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Citation

Frenkel, Edward, and Dennis Gaitsgory. 2010. "Weyl Modules and Opers without Monodromy." In *Arithmetic and geometry around quantization*, ed. O. Ceyhan, Y. I. Manin, M. Marcolli, Vol. 279, *Progress in Mathematics*, 101-121. Cambridge, MA: Birkhauser Boston. doi:10.1007/978-0-8176-4831-2_5.

Published version

https://doi.org/10.1007/978-0-8176-4831-2_5

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WEYL MODULES AND OPERS WITHOUT MONODROMY

EDWARD FRENKEL AND DENNIS GAITSGORY

ABSTRACT. We prove that the algebra of endomorphisms of a Weyl module of critical level is isomorphic to the algebra of functions on the space of monodromy-free opers on the disc with regular singularity and residue determined by the highest weight of the Weyl module. This result may be used to test the local geometric Langlands correspondence proposed in our earlier work.

1. INTRODUCTION

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. For an invariant inner product κ on $\mathfrak{g}$ (which is unique up to a scalar) define the central extension $\widehat{\mathfrak{g}}_\kappa$ of the formal loop algebra $\mathfrak{g} \otimes \mathbb{C}((t))$ which fits into the short exact sequence

$$0 \rightarrow \mathbb{C}\mathbf{1} \rightarrow \widehat{\mathfrak{g}}_\kappa \rightarrow \mathfrak{g} \otimes \mathbb{C}((t)) \rightarrow 0.$$

This sequence is split as a vector space, and the commutation relations read

$$(1.1) \quad [x \otimes f(t), y \otimes g(t)] = [x, y] \otimes f(t)g(t) - (\kappa(x, y) \operatorname{Res} fdg)\mathbf{1},$$

and $\mathbf{1}$ is a central element. The Lie algebra $\widehat{\mathfrak{g}}_\kappa$ is the *affine Kac–Moody algebra* associated to κ . We will denote by $\widehat{\mathfrak{g}}_\kappa\text{-mod}$ the category of *discrete* representations of $\widehat{\mathfrak{g}}_\kappa$ (i.e., such that any vector is annihilated by $\mathfrak{g} \otimes t^n \mathbb{C}[[t]]$ for sufficiently large n), on which $\mathbf{1}$ acts as the identity.

Let $U_\kappa(\widehat{\mathfrak{g}})$ be the quotient of the universal enveloping algebra $U(\widehat{\mathfrak{g}}_\kappa)$ of $\widehat{\mathfrak{g}}_\kappa$ by the ideal generated by $(\mathbf{1} - 1)$. Define its completion $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ as follows:

$$\widetilde{U}_\kappa(\widehat{\mathfrak{g}}) = \varprojlim U_\kappa(\widehat{\mathfrak{g}})/U_\kappa(\widehat{\mathfrak{g}}) \cdot (\mathfrak{g} \otimes t^n \mathbb{C}[[t]]).$$

It is clear that $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$ is a topological algebra, whose discrete continuous representations are the same as objects of $\widehat{\mathfrak{g}}_\kappa\text{-mod}$.

Let κ_{crit} be the *critical* inner product on $\mathfrak{g}$ defined by the formula

$$\kappa_{\text{crit}}(x, y) = -\frac{1}{2} \operatorname{Tr}(\operatorname{ad}(x) \circ \operatorname{ad}(y)).$$

In what follows we will use the subscript “crit” instead of κ_{crit} .

Let $\check{G}$ be the group of adjoint type whose Lie algebra $\check{\mathfrak{g}}$ is Langlands dual to $\mathfrak{g}$ (i.e., the Cartan matrix of $\check{\mathfrak{g}}$ is the transpose of that of $\mathfrak{g}$).

Date: June 2007.

The research of E.F. was supported by the DARPA Program “Focus Areas in Fundamental Mathematics”.

Let $\mathfrak{Z}_{\mathfrak{g}}$ be the center of $\tilde{U}_{\text{crit}}(\widehat{\mathfrak{g}})$. According to a theorem of [FF, F2], $\mathfrak{Z}_{\mathfrak{g}}$ is isomorphic to the algebra $\text{Fun Op}_{\mathfrak{g}}(\mathcal{D}^{\times})$ of functions on the space of $\mathfrak{g}$ -opers on the punctured disc $\mathcal{D}^{\times} = \text{Spec}(\mathbb{C}((t)))$ (see [BD1, FG2] and Sect. 2 for the definition of opers).

It is interesting to understand how $\mathfrak{Z}_{\mathfrak{g}}$ acts on various $\widehat{\mathfrak{g}}_{\text{crit}}$ -modules. The standard modules are the Verma modules and the Weyl modules. They are obtained by applying the induction functor

$$\begin{aligned} \text{Ind} : \mathfrak{g}\text{-mod} &\rightarrow \widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}, \\ M &\mapsto U(\widehat{\mathfrak{g}}_{\text{crit}}) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbf{1})} M, \end{aligned}$$

where $\mathfrak{g}[[t]]$ acts on M via the projection $\mathfrak{g}[[t]] \rightarrow \mathfrak{g}$ and $\mathbf{1}$ acts as the identity.

For $\lambda \in \mathfrak{h}^*$ let M_{λ} be the Verma module over $\mathfrak{g}$ with highest weight λ . The corresponding $\widehat{\mathfrak{g}}_{\text{crit}}$ -module $\mathbb{M}_{\lambda} = \text{Ind}(M_{\lambda})$ is the *Verma module* of critical level with highest weight λ .

For a dominant integral weight λ let V_{λ} be the irreducible finite-dimensional $\mathfrak{g}$ -module with highest weight λ . The corresponding $\widehat{\mathfrak{g}}_{\text{crit}}$ -module $\mathbb{V}_{\lambda} = \text{Ind}(V_{\lambda})$ is the *Weyl module* of critical level with highest weight λ . The module $\mathbb{V}_0 = \text{Ind}(\mathbb{C}_0)$ is also called the vacuum module.

It was proved in [FF, F2] that the algebra of $\widehat{\mathfrak{g}}_{\text{crit}}$ -endomorphisms of $\mathbb{V}_0$ is isomorphic to the algebra $\text{Fun Op}_{\mathfrak{g}}^{\text{reg}}$ of functions on the space $\text{Op}_{\mathfrak{g}}^{\text{reg}}$ of $\mathfrak{g}$ -opers on the disc $\mathcal{D} = \text{Spec}(\mathbb{C}[[t]])$. Moreover, there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{Z}_{\mathfrak{g}} & \xrightarrow{\sim} & \text{Fun Op}_{\mathfrak{g}}(\mathcal{D}^{\times}) \\ \downarrow & & \downarrow \\ \text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}}(\mathbb{V}_0) & \xrightarrow{\sim} & \text{Fun Op}_{\mathfrak{g}}^{\text{reg}} \end{array}$$

We have shown in [FG2], Corollary 13.3.2, that a similar result holds for the Verma modules as well: the algebra of $\widehat{\mathfrak{g}}_{\text{crit}}$ -endomorphisms of $\mathbb{M}_{\lambda}$ is isomorphic to $\text{Fun Op}_{\mathfrak{g}}^{\text{RS}, \varpi(-\lambda-\rho)}$, where $\text{Op}_{\mathfrak{g}}^{\text{RS}, \varpi(-\lambda-\rho)}$ is the space of $\mathfrak{g}$ -opers on $\mathcal{D}^{\times}$ with regular singularity and residue $\varpi(-\lambda-\rho)$, ϖ being the natural projection $\mathfrak{h}^* \rightarrow \text{Spec Fun}(\mathfrak{h}^*)^W$ (see [FG2], Sect. 2.4, for a precise definition). In addition, there is an analogue of the above commutative diagram for Verma modules.

In this paper we consider the Weyl modules $\mathbb{V}_{\lambda}$. In [FG2], Sect. 2.9, we defined the subspace $\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}} \subset \text{Op}_{\mathfrak{g}}^{\text{RS}, \varpi(-\lambda-\rho)}$ of λ -regular opers (we recall this definition below). Its points are those opers in $\text{Op}_{\mathfrak{g}}^{\text{RS}, \varpi(-\lambda-\rho)}$ which have trivial monodromy and are therefore $\check{G}(\hat{\mathcal{K}})$ gauge equivalent to the trivial local system on $\mathcal{D}^{\times}$. In particular, $\text{Op}_{\mathfrak{g}}^{0, \text{reg}} = \text{Op}_{\mathfrak{g}}^{\text{reg}}$. According to Lemma 1 below, the disjoint union of $\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}}$, where λ runs over the set P^+ of dominant integral weights of $\mathfrak{g}$, is precisely the locus of $\mathfrak{g}$ -opers on $\mathcal{D}^{\times}$ with trivial monodromy. The main result of this paper is the following theorem, which generalizes the description of $\text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{V}_0$ from [FF, F2] to the case of an arbitrary dominant integral weight λ .

Theorem 1. *For any dominant integral weight λ the center $\mathfrak{Z}_{\mathfrak{g}}$ maps surjectively onto $\text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{V}_{\lambda}$, and we have the following commutative diagram*

$$(1.2) \quad \begin{array}{ccc} \mathfrak{Z}_{\mathfrak{g}} & \xrightarrow{\sim} & \text{Fun Op}_{\check{G}}(\mathcal{D}^{\times}) \\ \downarrow & & \downarrow \\ \text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{V}_{\lambda} & \xrightarrow{\sim} & \text{Fun Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{reg}} \end{array}$$

For $\mathfrak{g} = \mathfrak{sl}_2$ this follows from Prop. 1 of [F1]. This statement was also independently conjectured by A. Beilinson and V. Drinfeld (unpublished).

In addition, we prove that $\mathbb{V}_{\lambda}$ is a free module over $\text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{V}_{\lambda} \simeq \text{Fun Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{reg}}$.

Theorem 1 has important consequences for the local geometric Langlands correspondence proposed in [FG2]. According to our proposal, to each “local Langlands parameter” σ , which is a $\check{G}$ -local system on the punctured disc $\mathcal{D}^{\times}$ (or equivalently, a $\check{G}$ -bundle with a connection on $\mathcal{D}^{\times}$), there should correspond a category $\mathcal{C}_{\sigma}$ equipped with an action of $G((t))$.

Now let χ be a fixed $\check{\mathfrak{g}}$ -oper on $\mathcal{D}^{\times}$, which we regard as a character of the center $\mathfrak{Z}_{\mathfrak{g}}$. Consider the full subcategory $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\chi}$ of the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}$ whose objects are $\widehat{\mathfrak{g}}_{\text{crit}}$ -modules, on which the $\mathfrak{Z}_{\mathfrak{g}}$ acts according to this character. This category carries a canonical action of the ind-group $G((t))$ via its adjoint action on $\widehat{\mathfrak{g}}_{\text{crit}}$. We proposed in [FG2] that $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\chi}$ should be equivalent to the sought-after category $\mathcal{C}_{\sigma}$, where σ is the $\check{G}$ -local system underlying the oper χ . This entails a far-reaching corollary that the categories $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\chi_1}$ and $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\chi_2}$ for two different opers χ_1 and χ_2 are equivalent if the underlying local systems of χ_1 and χ_2 are isomorphic to each other.

In particular, consider the simplest case when σ is the trivial local system. Then, by Lemma 1, χ must be a point of $\text{Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{reg}}$ for some $\lambda \in P^{+}$. Theorem 1 implies that the quotient $\mathbb{V}_{\lambda}(\chi)$ of the Weyl module $\mathbb{V}_{\lambda}$ by the central character corresponding to χ is non-zero. Therefore $\mathbb{V}_{\lambda}(\chi)$ is a non-trivial object of $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\chi}$ and also of the corresponding $G[[t]]$ -equivariant category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\chi}^{G[[t]]}$. In the case when $\lambda = 0$ we have proved in [FG1] that this is a unique irreducible object of $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\chi}^{G[[t]]}$ and that this category is in fact equivalent to the category of vector spaces. Therefore we expect the same to be true for all other values of λ . This will be proved in a follow-up paper.

The paper is organized as follows. In Sect. 2 we recall the relevant notions of opers, Cartan connections and Miura transformation. In Sect. 3 we explain the strategy of the proof of the main result, Theorem 1, and reduce it to two statements, Theorem 2 and Proposition 1. We then prove Proposition 1 assuming Theorem 2 in Sect. 4. Our argument is based on the exactness of the functor of quantum Drinfeld–Sokolov reduction, which we derive from [Ar]. In Sect. 5 we compute the characters of the algebra of functions on $\text{Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{reg}}$ and of the semi-infinite cohomology of $\mathbb{V}_{\lambda}$ (they turn out to be the same). We then give two different proofs of Theorem 2 in Sect. 6. This completes the proof of the main result. We also show that the natural map from the Weyl module $\mathbb{V}_{\lambda}$ to the corresponding Wakimoto module is injective and that $\mathbb{V}_{\lambda}$ is a free module over its endomorphism algebra.

2. SOME RESULTS ON OPERS

In this section we recall the relevant notions of opers, Cartan connections and Miura transformation, following [BD1, F2, FG2], where we refer the reader for more details.

Let $\mathfrak{g}$ be a simple Lie algebra and G the corresponding algebraic group of adjoint type. Let B be its Borel subgroup and $N = [B, B]$ its unipotent radical, with the corresponding Lie algebras $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$.

Let X be a smooth curve, or the disc $\mathcal{D} = \text{Spec}(\hat{\mathcal{O}})$, where $\hat{\mathcal{O}}$ is a one-dimensional smooth complete local ring, or the punctured disc $\mathcal{D}^\times = \text{Spec}(\hat{\mathcal{K}})$, where $\hat{\mathcal{K}}$ is the field of fractions of $\hat{\mathcal{O}}$.

Following Beilinson and Drinfeld (see [BD1], Sect. 3.1, and [BD2]), one defines a $\mathfrak{g}$ -oper on X to be a triple $(\mathcal{F}_G, \nabla, \mathcal{F}_B)$, where $\mathcal{F}_G$ is a principal G -bundle $\mathcal{F}_G$ on X , ∇ is a connection on $\mathcal{F}_G$, and $\mathcal{F}_B$ is a B -reduction of $\mathcal{F}_G$ which is transversal to ∇ , in the sense explained in the above references and in [FG2], Sect. 1.1. We note that the transversality condition allows us to identify canonically the B -bundles $\mathcal{F}_B$ underlying all opers.

More concretely, opers on the punctured disc $\mathcal{D}^\times$ may be described as follows. Let us choose a trivialization of $\mathcal{F}_B$ and a coordinate t on the disc $\mathcal{D}$ such that $\hat{\mathcal{O}} = \mathbb{C}[[t]]$ and $\hat{\mathcal{K}} = \mathbb{C}((t))$. Let us choose a nilpotent subalgebra $\mathfrak{n}_-$, which is in generic position with $\mathfrak{b}$ and a set of simple root generators $f_i, i \in I$, of $\mathfrak{n}_-$. Then a $\mathfrak{g}$ -oper on $\mathcal{D}^\times$ is, by definition, an equivalence class of operators of the form

$$(2.1) \quad \nabla = \partial_t + \sum_{i \in I} f_i + \mathbf{v}, \quad \mathbf{v} \in \mathfrak{b}(\hat{\mathcal{K}}),$$

with respect to the action of the group $N(\hat{\mathcal{K}})$ by gauge transformations. It is known that this action is free and the resulting set of equivalence classes is in bijection with $\hat{\mathcal{K}}^{\oplus \ell}$, where $\ell = \text{rank}(\mathfrak{g})$. Opers may be defined in this way over any base, and this allows us to define the ind-affine scheme $\text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)$ of $\mathfrak{g}$ -opers on $\mathcal{D}^\times$ (it is isomorphic to an inductive limit of affine spaces).

Let $\check{P}^+$ be the set of dominant integral coweights of $\mathfrak{g}$. For $\check{\lambda} \in \check{P}^+$ we define a $\mathfrak{g}$ -oper with $\check{\lambda}$ -nilpotent singularity as an equivalence class of operators

$$(2.2) \quad \nabla = \partial_t + \sum_{i \in I} t^{\langle \alpha_i, \check{\lambda} \rangle} \cdot f_i + \mathbf{v}(t) + \frac{v}{t}, \quad \mathbf{v}(t) \in \mathfrak{b}(\hat{\mathcal{O}}), v \in \mathfrak{n},$$

with respect to the action of the group $N(\hat{\mathcal{O}})$ by gauge transformations (see [FG2], Sect. 2.9). The corresponding scheme is denoted by $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{nilp}}$. According to Theorem 2.9.1 of [FG2], the natural map $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{nilp}} \rightarrow \text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)$ is injective, and its image is equal to the space of $\mathfrak{g}$ -opers with regular singularity and residue $\varpi(-\check{\lambda} - \check{\rho})$ (here $\check{\rho}$ is the half-sum of positive coroots of $\mathfrak{g}$ and ϖ is the projection $\mathfrak{h} \rightarrow \text{Spec Fun}(\mathfrak{h})^W$).

Now we define the space $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ of $\check{\lambda}$ -regular opers as the subscheme of $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{nilp}}$ corresponding to those operators (2.2) which satisfy $v = 0$ (so that ∇ is regular at $t = 0$). In particular, if $\check{\lambda} = 0$, then $\text{Op}_{\mathfrak{g}}^{0, \text{reg}}$ is just the space of regular $\mathfrak{g}$ -opers on the disc $\mathcal{D}$. The geometric significance of $\check{\lambda}$ -opers is explained by the following

Lemma 1. *Suppose that a $\mathfrak{g}$ -oper $\chi = (\mathcal{F}_G, \nabla, \mathcal{F}_B)$ on $\mathcal{D}^\times$ is such that the corresponding G -local system is trivial (in other words, the corresponding operator (2.1) is in the $G(\hat{\mathcal{K}})$ gauge equivalence class of $\nabla_0 = \partial_t$). Then $\chi \in \text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ for some $\check{\lambda} \in \check{P}^+$.*

Proof. It is clear from the definition that any oper in $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ is regular on the disc $\mathcal{D}$ and is therefore $G(\hat{\mathcal{K}})$ gauge equivalent to the trivial connection $\nabla_0 = \partial_t$.

Now suppose that we have an oper $\chi = (\mathcal{F}_G, \nabla, \mathcal{F}_B)$ on $\mathcal{D}^\times$ such that the corresponding G -local system is trivial. Then ∇ is $G(\hat{\mathcal{K}})$ gauge equivalent to a regular connection on $\mathcal{D}$. We have the decomposition $G(\hat{\mathcal{K}}) = G(\hat{\mathcal{O}})B(\hat{\mathcal{K}})$. The gauge action of $G(\hat{\mathcal{O}})$ preserves the space of regular connections (in fact, it acts transitively on it). Therefore if an oper connection ∇ is gauge equivalent to a regular connection under the action of $G(\hat{\mathcal{K}})$, then its $B(\hat{\mathcal{K}})$ gauge equivalence class must contain a regular connection. The oper condition then implies that this gauge class contains a connection operator of the form (2.2) with $\mathbf{v}(0) = 0$, for some dominant integral coweight $\check{\lambda}$. Therefore $\chi \in \text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$. $\square$

Let us choose a coordinate t on $\mathcal{D}$. The vector field $L_0 = -t\partial_t$ then acts naturally on $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ and defines a $\mathbb{Z}$ -grading on the algebra of functions on it. In Sect. 5 we will compute the character of the algebra $\text{Fun Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ of functions on $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ with respect to this grading.

Next, we introduce the space of H -connections and the Miura transformation.

Let X be as above. Denote by ω_X the $\mathbb{C}^\times$ -torsor corresponding to the canonical line bundle on X . Let $\omega_X^\check{\rho}$ be the push-forward of ω_X to an H -torsor via the homomorphism $\check{\rho} : \mathbb{C}^\times \rightarrow H$. We denote by $\text{Conn}_H(\omega_X^\check{\rho})$ the affine space of all connections on $\omega_X^\check{\rho}$. In particular, $\text{Conn}_H(\omega_{\mathcal{D}^\times}^\check{\rho})$ is an inductive limit of affine spaces $\text{Conn}_H(\omega_{\mathcal{D}}^\check{\rho})^{\text{ord}_k}$ of connections with pole of order $\leq k$. We will use the notation $\text{Conn}_H(\omega_{\mathcal{D}}^\check{\rho})^{\text{RS}}$ for $\text{Conn}_H(\omega_{\mathcal{D}}^\check{\rho})^{\text{ord}_1}$. A connection $\bar{\nabla} \in \text{Conn}_H(\omega_{\mathcal{D}}^\check{\rho})^{\text{RS}}$ has a well-defined residue, which is an element of $\mathfrak{h}$. For $\check{\mu} \in \mathfrak{h}$ we denote by $\text{Conn}_H(\omega_{\mathcal{D}}^\check{\rho})^{\text{RS}, \check{\mu}}$ the subspace of $\text{Conn}_H(\omega_{\mathcal{D}}^\check{\rho})^{\text{RS}}$ consisting of connections with residue $\check{\mu}$.

The *Miura transformation* is a morphism

$$(2.3) \quad \text{MT} : \text{Conn}_H(\omega_{\mathcal{D}^\times}^\check{\rho}) \rightarrow \text{Op}_{\mathfrak{g}}(\mathcal{D}^\times),$$

introduced in [DS] (see also [F2] and [FG2], Sect. 3.3). It can be described as follows. If we choose a coordinate t on $\mathcal{D}$, then we trivialize $\omega_{\mathcal{D}}$ and hence $\omega_{\mathcal{D}}^\check{\rho}$. A point of $\text{Conn}_H(\omega_{\mathcal{D}^\times}^\check{\rho})$ is then represented by an operator

$$\bar{\nabla} = \partial_t + \mathbf{u}(t), \quad \mathbf{u}(t) \in \mathfrak{h}(\hat{\mathcal{K}}).$$

We associate to $\bar{\nabla}$ the $\mathfrak{g}$ -oper which is the $N(\hat{\mathcal{K}})$ gauge equivalence class of the operator

$$\nabla = \partial_t + \sum_{i \in I} f_i + \mathbf{u}(t).$$

The following result is a corollary of Proposition 3.5.4 of [FG2]. Let $\mathcal{F}_{B,0}$ be the fiber at $0 \in \mathcal{D}$ of the B -bundle $\mathcal{F}_B$ underlying all $\mathfrak{g}$ -opers. We denote by $N_{\mathcal{F}_{B,0}}$ the $\mathcal{F}_{B,0}$ -twist of N .

Lemma 2. *Let $\check{\lambda}$ be a dominant integral coweight of $\mathfrak{g}$. The image of $\text{Conn}_H(\omega_{\mathcal{D}}^{\check{\rho}})^{\text{RS}, -\check{\lambda}}$ in $\text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)$ under the Miura transformation is equal to $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$. Moreover, the map $\text{Conn}_H(\omega_{\mathcal{D}}^{\check{\rho}})^{\text{RS}, -\check{\lambda}} \rightarrow \text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ is a principal $N_{\mathcal{F}_{B,0}}$ -bundle over $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$.*

In particular, this implies that the scheme $\text{Op}_{\mathfrak{g}}^{\check{\lambda}, \text{reg}}$ is smooth and in fact isomorphic to an infinite-dimensional (pro)affine space.

3. PROOF OF THE MAIN THEOREM

Our strategy of the proof of Theorem 1 will be as follows: we will first construct natural maps

$$(3.1) \quad \text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{M}_\lambda \rightarrow \text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{V}_\lambda \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0).$$

We already know from [FG2] that

$$\text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{M}_\lambda \simeq \text{Fun Op}_{\mathfrak{g}}^{\lambda, \text{nilp}}.$$

We will show that the corresponding composition

$$\text{Fun Op}_{\mathfrak{g}}^{\lambda, \text{nilp}} \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$$

factors as follows:

$$\text{Fun Op}_{\mathfrak{g}}^{\lambda, \text{nilp}} \twoheadrightarrow \text{Fun Op}_{\mathfrak{g}}^{\lambda, \text{reg}} \xrightarrow{\sim} H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0),$$

and that the map

$$\mathbb{V}_\lambda \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$$

is injective. This will imply Theorem 1.

As a byproduct, we will obtain an isomorphism

$$H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0) \simeq \text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{V}_\lambda$$

and find that the first map in (3.1) is surjective.

3.1. Homomorphisms of $\widehat{\mathfrak{g}}_{\text{crit}}$ -modules. Let us now proceed with the proof and construct the maps (3.1).

Note that a $\widehat{\mathfrak{g}}_{\text{crit}}$ -endomorphism of $\mathbb{M}_\lambda$ is uniquely determined by the image of the highest weight vector, which must be a vector in $\mathbb{M}_\lambda$ of weight λ annihilated by the Lie subalgebra

$$\widehat{\mathfrak{n}}_+ = (\mathfrak{n}_+ \otimes 1) \oplus (\mathfrak{g} \otimes t\mathbb{C}[[t]]).$$

This is the Lie algebra of the prounipotent proalgebraic group $I^0 = [I, I]$, where I is the Iwahori subgroup of $G((t))$. For a $\widehat{\mathfrak{g}}_{\text{crit}}$ -module M we denote the space of such vectors by $M_\lambda^{\widehat{\mathfrak{n}}_+}$.

According Corollary 13.3.2 of [FG2], we have

$$\text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{M}_\lambda = (\mathbb{M}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \simeq \text{Fun Op}_{\mathfrak{g}}^{\lambda, \text{nilp}} = \text{Fun Op}_{\check{G}}^{\text{RS}, \varpi(-\lambda-\rho)}$$

(see Sect. 2).

Likewise, any endomorphism of $\mathbb{V}_\lambda$ is uniquely determined by the image of the generating subspace V_λ . This subspace therefore defines a $\mathfrak{g}[[t]]$ -invariant vector in $(\mathbb{V}_\lambda \otimes V_\lambda^*)^{\mathfrak{g}[[t]]}$. Note that for any $\mathfrak{g}$ -integrable module M we have an isomorphism

$$(M \otimes V_\lambda^*)^{\mathfrak{g}[[t]]} \simeq M_\lambda^{\widehat{\mathfrak{n}}_+}.$$

Therefore we have

$$(3.2) \quad \text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{V}_\lambda = (\mathbb{V}_\lambda \otimes V_\lambda^*)^{\mathfrak{g}[[t]]} = (\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+}.$$

The canonical surjective homomorphism

$$\mathbb{M}_\lambda \twoheadrightarrow \mathbb{V}_\lambda$$

of $\widehat{\mathfrak{g}}_{\text{crit}}$ -modules gives rise to a map $(\mathbb{M}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow (\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+}$. We obtain the following commutative diagram:

$$(3.3) \quad \begin{array}{ccccc} \mathfrak{Z}_{\mathfrak{g}} & \longrightarrow & \text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{M}_\lambda & \longrightarrow & \text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{V}_\lambda \\ \downarrow & & \downarrow \sim & & \downarrow \sim \\ \text{Fun Op}_{\widehat{\mathfrak{g}}}(\mathcal{D}^\times) & \longrightarrow & \text{Fun Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{nilp}} & \longrightarrow & ? \end{array}$$

3.2. The functor of semi-infinite cohomology. Define the character

$$(3.4) \quad \Psi_0 : \mathfrak{n}_+((t)) \rightarrow \mathbb{C}$$

by the formula

$$\Psi_0(e_{\alpha, n}) = \begin{cases} 1, & \text{if } \alpha = \alpha_i, n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

We have the functor of semi-infinite cohomology (the + quantum Drinfeld–Sokolov reduction) from the category of $\widehat{\mathfrak{g}}_{\text{crit}}$ -modules to the category of graded vector spaces,

$$(3.5) \quad M \mapsto H^{\frac{\infty}{2} + \bullet}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], M \otimes \Psi_0),$$

introduced in [FF, FKW] (see also [FB], Ch. 15, and [FG2], Sect. 18; we follow the notation of the latter).

Let M be a $\widehat{\mathfrak{g}}_{\text{crit}}$ -module. Consider the space $M_\lambda^{\widehat{\mathfrak{n}}_+}$ of $\widehat{\mathfrak{n}}_+$ -invariant vectors in M of highest weight λ .

Lemma 3. *We have functorial maps*

$$(3.6) \quad M_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], M \otimes \Psi_0).$$

Proof. Consider the complex $C^\bullet(M)$ computing the above semi-infinite cohomology (see, e.g., [FB], Sect. 15.2). It follows from the definition that the standard Chevalley complex computing the cohomology of the Lie algebra $\mathfrak{n}_+[[t]]$ with coefficients in M embeds into $C^\bullet(M)$. Therefore we obtain functorial maps

$$M_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow M^{\mathfrak{n}_+[[t]]} \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0).$$

□

Introduce the notation

$$\begin{aligned}\mathfrak{z}^{\lambda, \text{nilp}} &= \text{Fun Op}_{\hat{\mathfrak{g}}}^{\lambda, \text{nilp}}, \\ \mathfrak{z}^{\lambda, \text{reg}} &= \text{Fun Op}_{\hat{\mathfrak{g}}}^{\lambda, \text{reg}}\end{aligned}$$

Our goal is to prove that

$$(\mathbb{V}_\lambda)_\lambda^{\hat{\mathfrak{n}}_+} \simeq \mathfrak{z}^{\lambda, \text{reg}}.$$

The proof will be based on analyzing the composition

$$(3.7) \quad \mathfrak{z}^{\lambda, \text{nilp}} \rightarrow (\mathbb{V}_\lambda)_\lambda^{\hat{\mathfrak{n}}_+} \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0),$$

where the first map is obtained from the diagram (3.3), and the second map from Lemma 3. We will use the following two results.

Theorem 2. *The composition (3.7) factors as*

$$(3.8) \quad \mathfrak{z}^{\lambda, \text{nilp}} \twoheadrightarrow \mathfrak{z}^{\lambda, \text{reg}} \simeq H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0).$$

Proposition 1. *The map*

$$(3.9) \quad (\mathbb{V}_\lambda)_\lambda^{\hat{\mathfrak{n}}_+} \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$$

is injective.

Assuming these two assertions, we can now prove our main result.

Proof of Theorem 1. By Theorem 2, we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{z}^{\text{nilp}, \lambda} & \longrightarrow & (\mathbb{V}^\lambda)_\lambda^{\mathfrak{n}_+} \\ \downarrow & & \downarrow \\ \mathfrak{z}^{\text{reg}, \lambda} & \xrightarrow{\sim} & H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0) \end{array}$$

The left vertical arrow is surjective, and the right vertical arrow is injective by Proposition 1. This readily implies that we have an isomorphism

$$(\mathbb{V}^\lambda)_\lambda^{\mathfrak{n}_+} \simeq \mathfrak{z}^{\text{reg}, \lambda}.$$

The assertion of Theorem 1 follows from this, the isomorphism (3.2), and the commutative diagram (3.3). $\square$

The rest of this paper is devoted to the proof of Theorem 2 and Proposition 1.

4. EXACTNESS

In this section we prove Proposition 1 assuming Theorem 2. The proof will rely on some properties of the semi-infinite cohomology functors.

Let $\hat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I, \mathbb{Z}}$ be the category of $\hat{\mathfrak{g}}_{\text{crit}}$ -modules which are equivariant with respect to the Iwahori subgroup $I \subset G((t))$ and equipped with a $\mathbb{Z}$ -grading with respect to the operator $L_0 = -t\partial_t$ which commutes with $\hat{\mathfrak{g}}_{\text{crit}}$ in the natural way. Since $I = H \ltimes I^0$, where $I^0 = [I, I]$, the first condition means that $\hat{\mathfrak{n}}_+ = \text{Lie}(I^0)$ acts locally nilpotently, and the constant Cartan subalgebra $\text{Lie}(H) = \mathfrak{h} \otimes 1 \subset \mathfrak{g} \otimes 1 \subset \hat{\mathfrak{g}}_{\text{crit}}$ acts semi-simply with eigenvalues corresponding to integral weights. The second condition means that we have an action of the extended affine Kac–Moody algebra $\hat{\mathfrak{g}}'_\kappa = \mathbb{C}L_0 \ltimes \hat{\mathfrak{g}}_\kappa$. The

category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I, \mathbb{Z}}$ is therefore the product of the blocks of the usual category $\mathcal{O}_{-h^\vee}$ of modules over the extended affine Kac–Moody algebra at the critical level $k = -h^\vee$, corresponding to the (finite) Weyl group orbits in the set of integral weights.

Let $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]], \mathbb{Z}}$ be the category of $\widehat{\mathfrak{g}}_{\text{crit}}$ -modules which are equivariant with respect to the subgroup $G[[t]]$ and equipped with a $\mathbb{Z}$ -grading with respect to the operator $L_0 = -t\partial_t$. This is the full subcategory of the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I, \mathbb{Z}}$, whose objects are modules integrable with respect to the constant subalgebra $\mathfrak{g} = \mathfrak{g} \otimes 1 \subset \widehat{\mathfrak{g}}_{\text{crit}}$.

We define $\mathbb{Z}$ -gradings on the modules $\mathbb{M}_\lambda$ and $\mathbb{V}_\lambda$ in the standard way, by setting the degrees of the generating vectors to be equal to 0 and using the commutation relations of L_0 and $\widehat{\mathfrak{g}}_{\text{crit}}$ to define the grading on the entire modules. Thus, $\mathbb{M}_\lambda$ and $\mathbb{V}_\lambda$ become objects of the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I, \mathbb{Z}}$, and $\mathbb{V}_\lambda$ also an object of the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]], \mathbb{Z}}$. Moreover, the homomorphism $\mathbb{M}_\lambda \rightarrow \mathbb{V}_\lambda$, and therefore the map $(\mathbb{M}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow (\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+}$, preserve these gradings.

We will now derive Proposition 1 from Theorem 2 and the following statement.

Proposition 2. *The functor (3.5) is right exact on the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I, \mathbb{Z}}$ and is exact on the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]], \mathbb{Z}}$.*

Introduce a $\mathbb{Z}$ -grading operator on the standard complex of semi-infinite cohomology $H^{\frac{\infty}{2} + \bullet}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], M \otimes \Psi_0)$, where $M = \mathbb{M}_\lambda$ or $\mathbb{V}_\lambda$, by the formula

$$L_0 - \check{\rho} \otimes 1 + \langle \lambda, \check{\rho} \rangle.$$

Here L_0 is the natural grading operator and $\check{\rho} \in \mathfrak{h}$ is such that $\langle \alpha_i, \check{\rho} \rangle = 1$ for all $i \in I$. In the same way as in [FB], Sect. 15.1.8, we check that this $\mathbb{Z}$ -grading operator commutes with the differential of the complex and hence induces a $\mathbb{Z}$ -grading operator on the cohomology. Observe that $\check{\rho} \otimes 1$ acts by multiplication by $\langle \lambda, \check{\rho} \rangle$ on any element in $(\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+}$. Therefore the map (3.9) preserves $\mathbb{Z}$ -gradings.

Proof of Proposition 1. Let $A \in (\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+}$ be an element in the kernel of the map (3.9). Since this map preserves $\mathbb{Z}$ -gradings, without loss of generality we may, and will, assume that A is homogeneous. Under the identification

$$(\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \simeq \text{End}_{\widehat{\mathfrak{g}}_{\text{crit}}} \mathbb{V}_\lambda$$

it gives rise to a homogeneous $\widehat{\mathfrak{g}}_{\text{crit}}$ -endomorphism E of $\mathbb{V}_\lambda$. Then the induced map $H(E)$ on $H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$ is identically zero. Indeed, the image of the generating vector of $(\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+}$ under the map (3.9) is identified with the element

$$1 \in \mathfrak{z}^{\lambda, \text{reg}} \simeq H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0),$$

where we use the isomorphism of Theorem 2. Therefore the image of this element 1 of $H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$ under $H(E)$ is equal to the image of A in $H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$, which is 0. By Theorem 2, $H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$ is a free $\mathfrak{z}^{\lambda, \text{reg}}$ -module generated by the element 1. Therefore we find that $H(E) \equiv 0$.

Now let M and N be the kernel and cokernel of E ,

$$0 \rightarrow M \rightarrow \mathbb{V}_\lambda \xrightarrow{E} \mathbb{V}_\lambda \rightarrow N \rightarrow 0.$$

Note that both M and N , as well as $\mathbb{V}_\lambda$, are objects of the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]],\mathbb{Z}}$. By Proposition 2, the functor of semi-infinite cohomology is exact on this category. Therefore we obtain an exact sequence

$$0 \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], M \otimes \Psi_0) \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0) \xrightarrow{H(E)} H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0) \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], N \otimes \Psi_0) \rightarrow 0,$$

where the middle map $H(E)$ is equal to zero. If $A \neq 0$, then M is a proper submodule of $\mathbb{V}_\lambda$ which does not contain the generating vector of $\mathbb{V}_\lambda$. We obtain that the values of the $\mathbb{Z}$ -grading on M are strictly greater than those on $\mathbb{V}_\lambda$. Therefore the above sequence cannot be exact. Hence $A = 0$ and we obtain the assertion of the proposition. $\square$

In the rest of this section we prove Proposition 2. Introduce the second semi-infinite cohomology functor (the $-$ quantum Drinfeld–Sokolov reduction of [FKW])

$$(4.1) \quad M \mapsto H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}_-((t)), t\mathfrak{n}_-[[t]], M \otimes \Psi_{-\check{\rho}}),$$

where

$$(4.2) \quad \Psi_{-\check{\rho}}^- : \mathfrak{n}_-((t)) \rightarrow \mathbb{C}$$

is given by the formula

$$\Psi_{-\check{\rho}}^-(f_{\alpha,n}) = \begin{cases} 1, & \text{if } \alpha = \alpha_i, n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following important result due to Arakawa [Ar], Main Theorem 1,(1) (note that the functor (4.1) is the functor $H_-^\bullet$ in the notation [Ar]):

Theorem 3. *The functor (4.1) is exact on the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I,\mathbb{Z}}$.*

We now derive Proposition 2 from Theorem 3.

Proof of Proposition 2. Recall that we have the convolution functors

$$M \mapsto \mathcal{F} \star M$$

on the category of I -equivariant $\widehat{\mathfrak{g}}_{\text{crit}}$ modules, for each I -equivariant right D-module $\mathcal{F}$ on $G((t))/I$ (see [FG2], Sect. 22, for the precise definition).

According to Proposition 18.1.1 of [FG2], we have

$$(4.3) \quad H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], M \otimes \Psi_0) \simeq H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}_-((t)), t\mathfrak{n}_-[[t]], j_{w_0\check{\rho},*} \star M \otimes \Psi_{-\check{\rho}})$$

for any I -equivariant $\widehat{\mathfrak{g}}_{\text{crit}}$ -module M . We recall that the D-module $j_{w_0\check{\rho},*}$ is defined as the $*$ -extension of the “constant” D-module on the I -orbit in the affine flag scheme $G((t))/I$ corresponding to the element $w_0\check{\rho}$ of the affine Weyl group. Hence the functor

$$M \mapsto j_{w_0\check{\rho},*} \star M$$

is right exact. Combining this with Theorem 3, we obtain that the functor

$$M \mapsto H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}_-((t)), t\mathfrak{n}_-[[t]], j_{w_0\check{\rho},*} \star M \otimes \Psi_{-\check{\rho}})$$

is right exact on the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I, \mathbb{Z}}$ (note that the convolution with $j_{w_0\check{\rho},*}$ sends $\mathbb{Z}$ -graded modules to $\mathbb{Z}$ -graded modules). The isomorphism (4.3) then implies that the functor (3.5) is right exact on the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I, \mathbb{Z}}$.

On the other hand, let $j_{w_0\check{\rho},!}$ be the $!$ -extension of the “constant” D-module on the same I -orbit. The convolution functor with $j_{w_0\check{\rho},!}$ is both left and right adjoint to the convolution with $j_{w_0\check{\rho},*}$. Therefore we find that the functor

$$M \mapsto j_{w_0\check{\rho},!} \star M$$

is left exact. Combining this with Theorem 3, we obtain that the functor

$$M \mapsto H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}_-((t)), t\mathfrak{n}_-[[t]], j_{w_0\check{\rho},!} \star M \otimes \Psi_{-\check{\rho}})$$

is left exact on the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I, \mathbb{Z}}$ (again, note that the convolution with $j_{w_0\check{\rho},!}$ sends $\mathbb{Z}$ -graded modules to $\mathbb{Z}$ -graded modules).

Now consider the homomorphism

$$(4.4) \quad j_{w_0\check{\rho},!} \star M \rightarrow j_{w_0\check{\rho},*} \star M$$

of $\widehat{\mathfrak{g}}_{\text{crit}}$ -modules induced by the morphism

$$j_{w_0\check{\rho},!} \rightarrow j_{w_0\check{\rho},*}$$

of D-modules on $G((t))/I$. Suppose in addition that M is $G[[t]]$ -equivariant. Then we have

$$j_{w_0\check{\rho},!} \star M \simeq j_{w_0\check{\rho},!} \star \delta_{1_{Gr_G}} \star_{G[[t]]} M, \quad j_{w_0\check{\rho},*} \star M \simeq j_{w_0\check{\rho},*} \star \delta_{1_{Gr_G}} \star_{G[[t]]} M,$$

where $\delta_{1_{Gr_G}}$ is the “delta-function” D-module on the affine Grassmannian $Gr_G = G((t))/G[[t]]$ supported at the identity coset. It then follows from Lemma 15.1.2 of [FG2] that the kernel and the cokernel of the map (4.4) are partially integrable $\widehat{\mathfrak{g}}_{\text{crit}}$ -modules. We recall from [FG2], Sect. 6.3, that a $\widehat{\mathfrak{g}}_{\text{crit}}$ -module is called partially integrable if it admits a filtration such that each successive quotient is equivariant with respect to the parahoric Lie subalgebra $\mathfrak{p}^\iota = \text{Lie}(I) + \mathfrak{sl}_2^\iota$ for some vertex ι of the Dynkin diagram of $\mathfrak{g}$, $\iota \in I$.

But, according to Lemma 18.1.2 of [FG2], if M is a partially integrable $\widehat{\mathfrak{g}}_{\text{crit}}$ -module, then

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}_-((t)), t\mathfrak{n}_-[[t]], M \otimes \Psi_{-\check{\rho}}) = 0$$

for all $i \in \mathbb{Z}$. Therefore we obtain that the map

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}_-((t)), t\mathfrak{n}_-[[t]], j_{w_0\check{\rho},!} \star M \otimes \Psi_{-\check{\rho}}) \rightarrow H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}_-((t)), t\mathfrak{n}_-[[t]], j_{w_0\check{\rho},*} \star M \otimes \Psi_{-\check{\rho}})$$

induced by (4.4) is an isomorphism for any $G[[t]]$ -equivariant $\widehat{\mathfrak{g}}_{\text{crit}}$ -module M . Since the former is right exact and the latter is left exact on the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]], \mathbb{Z}}$, we obtain that both functors are exact on this category. Combining this with the isomorphism (4.3), we find that the functor

$$M \mapsto H^{\frac{\infty}{2}+\bullet}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], M \otimes \Psi_0)$$

is exact on the category $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]], \mathbb{Z}}$.

This completes the proof of Proposition 2. $\square$

Remark 1. There are obvious analogues of the categories $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{I, \mathbb{Z}}$ and $\widehat{\mathfrak{g}}_{\text{crit}}\text{-mod}^{G[[t]], \mathbb{Z}}$ for an arbitrary level κ . The same proof as above works for any κ , so Proposition 2 actually holds for an arbitrary level. $\square$

It remains to prove Theorem 2. We will give two proofs: one relies on the results of [FG3], and the other uses the Wakimoto modules. Both proofs use the computation of the characters of $\mathfrak{z}^{\lambda, \text{reg}}$ and $H^{\frac{\infty}{2} + \bullet}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$ which is performed in the next section.

5. COMPUTATION OF CHARACTERS

5.1. Character of $\mathfrak{z}^{\lambda, \text{reg}}$. Let us compute the character of the algebra

$$\mathfrak{z}^{\lambda, \text{reg}} = \text{Fun Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{reg}}$$

of functions on $\text{Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{reg}}$ with respect to the $\mathbb{Z}$ -grading by the operator L_0 . This space was defined in Sect. 2, but note that we now switch to the Langlands dual Lie algebra $\check{\mathfrak{g}}$.

We give another, more convenient, realization of the space $\text{Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{reg}}$. Suppose that we are given an operator of the form

$$(5.1) \quad \nabla = \partial_t + \sum_{i \in I} t^{\langle \check{\alpha}_i, \lambda \rangle} f_i + \mathbf{v}(t), \quad \mathbf{v}(t) \in \check{\mathfrak{b}}(\hat{\mathcal{O}}).$$

Applying gauge transformation with $(\lambda + \rho)(t)$, we obtain an operator of the form

$$(5.2) \quad \nabla' = \partial_t + \frac{1}{t} \left(\sum_{i \in I} f_i - (\lambda + \rho) \right) + \mathbf{v}(t), \quad \mathbf{v}(t) \in (\lambda + \rho)(t) \check{\mathfrak{b}}(\hat{\mathcal{O}})(\lambda + \rho)(t)^{-1}.$$

The space $\text{Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{reg}}$ is defined as the space of $\check{N}(\hat{\mathcal{O}})$ -equivalence classes of operators (5.1). Equivalently, this is the space of $(\lambda + \rho)(t) \check{N}(\hat{\mathcal{O}})(\lambda + \rho)(t)^{-1}$ -equivalence classes of operators (5.2). It follows from Theorem 2.21 of [FG2] that the action of the group $(\lambda + \rho)(t) \check{N}(\hat{\mathcal{O}})(\lambda + \rho)(t)^{-1}$ on this space is free. Therefore the character of $\text{Fun Op}_{\widehat{\mathfrak{g}}}^{\lambda, \text{reg}}$ is equal to the character of the algebra of functions on the space of operators (5.2) divided by the character of the algebra $\text{Fun}((\lambda + \rho)(t) \check{N}(\hat{\mathcal{O}})(\lambda + \rho)(t)^{-1})$.

By definition, a character of a $\mathbb{Z}_+$ -graded vector space $V = \bigoplus_{n \in \mathbb{Z}_+} V_n$, where each V_n is finite-dimensional, is the formal power series

$$\text{ch } V = \sum_{n \geq 0} \dim V_n \cdot q^n.$$

Applying the dilation $t \mapsto at$ to (5.2), we obtain the action $\mathbf{v}(t) \mapsto a\mathbf{v}(at)$. Therefore the character of the algebra of functions on the space of operators (5.2) is equal to

$$\prod_{n > 0} (1 - q^n)^{-\ell} \cdot \prod_{\check{\alpha} \in \check{\Delta}_+} \prod_{n > 0} (1 - q^{n + \langle \check{\alpha}, \lambda + \rho \rangle})^{-1}.$$

On the other hand, the character of $\text{Fun}((\lambda + \rho)(t)\check{N}(\hat{\mathcal{O}})(\lambda + \rho)(t)^{-1})$ is equal to

$$\prod_{\check{\alpha} \in \check{\Delta}_+} \prod_{n \geq 0} (1 - q^{n + \langle \check{\alpha}, \lambda + \rho \rangle})^{-1}.$$

Therefore the character of $\mathfrak{z}^{\lambda, \text{reg}} = \text{Fun Op}_{\check{\mathfrak{g}}}^{\lambda, \text{reg}}$ is equal to

$$\prod_{n > 0} (1 - q^n)^{-\ell} \cdot \prod_{\check{\alpha} \in \check{\Delta}_+} (1 - q^{\langle \check{\alpha}, \lambda + \rho \rangle}).$$

We rewrite this in the form

$$(5.3) \quad \text{ch } \mathfrak{z}^{\lambda, \text{reg}} = \prod_{\check{\alpha} \in \check{\Delta}_+} \frac{1 - q^{\langle \check{\alpha}, \lambda + \rho \rangle}}{1 - q^{\langle \check{\alpha}, \rho \rangle}} \prod_{i=1}^{\ell} \prod_{n_i \geq d_i + 1} (1 - q^{n_i})^{-1},$$

using the identify

$$\prod_{\check{\alpha} \in \check{\Delta}_+} (1 - q^{\langle \check{\alpha}, \rho \rangle}) = \prod_{i=1}^{\ell} \prod_{m_i=1}^{d_i} (1 - q^{m_i}),$$

where $d_1, \dots, d_{\ell}$ are the exponents of $\mathfrak{g}$.

5.2. Computation of semi-infinite cohomology. Let us now compute the semi-infinite cohomology of $\mathbb{V}_{\lambda}$. The complex $C^{\bullet}(\mathbb{V}_{\lambda})$ computing this cohomology is described in [FB], Ch. 15. In particular, as explained in Sect. 4, it carries a $\mathbb{Z}$ -grading operator which commutes with the differential and hence gives rise to a grading operator on the cohomology. We will compute the character with respect to this grading operator.

Theorem 4. *We have*

$$\text{ch } H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_{\lambda} \otimes \Psi_0) = \text{ch } \mathfrak{z}^{\lambda, \text{reg}}$$

given by formula (5.3), and

$$H^{\frac{\infty}{2}+i}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_{\lambda} \otimes \Psi_0) = 0, \quad i \neq 0.$$

Proof. The vanishing the i th cohomology for $i \neq 0$ follows from Proposition 2. We will give an alternative proof of this, as well as the computation of the character of the 0th cohomology, using the argument of [FB], Sect. 15.2.

Consider the complex $C^{\bullet}(\mathbb{V}_{\lambda})$ computing our semi-infinite cohomology. This complex was studied in detail in [FB], Sect. 15.2, in the case when $\lambda = 0$. We decompose $C^{\bullet}(\mathbb{V}_{\lambda})$ into the tensor product of two subcomplexes as in [FB], Sect. 15.2.1,

$$C^{\bullet}(\mathbb{V}_{\lambda}) = C^{\bullet}(\mathbb{V}_{\lambda})_0 \otimes C^{\bullet}(\mathbb{V}_{\lambda})'$$

defined in the same way as the subcomplexes $C^{\bullet}_{-h^{\vee}}(\mathfrak{g})_0$ and $C^{\bullet}_{-h^{\vee}}(\mathfrak{g})'$, respectively. In fact, $C^{\bullet}(\mathbb{V}_{\lambda})_0$ is equal to $C^{\bullet}_{-h^{\vee}}(\mathfrak{g})_0$, and

$$C^{\bullet}(\mathbb{V}_{\lambda})' \simeq V_{\lambda} \otimes U(t^{-1}\mathfrak{b}_-[t^{-1}]) \otimes \bigwedge^{\bullet}(\mathfrak{n}_+[[t]]^*).$$

In particular, its cohomological grading takes only non-negative values on $C^{\bullet}(\mathbb{V}_{\lambda})'$.

We show, in the same way as in [FB], Lemma 15.2.5, that the cohomology of $C^{\bullet}(\mathbb{V}_{\lambda})$ is isomorphic to the tensor product of the cohomologies of the subcomplexes $C^{\bullet}(\mathbb{V}_{\lambda})_0$

and $C^\bullet(\mathbb{V}_\lambda)'$. The former is one-dimensional, according to [FB], Lemma 15.2.7, and hence we find that our semi-infinite cohomology is isomorphic to the cohomology of the subcomplex $C^\bullet(\mathbb{V}_\lambda)'$.

Following verbatim the computation in [FB], Sect. 15.2.9, in the case when $\lambda = 0$, we find that the 0th cohomology of $C^\bullet(\mathbb{V}_\lambda)'$ is isomorphic to

$$H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0) \simeq V_\lambda \otimes V(\mathfrak{a}_-)$$

(where $V(\mathfrak{a}_-)$ is defined in [FB], Sect. 15.2.9), and all other cohomologies vanish.

In particular, we find that the character of $H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$ is equal to

$$\text{ch } V_\lambda \cdot \text{ch } V(\mathfrak{a}_-),$$

where $\text{ch } V_\lambda$ is the character of V_λ with respect to the principal grading. By [FB], Sect. 15.2.9, we have

$$\text{ch } V(\mathfrak{a}_-) = \prod_{i=1}^{\ell} \prod_{m_i \geq d_i+1} (1 - q^{m_i})^{-1}.$$

According to formula (10.9.4) of [Kac], $\text{ch } V_\lambda$ is equal to

$$(5.4) \quad \text{ch } V_\lambda = \prod_{\check{\alpha} \in \check{\Delta}_+} \frac{1 - q^{\langle \check{\alpha}, \lambda + \rho \rangle}}{1 - q^{\langle \check{\alpha}, \rho \rangle}}.$$

Therefore the character

$$\text{ch } H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0)$$

is given by formula (5.3), which coincides with the character of $\mathfrak{z}^{\lambda, \text{reg}}$. $\square$

6. PROOF OF THEOREM 2

6.1. First proof. The following result is proved in [FG3], Lemma 1.7.

Proposition 3. *The action of the center $\mathfrak{Z}_{\mathfrak{g}}$ on $\mathbb{V}_\lambda$ factors through $\mathfrak{z}^{\lambda, \text{reg}}$.*

Let I_λ be the ideal of $\text{Op}_{\mathfrak{g}}^{\text{reg}, \lambda} = \text{Spec } \mathfrak{z}^{\text{reg}, \lambda}$ in the center $\mathfrak{Z}_{\mathfrak{g}} = \text{Fun}(\text{Op}_{\mathfrak{g}}(\mathcal{D}^\times))$. As explained in [FG2], Sect. 4.6 (see [BD1], Sect. 3.6, in the case when $\lambda = 0$), the Poisson structure on $\mathfrak{Z}_{\mathfrak{g}}$ gives rise to the structure of a Lie algebroid on the quotient $I_\lambda / (I_\lambda)^2$, which we denote by $N_{\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}} / \text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)}^*$. Recall from [FF, F2] that the Poisson structure on $\mathfrak{Z}_{\mathfrak{g}}$ is obtained by deforming the completed enveloping algebra of $\widehat{\mathfrak{g}}$ to non-critical levels. By Proposition 3, I_λ annihilates the module $\mathbb{V}_\lambda$. Since this module may be deformed to the Weyl modules at non-critical levels, we obtain that the Lie algebroid $N_{\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}} / \text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)}^*$ naturally acts on $\mathbb{V}_\lambda$ (see [BD1], Sect. 5.6, in the case when $\lambda = 0$) and on its semi-infinite cohomology. This action is compatible with the action of $\text{Fun}(\text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)) / I_\lambda = \mathfrak{z}^{\text{reg}, \lambda}$.

Using the commutative diagram (3.3) and Proposition 3, we obtain that the composition (3.7) factors through a map

$$\mathfrak{z}^{\lambda, \text{reg}} \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0).$$

By applying the same argument as in the proof of Proposition 18.3.2 of [FG2], we obtain that the above map is a homomorphism of modules over the Lie algebroid $N_{\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}} / \text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)}^*$. This homomorphism is clearly non-zero, because we can identify the image of the generator $1 \in \mathfrak{z}^{\lambda, \text{reg}}$ with the cohomology class represented by the highest weight vector of $\mathbb{V}_\lambda$. Since $\mathfrak{z}^{\text{reg}, \lambda}$ is irreducible as a module over $N_{\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}} / \text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)}^*$, this homomorphism is injective. Moreover, it is clear that this map preserves the natural $\mathbb{Z}$ -gradings on both modules. Therefore the equality of the two characters established in Sect. 5 shows that it is an isomorphism. $\square$

6.2. Second proof. Let us recall some results about Wakimoto modules of critical level from [F2] (see also [FG2]). Specifically, we will consider the module which was denoted by W_{λ, κ_c} in [F2] and by $\mathbb{W}_{\text{crit}, \lambda}^{w_0}$ in [FG2]. Here we will denote it simply by W_λ . As a vector space, it is isomorphic to the tensor product

$$(6.1) \quad W_\lambda = M_{\mathfrak{g}} \otimes \mathfrak{H}^\lambda,$$

where we use the notation

$$\mathfrak{H}^\lambda = \text{Fun Conn}_{\tilde{H}}(\omega_{\mathcal{D}}^{\rho})^{\text{RS}, -\lambda}$$

(see Sect. 2), and $M_{\mathfrak{g}}$ is the Fock representation of a Weyl algebra with generators $a_{\alpha, n}, a_{\alpha, n}^*, \alpha \in \Delta_+, n \in \mathbb{Z}$.

We now construct a map $\mathbb{V}_\lambda \rightarrow W_\lambda$. Let us observe that the action of the constant subalgebra $\mathfrak{g} \subset \widehat{\mathfrak{g}}_\kappa$ on the subspace

$$W_\lambda^0 = \mathbb{C}[a_{\alpha, 0}^*]_{\alpha \in \Delta_+} |0\rangle \simeq \text{Fun } N \subset M_{\mathfrak{g}}$$

coincides with the natural action of $\mathfrak{g}$ on the contragredient Verma module M_λ^* realized as $\text{Fun } N$. In addition, the Lie subalgebra $\mathfrak{g} \otimes t \otimes \mathbb{C}[[t]] \subset \widehat{\mathfrak{g}}_{\text{crit}}$ acts by zero on the subspace W_λ^0 , and $\mathbf{1}$ acts as the identity.

Therefore the injective $\mathfrak{g}$ -homomorphism $V_\lambda \hookrightarrow M_\lambda^*$ gives rise to a non-zero $\widehat{\mathfrak{g}}_{\text{crit}}$ -homomorphism

$$\iota_\lambda : \mathbb{V}_\lambda \rightarrow W_\lambda$$

sending the generating subspace $V_\lambda \subset \mathbb{V}_\lambda$ to the image of V_λ in $W_\lambda^0 \simeq M_\lambda^*$.

Now we obtain a sequence of maps

$$(6.2) \quad \mathbb{M}_\lambda \twoheadrightarrow \mathbb{V}_\lambda \rightarrow W_\lambda.$$

Recall from Sect. 3.1 that

$$(\mathbb{M}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \simeq \mathfrak{z}^{\lambda, \text{nilp}}.$$

We also prove, by using the argument of Lemma 6.5 of [F2] that

$$(W_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} = \mathfrak{H}^\lambda,$$

where $\mathfrak{H}^\lambda$ is identified with the second factor of the decomposition (6.1).

We obtain the following commutative diagram, in which all maps preserve $\mathbb{Z}$ -gradings:

$$(6.3) \quad \begin{array}{ccc} \mathfrak{z}^{\lambda, \text{nilp}} & \xrightarrow{b} & \mathfrak{H}^\lambda \\ a \downarrow & & c \downarrow \\ H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0) & \xrightarrow{d} & H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], W_\lambda \otimes \Psi_0) \end{array}$$

Here the map a is obtained as the composition

$$\mathfrak{z}^{\lambda, \text{nilp}} \simeq (\mathbb{M}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow (\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0),$$

the map c as the composition

$$\mathfrak{H}^\lambda \simeq (W_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], W_\lambda \otimes \Psi_0)$$

(see Lemma 3), and the map b as the composition

$$\mathfrak{z}^{\lambda, \text{nilp}} \simeq (\mathbb{M}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow (W_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \simeq \mathfrak{H}^\lambda.$$

Using Theorem 12.5 of [F2], we identify the map b with the homomorphism of the algebras of functions corresponding to the map

$$\text{Conn}_{\tilde{H}}(\omega_{\mathcal{D}}^\rho)^{\text{RS}, -\lambda} \rightarrow \text{Op}_{\mathfrak{g}}^{\lambda, \text{nilp}}$$

obtained by restriction from the Miura transformation (2.3). Therefore we obtain from Lemma 2 that the image of this map is precisely $\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}}$, and so the image of the homomorphism b is equal to

$$\mathfrak{z}^{\lambda, \text{reg}} = \text{Fun Op}_{\mathfrak{g}}^{\lambda, \text{reg}}.$$

On the other hand, we have the decomposition (6.1) and $\mathfrak{n}_+((t))$ acts along the first factor $M_{\mathfrak{g}}$. It follows from the definition of $M_{\mathfrak{g}}$ that there is a canonical identification

$$H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], M_{\mathfrak{g}} \otimes \Psi_0) \simeq \mathbb{C}[0].$$

Therefore we obtain that

$$H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], W_\lambda \otimes \Psi_0) \simeq \mathfrak{H}^\lambda$$

and so the map c is an isomorphism.

This implies that the image of the composition $c \circ b$ is $\mathfrak{z}^{\lambda, \text{reg}} \subset \mathfrak{H}^\lambda$. Therefore d factors as follows:

$$H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0) \rightarrow \mathfrak{z}^{\lambda, \text{reg}} \hookrightarrow \mathfrak{H}^\lambda.$$

But the characters of the first two spaces coincide, according to Theorem 4. Hence

$$H^{\frac{\infty}{2}}(\mathfrak{n}_+((t)), \mathfrak{n}_+[[t]], \mathbb{V}_\lambda \otimes \Psi_0) \simeq \mathfrak{z}^{\lambda, \text{reg}}.$$

This proves Theorem 2. □

We obtain the following corollary, which for $\lambda = 0$ was proved in [F2], Prop. 5.2.

Corollary 1. *The map $\iota_\lambda : \mathbb{V}_\lambda \rightarrow W_\lambda$ is injective for any dominant integral weight λ .*

Proof. Let us extend the action of $\widehat{\mathfrak{g}}_\kappa$ to an action of $\widehat{\mathfrak{g}}'_\kappa = \mathbb{C}L_0 \ltimes \widehat{\mathfrak{g}}_\kappa$ in the same way as above. Denote by K_λ the kernel of the map ι_λ . Suppose that $K_\lambda \neq 0$. Since both $\mathbb{V}_\lambda$ and $\mathbb{M}_\lambda$ become graded with respect to the extended Cartan subalgebra $\widetilde{\mathfrak{h}} = (\mathfrak{h} \otimes 1) \oplus \mathbb{C}L_0$, we find that the $\widehat{\mathfrak{g}}_{\text{crit}}$ -module K_λ contains a non-zero highest weight vector annihilated by the Lie subalgebra $\widehat{\mathfrak{n}}_+$. Let $\widetilde{\mu}$ be its weight. Since $\mathbb{V}_\lambda$ is $\mathfrak{g}$ -integrable, so is K_λ , and therefore the restriction of $\widetilde{\mu}$ to $\mathfrak{h} \subset \widetilde{\mathfrak{h}}$ is a dominant integral weight μ .

Now, since $\mathbb{V}_\lambda$ is a quotient of the Verma module $\mathbb{M}_\lambda$, and the action of the center $\mathfrak{Z}_{\mathfrak{g}}$ on $\mathbb{M}_\lambda$ factors through $\text{Fun Op}_{\mathfrak{g}}^{\lambda, \text{nilp}}$ (see the diagram (3.3)), we find that the same is true for $\mathbb{V}_\lambda$.¹ According to [F2], Prop. 12.8, the degree 0 part of $\text{Fun Op}_{\mathfrak{g}}^{\lambda, \text{nilp}}$ is isomorphic to $(\text{Fun } \mathfrak{h}^*)^W$, and it acts on any highest weight vector v through the quotient by the maximal ideal in $(\text{Fun } \mathfrak{h}^*)^W$ corresponding to $-\lambda - \rho$. Moreover, its action coincides with the action of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ on the vector v under the Harish-Chandra isomorphism $Z(\mathfrak{g}) \simeq (\text{Fun } \mathfrak{h}^*)^W$ followed by the sign isomorphism $x \mapsto -x$. Therefore if v has $\mathfrak{h}$ -weight μ , the action of $(\text{Fun } \mathfrak{h}^*)^W$ factors through the quotient by the maximal ideal in $(\text{Fun } \mathfrak{h}^*)^W$ corresponding to $-\mu - \rho$. Since μ is dominant integral, this implies that $\mu = \lambda$.

Thus, K_λ contains a non-zero highest weight vector of $\mathfrak{h}$ -weight λ . This vector then lies in $(\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+}$. But Theorem 1 and the diagram (6.3) imply that the map

$$(\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow W_\lambda$$

is injective. Indeed, the image of the composition

$$(\mathbb{M}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow (\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+} \rightarrow W_\lambda$$

is equal to $\mathfrak{z}^{\lambda, \text{reg}} \subset \mathfrak{H}^\lambda$, which is isomorphic to $(\mathbb{V}_\lambda)_\lambda^{\widehat{\mathfrak{n}}_+}$.

This leads to a contradiction and hence proves the desired assertion. $\square$

The following result is also useful in applications. Note that for $\lambda = 0$ it follows from the corresponding statement for the associated graded module proved in [EF] or from the results of [BD1], Sect. 6.2.

Theorem 5. *For any dominant integral weight λ the Weyl module $\mathbb{V}_\lambda$ is free as a $\text{Fun Op}_{\mathfrak{g}}^{\lambda, \text{reg}}$ -module.*

Proof. Recall from Sect. 6.1 that $\mathbb{V}_\lambda$ carries an action of the Lie algebroid $N_{\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}} / \text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)}^*$ compatible with the action of $\mathfrak{z}^{\text{reg}, \lambda} = \text{Fun}(\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}})$. According to the results of [FG2], Sect. 4.6, this Lie algebroid is nothing but the Atiyah algebroid of the universal $\check{G}$ -bundle on $\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}}$ whose fiber at $\chi \in \text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}}$ is the fiber of the $\check{G}$ -torsor on $\mathcal{D}$ underlying χ at $0 \in \mathcal{D}$. This bundle is isomorphic (non-canonically) to the trivial $\check{G}$ -bundle on $\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}}$. Let us choose such an isomorphism. Then $N_{\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}} / \text{Op}_{\mathfrak{g}}(\mathcal{D}^\times)}^*$ splits as a direct sum of the Lie algebra $\text{Vect}(\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}})$ of vector fields on $\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}}$ and the Lie algebra $\mathfrak{g} \otimes \text{Fun}(\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}})$.

¹We have already determined in Proposition 3 that the support of $\mathbb{V}_\lambda$ is contained in $\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}}$, but it is not necessary to use this result here.

Thus, we obtain an action of $\text{Vect}(\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}})$ on $\mathbb{V}_\lambda$ compatible with the action of $\text{Fun}(\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}})$. Note that the algebra $\mathfrak{z}^{\text{reg}, \lambda}$ and the Lie algebra $\text{Vect}(\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}})$ are $\mathbb{Z}$ -graded by the operator $L_0 = -t\partial_t$. According to Lemma 2, $\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}}$ is an infinite-dimensional affine space, and there exists a system of coordinates $x_i, i = 1, 2, \dots$, on it such that these coordinates are homogeneous with respect to L_0 . The character formula (5.3) shows that the degrees of all of these generators are strictly positive.

Thus, we find that the action on $\mathbb{V}_\lambda$ of the polynomial algebra $\text{Fun}(\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}})$ generated by the x_i 's extends to an action of the Weyl algebra generated by the x_i 's and the $\partial/\partial x_i$'s. Recall that the $\mathbb{Z}$ -grading on $\mathbb{V}_\lambda$ with respect to the operator L_0 takes non-negative integer values. Repeating the argument of Lemma 6.2.2 of [BD1] (see Lemma 9.13 of [Kac]), we obtain that $\mathbb{V}_\lambda$ is a free module over $\mathfrak{z}^{\text{reg}, \lambda} = \text{Fun}(\text{Op}_{\mathfrak{g}}^{\lambda, \text{reg}})$. $\square$

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