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Probing the onset of quantum avalanches in a many-body-localized system

Julian Léonard^{1,*†}, Sooshin Kim^{1,†}, Matthew Rispoli¹, Alexander Lukin¹,
Robert Schittko¹, Joyce Kwan¹, Eugene Demler^{1,‡}, Dries Sels^{2,3}, and Markus Greiner^{1,§}

¹*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*

²*Center for Computational Quantum Physics, Flatiron Institute, New York, New York 10010, USA*

³*Department of Physics, New York University, New York, New York 10003, USA*

Strongly correlated systems can exhibit unexpected phenomena when brought in a state far from equilibrium. An example is many-body localization, which prevents generic interacting systems from reaching thermal equilibrium even at long times [1, 2]. The stability of the many-body-localized phase has been predicted to be hindered by the presence of small thermal inclusions that act as a bath, leading to the delocalization of the entire system through an avalanche propagation mechanism [3–8]. Here we study the dynamics of a thermal inclusion of variable size when it is coupled to a many-body-localized system. We find evidence for accelerated transport of the thermal inclusion into the localized region. We monitor how the avalanche spreads through the localized system and thermalizes it site by site by measuring the site-resolved entropy over time. Furthermore, we isolate the strongly correlated bath-induced dynamics with multipoint correlations between the bath and the system. Our results have implications on the robustness of many-body localized systems and their critical behavior.

One of the founding principles of statistical physics is that a generic macroscopic system can equilibrate on its own. This means that local fluctuations of energy, magnetization, or particle density can relax towards thermal equilibrium because interactions allow different parts of the system to serve as reservoirs to each other. This universal picture has been challenged by the idea of many-body localization (MBL), which suggests that systems with strong disorder can evade thermalization even in the presence of interactions [1, 2, 9–15]. In one-dimensional systems, a stable MBL phase can be argued as follows: matrix elements of local operators decay exponentially with separation between two points, whereas the density of states increases exponentially with the system size. For strong disorder, matrix elements can thus be argued to decay faster than the density of states increases, ultimately inhibiting relaxation.

However, the existence of MBL remains a subject of debate, since it is unclear when those conditions are fulfilled [16–24]. For instance, by introducing a small region with weak disorder, part of the system may be delocalized and thus give rise to local operators with non-exponential decay [25–36]. Those local weakly disordered

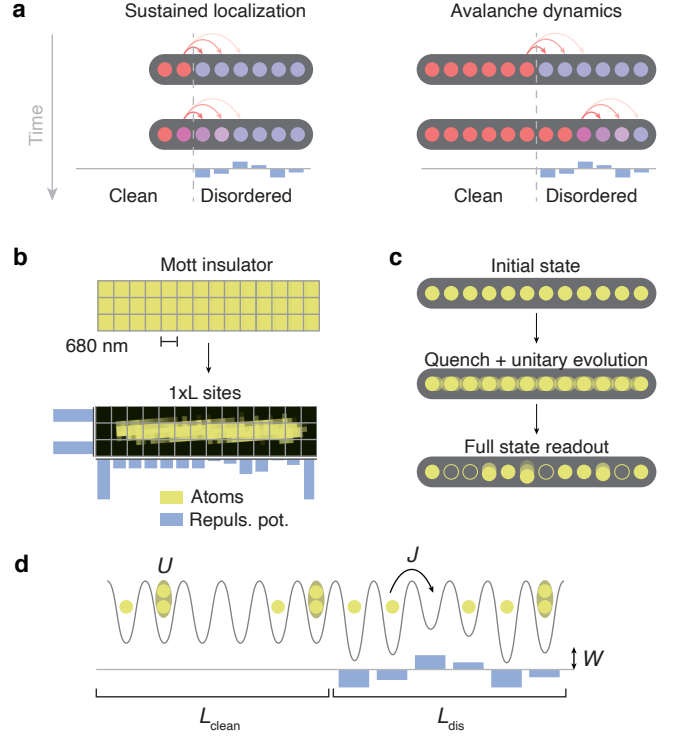


FIG. 1. Bath-induced quantum avalanches. **a**, Two scenarios at an interface of a thermal bath (clean) and a localized (disordered) region: the bath penetrates logarithmically slow and localization remains robust (left), or an avalanche thermalizes the disordered region site by site (right). **b**, Fluorescence pictures of a two-dimensional Mott insulator at unity filling, and of the initialized one-dimensional system of L sites. Projected optical potentials isolate the system and apply site-resolved offsets onto the disordered region (blue). **c**, Protocol. The initial state is brought far from equilibrium through a quantum quench by abruptly enabling tunneling along all neighbouring links, then evolved under the Hamiltonian, until we detect the site-resolved atom number with a fluorescence picture. **d**, The system's dynamics are governed by the Bose-Hubbard model with tunneling energy J and on-site interaction energy U , extended by a disorder potential with amplitude W in the disordered region.

regions occur naturally in randomly disordered systems, when potential offsets on consecutive lattice sites accidentally coincide [25, 28, 29, 37, 38]. The dynamics in MBL systems in the presence of a thermal region have been predicted to occur in so-called quantum avalanches, which imply that these regions grow by absorbing nearby

disordered regions [3–7]. Under which conditions quantum avalanches can arise, run out of steam, or propagate without halt determines the fate of MBL at long evolution times. Their understanding is thus closely connected to discerning thermalization in interacting many-body systems.

Perturbative bath-induced relaxation can often be captured in the context of Fermi’s golden rule (Fig. 1a, left). In this picture, the relaxation rate $\Gamma_i = g_i^2 \rho_{\text{bath}}$ of site i is given by the product of the bath’s density of states ρ_{bath} and the coupling rate $g_i \propto J e^{-i/\xi_{\text{loc}}}$, where ξ_{loc} is the localization length of the MBL system, and J is the tunneling rate between neighbouring sites. Consequently, within a perturbative description MBL remains robust against a local bath, with a bath penetration into the MBL region that increases logarithmically in time.

Quantum avalanches, in contrast, are predicted to emerge from dynamics beyond this simple picture (Fig. 1a, right). A more accurate description ought to take into account that the density of states of the bath grows when the first disordered site thermalizes and hence merges with the bath. This feedback effect enhances the relaxation rate Γ_i for the next MBL sites, giving rise to accelerated bath penetration into the MBL region faster than a logarithmic in time. Eventually, these non-perturbative relaxation processes may lead to a full delocalization of the system if the density of states grows faster than the decay in the coupling rates.

Studying quantum avalanches within disordered systems remains a challenge due to both the statistical rareness of a sufficiently large thermal inclusion, and the large time scales over which the inclusion spreads through the system. Consequently, theoretical approaches often consider disordered systems that are locally coupled to a thermal bath that represents the rare region [5]. Within this canonical setting, several signatures have been proposed to identify quantum avalanches through their short-term dynamics, including a speedup compared to a logarithmic spreading [7], and a backaction on the bath [3]. However, high demands in local control have so far hindered their experimental observation.

In this work we explore the dynamics of an MBL system coupled to a thermal inclusion (Fig. 1) and observe phenomena that suggest the presence of non-perturbative avalanche processes. Our experimental protocol starts by preparing a Mott-insulating state with one ^{87}Rb atom on each site of a two-dimensional optical lattice (Fig. 1b) (SI). The system is placed in the focus of a high-resolution imaging system through which we project site-resolved repulsive potentials on individual lattice sites. We isolate a one-dimensional system of L lattice sites from the Mott insulator and add potential offsets to the lattice sites. At this point, the system remains in a product state of one atom per lattice site. We then perform a quantum quench by abruptly reducing the lattice depth (Fig. 1c). The subsequent non-equilibrium dynamics are

described by the Bose-Hubbard Hamiltonian:

$$\hat{\mathcal{H}} = -J \sum_i \left(\hat{a}_i^\dagger \hat{a}_{i+1} + h.c. \right) + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) + W \sum_{i \in L_{\text{dis}}} h_i \hat{n}_i,$$

where $\hat{a}_i^\dagger$ ($\hat{a}_i$) is the creation (annihilation) operator for a boson on site i , and $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ is the particle number operator. The first term describes the tunneling between all neighboring lattice sites, and the second term represents the on-site repulsive interactions. The last term introduces a site-resolved energy offset. We set $h_i = 0$ for all lattice sites in the clean region of size L_{clean} , whereas the energy offsets in the disordered region of size L_{dis} follow a quasi-periodic disorder distribution $h_i = \cos(2\pi\beta i + \phi)$ with $1/\beta \approx 1.618$, phase ϕ and amplitude W . The quasi-periodic distribution avoids nearby lattice sites to co-incidentally have similar energy offsets, which inhibits the presence of secondary rare regions within the disordered region (SI) [39]. After a variable evolution time, we read out the site-resolved atom number by fluorescence imaging. The applied unitary evolution preserves the initial purity of 99.1(2)% per site [13, 40]. All observables are averaged over 200 disorder realizations with different ϕ . The tunneling time $\tau = \hbar/J = 4.3(1)$ ms (with the reduced Planck constant $\hbar$), the interaction strength $U = 2.87(3) J$, and the number of disordered sites $L_{\text{dis}} = 6$ remain constant in all experiments.

We first use the full site-resolved readout of our microscope to investigate the local transport dynamics in the system. The connected density-density correlations $\langle \hat{n}_i \hat{n}_j \rangle_c = \langle \hat{n}_i \hat{n}_j \rangle - \langle \hat{n}_i \rangle \langle \hat{n}_j \rangle$ detect correlations between the particle numbers on site i and j [15]. Negative values of $\langle \hat{n}_i \hat{n}_j \rangle_c$ signal anti-correlated density fluctuations, and thus particles motion between the involved sites (Fig. 2a). In the following, we consider a system with $L_{\text{clean}} = 6$ at disorder strength $W = 9.1 J$ after different evolution times T after the quantum quench. At the beginning of the evolution ($T = 0\tau$), we do not detect any correlations, because the initial state is a product state. After short evolution times ($T \lesssim \tau L$), we observe the buildup of spatially dependent anti-correlations in the system. Within the clean region all lattice sites develop mutual anti-correlations, signaling delocalized particles. In contrast, the anti-correlations in the disordered region remain short-ranged, indicating localized particles. These properties overall persist up to long evolution times ($T \gg \tau L$). In order to quantify the separate dynamics in the clean and the disordered region, we extract the mean of the off-diagonal correlations within each region (Fig. 2b). Within few tunneling times the clean region has reached its steady state and starts to act as a thermal bath to the disordered region.

For long evolution times ($T \gg \tau L$) we additionally

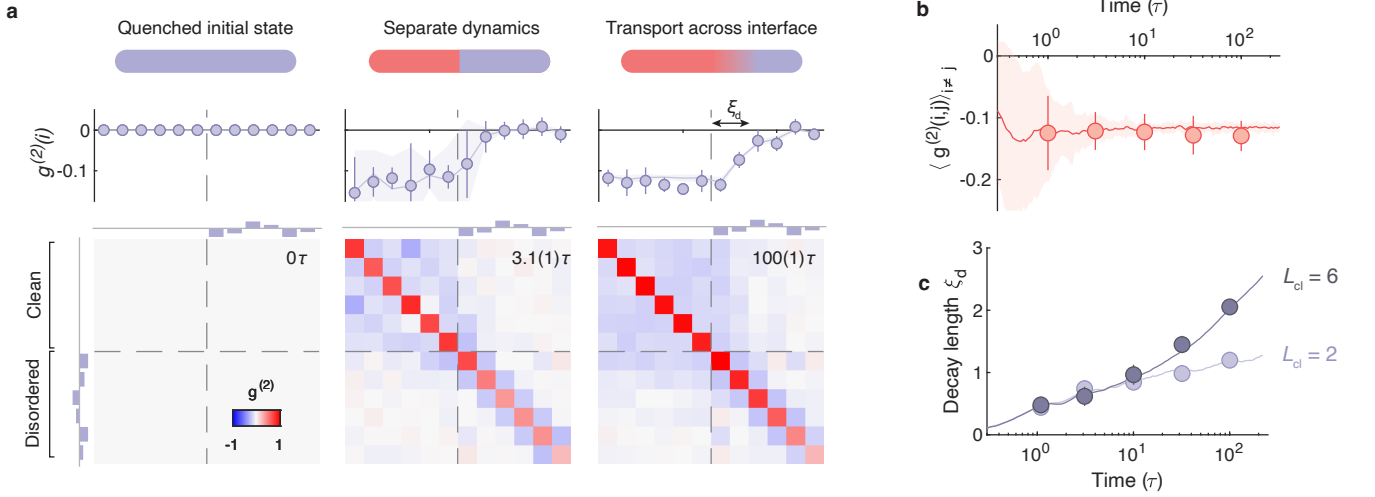


FIG. 2. **Accelerated transport across the clean-disorder interface.** **a**, Density correlations for all pairs of sites in a system consisting of $L_{\text{clean}} = L_{\text{dis}} = 6$ at disorder strength $W = 9.1 J$. After a quantum quench, an uncorrelated initial state (left) develops separate dynamics within each subsystem (center), followed by slow particle transport across the clean-disorder interface (grey dashed lines) for evolution times $\gg L_{\text{clean}}, L_{\text{dis}}$ (right). Cuts show the total density correlations $g^{(2)}(i)$ of the clean region with site i (i.e. average of top six rows, excluding diagonal entries), featuring homogeneous coupling among the clean sites, and approximately exponentially decaying anti-correlations with the distance of the disordered site from the interface. **b**, The mean (red data points) and the s.d. (error bars) of the off-diagonal correlations within the clean region reach a steady state within few tunneling times. **c**, The decay length ξ_d of the total density correlations increases first logarithmically in time and accelerates at long evolution times. For a smaller clean region of $L_{\text{clean}} = 2$, contrarily, the correlations continue to grow logarithmically. Solid lines show the prediction from exact numerics without free parameters, shaded areas in panel b show its s.d. (SI). Error bars in panels a and c denote the s.e.m. (below the marker size in panel a).

observe the buildup of anti-correlations between lattice sites in the clean and the disordered region, evidence for transport dynamics across the interface (right panel in Fig. 2a). Each of the disordered sites is equally anti-correlated to all clean sites, which confirms that the clean region acts as a heat bath for the disordered region. Motivated by this picture, we extract the total correlations of the clean region $\sum_{j \in L_{\text{clean}}} \langle \hat{n}_i \hat{n}_j \rangle_c$ by taking the sum of the correlations of each site with all clean sites (Fig. 2a cuts). The results are consistent with an exponential decay with distance from the clean region, in agreement with the Fermi golden rule picture of exponentially decaying couplings between bath and MBL.

While a static bath spectrum causes bath correlations to penetrate MBL logarithmically in time, a signature of the quantum avalanche is an accelerated increase, faster than logarithmically in time. In order to test this picture, we quantify the correlation decay into the disordered region by measuring the average distance ξ_d from the clean region over which anti-correlations form (Fig. 2c). At short times the decay length ξ_d increases logarithmically in time, but accelerates at long evolution times. We contrast this observation with a system with $L_{\text{clean}} = 2$, where we do not find any accelerating transport dynamics.

We next examine the local thermalization dynamics across the system. The site-resolved full atom number

readout enables us to measure the atom number distribution on a site-resolved level (Fig. 3a). Lattice sites in the clean region show a distribution corresponding to a thermal ensemble, whereas lattice sites in the disordered region show a distribution with enhanced probability for one particle, the initial state of the system. We quantify the site-resolved thermalization dynamics with the entropy per particle $s_i = -\sum_{n_i} p(n_i) \log p(n_i) / \langle \hat{n}_i \rangle$ on site i from the atom number distributions. We observe reduced thermalization dynamics of the disordered sites with increasing distance from the interface (Fig. 3b,c). Moreover, the data suggest that the dynamics are first stationary until thermalization sets in with a delay that is exponential in the site's distance from the interface. This picture is confirmed by our exact numerical calculations.

The accelerated transport suggests the long-term dynamics are driven by processes that go beyond a perturbative coupling to the bath. We investigate this effect through multipoint correlations [15, 41]. The presence of non-zero three-point connected correlations $\langle \hat{n}_i \hat{n}_j \hat{n}_k \rangle_c$ signals the presence of entanglement among all involved lattice sites, which cannot be explained in a perturbative, semiclassical description. We evaluate the connected correlations $\langle \hat{n}_i \hat{n}_j \hat{n}_{\text{clean}} \rangle_c$ among two disordered sites i, j and a clean sites k , averaged over all possible k (Fig. 4a). We find a strong dependence on the involved disordered

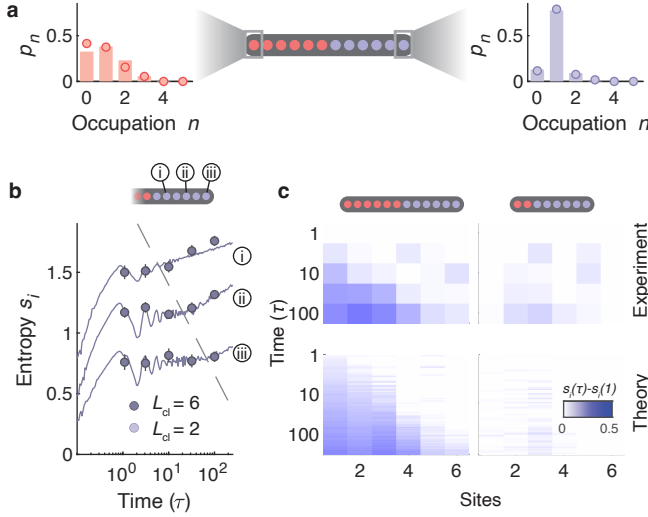


FIG. 3. **Site-resolved thermalization dynamics.** **a**, The atom number probability distribution for the edge sites in the clean region (left) and the disordered region (right), measured after 100τ in a system consisting of $L_{\text{clean}} = L_{\text{dis}} = 6$ at disorder strength $W = 9.1J$. **b**, Local entropy per particle $s_i = -\sum_n p_n \log p_n / \langle \hat{n}_i \rangle$ for selected disordered sites, extracted from the atom number distribution on each site i . The entropy grows after a stationary evolution whose length depends on the distance from the interface. For a smaller clean region of $L_{\text{clean}} = 2$, in contrast, the growth is absent. Traces are vertically offset for better visibility. **c**, Local entropy s_i , offset by $s_i(1\tau)$ for all disordered sites. Solid lines (bars in panel a) show the prediction from exact numerics without free parameters (SI). Error bars denote the s.e.m. (below the marker size in panel a).

sites: close to the interface correlations are strong, whereas they become weaker for distant sites. We quantify this behaviour by considering the correlations as a function of the mean distance $\bar{d} = (i + j)/2$ of the two disordered sites from the clean region (Fig. 4b). Indeed, the correlations decrease with increasing distance from the clean region, comparable to the decay length ξ_d . This demonstrates that the accelerated transport is driven by many-body processes, a key property for quantum avalanches. We quantify the presence of many-body correlations at different disorder strengths by taking their total average over all off-diagonal elements (Fig. 4c,d). The correlations are present throughout the covered disorder range with a maximum at intermediate strengths, close to the estimated critical point of the system [15].

In conclusion, we experimentally realized a clean-disordered interface and studied the emerging thermalization dynamics. We observed an accelerated intrusion of the bath in the MBL system, its evolution to thermal equilibrium site after site, and the many-body correlations between the two subsystems – the hallmarks of quantum avalanches. In future, our experiments can be readily extended in many ways. For example, one

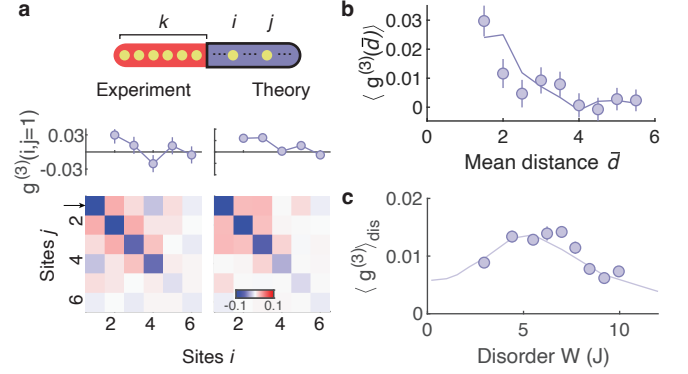


FIG. 4. **Bath-induced many-body correlations.** **a**, Three-point correlations $\langle \hat{n}_i \hat{n}_j \hat{n}_k \rangle_c$ among pairs of disordered sites i, j and one clean site k (summed over all clean sites k) in a system with $L_{\text{clean}} = L_{\text{dis}} = 6$ at disorder strength $W = 9.1J$ and evolution time $T = 100(1)$. The presence of multi-point correlations demonstrates non-perturbative dynamics: delocalization is driven through many-body processes between the disordered region and the clean region. **b**, Averaging over entries with equal mean distance $\bar{d} = (i + j)/2$ from the clean region shows a decay of correlations with the distance from the bath over several lattice sites, comparable to the decay length ξ_d . **c**, The average over all off-diagonal entries displays a maximum for intermediate disorder for the MBL-bath entanglement. Solid lines show the prediction from exact numerics without free parameters (SI). Error bars denote the s.e.m.

could more systematically study the fate of quantum avalanches as a function of bath size and localization length. By increasing both the system size of the disordered region, one could explore the interplay at intermediate disorder strengths in a quantitative way through its scaling behaviour, i.e. by increasing the system size at constant ratio of L_{clean} and L_{dis} , which may provide insight into the critical behaviour of the transition. An interesting extension would also be the influence of the statistical distribution of the disorder on the critical behaviour of the system. Furthermore, engineering other heterostructures with quantum gas microscopes may provide an avenue to studying phenomena in the physics of interfaces, or to building atomtronic devices.

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collecting and analyzing the data. S. K. performed the numerical simulations. The theoretical concepts were developed together with D. S. and E. D. M. G. supervised the work. All authors contributed to writing the manuscript and discussions.

The authors declare no competing interests.

* current address: Vienna Center for Quantum Science and Technology, Atominstitut, TU Wien, Vienna, Austria; [†] These authors contributed equally to this work; [‡] current address: Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland; [§] email: greiner@physics.harvard.edu

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