} = 6\frac{3}{4}$

   Whole number with remainder: 6 remainder 6

   Improper fraction: $\frac{54}{8} = \frac{27}{4}$

   Decimal: 6.75

3. 2.5, 2 ½, 5 ÷ 2

4. 81÷3 = 27 → 26, 27, 28

5. a) 14+15+16+17+18 = 16×5 = 80

   b) 51+53+55+57+59 = 55×5=275

   c) 1+2+3+4+…+48+49+50 =51×25 = 1,275

6. 95 students ÷ 24 students per bus = 3 buses with left over students → 4 buses

7. 27 miles ÷ 4 days = 6.75 miles per day

8. 50 mod 24 ≡ 2. 11 pm + 2 hours = 1 am

9. 9×4 = 36 ≡ 3 (mod 11) since 36/11 = 3 remainder 3

10. Answer: 14. Listing factor pairs is helpful.

Year 3 Lessons

The following 12 lessons make up the Year 3 course of study.

Lesson 3.1. Number Theory and Modular Arithmetic Review

Instructors are encouraged to begin the class with a brief review of number theory, revisiting what it is and how it is used in familiar mathematical contexts. The lesson then transitions to a review of modular arithmetic, reinforcing key ideas through guided examples and practice. Class practice problems are designed to help students strengthen their understanding of modular arithmetic in an engaging way.

Number Theory Review

What is "number theory" and why does it matter? Students should share their definitions with the class and discuss this first. To help frame the discussion, recall the introduction to number theory from Year 1 of the course, which was repeated in Year 2.

Number theory studies positive whole numbers (natural numbers). It examines relationships between different types of numbers and studies how smaller numbers combine to form bigger ones. For example,

1. Number theory helps us know how numbers work. If you know how LEGO pieces fit together, you can build things. In the same way, if you know how numbers connect, you can solve hard problems. For example, number theory helps us understand prime numbers, which are like the basic LEGO blocks of all numbers.

2. Number theory keeps information safe. When you send a text, play an online game, or use a website, number theory helps protect your information. It is used in making codes (encryption). Large prime numbers help lock your information so only you can open it.
3. Number theory helps computers work better. Computers use number patterns to do math quickly. Number theory makes shortcuts, so computers can solve problems in seconds instead of hours.
4. Number theory has many fun puzzles, like finding patterns, checking if a number is prime, or finding tricks with digits. If you like solving puzzles or brain teasers, you already think like a number theorist.

Spend the rest of class working on the Modular Arithmetic review to remind students about dividing with remainders and repeating patterns.

Modular Arithmetic Review

Sometimes, we care more about the remainder of a division than the answer (i.e., the quotient).

For example, when we work on clock arithmetic problems, we want to know what day it will be in some number of days. In other words, we want to know what is left over after dividing that number of days by 7; we do not really care how many weeks pass, just what day we end up on.

Class Practice Problems

Calculate the following:

1. 12 mod 3
2. 66 mod 12
3. 37 mod 6
4. 136 mod 5

Answers:

1. 12 mod 3 ≡ 0; 12 ÷ 3 = 4, remainder 0
2. 66 mod 12 ≡ 6; since 60 is the closest smaller multiple of 12 to 66, the remainder is 66-60 = 6.
3. 37 mod 6 ≡ 1; 37 ÷ 6 = 6, remainder 1
4. 136 mod 5 ≡ 1; since 135 is the closest smaller multiple of 5 to 136, the remainder is 136-135 = 1.

Modular Equivalence

7 mod 5 and 12 mod 5 are both equal to 2, so we can say that 2, 7, and 12 are congruent modulo 5. We write this using the symbol ≡. In other words, this means that these integers have the same remainder modulo 5: $2 \equiv 7 \equiv 12 \pmod 5$.

The (mod 5) part just tells us that we are working with the remainders when we divide by 5. The parentheses are optional.

Class Practice Problems

1. Find 3 other integers that are congruent to 2, 7, and 12 (mod 5).
2. True or False:
    a. $31 \equiv 1 \pmod{10}$ (T/F)
    b. $43 \equiv 23 \pmod 7$ (T/F)
    c. $139 \equiv 0 \pmod 2$ (T/F)

Answers:

1. 17, 22, 27, 32, 37, ... ≡ 2, 7, and 12 (mod 5).
2. True or False:
    a. True: 30 is divisible by 10, and 31-30 = 1.
    b. False: 43 mod 7 ≡ 1 but 23 mod 7 ≡ 2.
    c. False: 139 is odd, so it is not divisible by 2; 139 ≡ 1 mod 2.

Lesson 3.2. The Division Algorithm

This lesson reviews the division algorithm and the steps of long division. Students practice dividing numbers and interpreting each step, reinforcing the relationship between dividend, divisor, quotient, and remainder. The class concludes with an overview of division terminology.

Understanding the Division Algorithm

Consider the division problem, 435÷3. If you remember the division algorithm, i.e., long division, the process will look like this (or similar).

$$
\begin{array}{r}
145 \\
3\overline{)\,435} \\
-3 \\
\hline
135 \\
-12 \\
\hline
15 \\
-15 \\
\hline
0
\end{array}
$$

What is happening at every step? Here is one way to think of this process.

- When we figure out how many 3's fit into 4, we are actually deciding how many 300's fit into 400 since 4 is in the hundreds place. We can fit 1 three hundred into 400 and have 100 left over.
- We then ask how many 30's fit into 135. We can fit 4 thirties into 135 and have 15 left over.
- We then ask how many 3's fit into 15. Finally, we have 5 threes that go into 15, with nothing left over.
- The answer "145" represents the **1** three hundred, **4** thirties, and **5** threes that fit into 435.

Class Practice Problems

Divide and explain what happens at each step:

1. 4791 ÷ 7
2. 537 ÷ 6
3. 92,674 ÷ 9
4. 11,131 ÷ 5

Answers:

1. 4791 ÷ 7

   4/7 = 0 (no seven thousands fit into 4,000)

   47/7 = **6** R 5 (6 seven hundreds fit into 4,700)

   59/7 = **8** R 3 (8 seventies fit into 590)

   31/7 = **4 R 3** (4 sevens fit into 31)

   Answer: 684 R 3

2. 537 ÷ 6

   5/6 = 0 (no six hundreds fit into 500)

   53/6 = **8** R 5 (8 sixties fit into 530)

   57/6 = **9 R 3** (9 sixes fit into 57)

   Answer: 89 R 3

3. 92,674 ÷ 9

   9/9 = **1** R 0 (1 ninety thousand fits into 90,000)

   2/9 = **0** (no nine thousands fit into 2,000)

   26/9 = **2** R 8 (2 nine hundreds fit into 2,600)

   87/9 = **9** R 6 (9 nineties fit into 870)

64/9 = **7 R 1** (7 nines fit into 64)

Answer: 10,297 R 1

4. 11,131 ÷ 5

   1/5 = 0 (no fifty thousands fit into 10,000)

   11/5 = **2** R 1 (2 five thousands fit into 11,000)

   11/5 = **2** R 1 (2 five hundreds fit into 1,100)

   13/5 = **2** R 3 (2 fifties fit into 130)

   31/5 = **6 R 1** (6 fives fit into 31)

   Answer: 2,226 R 1

Division Terminology

- If you divide a number by another number, the first is called the dividend and the second is called the divisor.
- The whole number part of the answer is called the quotient**.**
- The number left over (always smaller than the divisor) is called the remainder**.**
- In modular arithmetic, the divisor is called the modulus.

$$\text{Dividend} \div \text{Divisor} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

OR

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

Class Practice Problems

Write a definition or an example of each of the following:

1. "A divides B"
2. "A mod B"
3. "A is divisible by B"
4. "A is a factor of B"
5. "A is a multiple of B"

Answers:

1. "A divides B"

- The number A is a factor of the number B
- B can be divided by A with no remainder
- A can be multiplied by a whole number to get B

- Example: 4 divides 28 because $4 \times 7 = 28$

2. “A mod B”

    - The remainder when the number A is divided by the number B
    - How far the number A is from a multiple of B that is smaller than A
    - Example: 24 mod 7 is 3 because 24 is 3 away from a multiple of 7

3. “A is divisible by B”

    - $A \div B$ has a remainder of zero
    - $A \div B$ is a whole number
    - A can be broken up into B equal groups
    - Example: 50 is divisible by 10 because $50 \div 10 = 5$, a whole number with no remainder

4. “A is a factor of B”

    - The number A can be multiplied by another whole number to result in B
    - Example: 3 is a factor of 12 because $3 \times 4 = 12$

5. “A is a multiple of B”

    - The number A is the result of multiplying the number B by another whole number
    - Example: 21 is a multiple of 7 because $7 \times 3 = 21$

Lesson 3.3. Divisibility Rules

This lesson introduces divisibility rules for numbers 2-11 (excluding 7). Students apply these rules through a hands-on dice game, determining whether the three-digit numbers they create are divisible by specific values.

Establishing Divisibility Rules

Multiples of 2 are even, by definition. We can spot them quickly by looking at the last digit and making sure that it is even. For example, 53,841 is not a multiple of 2 because it ends in an odd digit, 1.

Class Practice Problem

Find a way to identify multiples of each of these numbers.

- Multiples of 2: the last digit is even
- Multiples of 3: _____
- Multiples of 4: _____
- Multiples of 5: _____
- Multiples of 6: _____
- Multiples of 7: _____
- Multiples of 8: _____
- Multiples of 9: _____
- Multiples of 10: _____
- Multiples of 11: _____

Answers:

- Multiples of 2: the last digit is even.
- Multiples of 3: the sum of the digits is divisible by 3.
- Multiples of 4: the number formed by the last 2 digits is divisible by 4.
- Multiples of 5: end in 0 or 5.
- Multiples of 6: divisible by 2 AND 3, the prime factors of 6.
- Multiples of 7: N/A - no need for students to know this!
    - For teachers: take the last digit, double it, and then subtract it from the remaining number. If the result is divisible by 7, so is the original number. For example, to check if 589 is divisible by 7, take the last digit (9), double it (18), then subtract the answer from the original number without the last digit: 58-18=40. Since 40 is not divisible by 7, neither is 589.
- Multiples of 8: the last 3 digits are divisible by 8, i.e., it is divisible by 2 three times. How does this relate to prime factorization?
- Multiples of 9: the sum of the digits is divisible by 9.
- Multiples of 10: end in 0.
- Multiples of 11: 2-digit → repeated digit, 3-digit → first and last digits add up to the middle digit (kind of).

Divisibility Dice Game

In groups, students should roll three dice to generate a three-digit number. Using the Divisibility Rules they wrote, they can determine whether their number is divisible by certain listed numbers (2, 3, 4, 5, 6, 9, 10, and 11). Note that it might be tricky to decide if some numbers are divisible by 11.

If time allows, encourage students to create their own three-digit number and quiz their peers on its divisibility.

Lesson 3.4. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

1. Complete these modular arithmetic calculations. Show your work!

   a) Find 66 mod 12

   b) Find 125 mod 7

2. Find three other integers congruent to:

   a) 2 mod 5

   b) 7 mod 5

   c) 12 mod 5

3. True or False

   a) 31 is congruent to 1 modulo 10

   b) 48 is congruent to 3 modulo 9

4. Use long division to divide:

   a) 4791 divided by 7

   b) 562 divided by 8

5. For each statement, write the correct term from this list: “divides, “mod,” “is divisible by,” “is a factor of,” “is a multiple of.”

   a) 4 _____ 20

   b) 20 _____ 6 is 2

   c) 5 _____ 35

6. Determine if each number is divisible by the given numbers:

   a) Write the numbers that 234 is divisible by from this list: 2, 3, 5, 9, 10, 11

   b) Write the numbers that 495 is divisible by from this list: 2, 3, 5, 9, 10, 11

7. If a number n is divisible by both 3 and 4, is it always divisible by 12? Explain.

8. Fill in the blanks:

   a) 8 is congruent to ___ modulo 5

   b) 17 is congruent to ___ modulo 6

9. Find the smallest positive integer greater than 0 that is divisible by both 4 and 6.

10. True or False? Explain:

a) 3 plus 4 is congruent to 2 modulo 5.

b) If 12 is divisible by 6, then 6 is divisible by 12.

Check Your Understanding Answer Key

1. a) 66 mod 12 ≡ 6 (66 divided by 12 gives remainder 6)

   b) 125 mod 7 ≡ 6 (125 divided by 7 gives remainder 6)

2. a) 2, 7, 12, 17, 22...

   b) 7, 12, 17, 22, 27...

   c) 12, 17, 22, 27, 32...

3. a) True (31 divided by 10 gives a remainder of 1)

   b) True (48 divided by 9 gives a remainder of 3)

4. a) 4791 divided by 7 = 684, remainder 3

   b) 562 divided by 8 = 70, remainder 2

5. a) 4 divides 20, or 4 is a factor of 20

   b) 20 mod 6 ≡ 2

   c) 5 is a factor of 35

6. a) 234: divisible by 2 ✓, 3 ✓, 5 ✗, 9 ✓, 10 ✗, 11 ✗

   b) 495: divisible by 2 ✗, 3 ✓, 5 ✓, 9 ✓, 10 ✗, 11 ✓

7. Yes. If a number can be divided evenly by both 3 and 4, it can also be divided evenly by 12. For example, 24 is divisible by 3 (24 ÷ 3 = 8) and by 4 (24 ÷ 4 = 6), and it is also divisible by 12 (24 ÷ 12 = 2).

8. a) 8 mod 5 ≡ 3

   b) 17 mod 6 ≡ 5

9. The smallest positive integer divisible by both 4 and 6 is 12 (the LCM of 4 and 6).

10. a) True (3 + 4 = 7, 7 mod 5 ≡ 2)

    b) False (6 is not divisible by 12)

Lesson 3.5. Sequences and Pattern Recognition

This lesson focuses on sequences and pattern recognition, emphasizing applications of remainders. Students collectively complete a remainder warm-up, then learn about and practice identifying digits in repeating decimals and determining the last digit of cyclic powers.

Class Practice Problems

1. If today is Tuesday, what day will it be 359 days from now?
2. I have 34 cupcakes. They are packaged in packages of 6. How many packages will I need?
3. When a secret two-digit number is divided by 5, the remainder is 3. When the same secret number is divided by 6, the remainder is 2. What could the number be? Hint: there are multiple correct answers.

Answers:

1. Every 7 days, it will be Tuesday again. 359 mod 7 ≡ 2 → In 359 days, the weekday will be 2 days away from Tuesday. Answer: Thursday.
2. 34 cupcakes ÷ 6 cupcakes per package = 5 packages, remainder 4 cupcakes → round up to 6 packages since we need somewhere to put the remaining cupcakes. Answer: 6 packages.
3. If the remainder is 3 when divided by 5, the number must end in 3 or 8.

   If the remainder is 2 when divided by 6, the number is 2 more than a multiple of 6. Multiples of 6 are even, so adding 2 to a multiple of 6 gives us another even number, meaning this number must also be even → it ends in 8, not 3.

Then, the closest multiple of 6 to our number ends in 8-2 = 6.

36, 66, and 96 are all two-digit multiples of 6 ending in 6, so the number can be 36+2 = 38, 66+2 = 68, or 96+2 = 98. Answer: 38, 68, or 98.

Repeating Decimals

$\frac{1}{7}$ can be written as $0.\overline{142857}$. What is the 333rd digit after the decimal?

The repeated decimals have a cycle of length 6, so every digit that is a multiple of 6 will land at 7. For example, the 30th digit is 7, and so is the 606th digit. Just like this circular number line, we restart counting every time we pass a multiple of 6, so the question we are asking is really, what is 333 mod 6?

333÷6 = 55 remainder 3, so 333 mod 6 ≡ 3.

This means that the 333rd digit is the same as the 3rd digit after the decimal → 2.

Class Practice Problems

1. 2/11 = 0.181818… What is the 503rd digit after the decimal?
2. 4/7 = 0.571428571428… What is the 100th digit after the decimal?
3. What is the 349th digit after the decimal in $2.\overline{9876}$?

Answers:

1. The "18" repeats, so every odd digit is 1 and every even digit is 8. 503 is odd, so the digit must be 1.
2. The "571428" repeats, so every multiple of 6 digit is an 8. 101 mod 6 is 5, so the 100th digit is the same as the 5th digit → 2.
3. The "9876" repeats, so every multiple of 4 digit is a 6. 349 mod 4 is 1, so the 349th digit is the same as the 1st digit → 9.

Cyclic Powers

We have seen that repeating cycles are often remainder problems. Let's look for a pattern below.

$$2^1 = 2$$

$$2^2 = 4$$

$$2^3 = 8$$

$$2^4 = 16$$

$$2^5 = 32$$

$$2^6 = 64$$

$$2^7 = 128$$

What digit do you think that 2^8 ends in? What about 2^9?

Answer: 2^8 ends in 6 and 2^9 ends in 2. The pattern for powers of 2 is that they end in 2, 4, 8, 6.

Class Practice Problems

1. Find the last digit of 3^{13}. Hint: find the cycle for the last digit of powers of 3.
2. Find the last digit of 4^{10}.
3. Find the last digit of 7^{43}.

Answers:

1. Here are the first few powers of 3: 3, 9, 27, 81, 243, 729, ...

   The last digit cycles between 3, 9, 7, and 1. Since the repeated cycles have 4 options, every 4th power ends in 1. We need to find 13 mod 4 to see how far 13 is from a multiple of 4.

   13 mod 4 ≡ 1, so 3^{13} ends in the same digit as 3^1 → 3.

2. Here are the first few powers of 4: 4, 16, 64, 256, 1024, 4096, ...

   The last digit alternates between 4 and 6, so odd powers end in 4, and even powers end in 6. Since 10 is even, 4^{10} ends in the same digit as 4^2 → 6.

3. The first few powers of 7: 7, 49, 343, 2401, 16807, 117649, ...

   Last digit cycle: 7, 9, 3, 1

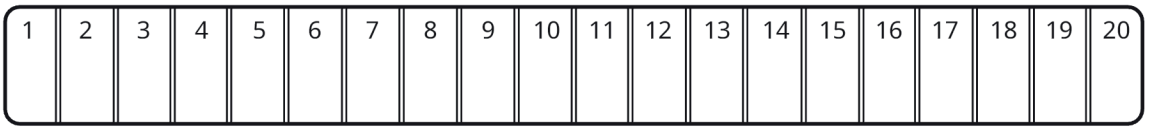
   Cycle length: 4

   43 mod 4 ≡ 3, so 7^{43} ends in the same digit as 7^3 → 3.

Lesson 3.6. Prime Numbers and Pattern-Recognition

Students will explore number patterns by “toggling” lockers open and closed in sequence. They’ll investigate how factors determine which lockers remain open and make predictions along the way. The activity promotes reasoning, pattern recognition, and discussion. Instructors can guide students step-by-step through the game, then pose questions for small groups to solve, encouraging them to analyze factors and explain their thinking.

The Locker Game



Mr. Wolf's 4th-grade class has 20 students, and the hallway has 20 lockers in a row. The students play a game where they take turns changing the lockers’ states (open or closed) as follows:

- Student 1: Starts at locker 1 and opens every locker in the hallway.
- Student 2: Starts at locker 2 and changes the state of every second locker (if it is open, they close it; if it is closed, they open it).
- Student 3: Starts at locker 3 and changes the state of every third locker.
- Student 4: Starts at locker 4 and changes the state of every fourth locker.

This pattern continues until all 20 students have taken their turn. Each student changes lockers that are multiples of their student number (e.g., the 5th student changes every 5th locker).

Questions

Students should discuss this in groups and come up with arguments.

1. Which lockers are open at the end of the game? Explain your reasoning.
2. Which lockers were changed by exactly two students? Explain your reasoning.
3. Which lockers were changed by exactly three students? Explain your reasoning.
4. Which lockers were changed the most times?

Solutions

1. The lockers that are still open are perfect squares, specifically 1×1, 2×2, 3×3, and 4×4. To see why this is true, first note that:

The 1st student changes all the lockers with a locker number that is a multiple of 1.
The 2nd student changes all the lockers with a locker number that is a multiple of 2.
The 3rd student changes all the lockers with a locker number that is a multiple of 3.
The 4th student changes all the lockers with a locker number that is a multiple of 4.
This pattern continues, so a locker is touched by a student if that student's number is a factor of that locker's number.

Now consider the factors of 16: 1, 2, 4, 8, and 16. This means the 16th locker was opened by the 1st student, then closed by the 2nd student, then opened by the 4th student, then closed by the 8th student, and finally opened one last time by the 16th student. Now note that the square numbers are the only numbers with an odd number of factors; all other numbers have factors that come in pairs. Since an odd number of factors means the locker will be changed an odd number of times, it must be open at the end.
With only 20 lockers, students can observe this fact about the number of factors for the numbers 1 to 20, which provides practice in finding factors of whole numbers.

2. The lockers that were touched by only two students are the prime numbers, since prime numbers only have two factors. In our case, they are 2, 3, 5, 7, 11, 13, 17, and 19.

3. The lockers that are touched by only three students are the squares of prime numbers, because these are the only numbers that have three different factors. In this 20-locker activity, these are 4 and 9. For example, 4 has three factors: 1, 2, and 4.
4. The lockers that are touched the most are those lockers whose number has the most factors. In this 20-locker activity, 12, 18, and 20 all have 6 factors, so they were each touched 6 times.

$$12 = 1 \times 12 = 2 \times 6 = 3 \times 4$$

$$18 = 1 \times 18 = 2 \times 9 = 3 \times 6$$

$$20 = 1 \times 20 = 2 \times 10 = 4 \times 5$$

Lesson 3.7. Greatest Common Factor and Prime Factorization

This lesson begins with a word problem application of the Greatest Common Factor (GCF). Students will then explore GCF through guided practice, defining it and finding it from lists of factors while summarizing key terminology. Next, the class will review two methods for finding GCF: listing factors and prime factorization. Finally, students will work in small groups to apply what they've learned while completing a worksheet.

Class Practice Problems

Alex has 48 small boxes and 64 pencils. He wants to pack them into sets so that each set has the same number of boxes and the same number of pencils.

1. How can Alex put all 48 boxes into sets with the same number of boxes in each set? List all possible ways to do this, including how many sets and how many boxes per set.
2. How can Alex put all 64 pencils into sets with the same number of pencils in each set? List all possible ways to do this, including how many sets and how many pencils per set.
3. Alex wants each set to have the same number of boxes and pencils. How many sets can they make, and how many of each item will go into each set? List all possible combinations.
4. What is the biggest number of sets Alex can make without having any leftover boxes or pencils? Explain why this is the maximum.

Answers:

1. The number of boxes in each set must be a factor of 48. Possible arrangements:

Sets	Boxes Per Set
48 sets	1 box
24 sets	2 boxes
16 sets	3 boxes
12 sets	4 boxes
8 sets	6 boxes
6 sets	8 boxes
4 sets	12 boxes
3 sets	16 boxes
2 sets	24 boxes
1 set	48 boxes

2. The number of pencils in each set must be a factor of 64. Possible arrangements:

Sets	Pencils Per Set
64 sets	1 pencil
32 sets	2 pencils
16 sets	4 pencils
8 sets	8 pencils
4 sets	16 pencils
2 sets	32 pencils
1 set	64 pencils

3. The number of sets must divide evenly into both 48 and 64.

   The overlap in the possible number of sets is 16, 8, 4, 2, and 1.

   Possible arrangements:

Sets	Boxes Per Set	Pencils Per Set
16 sets	3 boxes	4 pencils
8 sets	6 boxes	8 pencils
4 sets	12 boxes	16 pencils
2 sets	24 boxes	32 pencils
1 set	48 boxes	64 pencils

4. The biggest number of sets possible is 16, because it is the biggest number that divides both 48 and 64 evenly.

Class Practice Problems

1. Explain what a “greatest common factor” is.
2. List all the factors of 21 and 6. Then, determine the greatest common factor of these two numbers.
3. List all the factors of 28 and 12. Then, identify the greatest common factor of 28 and 12.
4. A rectangular bulletin board measures 12 inches in height and 27 inches in width. Sofia wants to cover the entire board with square pieces of colored paper. Each square must have a side length that is a whole number of inches.
    a. What is the largest possible side length of the square that Sofia could use to cover the board completely without leaving gaps or overlaps? Show your reasoning.
    b. How does this problem connect to the concept of the greatest common factor?

Answers:

1. The greatest common factor is the biggest number that divides a group of numbers with no remainder.
2. Factors of 21: 1, 3, 7, 21. Factors of 6: 1, 2, 3, 6. Greatest common factor: 3
3. Factors of 28: 1, 2, 4, 7, 14, 28. Factors of 12: 1, 2, 3, 4, 6, 12. Greatest common factor: 4
4. The bulletin board is rectangular and will be covered by square pieces of paper.

a. To cover the board completely, the side length of the square must divide evenly into both the height (12 inches) and the width (27 inches).

   Factors of 12: 1, 2, 3, 4, 6, 12

   Factors of 27: 1, 3, 9, 27

   Common factors of 12 and 27: 1, 3

   The largest common factor is 3, so the biggest square Sofia can use is 3 inches on each side.

b. This problem is an example of the greatest common factor. The side length of the square must be a number that divides evenly into both dimensions of the board. Finding the largest possible square means finding the greatest of the common factors.

Summary and Terminology

- A **factor** of a whole number $\boldsymbol{n}$ is a whole number that divides $\boldsymbol{n}$ exactly, leaving no remainder. For example, the numbers 1, 2, 3, 4, 6, and 12 are all factors of 12 because each divides 12 evenly.
- A **common factor** of two whole numbers is a number that is a factor of both. For instance, 1, 3, 5, and 15 are factors of 45, and they are also factors of 60. These numbers are the common factors of 45 and 60.
- The **greatest common factor** (GCF) is the largest number among the common factors of two numbers. In the example above, 15 is the greatest common factor of 45 and 60.
- If two numbers have no common factors other than 1, they are called **relatively prime**.

Methods for Finding the Greatest Common Factor (GCF)

Method 1: Listing Factors

One way to find the GCF of two numbers is to list all their factors and pick the largest factor they share.

Example: Find the GCF of 18 and 24.

Factors of 18: 1, 2, 3, 6, 9, 18

Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24

The common factors are 1, 2, 3, and 6. The largest of these is 6, so the GCF of 18 and 24 is 6.

Method 2: Using Prime Factorization

Another approach is to break each number down into its prime factors and then compare them. This method is especially useful for larger numbers where listing all factors is difficult.

Example: Find the GCF of 72 and 96 using prime factorization.

Prime factors of 72: $2 \times 2 \times 2 \times 3 \times 3$

Prime factors of 96: $2 \times 2 \times 2 \times 2 \times 2 \times 3$

Next, find the common prime factors: $2 \times 2 \times 2 \times 3 = 24$

So, the GCF of 72 and 96 is 24.

This method is helpful because it allows us to calculate the greatest factor that both numbers share, even if it is not immediately obvious. In this case, the prime factorization of 96 might be easier to find than its factor pairs.

Another example of the prime factorization method:

Prime factors of 64: **2 × 2 × 2** × 2 × 2 × 2

Prime factors of 72: **2 × 2 × 2** × 3 × 3

Next, find the common prime factors: 2 × 2 × 2 = 8

So, the GCF of 64 and 72 is 8.

What is the GCF of 25 and 18?

25 = 5×5

18 = 2×3×3

25 and 18 have no common prime factors, so they are relatively prime.

Class Practice Problems

Practice using prime factorization to find the GCF of two numbers.

1. 96 =

   48 =

   GCF (96, 48) =

2. 36 =

   90 =

   GCF (36, 90) =

3. 60 =

   44 =

   GCF (60, 44) =

4. 12 =

   56 =

   GCF (12, 56) =

Answers:

1. 96 = **2×2×2×2**×2×**3**

   48 = **2×2×2×2×3**

   GCF = 2×2×2×2×3 = 48

2. 36 = **2**×2×**3×3**

   90 = **2**×**3×3**×5

   GCF = 2×3×3 = 18

3. 60 = **2×2**×3×5

   44 = **2×2**×11

   GCF = 2×2 = 4

4. 12 = **2×2**×3

   56 = **2×2**×2×7

   GCF = 2×2 = 4

Greatest Common Factor Worksheet

Please answer the problems below, showing all calculations clearly.

1. A teacher is making gift bags. Each bag is to be filled with pencils and stickers. The teacher has 24 pencils and 36 stickers to use. Each bag will have the same number of each item, with no items left over. For example, she could make 2 bags with 12 pencils and 18 stickers each. What are the other possibilities? Explain or show your reasoning.

2. A. List all the factors of 42.

   B. What is the greatest common factor of 42 and 15?

   C. What is the greatest common factor of 42 and 50?

3. A school chorus has 90 sixth-grade students and 75 seventh-grade students. The music director wants to make groups of performers, with the same combination of sixth- and seventh-grade students in each group. She wants to form as many groups as possible.

   A. What is the largest number of groups that could be formed? Explain or show your reasoning.

   B. If that many groups are formed, how many students of each grade level would be in each group?

4. What is the GCF of 36 and 120? Find the answer using both methods. Which was easier?

Greatest Common Factor Answer Key

1. 1 bag: 24 pencils and 36 stickers, 3 bags: 8 pencils and 12 stickers, 4 bags: 6 pencils and 9 stickers, 6 bags: 4 pencils and 6 stickers, 12 bags: 2 pencils and 3 stickers
2. A. Factors of 42: 1, 42, 2, 21, 3, 14, 6, 7

   B. Check if each of the factors of 42 divides 15, starting with the largest. 42 and 21 are too big; 14 does not work; 7, 6, and 4 do not work → the answer is 3.

   Alternatively, compare factor lists. The factors of 15 are 1, 3, 5, and 15.

   C. Factors of 50: 1, 50, 2, 25, 5, 10. Greatest common factor: 2.
3. Find the GCF of 90 and 75

   A. 15 groups

   B. 6 sixth graders and 5 seventh graders per group
4. Factors of 36: 1, 36, 2, 18, 3, 12, 4, 9, 6.

   Factors of 120: 1, 120, 2, 60, 3, 40, 4, 30, 5, 24, 6, 20, 8, 15, 10, 12

   GCF(36, 120) = 12

   Other method:

   Prime factorization of 36: $2 \times 2 \times 3 \times 3$

   Prime factorization of 120: $2 \times 2 \times 2 \times 3 \times 5$

   GCF(36,120) = $2 \times 2 \times 3 = 12$

   It may help to discuss the preference for each method and encourage different opinions!

Lesson 3.8. Least Common Multiple and Prime Factorization

This lesson begins with a review of multiples, then a word problem application of the Least Common Multiple (LCM). Students will then explore LCM through guided practice, defining it and finding it from lists of multiples while summarizing key terminology. Next, the class will review two methods for finding LCM: listing multiples and using prime factorization. Finally, students will work in small groups to apply what they've learned while completing a worksheet.

Class Practice Problems

1. Identify all of the multiples of 4 in this list.

   1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

2. Identify all of the multiples of 6 in this list.

   1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

3. What patterns do you notice?

Answers:

1. Multiples of 4 on the list: 4, 8, 12, 16, 20, 24
2. Multiples of 6 on the list: 6, 12, 18, 24
3. Patterns: 12 and 24 are both multiples of 4 and 6. Multiples go on forever, so there are a lot more common multiples of 4 and 6 than we see above.

Class Practice Problems

Alex collects coins in two types of special packs: Gold coins come in packs of 12, while silver coins come in packs of 8. Last month, Alex bought the same total number of gold coins and silver coins.

1. If Alex bought no more than 100 of each type of coin, what are all the possible numbers of packs he could have purchased?
2. What is the smallest possible order for gold coins? What about silver coins? Explain your reasoning.

Answers:

1. Possible answers are the common multiples of 12 and 8: 24, 48, 72, or 96 of each flower type.
2. 24 gold coins ÷ 12 gold coins per pack = 2 packs of gold coins
   24 silver coins ÷ 8 silver coins per pack = 3 packs of silver coins

Class Practice Problems

1. Explain what "least common multiple" means.
2. List all multiples of 10 and 8 that are less than 100. Then find the least common multiple of 10 and 8.
3. List all multiples of 7 and 9 that are less than 100. Then find the least common multiple of 7 and 9.
4. Find the LCM of 10 and 20.
5. Find the LCM of 4 and 24.
6. In the previous two questions, one number is a multiple of the other. What do you notice about the LCM? Will this always happen when one number is a multiple of the other? Explain your reasoning.

Answers:

1. The least common multiple is the smallest number that both numbers share as a multiple.
2. Multiples of 10: 10, 20, 30, 40, 50, 60, 70, 80, 90

   Multiples of 8: 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96

   Common multiples: 40, 80…

   LCM: 40
3. Multiples of 7: 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98

   Multiples of 9: 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99

   Common multiples: 63, 126…

   LCM: 63
4. LCM of 10 and 20: 20
5. LCM of 4 and 24: 24
6. When one number is a multiple of the other, the LCM is always the larger number. This is because a number is always the smallest positive multiple of itself.

Class Practice Problem

Riley's favorite arcade is celebrating its anniversary. On this special day, the first 50 players who enter the arcade will get prizes:

Every 5th player gets a free token pack.

Every 9th player gets a sticker.

Every 12th player gets a small toy.

1. Jordan is the 23rd player in line. Should he move to get a prize? If so, how far back should he go to receive at least one prize? Explain your reasoning.

2. Casey is the 36th player in line. Will she get a prize? If so, which prize(s) will she receive?
3. How many prizes in total will the arcade give out? Explain your reasoning.

Answers:

1. Since 23 is not a multiple of 5, 9, or 12, Jordan should move farther back in line to a spot that is a multiple of one of those numbers. If he moves one spot back to being the 24th player, he can get a small toy.
2. Casey will get a sticker and a small toy since 36 is a multiple of both 9 and 12.
3. Multiples of 5 up to 50: 5, 10, 15, 20, 25, 30, 35, 40, **45**, 50 → 10 token packs

   Multiples of 9 up to 50: 9, 18, 27, **36**, **45** → 5 stickers

   Multiples of 12 up to 50: 12, 24, **36**, 48 → 4 small toys

   The arcade will give out 10+5+4 = 19 prizes in total.

Summary and Terminology

A multiple of a whole number is the result of multiplying that number by another whole number. For example, 20 is a multiple of 4 because $20 = 4 \times 5$.

A **common multiple** of two whole numbers is a number that is a multiple of both. For example, 20 is a multiple of 2 and also a multiple of 5, so it is a common multiple of 2 and 5.

The **least common multiple** (LCM) of two whole numbers is the smallest number that is a multiple of both. For example, 30 is the least common multiple of 6 and 10.

Methods for Finding the Least Common Multiple (LCM)

Method 1: Listing Multiples

One way to find the least common multiple of two numbers is to list multiples of each in order until we find the smallest multiple they have in common. Let's find the least common multiple of 4 and 10. First, we list some multiples of each number.

Multiples of 4: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44 . . .

Multiples of 10: 10, 20, 30, 40, 50, . . .

20 and 40 are both common multiples of 4 and 10 (as are 60, 80, …), but 20 is the smallest number that is on both lists, so 20 is the least common multiple.

Method 2: Prime Factorization

Another method is to compare the prime factors of each number. This is more helpful when the original numbers are large and listing their multiples goes on and on.

Prime factorization of 4: **2**×**2**

Prime factorization of 10: 2×**5**

For a number to be a multiple of both 4 and 10, it must include each of their prime factors <u>at least</u> as many times as they appear in the prime factorizations.

Least common multiple of 4 and 10: 2×2×5 = 20

Another example of the prime factorization method:

Prime factorization of 12: **2**×**2**×3

Prime factorization of 90: 2×**3**×**3**×**5**

Least common multiple of 12 and 90: 2×2×3×3×5 = 180

Class Practice Problems

Practice using prime factorization to find the LCM of two numbers.

1. 90 =

   39 =

   LCM (90, 39) =

2. 16 =

   28 =

   LCM (16, 28) =

3. 70 =

   60 =

   LCM (70, 60) =

4. 20 =

   48 =

   LCM (20, 48) =

Answers:

1. 90 = **2**×**3**×**3**×**5**

   39 = 3×**13**

   LCM (90, 39) = 2×3×3×5×13 = 1170

2. 16 = **2**×**2**×**2**×**2**

   28 = 2×2×**7**

   LCM (16, 28) = 2×2×2×2×7 = 112

3. 70 = 2×**5**×**7**

   60 = **2**×**2**×**3**×5

LCM (70, 60) = 2×2×3×5×7 = 420

4. 20 = 2×2×**5**

   48 = **2**×**2**×**2**×**2**×**3**

   LCM (20, 48) = 2×2×2×2×3×5 = 24

Least Common Multiple Worksheet

Please answer the problems below, showing all calculations clearly.

1. A. A green light blinks every 4 seconds, and a yellow light blinks every 5 seconds. When will both lights blink at the same time?

   B. A red light blinks every 12 seconds, and a blue light blinks every 9 seconds. When will both lights blink at the same time?

   C. Explain how to determine when 2 lights blink together.

2. Find the least common multiple of 10 and 15 using both methods. Which do you prefer?

3. Cups are sold in packages of 8. Napkins are sold in packages of 12.

   What is the fewest number of packages of cups and the fewest number of packages of napkins that can be purchased so there will be the same number of cups as napkins? How many cups and how many napkins will there be?

Least Common Multiple Answer Key

1. A. Green light blinks after 4, 8, 12, 16, 20, … seconds.

   Yellow light blinks after 5, 10, 15, 20, … seconds.

   Overlap: they both blink after 20 seconds.

   B. Red light blinks after 12, 24, 36, 48, … seconds.

   Blue light blinks after 9, 18, 27, 36, 45, … seconds.

   Overlap: they both blink after 36 seconds.

   C. They blink at the same time at the least common multiple of the two amounts of time.

2. Multiples of 10: 10, 20, 30, 40, …

   Multiples of 15: 15, 30, 45, 60, …

   LCM(10, 15) = 30

   Prime factorization of 10: 2×5

   Prime factorization of 15: 3×5

   LCM(10, 15) = 2×3×5 = 30

   It is helpful to discuss the preference for each method and encourage different opinions!

3. LCM(8, 12) = 24 of each item (24 cups and 24 napkins)

   24 cups ÷ 8 cups per package = 3 packages of cups

   24 napkins ÷ 12 napkins per package = 2 packages of napkins

Lesson 3.9. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz. A calculator is needed for this assessment, but please show all work.

1. Write the next three numbers in the sequence:

   a) 0.2, 0.4, 0.6, 0.8, _____, _____, _____

   b) 30, 60, 120, 240, _____, _____, _____

2. If today is Monday, what day of the week will it be:

   a) 50 days from now?

   b) 123 days from now?

3. Find the last digit of:

   a) 7^{17}

   b) 3^{40}

   c) 2^{15}

4. a) List all the numbers between 1 and 12 that have an odd number of factors.

   b) What do these numbers have in common that makes them different from numbers with an even number of factors?

5. Find the greatest common factor (GCF) of:

   a) 24 and 36

   b) 18 and 45

6. Find the GCF of:

   a) 40 and 60

   b) 28 and 42

7. Find the least common multiple (LCM) of:

   a) 6 and 8

   b) 9 and 12

8. Find the LCM of:

   a) 15 and 20

   b) 18 and 24

9. Find the next number in the sequence:

   a) 2, 4, 8, 16, _____

   b) 1, 1, 2, 3, 5, 8, _____

10. Decide if each statement is true or false:

    a) 3 plus 4 is congruent to 2 modulo 5

    b) If 12 is divisible by 6, then 6 is divisible by 12

Check Your Understanding Answer Key

1. a) 1.0, 1.2, 1.4

   b) 480, 960, 1920

2. a) Tuesday ($50 \equiv 1 \bmod 7$)

   b) Friday ($123 \equiv 4 \bmod 7$)

3. a) 7

   Powers of 7: 7, 49, 243, 2401, 16807, … last digit cycle length is 4

   17 mod 4 is 1, so 7^{17} ends like 7^1

   b) 1

   Powers of 3: 3, 9, 27, 81, 243, … last digit cycle length is 4

   40 mod 4 is 0, so 3^{40} ends like 3^4

   c) 8

   Powers of 2: 2, 4, 8, 16, 32, … last digit cycle length is 4

   15 mod 4 is 3, so 2^{15} ends like 2^3

4. a) 1, 4, 9

   b) They are perfect squares.

5. a) GCF = 12

   b) GCF = 9

6. a) GCF = 20

   b) GCF = 14

7. a) LCM = 24

   b) LCM = 36

8. a) LCM = 60

   b) LCM = 72

9. a) 32

   b) 13

10. a) True

    b) False

Lesson 3.10. More GCF and LCM

Instructors are encouraged to dedicate most of the class period to having students work in small groups on the LCM and GCF word problems below. This is also an opportunity to revisit any concepts from the previous lessons that need more time. At the end of the lesson, instructors should have students complete the provided exit ticket independently to demonstrate their understanding.

GCF and LCM Word Problem Worksheet

Please answer the problems below, showing all calculations clearly.

1. A red light flashes every 28 seconds. A green light flashes every 24 seconds. They both flash at the same time. After how many seconds will they both flash at the same time again?

2. A bus heading to Brooklyn leaves Manhattan every 36 minutes. A bus heading to Queens leaves Manhattan every 45 minutes. At 10:00 a.m., a bus to Brooklyn and a bus to Queens both depart from the Manhattan Bus Terminal. What is the next time both buses leave at the same time?

3. The least common multiple of two numbers is 130. The greatest common factor of the same two numbers is 13. Both numbers are less than 100. Write down two possible numbers.

4. Fred says that 20 and 21 have a greatest common factor of 0. Explain why Fred is wrong.

5. Abby and Annie have the same number of coins. Abby has sorted her coins into groups of 80. Annie has sorted her coins into groups of 75. They each have less than 2000 coins. How many coins do they have altogether?

6. A high-flying event manager, Laura, organizes an event every 2 days. A small-time event organizer, Kevin, hosts one event per week. If both Laura and Kevin are organizing events today, how long will it be until the two work on projects on the same day again?

7. Lisa has 48 red balloons and 60 blue balloons for a party. She wants to tie them into groups with the same number of balloons in each group, using only one color per group. What is the greatest number of balloons she can have in each group?

8. Boxes that are 12 inches tall are being stacked next to boxes that are 18 inches tall. What is the shortest height at which the two stacks will be the same height?

GCF and LCM Word Problems Answer Key

1. A red light flashes every 28 seconds. A green light flashes every 24 seconds. They both flash at the same time. After how many seconds will they both flash at the same time again?

   Prime Factorization of 24 = 2×2×2×3

   Prime Factorization of 28 = 2×2×7

   LCM (28, 24) = 2×2×2×3×7 = 168

   Answer: After 168 seconds

2. A bus heading to Brooklyn leaves Manhattan every 36 minutes. A bus heading to Queens leaves Manhattan every 45 minutes. At 10:00 a.m., a bus to Brooklyn and a bus to Queens both depart from the Manhattan Bus Terminal. What is the next time both buses leave at the same time?

   Prime Factorization of 36 = 2×2×3×3

   Prime Factorization of 45 = 3×3×5

   LCM (36, 45) = 2×2×3×3×5 = 180 minutes

   180 ÷ 60 = 3 hours

   10 am + 3 hours = 1 pm

   Answer: 1 pm

3. The least common multiple of two numbers is 130. The greatest common factor of the same two numbers is 13. Both numbers are less than 100. Write down two possible numbers.

   GCF = 13, so the numbers are multiples of 13

   LCM = 130 = 2×5×13

The numbers have to be $13\times5=65$ and $13\times2=26$

Answer: 26 and 65

4. Fred says that 20 and 21 have a greatest common factor of 0. Explain why Fred is wrong.

   Answer: The only multiple of zero is zero, so 0 is not a factor of any number except itself. Fred meant that the GCF of 20 and 21 is 1.

5. Abby and Annie have the same number of coins. Abby has sorted her coins into groups of 80. Annie has sorted her coins into groups of 75. They each have less than 2000 coins. How many coins do they have altogether?

   $80 = 2\times2\times2\times2\times5$

   $75 = 3\times5\times5$

   LCM (80, 75) $= 2\times2\times2\times2\times3\times5\times5 = 1200$

   Answer: They each have 1,200 coins.

6. A high-flying event manager, Laura, organizes an event every 2 days. A small-time event organizer, Kevin, hosts one event per week. If both Laura and Kevin are organizing events today, how long will it be until the two work on projects on the same day again?

   Laura: Every 2 days

   Kevin: Every 7 days

   Common day = LCM(2,7) = 14

   Answer: 14 days

7. Lisa has 48 red balloons and 60 blue balloons for a party. She wants to tie them into groups with the same number of balloons in each group, using only one color per group. What is the greatest number of balloons she can have in each group?

   $48 = 2 \times 2 \times 2 \times 2 \times 3$

   $60 = 2 \times 2 \times 3 \times 5$

   GCF (48, 60) = $2 \times 2 \times 3 = 12$

   Answer: 12 balloons per group

8. Boxes that are 12 inches tall are being stacked next to boxes that are 18 inches tall. What is the shortest height at which the two stacks will be the same height?

   $12 = 2 \times 2 \times 3$

   $18 = 2 \times 3 \times 3$

   LCM (12, 18) = $2 \times 2 \times 3 \times 3 = 36$

   Answer: 36 inches

Factors and Multiples Exit Ticket

Please answer the problems below, showing all calculations clearly.

1. Which equation shows that 8 is a factor of 32?

   (A) $32 = 8 \times 4$

   (B) $32 = 256 \div 8$

   (C) $32 = 8 + 24$

   (D) $32 = 40 - 8$

2. Which list shows all factors of 24?

   (A) 24, 48, 72, 96

   (B) 1, 2, 4, 6, 8, 10, 12, 24

   (C) 1, 2, 3, 4, 6, 8, 12, 24

   (D) 2, 12, 24, 48

3. Which option only includes multiples of 12?

   (A) 24, 36, 48, 60, 72

   (B) 12, 18, 24, 30, 36

   (C) 1, 2, 3, 4, 6, 12

   (D) 0, 6, 12, 18, 24

4. Which equation shows that 6 is a factor of 18?

   (A) $3 = 18 \div 6$

   (B) $6 = 18 \times 3$

   (C) $18 \times 6 = 108$

   (D) $18 - 12 = 6$

5. Which of the following is a multiple of 8?

   (A) 60

   (B) 4

   (C) 72

   (D) 5

6. Which of the following pairs of numbers are relatively prime?

   (A) 2 and 348

   (B) 15 and 72

   (C) 99 and 117

   (D) 111 and 152

7. Find the LCM of 10 and 12 using any method.

8. Find the GCF of 15 and 36 using any method.

Factors and Multiples Exit Ticket Answer Key

1. Which equation shows that 8 is a factor of 32?

   (A) 32 = 8×4

2. Which list shows all factors of 24?

   (C) 1, 2, 3, 4, 6, 8, 12, 24

3. Which option only includes multiples of 12?

   (A) 24, 36, 48, 60, 72

4. Which equation shows that 6 is a factor of 18?

   (A) 3 = 18÷6

5. Which of the following is a multiple of 8?

   (C) 72

6. Which of the following pairs of numbers are relatively prime?

   (D) 111 and 152

7. Find the LCM of 10 and 12.

   10 = 2×**5**

   12 = **2**×**2**×**3**

   LCM (10, 12) = 2×2×3×5 = 60

8. Find the GCF of 15 and 36.

   15 = **3**×5

   36 = 2×2×**3**×3

   GCF (15, 36) = 3

Lesson 3.11. Fraction Arithmetic and Prime Factors

The objective of this lesson's activities is to strengthen students' understanding of fraction arithmetic through the use of prime factorization. Because these tasks can be challenging, instructors are encouraged to have students work collaboratively in groups as they apply prime factorization, with the optional use of 4-function calculators. To begin, the class should review how to find the least common denominator (LCD) for groups of fractions, emphasizing that this is the same as finding their least common multiple (LCM). If time allows, instructors can also teach students an additional visual method for finding LCM and GCF by pairing prime factorization with Venn diagrams.

Class Practice Problems

1. Find the LCD of 5/12, 2/15, and 7/18.
2. Find the LCD of 5/4, 1/6, 7/10, and 8/15.

Answers

1. Write the prime factorization of each denominator:

   $12=2\times2\times3$

   $15=3\times5$

   $18=2\times3\times3$

   Multiply the highest amount of all factors:

   The least common denominator is $2\times2\times3\times3\times5 = 180$.

2. Write the prime factorization of each denominator:

   $4=2\times2$

   $6=2\times3$

   $10=2\times5$

   $15=3\times5$

   Multiply the highest powers of all factors:

   $\text{LCD} = 2\times2\times3\times5 = 60$

Students may also do these problems by listing multiples of each denominator and finding the smallest common multiple.

Adding and Subtracting Fractions Worksheet

These problems are intended to be challenging! All denominators are products of primes up to 13. You may use a 4-function calculator to find the prime factorization of the denominators. Please show your calculations clearly.

1. $\frac{1}{245} + \frac{4}{231}$

2. $\frac{5}{84} - \frac{7}{585}$

3. $\frac{1}{66} + \frac{1}{15} - \frac{1}{110}$

4. $\frac{5}{14} - \frac{3}{49} + \frac{2}{5}$

5. $\frac{2}{35} + \frac{3}{64} + \frac{4}{7}$

6. $\frac{1}{60} - \frac{1}{100} + \frac{2}{27}$

Adding and Subtracting Fractions Answer Key

1. $\frac{1}{245}+\frac{4}{231}$

$\frac{1}{5\times7\times7}+\frac{4}{3\times7\times11}$

$\frac{1}{5\times7\times7}\times\frac{3\times11}{3\times11}+\frac{4}{3\times7\times11}\times\frac{5\times7}{5\times7}$

$\frac{1\times3\times11}{3\times5\times7\times7\times11}+\frac{4\times5\times7}{3\times5\times7\times7\times11}$

$\frac{33}{8085}+\frac{140}{8085}=\frac{173}{8085}$

2. $\frac{5}{84}-\frac{7}{585}$

$\frac{5}{2\times2\times3\times7}-\frac{7}{3\times3\times5\times13}$

$\frac{5}{2\times2\times3\times7}\times\frac{3\times5\times13}{3\times5\times13}-\frac{7}{3\times3\times5\times13}\times\frac{2\times2\times7}{2\times2\times7}$

$\frac{5\times3\times5\times13}{2\times2\times3\times3\times5\times7\times13}+\frac{7\times2\times2\times7}{2\times2\times3\times3\times5\times7\times13}$

$\frac{975}{16380}-\frac{196}{16380}=\frac{779}{16380}$

3. $\frac{1}{66}+\frac{1}{15}-\frac{1}{110}$

$\frac{1}{2\times3\times11}+\frac{1}{3\times5}-\frac{1}{2\times5\times11}$

$\frac{1}{2\times3\times11}\times\frac{5}{5}+\frac{1}{3\times5}\times\frac{2\times11}{2\times11}-\frac{1}{2\times5\times11}\times\frac{3}{3}$

$\frac{1\times5}{2\times3\times5\times11}+\frac{1\times2\times11}{2\times3\times5\times11}-\frac{1\times3}{2\times3\times5\times11}$

$\frac{5}{330}+\frac{22}{330}-\frac{3}{330}=\frac{24}{330}=\frac{4}{55}$

4. $\frac{5}{14}-\frac{3}{49}+\frac{2}{5}$

$\frac{5}{2\times7}-\frac{3}{7\times7}+\frac{2}{5}$

$\frac{5}{2\times7}\times\frac{5\times7}{5\times7}-\frac{3}{7\times7}\times\frac{2\times5}{2\times5}+\frac{2}{5}\times\frac{2\times7\times7}{2\times7\times7}$

$\frac{5\times5\times7}{2\times5\times7\times7}-\frac{3\times2\times5}{2\times5\times7\times7}+\frac{2\times2\times7\times7}{2\times5\times7\times7}$

$\frac{175}{490}-\frac{30}{490}+\frac{196}{490}=\frac{341}{490}$

5. $\frac{2}{35}+\frac{3}{64}+\frac{4}{7}$

$\frac{2}{5\times7}+\frac{3}{2\times2\times2\times2\times2\times2}+\frac{4}{7}$

$\frac{2}{5\times7}\times\frac{2\times2\times2\times2\times2\times2}{2\times2\times2\times2\times2\times2}+\frac{3}{2\times2\times2\times2\times2\times2}\times\frac{5\times7}{5\times7}+\frac{4}{7}\times\frac{2\times2\times2\times2\times2\times2\times5}{2\times2\times2\times2\times2\times2\times5}$

$\frac{2^7}{2^6\times5\times7}+\frac{3\times5\times7}{2^6\times5\times7}+\frac{4\times2^6\times5}{2^6\times5\times7}$

$\frac{128}{2240}+\frac{105}{2240}+\frac{1280}{2240}=\frac{1513}{2240}$

6. $\frac{1}{60}-\frac{1}{100}+\frac{2}{27}$

$\frac{1}{2\times2\times3\times5}-\frac{1}{2\times2\times5\times5}+\frac{2}{3\times3\times3}$

$\frac{1}{2\times2\times3\times5}\times\frac{3\times3\times5}{3\times3\times5}-\frac{1}{2\times2\times5\times5}\times\frac{3\times3\times3}{3\times3\times3}+\frac{2}{3\times3\times3}\times\frac{2\times2\times5\times5}{2\times2\times5\times5}$

$\frac{1\times3\times3\times5}{2\times2\times3\times3\times3\times5\times5}-\frac{1\times3\times3\times3}{2\times2\times3\times3\times3\times5\times5}+\frac{2\times2\times2\times5\times5}{2\times2\times3\times3\times3\times5\times5}$

$\frac{45}{2700}-\frac{27}{2700}+\frac{200}{2700}=\frac{218}{2700}=\frac{109}{1350}$

Optional: Venn Diagram Factor Sorting Activity

If time allows and additional practice on GCF and LCM is needed, the activity below may be helpful.

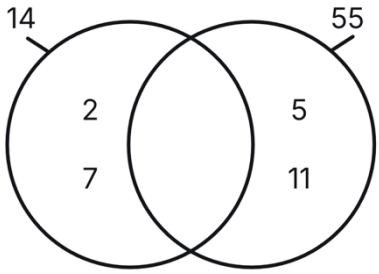
Instructions

1. Use prime factorization to sort each number's factors into a Venn diagram.
2. Put the prime factors that both numbers share in the overlap.
3. Put unique prime factors in the correct circle.
4. Then find (a) the GCF = product of the overlap, and (b) the LCM = product of all factors in the diagram
5. Finally, verify the identity: LCM×GCF = (first number)×(second number)

Examples

A. 14 and 55

14 = 2×7, 55 = 5×11



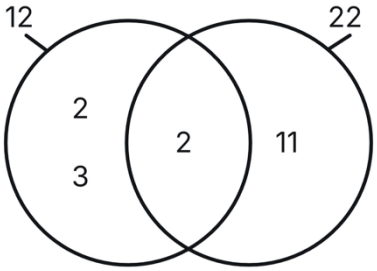
GCF = 1 (middle)

LCM = 2×7×5×11 = 770 (multiply all of the numbers)

Check: GCF×LCM = 1×770 = 770 = 14×55 ✓

B. 12 and 22

12 = 2×2×3, 22 = 2×11 - *place the overlapping 2 in the middle, then fill in the remaining factors.*



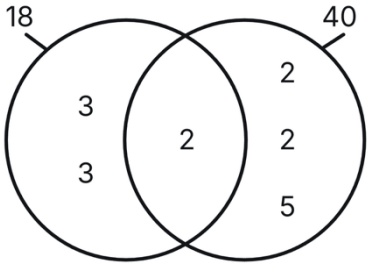
GCF = 2

LCM = 2×2×3×11 = 132

Check: GCF×LCM = 2×132 = 264 = 12×22 ✓

C. 18 and 40

18 = 2×3×3, 40 = 2×2×2×5



GCF = 2

LCM = 3×3×2×2×2×5 = 360

Check: GCF×LCM = 2×360 = 720 = 18×40 ✓

GCF and LCM using a Venn Diagram Worksheet

Find the GCF and LCM of each pair of numbers. Please show all calculations clearly. Try the Venn diagram method for at least 3 problems.

1. 12 and 18

2. 20 and 30

3. 14 and 21

4. 28 and 42

5. 16 and 24

6. 22 and 55

7. 27 and 33

8. 18 and 45

9. 11 and 44

10. 30 and 50

11. 9 and 15

12. 25 and 30

GCF and LCM using a Venn Diagram Answer Key

1. 12 and 18

   12 = 2×2×3, 18 = 2×3×3

   Venn: Left-only: 2 | Overlap: 2, 3 | Right-only: 3

   GCF = 2×3 = 6

   LCM = 2×2×3×3 = 36

   Check: 6×36 = 216 = 12×18

2. 20 and 30

   20 = 2×2×5, 30 = 2×3×5

   Venn: Left-only: 2 | Overlap: 2, 5 | Right-only: 3

   GCF = 2×5 = 10

   LCM = 2×2×5×3 = 60

   Check: 10×60 = 600 = 20×30

3. 14 and 21

   14 = 2×7, 21 = 3×7

   Venn: Left-only: 2 | Overlap: 7 | Right-only: 3

   GCF = 7

   LCM = 2×7×3 = 42

   Check: 7×42 = 294 = 14×21

4. 28 and 42

   $28 = 2\times2\times7$, $42 = 2\times3\times7$

   Venn: Left-only: 2 | Overlap: 2, 7 | Right-only: 3

   $GCF = 2\times7 = 14$

   $LCM = 2\times2\times7\times3 = 84$

   Check: $14\times84 = 1176 = 28\times42$

5. 16 and 24

   $16 = 2\times2\times2\times2$, $24 = 2\times2\times2\times3$

   Venn: Left-only: 2 | Overlap: 2, 2, 2 | Right-only: 3

   $GCF = 2\times2\times2 = 8$

   $LCM = 2\times2\times2\times2\times3 = 48$

   Check: $8\times48 = 384 = 16\times24$

6. 22 and 55

   $22 = 2\times11$, $55 = 5\times11$

   Venn: Left-only: 2 | Overlap: 11 | Right-only: 5

   $GCF = 11$

   $LCM = 2\times11\times5 = 110$

   Check: $11\times110 = 1210 = 22\times55$

7. 27 and 33

   $27 = 3\times3\times3$, $33 = 3\times11$

   Venn: Left-only: 3, 3 | Overlap: 3 | Right-only: 11

   GCF = 3

   LCM = $3\times3\times3\times11 = 297$

   Check: $3\times297 = 891 = 27\times33$

8. 18 and 45

   $18 = 2\times3\times3$, $45 = 3\times3\times5$

   Venn: Left-only: 2 | Overlap: 3, 3 | Right-only: 5

   GCF = $3\times3 = 9$

   LCM = $2\times3\times3\times5 = 90$

   Check: $9\times90 = 810 = 18\times45$

9. 11 and 44

   $11 = 11$, $44 = 2\times2\times11$

   Venn: Left-only: - | Overlap: 11 | Right-only: 2, 2

   GCF = 11

   LCM = $11\times2\times2 = 44$

   Check: $11\times44 = 484 = 11\times44$

10. 30 and 50

    30 = 2×3×5, 50 = 2×5×5

    Venn: Left-only: 3 | Overlap: 2, 5 | Right-only: 5

    GCF = 2×5 = 10

    LCM = 3×2×5×5 = 150

    Check: 10×150 = 1500 = 30×50

11. 9 and 15

    9 = 3×3, 15 = 3×5

    Venn: Left-only: 3 | Overlap: 3 | Right-only: 5

    GCF = 3

    LCM = 3×3×5 = 45

    Check: 3×45 = 135 = 9×15

12. 25 and 30

    25 = 5×5, 30 = 2×3×5

    Venn: Left-only: 5 | Overlap: 5 | Right-only: 2, 3

    GCF = 5

    LCM = 5×5×2×3 = 150

    Check: 5×150 = 750 = 25×30

Lesson 3.12. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

1. Find 87 mod 9.

2. Determine whether each number is divisible by 2, 3, 5, 9, and 11. Explain your reasoning for at least one number.

   a) 198

   b) 275

3. List all the factors of 36 and identify which are prime. Explain how you know a number is prime.

4. Find the greatest common factor (GCF) of 48 and 72 by listing the factors of each number. Explain why it is the GCF.

5. Find the least common multiple (LCM) of 12 and 18 using prime factorization.

6. Fill in the next three numbers in this sequence and show or explain the pattern:

   5, 10, 20, 40, ___, ___, ___

7. Find the last digit of 3^7 and explain how you used patterns or cyclic powers to figure it out.

8. Decide if each statement is true or false and explain your answer:

   a) 17 is divisible by 1 and itself only.

   b) 20 is a prime number.

9. Write a number that is divisible by 2, 3, and 5. Show how you found it.

10. A teacher has 24 red pencils and 36 blue pencils. She wants to divide them into identical packs with no pencils left over.

    a) What is the largest number of pencils she can put in each pack?

    b) Explain how you found your answer and why it works.

Check Your Understanding Answer Key

1. 87 mod 9 = 6
2. a) 198:

   Divisible by 2 ✓ (even)

   Divisible by 3 ✓ (1+9+8=18, which is divisible by 3)

   Divisible by 5 ✗ (does not end in 0 or 5

   Divisible by 9 ✓ (18 ÷ 9 = 2)

   Divisible by 11 ✓ (outer digits add up to the middle one: 1+8 = 9)

   b) 275: divisible by 2 ✗, by 3 ✗, by 5 ✓, by 9 ✗, by 11 ✓
3. Factors of 36: 1, 2, 3, 4, 6, 9, 12, 18, 36. Prime factors: 2, 3.

   Explanation: A prime number's only factors are 1 and itself.
4. Factors of 48: 1, 2, 3, 4, 6, 8, 12, 16, 24, 48

   Factors of 72: 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72

   GCF = 24 (the largest number found on both lists)
5. 12 = **2** × **2** × 3, 18 = 2 × **3** × **3** → LCM = 2 × 2 × 3 × 3 = 36.
6. 80, 160, 320. Each number is multiplied by 2 to find the next.
7. Answer: 7. Powers of 3 end in: 3, 9, 7, 1

   3^7 ends in 7 based on the repeated cycle for the last digit of powers of 3.
8. a) True, 17 has only 1 and 17 as factors → it is prime.

   b) False, 20 has factors 1, 2, 4, 5, 10, 20 → not prime.
9. Example: 30. This is the LCM of 2, 3, and 5: 2 × 3 × 5 = 30.
10. a) GCF of 24 and 36 = 12 → the largest number of pencils per pack is 12.

b) Explanation: 24 ÷ 12 = 2 packs of red, 36 ÷ 12 = 3 packs of blue, no pencils left over.

Year 4 Lessons

The following 12 lessons make up the Year 4 course of study.

Lesson 4.1. Number Theory and Prime Number Review

Instructors are encouraged to begin the class with a brief review of number theory, revisiting its definition and common applications in familiar mathematical contexts. The lesson then transitions to a refresher on prime numbers through an interactive prime number bingo activity, reinforcing prior knowledge in an engaging format. If time permits, instructors may proceed to the introduction of Lesson 4.2, which provides an overview of cryptography and its foundational concepts.

Number Theory Review

Number theory is a branch of mathematics that studies whole numbers (1, 2, 3, 4, 5, …), also called natural numbers, and the ways they relate to each other. If you think of numbers like LEGO blocks, number theory is the part of math that studies how those blocks fit together, what patterns they make, and what we can build with them.

Here are some reasons why number theory matters:

1. Understanding numbers: Number theory helps us understand how numbers behave and interact. Just like knowing how LEGO pieces snap together lets you build complex creations, understanding numbers helps us solve tricky math problems. For example, prime numbers are like the “basic building blocks” of all other numbers.

They cannot be broken down any further, but they combine in interesting ways to create all the rest.

2. Keeping information safe: Number theory is the secret behind online security. Every time you send a text, play an online game, or log in to an account, number theory is helping encrypt your information. Big prime numbers act like super-strong locks that keep your personal data safe.
3. Helping computers work faster: Computers solve problems using patterns in numbers. Number theory helps programmers find shortcuts that let computers calculate things much faster than doing everything step by step.
4. Solving puzzles: Number theory is full of brain-teasing puzzles, like spotting number patterns, testing whether numbers are prime or discovering surprising tricks with digits. If you like challenges and puzzles, you are already thinking like a number theorist.

Prime Number Bingo

Next, the class can move on to playing prime number bingo as a refresher on primes. 16 cards are included below, so students should be split into pairs or groups as needed.

Prime Number Bingo (Sheet 1)

13	41	7	59	23
31	11	37	67	5
19	97	Free	47	29
61	83	79	3	43
17	71	89	73	2

5	79	29	61	67
23	59	43	3	13
7	2	Free	41	31
73	37	47	19	89
53	97	71	17	83

53	17	73	89	59
43	83	5	11	67
37	79	Free	31	23
3	7	13	71	47
2	29	19	61	97

11	7	13	83	5
89	97	47	73	43
53	67	Free	37	17
41	61	71	3	29
19	2	23	59	79

Prime Number Bingo (Sheet 2)

113	41	29	97	43
89	31	73	127	101
59	83	Free	71	47
53	107	17	61	37
103	23	67	109	19

59	109	17	73	127
41	29	89	79	83
61	23	Free	47	71
101	67	31	37	97
107	103	53	113	43

79	41	17	73	23
103	83	97	37	19
67	61	Free	43	31
59	109	47	53	29
127	71	101	113	89

71	17	29	53	97
103	89	109	19	59
101	37	Free	43	107
79	113	23	47	127
83	61	73	67	31

Prime Number Bingo (Sheet 3)

79	103	131	97	17
31	109	73	157	37
19	137	Free	127	41
89	149	83	23	101
43	113	107	139	151

149	127	41	37	83
17	89	103	109	43
73	101	Free	23	31
157	79	19	139	113
137	107	97	29	151

157	137	107	103	149
97	113	19	139	79
109	131	Free	101	41
17	89	29	83	73
23	37	43	127	31

101	89	149	137	19
79	73	83	23	43
139	103	Free	41	107
109	29	97	127	113
31	37	131	157	151

Prime Number Bingo (Sheet 4)

149	179	89	139	23
163	79	181	83	73
97	167	Free	31	157
151	131	173	29	191
17	37	43	41	19

89	17	167	31	79
157	139	83	131	43
19	97	Free	73	191
29	173	23	41	181
163	137	149	179	37

89	131	173	79	17
149	163	167	19	137
29	151	Free	181	31
43	83	97	37	139
73	179	23	157	191

131	149	179	181	43
191	29	151	167	97
37	83	Free	89	173
157	137	73	23	79
17	139	41	163	31

Prime Bingo Call-Out List

Since we are working with primes under 200, here is a full list with hints to call out.

2 – I'm the only even prime number.

3 – I'm the first odd prime number.

5 – I'm an odd factor of 10.

7 – I'm the last single-digit prime.

11 – I'm a two-digit prime with repeated digits.

13 – I'm considered unlucky by some, but I'm prime!

17 – I'm the next prime after 13.

19 – I'm the largest teen prime number.

23 – I'm a prime in the 20s with digits that add to 5.

29 – I'm the last prime in the 20s.

31 – I'm a prime in the 30s with digits that add to 4.

37 – I'm a prime in the 30s with a reverse that is also prime!

41 – I'm the smallest prime over 40.

43 – I'm the first prime after 41.

47 – I'm the biggest prime in the 40s.

53 – I'm the smallest prime in the 50s.

59 – I'm the biggest prime in the 50s.

61 – I'm the smallest prime in the 60s.

67 – I'm a prime in the 60s, and my digits add to 13.

71 – I'm the first prime in the 70s.

73 – I'm the second prime in the 70s.

79 – I'm the last prime in the 70s.

83 – I'm the first prime in the 80s.

89 – I'm the last prime in the 80s.

97 – I'm the largest two-digit prime under 100.

101 – I'm the first three-digit prime.

103 – I'm the first prime after 101.

107 – I'm a prime in the 100s with digits that add to 8.

109 – I'm a prime in the 100s, with digits that add to 10.

113 – I'm the smallest prime in the 110s.

127 – I'm a prime in the 120s.

131 – I'm a prime in the 130s with repeated digits.

137 – I'm a prime in the 130s, not the smallest or biggest.

139 – I'm the biggest prime in the 130s.

149 – I'm a prime in the 140s.

151 – I'm a prime with repeated digits in the 150s.

157 – I'm the biggest prime in the 150s.

163 – I'm the smallest prime in the 160s.

167 – I'm the last prime in the 160s.

173 – I'm the smallest prime in the 170s.

179 – I'm the biggest prime in the 170s.

181 – I'm a prime with repeated digits in the 180s.

191 – I'm the smallest prime with repeated digits in the 190s.

193 – I'm the smallest prime in the 190s with three different digits.

197 – I’m a prime between 190 and 200, but not the largest.

199 – I’m the largest prime under 200.

Lesson 4.2. Introduction to Cryptography

In this lesson, instructors introduce the concept of cryptography by distinguishing between codes and ciphers, establishing foundational vocabulary and ideas. The lesson explores classical ciphers through the Caesar Cipher, including a worked example to demonstrate how shift ciphers operate in practice. Instructors are encouraged to guide students through the concept of frequency analysis as a method for deciphering encrypted messages, followed by a class practice problem that allows students to apply these techniques collaboratively.

The following lesson combines and adapts content from two Khan Academy articles (Khan Academy, n.d.), with examples and explanations rewritten for this thesis. All Khan Academy content is available for free at www.khanacademy.org

What is Cryptography?

Imagine two people who share an important secret, Alice and Bob, have to split up. This requires them to communicate private information from a distance. However, an eavesdropper named Eve also wants this information and has the ability to intercept their messages. So, Alice decides to communicate using letters written in some kind of secret code.

First, Alice locks her message in a box, using a lock that only she and Bob know the combination to. This is known as **encryption**. Then, the locked message is sent to Bob. When Bob receives the box, he opens it using the code they shared in advance. This is called **decryption**.

Cryptography begins when we abandon physical locks and use ciphers instead. Think of **ciphers** as virtual locks. Ciphers allow Alice and Bob to scramble and descramble their messages so that they would appear meaningless if Eve intercepted them.

Cryptography has been around for thousands of years. It has decided wars and is at the heart of communication today. The fascinating story of cryptography requires us to understand two very old ideas related to number theory and probability theory.

Summary of Key Words

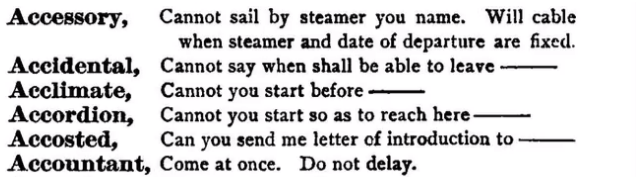
Encryption: locking the data

Decryption: unlocking the data

Ciphers: the virtual key used to lock the data

Codes Versus Ciphers

Codes require codebooks. A codebook is simply a list of these mappings.



Accessory, Cannot sail by steamer you name. Will cable when steamer and date of departure are fixed.
Accidental, Cannot say when shall be able to leave ———
Acclimate, Cannot you start before ———
Accordion, Cannot you start so as to reach here ———
Accosted, Can you send me letter of introduction to ———
Accountant, Come at once. Do not delay.

Source: Khan Academy

For example, in the codebook above, the word Accountant is code for "Come at once. Do not delay." A code is a mapping from some meaningful unit, such as a word, sentence, or phrase, into something else, usually a shorter group of symbols.

Codes generally operate on semantics and meaning, while ciphers operate on syntax and symbols. A code is stored as a mapping in a codebook, while ciphers transform individual symbols according to an algorithm. Ciphers are mechanical operations, known as algorithms, that are performed on individual or small chunks of letters. Consider these two common simple ciphers:

- Substitution cipher: Replace each character or symbol with a different one based on a mapping.
- Shift cipher: A type of substitution cipher where each letter is replaced with the letter shifted some number of positions away. For example, the mapping could be A→D, B→E, C→F, and so on, when we are using a shift of four. In this case, the key to unlock an encrypted message is four, and knowing that would allow two parties to encrypt messages: HELLO→KHOOR, and decrypt messages: KHOOR→HELLO.

Classical Ciphers: The Caesar Cipher

The first well-known cipher, a substitution cipher, was used by Julius Caesar around 58 BC. Caesar shifted each letter in his military commands by three in order to make them appear meaningless if the enemy intercepted them. Now, we call other shift ciphers where the shift amount is the same for all letters a Caesar Cipher.

Imagine Alice and Bob decided to communicate using the Caesar Cipher. First, they would need to agree in advance on a shift to use, say, three. So, to encrypt her message, Alice would need to apply a shift of three to each letter in her original message.

Original letter	A	B	C	…	W	X	Y	Z
Shift of 3	D	E	F	…	Z	A	B	C

So, A becomes D, B becomes E, C becomes F, and so on. This unreadable, or encrypted message, is then sent to Bob openly. Bob subtracts the shift of three from each letter in order to read the original message. Incredibly, this basic cipher was used by military leaders for hundreds of years after Caesar.

However, a lock is only as strong as its weakest point. A lock breaker may look for mechanical flaws, or they may extract information to narrow down the correct combination. The weakness of the Caesar Cipher was published 800 years later by an Arab mathematician named Al-Kindi. He broke the Caesar Cipher by using a clue based on an important property of the language in which a message is written. If you scan text from any book and count how often each letter shows up, you will find a fairly consistent pattern.

For example, these are the letter frequencies of English.

Letter	Frequency (%)
E	12.02
T	9.10
A	8.12
O	7.68
I	7.31
N	6.95
S	6.28
R	6.02
H	5.92
D	4.32
L	3.98
U	2.88
C	2.71
M	2.61
F	2.30
Y	2.11
W	2.09
G	2.03
P	1.82
B	1.49
V	1.11
K	0.69
X	0.17
Q	0.11
J	0.10
Z	0.07

This can be thought of as a fingerprint of English. We leave this fingerprint when we communicate without realizing it. This clue is one of the most valuable tools for a codebreaker. To break this cipher, they count up the frequencies of each letter in the encrypted text and check how far the fingerprint has shifted. For example, if H is the most popular letter in the encrypted message instead of E, then the shift was likely three. So, they reverse the shift in order to reveal the original message. This is called **frequency analysis**, and it was a blow to the security of the Caesar cipher.

Shift Ciphers in Practice

The Caesar Cipher is a type of shift cipher. Shift Ciphers work by using the modulo operator to encrypt and decrypt messages. The Shift Cipher has a key K, which is an integer from 0 to 25. We will only share this key with people who we want to see our message.

How to encrypt:

For every letter in the message M:

Convert the letter into the number that matches its order in the alphabet, starting from 0, and call this number X. (A=0, B=1, C=2, ...,Y=24, Z=25).

Add the shift to this number: $Y = (X + K) \bmod 26$.

Convert the number Y into a letter of the alphabet based on its order, starting from 0. (A=0, B=1, C=2, ...,Y=24, Z=25).

Example:

We agree with our friend to use the Shift Cipher with key K=19 for our message. We encrypt the message “KHAN,” as follows:

ENCRYPTION

	K	H	A	N	
	10	7	0	13	
+	19	19	19	19	
(	29	26	19	32	) mod 26
	3	0	19	6	
	D	A	T	G	

So, after applying the Shift Cipher with key K=19, our message text "KHAN" gave us the cipher text "DATG". We give the message "DATG" to our friend.

How to decrypt:

For every letter in the cipher text C:

Convert the letter into the number that matches its order in the alphabet, starting from 0, and call this number Y. (A=0, B=1, C=2, ..., Y=24, Z=25).

Subtract the shift: X = (Y - K) mod 26.

Convert the number X into a letter of the alphabet based on its order, starting from 0. (A=0, B=1, C=2, ..., Y=24, Z=25).

Our friend now decodes the message using our agreed-upon key K=19 as follows:

DECRYPTION

	D	A	T	G	
	3	0	19	6	
-	19	19	19	19	
(	-16	-19	0	-13	) mod 26
	10	7	0	13	
	K	H	A	N	

So, after decrypting the Shift Cipher with key K=19, our friend deciphers the cipher text "DATG" into the message text "KHAN".

Class Practice Problem

A message was encrypted using a Caesar cipher with a key of 7.

Ciphertext: **Kvu'a qbknl h ivvr if paz jvcly!**

What was the original plaintext?

Hint: think of modular arithmetic. If we know that the alphabet has 26 letters, instead of moving back 7, how many letters can we move forward to end up in the same spot?

Answer:

Original plaintext: **Don't judge a book by its cover!**

Instead of moving back 7 letters, move forward 26 - 7 = 19 letters!

Why is the Shift Cipher insecure?

A cipher should prevent an attacker, who has a copy of the ciphertext but does not know the key, from discovering the messages contents. Since we only have 26 keys, someone can easily try each one, one by one, until they recover the message. This type of attack is called a brute force attack.

Lesson 4.3. Classical Ciphers: Substitution

In this lesson, instructors expand on classical cryptography by introducing substitution ciphers and revisiting the Caesar cipher through guided class practice problems. The lesson broadens to include other forms of substitution ciphers, allowing students to compare methods and identify patterns. Instructors are encouraged to incorporate multiple class practice problems throughout the lesson to reinforce decoding strategies and support student understanding through hands-on application.

Classical Cipher Review

We have discussed one well-known classical cipher, the Caesar Cipher. Recall that this cipher works by shifting the letters in a message by a certain amount to encrypt the message. A receiver can decrypt the message if they know the key, i.e., how many positions each letter is shifted by.

We will learn about other notable classical ciphers.

Class Practice Problems

1. A plaintext was encrypted with a Caesar cipher, resulting in the following:

   DOOV ZHOO WKDW HQGV ZHOO

   Can you work out what the plaintext was?

   Hint: Note that there are several occurrences of "OO" appearing within the enciphered words. This must correspond to a double letter in the plaintext. Only some letters in English can be doubled up, so you can try some guesses.

2. A plaintext had its punctuation and spaces removed and was then encrypted with a Caesar cipher. The resulting ciphertext is:

   cxknxawxccxkncqjcrbcqnzdnbcrxwfruurjvbqjtnbynjan

   Use frequency analysis to identify which ciphertext characters are most common. What was the original plaintext?

   Hint: In plaintext English, the most common letter is E. Using frequency analysis, a sensible guess would be to try a shift which maps E to one of the most common ciphertext characters.

Answers:

1. That is the famous Shakespeare line. The ciphertext is a Caesar shift of 3. Shifting every letter back 3 (or forward 23) gives:

   DOOV: D→A, O→L, O→L, V→S; ALLS

   ZHOO: Z→W, H→E, O→L, O→L; WELL

   WKDW: W→T, K→H, D→A, W→T; THAT

   HQGV: H→E, Q→N, G→D, V→S; ENDS

   ZHOO: WELL

   Plaintext: **All's well that ends well.**

2. Using frequency analysis, we see that the most common ciphertext letters are c and n.

   - First, we consider if "c" is the ciphertext for "e" → shifting "e" to "c" is a shift of 24 forward (or 2 back), so we undo it by moving 24 back (or 2 forward).
     - cxknxawxccxkncqjcrbcqnzdnbcrxwfruurjvbqjtnbynjan becomes ezmp… which is clearly nonsensical.
   - Next, we check if "n" is the ciphertext for "e," which is a forward shift of 9. Undo it by shifting back 9 (or forward 26-9=17).
     - cxknxawxccxkncqjcrbcqnzdnbcrxwfruurjvbqjtnbynjan becomes tobeornottobethatisthequestionwilliamshakespeare.

   Adding spaces and punctuation, the plaintext is, "**To be or not to be, that is the question." William Shakespeare**

   This corresponds to a Caesar shift of 9 (ciphertext letters are shifted forward 9 from plaintext, while decryption shifts backward 9).

Other Substitution Ciphers

One issue with the Caesar cipher is that as soon as the key is known, the message can easily be decrypted. To make a brute force attack harder, we could make a more complex substitution cipher by using something other than a shift of the alphabet. By choosing a **random mapping**, we could get a more secure cipher, with the tradeoff that the encryption key is harder to describe; the key would now be the entire mapping, rather than just the shift amount.

Class Practice Problems (adapted from Lippman, n.d., CC BY-SA 3.0)

1. Use the substitution mapping below to encrypt the message "MARCH 12, 1920."

   Mapping:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	0	1	2	3	4	5	6	7	8	9
2	B	Q	F	5	W	R	T	D	8	I	J	6	H	L	C	O	S	U	V	K	3	A	0	X	9	Y	Z	N	1	G	4	M	E	7	P

   Answer: Using the mapping, the message would encrypt to "62SQT ZN ZPNY."

2. Now, try decoding this message using the same mapping: C2SVX2VP

   Answer: Party at 9

3. Is the general substitution cipher more secure than a shift cipher, such as the Caesar cipher?

   Answer: Yes, since each letter could map to one of 26 options, as we go through the table to figure out what each letter maps to, we have $26\times25\times24\times23\times...\times3\times2\times1$ options for the mapping. In fact, this type of cipher was initially considered unbreakable!

Since languages have predictable patterns, though, these ciphers become easier to break. For example, English has these features:

- Vowels, which constitute 40% of plaintext, are often separated by consonants.
- The letter “A” is often found at the beginning of a word or second from last.
- The letter “I” is often the third from the end of a word.
- “Q” is always followed by “U,” unless the word is anglicized from another language.

Cryptogram Worksheet

In the given cryptograms, each distinct letter in the original phrase has been substituted with another letter from the alphabet. Decipher each cryptogram by first figuring out the letter mapping in the given table. It may help to start by identifying two- or three-letter words.

Cryptogram 1 (ciphertext)

FHH RGVFD OBGX GVJSZGVL DMV LFWV VNNBGD.

Try to figure out the letter mapping. Hint: “V” in the cryptogram corresponds to “E” in the original plaintext.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
				V																					

What is the solution (plaintext)?

_____

_____

Cryptogram 2 (ciphertext)

ZU IAT TUQ FT EUVS OTGOTI IAT MAHUMTJ FT QZQU'I IHKT.

Try to figure out the letter mapping. Hint: "F" in the cryptogram corresponds to "W" in the original plaintext. There is also one ciphertext letter that is the same as its plaintext letter (it didn't change).

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
																						F			

What is the solution (plaintext)?

_____

_____

Cryptogram Answer Key

Cryptogram 1

FHH RGVFD OBGX GVJSZGVL DMV LFWV VNNBGD.

ALL GREAT WORK REQUIRES THE SAME EFFORT.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
F	Q	T	P	V	N	R	M	Z	L	X	H	W	K	B	C	J	G	Y	D	S	I	O	A	E	U

Cryptogram 2

ZU IAT TUQ FT EUVS OTGOTI IAT MAHUMTJ FT QZQU'I IHKT.

IN THE END WE ONLY REGRET THE CHANCES WE DIDN'T TAKE.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
H	X	M	Q	T	L	G	A	Z	R	K	V	Y	U	E	W	B	O	J	I	N	D	F	C	S	P

Lesson 4.4. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix A.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

Multiple Choice:

1. Which statement best describes the purpose of a cipher?

   a) To turn numbers into fractions

   b) To hide the meaning of a message using a rule

   c) To find the greatest common factor of two numbers

   d) To translate a message into a different language

2. In a Caesar cipher, every letter in the message is shifted forward by the same number of positions. What important structural idea does this illustrate?

   a) A cipher must use multiple alphabets

   b) A rule can be applied repeatedly to create a predictable pattern

   c) Every shift makes the message more random

   d) Only prime numbers can be used in a Caesar shift

3. Which of the following is the best description of plaintext?

   a) A scrambled message you cannot read

   b) A message after it has been encrypted

   c) The original message before any encryption happens

   d) A message written using multiple alphabets

4. A substitution cipher replaces each letter with a different letter. Why is this harder to break than a Caesar cipher?

   a) It uses numbers instead of letters

   b) The shift changes each time you write a new word

   c) It allows many more possible mappings between letters

   d) It requires prime numbers to work correctly

5. Why are prime numbers considered useful in modern encryption?

   a) They make addition easier

   b) They cannot be used in modular arithmetic

   c) They create structures that are difficult to reverse without special information

   d) They always create repeating decimals

6. What is the main reason cryptography is important in everyday life?

   a) It helps simplify long division

   b) It hides information so only the intended people can read it

   c) It shows how prime numbers behave in nature

   d) It allows computers to run faster

7. Suppose a friend sends you a message and says, “The ciphertext is ready!” What does this tell you?

   a) The message is still in its original form

   b) The message has not been written yet

   c) The message has been encrypted

   d) The message uses only prime numbers

8. Which everyday situation relies on cryptography?

   a) Solving a multi-step word problem

   b) Estimating a grocery bill

   c) Logging into an online account securely

   d) Calculating the area of a rectangle

9. A simple encryption technique changes each letter based on a rule that the sender and receiver both know. Which idea does this connect to?

   a) Communication can be made secure by a shared structure

   b) Prime factorization is the only way to decrypt messages

   c) All ciphers require multiple keys

   d) Encrypting a message removes its meaning

Free Response:

Choose one of the questions below and answer in complete sentences. Your response should explain your thinking, not just restate definitions.

1. Explain one example from the assessment where a pattern or structure helped you understand how a cipher works.
2. Why do you think shifting letters or substituting letters can hide meaning?
3. How do plaintext and ciphertext relate to each other? Why does that relationship matter?
4. Where do you see cryptography in everyday life?
5. Why might prime numbers be useful in modern encryption?

Chosen question number: _____

Response:

Check Your Understanding Answer Key

Multiple Choice

1. B
2. B
3. C
4. C
5. C
6. B
7. C
8. C
9. A

Free Response Sample Answers

1. Explain one example from the assessment where a pattern or structure helped you understand how a cipher works.

   A Caesar cipher uses a repeated shift to transform messages predictably.

2. Why do you think shifting letters or substituting letters can hide meaning?

   These rules change the structure of the original message in a consistent way that disguises it from anyone who does not know the pattern.

3. How do plaintext and ciphertext relate to each other, and why does that matter?

   Plaintext is the original message, and ciphertext is the encrypted version, so understanding their relationship is key to knowing what a cipher actually does.

4. Where do you see cryptography in everyday life?

   Cryptography is used when logging in to accounts, sending messages, or making online purchases because it protects sensitive information behind the scenes.

5. Why might prime numbers be useful in modern encryption?

   Prime numbers create mathematical structures that are easy to compute in one direction but very hard to reverse, which makes them helpful for secure communication.

Lesson 4.5. Puzzles and Symmetric Encryption

This lesson begins with a set of cryptography puzzles designed to be solved in small groups, encouraging collaboration and problem-solving while activating prior knowledge and curiosity about encryption concepts. Instructors are encouraged to facilitate discussion during this activity and allow students to compare strategies before transitioning to a more structured introduction to symmetric encryption. The lesson then shifts to direct instruction on symmetric encryption, providing foundational terminology, core principles, and examples that build on the patterns and ideas encountered in the initial puzzle-based activity.

Cryptography Puzzle Worksheet

You are provided with a letter-translation table and a set of coded questions. Use the table to decode the missing words in each question. Once you've filled in the blanks, answer the questions using your knowledge of science.

Letter table:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
@	☺	♦	1	□	☆	∞	7	☾	⬇	8	μ	☪	△	Σ	Ω	🗲	•	✡	6	ℜ	∃	2	⊥	Φ	3

1. What is the process by which Ωμ@△6✡ make their own ☆ΣΣ1 using ✡ℜ△μ☾∞76?

   Answer: _____

2. Which planet is known as the •□1 Ωμ@△□6?

   Answer: _____

3. What Ωℜμμ✡ objects toward the ♦□△6□• of the □@•67?

   Answer: _____

Cryptography Puzzle Answer Key

Solving the challenge involved using the provided key to substitute letters in the questions, which revealed the missing words. Once the words were filled in, the correct answer could be determined using scientific knowledge.

1. What is the process by which ***plants*** make their own ***food*** using ***sunlight***?

   Answer: **Photosynthesis**

2. Which planet is known as the ***Red Planet***?

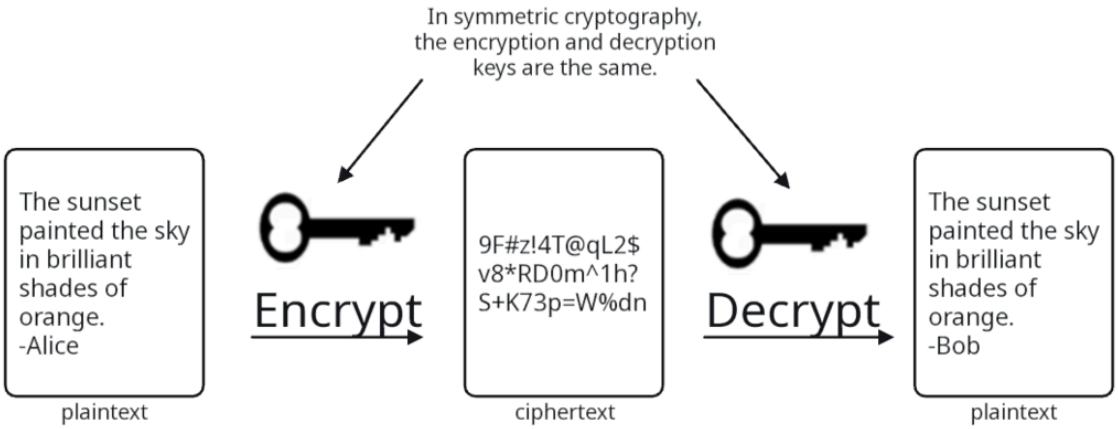
   Answer: **Mars**

3. What ***pulls*** objects toward the ***center*** of the ***Earth***?

   Answer: **Gravity**

Symmetric Encryption

Alice and Bob have a history of illicit dealings. We are not really sure what they are up to, but they do not want us, or the ever-curious Eve, to know. Before the internet, Alice and Bob could exchange secret messages by encrypting their messages with an agreed-upon cipher. Maybe that was through letter substitution, shifting, or other sophisticated methods. They agreed on the method in advance, and they both knew how to encode and decode the end message.



This is known as **symmetric cryptography**: the same key is used in both directions. This worked well for World War II spies, but what happens when our secret messengers cannot agree on the key in advance?

Alice and Bob need a way to communicate securely without first meeting to agree on a shared encryption key or risking key interception by an eavesdropper like Eve on unsecured channels. This advancement is crucial, especially because of the internet. Soon, we will discuss **asymmetric (public-key) cryptography**.

Lesson 4.6. Public Key Cryptography: Diffie-Hellman

This lesson introduces the Diffie-Hellman Key Exchange through a conceptually focused progression that emphasizes understanding over computation. Instructors are encouraged to begin with a color analogy, building in frequent pauses to check for comprehension and ensuring that the analogy clearly conveys the idea of shared secrets before moving on. Once students demonstrate comfort with the analogy, the lesson transitions to the same process expressed with numbers, where instructors should highlight the roles of prime numbers and modular arithmetic rather than the mechanics of the calculations themselves. Throughout the lesson, instructors should prioritize conceptual clarity, reinforce why these mathematical properties are essential to secure key exchange, and avoid letting arithmetic details distract from the underlying ideas.

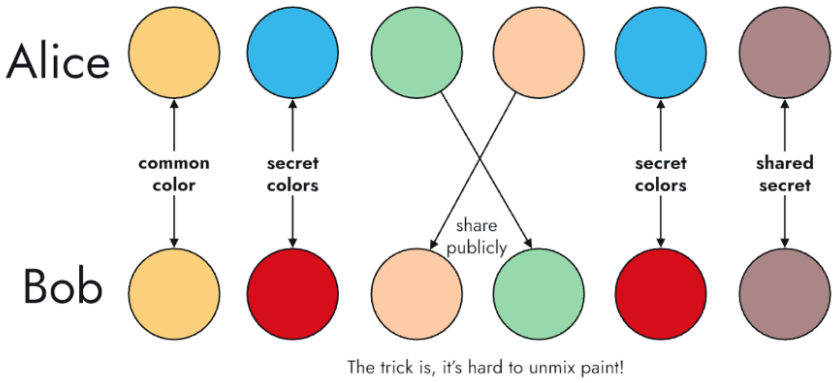
The Diffie-Hellman Key Exchange

Almost everything we do on the internet requires encryption to remain secure. Early on, this required two parties to already share a secret key, so the question became: how can two strangers agree on a secret key without anyone else learning it? In 1976, Whitfield Diffie and Martin Hellman introduced a clever solution.

A Color Analogy

Suppose Alice and Bob want to agree on a secret color without Eve discovering it. Mixing colors is easy, but figuring out the exact original colors from a mixed color is hard. This “easy one way, hard the other way” idea is the basis of the trick. First, Alice and Bob publicly agree on a starting color, like yellow. Then each secretly picks a private

color and mixes it with the yellow. They send these mixed colors to each other. Finally, each adds their own secret color to the mixture they received. Both end up with the same final color, their shared secret that Eve cannot determine without knowing one of their private colors.



The Same Trick with Numbers

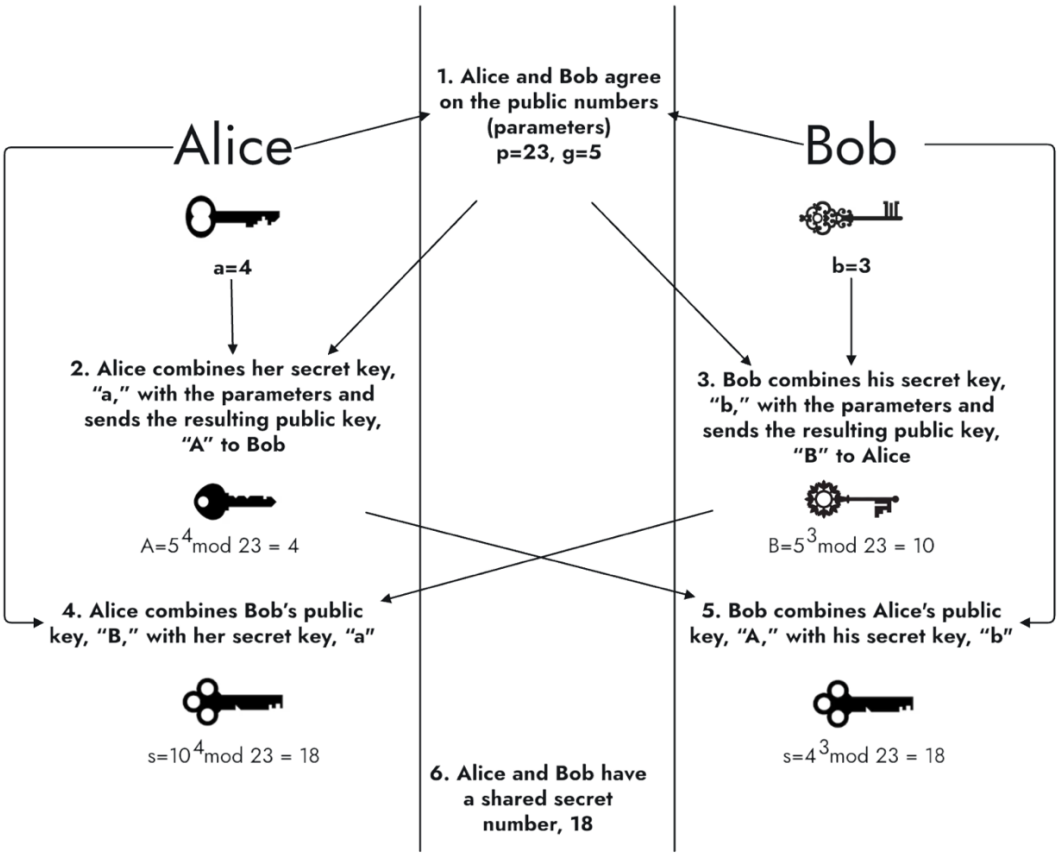
To make the color trick work with numbers, we use modular arithmetic, since it is easy to do in one direction but very hard to undo. Alice and Bob start by agreeing on **two public numbers**: a prime modulus, like 23, and another number that works well with it, which we call a generator. Finding this second number, the generator, uses slightly more advanced math, so we will not go into the details here. Our generator for 23 is 5.

Each person then picks a secret number. They each take the generator, raise it to the power of their secret number, and reduce the result mod 23. This is easy to calculate. They send these results to each other.

The important idea is that reversing this process is very hard. If someone only sees the final result, like 10 mod 23, it is extremely difficult to figure out what power produced it.

With small numbers, you could try all possibilities, but with the large primes used in real systems, trying every exponent would take far too long.

Now for the key step. Alice takes Bob's public result and raises it to the power of her secret number, mod 23. Bob does the same with Alice's public result and his secret number. Both end up with the exact same final value, which becomes their shared secret key.



Step-by-step:

Alice's secret number: 4 → Encryption: $5^4 \bmod 23 \equiv 4$ → Send to Bob (public)

Bob's secret number: 3 → Encryption: $5^3 \bmod 23 \equiv 10$ → Send to Alice (public)

Shared secret – Alice's perspective: $10^4 \bmod 23 \equiv 18 \rightarrow (5^3)^4 \bmod 23$

Shared secret – Bob's perspective: $4^3 \bmod 23 \equiv 18 \rightarrow (5^4)^3 \bmod 23$

Notice they did the same calculation with the exponents in a different order: $(5^3)^4$ and $(5^4)^3$. When you flip the exponent order, the result does not change.

Without one of these secret numbers, 4 or 3, Eve will not be able to find the solution. So, while Eve is stuck grinding away at this decryption problem, with large enough numbers,

we can say it is practically impossible for her to break the encryption in a reasonable amount of time. This solves the key exchange problem. It can be used along with a pseudorandom generator to encrypt messages between people who have never met. This is called the **Diffie-Hellman key exchange**.

Lesson 4.7. Public Key Cryptography: RSA

This lesson introduces RSA encryption as a core example of asymmetric key exchange, with an emphasis on how public and private keys function together to enable secure communication. Instructors are encouraged to clearly distinguish asymmetric encryption from previously studied symmetric methods, focusing on the conceptual roles of key pairs rather than procedural detail. The lesson concludes with a guided class practice problem, during which instructors should model the reasoning process, check for understanding at each step, and use the activity to reinforce why the mathematical structure of RSA supports secure key exchange in practice.

RSA Encryption

Before the 1970s, cryptography used symmetric keys, which meant both people needed the same secret key to send and read messages. The big problem was that Alice and Bob had to somehow share this key first, which is hard if they cannot meet in person. Diffie-Hellman introduced a new idea called asymmetric key exchange, but it still requires both people to agree on a large prime number and a special generator number to make it work.

Plus, if Alice needs to communicate with multiple people, perhaps she is a bank, then she will have to exchange different keys with each person. Now she will have to manage all of these keys and send thousands of messages just to establish them. Could there be a simpler way?

In the 1970s, people began searching for a way to communicate securely without first sharing a secret key. One early idea was that you could “lock” a message with one key and “unlock” it with a different key. Anyone could have the locking key, but only the receiver would have the unlocking key.

To imagine this, think of Alice giving everyone an open padlock. Anyone can put a message inside a box and snap the lock shut, but only Alice has the key that can open it. No key ever needs to be exchanged.

A Color Analogy

Alice starts by choosing a private color, for example, red. She then uses a special color machine that only she can access to find the exact complementary color to her red. This complementary color is cyan. Alice sends this cyan to Bob as her public key. Even if someone sees the cyan, they cannot figure out the exact shade of red she used.

Now suppose Bob wants to send Alice a secret color, like yellow. He mixes his yellow with Alice’s public color (cyan) and sends the mixture back to her. When Alice receives it, she adds her private red. Adding her red cancels out the cyan she originally sent, so she is left with Bob’s original yellow.

Eve cannot find Bob’s yellow because she would need Alice’s private red to undo the mixture. This shows how a system with a public key and a private key could work in theory, though a real mathematical version was still needed to make it practical.

The Same Trick with Numbers

The breakthrough idea is simple to state: multiplying two large primes is easy, but factoring the result to recover the original primes is extremely hard. If Alice keeps her two primes secret and shares only their product, no one else can easily reverse it.

Using this idea, Alice can create two keys: a public key that anyone can use to encrypt messages to her, and a private key that only she can use to decrypt them. Even if Eve sees the encrypted message, she cannot read it without the private key.

This method was later rediscovered by Ron Rivest, Adi Shamir, and Leonard Adleman in 1977, and became known as RSA. Today, RSA is one of the most widely used encryption methods in the world, and its security depends on how hard it is to factor very large numbers.

RSA is used to provide privacy and ensure the authenticity of digital data across many areas of our daily lives: websites, phone applications, secure messaging, email, and digital payment systems.

Class Practice Problem

Work through this RSA example together as a class. In the next lesson, students will practice additional RSA problems in small groups. The key ideas to emphasize are:

- Prime numbers and modular arithmetic play an important role in modern encryption.
- Learning to follow a clear set of algorithmic steps is an essential skill for solving these problems.

Step 1: Choose two small prime numbers, p and q

$$p = 3$$

$$q = 11$$

Note: These numbers should be much bigger, but we chose small numbers so we can do these calculations by hand.

Step 2: Compute n and $\varphi(n)$

$$n = p * q = 3 * 11 = 33$$

$$\varphi(n) = (p - 1)(q - 1) = 2 * 10 = 20$$

Note: n is called the modulus and $\varphi(n)$ is called the Euler Totient Function

Step 3: Choose e (public exponent)

Choose e between 1 and $\varphi(n)$ with gcd(e,$\varphi(n)$)=1

Let's pick: $e = 3$ (other possibilities are 7,9,11,13,17, and 19)

Note: Prime numbers that are not factors of $\varphi(n)$ are good candidates for e.

Step 4: Compute d (private key)

Find d such that $d * e \bmod \varphi(n) = 1$

Try small values for d.

Let's try $d = 7$: $d * e = 7 * 3 = 21$.

Since 21 mod 20 is 1, this works.

Note:

Public key: $(e = 3, n = 33)$

Private key: $(d = 7, n = 33)$

Step 5: Encrypt the message

Since RSA encrypts numbers, if a message starts as text, it has to be changed to a number. For simplicity, we will encrypt a very simple message: the number 7.

We call the message "m," so $m = 7$.

Cipher: $c = m^e \bmod n = 7^3 \bmod 33 = 343 \bmod 33 = 13$.

Encrypted message: $c = 13$.

Note:

RSA encryption transforms a message into something that looks random (e.g., 13). Only someone with the private key can undo this.

Step 6: Decryption

Now we want to get the original message (7) back from the encrypted number (13), using the private key (7):

Start with the encrypted message, $c = 13$.

Decryption process to get back m:

$m = c^d \bmod n = 13^7 \bmod 33$

This is difficult to calculate, so we can work with smaller powers of 13 and reduce them mod 33.

Keep in mind some multiples of 33 to help with the modular arithmetic:

33×5=165, 33×6=198, 33×25=825

$13^2 \bmod 33$

$= 169 \bmod 33$

$= 4 \bmod 33$

$13^7 \bmod 33$

$= 13^2 \times 13^2 \times 13^2 \times 13 \bmod 33$

$=4\times4\times4\times13 \bmod 33$

$=64\times13 \bmod 33$

$=832 \bmod 33$

$= 7$

Decrypted message: $m = 7$.

This problem shows how powerful simple ideas like primes and modular arithmetic can be when used in the right way. Even though the numbers we used were small, the method is the same one that helps protect information all over the internet.

Lesson 4.8. RSA In Practice and Modern-Day Encryption

This lesson builds on prior instruction in RSA encryption by reinforcing the overall process while de-emphasizing computational complexity. Instructors are encouraged to remind students that the underlying mathematics relies on familiar skills (e.g., prime factorization, multiplication, and modular arithmetic) and to frame the class practice problems as an opportunity to strengthen conceptual understanding through paired or small-group work.

The lesson then transitions to a discussion-based wrap-up and review, allowing instructors to check for understanding, address common misconceptions, and synthesize key ideas.

The final segment looks ahead to the future of encryption, prompting instructors to facilitate reflection on how cryptographic methods continue to evolve and why these concepts remain relevant in real-world applications.

Class Practice Problems

Fill the table below by identifying the prime factors of the modulus, p and q, then completing the remaining calculations.

Encrypted # $\boldsymbol{c}$	Public key $\boldsymbol{e}$	Modulus $\boldsymbol{n}$	Primes $\boldsymbol{p, q}$	Euler $\boldsymbol{\phi(n)}$	Private key $\boldsymbol{d}$	Decrypted # $\boldsymbol{m}$
$\boldsymbol{m^e}$ mod $\boldsymbol{n}$	$1 < e < \varphi(n)$ gcd $(e,\varphi(n))=1$	$\boldsymbol{p \times q}$		$(p-1)(q-1)$	$d \times e \bmod \varphi(n) = 1$	$m = c^d \bmod n$
9	3	55			27	
25	5	91			29	
64	3	187			107	
67	7	143			103	

To decrypt:

- Figure out the factors of n: p and q
- Calculate $\phi(n) = (p-1)(q-1)$
- Calculate m: $m = c^d \bmod n$

Answers:

Encrypted # c	Public key e	Modulus n	Primes p, q	Euler $\phi(n)$	Private key d	Decrypted # m
9	3	55	**5, 11** $5 * 11 = 55$	**40** $4 * 10 = 40$	27 $27 * 3 = 81$ $81 \bmod 40 = 1$	**4** $9^{27} \bmod 55 = 4$
25	5	91	**7, 13** $7 * 13 = 91$	**72** $6 * 12 = 72$	29 $29 * 5 = 145$ $145 \bmod 72 = 1$	**4** $25^{29} \bmod 91 = 4$
64	3	187	**11, 17** $11 * 17 = 187$	**160** $10 * 16 = 160$	107 $107 * 3 = 321$ $321 \bmod 160 = 1$	**4** $64^{107} \bmod 187 = 4$
67	7	143	**11, 13** $11 * 13 = 143$	**120** $10 * 12 = 120$	103 $103 * 7 = 721$ $721 \bmod 120 = 1$	**4** $67^{103} \bmod 143 = 4$

Class Encryption Wrap-Up and Review

1. What is the main purpose of encryption?

   Answer: To keep information private and unreadable to unauthorized users.

2. Why are simple ciphers like the Caesar cipher no longer secure?

   Answer: They are easily broken by analyzing patterns or using trial and error.

3. How is asymmetric (public-key) encryption different from symmetric encryption?

   Answer: Symmetric uses one key to encrypt and decrypt. Asymmetric uses a public key to encrypt and a private key to decrypt.

4. Why does RSA encryption rely on large prime numbers?

   Answer: Multiplying large primes is easy, but factoring the result is extremely hard. This keeps messages secure.

5. How does learning about encryption help us understand real-world security?

   Answer: It shows how messages, files, and online communications stay private. It also explains why secure keys and primes are important for protecting information.

The Future of Encryption

Cyberattacks are constantly evolving, so security experts need to develop new ways to keep information safe. They test these methods carefully to ensure hackers cannot easily break them.

In the future, encryption could face big challenges. Many current systems, like RSA, rely on the fact that even the fastest computers take a very long time to do certain calculations, like factoring large numbers. But what if computers became much faster? This could make current encryption easier to break.

Quantum computers are a possible example. You can think of them like super-powerful puzzle solvers. Right now, they have limits, like needing a huge amount of memory, so they cannot yet break encryption. But if they get stronger, some of the encryption we use today could be cracked quickly. That is why scientists are already working on new types of encryption that can stay safe even against these future supercomputers.

Lesson 4.9. Check-Your-Understanding Assessment

Dedicate the first half of class to having students complete the assessments quietly and independently, allowing them to focus fully on their own work. The assessment and answer key can be found below.

Correct the assessment together as a class. Students should grade their own assessments, then fill out a reflection journal. The journal can be found in Appendix E.

Suggested self-grading instructions for students:

- Each student should carefully self-grade their work using a two-color system: write your original answers in pencil and mark your self-grading in pen to clearly separate your work from your evaluation.
- After grading, complete your reflection journal, thoughtfully analyzing any mistakes, identifying patterns or strategies that helped, and recording insights to guide your future learning.
- This process is meant to help you understand your thinking, recognize areas for growth, and create a personal reference for future assessments.

Check Your Understanding

Use a pencil to take this quiz, show your reasoning clearly, and remember that calculators are not allowed.

Multiple Choice:

1. In shared-key encryption, both people use the same secret key. What is one major drawback of this method?

   a) The key cannot be written down

   b) Both people need the same key, and sharing it can be risky

   c) It only works with very small numbers

   d) It does not use modular arithmetic

2. The fundamental theorem of arithmetic says that every whole number greater than 1 can be written as a product of prime numbers. Why does this matter for encryption?

   a) It guarantees that primes always come in pairs

   b) It makes addition easier

   c) It ensures that prime factorizations have a predictable structure

   d) It lets every number have several different factorizations

3. Why is public/private key encryption thought to be safer than shared-key encryption?

   a) The private key changes every few seconds

   b) Only the private key can unlock messages, and it never has to be shared

   c) The public key disappears after one use

   d) It uses substitution ciphers instead of modular arithmetic

4. In a public key system, what can someone do with the public key?

   a) Decrypt any message

   b) Break the encryption quickly

   c) Encrypt messages so that only the person with the private key can read them

   d) Factor large numbers with more simplicity

5. What does the private key allow the owner to do?

   a) Encrypt messages for anyone

   b) Decrypt messages that were encrypted using the public key

   c) Change the size of prime numbers

   d) Find the greatest common factor more quickly

6. Why are large primes useful in public key encryption?

   a) They are easy for computers to factor

   b) It is easy to multiply them, but hard to factor their product

   c) They make addition faster

   d) They guarantee that every message is unbreakable

7. Which statement best describes why some encryption methods are secure?

   a) Some math problems take a long time for computers to solve

   b) Computers do not use multiplication

   c) All prime numbers are extremely rare

   d) Computers cannot perform modular arithmetic

8. When encrypting with a public key and decrypting with a private key, which idea is most important?

   a) Encryption and decryption must use different alphabets

   b) The decryption step must be easier than the encryption step

   c) The relationship between the keys must be connected but not reversible

   d) The private key must always be a prime number

9. Why does modular arithmetic play a major role in public key encryption?

   a) Modulo operations erase all information

   b) Modulo operations make repeated multiplication cycle in predictable ways

   c) Modular arithmetic only works with primes

   d) Modulo operations guarantee no one can decode a message

10. Which statement best explains why encryption keeps information safe but not perfectly safe?

    a) No encrypted message can ever be decrypted

    b) Strong encryption has no weaknesses

    c) Given enough time and computing power, an attacker could break it

    d) Encryption only protects messages that use small numbers

Free Response:

Choose any one of the questions below and answer in complete sentences. Your response should explain your thinking, not just restate definitions.

1. Shared-key encryption requires both people to use the same key. Why can this become a security problem in real situations?
2. Why is it helpful that public and private keys are connected in one direction but not easy to reverse?
3. How does modular arithmetic help create predictable patterns that are still difficult for outsiders to undo?
4. Which idea from this assessment do you think is most important for keeping digital information safe, and why?

Chosen question number: _____

Response:

Check Your Understanding Answer Key

Multiple Choice:

1. B
2. C
3. B
4. C
5. B
6. B
7. A
8. C
9. B
10. C

Free Response Sample Answers:

1. Shared-key encryption requires both people to use the same key. Why can this become a security problem in real situations?

   Sharing the same key means that if someone gets hold of it, every message can be read. The system's security depends entirely on keeping this one secret safe.

2. Why is it helpful that public and private keys are connected in one direction but not easy to reverse?

   Anyone can encrypt messages with the public key, but only the private key holder can decrypt them. Because decryption is one-way, the public key can be shared openly without risk.

3. How does modular arithmetic help create predictable patterns that are still difficult for outsiders to undo?

   Modulo operations create cycles that follow mathematical rules, which makes encryption steps repeatable for those with the key. Without the key, reversing these cycles can become extremely hard.

4. Which idea from this assessment do you think is most important for keeping digital information safe, and why?

   Hard math problems are essential, since encryption depends on following steps that computers cannot quickly reverse. This difficulty underpins secure communication.

Lesson 4.10. Number Theory Concept Review

This lesson serves as a comprehensive review of number theory concepts introduced throughout the four-year course. Instructors are encouraged to use the key ideas and guided examples to reinforce foundational principles, making explicit connections between concepts and their applications in cryptography. Class practice problems should be used strategically to assess understanding and address gaps, with an emphasis on clarity, reasoning, and long-term retention rather than procedural speed.

Classifying Numbers

Key Ideas

- Natural numbers: counting numbers starting at 1
- Whole numbers: natural numbers + 0
- Integers: whole numbers and their negatives
- Rational numbers: any number that can be written as a fraction
- Even: divisible by 2
- Odd: not divisible by 2

Guided Example

Classify the number -4:

It is an integer and a rational number (because -4 = -4/1).

It is even because it is divisible by 2.

Not natural or whole.

Class Practice Problems

Classify each number using as many labels as apply.

- 0
- 7
- 3/5
- 18

Divisibility, Factors, Multiples

Key Ideas

- A number is divisible by another if it divides with no remainder.
- Factors multiply together to give a number.
- Multiples are what you get when you multiply a number by whole numbers.
- Divisibility Reminders
    - By 2: ends in 0, 2, 4, 6, 8
    - By 3: sum of digits divisible by 3
    - By 5: ends in 0 or 5
    - By 9: sum of digits divisible by 9
    - By 10: ends in 0

Guided Example

Is 96 divisible by 3?

Sum of digits: 9 + 6 = 15

15 is divisible by 3, so yes.

Class Practice Problems

Determine whether each number is divisible by 2, 3, 5, 9, and 10:

- 45
- 120
- 38

List factors:

- Factors of 24
- Factors of 54

Prime and Composite Numbers

Key Ideas

- Prime: exactly two factors (1 and itself).
- Composite: more than two factors.
- 1 is neither prime nor composite.

Class Practice Problems

Identify whether each number is prime or composite:

- 19
- 51
- 37
- 65

The Sieve of Eratosthenes

Key Ideas

This method finds all primes up to a given number by crossing out multiples of each prime.

Guided Example

To find primes up to 30:

- Start with 2, cross out all multiples of 2
- Next uncrossed number is 3, cross out multiples of 3
- Continue with 5, then 7
- Remaining un-crossed numbers are primes.

Class Practice Problems

- Use the Sieve of Eratosthenes to list all primes up to 40.
- List all primes between 50 and 70.

Prime Factorization

Key Ideas

Break a number into its prime factors using a factor tree or repeated division.

Guided Example

Prime factorization of 84:

84 = **2** × 42

42 = **2** × 21

21 = **3** × **7**

So 84 = 2 × 2 × 3 × 7

Class Practice Problems

Find the prime factorization of:

- 72
- 150
- 210

Greatest Common Factor (GCF) and Least Common Multiple (LCM)

Key Ideas

GCF: Largest number that divides two numbers.

LCM: Smallest number that is a multiple of both numbers.

Guided Example

Find the GCF and LCM of 12 and 30:

Prime factorization:

12 = 2 × 2 × 3

30 = 2 × 3 × 5

GCF: overlap = 2 × 3 = 6

LCM: combine all = 2 × 2 × 3 × 5 = 60

Class Practice Problems

- Find the GCF of 18 and 45
- Find the LCM of 8 and 12
- Find both the GCF and LCM of 20 and 50

Relationship Between GCF and LCM

Key Ideas

For any two positive integers, $GCF(a,b) \times LCM(a,b) = a \times b$

Guided Example

Check the relationship for 12 and 30:

$GCF \times LCM = 6 \times 60 = 360$

$12 \times 30 = 360$ → it works!

Class Practice Problems

- Check the GCF-LCM relationship for 9 and 21
- Check the GCF-LCM relationship for 10 and 15

Lesson 4.11. Number Theory Application Review

This lesson provides a comprehensive review of the number theory applications explored throughout the four-year course. Instructors are encouraged to connect each application back to the underlying mathematical ideas, emphasizing how abstract concepts are used to solve practical problems. Guided examples and class practice problems should be used to reinforce these connections, support synthesis across topics, and help students reflect on the relevance and impact of the material beyond the classroom.

Fractions, GCF, and LCM

Key Ideas

- GCF helps simplify fractions.
- LCM helps compare fractions or add/subtract fractions with unlike denominators.

Guided Example

Simplify the fraction: $\frac{36}{48}$

GCF(36, 48) = 12

Divide the numerator and denominator by 12: $\frac{36\div12}{48\div12} = \frac{3}{4}$

Add the fractions: $\frac{1}{6} + \frac{1}{8}$

LCM(6, 8) = 24 → this is the least common denominator

Rewrite: $\frac{1\times4}{6\times4} + \frac{1\times3}{8\times3} = \frac{4}{24} + \frac{3}{24} = \frac{7}{24}$

Class Practice Problems

Simplify $\frac{45}{60}$.

What is the LCM of 9 and 12? Use it to rewrite $\frac{2}{9}$ and $\frac{1}{12}$ with a common denominator.

Simplify $\frac{84}{126}$ using GCF.

Modular Arithmetic and Remainders

Key Ideas

- Modular arithmetic focuses on remainders.
- "a mod n" means the remainder when a is divided by n.
- Modular arithmetic is useful for patterns, calendars, clock arithmetic, ciphers, and encryption.

Guided Example

What is 37 mod 6?

37÷6 = 6 remainder 1 → Answer: 1

Reminder:

The first few powers of 7 are 7, 49, 343, 2401, …

The last digits 7, 9, 3, and 1 repeat, so we can use this pattern to find the last digit of any power of 7.

Class Practice Problems

- Find: 58 mod 5
- Find: 103 mod 7
- On a 12-hour clock, what time will it be 19 hours after 5:00?
- What is the last digit of 7^{23}?

Encoding and Decoding Classical Ciphers

Key Ideas

- Caesar Cipher: Shift each letter by a fixed number.
- Substitution Cipher: Each letter is replaced by a different letter using a fixed mapping.

Guided Example

Shift by +3: HELLO → KHOOR

Class Practice Problems

Caesar:

- Encode with a shift of +4: MATH
- Decode a shift of -2: OKFF

Substitution mapping:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	0	1	2	3	4	5	6	7	8	9
2	B	Q	F	5	W	R	T	D	8	I	J	6	H	L	C	O	S	U	V	K	3	A	0	X	9	Y	Z	N	1	G	4	M	E	7	P

- Encode: CAT
- Decode: A2RLH

Principles Behind Diffie-Hellman and RSA

Key Ideas

- Some problems are easy to do but hard to undo.
- Large prime numbers help build systems like RSA and Diffie-Hellman.
- Modular arithmetic creates patterns that outsiders cannot reverse without a key.
- Public key: used to encrypt
- Private key: used to decrypt
- Diffie-Hellman: Two people create a shared secret without sending the secret itself. They share a base and a modulus but keep exponents private.
- RSA: multiplying two large primes creates a number (public key) that is very hard to factor. The private key uses special information that depends on the prime factors.

Class Practice Problems

- Diffie-Hellman: Suppose two people use base 5 and modulus 14

  Person A chooses secret 3 → sends 5^3 mod 14

  Person B chooses secret 4 → sends 5^4 mod 14.

  Compute both values.

- RSA: Let the "RSA number" be 77 (7×11).

  Why is knowing the factors important for decoding?

  Why is 77 difficult enough for a computer only when the primes are huge?

Applying Number Theory and Cryptography to Real Problems

Key Ideas/Guided Example

A website encrypts login information with a public key. Explain why someone can send information securely even without knowing the private key.

Answer: The public key only encrypts. The private key decrypts.

Class Practice Problems

- A message is encoded using a Caesar shift of +7. The decoder knows modular arithmetic mod 26. How does mod 26 help?
- A student claims that primes and modular arithmetic have nothing to do with each other. Give one reason this is incorrect.
- Explain how simplifying fractions and finding prime factorizations connect to breaking or analyzing classical ciphers.
- Give one real-world example where cryptography protects information.

Lesson 4.12. A Final Game

For the final class of the curriculum, students will engage in a collaborative class puzzle called The Vault, designed to integrate and reinforce the key concepts learned over four years of number theory instruction. Instructors are encouraged to facilitate teamwork, monitor group progress, and provide guidance as needed, ensuring that students apply both their problem-solving skills and conceptual understanding. The activity serves as a capstone experience, emphasizing synthesis, collaboration, and reflection on how the course content connects across topics and real-world applications.

The Vault

For the final class of the curriculum, students will work on a collaborative class puzzle adventure that ties together four years of number theory.

Teacher Instruction Sheet

Duration: 60-90 minutes

Goal: Wrap up the course by applying number theory, modular arithmetic, and classical and modern cryptography in a collaborative and engaging activity.

Overview

Students work in teams to unlock a "vault" by completing a series of puzzles. Each station reinforces a concept from the course:

- Prime factorization, GCF, and LCM
- Modular arithmetic and remainders

- Classical ciphers (Caesar, substitution)
- Mini Diffie-Hellman exchange
- Mini RSA problem

Teams must collect clues from each station and combine them to solve a final puzzle. The activity emphasizes teamwork, problem-solving, and the application of number theory in real-world cryptography.

Materials Needed

General Materials:

- Hacker cards/envelopes for each team to collect clues
- Pencils and scrap paper
- Teacher answer key (provided below)
- Vault prop (box, envelope, or printed image)
- Optional: prizes (snacks, certificates, stickers) and timer

Station Materials:

- Station 1: Worksheet with numbers for GCF, LCM, and prime factorization; clues in envelopes or on slips
- Station 2: Worksheet or cards with modular arithmetic problems; clues printed
- Station 3: Pre-written cipher messages (Caesar, substitution); cipher worksheets; clue letters or phrases
- Station 4: Base/modulus cards; worksheet for calculating shared secret; clue letters or numbers
- Station 5: Tiny RSA examples with small n and e; worksheet for decoding; clue letters or numbers

Student Worksheets

Station 1: Prime Factorization, GCF, LCM

Instructions:

Calculate the greatest common factor (GCF), least common multiple (LCM), and prime factorization of the given numbers.

Problems:

1. Find the GCF of 24 and 36
2. Find the LCM of 8 and 12

Find the prime factorization of 60

Clue 1 Unlocked: _____

Station 2: Modular Arithmetic

Instructions:

Solve these remainder problems and identify repeating patterns. Each correct answer helps with Station 3.

Problems:

1. Calculate 37 mod 6
2. Calculate 143 mod 18
3. What time will it be on a 12-hour clock, 19 hours after 5:00?

When all questions are completed correctly, the clue is unlocked.

Clue 2 Unlocked: _____

Station 3: Classical Cipher

Instructions:

Decode the messages using the given cipher. The shift/key may come from Station 2.

Problems:

1. Caesar cipher (+4): MATH → _____
2. Substitution cipher: A → Q, B → R, … Encode CAT → _____

When all questions are completed correctly, the clue is unlocked.

Clue 3 Unlocked: _____

Station 4: Mini Diffie-Hellman

Instructions:

Simulate a Diffie-Hellman key exchange with your partner. This group needs to split into two (A and B) to mimic secret information exchange.

Base = 5, modulus = 14

- Students in Group A choose a secret exponent "a."
- Students in Group A compute (with calculator help): 5^a mod 14
- Students in Group B choose a partner exponent "b."
- Students in Group B "send" 5^b mod 14
- Both groups compute a shared secret $(5^b)^a = (5^a)^b$ mod 14

Once the groups have completed the calculations correctly and arrive at the same answer, this key exchange is complete, and the clue is unlocked.

Clue 4 Unlocked: _____

Station 5: Mini RSA

Instructions:

Decrypt the encrypted letter using RSA.

- Encrypted letter: E (hint: what is the number corresponding to E in the alphabet?)
- Decrypt using d = 43 and n = 77
- Set up the RSA decryption formula: (letter number) = (encrypted number)d mod n
- This is a tricky calculation! Set up the formula up correctly, and your teacher will give you the letter number.
- Find the corresponding (decrypted) letter.

Clue 5 Unlocked: _____

Final Code

Combine all 5 clues in order to unlock the vault:

Vault Code: _____

Teacher Notes and Answer Key

- Teams can rotate stations or do them in sequence.
- Encourage teamwork: different students may contribute strengths in calculations, modular arithmetic, or ciphers.
- Optional: use a timer to add urgency.
- If teams get stuck, give hint cards with partial calculations or cipher keys.

Station 1

GCF(24, 36) = 12

LCM(8, 12) = 24

Prime factorization of 60 = 2 × 2 × 3 × 5

Clue: PRIMES

Station 2

37 mod 6 = 1

143 mod 18 = 17

19 hours after 5:00 on a 12-hour clock → 12:00

Clue: KEEP

Station 3

Caesar shift +4: MATH → QEXL

Substitution: CAT → QRS

Clue: SECRETS

Station 4

Base 5, modulus 14

Example: secret a = 3, secret b = 4

Compute: 5^3 mod 14 = 125 mod 14 = 13

Teacher's value: 5^4 mod 14 = 625 mod 14 = 9

Shared secret: 9^3 mod 14 = 729 mod 14 = 1 (or similar)

Clue: SAFE

Station 5

Encrypted letter: E → number = 5 (A=1, B=2, …, L=12)

Private key: d=43

Modulus: n=77

Decryption formula: (letter number) = (encrypted number)^d mod n = 5^{43} mod 77

5^{43} mod 77 = 26 → Z

Clue: TODAY

Final Code

Combine clues in order:

PRIMES KEEP SECRETS SAFE TODAY

Appendix A.

Check Your Understanding Reflection Journal

Please see the next page for a sample reflection journal for students to complete after each assessment.

Name: _____ Date: _____

Check Your Understanding Reflection Journal

Purpose: After grading your own assessment, use this journal to reflect on your understanding, notice patterns in mistakes, and record strategies for improvement. These notes are yours. They can help you study, remember concepts, and approach future problems with confidence.

Overall Score (self-graded): _____ / _____

Journal

Put each question you got wrong or found difficult in this table and write some notes down that can help you in the future.

- What mistake did I make?
- How can I correct it if I see a similar problem again?
- If I see this type of problem again, here is what I will do differently.

Question	Notes

Question	Notes

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