Applications of Combinatorics to Problems of Geometry

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Abstract

This thesis concerns applications of **combinatorics** to problems of **geometry**. Combinatorics and geometry are almost antipodal in the world of mathematics. Through the lens of combinatorics, mathematics is discrete and finitary, and we study graphs, sequences, trees and the processes which cause them to grow and modify and subdivide. Through the lens of geometry however, mathematics is a wild and continuous beast, and we study shapes, curves, surfaces and the processes which cause them to evolve and change and break apart.

The intersection of combinatorics and geometry is the study of discrete, finitary processes which govern continuous, dynamical behaviour. Here, we will focus on exploiting these processes to solve two problems in Euclidean geometry, and two problems in algebraic geometry.

Euclidean geometry is the study of regions in Euclidean n-dimensional space $\mathbb{R}^n$. With so few constraints placed on the objects under consideration, this area is the home to some of the most notorious outstanding problems in all of mathematics. I believe that the best combinatorics problems are ones that anyone can understand but resist many good attempts at them. The Euclidean problems in this thesis certainly have this flavour, and one of them in particular lies at the center of an extremely active area of research surrounding stability of geometric inequalities.

Algebraic geometry is the study of regions cut out by polynomial equations in $\mathbb{C}^n$. Such regions display certain rigidities that arbitrary surfaces do not have, but they still vary continuously in families by varying the underlying equations. There are many profound open problems in the field, such as the Hodge conjecture and the Jacobian conjecture. At the same time, there are very basic questions about computing invariants of algebraic varieties which are still open and have important enumerative consequences — we will focus on this latter type of problem here. I hope that the breadth of topics covered, spanning equivariant intersection theory, quantum cohomology, matroid theory, polytopes (occasionally infinite) and their subdivisions, lattice point enumeration, and the representation theory of S_n with $n \in \mathbb{C}$, will interest both the casual, and the ardent reader.

Contents

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Chapter 1

Outline of Thesis

1.1 Euclidean geometry

The first part of this thesis resolves two open problems in Euclidean geometry, the former an old chestnut, and the latter extremely recent but at the center of a very active area of current research.

1.1.1 Local maxima of quadratic boolean functions

This project is single author, adapted from [58], and later published in revised form in *Combinatorics, Probability, and Computing* [59], reproduced with permission. A quadratic function $\Theta : \{0,1\}^n \to \mathbb{R}$ is called a *quadratic boolean function*. We say that $v \in \{0,1\}^n$ is a *strict local maximum* if $\Theta(v) > \Theta(v')$ for all $v' \in \{0,1\}^n$ which differ in exactly one coordinate from v. Generalizing Kleitman's result [46] on concentration of measure of sums of Bernoulli random variables (that a paralellepiped with all sides of length > 2 has at most $\binom{n}{\lfloor n/2 \rfloor}$ vertices inside any ball of radius 1), Ron Holzman in the late 1980s (see [55] and Section 2.1) conjectured that the number of strict local maxima of any quadratic boolean function is at most $\binom{n}{\lfloor n/2 \rfloor}$.

Theorem 1.1.1. *(S.) A quadratic boolean function has at most* $\binom{n}{\lfloor n/2 \rfloor}$ *strict local maxima.*

Inspired by Kleitman's symmetric chain decomposition techniques, we create a discrete recursive process which governs the evolution of potential sites of strict local maxima as we reveal coefficients of f. This is accomplished via a generalization of Sperner's theorem which may be of independent interest.

1.1.2 Sharp stability of Brunn-Minkowski for homothetic regions

The next question that we answer (joint with Peter van Hintum and Marius Tiba, adapted from [61], reproduced with permission) is a conjecture of David Jerison and recent Field's medalist Alessio Figalli concerning stability of the Brunn-Minkowksi inequality. Recall that the Brunn-Minkowski inequality asserts that

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}$$

for any measurable subsets $A, B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the outer Lebesgue measure. The quantity $\delta := \frac{|A+B|^{\frac{1}{n}}}{|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}} - 1$ measures the deviation of the Brunn-Minkowski inequality from its equality case, which is known to occur precisely when A, B are positive homothetic copies of the same convex set, less a measure 0 set. The stability question for the Brunn-Minkowski inequality asks how close A, B are to being homothetic copies of the same convex set in terms of δ. Sharp stability results were known in the case that one of A, B is a convex set, but beyond this nothing was known. The conjecture of Figalli and Jerison was that when $A = B$ we have $|\operatorname{co}(A) \setminus A| \leq C_n \delta$ for $\operatorname{co}(A)$ the convex hull of A, or after some simplification $|\operatorname{co}(A) \setminus A| \leq C_n |\frac{A+A}{2} \setminus A|$ for some universal constant C_n depending only on the dimension, provided δ (or equivalently $|\frac{A+A}{2} \setminus A|$) is sufficiently small.

Theorem 1.1.2. *(van Hintum,S., Tiba) There exist constants C_n, Δ_n such that if $|\frac{A+A}{2} \setminus A| \leq \Delta_n |A|$ then $|\operatorname{co}(A) \setminus A| \leq C_n |\frac{A+A}{2} \setminus A|$ for any measurable $A \subset \mathbb{R}^n$ of positive measure.*

In other words, when $\frac{A+A}{2}$ is close to A, then the semi-sum $\frac{A+A}{2}$ fills in a positive fraction of the volume between A and $\operatorname{co}(A)$.

1.2 Algebraic geometry

The second part of this thesis resolves two open problems in algebraic geometry. By reducing the problems to their underlying combinatorial processes, we find unexpected connections to disparate fields, and build frameworks to both solve and properly contextualize our results.

1.2.1 GL_{r+1}-orbits in $(\mathbb{P}^r)^n$ via quantum cohomology

This project (joint with Dennis Tseng, Mitchell Lee, Anand Patel), adapted from [51], to appear in *Advances in Mathematics* [52], reproduced with permission, marries quantum cohomology, matroid theory, and the combinatorics of polytopes to complete in an equivariant setting a programme outlined by Kapranov [42] and by Aluffi and Faber [3, 4] of computing cohomology classes of GL_{r+1}-orbit closures in $(\mathbb{P}^r)^n$. Through a new connection to small quantum cohomology [36, 44, 37], we can bundle the answers to enumerative questions associated to these orbit closures into operations which satisfy recursions analogous to the ones implied by the associativity of the small quantum product.

First, recall the goal of small quantum cohomology. Given subvarieties $X_1, \ldots, X_n$

of $\mathbb{P}^r$, suppose we want to compute

$$a_k(X_1, \ldots, X_n) = \# \text{ of degree } k \text{ maps } \mathbb{P}^1 \to \mathbb{P}^r \text{ passing through } X_1, \ldots, X_n$$
$$\text{with a fixed moduli of intersection on } \mathbb{P}^1.$$

Small quantum cohomology asserts that there is a commutative, associative operation $\star : H^\bullet(\mathbb{P}^r)^{\otimes 2} \to H^\bullet(\mathbb{P}^r)[\hbar]$ which deforms the usual multiplication on $H^\bullet(\mathbb{P}^r)$, such that if we write

$$[X_1] \star \ldots \star [X_n] = b_0(\{X_i\}) + b_1(\{X_i\})\hbar^1 + b_2(\{X_i\})\hbar^2 + \ldots$$

for cohomology classes $b_i(\{X_i\})$, then $\int_{\mathbb{P}^r} b_i(\{X_i\}) = a_i(\{X_i\})$. The class $b_i(\{X_i\})$ is more generally the class of the variety swept out by an additional $n+1$st point on $\mathbb{P}^1$ with fixed moduli relative to the original n points.

It turns out that a_d equivalently counts the number of projective d-planes which intersect $X_1, \ldots, X_n$ in a fixed generic GL_{d+1}-moduli. What if the moduli is not generic? For a $(d+1) \times n$ matrix M whose columns are a vector configuration in this moduli, define

$$a_d(X_1, \ldots, X_n; M) = \# \text{ of projective } d\text{-planes passing through } X_1, \ldots, X_n$$
$$\text{with a fixed } GL_{d+1}\text{-moduli of intersection } M,$$

and b_d analogously to before when M has $n+1$ columns. Let $M \oplus *$ denote the block direct sum of M and a nonzero 1×1 matrix, and $\tau^{\leq k+1}(M \oplus *)$ denote a generic rank $k+1$-projection of the resulting vector configuration. We show that the n-ary operation

$$[M]_\hbar(X_1, \ldots, X_n) = \sum b_k(X_1, \ldots, X_n; \tau^{\leq k+1}(M \oplus *))\hbar^k$$

depends only on the matroid of the vector configuration M and satisfies non-trivial recursive properties analogous to (and specializing to) the ones arising from small quantum cohomology, governed by certain matroid operations. Integrating out b_k then allows us to recover the enumerative answer a_k.

Through an explicit degeneration tree involving certain infinite polytopal regions, we are able to perturb every GL_{r+1}-orbit closure (whose partial convolutions compute the b_i classes) to a linear combination of orbit closures which are computed by the operations of small quantum cohomology.

Finally, we show that the GL_{r+1}-equivariant cohomology classes of these orbit closures are an equivariant deformation of the lattice point enumeration formula of Brion. For example, the diagonal in $H^\bullet((\mathbb{P}^r)^2) \cong \mathbb{Z}[H_1, H_2]/(H_1^{r+1}, H_2^{r+1})$ is given by $\sum_{i=0}^r H_1^{r-i}H_2^i =$ "$\frac{H_1^{r+1}-H_2^{r+1}}{H_1-H_2}$", which arises from Brion's formula applied to the line segment connecting $(r, 0)$ to $(0, r)$. The diagonal class in $H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^2) \cong \mathbb{Z}[c_1, \ldots, c_{r+1}][H_1, H_2]/(F(H_1), F(H_2))$, where $F(z) = z^{r+1} + c_1 z^r + \ldots + c_{r+1}$ is the universal Chern polynomial, is then given by "$\frac{F(H_1)-F(H_2)}{H_1-H_2}$".

1.2.2 Incidence strata of affine varieties with complex multiplicities

This project is joint with Dennis Tseng, adapted from [60], reproduced with permission. We construct an algebraic moduli space of colored $\mathbb{C}$-weighted point configurations on an affine variety using the Deligne categories $\mathrm{Rep}(S_n)$ [19] with $n \in \mathbb{C}$. Analyzing the singularities in the case of affine curves generalizes previously known singularity results for incidence strata on curves in the uncolored $\mathbb{N}$-weighted setting, and using this we resolve a question of Farb and Wolfson [24].

Let

$$Rat^*_{d,n} = \{\text{Degree } d \text{ pointed maps } \mathbb{P}^1 \to \mathbb{P}^n\}, \text{ and}$$
$$Poly^{d(n+1),1}_{n+1} = \{\text{Monic degree } d(n+1)\text{-polynomial with no multiplicity } n+1 \text{ root}\}.$$

Farb and Wolfson computed a large number of invariants of these two spaces, which all agreed with each other, and they showed the spaces were isomorphic when $d = 1$, $n \geq 2$. They conjectured for $n \geq 2$ that these varieties were in fact always isomorphic. We disprove this conjecture.

Theorem 1.2.1. *(S., Tseng) For all $n, d \geq 2$ we have $Rat^*_{d,n} \not\cong Poly^{d(n+1),1}_{n+1}$.*

These particular moduli spaces can be interpreted as moduli spaces of colored point configurations on $\mathbb{A}^1$. We show that for any affine variety X, there is a finite-type family

$$\Delta_{k,n}(X) \to \mathbb{C}^n \setminus \bigcup_{A \subset \{1,\ldots,n\}} \{\sum_{i\in A} x_i = 0\}$$

whose reduced fiber over $(n_1, \ldots, n_k) \in \mathbb{N}^k$ is the closed $(n_1, \ldots, n_k)$-incidence stratum in $\mathrm{Sym}^{n_1+\ldots+n_k}(X)$. This is surprising because we are interpolating between subvarieties of different spaces as $\sum n_i$ varies, and as a corollary we get that the singularities that occur in k-part incidence strata are of bounded complexity.

In the case that X is a curve, we are able to compute the singularity locus of any colored incidence stratum (which lies inside a product of symmetric powers of X, one for each color), and the condition generalizes naturally to the colored $\mathbb{C}$-weighted setting.

The finite-typeness (and effective elimination bound) for our construction requires genuine combinatorial arguments. Abstractly, the construction can be phrased via the Deligne-category $\mathrm{Rep}(S_{n_1}) \boxtimes \ldots \boxtimes \mathrm{Rep}(S_{n_k})$, with $n_1, \ldots, n_k \in \mathbb{C}$, interpolating between the representation categories of symmetric groups S_n with $n \in \mathbb{N}$. Thus for example to compute the (π, e)-incidence strata, we attempt to interpret as literally as possible the notion of "$\pi + e$ points" on X with π of the points grouped at one place and e points at the other, by considering "$\pi + e$ distinguishable points" in X grouped in this fashion, and quotienting by "$S_{\pi+e}$".

Part I

Euclidean Geometry

Chapter 2

Local maxima of quadratic boolean functions

This project is single author, adapted from [58], later published in a revised form in *Combinatorics, Probability, and Computing* [59], reproduced with permission. In this project, we prove that the number of strict local maxima that a quadratic boolean function $\Theta : \{0,1\}^n \to \mathbb{R}$ can have is at most $\binom{n}{\lfloor n/2 \rfloor}$, confirming a conjecture of Ron Holzman ([55], see Section 2.1). Our approach is via a generalization of Sperner's theorem that may be of independent interest.

2.1 Introduction

Let $\Theta : \{0,1\}^n \to \mathbb{R}$ be a real quadratic function (that is, a polynomial of total degree ≤ 2 in n variables $x_1, \ldots, x_n$). A *strict local maximum* (or just *local maximum*) of Θ on the discrete cube $Q_n = \{0,1\}^n$ is an element $v \in Q_n$ such that $\Theta(v) > \Theta(v')$ for all $v' \in Q_n$ differing by v in exactly one coordinate. We will prove the following conjecture attributed to Ron Holzman (see [55]; to the best of our knowledge it appeared first in the late 1980s, and was never formally published).

Theorem 2.1.1. *Let Θ be a quadratic function on Q_n. Then Θ has at most $\binom{n}{\lfloor n/2 \rfloor}$ local maxima.*

Example 2.1.2. *This bound is attained for example when $\Theta = -(x_1 + \ldots + x_n - \lfloor n/2 \rfloor)^2$. Indeed, $\Theta(x_1, \ldots, x_n) \leq 0$ with equality if and only if exactly $\lfloor n/2 \rfloor$ of the x_i take the value 1. We still attain equality if we perturb the coefficients of Θ slightly and apply automorphisms $x_i \mapsto 1 - x_i$ of Q_n.*

A special case of this is Kleitman's theorem [46] on the Littlewood-Offord problem, which states that the number of vertices of a parallelepiped in $\mathbb{R}^n$ with all side lengths > 2 in a ball of radius 1 is at most $\binom{n}{\lfloor n/2 \rfloor}$. Indeed, the negative square of the distance from the ball's center to the vertices of the parallelepiped can be interpreted as a quadratic boolean function on Q_n, so there are at most $\binom{n}{\lfloor n/2 \rfloor}$ points closer to the center of the ball than all of their neighbors (which is a necessary condition to be inside the ball).

Let $[n] = \{1, \ldots, n\}$ and let Δ denote symmetric set difference. We introduce the following new notion.

Definition 2.1.3. *Say that a family $\mathcal{F}$ of subsets of $[n]$ is* weakly Sperner *with respect to subsets $S_1, \ldots, S_n \subset [n]$ if for every i and every $A, B \in \mathcal{F}$ with $i \in A \Delta S_i$ and $i \notin B \Delta S_i$, we have $A \Delta S_i \not\supset B \Delta S_i$..*

An antichain is weakly Sperner with respect to $S_i = \emptyset$ for all $i \in [n]$.

We will find that a key step is to prove the following purely combinatorial result, which we reduce to in Subsection 2.2.1.

Theorem 2.1.4. *Let $\mathcal{F}$ be a weakly Sperner family of subsets in $[n]$ with respect to subsets $S_1, \ldots, S_n \subset [n]$. Then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.*

This generalizes Sperner's theorem on antichains (which is the case $S_i = \emptyset$). Moreover, these theorems seem closely related to the influence of boolean functions, specifically the $k = 2$ case of the Gotsman-Linial conjecture [38] on the maximal total influence of a polynomial threshold function $\mathrm{sgn}(p(x))$ on Q_n, where p is a real degree k polynomial.

2.2 Proof

We frequently view the discrete cube as the family of subsets of $[n]$ in the obvious way (using x_i as indicator functions). As we are only concerned with the value of Θ on Q_n, we may assume that all monomials in Θ are of degree 2, as the constant term is irrelevant, and we can replace x_i with x_i^2 if necessary. Thus we can assume $\Theta(x) = \sum_{i,j} q_{ij} x_i x_j$ for some symmetric matrix $\begin{bmatrix} q_{ij} \end{bmatrix}$.

2.2.1 Reduction to Theorem 2.1.4

We first reduce Theorem 2.1.1 to Theorem 2.1.4. The idea is that for certain pairs of elements A and B, the signs of the q_{ij} imply a contradiction from the pair of assertions that A is strictly larger than A' and B is strictly larger than B', where A', B' are obtained by switching the i'th coordinate from 0 to 1 or vice versa.

Claim 2.2.1. *Let $\Theta : Q_n \to \mathbb{R}$ be a real quadratic function, and set*

$$\mathcal{F} = \{A \subset [n] : A \textit{ is a local maximum of } \Theta\}.$$

Then $\mathcal{F}$ is weakly Sperner with respect to

$$S_i = \{j \in [n] \setminus \{i\} \mid q_{ij} > 0\}.$$

Proof. Fix $i \in [n]$. The difference between the value of Θ on the $x_i = 1$ plane and the $x_i = 0$ plane as a function of the remaining coordinates is the linear function

$$\theta_i(x) = q_{ii} + 2\sum_{j\neq i} q_{ij}x_j.$$

Since $i \notin S_i$, we need to show that we cannot have $A, B \in \mathcal{F}$ with $i \in A$, $i \notin B$, and $A\Delta S_i \supset B\Delta S_i$. Indeed, as A, B are local maxima, we must have

$$2\sum_{i\neq j\in A} q_{ij} - 2\sum_{i\neq j\in B} q_{ij} = 2\left(\sum_{i\neq j\in S_i^c\cap(A\setminus B)} q_{ij} - \sum_{i\neq j\in S_i\cap(B\setminus A)} q_{ij}\right) > 0,$$

which is clearly false as both terms are non-positive. Here we used that $A\Delta S_i \supset B\Delta S_i$ implies $B \setminus A \subset S_i$ and $A \setminus B \subset S_i^c$. □

2.2.2 Proof of Theorem 2.1.4

For the remainder of the proof, we view Q_n as the family of subsets of $[n]$. Recall $\mathcal{F}$ is a weakly Sperner family with respect to sets $S_1, \ldots, S_n \in Q_n$, and we want to show $|\mathcal{F}| \leq \binom{n}{\lfloor n/2\rfloor}$.

Inspired by Kleitman's proof of the Littlewood-Offord problem [46] (or see [9, Ch.4] for general background), we seek an inductive decomposition of Q_n (described below) into $\binom{n}{\lfloor n/2\rfloor}$ parts (which we will call quasichains), such that we can guarantee that at most one element of $\mathcal{F}$ lies inside each part. The heart of this proof is the definition of a "quasichain", which allows Kleitman's symmetric chain decomposition method to go through. This definition is rather surprising and seems contrived, however as we will see, once we have this definition the proof is straightforward.

Recall that a *tournament* is a complete directed graph. For sets $B, C \subset [n]$, we write $B \supset_{S_i} C$ to mean $B\Delta S_i \supset C\Delta S_i$.

Definition 2.2.2. *If $A, B \in Q_n$, $i \in [n]$, $S_1, \ldots, S_n \in Q_n$, we say that $A \xrightarrow{i} B$ is an* admissible arrow *with respect to* $S_1, \ldots, S_n$ *if* $i \in A \Delta S_i$*,* $i \notin B \Delta S_i$*, and* $A \supset_{S_i} B$*.*

Definition 2.2.3. *We define a* quasichain $\mathcal{C}$ *(with respect to* $S_1, \ldots, S_n$*) to be a coloured tournament (with colours in* $[n]$*) with vertex set a family* $\mathcal{G} = \{G_1, \ldots, G_k\} \subset Q_n$*, such that:*

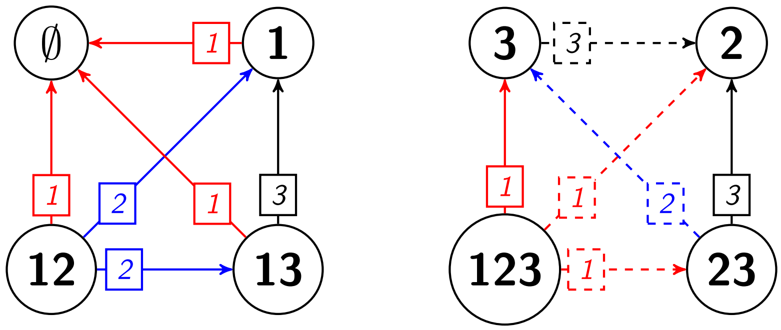
(i) whenever there is a directed edge from G_x *to* G_y *of colour* i*, then* $G_x \xrightarrow{i} G_y$ *is an admissible arrow with respect to* $S_1, \ldots, S_n$*, and*

(ii) for any subset of the colours (including the empty set), if we swap the direction of edges associated with those colours, then the resulting tournament is acyclic (or equivalently transitive).

It is easy to check that the acyclicity condition is equivalent to saying that no triangle has three distinct colours, any monochromatic triangle is acyclic, and any triangle with two distinct colours has the two edges with the same colour either both leaving, or both entering, the same vertex.

Note that a quasichain does *not* contain all possible information about $\supset_{S_i}$ containment between its various members, but rather remembers only one such containment for every pair (just enough information to guarantee that at most one element from the pair can lie in $\mathcal{F}$, and hence at most one element from each quasichain can lie in $\mathcal{F}$).

Example 2.2.4. *The following is a decomposition of the elements of* Q_3 *into two quasichains with respect to* $S_1 = \emptyset$*,* $S_2 = \{3\}$*, and* $S_3 = \{2\}$*. The solid arrows indicate a decomposition into* 3 *quasichains which arises from an inductive* symmetric quasichain decomposition*, described next.*



Definition 2.2.5. *Define a* symmetric quasichain decomposition of Q_n *(with respect to $S_1, \ldots, S_n$) to be a partition of Q_n into quasichains with respect to $S_1, \ldots, S_n$, inductively built up from the trivial decomposition of Q_0 as follows.*

Suppose we have such a partition of Q_{k-1} into quasichains $\{\mathcal{C}_1, \ldots, \mathcal{C}_r\}$ (with respect to $S_1 \cap [k-1], \ldots, S_{k-1} \cap [k-1]$). Then for each $\mathcal{C}_j$ with vertex set $\{A_1, \ldots, A_s\}$, we find an A_t such that $\{A_1, A_2, \ldots, A_s, A_t \cup \{k\}\}$ and $\{A_1 \cup \{k\}, \ldots, A_{t-1} \cup \{k\}, A_{t+1} \cup \{k\}, \ldots, A_s \cup \{k\}\}$ both support quasichain structures with respect to $S_1 \cap [k], \ldots, S_k \cap [k]$. Discarding any resulting quasichains with empty vertex set, we obtain a partition of Q_k into quasichains with respect to $S_1 \cap [k], \ldots, S_k \cap [k]$.

Finding such an A_t to create two new quasichains is the main difficulty.

We claim that this process will always result in $\binom{n}{\lfloor n/2 \rfloor}$ subsets of Q_n. Indeed, if we let $b_{n,r}$ be the number of parts of size r in Q_n, then $b_{n,r} = 1_{r \neq 1} b_{n-1,r-1} + b_{n-1,r+1}$, and $b_{0,r} = 1_{r=1}$. One can directly check that setting $b_{n,n+1-2r} = \binom{n}{r} - \binom{n}{r-1}$ for $1 \leq r \leq \lfloor n/2 \rfloor$, and otherwise setting $b_{n,k} = 0$, satisfies the recurrence, and the sum of the $b_{n,r}$ for fixed n is a telescoping sum evaluating to $\binom{n}{\lfloor n/2 \rfloor}$ (see e.g. [9, p.17-20]).

As a quasichain can intersect our weakly Sperner set $\mathcal{F}$ in at most one element, producing a symmetric quasichain decomposition for Q_n with respect to $S_1, \ldots, S_n$ shows $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ as desired. The following lemma can be checked directly by splitting into cases depending on whether n belongs to A or B.

Lemma 2.2.6. *Suppose $i \in [n-1]$, and $A \xrightarrow{i} B$ is an admissible arrow in Q_{n-1} with respect to $S_1 \cap [n-1], \ldots, S_{n-1} \cap [n-1]$. Then the following are admissible arrows with respect to $S_1, \ldots, S_n$ in Q_n:*

(i) $A \xrightarrow{i} B$,

(ii) $A \cup \{n\} \xrightarrow{i} B \cup \{n\}$

(iii) $A \cup \{n\} \xrightarrow{i} B$ *if* $n \notin S_i$,

(iv) $A \xrightarrow{i} B \cup \{n\}$ *if* $n \in S_i$.

Proof. Omitted. □

Proof of Theorem 2.1.4. We will prove the theorem by induction on n, showing that Q_n admits a symmetric quasichain decomposition with respect to any collection of n sets $S_1, \ldots, S_n$. For $n = 0$, we simply take the whole of Q_0 as our quasichain, so now assume that $n \geq 1$.

Given $S_1, \ldots, S_n \in Q_n$, we have by the induction hypothesis a symmetric quasichain decomposition for Q_{n-1} with respect to $S_1 \cap [n-1], \ldots, S_{n-1} \cap [n-1]$. We will extend this decomposition to one of Q_n with respect to $S_1, \ldots, S_n$. If we view a quasichain $\mathcal{C}$ with underlying vertex set $\{A_1, \ldots, A_r\}$ in Q_{n-1} with respect to $S_1 \cap [n-1], \ldots, S_{n-1} \cap [n-1]$ as a coloured tournament in Q_n, then all arrows remain admissible with respect to $S_1, \ldots, S_n$ by part (i) of Lemma 2.2.6, and the second quasichain condition is clearly preserved, so it remains a quasichain. Similarly, $\mathcal{C}^n = \{A_1 \cup \{n\}, \ldots, A_r \cup \{n\}\}$ is also a quasichain with respect to $S_1, \ldots, S_n$ (with the same coloured arrows) by part (ii) of Lemma 2.2.6. To complete the symmetric quasichain decomposition, we need to exhibit an $A_t \cup \{n\}$ we can transfer from $\mathcal{C}^n$ to $\mathcal{C}$ in such a way that they can be made into quasichains with respect to $S_1, \ldots, S_n$.

Let $R = \{i \in [n-1] \mid n \text{ lies in } S_i\}$. By the second quasichain condition for $\mathcal{C}$, if all arrows with colours in R are reversed, then the resulting tournament is acyclic; let A be a maximal element for this new graph. This means that in $\mathcal{C}$, the colour of all arrows pointing into A belong to R, and the colour of all arrows out do not belong to R. We choose $A \cup \{n\}$ as the element to transfer from $\mathcal{C}^n$ to $\mathcal{C}$. If we remove a vertex from a quasichain, then all arrows remain admissible, and a subgraph of an acyclic graph is acyclic, so $\mathcal{C}^n - (A \cup \{n\})$ is a quasichain if we keep the same directed edges on the remaining vertices. Thus it suffices to show that $\mathcal{C} \cup \{A \cup \{n\}\}$ is in fact a quasichain once we appropriately colour and direct edges containing $A \cup \{n\}$ (we retain the same edge colourings on all edges which were already in $\mathcal{C}$).

If $A \xrightarrow{i} X$, then $i \notin R$, so $n \notin S_i$, so $A \cup \{n\} \xrightarrow{i} X$ by part (iii) of Lemma 2.2.6. If $X \xrightarrow{i} A$, then $i \in R$, so $n \in S_i$, so $X \xrightarrow{i} A \cup \{n\}$ is admissible by part (iv) of Lemma 2.2.6. Thus if we duplicate the direction and colour of all arrows incident to A in $\mathcal{C}$ for $A \cup \{n\}$, then these new arrows will all be admissible. Finally, $A \cup \{n\} \to A$ is admissible if $n \notin S_n$, and $A \xrightarrow{n} A \cup \{n\}$ is admissible if $n \in S_n$, so one of the two directions for $\xrightarrow{n}$ between $A \cup \{n\}$ and A gives us the final admissible arrow for $\mathcal{C} \cup \{A \cup \{n\}\}$.

To complete the proof, we must show that the newly constructed graph satisfies the acyclicity condition. After swapping some directions associated with a subset of the colours, we have a tournament H, with vertices x, y corresponding to $A, A \cup \{n\}$ respectively. For all z, then, we have either $x \to z$, $y \to z$, or $z \to x$, $z \to y$. If we have a cycle, then identifying x and y yields a cycle in the original graph (since x and y have the same incoming/outgoing edges, this identification is well-defined), which is a contradiction. □

Chapter 3

Sharp Brunn-Minkowski stability for homothetic regions

This project is joint with Peter van Hintum and Marius Tiba, adapted from [61], reproduced with permission. We prove a sharp stability result concerning how close homothetic sets attaining near-equality in the Brunn-Minkowski inequality are to being convex. In particular, we show there are universal constants $C_n, d_n > 0$ such that for $A \subset \mathbb{R}^n$ of positive measure, if $|\frac{A+A}{2} \setminus A| \leq d_n|A|$, then $|\operatorname{co}(A) \setminus A| \leq C_n|\frac{A+A}{2} \setminus A|$ for $\operatorname{co}(A)$ the convex hull of A, resolving a conjecture of Figalli and Jerison [27].

3.1 Introduction

The Brunn-Minkowski inequality states that for non-empty measurable subsets[1] A, B of $\mathbb{R}^n$,

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}$$

[1] We will always assume A, B have positive measure throughout.

for $|\cdot|$ the outer Lebesgue measure. Equality is known to hold if and only if A and B are homothetic copies of the same convex body (less a measure 0 set). A natural question is whether this inequality is stable: if we are close to equality in the Brunn-Minkowski inequality, are A and B close to homothetic copies of the same convex body? More precisely, we want to know if

$$\omega = \min_{\substack{K_A \supset A, K_B \supset B \\ K_A, K_B \text{ homothetic convex sets}}} \max \left\{ \frac{|K_A \setminus A|}{|A|}, \frac{|K_B \setminus B|}{|B|} \right\}$$

is bounded above in terms of the quantities

$$\delta' = \frac{|A+B|^{\frac{1}{n}}}{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}} - 1, \text{ and } t = \frac{|A|^{\frac{1}{n}}}{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}}.$$

The bound should be positively correlated with δ', and negatively correlated with $\min(t, 1-t)$ (as when $\min(t, 1-t)$ is smaller the volumes of A, B are more disproportionate).

We prove the following sharp stability result for the Brunn-Minkowski inequality in the particular case that A, B are homothetic sets. Taking $A = B$ resolves a conjecture of Figalli and Jerison [27].

Theorem 3.1.1. *For all $n \geq 2$, there is a (computable) constant $C'_n > 0$ and (computable) constants $d_n(\tau) > 0$ for each $\tau \in (0, \frac{1}{2}]$ such that the following is true. With the notation above, if $\tau \in (0, \frac{1}{2}]$ and $A, B \subset \mathbb{R}^n$ are measurable homothetic sets such that $t \in [\tau, 1-\tau]$ and $\delta' \leq d_n(\tau)$, then*

$$\omega \leq C'_n \tau^{-1} \delta'.$$

For these optimal exponents, we also show that $e^{\Omega(n)} \leq C'_n \leq e^{O(n \log n)}$ with explicit constants. We discuss this further in Section 3.5.

The most general stability result for the Brunn-Minkowski inequality was proved in a landmark paper by Figalli and Jerison [26, Theorem 1.3]. There they showed that for arbitrary measurable sets $A, B \subset \mathbb{R}^n$, there exists (computable) constants $\beta_n, C_n > 0$ and $\alpha_n(\tau), d_n(\tau) > 0$ for each $\tau \in (0, \frac{1}{2}]$ such that if $t \in [\tau, 1-\tau]$ and $\delta' \leq d_n(\tau)$ then

$$\omega \leq C_n \tau^{-\beta_n} \delta'^{\alpha_n(\tau)}$$

(prior to this result, Christ [17] had proved a non-computable non-polynomial bound involving δ' and τ via a compactness argument). A natural question is therefore to find the optimal exponents of δ' and τ, prioritized in this order. This question, with A, B restricted to various sub-classes of geometric objects, is the subject of a large body of literature. These optimal exponents potentially depend on which class of objects is being considered. For arbitrary measurable A, B the question is still wide open. Our result is the first sharp stability result of its kind which does not require one of A, B to be convex.

Most of the literature focuses on upper bounding a measure closely related to ω for how close A, B are to the same convex set, namely the *asymmetry index* [30]

$$\alpha(A, B) = \inf_{x \in \mathbb{R}^n} \frac{|A \,\Delta\, (s \cdot \mathrm{co}(B) + x)|}{|A|}$$

where $\mathrm{co}(B)$ is the convex hull of B, and s satisfies $|A| = |s \cdot \mathrm{co}(B)|$. We always have $\alpha(A, B) \leq 2\omega$, so bounding the asymmetry index is weaker than bounding ω. When A and B are convex, the optimal inequality $\alpha(A, B) \leq C_n \tau^{-\frac{1}{2}} \delta'^{\frac{1}{2}}$ was obtained by Figalli, Maggi, and Pratelli in [29, 30]. When B is a ball and A is arbitrary, the optimal inequality $\alpha(A, B) \leq C_n \tau^{-\frac{1}{2}} \delta'^{\frac{1}{2}}$ was obtained by Figalli, Maggi, and Mooney in [28]. We note that this particular case is intimately connected with stability for the isoperimetric inequality. When just B is convex the (non-optimal) inequality

$\alpha(A, B) \leq C_n \tau^{-(n+\frac{3}{4})} \delta'^{\frac{1}{4}}$ was obtained by Carlen and Maggi in [14]. Finally, Barchiesi and Julin [5] showed that when just B is convex, we have the optimal inequality $\alpha(A, B) \leq C_n \tau^{-\frac{1}{2}} \delta'^{\frac{1}{2}}$, subsuming these previous results.

In [25] Figalli and Jerison gave an upper bound for ω when $A = B$, and later in [27] they conjectured the sharp bound $\omega \leq C_n \delta'$ when $A = B$. This conjecture was proved in [27] for $n \leq 3$ using an intricate analysis which unfortunately does not extend beyond this case. Later on, Figalli and Jerison suggested a stronger conjecture that $\omega \leq C_n \tau^{-1} \delta'$ for A, B homothetic regions, which we will prove in this paper.

Because previous sharp exponent results have taken at least one of A, B to be convex, allowing for the use of robust techniques from convex geometry, the implicit hope was that solving this special case would shine a light on the general case where previous methods are not as applicable. The methods we use are indeed very different from the ones from convex geometry and, after an initial reduction, from [27]. We hope that these new techniques, particularly the fractal and the boundary covering detailed in Subsection 3.1.2, can provide new insight into finding optimal exponents for general A, B.

3.1.1 Main theorem

As we are considering homothetic regions A, B, we can replace A with tA and B with $(1-t)A$. Note that t retains its earlier meaning as $t = \frac{|tA|^{\frac{1}{n}}}{|tA|^{\frac{1}{n}} + |(1-t)A|^{\frac{1}{n}}}$. Define the *interpolated sumset* of A as

$$D(A;t) := tA + (1-t)A = \{ta_1 + (1-t)a_2 \mid a_1, a_2 \in A\}.$$

Note that we always have $A \subset D(A;t)$. To quantify how small $D(A;t)$ is, we introduce the expression

$$\delta(A;t) := |D(A;t) \setminus A|.$$

As a further simplification, we note that

$$\delta' = \frac{|D(A;t)|^{\frac{1}{n}}}{|A|^{\frac{1}{n}}} - 1 = \left(\frac{1}{n} + o(1)\right)\left(\frac{|D(A;t)|}{|A|} - 1\right) = \left(\frac{1}{n} + o(1)\right)\frac{\delta(A;t)}{|A|}$$

where $o(1)$ depends on the upper bound on δ'. Since the exponent of δ' is always at most 1 (as shown by Example 3.1.4), we may work with $\frac{\delta(A;t)}{|A|}$ in place of δ' by absorbing the $\frac{1}{n} + o(1)$ term into C_n' to make a new constant C_n.

The following is the specialization of [26, Theorem 1.3] to homothetic A, B, which we restate for the reader's convenience.

Theorem 3.1.2. *(Figalli and Jerison [26]) For $n \geq 2$ there are (computable) constants $\beta_n, C_n > 0$, and (computable) constants $\alpha_n(\tau), d_n(\tau) > 0$ for each $\tau \in (0, \frac{1}{2}]$, such that the following is true. If $A \subset \mathbb{R}^n$ is a measurable set, $\tau \in (0, \frac{1}{2}]$ and $t \in [\tau, 1-\tau]$, then*

$$|\operatorname{co}(A) \setminus A| \leq C_n |A| \tau^{-\beta_n} \left(\frac{\delta(A;t)}{|A|}\right)^{\alpha_n(\tau)}$$

whenever $\delta(A;t) \leq d_n(\tau)|A|$.

Our main result optimizes the exponents to be $\alpha_n = \beta_n = 1$ in Theorem 3.1.2, verifying the conjecture of [27] and the further generalization to homothetic sets suggested by Figalli and Jerison.

Theorem 3.1.3. *(Theorem 3.1.1 reformulated) For all $n \geq 2$, there is a (computable) constant $C_n > 0$ (we can take $C_n = (4n)^{5n}$), and (computable) constants $\Delta_n(\tau) > 0$ for each $\tau \in (0, \frac{1}{2}]$ such that the following is true. If $A \subset \mathbb{R}^n$ is a measurable set,*

$\tau \in (0, \frac{1}{2}]$ and $t \in [\tau, 1-\tau]$, then

$$|\operatorname{co}(A) \setminus A| \le C_n \tau^{-1} \delta(A;t)$$

whenever $\delta(A;t) \le \Delta_n(\tau)|A|$.

Example 3.1.4. *To see that the exponents on δ and τ are sharp, suppose we have some inequality of the form*

$$|\operatorname{co}(A) \setminus A| \le C_n |A| \tau^{-\rho_1} (\delta(A;t)|A|^{-1})^{\rho_2}$$

whenever $\delta(A;t) \le \Delta_n(\tau)|A|$. Take $A = \{(0,0) \cup [\lambda, 1+\lambda] \times [0,1]\} \times [0,1]^{n-2}$, with $\lambda \ll \frac{\Delta_n(\tau)}{2\tau}$, and $t = \tau$. The inequality then becomes $\frac{\lambda}{2} \le C_n \tau^{-\rho_1}(\tau\lambda(2-3\tau))^{\rho_2}$. Because we can take λ arbitrarily small, it follows that $\rho_2 \le 1$, so $\rho_2 = 1$ would be the optimal exponent. Given $\rho_2 = 1$, we then have $\rho_1 \ge 1$, so $\rho_1 = 1$ would be the optimal exponent.

Remark 3.1.5. *When $n = 1$, Theorem 1.1 from [25] with A replaced with tA and B replaced with $(1-t)A$ shows that the optimal exponents are actually $\tau^0 \delta(A;t)^1$ in contrast to the case $n \ge 2$.*

Example 3.1.6. *Given exponents $\rho_1 = \rho_2 = 1$, the constant C_n grows at least exponentially as shown by the following example. Let $R \ge 2$. Consider the set $A \subset \mathbb{R}^n$, $A = [0,2]^{n-1} \times [-R, 0] \cup \{(0, \dots, 0, 2)\}$. Then $\operatorname{co}(A) = A \cup \bigcup_{x \in [0,2]} [0, 2-x]^{n-1} \times \{x\}$ and $\frac{A+A}{2} = A \cup [0,1]^n$. Hence, $\delta(A, \frac{1}{2}) = 1$ and $|\operatorname{co}(A) \setminus A| = \int_{x \in [0,2]} (2-x)^{n-1} dx = \frac{2^n}{n}$. This example shows that $C_n \ge \frac{2^{n-1}}{n}$.*

3.1.2 Outline

By replacing t with $1-t$ we may assume that $t \le \frac{1}{2}$.

Initial reduction

We first carry out a straightforward reduction along the lines of the reduction in [27] to [27, Lemma 2.2], reducing to the case that $\mathrm{co}(A)$ is a simplex T, and A contains all of the vertices of T. In this reduction we use Theorem 3.1.2, though we need only the following much weaker statement due to Christ [17]: $|\mathrm{co}(A) \setminus A||A|^{-1}$ is bounded above by a (computable) function of the parameters $\delta(A;t)|A|^{-1}$ and τ which, for fixed τ, tends to 0 as $\delta(A;t)|A|^{-1}$ tends to 0.

Fractal structure

Next we show that if $\delta(A;t)|A|^{-1}$ is small, then A contains an approximate fractal structure. For each i we recursively construct a nested sequence of families of simplices $\mathcal{T}_{i,0} \subset \mathcal{T}_{i,1} \subset \ldots$; each family $\mathcal{T}_{i,k}$ consists of translates of $(1-t)^i T$ contained inside T, and in the limit $\cup_k \mathcal{T}_{i,k}$ is dense among the translates of $(1-t)^i T$ contained inside T. We show that there exist universal constants $c_{i,k,n} = i + 2k$ such that for translates $T' \in \mathcal{T}_{i,k}$,

$$|((1-t)^i A)_{T'} \cap A| \geq |((1-t)^i A)_{T'}| - c_{i,k,n}\delta(A;t),$$

where $((1-t)^i A)_{T'}$ is the translate of $(1-t)^i A$ induced by the translation that identifies $(1-t)^i T$ with T'. Though we need the fractal structure in order to prove this inequality recursively, we only use the corollary that $|T' \cap A| \geq \frac{|T'|}{|T|}|A| - c_{i,k,n}\delta(A;t)$. This corollary quantitatively establishes that A becomes more homogeneous in T as $\delta(A;t)|A|^{-1} \to 0$.

Covering a thickened ∂T with small total volume

Next, we consider a large homothetic scaled copy $R := (1-\zeta)T$ inside T for $\zeta \approx \frac{1}{n^4}$ and we produce a cover $\mathcal{A} \subset \mathcal{T}_{i,k}$ of $T \setminus R$ for $i \approx 5\log(n)/t$ and $k \approx n\log(n)/t$. The cover $\mathcal{A}$ consists of translates of $(1-t)^i T \approx \frac{1}{n^5}T$ and has the property that the size of $\mathcal{A}$ is at most $(2n)^{5n}$ and the total volume of the simplicies in $\mathcal{A}$ is less than $\frac{1}{2}|T|$. We note that $|\mathcal{A}|, i, k$ affect the complexity of C_n, whereas ζ affects only the complexity of $\Delta_n(\tau)$ and not C_n.

In order to produce the covering $\mathcal{A}$ above we proceed in two steps. First, we use a covering result of Rogers [56] to produce an efficient covering $\mathcal{B}$ of $T \setminus R$ with translates of $n^{-\frac{1}{n}}(1-t)^i T$ contained inside T. The covering $\mathcal{B}$ has the property that the size of $\mathcal{B}$ is at most $(2n)^{5n}$ and the total volume of the simplicies in $\mathcal{B}$ is less than $\frac{1}{2n}|T|$. Second, we show that for each translate T' of $n^{-\frac{1}{n}}(1-t)^i T$ contained inside T, there exists a simplex $T'' \in \mathcal{T}_{i,k}$ such that $T' \subset T''$. This naturally gives the desired cover $\mathcal{A}$.

Putting it all together

We may assume that $R \subset D(A;t)$ since a straightforward argument shows this holds whenever $|T \setminus A||A|^{-1}$ is sufficiently small, and $|T \setminus A||A|^{-1} \to 0$ as $\delta(A;t)|A|^{-1} \to 0$ by Theorem 3.1.2. Rephrasing the homogeneity statement for A, for each $T' \in \mathcal{T}_{i,k}$ we have

$$|T' \setminus A| \leq \frac{|T \setminus A|}{|T|}|T'| + c_{i,k,n}\delta(A;t).$$

Because $\mathcal{A}$ covers $T \setminus R$ and $A \subset D(A;t)$, we have $|T \setminus D(A;t)| \leq \sum_{T' \in \mathcal{A}} |T' \setminus A|$, and by construction $\sum_{T' \in \mathcal{A}} |T'| \leq \frac{1}{2}|T|$. Combining these facts, we immediately deduce

$$|T \setminus D(A;t)| \leq \frac{1}{2}|T \setminus A| + |\mathcal{A}|c_{i,k,n}\delta(A;t),$$

i.e.

$$|T \setminus A| \leq 2(1 + |\mathcal{A}|c_{i,k,n})\delta(A;t).$$

Because $|\mathcal{A}| \leq (2n)^{5n}$ and $c_{i,k,n} \approx \frac{n\log(n)}{t}$, we see that with $C_n = (4n)^{5n}$ we have

$$|T \setminus A| \leq C_n \tau^{-1} \delta(A; t).$$

3.2 Initial reduction

In this section, we will reduce Theorem 3.1.3 to Theorem 3.2.1, similar to the initial reduction in [27] to [27, Lemma 2.2].

Theorem 3.2.1. *For all $n \geq 2$ there are (computable) constants $C_n > 0$ (we can take $C_n = (4n)^{5n}$) and constants $0 < \delta_n(\tau) < 1$ for each $\tau \in (0, \frac{1}{2}]$ such that the following is true. Let $\tau \in (0, \frac{1}{2}]$, $t \in [\tau, 1-\tau]$, and suppose $T \subset \mathbb{R}^n$ is a simplex with $|T| = 1$, $A \subset T$ a measurable subset containing all vertices of T, and $|A| = 1 - \delta$ with $0 < \delta \leq \delta_n(\tau)$. Then*

$$|T \setminus A| \leq C_n \tau^{-1} \delta(A; t).$$

We first need the following geometric lemma.

Lemma 3.2.2. *For every convex polytope P, there exists a point $o \in P$ (which we set to be the origin) such that the following is true. For any constant $b_n(\tau) \in (0, 1)$, there exists a constant $\epsilon_n(\tau)$ such that for any $A \subset P$, if $t \in [\tau, 1-\tau]$ and $|P \setminus A| \leq \epsilon_n(\tau)|P|$, then $(1 - b_n(\tau))P \subset D(A; t)$.*

Proof. We may assume that $t \leq \frac{1}{2}$ as the statement is invariant under replacing t with $1 - t$. Without loss of generality we may assume that $|P| = 1$. By a lemma of John [41], after a volume-preserving affine transformation, there exists a ball $B \subset P$ of radius n^{-1}. Denote o for the center of B, and set o to be the origin.

We will show that $(1 - b_n(\tau))P \subset D(A; t)$. Take $x \in (1 - b_n(\tau))P$, and let y be the intersection of the ray ox with ∂P. Note that the ratio $r = |xy|/|oy| \geq b_n(\tau)$.

Let H be the homothety with center y and ratio r. This homothety sends o to x and P to $H(P)$. Note that $H(P) \subset P$ because P is convex. Denoting

$$A' = A \cap H(P),$$

we have

$$|A'| \geq r^n - \epsilon_n(\tau).$$

The statement $x \in D(A'; t)$ is implied by the statement that $o \in D(C; t)$ for $C = H^{-1}(A') \subset P$, which we will now show (in fact we will show $o \in D(C \cap B; t)$).

Note that $|C| \geq 1 - r^{-n}\epsilon_n(\tau)$, so $|B \setminus C| \leq r^{-n}\epsilon_n(\tau)$. Consider the negative homothety H' scaling by a factor of $-\frac{t}{1-t} \in [-1, 0)$ about o. If $o \notin D(C \cap B; t)$, then at least one of y and $H'(y)$ is not in $C \cap B$ for every $y \in B$. A simple volume argument shows that this would imply $|B \setminus C| \geq \frac{1}{2}|H(B')|$, and as B contains a cube of side length $2/n^{3/2}$ we would have

$$r^{-n}\epsilon_n(\tau) \geq |B \setminus C| \geq \frac{1}{2}|H'(B)| = \frac{1}{2}\left(\frac{t}{1-t}\right)^n |B| \geq \frac{1}{2}\left(\frac{\tau}{1-\tau}\right)^n \left(\frac{2}{n^{3/2}}\right)^n.$$

Therefore as $b_n(\tau)^{-n} \geq r^{-n}$, taking

$$\epsilon_n(\tau) < b_n(\tau)^n \frac{1}{2}\left(\frac{\tau}{1-\tau}\right)^n \left(\frac{2}{n^{3/2}}\right)^n,$$

we deduce that $o \in D(C \cap B; t)$ and therefore in particular $x \in D(A'; t)$.

□

Observation 3.2.3. *If P is a (regular) simplex T, we can take o to be the barycenter of T.*

Proof that Theorem 3.2.1 implies Theorem 3.1.3. We may assume that $t \leq \frac{1}{2}$ since

Theorem 3.1.3 is invariant under replacing t with $1-t$. By approximation, we can assume that A has polyhedral convex hull $\operatorname{co}(A)$ with the vertices of $\operatorname{co}(A)$ lying in A (see e.g. [27, p.3 footnote 2]).

Take $b_n(\tau)$ to be the minimum of τ and the constant such that

$$\delta_n(\tau)^{-1}(1-(1-b_n(\tau))^n) = 1 - C_n^{-1}\tau,$$

and take $\epsilon_n(\tau)$ as in Lemma 3.2.2.

From Theorem 3.1.2, we see that we can choose $\Delta_n(\tau)$ sufficiently small so that

$$|\operatorname{co}(A) \setminus A| \leq \epsilon_n(\tau)|A| \leq \epsilon_n(\tau)|\operatorname{co}(A)|,$$

and therefore by Lemma 3.2.2 there is a translate of $(1-b_n(\tau))\operatorname{co}(A) \subset D(A;t)$. Let o be the center of homothety relating this translate of $(1-b_n(\tau))\operatorname{co}(A)$ and $\operatorname{co}(A)$. Because $b_n(\tau) \leq \tau$, the region $to + (1-t)\operatorname{co}(A)$ is contained in $D(A;t)$, so from this we deduce that $D(A \cup \{o\}; t) = D(A;t)$. Therefore we may assume without loss of generality that $o \in A$.

Note that the inequality in Theorem 3.1.3 that we want to deduce is equivalent to

$$|\operatorname{co}(A) \setminus D(A;t)| \leq (1 - C_n^{-1}\tau)|\operatorname{co}(A) \setminus A|.$$

Triangulate $\operatorname{co}(A)$ into simplices T_i by triangulating $\partial\operatorname{co}(A)$ and coning off each facet at o. Then in each simplex T_i, we claim that

$$|T_i \setminus D(A;t)| \leq (1 - C_n^{-1}\tau)|T_i \setminus A|.$$

Provided $|T_i \setminus A| \leq \delta_n(\tau)|T_i|$, applying Theorem 3.2.1 to $T_i, A \cap T_i$ yields the stronger inequality

$$|T_i \setminus D(A \cap T_i; t)| \leq (1 - C_n^{-1}\tau)|T_i \setminus A|.$$

On the other hand, if $|T_i \setminus A| \geq \delta_n(\tau)|T_i|$, then as $b_n(\tau)o + (1 - b_n(\tau))T_i \subset D(A; t) \cap T_i$, we have

$$\begin{aligned} |T_i \setminus D(A; t)| \leq |T_i|(1 - (1 - b_n(\tau))^n) &\leq \delta_n(\tau)^{-1}(1 - (1 - b_n(\tau))^n)|T_i \setminus A| \\ &\leq (1 - C_n^{-1}\tau)|T_i \setminus A|. \end{aligned}$$

We conclude by noting

$$|\operatorname{co}(A) \setminus D(A; t)| = \sum |T_i \setminus D(A; t)| \leq \sum (1 - C_n \tau^{-1})|T_i \setminus A| = (1 - C_n^{-1}\tau)|\operatorname{co}(A) \setminus A|.$$

$\square$

3.3 Setup and technical lemmas

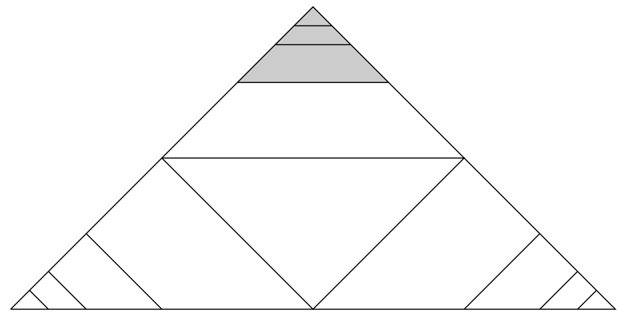
We may assume that $t \leq \frac{1}{2}$ since Theorem 3.2.1 is invariant under replacing t with $1-t$. It suffices to prove the statement for a particular choice of T since all simplices of volume 1 in $\mathbb{R}^n$ are equivalent under volume-preserving affine transformations. Hence we work in a fixed regular simplex $T \subset \mathbb{R}^n$ from now on. Let $x_0, \ldots, x_n$ denote the vertices of T, and define the *corner λ^i-scaled simplices* to be

$$S_i^j(\lambda) = (1 - \lambda^i)x_j + \lambda^i T \text{ for } 0 \leq j \leq n$$

and set

$$\mathcal{S}_i(\lambda) := \{S_i^0(\lambda), \ldots, S_i^n(\lambda)\}.$$

In the picture below, we've shaded one of the $S_2^j(\frac{1}{2})$'s inside T when $n = 2$.



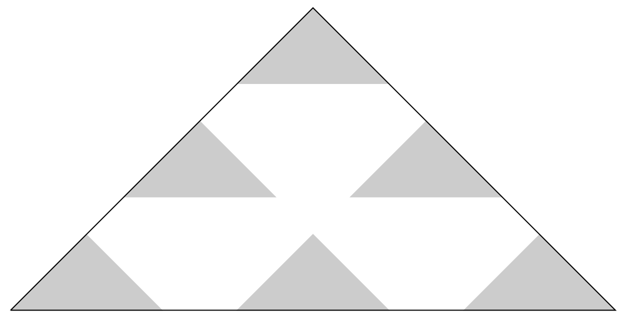
Define the λ^i*-scaled* k*-averaged simplices* $\mathcal{T}_{i,k}(\lambda)$ iteratively by

$$\mathcal{T}_{i,0}(\lambda) = \mathcal{S}_i(\lambda)$$
$$\mathcal{T}_{i,k+1}(\lambda) = \{\lambda B_1 + (1-\lambda)B_2 \mid B_1, B_2 \in \mathcal{T}_{i,k}(\lambda)\}.$$

Note that all simplices in $\mathcal{T}_{i,k}(\lambda)$ are translates of $\lambda^i T$, and we have the inclusions

$$\mathcal{T}_{i,0}(\lambda) \subset \mathcal{T}_{i,1}(\lambda) \subset \mathcal{T}_{i,2}(\lambda) \subset \dots$$

For fixed i, λ, the simplices in the family $\mathcal{T}_{i,k}(\lambda)$ eventually cover all of T and heavily overlap each other as $k \to \infty$ (in fact the translates become dense among all possible translates of $\lambda^i T$ which lie inside T). Shaded below are the simplices in $\mathcal{T}_{2,1}(\frac{1}{2})$ when $n = 2$.



Lemma 3.3.1 is the crux of our argument. The proof of Lemma 3.3.1 shows that for all $T' \in \mathcal{T}_{i,k}(1-t)$, the set $|T' \cap A|$ contains a translated copy of $(1-t)^i A$ (up to a

bounded error). This fractal structure allows us to conclude that $|T' \cap A|$ is bounded below by $|T'|(1-\delta)$ (up to a bounded error).

Lemma 3.3.1. *The constants* $c_{i,k,n} = i + 2k$ *are such that for every* $T' \in \mathcal{T}_{i,k}(1-t)$ *we have*

$$|T' \cap A| \geq |T'|(1-\delta) - c_{i,k,n}\delta(A;t).$$

Proof. For the remainder of this proof, we will denote

$$\lambda = 1 - t,$$

and write for notational convenience S_i^j instead of $S_i^j(\lambda)$. The following notation will be useful for us: consider the translation that brings $\lambda^i T$ to T' and denote by $(\lambda^i A)_{T'}$ the shift of the set $\lambda^i A$ under this translation.

We shall actually show the stronger inequalities

$$|(\lambda^i A)_{T'} \setminus A| \leq c_{i,k,n}\delta(A;t)$$

(which are stronger as $|(\lambda^i A)_{T'}| = |T'|(1-\delta)$).

First, we show the inequality when $k = 0$. Recall that if $T' \in \mathcal{T}_{i,0}(\lambda)$ then $T' = S_i^j$ for some j. The inequality is trivial for $(i,k) = (0,0)$ by definition of δ.

We now show the inequality for $(i,k) = (1,0)$. Note $(\lambda A)_{S_1^j} = \lambda x_j + (1-\lambda)A \subset D(A;t)$, so

$$|(\lambda A)_{S_1^j} \setminus A| \leq |D(A;t) \setminus A| = \delta(A;t).$$

Suppose we know the result for $(i,0)$, we now prove the result for $(i+1,0)$. Then

$(\lambda^{i+1}A)_{S^j_{i+1}} = (1-\lambda^{i+1})x_j + \lambda^{i+1}A$, and we have

$$\begin{aligned} |(\lambda^{i+1}A)_{S^j_{i+1}} \setminus A| &\le |(\lambda^{i+1}A)_{S^j_{i+1}} \setminus (\lambda A)_{S^j_1}| + |(\lambda A)_{S^j_1} \setminus A| \\ &= \lambda^n |(\lambda^i A)_{S^j_i} \setminus A| + |(\lambda A)_{S^j_1} \setminus A| \\ &\le (\lambda^n c_{i,0,n} + c_{1,0,n})\delta(A;t) \\ &\le c_{i+1,0,n}\delta(A;t). \end{aligned}$$

Finally, we induct on k. We have proved the base case $k = 0$, so assume the inequality for (i,k). We will now prove the inequality for $(i,k+1)$.

Thus we suppose that $T' \in \mathcal{T}_{i,k+1}$, which by definition means that there exists $T'_1, T'_2 \in \mathcal{T}_{i,k}$ such that

$$T' = \lambda T'_1 + (1-\lambda)T'_2.$$

We now prove an easy claim before returning to the proof of the lemma.

Claim 3.3.2. *Let X, X' be translates of each other in $\mathbb{R}^n$ with common volume $V = |X| = |X'|$, and let $Y \subset X$, $Y' \subset X'$. Then if V' is a constant such that $|X \setminus Y|, |X' \setminus Y'| \le V'$, we have*

$$|\lambda Y + (1-\lambda)Y'| \ge V - V'.$$

Proof. We have $|Y|, |Y'| \ge V - V'$, so the result follows from the Brunn-Minkowski inequality. □

Returning to the proof of the lemma, we have by the induction hypothesis that both

$$\begin{aligned} &|(\lambda^i A)_{T'_1} \setminus A| \le c_{i,k,n}\delta(A;t), \text{ and} \\ &|(\lambda^i A)_{T'_2} \setminus A| \le c_{i,k,n}\delta(A;t). \end{aligned}$$

Because $(\lambda^i A)_{T_1'}$ and $(\lambda^i A)_{T_2'}$ are translates of each other with common volume $(1-\delta)|T'|$, setting $X = (\lambda^i A)_{T_1'}$, $X' = (\lambda^i A)_{T_2'}$, $Y = A \cap (\lambda^i A)_{T_1'}$, $Y' = A \cap (\lambda^i A)_{T_2'}$ we deduce from the claim that

$$|\lambda(A \cap (\lambda^i A)_{T_1'}) + (1-\lambda)(A \cap (\lambda^i A)_{T_2'})| \geq |T'|(1-\delta) - c_{i,k,n}\delta(A;t).$$

Because $D(A;t) = \lambda A + (1-\lambda)A$ and $(\lambda^i D(A;t))_{T'} = \lambda(\lambda^i A)_{T_1'} + (1-\lambda)(\lambda^i A)_{T_2'}$, we have

$$\begin{aligned}
|D(A;t) \cap (\lambda^i A)_{T'}| &\geq |D(A;t) \cap (\lambda^i D(A;t))_{T'}| - |\lambda^i D(A;t) \setminus \lambda^i A| \\
&\geq |D(A;t) \cap (\lambda^i D(A;t))_{T'}| - \delta(A;t) \\
&\geq |\lambda(A \cap (\lambda^i A)_{T_1'}) + (1-\lambda)(A \cap (\lambda^i A)_{T_2'})| - \delta(A;t) \\
&\geq |T'|(1-\delta) - (c_{i,k,n}+1)\delta(A;t),
\end{aligned}$$

which as $|(\lambda^i A)_{T'}| = (1-\delta)|T'|$ is equivalent to

$$|(\lambda^i A)_{T'} \setminus D(A;t)| \leq (c_{i,k,n}+1)\delta(A;t).$$

We conclude that

$$\begin{aligned}
|(\lambda^i A)_{T'} \setminus A| &\leq |(\lambda^i A)_{T'} \setminus D(A;t)| + \delta(A;t) \\
&\leq (c_{i,k,n}+2)\delta(A;t) \\
&= c_{i,k+1,n}\delta(A;t).
\end{aligned}$$

$\square$

The following lemma shows that given $\alpha < 1$ and $\frac{1}{2} \leq \lambda < 1$, any covering of T by translates of $\alpha^n \lambda T$ contained inside T can be approximated by a covering consisting of elements of $\mathcal{T}_{i,k}(\lambda)$ for fixed small values i, k. The parameters i, k are positively correlated with λ, α.

Before we proceed, we need the following notation. Let $\mathcal{T}_k(\lambda; \lambda'; T)$ be recursively defined by setting

$$\mathcal{T}_0(\lambda; \lambda'; T) = \{\lambda' T + (1 - \lambda')x_j \mid j \in \{0, \dots, n\}\}$$
$$\mathcal{T}_k(\lambda; \lambda'; T) = \{\lambda B_1 + (1 - \lambda)B_2 \mid B_1, B_2 \in \mathcal{T}_{k-1}(\lambda; \lambda; T)\}.$$

Note that by definition, $\mathcal{T}_{i,k}(\lambda) = \mathcal{T}_k(\lambda; \lambda^i; T)$.

Lemma 3.3.3. *For $\alpha, \mu \in (0,1), \lambda \in [\frac{1}{2}, 1)$, every translate $T' \subset T$ of $\alpha^n \mu T$ is completely contained in some element of $\mathcal{T}_{k'}(\lambda; \mu; T)$ with*

$$k' = \sum_{j=1}^{n} \lceil \log(\alpha^{j-1}(1 - \alpha)\mu) / \log(\lambda) \rceil.$$

Proof. To prove this we need the following claim, which is essentially the result for $n = 1$.

Claim 3.3.4. *Every weighted average of two (corner) simplices in $\mathcal{T}_0(\lambda; \alpha\mu; T)$ lies in some simplex of $\mathcal{T}_\ell(\lambda; \mu; T)$ with $\ell = \lceil \log((1 - \alpha)\mu) / \log(\lambda) \rceil$*

Proof. Suppose the two corner simplices are at the corners x_a and x_b. Then every homothetic copy $T' \subset T$ of T is determined by the corresponding edge $x'_a x'_b$. Thus the claim is implied by the one-dimensional version of the claim by intersecting all simplices with $x_a x_b$. Hence we may assume that $T = [0, 1]$, so that $\mathcal{T}_0(\lambda; \mu; T) = \{[0, \mu], [1 - \mu, 1]\}$, and we want to show that every sub-interval of $[0, 1]$ of length $\alpha\mu$ is contained in an element of $\mathcal{T}_\ell(\lambda; \mu; T)$.

We will now proceed by showing that the largest distance between consecutive midpoints of intervals in $\mathcal{T}_{j+1}(\lambda; \mu; T)$ is at most λ times the largest such distance in

$\mathcal{T}_j(\lambda;\mu;T)$. Let I_1, I_2 be two consecutive intervals in $\mathcal{T}_j(\lambda;\mu;T)$ for some j. Then in $\mathcal{T}_{j+1}(\lambda;\mu;T)$ we also have the intervals $J = \lambda I_1 + (1-\lambda)I_2$ and $K = (1-\lambda)I_1 + \lambda I_2$, and the intervals I_1, J, K, I_2 appear in this order from left to right as $\lambda \geq \frac{1}{2}$. If d is the distance between the midpoints of I_1, I_2, then the distances between the consecutive midpoints of I_1, J, K, I_2 are $(1-\lambda)d, (2\lambda-1)d, (1-\lambda)d$ respectively. Therefore, the largest distance between two midpoints in $\mathcal{T}_{j+1}(\lambda;\mu;T)$ is at most $\max(1-\lambda, 2\lambda-1, 1-\lambda) \leq \lambda$ times the largest distance between two consecutive midpoints in $\mathcal{T}_j(\lambda;\mu;T)$. Therefore, the distance between two consecutive midpoints in $\mathcal{T}_\ell(\lambda;\mu;T)$ is at most $\lambda^\ell \leq (1-\alpha)\mu$.

Given an interval I of length $\alpha\mu$, then either the midpoint lies in $[0,\mu/2] \cup [1-\mu/2, 1]$, in which case I is already contained in one of $[0,\mu]$ or $[1-\mu, 1]$ belonging to $\mathcal{T}_0(\lambda;\mu;T)$, or else we can find an interval $I' \in \mathcal{T}_\ell(\lambda;\mu;T)$ of length μ such that the distance between the midpoints of I and I' is at most $\frac{1}{2}(1-\alpha)\mu$, which implies $I \subset I'$. □

We prove our desired statement by induction on the dimension n. The claim above proves the base case $n = 1$, so now assuming the statement is true for dimensions up to $n-1$, we will show it to be true for n.

Let $T' \subset T$ be a fixed translate of $\alpha^n \mu T$, with corresponding vertices $x'_0, \ldots, x'_n$. Denote by F the facet of T opposite x_n, and denote by F' the facet of T' opposite the corresponding vertex x'_n. Denote by H the hyperplane spanned by F'. Then $S = H \cap T$ is an $n-1$-simplex, with vertices $y_0, \ldots, y_{n-1}$ such that y_i is on the edge of T connecting x_i to x_n.

If the common ratio $r := |y_j x_n|/|x_j x_n| \leq \alpha\mu$, then T' is already contained in an element of $\mathcal{T}_0(\lambda;\mu;T)$ and we are done. Otherwise, denote by $T_0, \ldots, T_{n-1} \subset T$ the translates of $\alpha\mu T$ that sit on H and have corners at $y_0, \ldots, y_{n-1}$ respectively. Denote the facet $T_i \cap H$ of T_i by F_i. We remark that each F_i is a translate of $\mu' S$ for some fixed $\mu' \geq \alpha\mu$.

By the claim, the simplices $T_0, \ldots, T_{n-1}$ are completely contained in elements of $\mathcal{T}_\ell(\lambda; \mu; T)$ with

$$\ell = \lceil \log((1-\alpha)\mu)/\log(\lambda) \rceil.$$

By the induction hypothesis applied to the $n-1$-simplex S, F' is completely contained in a simplex from the family $\mathcal{T}_{\ell'}(\lambda; \mu'; S)$ for

$$\ell' := \sum_{j=1}^{n-1} \lceil \log((1-\alpha)\alpha^{j-1}\mu')/\log(\lambda) \rceil \leq \sum_{j=1}^{n-1} \lceil \log((1-\alpha)\alpha^{j}\mu)/\log(\lambda) \rceil,$$

as $\mu' \geq \alpha\mu$. Note that F' is contained in a certain iterated weighted average of the facets $F_0, \ldots, F_{n-1}$ if and only if T' is contained in the analogously defined iterated weighted average of $T_0, \ldots, T_{n-1}$. Therefore $T' \in \mathcal{T}_{\ell+\ell'}(\lambda; \mu; T)$.

Finally, we have that $\ell + \ell' \leq k'$, so $T' \in \mathcal{T}_{k'}(\lambda; \mu; T)$ as desired. □

The following lemma helps to show that arbitrary coverings of T can be modified at no extra cost to coverings of T contained inside T.

Lemma 3.3.5. *Let $r \in (0, 1)$ and $rT + x$ a translate of rT. Then there exists a y such that $(rT + x) \cap T \subset rT + y \subset T$.*

Proof. The intersection of any two copies of the simplex T is itself homothetic to T. Therefore $(rT + x) \cap T$ is homothetic to T, and so must be a translate of $r'T$ for some $r' \leq r$. Because T is convex and $(rT + x) \cap T$ is a homothetic copy of T lying inside T, the center of homothety between $(rT + x) \cap T$ and T lies inside $(rT + x) \cap T$, and all intermediate homotheties lie inside T. In particular there is a homothety which produces a translate of rT which lies inside T, and this translate by construction contains $(rT + x) \cap T$. □

3.4 Proof of Theorem 3.2.1

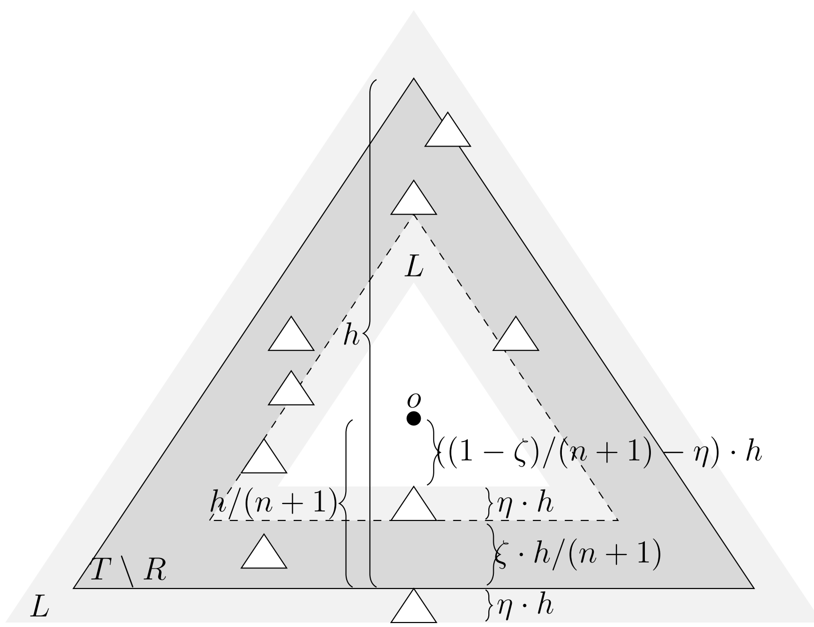
Recall that we may assume that $t \le \frac{1}{2}$.

Proof of Theorem 3.2.1. Let $i = \left\lceil \log\left(\frac{n^{\frac{1}{n}}}{(2n)^5}\right) / \log(1-t) \right\rceil$, so $(1-t)^i \in \left[\frac{n^{\frac{1}{n}}}{2(2n)^5}, \frac{n^{\frac{1}{n}}}{(2n)^5}\right]$ (as $t \le \frac{1}{2}$). Note that

$$i \le 1 + \log((2n)^5)/t \le 6\log(2n)/t.$$

Let $\eta = n^{-\frac{1}{n}}(1-t)^i \in [\frac{1}{2(2n)^5}, \frac{1}{(2n)^5}]$, and let $\zeta = (n+1)\eta$.

Recall T is a regular simplex of volume 1, denote by o the barycenter. By Lemma 3.2.2, setting o to be the origin, if we choose $\delta_n(\tau)$ sufficiently small, then $R := (1-\zeta)T$ is contained in $D(A;t)$. Let $L = (1+\eta(n+1))T \setminus (1-\zeta-\eta(n+1))T$. Note that $T \setminus R \subset L$ and for any T' a translate of ηT intersecting $T \setminus R$, we have $T' \subset L$.



Claim 3.4.1. *There exists a covering $\mathcal{B}$ of $T \setminus R$ by translates of ηT contained in T, such that $\sum_{T' \in \mathcal{B}} |T'| \le \frac{1}{2n}$ and $|\mathcal{B}| \le (2n)^{5n}$.*

Proof of claim. It follows from [56] that[2] for all $n \geq 2$, there exists $r \in \mathbb{R}$ and there exists a covering $\mathcal{F}$ of $\mathbb{R}^n/(r\mathbb{Z})^n$ by translates of ηT with average density at most $7n\log(n)$, i.e.

$$\frac{|\mathcal{F}|\eta^n}{r^n} \leq 7n\log n.$$

Passing to a multiple of r, we may assume that $T, L \subset [-r/2, r/2]^n$. Consider a uniformly random translate $\mathcal{F} + \mathbf{x}$. For any $T' \in \mathcal{F}$ and any point $t' \in T'$, we have

$$\mathbb{P}(T' + \mathbf{x} \subset L) \leq \mathbb{P}(t' + \mathbf{x} \in L) = \frac{|L|}{r^n}.$$

Therefore,

$$\mathbb{E}(|\{T' + \mathbf{x} \in \mathcal{F} + \mathbf{x} \mid T' + \mathbf{x_0} \subset L\}|) \leq \frac{|L|}{r^n}|\mathcal{F}| \leq |L|\eta^{-n}7n\log n,$$

so there exists an $\mathbf{x_0}$ such that

$$|\{T' + \mathbf{x_0} \in \mathcal{F} + \mathbf{x_0} \mid T' \subset L\}| \leq |L|\eta^{-n}7n\log n.$$

Define

$$\mathcal{B}' = \{T' + \mathbf{x_0} \in \mathcal{F} + \mathbf{x_0} \mid (T' + \mathbf{x_0}) \cap (T \setminus R) \neq \emptyset\},$$

then by the above discussion we have $\mathcal{B}'$ is a covering of $T \setminus R$, and

$$|\mathcal{B}'| \leq |L|\eta^{-n}7n\log n.$$

By Lemma 3.3.5, for each element $T' + \mathbf{x_0} \in \mathcal{B}'$ we can find a translate $T' + \mathbf{y_{T'}}$ such

[2] We note that $n = 2$ is not mentioned explicitly in [56] but follows easily.

that $(T' + \mathbf{x_0}) \cap T \subset T' + \mathbf{y_{T'}} \subset T$. Define

$$\mathcal{B} = \{T' + \mathbf{y_{T'}} \mid T' + \mathbf{x_0} \in \mathcal{B}'\},$$

then $\mathcal{B}$ is a cover of $T \backslash R$ by translates of ηT contained in T with $|\mathcal{B}| \leq |L|\eta^{-n} 7n \log n$.

We can calculate the upper bound

$$\begin{aligned}
|L| &= (1 + \eta(n+1))^n - (1 - \zeta - \eta(n+1))^n \\
&= (1 + \eta(n+1))^n - (1 - 2\eta(n+1))^n \\
&\leq 1 + 2\eta n(n+1) - (1 - 2\eta n(n+1)) \\
&= 4\eta n(n+1).
\end{aligned}$$

The inequality follows from the fact that $\eta^k (n+1)^k \binom{n}{k} \leq (1/2)^k 2\eta(n+1)$ and the convexity of $(1-x)^n$ for $x \in (0,1)$.

Therefore,

$$|\mathcal{B}| \leq 4\eta n(n+1)\eta^{-n}(7n \log n) \leq \frac{1}{2n}\eta^{-n} \leq (2n)^{5n}$$

and

$$\sum_{T' \in \mathcal{B}} |T'| = \eta^n |\mathcal{B}| \leq \eta^n \frac{1}{2n}\eta^{-n} \leq \frac{1}{2n}.$$

$\square$

Returning to the proof of Theorem 3.2.1, we apply Lemma 3.3.3 with $\alpha = n^{-\frac{1}{n}}$,

$\lambda = 1-t$, $\mu = (1-t)^i$. Let

$$\begin{aligned} k = \sum_{j=1}^{n} \lceil \log(\alpha^{j-1}(1-\alpha)\mu)/\log(\lambda)\rceil &\le n\lceil \log(\alpha^n(1-\alpha)\mu)/\log(\lambda)\rceil \\ &\le n\left(1+\log\left(\frac{\log n}{2n^2}\mu\right)/\log(\lambda)\right) \\ &\le n\left(1+\log\left(\frac{\log n}{(2n)^7}\right)/(-t)\right) \\ &\le 8n\log(2n)/t. \end{aligned}$$

This shows that every translate of $\eta T = n^{-\frac{1}{n}}(1-t)^i T$ inside T is contained in some element of $\mathcal{T}_k((1-t);(1-t)^i;T) = \mathcal{T}_{i,k}(1-t)$. For each simplex $T' \in \mathcal{B}$, we can therefore choose a simplex $f(T') \in \mathcal{T}_{i,k}(1-t)$ such that $T' \subset f(T')$. Let

$$\mathcal{A} = \{f(T') \mid T' \in \mathcal{B}\}.$$

Note that $\mathcal{A}$ is a cover of $T \setminus R$,

$$|\mathcal{A}| = |\mathcal{B}| \le (2n)^{5n},$$

and

$$\sum_{T'' \in \mathcal{A}} |T''| = (n^{\frac{1}{n}})^n \sum_{T' \in \mathcal{B}} |T'| \le \frac{1}{2}.$$

Note that since $\mathcal{A} \subset \mathcal{T}_{i,k}(1-t)$, Lemma 3.3.1 implies that for every $T'' \in \mathcal{A}$ we have

$$|T'' \setminus A| \le \frac{|T \setminus A|}{|T|}|T''| + c_{i,k,n}\delta(A;t).$$

Since $R \subset D(A;t)$, we have

$$|T \setminus D(A;t)| = |(T \setminus R) \setminus D(A;t)| \leq \sum_{T'' \in \mathcal{A}} |T'' \setminus D(A;t)| \leq \sum_{T'' \in \mathcal{A}} |T'' \setminus A|$$
$$\leq \frac{|T \setminus A|}{|T|} \sum_{T'' \in \mathcal{A}} |T''| + |\mathcal{A}| \cdot c_{i,k,n} \delta(A;t) \leq \frac{1}{2} |T \setminus A| + |\mathcal{A}| \cdot c_{i,k,n} \delta(A;t),$$

which after replacing $|T \setminus D(A;t)| = |T \setminus A| - \delta(A;t)$ yields

$$|T \setminus A| \leq 2(1 + |\mathcal{A}| \cdot c_{i,k,n}) \delta(A;t).$$

We estimate

$$c_{i,k,n} = i + 2k \leq 6 \log(2n)/t + 8n \log(2n)/t \leq 11n \log(2n)/t$$

Therefore

$$2(1 + |\mathcal{A}| c_{i,k,n}) \leq 2(1 + (2n)^{5n}(11n \log(2n)/t)) \leq (4n)^{5n}/\tau.$$

In conclusion, with $C_n = (4n)^{5n}$ we obtain

$$|T \setminus A| \leq C_n \tau^{-1} \delta(A;t)$$

as desired. □

3.5 Sharpness of C_n

In studying the asymptotic behaviour of the optimal value of C_n in Theorem 3.1.3, we note that there is still a gap of order $\log(n)$ in the exponent between the upper and lower bounds. Our proof shows the upper bound $C_n \leq (4n)^{5n} = e^{5n \log(4n)}$ and, the example mentioned in the introduction shows the lower bound $C_n \geq \frac{2^{n-1}}{n}$.

In our method the complexity of C_n is limited by the fact that $|\mathcal{A}| \leq C_n$, where $\mathcal{A}$ is a set of translates of ηT contained inside T with $\eta \leq \frac{1}{2}$ covering ∂T and satisfying $\sum_{T' \in \mathcal{A}} |T'| < |T|$. In fact, by a slight restructuring of our proof it is equivalent to covering just a single facet F of T. Taking $\mathcal{A}'$ to be the family of intersections of elements of $\mathcal{A}$ with the hyperplane containing F, we see that $|\mathcal{A}'| \leq C_n$ with $\mathcal{A}'$ a set of translates of ηF covering F and $\sum_{F' \in \mathcal{A}'} |F'| < \eta^{-1} |F|$.

Question 3.5.1. *Is it true that for every* $0 < \eta_0 \leq \frac{1}{2}$*, then for all sufficiently large* n *if* $F \subset \mathbb{R}^n$ *is a simplex and* $\mathcal{A}'$ *is a family of translates of* $\eta_0 F$ *covering* F *we have*

$$\sum_{F' \in \mathcal{A}'} |F'| > \eta_0^{-1} |F|?$$

Resolving this question would shed light on the correct growth rate of C_n. In particular, if the question has a negative answer with η_0^{-1} replaced with $\eta_0^{-1}(1-\epsilon)$ for some fixed ϵ, then our methods would show that C_n has exponential growth.

Part II

Algebraic Geometry

Chapter 4

GL_{r+1}-orbits in $(\mathbb{P}^r)^n$ via quantum cohomology

This is a heavily excerpted version of [51], later published in revised form in *Advances in Mathematics* [52], reproduced with permission, joint with Mitchell Lee, Anand Patel, and Dennis Tseng. We describe the degenerations of cycles related to the small equivariant quantum cohomology of $\mathbb{P}^r$ in terms of degenerations of more general cycles related to matrix projections (a special case of which are GL_{r+1}-orbits in $(\mathbb{P}^r)^n$). These are in turn governed by polytope subdivisions. Using this, we are able to compute the GL_{r+1}-equivariant cohomology classes of GL_{r+1}-orbits in $(\mathbb{P}^r)^n$, which was extensively studied non-equivariantly by Kapranov [42], and by Aluffi and Faber [3, 4], and which was only recently computed non-equivariantly by Li [53].

The culmination of our results is in Section 4.9, where we see that operations associated to convolutions with the cycles (which occur when considering enumerative questions) can be packaged into operations that satisfy non-trivial recursive relationships related to certain matroid operations. As an important special case, the associativity of small quantum cohomology follows immediately from the fact that the direct sum of 1×1 matrices is associative, and unwinding the proof in its entirety would exactly reproduce the standard proof using the degenerative theory of the Kontsevich mapping space $\overline{\mathcal{M}_{0,n+1}}(\mathbb{P}^r, d)$.

We refer to Section 4.4 for an extended worked example.

We omit from [52] how using the Gelfand-Macpherson correspondence these results complete the programme set out by Berget and Fink [6, 7, 8] to compute torus equivariant classes of torus orbits in the Grassmannian. We also omit here the geometrical interpretation of the operation we construct as a pull-back convolution operation via the wonderful compactification associated to a building set [54], and the enumerative application of counting the number of lines intersecting projective hypersurfaces of appropriate degree and dimension in a fixed unordered moduli of intersection, generalizing results of Cadman and Laza [13].

4.1 Background: small quantum cohomology

The starting point for us is the formula governing the small quantum cohomology of $\mathbb{P}^r$. Let $\mathcal{C}_3$ denote a $\mathbb{P}^1$ with 3 marked points. In $(\mathbb{P}^r)^3$, we consider

$$K_1(\mathcal{C}_3) = \{(p_1, p_2, p_3) \in (\mathbb{P}^r)^3 \mid p_1, p_2, p_3 \text{ collinear}\},$$

the closure of the locus in $(\mathbb{P}^r)^3$ swept out by the 3 marked points of $\mathcal{C}_3$ under all degree 1 maps to $\mathbb{P}^r$. Any cycle $Z \subset X \times Y$ in a product space under reasonable assumptions induces a convolution operation $\mathrm{conv}_Z : H^\bullet(X) \to H^\bullet(Y)$ given by

$$\alpha \mapsto (\pi_Y)_*(\pi_X^* \alpha \cup [Z]).$$

Note that by the Künneth formula, the operation $\mathrm{conv}_{K_1(\mathcal{C}_3)}$ can be considered as an operation

$$\mathrm{conv}_{K_1(\mathcal{C}_3)} : H^\bullet(\mathbb{P}^r)^{\otimes 2} \to H^\bullet(\mathbb{P}^r).$$

If $X_1, X_2 \subset \mathbb{P}^r$ are (generically situated) subvarieties, then $\mathrm{conv}_{K_1(\mathcal{C}_3)}([X_1] \otimes [X_2])$ can be described as follows. Consider all linear maps $\mathcal{C}_3 \to \mathbb{P}^r$ with the first two marked points constrained to lie on X_1, X_2 respectively. Then the locus swept out by the third point (with appropriate multiplicity) is the result. More colloquially, it is the variety swept out by all lines joining a point in X_1 to a point in X_2. To describe this operation algebraically, we recall that

$$H^\bullet(\mathbb{P}^r) \cong \mathbb{Z}[H]/H^{r+1}$$

where $H = \mathcal{O}(1)$ is the class of a hyperplane section in $\mathbb{P}^r$, and more generally that

$$H^\bullet((\mathbb{P}^r)^n) \cong \mathbb{Z}[H_1, \ldots, H_n]/(H_1^{r+1}, \ldots, H_n^{r+1})$$

where H_i is $\mathcal{O}(1)$ pulled back from the i'th factor of $\mathbb{P}^r$.

Lemma 4.1.1. *The operation*

$$\mathrm{conv}_{K_1(\mathcal{C}_3)} : \mathbb{Z}[H_1, H_2]/(H_1^{r+1}, H_2^{r+1}) \to \mathbb{Z}[H_3]/(H_3^{r+1})$$

is computed as follows. Denote the reduction mod H^{r+1} of $f(H)$ by $\overline{f}$. Then with z a formal variable, if we uniquely write

$$\overline{f_1}(z)\overline{f_2}(z) = a_0 + a_1 z^{r+1}$$

with $\deg a_0(z), a_1(z) \leq r$, *then*

$$\mathrm{conv}_{K_1(\mathcal{C}_3)}(f_1(H_1), f_2(H_2)) = a_1(H_3).$$

Proof. By linearity it suffices to check this for two linear spaces, where the result is obvious. □

Definition 4.1.2. *Define the small quantum cohomology ring*

$$QH^\bullet(\mathbb{P}^r) = \mathbb{Z}[z, \hbar]/(z^{r+1} - \hbar) \cong \mathbb{Z}[z],$$

and denote its multiplication by $\star$. *We define lifting and projection maps*

$$\begin{aligned} \ell : H^\bullet(\mathbb{P}^r) \to QH^\bullet(\mathbb{P}^r) \qquad & f(H) \mapsto \overline{f}(z) \\ p : QH^\bullet(\mathbb{P}^r) \to H^\bullet(\mathbb{P}^r) \qquad & \sum a_i (z^{r+1})^i \mapsto \sum a_i \hbar^i \ \textit{when } \deg(a_i) \leq r. \end{aligned}$$

Rather than explicitly working with the lifting and projection maps, it is convenient to abuse notation and denote

$$\star : H^\bullet(\mathbb{P}^r)^{\otimes 2} \to H^\bullet(\mathbb{P}^r)[\hbar]$$

the deformed multiplication on $H^\bullet(\mathbb{P}^r)$ given by transporting the multiplication on $QH^\bullet(\mathbb{P}^r)$ to an operation $H^\bullet(\mathbb{P}^r)^{\otimes 2} \to H^\bullet(\mathbb{P}^r)[\hbar]$ via ℓ and p, which we can extend $\hbar$-linearly to a multiplication operation on $H^\bullet(\mathbb{P}^r)[\hbar]$ (different from the usual multiplication).

With this notation, Lemma 4.1.1 says that

$$\mathrm{conv}_{K_1(\mathcal{C}_3)}(f_1, f_2) = [\hbar^1](f_1 \star f_2)$$

where $[\hbar^1]$ denotes the coefficient of $\hbar^1$. The power of these definitions is in the surprising geometric interpretation of the coefficients of $f_1 \star \ldots \star f_n$. Let $\mathcal{C}_{n+1}$ denote a fixed $\mathbb{P}^1$ equipped with $n+1$ marked points. Then for $d \leq n-1$ we denote the cycle

$$K_d(\mathcal{C}_{n+1}) \subset (\mathbb{P}^r)^n \times \mathbb{P}^r$$

the closure of the locations of the $n+1$ marked points on $\mathcal{C}_{n+1}$ as we vary over all degree d maps $\mathcal{C}_{n+1} \to \mathbb{P}^r$. Note for $d = 0$ we obtain the diagonal, so $\mathrm{conv}_{K_d(C_{n+1})}(f_1, \ldots, f_n)$ is equal to $(f_1 \cup \ldots \cup f_n)$. The following is the main theorem of small quantum cohomology of $\mathbb{P}^r$.

Theorem 4.1.3.

$$\sum \mathrm{conv}_{K_d(\mathcal{C}_{n+1})}(f_1, \ldots, f_n)\hbar^d = f_1 \star \ldots \star f_n,$$

i.e.

$$\mathrm{conv}_{K_d(\mathcal{C}_{n+1})}(f_1, \ldots, f_n) = [\hbar^d](f_1 \star \ldots \star f_n).$$

Note in particular that this implies that the class of the cycle $K_d(\mathcal{C}_{n+1})$ is independent of the choice of $\mathcal{C}_{n+1}$. This yields a cohomology class, but suppose we want to answer the following enumerative question.

How many degree d maps $\mathcal{C}_n \to \mathbb{P}^r$ send the marked points to $X_1, \ldots, X_n$?

In other words how many degree d genus 0 curves passing through $X_1, \ldots, X_n$ intersect the varieties in a collection of points of fixed moduli on the curve. To answer this, we adjoin an extra "output" point to $\mathcal{C}_n$ to make it $\mathcal{C}_{n+1}$, take the subvariety swept out by the output point, and if this variety is a finite collection of points we want to record the number of points since they are in bijection with maps sending the original

n marked points to $X_1, \ldots, X_n$. But this is simply

$$\int [\hbar^d](f_1 \star \ldots \star f_n),$$

where

$$\int : H^\bullet(\mathbb{P}^r) \to H^\bullet(\mathrm{pt}) = \mathbb{Z}$$
$$a_0 + a_1 H + \ldots + a_r H^r \mapsto a_r$$

is the pushforward map induced by the proper map $\mathbb{P}^r \to \mathrm{pt}$. Representing cohomology classes by differential forms, this is literally the integration map, and H^r, the class of a point in cohomology, corresponds to a volume form with total volume 1.

Example 4.1.4. *Suppose we have a collection of* 3 *generically situated conic sections* $X_1, X_2, X_3 \subset \mathbb{P}^2$ *and a generic point* $p_4 \in \mathbb{P}^2$*, and we want to know how many degree* 1 *maps* $\mathbb{P}^1 \to \mathbb{P}^2$ *intersect* X_1, X_2, X_3, p_4 *in a fixed* j*-invariant. The cohomology classes in* $\mathbb{Z}[H]/H^3$ *are* $2H, 2H, 2H, H^2$*, so we write*

$$2z \cdot 2z \cdot 2z \cdot z^2 = 0z^0 + (8z^2)z^3 = 0\hbar^0 + 8z^2\hbar^1,$$

and so obtain the answer $\int 8H^2 = 8$.

Geometrically, $K_d(\mathcal{C}_{n+1})$ arises from the following construction. Consider the Kontsevich compactification $\overline{\mathcal{M}_{0,n+1}}(\mathbb{P}^r, d)$ of $n+1$-pointed genus 0 degree d stable maps $\mathbb{P}^1 \to \mathbb{P}^r$. This space has two natural projections

$$\begin{array}{ccc} \overline{\mathcal{M}_{0,n+1}}(\mathbb{P}^r, d) & \xrightarrow{\mathrm{ev}} & (\mathbb{P}^r)^{n+1} \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}_{0,n+1}} & & \end{array}$$

where π forgets the map and stabilizes the curve, and $\mathrm{ev} = \prod_{i=1}^{n+1} \mathrm{ev}_i$ forgets the curve. The map π is flat [48, Remark 2.6.8] so the cycle $\pi^{-1}(\mathcal{C})$ has class independent of $\mathcal{C}$, and the cycle $K_d(\mathcal{C})$ is nothing more than $\mathrm{ev}_*(\pi^{-1}(\mathcal{C}))$ (the restriction $d \leq n-1$ corresponds to the fact that for $d \geq n$ the map ev no longer has finite fibers when restricted to $\pi^{-1}(\mathcal{C})$, so the pushforward is 0). The equivalence

$$\mathrm{conv}_{K_d(\mathcal{C})}(f_1, \ldots f_n) = (\mathrm{ev}_{n+1})_*(\pi^{-1}(\mathcal{C}) \cup \mathrm{ev}_1^* f_1 \cup \ldots \cup \mathrm{ev}_n^* f_n)$$

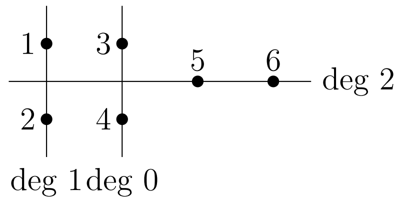
between $\mathrm{conv}_{K_d(\mathcal{C})}$ and the typical way of defining the coefficient of $\hbar^d$ in small quantum cohomology in the literature follows from the push-pull (adjunction) formula.

4.1.1 Components of $\pi^{-1}(\mathcal{C})$ and their convolution operations

We digress a bit to discuss the fibers of the projection π to $\overline{\mathcal{M}_{0,n+1}}$, and in particular their components. A genus 0 $n+1$-pointed degree d stable map is given by taking a tree of $\mathbb{P}^1$s with $n+1$ marked points distinct from the intersection points, and assigning to each $\mathbb{P}^1$ a map to $\mathbb{P}^r$, agreeing on intersections, such that the sum of the degrees is d, and such that the datum has finitely many automorphisms. Having finitely many automorphisms is equivalent to saying that every curve assigned a degree 0 map has at least 3 points which are either marked or intersections.

Certain components of $\pi^{-1}(\mathcal{C})$ have non-finite fibers under ev, and consequently do not contribute to the convolution formula. The precise condition is that the number of intersection and marked points combined on each $\mathbb{P}^1$ is at least 2 greater than the degree of the map associated to it. A moments thought shows that this implies in particular that no components of the stable map get collapsed under stabilization to $\overline{\mathcal{M}_{0,n+1}}$.

For each component $Z \subset \pi^{-1}(\mathcal{C})$ which does contribute to the convolution formula (i.e. with $\mathrm{ev}_* Z \neq 0$), we can describe the operation conv_Z in terms of the operations for each component. Indeed, the flatness of the gluing maps of Kontsevich mapping spaces implies that if for example a generic element of Z looks like



then the operation $\mathrm{conv}_{\mathrm{ev}_* Z}$ is

$$(f_1, f_2, f_3, f_4, f_5, f_6) \mapsto [\hbar^2]([\hbar^1](f_1 \star f_2) \star [\hbar^0](f_3 \star f_4) \star f_5 \star f_6)$$

where $[\hbar^i]$ means extract the $\hbar^i$-coefficient. If we vary the degrees of the maps in all possible ways so that they add to 3 and add up the results, we will obtain

$$[\hbar^3](f_1 \star f_2 \star f_3 \star f_4 \star f_5)$$

because the class of $\pi^{-1}(\mathcal{C})$ does not depend on the choice of $\mathcal{C}$, and this is the result for $\mathcal{C}$ in the generic open locus where it has only one component. More generally, as we vary $\mathcal{C}$, certain components may split up into other components, and the convolution formula associated to the original component before splitting is equal to the sum of the convolution formulas after splitting.

4.2 Key observation and main questions

Now we are ready to state the key observation, and the main questions we answer.

Consider for each $d \times n$ matrix M with no zero columns the partially defined map

$$\mu_M : \mathbb{P}(Mat_{(r+1)\times d}) \dashrightarrow (\mathbb{P}^r)^n$$
$$A \mapsto \text{projectivized columns of } AM$$

Because the map is induced by a linear map, it is easy to see that if μ_M is generically finite, then it is generically injective. Also if $rank(M) < d$ then μ_M is not generically injective. Denote

$$\mathcal{O}_M = \begin{cases} \overline{Im\mu_M} & \mu_M \text{ generically injective} \\ 0 & \text{otherwise.} \end{cases}$$

The following observation appears to be new in this level of generality, and expresses cycles arising from small quantum cohomology as $\mathcal{O}_M$ for certain matrices M.

Observation 4.2.1. *Let $Z \subset \overline{\mathcal{M}_{0,n}}(\mathbb{P}^r, d)$ be a component of a fiber $\pi^{-1}(\mathcal{C})$. Then* $\mathrm{ev}_* Z = \mathcal{O}_M$ *for some $(d+1) \times n$ matrix M.*

Proof. Suppose that $\mathcal{C} = [(C, p_1, \ldots, p_n)]$ and let $C_1, \ldots, C_k$ be the components of C. There is an open locus $Z^\circ \subset Z$ and integers $d_1, \ldots, d_k$ such that Z° parameterizes maps $C \to \mathbb{P}^r$ that restrict to degree d_i on each C_i.

Let L be the line bundle on C that restricts to $\mathcal{O}_{\mathbb{P}^1}(d_v)$ on each component $C_v \cong \mathbb{P}^1$ and let $V = H^0(C, L)$. Then $\dim(V) = d + 1$ and we have a map $f : C \to \mathbb{P}(V^\vee)$.

Let $W = \mathbb{C}^{r+1}$. Each rational map $C \to \mathbb{P}^r = \mathbb{P}(W^\vee)$ that restricts to degree d_v on each component C_v can be written uniquely as a composition $C \xrightarrow{f} \mathbb{P}(V^\vee) \dashrightarrow \mathbb{P}(W^\vee)$, where the map $\mathbb{P}(V^\vee) \dashrightarrow \mathbb{P}(W^\vee)$ is induced by a linear map $W \to V$. This is a regular map if and only if the linear system $W \to V = H^0(C, L)$ is basepoint-free.

Let $U \subset \mathbb{P}(\mathrm{Hom}(W, V))$ be the open locus of basepoint-free linear systems $W \to V \cong H^0(C, L)$. Then U is nonempty and by the above it embeds into Z°. We have a diagram

$$\begin{array}{ccc} U & \hookrightarrow & Z \\ \downarrow & & \downarrow \mathrm{ev}|_Z \\ \mathbb{P}(\mathrm{Hom}(W,V)) & \overset{\mu_M}{\dashrightarrow} & (\mathbb{P}^r)^n. \end{array}$$

So $\mathbb{P}(\mathrm{Hom}(W, V))$ and Z share the open set U, and the bottom map is μ_M where M is any matrix with projectivized columns $f(p_1), \ldots, f(p_n) \in \mathbb{P}(V^\vee)$. □

Note that a one-parameter degeneration $\mathcal{C}(t)$ in $\overline{M}_{0,n+1}$ induces, for any choice of component $Z \subset \pi^{-1}(\mathcal{C}(t'))$ for generic t', a one-parameter deformation $Z(t)$ of cycles, and we know by the degenerative theory of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{P}^r, d)$ what the corresponding degeneration looks like. It is not hard to show this leads to a degeneration of cycles

$$\mathcal{O}_{M(t)} \to \sum_i \mathcal{O}_{M_i}$$

for certain other matrices M_i corresponding to the components of the special fiber $Z(t)$ degenerated to, and these components appear with multiplicity 1 (for the multiplicity statement, see [32, Section 5] or [35, Section 4.1]).

Remark 4.2.2. *For $d \leq r+1$ the cycles $\mathcal{O}_M$ are the same as orbit closures under the diagonal GL_{r+1}-action of $(\mathbb{P}^r)^n$. We don't restrict ourselves to orbits closures however as for $d \geq r+2$ the cycles still have relevance to quantum cohomology as seen above.*

Question 4.2.3. *For $d \times n$-matrices M,*

1. *What is the degenerative theory of the cycles $\mathcal{O}_M$?*

2. *What are the convolution operations associated to $\mathcal{O}_M$?*

3. *What are the cohomology classes associated to $\mathcal{O}_M$ in $H^\bullet((\mathbb{P}^r)^n)$?*

The answers to these questions will lead us through a myriad of disparate combinatorial topics and constructions surrounding matroids and polytope theory. We will answer these questions more generally in the GL_{r+1}-equivariant setting, which the next section will elaborate on.

4.3 Working GL_{r+1}-equivariantly

Non-equivariantly the second and third question can be answered using existing techniques and results from the literature. Indeed, a special case of Li's result [53] computes $[\mathcal{O}_M]$ in terms of the "rank polytope" P_M associated to a matrix M (see Definition 4.5.1).

Theorem 4.3.1. *(Li [53])*

$$[\mathcal{O}_M] = \sum_{(i_1,\ldots,i_n)\in(r+1)P_M-\overline{\epsilon}} H_1^{r-i_1}\ldots H_n^{r-i_n}$$

where $\overline{\epsilon}$ is a vector of positive numbers adding to 1*.*

The formula for the GL_{r+1}-equivariant classes appears intractable to direct computation — indeed, even the formula for a general matrix, which was originally computed non-equivariantly by Kapranov [42] was not known equivariantly (an incorrect and non-fixable computation appears in Berget and Fink [8]). Via our connection to equivariant quantum cohomology, this is equivalent to describing the Kronecker dual in $H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^{n+1})$ of the equivariant quantum cohomology operation $\star : H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n) \to H^\bullet_{GL_{r+1}}(\mathbb{P}^r)$, but even under this alternate formulation the computation was never carried out.

Using our techniques, we are able to solve all of the questions outlined in the previous section GL_{r+1}-equivariantly, and relate the answers to the equivariant quantum cohomology of $\mathbb{P}^r$.

4.3.1 Construction of $H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n)$

We now recall the construction of equivariant cohomology in this particular setting, and then motivate why we would want to study these classes.

Definition 4.3.2. *Define EGL_{r+1} to be a contractible space on which GL_{r+1} acts freely, and let $EGL_{r+1} \to BGL_{r+1}$ be the principal GL_{r+1}-bundle induced by the quotient map.*

We can take EGL_{r+1} as the colimit of rank $(r+1)$ matrices of size $(r+1) \times k$ as $k \to \infty$, which expresses BGL_{r+1} as an infinite Grassmannian, the colimit of the Grassmannians $\mathrm{Gr}(r+1,k)$ of $(r+1)$-dimensional subspaces of $\mathbb{C}^k$ under a nested sequence of inclusions $\mathbb{C}^k \subset \mathbb{C}^{k+1} \subset \ldots$.

Definition 4.3.3. *If GL_{r+1} acts on X, then*

$$H^\bullet_{GL_{r+1}}(X) := H^\bullet(EGL_{r+1} \times_{GL_{r+1}} X).$$

As a special case, $H^\bullet_{GL_{r+1}}(\mathrm{pt}) = H^\bullet(BGL_{r+1})$, the limit of the cohomology rings of the $\mathrm{Gr}(r+1,k)$. The cohomology of $\mathrm{Gr}(r+1,k)$ is generated by the chern classes of the rank $(r+1)$ tautological bundle modulo certain relations, but these relations disappear when we take the colimit, and we obtain the following.

Observation 4.3.4. *We have $H^\bullet_{GL_{r+1}}(\mathrm{pt}) = \mathbb{Z}[c_1, \ldots, c_{r+1}]$ where c_i is the i'th chern class of the tautological rank $(r+1)$ vector bundle $EGL_{r+1} \times_{GL_{r+1}} \mathbb{C}^{r+1} \to BGL_{r+1}$.*

Now, we recall the formula for the cohomology ring of a projectivized vector bundle.

Observation 4.3.5. *Let* $\mathcal{V} \to B$ *be a rank* $(r+1)$ *vector bundle with chern classes* $c_1(\mathcal{V}), \ldots, c_{r+1}(\mathcal{V}) \in H^\bullet(B)$. *Then* $H^\bullet(\mathbb{P}(\mathcal{V})) = H^\bullet(B)[H]/(H^{r+1} + c_1(\mathcal{V})H^r + \ldots + c_{r+1}(\mathcal{V}))$, *where* H *is the class of relative* $\mathcal{O}(1)$, *and* $H^{r+1} + c_1(\mathcal{V})H^r + \ldots + c_{r+1}(\mathcal{V})$ *is the* Leray relation. *The pushforward map* $\int : H^\bullet(\mathbb{P}(\mathcal{V})) \to H^\bullet(B)$ *is the map which sends*

$$\int : a_0 + a_1 H + \ldots + a_r H^r \mapsto a_r$$

for any classes $a_i \in H^\bullet(B)$. *This formula applies in particular to the tautological rank* $(r+1)$ *vector bundle over* BGL_{r+1}, *yielding an induced map* $H^\bullet_{GL_{r+1}}(\mathbb{P}^r) \to H^\bullet_{GL_{r+1}}(\mathrm{pt})$.

Definition 4.3.6. *Let* $F(H)$ *denote the* universal Leray relation $H^{r+1} + c_1 H^r + \ldots + c_{r+1}$ *associated to the tautological rank* $(r+1)$ *vector bundle* $EGL_{r+1} \times_{GL_{r+1}} \mathbb{C}^{r+1} \to BGL_{r+1}$.

We now observe that $EGL_{r+1} \times_{GL_{r+1}} (\mathbb{P}^r)^n$ is an iterated projective bundle over BGL_{r+1}, and obtain the following expression.

Observation 4.3.7. *We have*

$$H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n) = \mathbb{Z}[c_1, \ldots, c_{r+1}][H_1, \ldots, H_n]/((F(H_1), \ldots, F(H_n)).$$

Note that there is a map $H^\bullet_{GL_{r+1}}((\mathbb{P}^n)^{r+1}) \to H^\bullet((\mathbb{P}^n)^{r+1})$ sending all c_i to 0. This is induced by the inclusion of $(\mathbb{P}^r)^n$ into any fiber of $EGL_{r+1} \times_{GL_{r+1}} (\mathbb{P}^r)^n \to BGL_{r+1}$.

4.3.2 Motivation for studying $H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n)$

We now describe the main motivation for studying $H^\bullet_{GL_{r+1}}(\mathbb{P}^r)$. Suppose we have a subvariety $X \subset (\mathbb{P}^r)^n$ invariant under the GL_{r+1}-action. Then for any rank $(r+1)$

vector bundle over a base $\mathcal{V} \to B$, we can construct the relative version

$$X_{\mathcal{V}} \subset \mathbb{P}(\mathcal{V})^n := \mathbb{P}(\mathcal{V}) \times_B \ldots \times_B \mathbb{P}(\mathcal{V}),$$

which is an X-bundle over the base B, and ask for $[X_{\mathcal{V}}] \in H^\bullet(\mathbb{P}(\mathcal{V})^n)$.

Definition 4.3.8. *For $X \subset (\mathbb{P}^r)^n$ a GL_{r+1}-invariant subvariety, we denote by $[X] \in H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n)$ the class of the relative version of X in $EGL_{r+1} \times_{GL_{r+1}} (\mathbb{P}^r)^n$.*

Theorem 4.3.9 (Main theorem of GL_{r+1}-equivariant cohomology)**.** *The class $[X_{\mathcal{V}}]$ is given by a universal polynomial in the chern classes $c_1(\mathcal{V}), \ldots, c_{r+1}(\mathcal{V})$, and the classes H_i, which is relative $\mathcal{O}(1)$ pulled back from the i'th copy of $\mathbb{P}(\mathcal{V})$. This polynomial is any expression for $[X] \in H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n)$.*

Proof. BGL_{r+1} classifies rank $r+1$-vector bundles, so we have a pullback square

$$\begin{array}{ccc} \mathbb{P}(\mathcal{V})^n & \longrightarrow & EGL_{r+1} \times_{GL_{r+1}} (\mathbb{P}^r)^n \\ \downarrow & & \downarrow \\ B & \longrightarrow & BGL_{r+1} \end{array}$$

and the class of $X_{\mathcal{V}}$ is the image of the class of $[X] \in H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n)$ under the pullback along the top horizontal map

$$H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n) \to H^\bullet(\mathbb{P}(\mathcal{V})^n)$$

sending $c_i \mapsto c_i(\mathcal{V})$ and $H_i \mapsto H_i$. □

4.3.3 Small equivariant quantum cohomology

Giventhal and Kim have developed an equivariant analogue of small quantum cohomology, and we outline the modifications here.

Definition 4.3.10. *The* small equivariant quantum cohomology ring *of* $\mathbb{P}^r$ *[36] is defined as*

$$QH^{\bullet}_{GL_{r+1}}(\mathbb{P}^r) = \mathbb{Z}[c_1, \ldots, c_r][z, \hbar]/(F(z) - \hbar) \cong \mathbb{Z}[c_1, \ldots, c_r][z].$$

We define $f \mapsto \overline{f}$ as the reduction modulo $F(z)$ instead of reduction modulo z^{r+1}. Writing $\overline{f}(z)\overline{g}(z) = a_0(z) + F(z)a_1(z)$ with $\deg a_0(z), a_1(z) \leq r$ in the z variable,

$$\mathrm{conv}_{K_1(\mathcal{C}_3)}(f(H_1), g(H_2)) = a_1(H_3).$$

The lifting map ℓ takes $f \mapsto \overline{f}$, and the projection map takes $\sum a_i F(z)^i \mapsto \sum a_i \hbar^i$, where $a_i(z)$ are polynomials in $\mathbb{Z}[c_1, \ldots, c_{r+1}][z]$ of degree at most r in z.

With these definitions, we have Theorem 4.1.3 holds verbatim.

Theorem 4.3.11. *(Givental, Kim [36, 44, 37]) We have the equality in* $H^{\bullet}_{GL_{r+1}}(\mathbb{P}^r)[\hbar]$

$$\sum \mathrm{conv}_{K_d(\mathcal{C}_{n+1})}(f_1, \ldots, f_n)\hbar^d = f_1 \star \ldots \star f_n,$$

i.e.

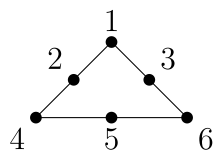
$$\mathrm{conv}_{K_d(\mathcal{C}_{n+1})}(f_1, \ldots, f_n) = [\hbar^d](f_1 \star \ldots \star f_n).$$

4.4 Worked example

In this section we present an extended example to demonstrate the key aspects of our results.

4.4.1 Setup

Let $p_1, \ldots, p_6 \in \mathbb{P}^2$ be the points depicted below (or any other six points with the same collinearity relations).



Let A be any 3×6 matrix whose projectivized columns are $p_1, \ldots, p_6$. For any $r \geq 2$, the cycle $[\mathcal{O}_A] \in H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^6)$ is the closure of the GL_{r+1}-orbit of $(p_1, \ldots, p_6) \in (\mathbb{P}^r)^6$, where each point p_i is considered as an element of $\mathbb{P}^r$ via any linear inclusion $\mathbb{P}^2 \hookrightarrow \mathbb{P}^r$.

4.4.2 Formula for $[\mathcal{O}_A]$

The rank polytope of A (see Definition 4.5.1) is the convex hull of the 17 points

$$\{e_i + e_j + e_k \mid p_i, p_j, p_k \text{ span } \mathbb{P}^2\} \subset \mathbb{R}^6.$$

and Theorem 4.7.4 expresses the class $[\mathcal{O}_M] \in H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^6)$ as a sum of 6! rational functions. However, such a large sum is difficult to manipulate in practice. Instead, we may understand $[\mathcal{O}_A]$ by computing the Kronecker dual

$$[\mathcal{O}_A]^\dagger : H^\bullet_{GL_{r+1}}(\mathbb{P}^r)^{\otimes 6} \to H^\bullet_{GL_{r+1}}(\mathrm{pt}),$$

which is defined (Definition 4.9.7) by

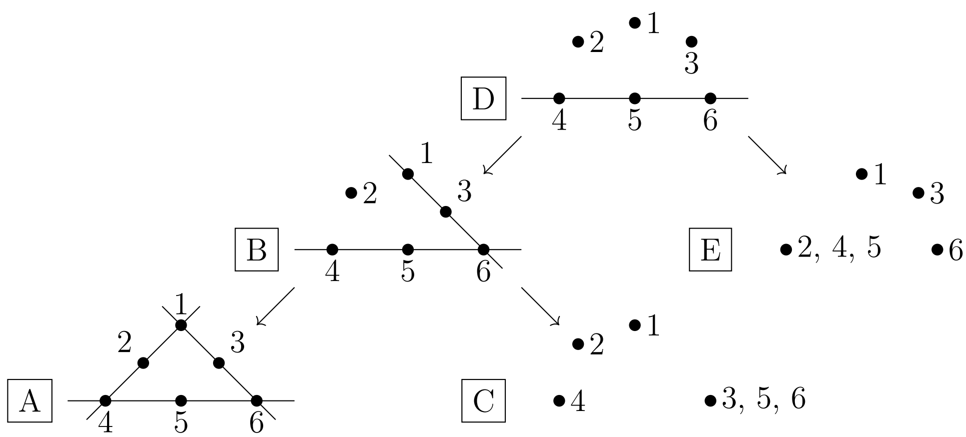
$$[\mathcal{O}_A]^\dagger(f_1(H), \ldots, f_6(H)) = \int_{(\mathbb{P}^r)^6} [\mathcal{O}_A] f_1(H_1) \ldots f_6(H_6)$$

for any polynomials $f_1, \ldots, f_6$ with coefficients in $H^\bullet_{GL_{r+1}}(\mathrm{pt})$.

4.4.3 Degenerations

In many cases (such as this one), we can compute Kronecker dual classes efficiently by breaking the problem into smaller pieces.

We can relate $[\mathcal{O}_A]$ with other classes through the following degeneration tree. Here B, C, D, E are 3×6 matrices whose projectivized columns are the depicted point configurations.



The rank polytopes of A, B, C, D, and E are related by subdivisions. The hyperplane $\{x_1 + x_2 + x_4 = 2\}$ separates the rank polytope P_B of B into the rank polytopes P_A and P_C of A and C respectively. Similarly, the hyperplane $\{x_1+x_3+x_6 = 2\}$ separates P_D into P_B and P_E.

Accordingly, the cycle $\mathcal{O}_B$ degenerates to a union $\mathcal{O}_A \cup \mathcal{O}_C$ and $\mathcal{O}_D$ degenerates to a union $\mathcal{O}_B \cup \mathcal{O}_E$ (Theorem 4.5.3), so we get

$$[\mathcal{O}_B] = [\mathcal{O}_A] + [\mathcal{O}_C] \qquad [\mathcal{O}_D] = [\mathcal{O}_B] + [\mathcal{O}_E].$$

Not all subdivisions of rank polytopes yield degenerations of the cycles $\mathcal{O}_M$, but they all yield relations between the generalized matrix orbit classes $[\mathcal{O}_M]$ (Theorem 4.5.4).

Therefore,

$$[\mathcal{O}_A] = [\mathcal{O}_D] - [\mathcal{O}_C] - [\mathcal{O}_E]$$

so

$$[\mathcal{O}_A]^\dagger = [\mathcal{O}_D]^\dagger - [\mathcal{O}_C]^\dagger - [\mathcal{O}_E]^\dagger.$$

4.4.4 Kronecker duals to Schubert matroids

To compute $[\mathcal{O}_A]^\dagger$, it remains to compute $[\mathcal{O}_C]^\dagger$, $[\mathcal{O}_D]^\dagger$, and $[\mathcal{O}_E]^\dagger$, for which we will use the operations (Definition 4.9.4)

$$[C]_\hbar, [D]_\hbar, [E]_\hbar : H^\bullet_{GL_{r+1}}(\mathbb{P}^r)^{\otimes 6} \to QH^\bullet_{GL_{r+1}}(\mathbb{P}^r) = H^\bullet_{GL_{r+1}}(\mathrm{pt})[z][\hbar]/(\hbar - F(z)).$$

To compute $[C]_\hbar, [D]_\hbar, [E]_\hbar$, we will use the fact that C, D, E are Schubert matrices (see Definition 4.6.2). For example, we may write

$$D = \mathrm{Sch}_6(3, 3).$$

Hence, we have by Observation 4.9.6 and Remark 4.9.10 (or Example 4.9.5) that

$$[D]_\hbar(f_1(H), \dots, f_6(H)) = f_1(z)f_2(z)f_3(z)(f_4(z)f_5(z)f_6(z) \bmod F(z)^2) \bmod F(z)^3$$

for any polynomials $f_1, \dots, f_6$ of degree at most r with coefficients in $H^\bullet_{GL_{r+1}}(\mathrm{pt})$. We have thus expressed $[D]_\hbar$ using polynomial multiplication and division with remainder.

By Observation 4.9.8, we can thus extract

$$\begin{aligned}[\mathcal{O}_D]^\dagger(f_1(H), \dots, f_6(H)) &= [z^r][F(z)^2][D]_\hbar(f_1(H), \dots, f_6(H)) \\ &= [z^r][F(z)^2](f_1(z)f_2(z)f_3(z)(f_4(z)f_5(z)f_6(z) \bmod F(z)^2)).\end{aligned}$$

We can perform a similar procedure for the matrices C and E, yielding

$$\begin{aligned}[\mathcal{O}_C]^\dagger(f_1(H),\dots,f_6(H)) &= [z^r][F(z)^2][C]_\hbar(f_1(H),\dots,f_6(H)) \\ &= [z^r][F(z)^2](f_1(z)f_2(z)f_4(z)(f_3(z)f_5(z)f_6(z) \bmod F(z))) \\ [\mathcal{O}_E]^\dagger(f_1(H),\dots,f_6(H)) &= [z^r][F(z)^2][E]_\hbar(f_1(H),\dots,f_6(H)) \\ &= [z^r][F(z)^2](f_1(z)f_3(z)f_6(z)(f_2(z)f_4(z)f_5(z) \bmod F(z))).\end{aligned}$$

Hence $[\mathcal{O}_A]^\dagger$ is given by

$$\begin{aligned}[\mathcal{O}_A]^\dagger(f_1(H),\dots,f_6(H)) = [z^r][F(z)^2](&f_1(z)f_2(z)f_3(z)(f_4(z)f_5(z)f_6(z) \bmod F(z)^2) \\ &- f_1(z)f_2(z)f_4(z)(f_3(z)f_5(z)f_6(z) \bmod F(z)) \\ &- f_1(z)f_3(z)f_6(z)(f_2(z)f_4(z)f_5(z) \bmod F(z))).\end{aligned}$$

for all polynomials $f_1,\dots,f_6$ of degree at most r with coefficients in $H^\bullet_{GL_{r+1}}(\mathrm{pt})$.

4.5 Degenerative theory: infinite polytope subdivisions

To each $d \times n$ matrix M, we will associate a polytope P_M called the *matroid rank polytope*, first defined by Gelfand, Gorsesky, Serganova, and Macpherson [34].

Definition 4.5.1. *Let*

$$\mathcal{B}_M = \{A \subset \{1,\dots,n\} \mid |A| = d, M_A \textit{ has rank } d\}.$$

where M_A is the restriction of M to the columns in A. Then we define

$$P_M = \mathrm{conv}(v \mid v \textit{ the indicator function of an element of } \mathcal{B}_M\} \subset \{\sum_{i=1}^{n} x_i = d\} \subset \mathbb{R}^n.$$

For M a general $d \times n$ matrix, we define the hypersimplex

$$\Delta_{d,n} \coloneqq P_M = [0,1]^n \cap \{\sum x_i = d\}.$$

Remark 4.5.2. *P_M is called the rank polytope, because it is alternatively defined as*

$$P_M = [0,1]^n \cap \{\sum x_i = d\} \cap \bigcap_{A \subset \{1,\dots,n\}} \{\sum_{i \in A} x_i \leq \mathrm{rank}(M_A)\}.$$

Our first result concerning degenerations is that if $M(t)$ is a one-parameter degeneration of $M = M(0)$, then we can relate the polytopes of a generic fiber $M(t')$ with the polytopes of M and the remaining components of the special fiber at $t = 0$.

Theorem 4.5.3. *If $\mathcal{O}_{M(t)}$ is a one-parameter degeneration, then there are matrices $M_0 = M(0), \dots, M_k$ such that the special fiber of the family is $\bigcup_i \mathcal{O}_{M_i}$. Furthermore, for generic t we have*

$$P_{M(t)} = P_{M_0} \cup \dots P_{M_k}$$

is a subdivision of $P_{M(t)}$.

Proof. Associated to each $A \in \mathcal{B}_M$, we associate the number $c_A = val(det(M_A))$, where we take the t-adic valuation (so e.g. $val(t^2(t+1)) = 2$). Then we consider the point set

$$\{(v_A, c_A) \mid v_A \text{ the vertex of } P_M \text{ associated to } A\} \subset \mathbb{R}^n \times \mathbb{R}$$

and denote its lower convex hull over $P_{M(t)}$ (for t generic) by $\widetilde{P_{M(t)}}$. Then the facets of $\widetilde{P_{M(t)}}$ project down to a subdivision of $P_{M(t)}$, and we claim that for each such projected facet F we can identify it with a matroid rank polytope P_{M_i} with $\mathcal{O}_{M_i}$ in the special fiber.

Pulling back the family under a map $t \mapsto t^m$ has the effect of multiplying each c_A by m, so we can assume that the supporting hyperplane $H_F = \sum_{i=1}^n b_i x_i + c$ of $\widetilde{F} \subset \widetilde{P_{M(t)}} \subset P_M \times \mathbb{R}$ has integral coefficients. Note that we can multiply $M(t)$ by any matrix in $D(t) \in T(\mathbb{C}(t))$ and obtain the same family. The effect of multiplying $M(t)$ by $D(t) = diag(t^{-b_1}, \ldots, t^{-b_n})$ is that $c_A \mapsto c_A - \sum_{i \in A} b_i = c + c_A - H_F(v_A)$, resulting in $c_A = c$ for v_A a vertex of F and $c_A > c$ for all other vertices v_A of $P_{M(t)}$.

Now that we have made this reduction, we show that we can find a path of matrices $B(t)$ in $GL_d(\mathbb{C}(t))$ such that $B(t)M(t)D(t)$ has a well-defined limit as $t \to 0$, which will be our M_i. Indeed, take $B(t)$ to be the inverse of a $d \times d$ submatrix $(MD)_A$ with c_A minimal. Considering submatrices with $d-1$ columns in A and 1 column outside of A allows us to show entry by entry that each element of BM has non-negative valuation, which means BMD has a well-defined limit M_i when $t \to 0$. Multiplying by $B(t)$ takes each $c_A \mapsto c_A - c$, so $P_{M_i} = F$.

That the matrices appear with the correct multiplicites and that no further components appear follows from Theorem 4.3.1, since this formula implies that non-equivariantly $[\mathcal{O}_M] = \sum [\mathcal{O}_{M_i}]$ because P_M is subdivided by the P_{M_i}, and any other cycle appearing in the fiber would contribute to the non-equivariant class on the right hand side.

□

Now we have a partial picture of degenerations of $\mathcal{O}_M$, but we will need to know for later use a complete understanding of relations between cohomology classes of cycles we can obtain through degenerations. Note that it is clear from the above description that the relations are a subset of the relations between indicator functions of the P_{M_i} modulo higher codimenion. The next result, our main result concerning degenerations, says that the converse is also true.

Theorem 4.5.4. *If we have an equality of indicator functions up to higher codimension*

$$\sum b_i 1_{P_{M_i}} \sim 0,$$

then there is an explicit tree of degenerations $\mathcal{O}_{M(t)}$ realizing this relation.

Corollary 4.5.5. *If $P_M = P_{M'}$ then $[\mathcal{O}_M] = [\mathcal{O}_{M'}] \in H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n)$ are equal.*

Proof of Theorem. We use a sequence of degenerations of infinite polytopes from Appendix A of Derksen and Fink [20], and show that intersecting each stage with $\Delta_{d,n}$ then we obtain a subdivision relation arising from an actual degeneration of matrices.

To that end, we recall the classical fact that P_M has the additional special property that all edges are parallel to vectors of the form $e_i - e_j$ (this is more generally true for polytopes associated to matroids [34]). We say that a cone C is *positive matroidal* if it is generated by a subset of vectors of the form $e_i - e_j$ with $i > j$.

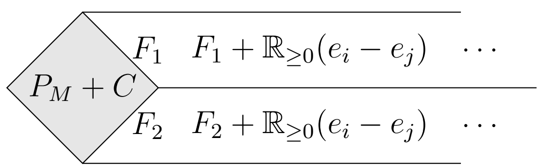
For each P_M, we will describe a sequence of subdivision relations between infinite polytopes, which formally equates $1_{P_M} \sim \sum b_i' 1_{C_i + v_i}$ where $v_i \in \{0,1\}^n \cap (\sum x_i = d)$ and C_i are positive matroidal cones. We will do this in such a way that each subdivision, when intersected with $\Delta_{d,n}$, is a subdivision of matroid polytopes associated to a degeneration involving the current collection of $\mathcal{O}_M$ cycles, show that there are no non-trivial relations between the indicator functions of $(C_i + v_i) \cap \Delta_{d,n}$, and then finally show that the $\mathcal{O}_M$ associated to the positive matroidal cone have the same class. Because of these last two steps, we can apply this sequence of degenerations separately for each M_i, and then are guaranteed formal cancellation at the end.

Basic facts about Minkowski sums shows that the sum of two positive matroidal cones is positive matoridal, and for C a positive matroidal cone, $P_M + C$ has all edges parallel to $e_i - e_j$ and all infinite edges extremal rays of C.

The key subdivision we will use is a "shadow facet" subdivision, arising as follows. Let $P_M + C$ be the Minkowski sum of a rank polytope P_M with a positive matroidal cone C, and take $i > j$ such that $e_i - e_j \notin C$. Then $P_M + C + \mathbb{R}_{\geq 0}(e_i - e_j)$ has a subdivision

$$P_M + C + \mathbb{R}_{\geq 0}(e_i - e_j) = P_M + C \cup \bigcup_i F_i + \mathbb{R}_{\geq 0}(e_i - e_j)$$

where F_i are the facets of $P_M + C$ such that $F_i + \epsilon(e_i - e_j)$ doesn't intersect $P_M + C$ for small $\epsilon > 0$. Depicted below is an example with P_M a square and $C = \{0\}$.



We say that a matrix M and a positive matroidal cone C are *compatible* if whenever $e_i - e_j \in C$ with $i > j$, the matrix M' obtained by adding a generic multiple of column j to column i has the same rank polytope. For any cone C, let B_C be the $n \times n$ matrix where

$$(B_C)_{i,j} = \begin{cases} \text{generic number} & e_i - e_j \in C \\ 0 & \text{otherwise.} \end{cases}$$

Then an equivalent definition of compatibility of (M, C) is that $P_{MB_C} = P_M$.

Given a compatible pair (M, C) with $e_i - e_j \notin C$, we consider the linear degeneration $M(t) = M(I + tB_{C+\mathbb{R}_{\geq 0}(e_i - e_j)})$. This has the effect of not only adding a generic multiple of column i to column j, but also a generic multiple of column i' to column j' whenever $e_i - e_{i'} \in C$ and $e_{j'} - e_j \in C$.

An exhuasting but ultimately straightforward verification (see our paper [52]) shows the following. Recall that we have a lower convex hull construction $\widetilde{P_{M(t)}}$ associated to the degeneration. First, the facets associated to $\widetilde{P_{M(t)}}$ other than $(P_M + C) \cap \Delta_{d,n} = P_M$ are precisely $(F_i + \mathbb{R}_{\geq 0}(e_i - e_j)) \cap \Delta_{d,n}$, so there are matrices M_i with $\mathcal{O}_{M_i}$ components of the special fiber such that

$$P_{M_i} = (F_i + \mathbb{R}_{\geq 0}(e_i - e_j)) \cap \Delta_{d,n}.$$

Second, M_i is compatible with the positive matroidal cone $C_i = \lim_{t \to 0} t(F_i + \mathbb{R}_{\geq 0}(e_i - e_j))$, and

$$P_{M_i} + C_i = F_i + \mathbb{R}_{\geq 0}(e_i - e_j).$$

Finally, the matrix $M(t)$ for generic t has rank polytope $(P_M + C + \mathbb{R}_{\geq 0}(e_i - e_j)) \cap \Delta_{d,n}$, the matrix $M(t)$ is compatible with the cone $C + \mathbb{R}_{\geq 0}(e_i - e_j)$, and

$$P_{M(t)} + C + \mathbb{R}_{\geq 0}(e_i - e_j) = P_M + C + \mathbb{R}_{\geq 0}(e_i - e_j).$$

We claim that repeatedly applying shadow facet subdivisions in any fashion, eventually all of the resulting (possibly infinite) polytopes will be positive matroidal cones. Indeed, assume we know the result for $P + C$ with all P of lower dimension. Then the shadow facet subdivision of $P + C$ formally replaces $P + C$ with the negative sum of infinite polytopes $F_i + C_i$, where we know the result by induction, and $P + (C + \mathbb{R}_{\geq 0}(e_i - e_j))$. After $\binom{n}{2}$ steps the cone cannot grow any further, but then $P + C$ is just a positive matroidal cone at the lexicographically first basis amongst the columns of M thanks to the Steinitz exchange lemma.

There are no non-trivial relations amongst the indicator functions of $(C_i + v_i) \cap \Delta_{d,n}$ up to higher codimension by the following argument. Consider by way of contradiction a minimal such relation. Among all $(C_i + v_i) \cap \Delta_{d,n}$ that appear, we consider the lexicographically first $v = v_i$ to appear. Then $(C_i + v_i) \cap \Delta_{d,n}$ does not intersect a tiny ball around v unless $v_j = v$, so restricting to this tiny ball, we get an induced relation between the $C_j + v_j$ with $v_j = v$, so in particular a relation between the $(C_j + v_j) \cap \Delta_{d,n}$ with $v_j = v$. By minimality of the relation, we thus can assume that $v_j = v$ for all j. Again, intersecting with a tiny ball around v shows us that we in fact have a relation

$$\sum b_i 1_{C_j+v} \sim 0,$$

and this is a minimal relation among all $C_j + v$ cones. Let $C_{big} = \sum_{i<j} \mathbb{R}_{\geq 0}(e_i - e_j)$. Then if we consider the relation in a small ball around $v + \epsilon(e_i - e_j)$ and intersect with C_{big}, we obtain

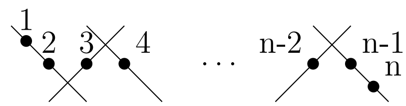
$$\sum_{e_i - e_j \in C} b_i 1_{C_j + v + \epsilon(e_i - e_j)} \sim 0,$$

which after translating by $-\epsilon(e_i - e_j)$ yields $\sum_{e_i - e_j \in C} b_i 1_{C_j+v} \sim 0$. Therefore by minimality of the relation either $e_i - e_j$ lies in all C or no C for every $i < j$. In particular all C_i are equal contradicting the non-triviality of the relation.

Putting this all together now, we start with our cycles $\mathcal{O}_M$, and augment the data of the cycle with a compatible positive-matroidal cone, initially the zero cone. Starting with $(M, 0)$, we repeatedly degenerate via shadow facet subdivisions of $P_{M'} + C'$ for each compatible pair (M', C') which appears along the way until we reduce to pairs (M', C') with $P_{M'} + C'$ a positive matroidal cone. By the independence of positive matroidal cones just proved, we must have that after all of these degenerations, the sum of coefficients associated to cycles $\mathcal{O}_{M'}$ with (M', C') a compatible pair such that $P_{M'} + C'$ is a positive matroidal cone is zero. It is straightforward to check (see [52]) that if $P_{M'} + C' = P_{M''} + C''$ is a fixed positive matroidal cone, then the direct linear interpolation between M' and M'' yields that $\mathcal{O}_{M'}$ is related to $\mathcal{O}_{M''}$ by this explicit degeneration, so using these we can complete our degeneration tree. □

4.6 Writing $1_{P_M} \sim \sum b_i 1_{P_{M_i}}$ with $\mathcal{O}_{M_i} = \mathrm{ev}_* Z_i$

Now for a $d \times n$ matrix M we will show that we can write $1_{P_M} \sim \sum b_i 1_{P_{M_i}}$ for certain integers b_i, with $\mathcal{O}_{M_i} = \mathrm{ev}_* Z_i$ for $Z_i \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d-1)$ components of $\pi^{-1}(\mathcal{C}_n^{\mathrm{chain}})$, with $\mathcal{C}_n^{\mathrm{chain}} \in \overline{\mathcal{M}_{0,n}}$ the unique n-pointed stable genus 0 curve given by the $\mathbb{P}^1$ chain



(see Observation 4.2.1). By Theorem 4.5.4, we thus obtain

$$[\mathcal{O}_M] = \sum b_i [\mathrm{ev}_* Z_i],$$

so finding the class of $\mathcal{O}_M$ reduces down to finding the classes of the $\mathrm{ev}_* Z_i$.

Note that the components Z of $\pi^{-1}(\mathcal{C}_n^{\mathrm{chain}})$ are given by degree labellings of the components of $\mathcal{C}_n^{\mathrm{chain}}$, and the ones corresponding to non-zero $\mathrm{ev}_* Z$ are precisely those with labellings all 0 or 1 (see the discussion in Subsection 4.1.1). This motivates the following definition.

Definition 4.6.1. *Let $\mathcal{S}_{n-1,d-1} = \{(s_1, \ldots, s_{n-1}) \mid s_i \in \{0,1\}, \sum s_i = d-1\}$. For $S \in \mathcal{S}_{n-1,d-1}$, we let $Z_{n,S}^{chain} \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d-1)$ be the component of $\pi^{-1}(\mathcal{C}_n^{chain})$ corresponding to labelling the $\mathbb{P}^1$'s in $\mathcal{C}_n^{chain}$ from left to right with degrees from S. Finally, we let M_S be a corresponding matrix under Observation 4.2.1 with $\mathcal{O}_{M_S} = \mathrm{ev}_* Z_{n,S}^{chain}$.*

As an intermediate step to showing $1_{P_M} \sim \sum b_i 1_{P_{M_i}}$, we note that by the main result of Derksen and Fink [20], we can explicitly write

$$1_{P_M} \sim \sum b_i' 1_{P_{M_i'}}$$

where M_i' are $d \times n$ matrices called "Schubert Matrices".

Definition 4.6.2. *We define* $\mathrm{Sch}_n(a_1, \ldots, a_d)$ *with* $a_1 + \ldots + a_d = n$ *and* $a_i \geq 0$ *to be a* $d \times n$ *matrix* M *with*

$$M_{i,j} = \begin{cases} \textit{generic element} & j > a_1 + \ldots + a_{i-1} \\ 0 & \textit{otherwise.} \end{cases}$$

Define a Schubert matrix to be any matrix M' *such that after possibly permuting the columns,* $P_{M'} = P_M$ *with* $M = \mathrm{Sch}_n(a_1, \ldots, a_d)$ *for some* $a_1, \ldots, a_d$.

For example, we have $\mathrm{Sch}_8(1, 2, 5)$ is a generic matrix of the form

$$\mathrm{Sch}_8(1,2,5) = \begin{bmatrix} * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * \end{bmatrix}.$$

Note that if we let v_i be the i'th column of a Schubert matrix M corresponding to $\mathrm{Sch}_n(a_1, \ldots, a_d)$, then $v_i \in \mathbb{C}^d$ are vectors such that after possibly permuting them, $v_1, \ldots, v_{a_1}$ are generic vectors in a dimension 1 space, and $v_{a_1+1}, \ldots, v_{a_1+a_2}$ are generic vectors in a dimension 2 space containing $\langle v_1, \ldots, v_{a_1} \rangle$. More generally $v_{a_1+\ldots+a_i+1}, \ldots v_{a_1+\ldots+a_{i+1}}$ are generic vectors in an $i+1$-dimensional space containing the span $\langle v_1, \ldots, v_{a_1+\ldots+a_i} \rangle$. Note by Remark 4.5.2 that

$$P_{\mathrm{Sch}_n(a_1,\ldots,a_d)} = [0,1]^n \cap \{\sum x_i = d\} \bigcap_{j=1}^{d-1} \{ \sum_{i \leq a_1 + \ldots + a_j} x_i \leq j \}.$$

Definition 4.6.3. *For* $a_1, \ldots, a_d \geq 0$, *define* $\mathcal{S}_{n-1,d-1}(a_1, \ldots, a_d) \subset \mathcal{S}_{n-1,d-1}$ *to be those sequences* $(s_1, \ldots, s_{n-1})$ *with* $\sum_{i=1}^{a_1+\ldots+a_j-1} s_i \leq j-1$ *for all* $1 \leq j \leq d-1$.

Theorem 4.6.4. *For* $\mathrm{Sch}_n(a_1, \ldots, a_d) \subset \Delta_{d,n}$ *of codimension* 0, *we have the subdivi-*

sion of polytopes

$$P_{\mathrm{Sch}_n(a_1,\ldots,a_d)} = \bigcup_{S \in S_{n-1,d-1}(a_1,\ldots,a_d)} P_{M_S}.$$

Proof. We proceed by induction. We split the polytope $P_{\mathrm{Sch}_n(a_1,\ldots,a_d)}$ by the hyperplane $x_{n-1} + x_n = 1$. As $P_{\mathrm{Sch}_n(a_1,\ldots,a_d)} \subset \{\sum x_i = d\}$, on the polytope $x_{n-1} + x_n \geq 1$ is equivalent to $x_1 + \ldots + x_{n-2} \leq d - 1$. Therefore, it suffices to show that

$$P_{\mathrm{Sch}_n(a_1,\ldots,a_d)} \cap \{x_{n-1} + x_n \leq 1\} = \bigcup_{S \in \mathcal{S}_{n-1,d-1}(a_1,\ldots,a_d)^0} P_{M_S}$$
$$P_{\mathrm{Sch}_n(a_1,\ldots,a_d)} \cap \{\sum_{i=1}^{n-2} x_i \leq d - 1\} = \bigcup_{S \in \mathcal{S}_{n-1,d-1}(a_1,\ldots,a_d)^1} P_{M_S}$$

where $\mathcal{S}_{n-1,d-1}(a_1, \ldots, a_d) = \mathcal{S}_{n-1,d-1}(a_1, \ldots, a_d)^0 \sqcup \mathcal{S}_{n-1,d-1}(a_1, \ldots, a_d)^1$ is the partition according to whether the last element of s_{n-1} of $S = (s_1, \ldots, s_{n-1})$ is 0 or 1. Denote the linear map

$$\Phi : \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$$
$$(v, x, y, z) \mapsto (v, x + y, z).$$

Then one can check directly that (noting $x_{n-1}+x_n \geq 1$ is equivalent to $x_1+\ldots+x_{n-2} \leq d - 1$ since $P_{\mathrm{Sch}_n(a_1,\ldots,a_d)} \subset \{\sum x_i = d\}$)

$$P_{\mathrm{Sch}_n(a_1,\ldots,a_d)} \cap \{x_{n-1} + x_n \leq 1\} = \Phi(P_{\mathrm{Sch}_{n-1}(a_1,\ldots,a_d-1)} \times P_{\mathrm{Sch}_3(3)}) \cap \{\sum x_i = d\}$$
$$P_{\mathrm{Sch}_n(a_1,\ldots,a_d)} \cap \{\sum_{i=1}^{n-2} x_i \leq d - 1\} = \Phi(P_{\mathrm{Sch}_{n-1}(a_1,\ldots,a_{d-2},a_{d-1}+a_d-1)} \times P_{\mathrm{Sch}_3(0,3)})$$
$$\cap \{\sum x_i = d\}.$$

The induction hypothesis on n allows us to replace the polytopes $P_{\mathrm{Sch}_{n-1}(a_1,\ldots,a_d-1)}$ and $P_{\mathrm{Sch}_{n-1}(a_1,\ldots,a_{d-2},a_{d-1}+a_d-1)}$ by the corresponding sum of P_{M_S} polytopes. There is a bijection between $\mathcal{S}_{n-1,d-1}(a_1,\ldots,a_d)^0$ and the set $\mathcal{S}_{n-1,d-1}(a_1,\ldots,a_d-1)$ taking S to $(s_1,\ldots,s_{n-2})$, and one can check that

$$P_{M_S} = \Phi(P_{M_{(s_1,\ldots,s_{n-2})}} \times P_{\mathrm{Sch}_3(3)}) \cap \{\sum x_i = d\}.$$

We obtain another similar bijection between the set $\mathcal{S}_{n-1,d-1}(a_1,\ldots,a_d)^1$ and the set $\mathcal{S}_{n-2,d-1}(a_1,\ldots,a_{d-2},a_{d-1}+a_d-1)$ taking $S \mapsto (s_1,\ldots,s_{n-2})$, and one can check

$$P_{M_S} = \Phi(P_{M_{(s_1,\ldots,s_{n-2})}} \times P_{\mathrm{Sch}_3(0,3)}) \cap \{\sum x_i = d\}.$$

This concludes the proof. $\square$

In particular, combining with Derksen and Fink's result mentioned above and Theorem 4.5.4, we obtain the following.

Theorem 4.6.5. *For M a $d \times n$ matrix with P_M of codimension 0 in $\Delta_{d,n}$,*

$$1_{P_M} = \sum_{\sigma \in S_n} \sum_{S \in \mathcal{S}_{n-1,d}} b_{\sigma,S}\,\sigma(1_{P_{M_S}})$$

for some choices of integers $b_{\sigma,S}$. In particular,

$$[\mathcal{O}_M] = \sum_{\sigma \in S_n} \sum_{S \in \mathcal{S}_{n-1,d}} b_{\sigma,S}\,\sigma([\mathrm{ev}_* Z^{chain}_{n,S}]).$$

4.7 The class of $\mathcal{O}_M$ via Brion's theorem

First, note that the non-equivariant formula in Theorem 4.3.1 is inherently geometric (in the Euclidean sense), as it is a sum of monomials inside a polyhedral region.

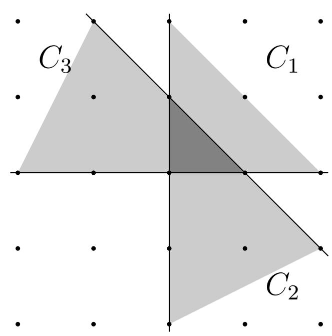
Theorem 4.7.1. *(Brion [11]) Let $P \subset \mathbb{R}^n$ be a polytope with vertices in $\mathbb{Z}^n$. If C_v is* *the vertex cone at a vertex v of P, then*

$$f_{C_v}(z_1,\dots,z_n) = \sum_{(i_1,\dots,i_n)\in P\cap\mathbb{Z}^n} z^{i_1}\dots z^{i_n},$$

is a rational function, and we have the equality

$$\sum_{(i_1,\dots,i_n)\in P\cap\mathbb{Z}^n} z^{i_1}z^{i_2}\dots z^{i_n} = \sum_v f_{C_v}(z_1,\dots,z_n) \in \mathbb{C}(z_1,\dots,z_n).$$

Example 4.7.2. *Let T be the triangle with vertices $v_1 = (0,0)$, $v_2 = (0,1)$, $v_3 = (1,0)$.*



Then $f_{C_1} = \frac{1}{(1-x)(1-y)}$, $f_{C_2} = \frac{y}{(1-y^{-1})(1-xy^{-1})}$, and $f_{C_3} = \frac{x}{(1-x^{-1})(1-x^{-1}y)}$, and the sum

$$\frac{1}{(1-x)(1-y)} + \frac{y}{(1-y^{-1})(1-xy^{-1})} + \frac{x}{(1-x^{-1})(1-x^{-1}y)} = 1 + x + y.$$

The shift by $\bar{\epsilon}$ doesn't impose a serious obstruction to applying Brion's theorem, but the vertex cones of P_M are difficult to handle. Instead, we can apply Brion's theorem to $P_M + t\Delta_{d,n}$, whose vertex cones are the same as the vertex cones of the hypersimplex $\Delta_{d,n}$, and let $t \to 0$. We do not reproduce it here, but in [52] using standard results about matroids we obtain the following.

Corollary 4.7.3. *Let M be a $d \times n$ matrix, and for each $\sigma \in S_n$ let $B(\sigma)$ be the lexicographically first d-element subset $A \subset \{1,\dots,n\}$ such that $A \in \mathcal{B}(M)$ with* *respect to the ordering* $\sigma(1) \prec \sigma(2) \prec \ldots \prec \sigma(n)$*. Then*

$$[\mathcal{O}_M] = \sum_\sigma \Big(\prod_{i\in\{1,\ldots,n\}\setminus B(\sigma)} H_i^{r+1}\Big)\frac{1}{(H_{\sigma(2)} - H_{\sigma(1)})\ldots(H_{\sigma(n)} - H_{\sigma(n-1)})} \in H^\bullet(\mathbb{P}^r).$$

Note that what we actually mean is that the rational function is formally a polynomial expression, and plugging in $H_1, \ldots, H_n$ into this polynomial yields the non-equivariant class. The first product in each term controls the location of each vertex, and the recipricol product is the sum of lattice points in the corresponding vertex cone translated to 0.

Our next result is the surprising analogue for the equivariant class.

Theorem 4.7.4. *Let* M *be a* $d \times n$ *matrix, and for each* $\sigma \in S_n$ *let* $B(\sigma)$ *be the lexicographically first* d*-element subset* $A \subset \{1, \ldots, n\}$ *such that* $A \in \mathcal{B}(M)$ *with respect to the ordering* $\sigma(1) \prec \sigma(2) \prec \ldots \prec \sigma(n)$*. Then*

$$[\mathcal{O}_M] = \sum_\sigma \Big(\prod_{i\in\{1,\ldots,n\}\setminus B(\sigma)} F(H_i)\Big)\frac{1}{(H_{\sigma(2)} - H_{\sigma(1)})\ldots(H_{\sigma(n)} - H_{\sigma(n-1)})} \in H^\bullet_{GL_{r+1}}(\mathbb{P}^r).$$

Note that this appears to be Brion's formula applied to (an affine transformation of) the polytope $(r+1)P_M$, with the vertex parameters deformed. This is the only case we are aware of where such a deformation can be carried out and still yield a polynomial expression, which is of independent interest.

Proof. We will first show that there is a unique universal rational function of $F(H_i)$ and H_i which formally simplifies to the reduced expression for $[\mathcal{O}_M] \in H^\bullet_{GL_{r+1}}(\mathbb{P}^r)$ (i.e. the degree of each power of H_i never exceeds r), independent of r. By Theorem 4.6.5, it suffices to show this for $M = M_S$ with $S \in \mathcal{S}_{d,n-1}$.

We will show by induction on n that the reduced expression $[\mathcal{O}_{M_S}]$ can be formally written

$$\sum_{i=1}^{n-1} \frac{F(H_n) - F(H_i)}{H_n - H_i} Q_{n,i}(H_1, \ldots, H_{n-1}, F(H_1), \ldots, F(H_{n-1}))$$

with $Q_{S,i}$ rational functions independent of r. To do this, we note the following method for passing between convolution operations and their Kronecker dual cohomology classes. If $f(H)$ is a poynomial in H, then

$$\int_{\mathbb{P}^r} f(H) \frac{F(H) - F(z)}{H - z} = \overline{f}(z).$$

For $\mathcal{C} \in \overline{\mathcal{M}}_{0,3}$ the unique 3-pointed curve, we have

$$\int_{(\mathbb{P}^r)^2} f_1(H_1) f_2(H_2) \frac{F(H_1) - F(z)}{H_1 - z} \frac{F(H_2) - F(z)}{H_2 - z} = \overline{f_1}(z)\overline{f_2}(z) = (f_1 \star f_2)(z)$$
$$= \overline{\mathrm{conv}_{\mathcal{O}_{K_0(\mathcal{C})}}(f_1(H_1), f_2(H_2))}(z) + F(z)\overline{\mathrm{conv}_{\mathcal{O}_{K_1(\mathcal{C})}}(f_1(H_1), f_2(H_2))}(z).$$

Thus if we can write $\frac{F(H_1)-F(z)}{H_1-z}\frac{F(H_2)-F(z)}{H_2-z} = a_0(H_1, H_2, z) + F(z)a_1(H_1, H_2, z)$ with $\deg a_0, a_1 \leq r$ in the z variable, then equating the $F(z)^0$ and $F(z)^1$ coefficients, we deduce that $a_0(H_1, H_2, H_3) = \mathcal{O}_{K_0(\mathcal{C})}$ and $a_1(H_1, H_2, H_3) = \mathcal{O}_{K_1(\mathcal{C})}$. To do this, we apply partial fraction decomposition, writing

$$\frac{F(z) - F(H_1)}{z - H_1} \frac{F(z) - F(H_2)}{z - H_2}$$
$$= (F(z) - F(H_1))(F(z) - F(H_2))\left(\frac{1}{(H_1 - H_2)(z - H_1)} - \frac{1}{(H_1 - H_2)(z - H_2)}\right)$$
$$= \left(-\frac{F(z) - F(H_1)}{z - H_1} \cdot \frac{F(H_2)}{H_1 - H_2} + \frac{F(z) - F(H_2)}{z - H_2} \cdot \frac{F(H_1)}{H_1 - H_2}\right)$$
$$+ F(z)\left(\frac{F(z) - F(H_1)}{z - H_1} \cdot \frac{1}{H_1 - H_2} - \frac{F(z) - F(H_2)}{z - H_2} \cdot \frac{1}{H_1 - H_2}\right).$$

Thus we have the result for $n = 3$. In general, suppose we have the result for M_S with $S \in \mathcal{S}_{d-1,n-1}$. Let $S^0 \in \mathcal{S}_{d-1,n}$ be obtained by adjoining a 0 onto the end of S, and let $S^1 \in \mathcal{S}_{d,n}$ be obtained by adjoining a 1 onto the end of S. Then

$$\overline{\mathrm{conv}_{\mathcal{O}_{M_{S^0}}}(f_1(H_1), \ldots, f_{n-1}(H_{n-1}))}(z) + F(z)\overline{\mathrm{conv}_{\mathcal{O}_{M_{S^1}}}(f_1(H_1), \ldots, f_{n-1}(H_{n-1})}(z)$$
$$= (\mathrm{conv}_{\mathcal{O}_M}(f_1(H_1), \ldots, f_{n-2}(H_{n-2})) \star f_{n-1}(H_{n-1}))(z)$$
$$= \int_{(\mathbb{P}^r)^{n-1}} \prod_{i=1}^{n-2} f_i(H_i) \sum_{i=1}^{n-2} \frac{F(H_{n-1}) - F(H_i)}{H_{n-1} - H_i} Q_{S,i} \frac{F(H_{n-1}) - F(z)}{H_{n-1} - z} f_n(H_n),$$

and a similar partial fraction decomposition yields the inductive step.

Thus for fixed M there is a universal rational function $f_M(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that $f_M(x_1, \ldots, x_n, F(x_1), \ldots, F(x_n))$ computes the reduced expression for $[\mathcal{O}_M]$ in $H^\bullet_{GL_{r+1}}(\mathbb{P}^r)$, independent of r. By Corollary 4.7.3, the function

$$f_{M'}(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_\sigma \Big(\prod_{i \in \{1,\ldots,n\} \setminus B(\sigma)} y_i \Big) \frac{1}{(x_{\sigma(2)} - x_{\sigma(1)}) \ldots (x_{\sigma(n)} - x_{\sigma(n-1)})}$$

has the property that $f'_M(x_1, \ldots, x_n, x_1^{r+1}, \ldots, x_n^{r+1})$ computes the reduced expression for the non-equivariant class $[\mathcal{O}_M] \in H^\bullet(\mathbb{P}^r)$.

Finally, this implies

$$f_M(x_1, \ldots, x_n, x_1^{r+1}, \ldots, x_n^{r+1}) = f'_M(x_1, \ldots, x_n, x_1^{r+1}, \ldots, x_n^{r+1})$$

for all r, which implies $f_M = f'_M$, completing the proof. □

4.8 Convolution with Schubert matrices

Now, we obtain one of the nicest corollaries of our results, namely the expression for convolution with Schubert matrices.

Theorem 4.8.1. *The partial convolution operation* $\mathrm{conv}_{\mathrm{Sch}_{n+1}(a_1,\ldots,a_d)}(f_1, \ldots, f_n)$:

$H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n) \to H^\bullet_{GL_{r+1}}(\mathbb{P}^r)$ *is given by*

$$[\hbar^{d-1}]((\ldots((f_1 \star \ldots \star f_{a_1} \ \mathit{mod}\ \hbar^1) \star f_{a_1+1} \star \ldots \star f_{a_1+a_2} \ \mathit{mod}\ \hbar^2)\ldots) \star f_{a_1+\ldots+a_{d-1}+1} \star \ldots \star f_n \ \mathit{mod}\ \hbar^d)$$

Proof. We have $P_{\mathrm{Sch}_{n+1}(a_1,\ldots,a_d)}$ is subdivided into P_{M_S} with $S \in \mathcal{S}_{n,d-1}(a_1,\ldots,a_d)$. Therefore,

$$[\mathcal{O}_{\mathrm{Sch}_{n+1}(a_1,\ldots,a_d)}] = \sum_{S \in \mathcal{S}_n(a_1,\ldots,a_d)} [\mathcal{O}_{M_S}] = \sum_{S \in \mathcal{S}_n(a_1,\ldots,a_d)} [\mathrm{ev}_* Z^{\mathrm{chain}}_{n+1,S}].$$

Therefore,

$$\mathrm{conv}_{\mathrm{Sch}_{n+1}(a_1,\ldots,a_d)}(f_1,\ldots,f_n) = \sum_{S \in \mathcal{S}_{n+1}(a_1,\ldots,a_d)} \mathrm{conv}_{Z^{\mathrm{chain}}_{n+1,S}}(f_1,\ldots,f_n).$$

For $S = (s_1,\ldots,s_n) \in \mathcal{S}_{n,d-1}$, we have the equality

$$\mathrm{conv}_{Z^{\mathrm{chain}}_{n+1,S}}(f_1,\ldots,f_n) = [\hbar^{s_n}](([\hbar^{s_{n-1}}](\ldots([\hbar^{s_2}](([\hbar^{s_1}](f_1 \star f_2)) \star f_3))\ldots)) \star f_n).$$

Multiplying out $((f_1 \star f_2) \star f_3) \star \ldots$ with this bracketing, we see that the sum of the convolution operations in $\mathcal{S}_{n,d-1}(a_1,\ldots,a_d)$ precisely eliminates the branches in the multiplication process in the same way that taking the mods and extracting the $\hbar^{d-1}$-coefficient does. □

4.9 Operations $[M]_\hbar$

To extend the result of small quantum cohomology to the more general setting of cycles $\mathcal{O}_M$, we first need to understand how these cycles behave under the $\star$-product.

Definition 4.9.1. *For an $a \times b$-matrix M, we denote by $\tau^{\leq k} M$ the result of multiplying AM for A a generic $k \times a$-matrix. We denote by $*$ for a generic 1×1-matrix, and* $M \oplus N = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$.

Definition 4.9.2. *Given a $d_1 \times n_1$ matrix M and a $d_2 \times n_2$ matrix N, the* series connection *$S(M, N)$ is a $(d_1 + d_2) \times (n_1 + n_2 - 1)$ matrix obtained by replacing the d_1 and $d_1 + 1$-columns in $M \oplus N$ with a generic vector in their span. The* parallel connection *$P(M, N)$ is a $(d_1 + d_2 - 1) \times (n_1 + n_2 - 1)$ matrix obtained by projecting the columns of $M \oplus N$ by a codimension 1 projection with kernel in the span of the d_1 and the $d_1 + 1$-columns of $M \oplus N$.*

We should think of parallel connection as gluing the point configurations in $\mathbb{P}^r$ associated to the columns of M and N at their last and first points respectively, and series connection as joining the two disjoint point configurations with a line connecting the last point of M and the first point of N, and then replacing these two points with a generic point on this line.

Theorem 4.9.3. *Let M, N be respectively $d_1 \times n_1$ and $d_2 \times n_2$ matrices. Then $[\mathcal{O}_M] \in H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^{n_1 - 1}) \otimes H^\bullet_{GL_{r+1}}(\mathbb{P}^r)$, and $[\mathcal{O}_N] \in H^\bullet_{GL_{r+1}}(\mathbb{P}^r) \otimes H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^{n_2 - 1})$, and*

$$\begin{aligned} [\mathcal{O}_M] \star [\mathcal{O}_N] =& [\mathcal{O}_{P(M,N)}] + \hbar [\mathcal{O}_{S(M,N)}] \\ & \in H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^{n_1 - 1}) \otimes H^\bullet_{GL_{r+1}}(\mathbb{P}^r)[\hbar] \otimes H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^{n_2 - 1}) \end{aligned}$$

Proof. By the informal description of $P(M, N)$ it is clear that $[\mathcal{O}_{P(M,N)}]$ should be obtained by equating all elements in $\mathcal{O}_M \times \mathcal{O}_N$ where the last coordinate of $\mathcal{O}_M$ agrees with the first coordinate of $\mathcal{O}_N$, which is obtained by taking the product of the two cycles with this factor in common. Similarly, the construction of $S(M, N)$ is analogous to the geometric description of the operation $\mathrm{conv}_{K_1(\mathcal{C}_3)}$ applied to $\mathcal{O}_M$

and $\mathcal{O}_N$. See our paper [52] for more details. □

Recall that convolution operations associted to small quantum cohomology could be expressed via the formula

$$\sum \hbar^d \operatorname{conv}_{K_d(\mathcal{C}_{n+1})}(f_1, \ldots, f_n) = f_1 \star \ldots \star f_n.$$

We can make a similar definition with the convolution operations associated to $\mathcal{O}_M$, and it turns out that a very analogous construction yields similar recursive properties. First, note that $K_d(\mathcal{C}_{n+1}) = \mathcal{O}_M$ for M a $(d+1) \times (n+1)$ matrix with no non-trivial linear relations between the columns. Therefore for the purposes of convolution, M might as well be a generic $(d+1) \times (n+1)$ matrix by Corollary 4.5.5, and we can obtain such matrices as d decreases via column projections, corresponding to the matroid operation of *truncation* $\tau^{\leq k}$.

Definition 4.9.4. *We let* $[M]_\hbar = \sum \hbar^k \operatorname{conv}_{\mathcal{O}_{\tau^{\leq k+1}(M \oplus *)}} : H^\bullet_{GL_{r+1}}(\mathbb{P}^r)^{\otimes n} \to H^\bullet_{GL_{r+1}}(\mathbb{P}^r)$.

Example 4.9.5. *If* $M = \operatorname{Sch}_n(a_1, \ldots, a_d)$*, then for* $k \geq d$ *we have* $\mathcal{O}_{\tau^{\leq k+1}(M \oplus *)} = 0$*, and otherwise* $\tau^{\leq k+1}(M \oplus *) = \operatorname{Sch}_{n+1}(a_1, \ldots, a_{k-1}, a_k + a_{k+1} + \ldots + a_n + 1)$*. Thus the result from the previous section shows that* $[M]_\hbar(f_1, \ldots, f_n)$ *is just*

$$\begin{aligned}((\ldots((f_1 \star \ldots \star f_{a_1} \ mod\ \hbar^1) \star f_{a_1+1} \star \ldots \star f_{a_1+a_2} \ mod\ \hbar^2) \ldots) \\ \star f_{a_1+\ldots+a_{d-1}+1} \star \ldots \star f_n \ mod\ \hbar^d).\end{aligned}$$

In particular, for a general $n \times n$ *matrix,* $[M]_\hbar(f_1, \ldots, f_n) = f_1 \star \ldots \star f_n$.

Observation 4.9.6. *We have* $[M]_\hbar$ *satisfies the easy properties that*

$$[*]_\hbar = id$$

$$[\tau^{\leq k} M]_\hbar = [M]_\hbar \ mod \ \hbar^k,$$

and if we permute the columns of M, then this corresponds to permuting the inputs of $[M]_\hbar$.

Definition 4.9.7. *Write $[\mathcal{O}_M]^\dagger$ for the convolution $H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n) \to H^\bullet_{GL_{r+1}}(\mathrm{pt})$ associated to $\mathcal{O}_M \subset (\mathbb{P}^r)^n \times \mathrm{pt}$, i.e.*

$$[\mathcal{O}_M]^\dagger(f_1(H_1), \ldots, f_n(H_n)) = \int_{(\mathbb{P}^r)^n} [\mathcal{O}_M] f_1(H_1) \ldots f_n(H_n).$$

Observation 4.9.8. *Composing $[M]_\hbar : H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n) \to H^\bullet_{GL_{r+1}}(\mathbb{P}^r)[\hbar]$ with the $\hbar$-linear extension of $\int : H^\bullet_{GL_{r+1}}(\mathbb{P}^r) \to H^\bullet_{GL_{r+1}}(\mathrm{pt})$ yields the operation*

$$\sum [\mathcal{O}_{\tau^{\leq k+1} M}]^\dagger \hbar^k : H^\bullet_{GL_{r+1}}((\mathbb{P}^r)^n) \to H^\bullet_{GL_{r+1}}(\mathrm{pt})[\hbar],$$

so we can extract the convolution operations associated to the various truncations of M from $[M]_\hbar$.

Analogously to the recursive properties encoded in the small quantum product, the following definition encodes a complicated recursive relationship.

Theorem 4.9.9.

$$[M \oplus N]_\hbar = [M]_\hbar \star [N]_\hbar.$$

Proof. (Sketch, see [52]) Taking Kronecker duals, we obtain classes

$$[M]^\dagger_\hbar = \sum \hbar^k [\mathcal{O}_{\tau^{\leq k+1}(M \oplus *)}] \in H^\bullet_{GL_{r+1}}(\mathbb{P}^r)^{\otimes n_1 + 1}[\hbar]$$

$$[N]^\dagger_\hbar = \sum \hbar^k [\mathcal{O}_{\tau^{\leq k+1}(N \oplus *)}] \in H^\bullet_{GL_{r+1}}(\mathbb{P}^r)^{\otimes n_2 + 1}[\hbar]$$

$$[M \oplus N]^\dagger_\hbar = \sum \hbar^k [\mathcal{O}_{\tau^{\leq k+1}(M \oplus N \oplus *)}] \in H^\bullet_{GL_{r+1}}(\mathbb{P}^r)^{\otimes n_1 + n_2 + 1}[\hbar],$$

and by Theorem 4.9.3 and Theorem 4.5.4, it suffices to show that

$$P_{\tau^{\leq k}(M\oplus *\oplus N)} = \bigcup_{k_1+k_2=k} P_{S(\tau^{\leq k_1}(M\oplus *),\tau^{\leq k_2}(*\oplus N))} \cup \bigcup_{k_1+k_2=k+1} P_{P(\tau^{\leq k_1}(M\oplus *),\tau^{\leq k_2}(*\oplus N))}$$

is a subdivision (we switched orders of factors to agree with our P, S definitions).

To show this, we need the following observation of Fink and Speyer [31, Proof of Theorem 7.3], that if Φ is the projection $\mathbb{R}^{n_1-1}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n_2-1}\to\mathbb{R}^{n_1-1}\times\mathbb{R}\times\mathbb{R}^{n_2-1}$ taking $(v, z_1, z_2, w)\mapsto(v, z_1+z_2, w)$, then Φ is injective on $P_M\times P_N$, and we have the subdivision

$$\Phi(P_M\times P_N) = P_{S(M,N)} \cup (P_{P(M,N)} + e_{n_1}).$$

We introduce coordinates now: let $x_1,\ldots,x_n, z_1$ be the coordinates on the ambient space $\mathbb{R}^{n_1+1}\supset P_{\tau^{\leq i}(M\oplus *)}$ for all i, let $z_2, y_1,\ldots,y_n$ be the coordinates on the ambient space of $\mathbb{R}^{n_2+1}\supset P_{\tau^{\leq i}(N\oplus *)}$ for all i, and identifying $\mathbb{R}^{n_1+n_2+1} = \Phi(\mathbb{R}^{n_1+1}\times\mathbb{R}^{n_2+1})$, we let $x_1,\ldots,x_{n_1}, z, y_1,\ldots,y_{n_2}$ be the coordinates on the ambient space $\mathbb{R}^{n_1+n_2+1}\supset P_{\tau^{\leq i}(M\oplus *\oplus N)}$ for all i. Then for $k_1+k_2=k$ we will show that

$$P_{S(\tau^{\leq k_1}(M\oplus *),\tau^{\leq k_2}(*\oplus N))} = P_{\tau^{\leq k}(M\oplus *\oplus N)}\cap\{\sum x_i\in[k_1-1,k_1]\}\cap\{\sum y_i\in[k_2-1,k_2]\}$$

and for $k_1+k_2=k+1$ we have

$$P_{P(\tau^{\leq k_1}(M\oplus *),\tau^{\leq k_2}(*\oplus N))} = P_{\tau^{\leq k}(M\oplus *\oplus N)}\cap\{\sum x_i\in[k_1-1,k_1]\}\cap\{\sum y_i\in[k_2-1,k_2]\},$$

which implies the subdivision as $\sum x_i + \sum y_i \in [k-1,k]$ and the above conditions on the right hand sides partition the set of all such pairs of $(\sum x_i, \sum y_i)$ whose sum lies in $[k-1,k]$.

For the first case, we note that the definition of series connection implies

$$P_{S(\tau^{\leq k_1}(M\oplus *),\tau^{\leq k_2}(*\oplus N)} = \Phi(P_{\tau^{\leq k_1}}(M \oplus *) \times P_{\tau^{\leq k_2}}(* \oplus N) \cap \{z_1 + z_2 \leq 1\}).$$

We have $P_{S(\tau^{\leq k_1}(M\oplus *),\tau^{\leq k_2}(*\oplus N))}$ is Φ applied to the polytope

$$\begin{aligned} P_{\tau^{\leq k}(M\oplus *)} \times P_{\tau^{\leq k_2}(*\oplus N)} \cap \{z_1 + z_2 \leq 1\} \\ = \bigcap_i \{x_i \geq 0\} \cap \bigcap_i \{y_i \geq 0\} \cap \{z_1, z_2 \geq 0\} \\ \cap \bigcap_A \{\sum x_i \leq rk(M_A)\} \cap \bigcap_B \{\sum y_i \leq rk(M_B)\} \\ \cap \{\sum x_i + z_1 = k_1\} \cap \{\sum y_i + z_2 = k_2\} \cap \{z_1 + z_2 \leq 1\}. \end{aligned}$$

$\Phi(x_1, \ldots, x_{n_1}, z_1, z_2, y_1, \ldots, y_{n_2}) = (x_1, \ldots, x_{n_1}, z, y_1, \ldots, y_{n_2})$ clearly still satisfies the rank inequalities associated to A and to B, and $\sum x_i + z + \sum y_i = k_1 + k_2$. Hence $\Phi(x_1, \ldots, x_{n_1}, z_1, z_2, y_1, \ldots, y_{n_2}) \in P_{\tau^{\leq k_1+k_2}(M\oplus *\oplus N)}$. Furthermore, as $z_1, z_2 \in [0, 1]$ we have $\sum x_i \in [k_1 - 1, k_1]$ and $\sum y_i \in [k_2 - 1, k_2]$, so

$$P_{S(\tau^{\leq k_1}(M\oplus *),\tau^{\leq k_2}(*\oplus N))} \subset P_{\tau^{\leq k}(M\oplus *\oplus N)} \cap \{\sum x_i \in [k_1 - 1, k_1]\} \cap \{\sum y_i \in [k_2 - 1, k_2]\}.$$

Conversely, given $(x_1, \ldots, x_n, z, y_1, \ldots, y_n)$ lying in the right hand side, we produce $z_1, z_2 \in [0, 1]$ as $k_1 - \sum x_i$ and $k_2 - \sum y_i$ respectively, and because $\sum x_i + z + \sum y_i = k_1 + k_2$, we have $z_1 + z_2 = z$, so in particular $z_1 + z_2 \leq 1$ and we conclude the reverse containment.

A similar argument works for the second case with parallel connection. $\square$

Remark 4.9.10. *Schubert matrices can constructed as* $(\ldots \tau^{\leq 2}(\tau^{\leq 1}(* \oplus \ldots \oplus *) \oplus * \oplus \ldots \oplus *) \ldots)$, *so the direct sum and truncation results imply the result from Example 4.9.5.*

Theorem 4.9.11. *If we have an exact relation of indicator functions* $\sum b_i 1_{P_{M_i}} = 0$*, then we have* $\sum b_i [M]_\hbar = 0$.

Proof. Let I_M be the *independence polytope* of M, the convex hull of all indicator functions of independent subsets of columns of M. Then $I_M = (P_M + \sum_i \mathbb{R}_{\geq 0}(-e_i)) \cap (\mathbb{R}_{\geq 0})^n$. Because relations between indicator functions are preserved under Minkowski sums with rays (the proof reduces to the one-dimensional case where it is a direct check), we thus obtain the relation $\sum b_i 1_{I_{M_i}} = 0$. Taking the product with $[0,1]$ gives $\sum b_i 1_{I_{M_i \oplus *}} = 0$, and intersecting with $\sum x_i = k$ then gives us

$$\sum b_i 1_{P_{\tau^{\leq k}(M_i \oplus *)}} = 0.$$

This in turn implies

$$\sum \mathcal{O}_{\tau^{\leq k}(M_i \oplus *)} = 0,$$

from which we may conclude that

$$[\hbar^k] \sum b_i [M_i]_\hbar = 0.$$

As this is true for all k the result follows. □

Remark 4.9.12. *By [20] we can explicitly write*

$$1_{P_{M_i}} = \sum b_i 1_{P_{S_i}}$$

for various Schubert matrices S_i*, so the above result and example allow us to explicitly write down* $[M]_\hbar$ *for any* M*.*

We end with a discussion of what the operation $[M]_\hbar$ is in the non-equivariant case. To understand $[M]_\hbar$, we look at $[M]_\hbar^\dagger = \sum[\mathcal{O}_{\tau^{\leq i+1}(M\oplus *)}]\hbar^i$. This is

$$\sum_i \sum_{(i_1,\ldots,i_n,j)\in(r+1)P_{\tau^{\leq i+1}(M\oplus *)}-\epsilon} H_1^{r-i_1}H_2^{r-i_2}\ldots H_n^{r-i_n}H^{r-j}\hbar^i.$$

This expression, when we substitute $H = z$ and $\hbar = z^{r+1}$, is precisely

$$\sum_{(i_1,\ldots,i_n)\in(r+1)I_M-\epsilon} H_1^{r-i_1}\ldots H_n^{r-i_n}z^{\sum_k i_k}.$$

We defer a more detailed discussion to our paper [52], but one can show in the non-equivariant case that the relation $I_M \times I_N = I_{M\oplus N}$ implies because of this last statement that $[M]_\hbar \star [N]_\hbar = [M \oplus N]_\hbar$, and using a similar non-equivariant to equivariant trick as in the proof of Theorem 4.7.4, we can recover the equivariant result. More generally, we can express

$$I_M \cap \{\sum x_i \in [k, k+1]\} = \phi(P_{\tau^{\leq k+1}(M\oplus *)}),$$

where ϕ is the projection away from the last coordinate corresponding to $*$, and the polytope subdivision from the proof of Theorem 4.9.9 actually follows from decomposing each of $I_M = \bigcup_k I_M \cap \{\sum x_i \in [k, k+1]\}$ and $I_N = \bigcup_k I_N \cap \{\sum x_i \in [k, k+1]\}$ and expressing $(I_M \times I_N) \cap \{\sum x_i \in [k, k+1]\} = I_{M\oplus N} \cap \{\sum x_i \in [k, k+1]\}$ in terms of these decompositions, along with the observation of Fink and Speyer stated in the proof. The operation $[M]_\hbar$ can actually be viewed as a pullback operation along a resolution of μ_M, where the connection to the independence polytope is more clear. For more details see our paper [52].

Chapter 5

Incidence strata of affine varieties with complex multiplicities

This project is joint with Dennis Tseng, adapted from [60], reproduced with permission. To each affine variety X and $m_1, \ldots, m_k \in \mathbb{C}$ such that no subset of the m_i add to zero, we construct a variety which for $m_1, \ldots, m_k \in \mathbb{N}$ specializes to the closed $(m_1, \ldots, m_k)$-incidence stratum of the symmetric power $\mathrm{Sym}^{m_1+\ldots+m_k} X$. These fit into a finite-type family, which is functorial in X, and which is topologically a family of $\mathbb{C}$-weighted configuration spaces. We verify our construction agrees with an analogous construction in the Deligne category $\mathrm{Rep}(S_d)$ for $d \in \mathbb{C}$.

We next classify the singularity locus and branching behaviour of colored incidence strata for arbitrary smooth curves. As an application, we negatively answer a question of Farb and Wolfson concerning the existence of an isomorphism between two natural moduli spaces.

5.1 Introduction

The moduli space of d unordered points on $\mathbb{A}^1$ over $\mathbb{C}$ is

$$\mathrm{Sym}^d\mathbb{A}^1 \cong \mathbb{A}^d = \{z^d + a_1 z^{d-1} + \ldots + a_d \mid a_1, \ldots, a_d \in \mathbb{C}\},$$

where the correspondence associates to a degree d monic polynomial $f(z)$ its unordered set of roots. For a partition $P = (m_1, \ldots, m_k)$ of d, there is an associated *incidence stratum*[1]

$$\Delta^k_P(\mathbb{A}^1) := \{(z - x_1)^{m_1} \ldots (z - x_k)^{m_k} \mid x_1, \ldots, x_k \in \mathbb{C}\} \subset \mathrm{Sym}^d\mathbb{A}^1,$$

which corresponds to the subvariety of d-point configurations on $\mathbb{A}^1$ whose incidences are at least as coarse as P. The study of the geometry and defining equations of the incidence strata $\Delta^k_P(\mathbb{A}^1)$ dates back to Cayley [15], and has been extensively studied since [62, 64, 63, 16, 1, 2, 49, 50].

Example 5.1.1. *Let $P = (2, 1)$. Then $\Delta^2_{(2,1)}(\mathbb{A}^1)$ is the cubic discriminant hypersurface*

$$a_1^2 a_2^2 + 18a_1a_2a_3 - 4a_2^3 - 4a_1^3a_3 - 27a_3^2 = 0.$$

The normalization is the bijective morphism

$$\begin{aligned}\Phi_{(2,1)} : \mathbb{A}^1 \times \mathbb{A}^1 &\to \Delta^3_{(2,1)}(\mathbb{A}^1)\\ (a, b) &\mapsto (z - a)^2(z - b),\end{aligned}$$

and despite being a homeomorphism in the Euclidean topology, $\Phi_{(2,1)}$ is not an iso-

[1] We always mean *closed* incidence stratum, and will always omit the word closed.

morphism as $\mathbb{A}^1 \times \mathbb{A}^1$ *is non-singular but*

$$Sing(\Delta^2_{(2,1)}) = \{(z-a)^3 \mid a \in \mathbb{C}\} = \Delta^1_{(3)},$$

occuring when the doubled and single root collide. This reflects a key difference between the topology and the algebraic geometry of incidence strata.

Even though the incidence strata $\Delta^k_P(\mathbb{A}^1)$ are defined with $m_1, \ldots, m_k$ positive integers, one can actually make sense of the case where $m_1, \ldots, m_k$ are complex numbers with no subset summing to zero [43, 57, 23], and these generalized incidence strata for fixed k fit into a finite-type family $\Delta^k(\mathbb{A}^1)$ over the space of allowable m_i [12, Proposition 2.6]. The goal of this paper is to expand upon this theory in a number of different directions with concrete applications to algebraic geometry in mind.

First, we show for fixed k that the incidence strata $\Delta^k_P(X)$ of arbitrary affine varieties X fit into a finite-type family $\Delta^k(X)$ exactly as with $\mathbb{A}^1$ (Theorem 5.1.3), and this construction is functorial in X. The construction from [12, Proposition 2.6] yields for fixed k a family of embeddings $\Delta^k_P(\mathbb{A}^1) \subset \mathbb{A}^{N_k}$ where N_k depends only on the number of parts k of the partition P. We will see a similar uniform embedding result holds for $\Delta^k_P(X)$ (Proposition 5.5.5). The known proof that N_k exists is not constructive [12, Remark 2.7 (1)]; we constructivize this result, conjecture $N_k = 2^k - 1$ (Conjecture 5.1.11), verify $N_k = 2^k - 1$ for $k \leq 3$, and show $N_4 \geq 15$.

We next extend the method of determining the singular locus of $\Delta^k_P(\mathbb{A}^1)$ [16, 49] to find the singular locus of colored incident strata in arbitrary smooth curves (Theorem 5.1.4). Using this we answer a question of Farb and Wolfson [24, Question 1.4] about whether certain moduli spaces are isomorphic (Theorem 5.1.7).

Finally, Pavel Etingof communicated to us an analogous construction using the Deligne category $\mathrm{Rep}(S_d)$ for $d \in \mathbb{C}$. The Deligne-categoric framework allows us to naturally consider d'th powers of varieties for $d \in \mathbb{C}$, and the construction mirrors the classical construction of $\Delta^k_{(m_1,\ldots,m_k)}(X)$ as the $S_{m_1+\ldots+m_k}$-quotient of the $(m_1,\ldots,m_k)$-incidence loci of ordered configurations in $X^{m_1+\ldots+m_k}$. In Section 5.9, we verify that our interpolated varieties $\Delta^k_{(m_1,\ldots,m_k)}(X)$ arise as the spectrum of the algebra produced by this construction.

Remark 5.1.2. *As we saw in Example 5.1.1, there are invariants of incidence strata which can be detected algebraically but not topologically. The topology of an incidence strata is determined by the lattice of partial sums of the m_i, yielding finitely many homeomorphism types, but as we will see if $k = 3$ and $X = \mathbb{A}^n$ there are infinitely many isomorphism types of k-part incidence strata (Proposition 5.5.13). Surprisingly when $k = 2$ there are only two isomorphism types of incidence strata occurring in $\mathbb{A}^n$, classified by whether or not m_1 and m_2 are equal (Proposition 5.5.9), but we believe that this is particular to $\mathbb{A}^n$.*

5.1.1 Statement of results

We briefly sketch how one can generalize the incidence stratum $\Delta^k_P(\mathbb{A}^1)$ to the case where the weights $m_1,\ldots,m_k$ are complex numbers. If $m_1,\ldots,m_k \in \mathbb{N}$, then we may express

$$(z - x_1)^{m_1} \ldots (z - x_k)^{m_k} = z^d - a_1 z^{d-1} + \ldots$$

where

$$a_i = e_i(\underbrace{x_1,\ldots,x_1}_{m_1},\ldots,\underbrace{x_k,\ldots,x_k}_{m_k}) = \sum_{i_1+\ldots+i_k=d} \binom{m_1}{i_1}\ldots\binom{m_k}{i_k} x_1^{i_1}\ldots x_k^{i_k}.$$

Note that for $m_i \in \mathbb{N}$, the expressions a_i vanish for $i > d$.

For complex weights, these expressions for the a_i still make sense, although the a_i are in general non-zero for all i. One can show [43, 57] that if we restrict the m_i to lie in the subset

$$(\mathbb{C}^k)^\circ = \mathbb{C}^k \setminus \bigcup_{S \subset \{1,\dots,k\}} \{\sum_{i \in S} m_i = 0\},$$

then despite not stabilizing to zero for sufficiently large i the a_i eventually depend polynomially on the previous a_j, so truncating at this point loses no information. Furthermore, there is a uniform bound for when this occurs [12], allowing one to use the truncated list of coefficients to define a family containing all k-part incidence strata on $\mathbb{A}^1$.

Formally, as long as no subset of the m_i sum to zero, the subalgebra $\mathbb{C}[a_1, a_2, \dots] \subset \mathbb{C}[x_1, \dots, x_k]$ is finite type. In fact, letting the m_i vary, we have a finite type subalgebra

$$\mathbb{C}[m_1, \dots, m_k][a_1, \dots][\{\frac{1}{\sum_{i \in S} m_i}\}_S] \subset \mathbb{C}[m_1, \dots, m_k, x_1, \dots, x_k][\{\frac{1}{\sum_{i \in S} m_i}\}_S]$$

whose inclusion is finite [12, Proposition 2.6] (here S ranges over nonempty subsets of $\{1, \dots, k\}$). Taking Spec shows these generalized incidence strata all fit together into a family, embedded in the same affine space $\mathbb{A}^{N_k}$ where N_k is the point at which $m_1, \dots, m_k, a_1, \dots, a_{N_k}$ generate the entire sub-algebra.

We show that these considerations can be generalized to arbitrary affine varieties.

Incidence strata of affine varieties with complex multiplicities

Theorem 5.1.3. *There is a functorial assignment $X \mapsto \Delta^k(X)$ of affine varieties to affine varieties over $(\mathbb{C}^k)^\circ$ such that the reduced fiber over any $P = (m_1, \dots, m_k) \in \mathbb{N}^k \subset \mathbb{C}^k$ is precisely the $(m_1, \dots, m_k)$-incidence strata*

$$\Delta^k_P(X) \subset \mathrm{Sym}^{m_1 + \dots + m_k} X.$$

Theorem 5.1.3 generalizes almost verbatim to *colored incidence strata*, where we will be especially interested in the case of smooth curves C. Consider again for simplicity the case $C = \mathbb{A}^1$, and suppose we have colors $1, \ldots, r$ and d_i points of each color i. Then given $\overline{P} = (\overline{m}_1, \ldots, \overline{m}_k) \in (\mathbb{Z}^r_{\geq 0} \setminus \{\overline{0}\})^k$ such that

$$\overline{m}_1 + \ldots + \overline{m}_k = (d_1, \ldots, d_r),$$

we have an analogously defined *r-colored incidence strata*

$$\Delta^{k,r}_{\overline{P}} \subset \prod_{i=1}^{r} \mathrm{Sym}^{d_i} \mathbb{A}^1 \cong \prod_{i=1}^{r} \mathbb{A}^{d_i}$$
$$\Delta^{k,r}_{\overline{P}} := \{(\prod_{i=1}^{k}(z - x_i)^{(\overline{m_i})_1}, \ldots, \prod_{i=1}^{k}(z - x_i)^{(\overline{m_i})_r}) \mid x_1, \ldots, x_k \in \mathbb{C}\}$$

where the $\overline{m}_i$ now not only keeps track of how many colored points are incident but also how many of each color. We show that the colored incidence strata $\Delta^{k,r}_{\overline{P}}(\mathbb{A}^1)$ can be similarly placed into a natural finite-type family

$$\begin{array}{l} \Delta^{k,r}(\mathbb{A}^1) \\ \quad\downarrow \\ ((\mathbb{C}^r)^k)^\circ = (\mathbb{C}^r)^k \setminus \bigcup_{S \subset \{1,\ldots,k\}} \sum_{i \in S}\{\overline{m}_i = 0\}, \end{array}$$

so we may talk about $\Delta^{k,r}_{\overline{P}}(\mathbb{A}^1)$ for arbitrary $\overline{P} \in ((\mathbb{C}^r)^k)^\circ$. The following theorem specializes to Theorem 5.1.3 when $r = 1$ (i.e. the "uncolored case").

Theorem 5.1.3'. *There is a functorial assignment $X \mapsto \Delta^{k,r}(X)$ of affine varieties to varieties over $((\mathbb{C}^r)^k)^\circ$ such that the reduced fiber over any $\overline{P} = (\overline{m_1}, \ldots, \overline{m_k}) \in (\mathbb{Z}^r_{\geq 0} \setminus \{\overline{0}\})^k \subset ((\mathbb{C}^r)^k)^\circ$ is precisely the r-colored $(\overline{m_1}, \ldots, \overline{m_k})$-incidence strata*

$$\Delta^{k,r}_{\overline{P}}(X) \subset \mathrm{Sym}^{(\overline{m_1})_1 + \ldots + (\overline{m_k})_1} X \times \cdots \times \mathrm{Sym}^{(\overline{m_1})_r + \ldots + (\overline{m_k})_r} X.$$

A topological model. One can construct the underlying topological spaces of $\Delta^k_P(X)$

and $\Delta^{k,r}_{\overline{P}}(X)$ in the Euclidean topology as the quotient of the space of configurations of k ordered points on X (which we think of as carrying multiplicities associated to P or $\overline{P}$ respectively), by the relation that two configurations $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent if at each point of $x \in X$, the sum of the weights associated to the two configurations are the same. In other words, when points collide their multiplicities add. This process also works relatively over the space of allowable weights, and creates a natural family of weighted configuration spaces over this base. In Subsection 5.4.1, we prove that our constructions agree with these topological notions.

Singularities of incidence strata of curves with complex multiplicities

Our next result determines the singularity and branching locus of the fiber $\Delta^{k,r}_{\overline{P}}(\mathbb{A}^1)$ (and more generally $\Delta^{k,r}_{\overline{P}}(C)$) with $\overline{P} \in ((\mathbb{C}^r)^k)^\circ$. Theorem 5.1.4 vastly generalizes the characterizations in [16, 49] of singularity loci and branching behaviour of uncolored incidence strata of $\mathbb{A}^1$ (see Remark 5.1.6).

Theorem 5.1.4. *Let C be a smooth curve, let $\overline{P} = (\overline{m_1}, \ldots, \overline{m_k}) \in ((\mathbb{C}^r)^k)^\circ$, and let $\overline{X} \in \Delta^{k,r}_{\overline{P}}(C)$ be a point with associated multiplicities $\overline{Q} = (\overline{q_1}, \ldots, \overline{q_\ell}) \in ((\mathbb{C}^r)^\ell)^\circ$.*

- *Preimages $\overline{Y}$ of $\overline{X}$ in the normalization of $\Delta^{k,r}_{\overline{P}}(C)$ (corresponding to branches at $\overline{X}$) are in bijection with systems of equations*

$$\begin{aligned}
\overline{q_1} &= \overline{p_{1,1}} + \ldots + \overline{p_{1,i_1}} \\
\overline{q_2} &= \overline{p_{2,1}} + \ldots + \overline{p_{2,i_2}} \\
&\vdots \\
\overline{q_\ell} &= \overline{p_{\ell,1}} + \ldots + \overline{p_{\ell,i_\ell}}
\end{aligned}$$

  where we consider for each j the collection of $\overline{p_{j,1}}, \ldots, \overline{p_{j,i_j}}$ only up to permutation, such that $i_1 + \ldots + i_\ell = k$, and the collection of all $\overline{p_{i,j}}$ constitute the *collection* $\overline{m_1}\ldots,\overline{m_k}$ *with multiplicities.*

- *The normalization of* $\Delta^{k,r}_{\overline{P}}(C)$ *is smooth, and the normalization map is not an immersion near a preimage* $\overline{Y}$ *of* $\overline{X}$ *(i.e. the corresponding branch is not smooth at* $\overline{X}$*) if and only if there exists a* j *such that the set of* $\overline{p_{j,u}}$ *with* $1 \leq u \leq i_j$ excluding repetitions *are linearly dependent.*

Example 5.1.5. *Consider the following examples:*

1. *If* $\overline{m}_1 = \begin{pmatrix}1\end{pmatrix}$, $\overline{m}_2 = \begin{pmatrix}2\end{pmatrix}$, $\overline{q}_1 = \begin{pmatrix}3\end{pmatrix}$, *then there is one preimage of* q_1 *in the normalization, and the normalization map is not an immersion at that point as the two vectors* $\begin{pmatrix}1\end{pmatrix}$ *and* $\begin{pmatrix}2\end{pmatrix}$ *are not linearly independent. This is Example 5.1.1 above when* $C = \mathbb{A}^1$.

2. *If* $\overline{m_1} = \begin{pmatrix}1\end{pmatrix}$, $\overline{m}_2 = \begin{pmatrix}1\end{pmatrix}$, $\overline{q}_1 = \begin{pmatrix}2\end{pmatrix}$, *then there is one preimage of* $\overline{q_1}$ *in the normalization, and the normalization map is an immersion at that point as the single vector* $\begin{pmatrix}1\end{pmatrix}$ *is linearly independent. In this case, the normalization is simply the identity map on the symmetric square of* C.

3. *If* $\overline{m_1},\ldots,\overline{m_5}$ *are the columns of* $\begin{pmatrix}1 & 1 & 2 & 0 & 100\\ 1 & 1 & 0 & 2 & 101\end{pmatrix}$ *and* $\overline{q_1},\overline{q_2}$ *are the columns of* $\begin{pmatrix}102 & 2\\ 103 & 2\end{pmatrix}$, *then there are two preimages in the normalization corresponding to*

$$\overline{q_1} = \overline{m_1} + \overline{m_2} + \overline{m_5} \qquad \overline{q_1} = \overline{m_3} + \overline{m_4} + \overline{m_5}$$

$$\overline{q_2} = \overline{m_3} + \overline{m_4} \qquad \overline{q_2} = \overline{m_1} + \overline{m_2}.$$

The expressions for $\overline{q_1}$ *and* $\overline{q_2}$ *on the left correspond to a preimage where the normalization map is an immersion and the expressions for* $\overline{q_1}$ *and* $\overline{q_2}$ *on the right* *correspond to a preimage where the normalization map is not an immersion.*

Remark 5.1.6. *Since the normalization of $\Delta_{\overline{P}}^{k,r}(C)$ is smooth (Section 5.7), its smooth locus coincides with its normal locus by Zariski's main theorem. Thus, a point $\overline{X} \in \Delta_{\overline{P}}^{k,r}(C)$ is non-singular if and only if it has exactly one preimage in the normalization and the normalization map is an immersion near that preimage. Theorem 5.1.4 thus gives a combinatorial description of the singular locus of $\Delta_{\overline{P}}^{k,r}(C)$.*

A conjecture of Farb and Wolfson. As an application, we negatively answer a question of Farb and Wolfson [24] concerning the existence of an isomorphism between two natural moduli spaces. In [24], it was asked whether the varieties

$$\begin{aligned} Rat^*_{d,n} &= \{\text{Degree } d \text{ pointed regular maps } \mathbb{P}^1 \to \mathbb{P}^n\} \\ Poly^{d(n+1),1}_{n+1} &= \{\text{Monic degree } d(n+1) \text{ polynomials with no multiplicity } n+1 \text{ root}\} \end{aligned}$$

are isomorphic when $n \geq 2$.

We prove the following.

Theorem 5.1.7. *The varieties $Rat^*_{d,n}(\mathbb{C})$, $Poly^{d(n+1),1}_{n+1}(\mathbb{C})$ are not isomorphic for any $d, n \geq 2$.*

Note that Remark 1.5.1 in [24] shows that for $n = 1$ and $d \geq 2$ the varieties have different fundamental groups. For $d = 1$ the varieties are isomorphic by an identical argument to the one given in [24, Remark 1.5.2]. Thus we in fact have a complete classification of when these varieties are isomorphic.

The proof also shows that they in fact can't even be biholomorphic to each other, but the following remains open.

Question 5.1.8. *Are $Rat^*_{d,n}(\mathbb{C})$ and $Poly^{d(n+1),1}_{n+1}(\mathbb{C})$ homeomorphic over $\mathbb{C}$?*

Connection to Deligne categories

Deligne categories $\text{Rep}(S_d)$ [19] for $d \in \mathbb{C}$ are rigid tensor categories which have a notion of "complex tensor power of an algebra", allowing us to consider objects like $\text{Spec}\, A^{\otimes d}$ parameterizing d ordered points on $\text{Spec}\, A$ for $d \in \mathbb{C}$. A construction for the interpolated varieties $\Delta^k_{(m_1,\ldots,m_k)}$ via $\text{Rep}(S_d)$ was communicated to us by Pavel Etingof, which we will verify agrees with our construction. The ideal $I(m_1, \ldots, m_k) \subset A^{\otimes d}$ used in the following theorem will be defined and motivated in Subsection 5.2.3.

Theorem 5.1.9. *Let A be a reduced finite-type $\mathbb{C}$-algebra, and let $(m_1, \ldots, m_k) \in (\mathbb{C}^k)^\circ$ be such that $m_1, \ldots, m_k$ and $d = m_1 + \ldots + m_k$ are all not in $\mathbb{Z}_{\geq 0}$. Then*

$$\text{Spec}(\text{Hom}_{\text{Rep}(S_d)}(\mathbf{1}, A^{\otimes d}/I(m_1, \ldots, m_k))) \cong \Delta^k_{(m_1,\ldots,m_k)}(\text{Spec} A)$$

The condition on d and the m_i not being in $\mathbb{Z}_{\geq 0}$ does not present an obstacle, since one can simply scale d and the $m_1, \ldots, m_k$ by the same number.

Remark 5.1.10. *When $d \in \mathbb{Z}_{\geq 0}$, $\text{Rep}(S_d)$ is no longer semisimple and one needs to take a further quotient to get the usual category of representations of S_d [18, Section 3.4]. We expect this does not present a serious issue for the direct construction, see [19, Proposition 5.1] and [21, Remark 4.1.2].*

Effective bounds for incidence strata

The construction from [12, Proposition 2.6] yields for fixed k a family of embeddings $\Delta^k_P(\mathbb{A}^1) \subset \mathbb{A}^{N_k}$ (see the beginning of Subsection 5.1.1 for a more precise definition of N_k). Here, we give results and conjectures towards computing N_k. Define

$$N_k(P) = \min\{n \mid \Delta^k_P \text{ embeds into } \mathbb{A}^n\}.$$

Then we have

$$\max\{N_k(P) \mid P \in \mathbb{N}^k\} \leq \max\{N_k(P) \mid P \in (\mathbb{C}^k)^\circ\} \leq N_k.$$

Conjecture 5.1.11. *These three quantities equal $2^k - 1$ for $k \geq 1$.*

Theorem 5.1.12. *Conjecture 5.1.11 is true for $k \leq 3$, and $N_4((1,2,4,8)) = 15$.*

We believe that the maximum of $N_k(P)$ is attained for $P = (2^0, 2^1, \ldots, 2^{k-1})$. As we will see (Proposition 5.5.5), $N_k(P)$ not only bounds the complexity of k-part incidence strata $\Delta^k_P(\mathbb{A}^1)$, but also the complexity of k-part incidence strata

$$\Delta^k_P(X) \subset \mathrm{Sym}^{m_1+\ldots+m_k} X$$

for affine X. Indeed, if $X \subset \mathbb{A}^n$, then we will see that we may embed

$$\Delta^k_P(X) \subset \mathbb{A}^{\binom{N_k(P)+n}{n}-1}.$$

Using $N_k(P) \leq N_k$ this yields for fixed k a uniform bound which drastically improves the dimension given by the Chow embedding

$$\Delta^k_P(X) \subset \mathrm{Sym}^{m_1+\ldots+m_k} X \subset \mathbb{A}^{\binom{m_1+\ldots+m_k+n}{n}-1}$$

which depends on the total multiplicity $d = \sum_{i=1}^k m_i$.

We are able to produce an upper bound of approximately $k!^k$ on $N_k(P)$ and $2^{k^2} k!^k$ on N_k using the effective Nullstellensatz [40, Theorem 1.3], but with a slightly worse bound of approximately $k!^{2k}$, we can explicitly eliminate coefficients in the Chow embedding to produce our embedding. These yield recursive identities similar in spirit to the Newton identities, but for symmetric polynomials whose variables are specialized to k groups of distinct values with m_i variables in each group (Section 5.6), which may be of independent interest.

5.1.2 Acknowledgements

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5.2 Three constructions for interpolating k-part incidence strata

In this section, we outline three ways to interpolate k-part incidence strata of an affine variety X, which we will later show in Section 5.9 yield the same result. For simplicity we describe here all of the constructions in the uncolored case, and in future sections only detail the extension of this first construction to the colored case. We end this section with a discussion of the subtleties of reducedness for the fibers of $\Delta^k(X)$ over $(\mathbb{C}^k)^\circ$. We could extend all of our constructions to $\Delta^k(\mathrm{Spec} A)$ for A non-reduced, but we do not understand the scheme-theoretic fibers over $P \in (\mathbb{C}^k)^\circ$ well enough to refine Theorem 5.1.3 in this direction.

5.2.1 Construction 1: Deforming the Chow embedding

This first construction is the most straightforward, and the most ad hoc construction. Our plan will be to first construct $\Delta^k(\mathbb{A}^n)$ by deforming the Chow embedding, and then define $\Delta^k(X)$ as the closed subvariety where the points are restricted to lie on X.

Recall that for an n-dimensional vector space V, the Chow embedding takes

$$\mathrm{Sym}^d \mathbb{A}(V) \hookrightarrow \prod_{i=1}^{d} \mathrm{Sym}^i V$$
$$(v_1, \ldots, v_d)^{S_d} \mapsto \text{Coefficients of } (z - v_1) \ldots (z - v_d),$$

in exact analogue to the identification of $\mathrm{Sym}^d \mathbb{A}^1$ with the set of degree d monic polynomials in an indeterminate z (see Lemma 5.3.3). Therefore the $(m_1, \ldots, m_k)$-incidence strata corresponds to

$$\Delta^k_{(m_1,\ldots,m_k)}(\mathbb{A}(V)) = \{(z - v_1)^{m_1} \ldots (z - v_k)^{m_k} \mid v_1, \ldots, v_k \in V\} \subset \prod_{i=1}^{d} \mathrm{Sym}^i V$$

where $d = m_1 + \ldots + m_k$. Exactly as we sketched in the introduction in the case of $\mathbb{A}^1$ from [12], $\Delta^k_{(m_1,\ldots,m_k)}(\mathbb{A}(V))$ projects isomorphically onto its image in $\prod_{i=1}^{N_k} \mathrm{Sym}^i V$, and now we have a bounded list of coefficients which we can interpolate to complex m_i. Writing a_i for the coefficient of z^{d-i} in $(z - v_1)^{m_1} \ldots (z - v_k)^{m_k}$, which depends polynomially on $m_1, \ldots, m_k, v_1, \ldots, v_k$, we will show the morphism

$$(\mathbb{C}^k)^\circ \times \mathbb{A}(V)^k \to (\mathbb{C}^k)^\circ \times \prod_{i=1}^{N_k} \mathrm{Sym}^i V$$
$$(m_1, \ldots, m_k, v_1, \ldots, v_k) \mapsto (m_1, \ldots, m_k, a_1, \ldots, a_{N_k})$$

is finite and birational, so in particular the image is closed. In particular, the reduced fiber of the image over any $(m_1, \ldots, m_k) \in (\mathbb{C}^k)^\circ$ is precisely $\Delta^k_{(m_1,\ldots,m_k)}(\mathbb{A}(V))$ projected to $\prod_{i=1}^{N_k} \mathrm{Sym}^i V$, which as we said before is isomorphic to $\Delta^k_{(m_1,\ldots,m_k)}(\mathbb{A}(V))$. Therefore we may take $\Delta^k(\mathbb{A}(V))$ to be the image.

Finally, given an affine variety X, we choose an embedding $X \hookrightarrow \mathbb{A}(V)$, and take

$\Delta^k(X)$ to be the image of the proper composite map

$$(\mathbb{C}^k)^\circ \times X^k \to (\mathbb{C}^k)^\circ \times \mathbb{A}(V)^k \to (\mathbb{C}^k)^\circ \times \prod_{i=1}^{N_k} \mathrm{Sym}^i V.$$

We will later show that this is functorial and works as expected topologically.

5.2.2 Construction 2: Interpolating the symmetric quotient

The next construction is manifestly functorial, and relies on the identification of $\Delta^k_{(m_1,\ldots,m_k)}(X)$ as the S_d-quotient of the corresponding ordered $(m_1,\ldots,m_k)$-incidence strata in X^d, where $d = m_1+\ldots+m_k$. Here, the ordered $(m_1\ldots,m_k)$-incidence strata in X^d is the union of all embedded copies of X^k formed by partitioning $\{1,\ldots,d\} = M_1 \sqcup \ldots \sqcup M_k$ with $|M_i| = m_i$, and taking

$$\underbrace{X \times \ldots \times X}_{k} \to X^{M_1} \times \ldots \times X^{M_k} = X^d$$

where the first map is the product of diagonal embeddings. Writing $X = \mathrm{Spec}A$, we now exploit the fact that the subalgebra $(A^{\otimes d})^{S_d} \subset A^{\otimes d}$ is the subalgebra generated by elements of the form $\sum_{i=1}^d 1^{\otimes i-1} \otimes a \otimes 1^{\otimes d-i}$. The ideal of each embedded copy of X^k has the same intersection with $(A^{\otimes d})^{S_d}$, so the algebra of functions on $\Delta^k_{(m_1,\ldots,m_k)}(X)$ is the cokernel of the composite

$$(A^{\otimes d})^{S_d} \hookrightarrow A^{\otimes d} = A^{\otimes m_1} \otimes \ldots \otimes A^{\otimes m_k} \to \underbrace{A \otimes \ldots \otimes A}_{k} = A^{\otimes k}$$

where the final map is the tensor product of the multiplication maps $A^{\otimes m_i} \to A$. Thus, we see that $\Delta^k_{(m_1,\ldots,m_k)}(X)$ is the spectrum of the subalgebra of $A^{\otimes k}$ generated by expressions of the form

$$\sum_{i=1}^{k} m_i(1^{\otimes i-1} \otimes a \otimes 1^{\otimes k-i})$$

with $a \in A$. Finally, we take $\Delta^k(X)$ to be the spectrum of the subalgebra of $\Gamma((\mathbb{C}^k)^\circ) \otimes A^{\otimes k}$ generated by $m_1, \ldots, m_k$ and all expressions of the above form.

Remark 5.2.1. *It is not clear from this construction that the resulting algebra is finite-type, nor that taking the reduced fiber over $(m_1, \ldots, m_k) \in (\mathbb{C}^k)^\circ$ yields the variety $\Delta^k_{(m_1,\ldots,m_k)}(X)$. However, it is easy to show that this construction agrees with Construction 1 (see Section 5.9), for which these properties are checked.*

5.2.3 Construction 3: Deligne categorical interpolation

The final construction, communicated to us by Pavel Etingof, uses the natural interpolation provided by the Deligne category $\mathrm{Rep}(S_d)$ to construct the algebra of functions on $\Delta^k_{(m_1,\ldots,m_k)}(X)$ with $(m_1, \ldots, m_k) \in (\mathbb{C}^k)^\circ$. It would be interesting to check the entire interpolated variety $\Delta^k(X)$ can be obtained through Deligne categories, but the verification and construction would be necessarily more complicated.

Given a commutative algebra A and $d \in \mathbb{C}$, we can construct the algebra $A^{\otimes d}$ in the Ind-completion of the Deligne category $\mathrm{Rep}(S_d)$ [22, Section 4.1]. There is a natural multiplication map of algebras $A^{\otimes d} \to A$ in $\mathrm{Ind}(\mathrm{Rep}(S_d))$, whose kernel is an ideal J_d. Intuitively, we should think of $A^{\otimes d}/J_d$ as the algebra of functions on the diagonal of "$(\mathrm{Spec} A)^d$" where "all d points are equal".

Given $m_1, \ldots, m_k \in \mathbb{C}$ with $m_1 + \ldots + m_k = d$, there is a natural restriction functor [22, Section 2.3] $\mathrm{Res} : \mathrm{Rep}(S_d) \to \mathrm{Rep}(S_{m_1}) \boxtimes \ldots \boxtimes \mathrm{Rep}(S_{m_k})$ taking

$$A^{\otimes d} \mapsto A^{\otimes m_1} \boxtimes \ldots \boxtimes A^{\otimes m_k}.$$

Definition 5.2.2. *Let $I(m_1,\ldots,m_k)$ be the sum of all ideals $J \subset A^{\otimes d}$ such that*

$$\mathrm{Res}(J) \subset J_{m_1} \boxtimes \ldots \boxtimes J_{m_k}.$$

The quotient $A^{\otimes d}/I(m_1,\ldots,m_k)$ should be thought of as the algebra of functions on the configurations of points in "$(\mathrm{Spec} A)^d$" where the d points that are grouped with multiplicities $m_1,\ldots,m_k$. Letting $\mathbf{1}$ be the object in $Rep(S_d)$ corresponding to the trivial representation, $\mathrm{Hom}(\mathbf{1},-)$ should be thought of as taking S_d-invariants, so our third construction will be the spectrum of

$$\mathrm{Hom}_{\mathrm{Rep}(S_d)}(\mathbf{1}, A^{\otimes d}/I(m_1,\ldots,m_k)),$$

which is an honest $\mathbb{C}$-algebra which should morally be the algebra of functions on $\Delta^k_{(m_1,\ldots,m_k)}(X)$. In fact, when $m_1,\ldots,m_k,d \notin \mathbb{Z}_{\geq 0}$, we will show in Section 5.9 that Constructions 2 and 3 agree for arbitrary (possibly nonreduced and non-finite-type) $\mathbb{C}$-algebras A.

5.2.4 On the fibers of $\Delta^k(X)$

Theorem 5.1.3 shows the reduced fibers of the family $\Delta^k(X) \to (\mathbb{C}^k)^\circ$ agree with $\Delta^k_P(X)$ for $P \in (\mathbb{C}^k)^\circ$. However, if we take scheme-theoretic fibers, some of the fibers will be nonreduced.

Example 5.2.3. *We will see in Proposition 5.5.9 that*

$$\Delta^2(\mathbb{A}^1) \cong \mathbb{C}[m_1, m_2, \frac{1}{m_1 m_1 (m_1+m_2)}][x,y]/((m_1-m_2)^2 x^3 = y^2).$$

Taking the fiber over $(m_1,m_2) = (a,a)$ yields the doubled line $\mathbb{C}[x,y]/(y^2)$. Concretely, this non-reducedness occurs because there is a relation between the coefficients of $(z-$

$x_1)^a(z-x_2)^a = z^{2a}+b_1z^{2a-1}+\dots$ *which does not occur through specializing a relation between the coefficients of* $(z-x_1)^{m_1}(z-x_2)^{m_2}$*, namely*

$$\binom{1/a}{1}b_3 + \binom{1/a}{2}(2b_1b_2) + \binom{1/a}{3}b_1^3 = 0.$$

One can check that the square of this equation does arise from such a specialization.

In spite of this, it is easy to see by the fact that geometric-reducedness of fibers is an open condition on the base [39, EGA IV33, 12.2.4] and from Noetherian induction that there exists a partition of $(\mathbb{C}^k)^\circ$ into locally closed sets so that $\Delta^k(X)$ has reduced fibers over $(\mathbb{C}^k)^\circ$ when restricted to the preimage of any of the parts. In particular, replacing $\Delta^k(X)$ with the disjoint union of these preimages yields a variety whose scheme-theoretic fiber over $P \in (\mathbb{C}^k)^\circ$ is precisely $\Delta^k_P(X)$. However, it is unclear whether one can take this partition to be independent of X, which would be necessary to extend the functoriality part of Theorem 5.1.3 in this direction.

Motivated by examples like Example 5.2.3, we conjecture that one could simply stratify $(\mathbb{C}^k)^\circ$ according to the combinatorics of point-collisions in the incidence strata, i.e. which partial sums of the m_i agree.

Conjecture 5.2.4. *The stratification of* $(\mathbb{C}^k)^\circ$ *induced by the equivalence relation* $(m_1,\dots,m_k)\sim(n_1,\dots,n_k)$ *if*

$$\sum_{i\in A} m_i = \sum_{i\in B} m_i \Leftrightarrow \sum_{i\in A} n_i = \sum_{i\in B} n_i \text{ for all } A,B\subset\{1,\dots,k\}$$

has the property that the preimage in $\Delta^k(X)$ *of any equivalence class has reduced fibers over* $(\mathbb{C}^k)^\circ$*.*

5.3 Interpolating colored incidence strata in $\mathbb{A}^n$

In this section we construct $\Delta^{k,r}(\mathbb{A}^n)$ as claimed in Theorem 5.1.3'. Our construction will be a generalization of [43, 57, 12]. For $\overline{P} = (\overline{m_1}, \ldots, \overline{m_k}) \in (\mathbb{N}^r)^k$, we define $\overline{d}$ by

$$\overline{d} = (d_1, \ldots, d_r) := \overline{m_1} + \ldots + \overline{m_k},$$

where $\overline{P}$ is dropped from the notation.

Definition 5.3.1. *Define the* weighted elementary symmetric polynomial e_n *to be*

$$e_n(y_1, \ldots, y_k, x_1, \ldots, x_k) := \sum_{i_1+\ldots+i_k=n} \binom{y_1}{i_1} \cdots \binom{y_k}{i_k} x_1^{i_1} \ldots x_k^{i_k}$$

and define the formal series

$$f_i(\overline{P}, x_1, \ldots, x_k) := \prod_{j=1}^{k} (z - x_j)^{(\overline{m_j})_i} = \sum_{j=0}^{\infty} (-1)^j z^{d_i - j} e_j((\overline{m_1})_i, \ldots, (\overline{m_k})_i, x_1, \ldots, x_k).$$

The exponents of z are complex numbers, but we introduce these expressions purely as a bookkeeping device. Define the weighted power sum p_n to be

$$p_n(y_1, \ldots, y_k, x_1, \ldots, x_k) := y_1 x_1^n + \ldots + y_k x_k^n.$$

We define $e_n(x_1, \ldots, x_k)$ and $p_n(x_1, \ldots, x_k)$ to be the usual elementary symmetric functions and power sums respectively, which are the specializations of the weighted versions to when all the weights are 1.

Remark 5.3.2. *Note that for $\overline{P} = (\overline{m_1}, \ldots, \overline{m_k}) \in (\mathbb{N}^r)^k$, only finitely many coeffi-*

cients of $f_i(\overline{P}, x_1, \dots, x_k)$ are non-zero, and in the terminology of the introduction,

$$\Delta^k_{\overline{P}}(\mathbb{A}^1) := \{(f_1, \dots, f_r) \mid x_1, \dots, x_k \in \mathbb{C}\} \subset \mathrm{Sym}^{d_1}\mathbb{A}^1 \times \dots \times \mathrm{Sym}^{d_r}\mathbb{A}^1.$$

Lemma 5.3.3 is given by restricting the Chow form $\mathrm{Sym}^d\mathbb{P}^n \to \mathbb{P}(H^0(\mathbb{P}^{n\vee}, \mathcal{O}_{\mathbb{P}^{n\vee}(d)}))$ to an affine chart, where the Chow form maps the zero-cycle $P_1+\dots+P_d$ to the product $L_1 \cdots L_d$, where L_i is the linear form on $\mathbb{P}^{n\vee}$ parameterizing hyperplanes through P_i.

Lemma 5.3.3. *Let V be an n-dimensional vector space. Then there is an embedding*

$$\begin{aligned} \mathrm{Sym}^d\mathbb{A}(V) &\hookrightarrow V \times \mathrm{Sym}^2 V \times \dots \times \mathrm{Sym}^d V \\ (v_1, \dots, v_d)^{S_d} &\mapsto (-e_1(v_1, \dots, v_k), e_2(v_1, \dots, v_k), \dots, (-1)^d e_d(v_1, \dots, v_k)). \end{aligned}$$

Proof. See [33, Chapter 4, Proposition 2.1 and Theorem 2.2]. □

Remark 5.3.4. *Confusingly, $\mathrm{Sym}^d\mathbb{A}(V)$ in Lemma 5.3.3 is meant as the symmetric quotient $\mathbb{A}(V)^d/S_d$ as a variety, while $\mathrm{Sym}^d V$ is meant to be the vector space of dimension $\binom{n+d-1}{n-1}$.*

As a shorthand we may write this map as

$$(v_1, \dots, v_d)^{S_d} \mapsto (z - v_1)(z - v_2)\dots(z - v_d)$$

where multiplication of the v_i's is taken in the appropriate $\mathrm{Sym}^i V$. Then

$$\Delta^{k,r}_{\overline{P}}(\mathbb{A}(V)) \subset \mathrm{Sym}^{d_1}\mathbb{A}(V) \times \dots \times \mathrm{Sym}^{d_r}\mathbb{A}(V)$$

for $\overline{m_i} \in (\mathbb{Z}^{\geq 0})^r \setminus \{\overline{0}\}$ corresponds to the locus

$$\Delta^{k,r}_{\overline{P}}(\mathbb{A}(V)) = \{(f_1(\overline{P}, v_1, \ldots, v_k), \ldots, f_r(\overline{P}, v_1, \ldots, v_k)) \mid v_1, \ldots, v_k \in V\}$$
$$\subset (V \times \cdots \times \mathrm{Sym}^{d_1} V) \times \cdots \times (V \times \cdots \times \mathrm{Sym}^{d_r} V)$$

Our plan is to define $\Delta^{k,r}(\mathbb{A}(V))$ to be the image of

$$((\mathbb{C}^r)^k)^\circ \times V^k \to ((\mathbb{C}^r)^k)^\circ \times (\prod_{i=1}^{\infty} \mathrm{Sym}^i V)^r$$
$$(\overline{m_1}, \ldots, \overline{m_k}, v_1, \ldots, v_k) \mapsto (\overline{m_1}, \ldots, \overline{m_k}, f_1(\overline{P}, v_1, \ldots, v_k), \ldots, f_r(\overline{P}, v_1, \ldots, v_k)).$$

with an appropriate scheme-structure placed on it. Here again z is used for book-keeping, and the map actually extracts the coefficients of the $f_i(\overline{P}, v_1, \ldots, v_k)$.

Our choice to define $\Delta^{k,r}(\mathbb{A}(V))$ in this way is a priori surprising since we are working with a locus in an infinite-dimensional vector space and it is unclear what scheme structure we would like to put on it. However, we will show that there is an N such that no information is lost after ignoring all $\mathrm{Sym}^{N'} V$ for $N' \geq N$, and after doing so the resulting map will be finite. From here $\Delta^{k,r}(\mathbb{A}(V))$ will be easily shown to have the desired properties in light of Lemma 5.3.3. The key technical lemmas adapted from [43, 57, 12] are as follows.

Lemma 5.3.5. *Suppose that R is a reduced finite-type algebra over $\mathbb{C}$, and $\mathcal{A} \subset R[x_1, \ldots, x_\ell]$ is a sequence of positive degree homogenous polynomials such that if all polynomials in $\mathcal{A}$ vanish then all x_i are equal to zero. Then $R[x_1, \ldots, x_\ell]$ is a finite $R[\mathcal{A}]$-module, and in particular $R[\mathcal{A}]$ is a finitely-generated R-algebra.*

Proof. By applying the graded Nakayama's Lemma to the graded ring $R[\mathcal{A}]$ and the graded $R[\mathcal{A}]$-module $R[x_1, \ldots, x_\ell]$, it suffices to show that $R[x_1, \ldots, x_\ell]/(\mathcal{A})$ is a finite R-module. But the hypotheses and the Nullstellensatz imply that the radical of $(\mathcal{A})$

is precisely $(x_1, \ldots, x_\ell)$. ☐

Lemma 5.3.6. *For every N we have*

$$\mathbb{C}[\{p_i(y_1, \ldots, y_k, x_1, \ldots, x_k)\}_{1 \leq i \leq N}] = \mathbb{C}[\{e_i(y_1, \ldots, y_k, x_1, \ldots, x_k)\}_{1 \leq i \leq N}].$$

Proof. The Newton identities hold in the weighted case when the weights are positive integers. Since $\mathbb{N}^k \subset \mathbb{C}^k$ is Zariski dense, they hold for all complex weights. ☐

Lemma 5.3.7. *Let* $\mathrm{Spec}R$ *be either a point of* $((\mathbb{C}^r)^k)^\circ$ *or an affine open subset of* $((\mathbb{C}^r)^k)^\circ$*. Let $V \cong \mathbb{C}^n$ be an n-dimensional vector space, and S be a polynomial ring over R so that* $\mathrm{Spec}S = \mathrm{Spec}R \times V^k$*. Let $\mathcal{A} \subset S$ be the subset of polynomials given by pulling back the coordinate functions under*

$$\mathrm{Spec}R \times V^k \to \mathrm{Sym}^i V$$
$$(\overline{m_1}, \ldots, \overline{m_k}, v_1, \ldots, v_k) \mapsto e_i((\overline{m_1})_j, \ldots, (\overline{m_k})_j, v_1, \ldots, v_k)$$

with $i \geq 1$ and $1 \leq j \leq n$. Then S is a finite $R[\mathcal{A}]$-module and $R[\mathcal{A}]$ is a finitely generated R-algebra.

Proof. We will show that R and $\mathcal{A}$ satisfy the hypothesis of Lemma 5.3.5. Suppose that for some choice of $(\overline{m_1}, \ldots, \overline{m_k}) \in \mathrm{Spec}R$ and $v_1, \ldots, v_k \in V$ that the polynomial $e_i((\overline{m_1})_j, \ldots, (\overline{m_k})_j, v_1, \ldots, v_k)$ vanishes for all $i \geq 1, 1 \leq j \leq r$. Lemma 5.3.6 then formally implies that for all such i, j we have

$$(\overline{m_1})_j v_1^i + \ldots + (\overline{m_k})_j v_k^i = \overline{0}.$$

In particular, this implies that

$$\frac{1}{z - v_1}\overline{m_1} + \ldots + \frac{1}{z - v_k}\overline{m_k} - \frac{1}{z}\sum \overline{m_i} = 0$$

by expanding in powers of $\frac{1}{z}$. But by hypothesis on R we have $\sum_{i \in A} \overline{m_i} \neq 0$ for all subsets A, so if some $v_i \neq 0$ it is impossible for there not to be a pole at $z = v_i$. $\square$

Now, we construct $\Delta^{k,r}(\mathbb{A}(V))$ as follows. Cover $((\mathbb{C}^r)^k)^\circ$ with basic affine opens $\mathrm{Spec}A_1, \ldots \mathrm{Spec}A_w$ in $(\mathbb{C}^r)^k$. By Lemma 5.3.7, for N sufficiently large the images of

$$\mathrm{Spec}A_i \times V^k \to \mathrm{Spec}A_i \times (V \times \ldots \times \mathrm{Sym}^{N'} V)^r$$

are all closed subvarieties for $N' \geq N$ and are isomorphic to each other by projection. Taking $N' = N$, these varieties also clearly glue together compatibly over the overlaps. This gives a natural scheme structure to the image of

$$((\mathbb{C}^r)^k)^\circ \times V^k \to ((\mathbb{C}^r)^k)^\circ \times (V \times \ldots \times \mathrm{Sym}^N V)^r.$$

The reduced fiber over any point in $\overline{P} \in ((\mathbb{C}^r)^k)^\circ$ is then precisely $\Delta^{k,r}_{\overline{P}}(\mathbb{A}(V))$.

Remark 5.3.8. *By construction*

$$((\mathbb{C}^r)^k)^\circ \times V^k \to \Delta^{k,r}_{\overline{P}}(\mathbb{A}(V))$$

is finite, and hence in particular proper. Also, a similar application of Lemma 5.3.6 shows that for fixed $\overline{m_1}, \ldots, \overline{m_k}$, $(v_1, \ldots, v_k)$ and $(v'_1, \ldots, v'_k)$ correspond to the same point if and only if

$$\frac{1}{z - v_1}\overline{m_1} + \ldots + \frac{1}{z - v_k}\overline{m_k} = \frac{1}{z - v'_1}\overline{m_1} + \ldots + \frac{1}{z - v'_k}\overline{m_k}.$$

5.4 Interpolating colored incidence strata of affine varieties

In this section we prove Theorem 5.1.3 and discuss the Euclidean topology of our families of interpolated incidence strata. To each affine variety X, we embed $X \subset \mathbb{A}(V_X)$ for some vector space V_X, and then take $\Delta^{k,r}(X)$ to be the image of

$$((\mathbb{C}^r)^k)^\circ \times X^k \hookrightarrow ((\mathbb{C}^r)^k)^\circ \times \mathbb{A}(V_X)^k \to \Delta^{k,r}(\mathbb{A}(V_X)). \tag{5.1}$$

Proof of Theorem 5.1.3. Note that the map defining $\Delta^{k,r}(X)$ is the composite of two proper maps, and hence $\Delta^{k,r}(X)$ is a closed subvariety of $\Delta^{k,r}(\mathbb{A}(V_X))$.

We first verify that the fibers are correct at $\overline{P} \in (\mathbb{Z}^r_{\geq 0} \setminus \{\overline{0}\})^k$. To do this, note that the embedding $X \hookrightarrow \mathbb{A}(V_X)$ induces an embedding $\prod \mathrm{Sym}^{d_i} X \hookrightarrow \prod \mathrm{Sym}^{d_i} \mathbb{A}(V_X)$ which identifies $\prod \mathrm{Sym}^{d_i} X$ with the subset of $\prod \mathrm{Sym}^{d_i} \mathbb{A}(V_X)$ where the points are constrained to lie on X. Therefore, the incidence strata $\Delta^{k,r}_{\overline{P}}(X) \subset \Delta^{k,r}_{\overline{P}}(\mathbb{A}(V_X))$ is simply the subset of $\Delta^{k,r}_{\overline{P}}(\mathbb{A}(V_X))$ where the points are constrained to lie on X. Hence by construction the fiber over $\overline{P} \in (\mathbb{Z}^r_{\geq 0} \setminus \{\overline{0}\})^k$ is indeed $\Delta^{k,r}_{\overline{P}}(X)$ as desired.

Now we verify functoriality. Given a map $\phi : X \to Y$, we define the induced map $\Delta^{k,r}(X) \to \Delta^{k,r}(Y)$ to be at the level of sets

$$(\prod_{i=1}^k (z-\overline{x_i})^{(\overline{m_i})_1}, \ldots, (\prod_{i=1}^k (z-\overline{x_i})^{(\overline{m_i})_r}) \mapsto (\prod_{i=1}^k (z-\phi(\overline{x_i}))^{(\overline{m_i})_1}, \ldots, (\prod_{i=1}^k (z-\phi(\overline{x_i}))^{(\overline{m_i})_r}).$$

First we check that this is a morphism. Indeed, suppose ϕ was induced from some identically named map $\phi : \mathbb{A}(V_X) \to \mathbb{A}(V_Y)$. Then it suffices to show that we have an induced morphism $\Delta^{k,r}(\mathbb{A}(V_X)) \to \Delta^{k,r}(\mathbb{A}(V_Y))$ since the image of $\Delta^{k,r}(X)$ clearly lies inside $\Delta^{k,r}(Y)$. Writing $\overline{x_i} = \sum_j r_{i,j} v_j$ for basis vectors $v_1, \ldots, v_n$ of V_X and letting $w_1, \ldots, w_m$ be a basis of V_Y, it suffices by Lemma 5.3.6 to show for each ℓ that we can write all of the coefficients of the coordinates in the $\prod w_i^{\mu_i}$ basis of

$$(\overline{m_1})_s\phi(\overline{x_1})^\ell + \ldots + (\overline{m_k})_s\phi(\overline{x_k})^\ell$$

as linear combinations of the coefficients of the coordinates of expressions in

$$\{(\overline{m_1})_s\overline{x_1}^i + \ldots + (\overline{m_k})_s\overline{x_k}^i\}_{1\leq i<\infty}.$$

The result follows from the following two observations.

- Every coefficient of $(\overline{m_1})_s\phi(\overline{x_1})^\ell + \ldots + (\overline{m_k})_s\phi(\overline{x_k})^\ell$ is a linear combination of elements of the form
$$\sum_t (\overline{m_t})_s r_{t,1}^{\lambda_1}\ldots r_{t,n}^{\lambda_n}.$$
- The expression $\sum_t (\overline{m_t})_s r_{t,1}^{\lambda_1}\ldots r_{t,n}^{\lambda_n}$ is a multiple of the coefficient of $v_1^{\lambda_1}\ldots v_n^{\lambda_n}$ in $(\overline{m_1})_s x_1^{\sum\lambda_i} + \ldots + (\overline{m_k})_s x_k^{\sum\lambda_i}$.

As the morphisms at the level of sets are manifestly functorial, it follows that the assignment is functorial as desired. □

5.4.1 A topological model for weighted incidence strata.

In this subsection, we will use the Euclidean topology instead of the Zariski topology. For a variety X we may topologically define a family of weighted configuration spaces on X as $(((\mathbb{C}^r)^k)^\circ \times X^k)/\sim$, formed as the quotient of the space of tuples $(\overline{m_1},\ldots,\overline{m_k},x_1,\ldots,x_k) \in ((\mathbb{C}^r)^k)^\circ \times X^k$ by the relation that

$$(\overline{m_1},\ldots,\overline{m_k},x_1,\ldots,x_k) \sim (\overline{m_1},\ldots,\overline{m_k},x_1',\ldots,x_k')$$

if for each $x \in X$ we have

$$\sum_{i \text{ such that } x_i = x} \overline{m_i} = \sum_{i \text{ such that } x'_i = x} \overline{m_i}.$$

Proposition 5.4.1. *The underlying topology of* $\Delta^{k,r}(X)$ *agrees with the family of weighted configuration spaces* $(((\mathbb{C}^r)^k)^\circ \times X^k)/\sim$ *above.*

Proof. We first consider the case $X = \mathbb{A}(V)$. Note that the condition $\sim$ is identical to the condition in Remark 5.3.8. Thus the topologically proper map (Remark 5.3.8)

$$((\mathbb{C}^r)^k)^\circ \times V^k \to \Delta^{k,r}(\mathbb{A}(V))$$

factors through the bijective continuous

$$(((\mathbb{C}^r)^k)^\circ \times V^k)/\sim \to \Delta^{k,r}(\mathbb{A}(V)).$$

This implies the bijective continuous map is also proper, and hence a homeomorphism as desired.

For arbitrary X, because the composite map (5.1) defining $\Delta^{k,r}(X)$ is proper, $\Delta^{k,r}(X)$ is topologically identified with the quotient $(((\mathbb{C}^r)^k)^\circ \times X^k)/\sim$. □

5.5 The constant N_k

Definition 5.5.1. *Define*

$$R_i = \mathbb{C}[m_1, \ldots, m_k][\{\frac{1}{\sum_{j \in S} m_j}\}][\{e_j(m_1, \ldots, m_k, x_1, \ldots, x_k)\}_{1 \le j \le i}]$$

and for fixed $P = (m_1, \dots, m_k) \in (\mathbb{C}^k)^\circ$*, define*

$$R_i(P) = \mathbb{C}[\{e_j(m_1, \dots, m_k, x_1, \dots, x_k)\}_{1 \le j \le i}].$$

Define $N_k, N_k(P)$ *to be the first points at which*

$$R_{N_k} = R_{N_k+1} = \dots$$
$$R_{N_k(P)}(P) = R_{N_k(P)+1}(P) = \dots$$

Note N_k and $N_k(P)$ exist by Lemma 5.3.7. Our first goal will be to show that $N_k(P)$ agrees with the definition from the introduction as the affine embedding dimension of $\Delta_P^k(\mathbb{A}^1)$. We start with a useful proposition.

Proposition 5.5.2. *If* $R_i = R_{i+1}$*, then* $R_i = R_{i+1} = R_{i+2} = \dots$*. Also, if* $R_i(P) = R_{i+1}(P)$ *then* $R_i(P) = R_{i+1}(P) = R_{i+2}(P) = \dots$*.*

Proof. Note that Lemma 5.3.6 allows us to consider the p_i's stabilizing rather than the e_i's. Let $p_j = p_j(m_1, \dots, m_k, x_1, \dots, x_k)$, and suppose we have an equation

$$p_{i+1} = g(p_1, \dots, p_i)$$

where g is a polynomial with coefficients in $\mathbb{C}[m_1, \dots, m_k][\{\frac{1}{\sum_{j \in S} m_i}\}]$ if we're working with R_i, or $\mathbb{C}$ if we're working with $R_i(P)$. In either case, substituting $x_u + \epsilon x_u^t$ into x_u for every $1 \le u \le k$ yields

$$p_{i+1} + \epsilon(i+1)p_{t+i} \equiv g(p_1, \dots, p_i) + \epsilon \sum_{j=1}^{i} (\partial_j g)(p_1, \dots, p_i) j p_{t+j-1} \pmod{\epsilon^2}.$$

Equating the ϵ terms then yields

$$p_{t+i} = \frac{1}{i+1}\sum_{j=1}^{i}(\partial_j g)(p_1,\ldots,p_i)jp_{t+j-1}.$$

$\square$

Thus, to find N_k or $N_k(P)$ it suffices to find the first point at which the corresponding rings start stabilizing.

Lemma 5.5.3. *For $f_1, f_2, \ldots, f_u$ homogenous polynomials in $\mathbb{C}[x_1,\ldots,x_\ell]$, such that no f_i is redundant as a generator of $\mathbb{C}[f_1,\ldots,f_u]$, then u is the minimum number of generators of $\mathbb{C}[f_1,\ldots,f_u]$ as a $\mathbb{C}$-algebra.*

Proof. Omitted. $\square$

Proposition 5.5.4. *Given $P = (m_1,\ldots,m_k) \in (\mathbb{C}^k)^\circ$, $N_k(P)$ from Definition 5.5.1 equals*

$$\min\{n \mid \Delta_P^k(\mathbb{A}^1) \textit{ embeds into } \mathbb{A}^n\}.$$

Proof. It is clear from Proposition 5.5.2 that $\{e_i(m_1,\ldots,m_k,x_1,\ldots,x_k)\}_{1\le i\le N_k(P)}$ satisfies the hypothesis of Lemma 5.5.3. $\square$

We now show that k-part r-colored incidence strata of varieties $X \subset \mathbb{A}^n$ are embeddedable in $\mathbb{A}^{r(\binom{n+N_k}{n}-1)}$.

Proposition 5.5.5. *Given a subvariety $X \subset \mathbb{A}^n$ and $\overline{P} \in ((\mathbb{C}^r)^k)^\circ$, there is an embedding*

$$\Delta_{\overline{P}}^{k,r}(X) \subset \mathbb{A}^{r(\binom{n+N_k}{n}-1)}.$$

Proof. Write $\mathbb{A}^n = \mathbb{A}(V)$, and let $R = \Gamma(\Delta_{\overline{P}}^{k,r}(\mathbb{A}(V)))$. As by construction $\Delta_{\overline{P}}^{k,r}(X) \subset$

$\Delta^{k,r}_{\overline{P}}(\mathbb{A}(V))$, it suffices to show that

$$\Delta^{k,r}_{\overline{P}}(\mathbb{A}(V)) \subset \mathbb{A}^{r(\binom{n+N_k}{n}-1)}.$$

By construction, we have R is generated as a $\mathbb{C}$-algebra by the coordinate projections of all maps of the form

$$(v_1, \ldots, v_k) \mapsto e_i((\overline{m_1})_j, \ldots, (\overline{m_k})_j, v_1, \ldots, v_k) \in \mathrm{Sym}^i V.$$

Lemma 5.3.6 formally implies that R is also generated by the coordinate projections of all maps of the form

$$(v_1, \ldots, v_k) \mapsto (\overline{m_1})_j v_1^i + \ldots + (\overline{m_k})_j v_k^i \in \mathrm{Sym}^i V.$$

Written in this form it is clear that applying a common invertible linear transformation to all $\overline{m_i}$'s maps the algebra to precisely the same algebra, so we may apply such a map to assume without loss of generality that for every j and $S \subset \{1, \ldots, k\}$ that

$$\sum_{i \in S} (\overline{m_i})_j \neq 0.$$

But for each j and $i > N_k$, we obtain by Proposition 5.5.2 a polynomial relation

$$\begin{aligned} &e_i((\overline{m_1})_j, \ldots, (\overline{m_k})_j, v_1, \ldots, v_k) \\ &\quad = g_i(e_1((\overline{m_1})_j, \ldots, (\overline{m_k})_j, v_1, \ldots, v_k), \ldots, e_{i-1}((\overline{m_1})_j, \ldots, (\overline{m_k})_j, v_1, \ldots, v_k)). \end{aligned}$$

Thus we only need to use polynomials up to e_{N_k} to generate R and the result now follows by counting the number of coordinate projections of such functions. $\square$

Remark 5.5.6. *For the case $r = 1$ we could have used $N_k(P)$ instead of N_k.*

Remark 5.5.7. *Following the proof of Proposition 5.5.5, it is clear that* $\Delta^{k,r}_{\overline{P}}(X)$ *is unchanged if we act by* GL_r *on* $\overline{P}$.

Proposition 5.5.8. *We have* $N_2((1,2)) = N_2 = 3$.

Proof. $N_2 \leq 3$ by Lemma 5.3.6, Proposition 5.5.2 and the relation

$$(m_1 + m_2)p_4 - 4p_1p_3 + 3p_2^2 = ((m_1 + m_2)p_2 - p_1^2)^2.$$

$N_2 \geq N_2((1,2)) = 3$ since taking $m_1 = 1$ and $m_2 = 2$, it is easy to see that there is no way of expressing $x_1^3 + 2x_2^3$ as a polynomial in $x_1 + 2x_2$ and $x_1^2 + 2x_2^2$, and we conclude by Lemma 5.3.6. □

Proposition 5.5.9. $\Delta^2(\mathbb{A}^n)$ *is isomorphic over* $(\mathbb{C}^2)^\circ$ *to the image of*

$$(\mathbb{C}^2)^\circ \times \mathbb{A}^n \times \mathbb{A}^n \to (\mathbb{C}^2)^\circ \times \mathbb{C}^n \times \mathrm{Sym}^2\mathbb{C}^n \times \mathrm{Sym}^3\mathbb{C}^n$$

taking

$$(m_1, m_2, v_1, v_2) \mapsto (m_1, m_2, v_1, v_2^2, (m_1 - m_2)v_2^3).$$

In particular, the family is isotrivial.

Proof. By Section 5.5 and Lemma 5.3.6, $\Delta^2(\mathbb{A}^n)$ is isomorphic to the image of the map

$$(m_1, m_2, v_1, v_2) \mapsto (m_1, m_2, p_1(m_1, m_2, v_1, v_2), p_2(m_1, m_2, v_1, v_2), p_3(m_1, m_2, v_1, v_2)).$$

We first do a polynomial change of coordinates on the codomain to obtain the new map

$$(m_1, m_2, v_1, v_2) \mapsto (m_1, m_2, m_1v_1 + m_2v_2, (v_1 - v_2)^2, (m_1 - m_2)(v_1 - v_2)^3).$$

Indeed, we observe that we have the following universal relations between the polynomials $p_i = p_i(m_1, m_2, x_1, x_2)$ with x_1, x_2 indeterminates:

$$(x_1 - x_2)^2 = \frac{1}{m_1 m_2}((m_1 + m_2)p_2 - p_1^2)$$
$$(m_1 - m_2)(x_1 - x_2)^3 = -\frac{1}{m_1 m_2}\left((m_1 + m_2)^2 p_3 - 3(m_1 + m_2)p_1 p_2 + 2p_1^3\right).$$

Now, we may change coordinates on the domain with $v_1' = mv_1 + nv_2$ and $v_2' = v_1 - v_2$ (which is an invertible linear transformation as $m + n \neq 0$). □

Remark 5.5.10. *In [1, 2], such 2-part incidence strata were studied in $\mathbb{P}^n$ rather than $\mathbb{A}^n$. One can show along the lines of Proposition 5.5.13 that 2-part incidence strata in $\mathbb{P}^n$ are in general not isomorphic to each other, despite being locally isomorphic by Proposition 5.5.9.*

Remark 5.5.11. *It would be tempting to try to conclude that there are only two distinct isomorphism classes of two-part incidence strata on* any *affine variety V by its embedding into $\mathbb{A}^n$, however as the final change of coordinates in the proof is not an automorphism of $V \times V \subset \mathbb{A}^n \times \mathbb{A}^n$, we are not allowed to apply it.*

Remark 5.5.12. *For $n = 1$ we see that all two-part incidence strata on $\mathbb{A}^1$ are equal to $\mathrm{Sym}^2\mathbb{A}^1 \cong \mathbb{A}^2$ if the parts are the same, and $\mathbb{A}^1 \times C$ if the parts are unequal, where C is the cuspidal cubic $y^2 = x^3$ in $\mathbb{A}^2$.*

Proposition 5.5.13. *The family $\Delta^3(\mathbb{A}^n)$ is not iso-trivial over $(\mathbb{C}^3)^\circ$. In fact, if $m_1, m_2, m_3 \in \mathbb{N}$ are distinct and $m_1', m_2', m_3' \in \mathbb{N}$ are distinct, such that (m_1, m_2, m_3) is not a multiple of (m_1', m_2', m_3') after possibly rearranging the coordinates, then $\Delta^3_{(m_1,m_2,m_3)}(\mathbb{A}^n) \ncong \Delta^3_{(m_1',m_2',m_3')}(\mathbb{A}^n)$.*

Proof. Suppose that there was an isomorphism. Then there is an isomorphism of normalizations of $\Delta^3_{(m_1,m_2,m_3)}(\mathbb{A}^n)$ and $\Delta^3_{(m_1',m_2',m_3')}(\mathbb{A}^n)$ fitting into the diagram

$$\begin{array}{ccc} \mathbb{A}^n \times \mathbb{A}^n \times \mathbb{A}^n & \longrightarrow & \Delta^n_{(m_1,m_2,m_3)}(\mathbb{A}^n) \\ \downarrow & & \downarrow \\ \mathbb{A}^n \times \mathbb{A}^n \times \mathbb{A}^n & \longrightarrow & \Delta^n_{(m_1',m_2',m_3')}(\mathbb{A}^n). \end{array}$$

For either incidence strata, the singular locus occurs when a point collision occurs, and the singular locus of the singular locus is when all 3 points collide. This corresponds to the image of the diagonal in $\mathbb{A}^n \times \mathbb{A}^n \times \mathbb{A}^n$, so the left vertical isomorphism $(v_1, v_2, v_2) \mapsto (f(v_1, v_2, v_3), g(v_1, v_2, v_3), h(v_1, v_2, v_3))$ must have the constant terms of f, g, h all equal. Applying the translation automorphism to the $\mathbb{A}^n$ that $\Delta^3_{m_1',m_2',m_3'}(\mathbb{A}^n)$ is the incidence strata for, we may assume that $f(0) = g(0) = h(0) = 0$. Letting f', g', h' represent the inverse isomorphism, the constant terms of f', g', h' are then also zero.

Now, the isomorphism implies that the maps

$$\begin{aligned} &(z - v_1)^{m_1}(z - v_2)^{m_2}(z - v_3)^{m_3} \mapsto \\ &(z - f(v_1, v_2, v_3))^{m_1'}(z - g(v_1, v_2, v_3))^{m_2'}(z - h(v_1, v_2, v_3))^{m_3'}, \text{ and} \\ &(z - v_1)^{m_1'}(z - v_2)^{m_2'}(z - v_3)^{m_3'} \mapsto \\ &(z - f'(v_1, v_2, v_3))^{m_1}(z - g'(v_1, v_2, v_3))^{m_2}(z - h'(v_1, v_2, v_3))^{m_3} \end{aligned}$$

are both morphisms. As coordinate projections of the coefficients of $(z - v_1)^{m_1}(z - v_2)^{m_2}(z - v_3)^{m_3}$ are homogenous, replacing each of f, g, h with their lowest order homogenous terms (i.e. their linear terms since (f, g, h) is an automorphism of $\mathbb{A}^n \times \mathbb{A}^n \times \mathbb{A}^n$) results in a morphism $\Delta^3_{(m_1,m_2,m_3)}(\mathbb{A}^n) \to \Delta^3_{(m_1',m_2',m_3')}(\mathbb{A}^n)$, and similarly replacing each of f', g', h' with their lowest order terms yields an inverse morphism $\Delta^3_{(m_1',m_2',m_3')}(\mathbb{A}^n) \to \Delta^3_{(m_1,m_2,m_3)}(\mathbb{A}^n)$. Thus, we may assume that f, g, h are linear.

Now, as the singular locus occurs during a point collision, we must have that $v_i = v_j$ implies two of $f(v_1, v_2, v_3), g(v_1, v_2, v_3), h(v_1, v_2, v_3)$ are equal. Since f, g, h are distinct linear forms, this implies that possibly after reordering f, g, h (which we are allowed to do by reordering m_1', m_2', m_3'), there exist linear R, L such that

$$f(v_1, v_2, v_3) = R(v_1, v_2, v_3) + L(v_1)$$
$$g(v_1, v_2, v_3) = R(v_1, v_2, v_3) + L(v_2)$$
$$h(v_1, v_2, v_3) = R(v_1, v_2, v_3) + L(v_3)$$

As (f, g, h) is a full rank linear map, $(f, g - f, h - f)$ is also a full rank linear map, so L must be invertible. Let w_1 be an eigenvector for L, so $L(w_1) = \lambda w_1$ for some $\lambda \neq 0$. Extend w_1 to a basis $w_1, \ldots, w_n$ of $\mathbb{C}^n$ and write $v_i = \sum_j r_{i,j} w_j$. In the basis $\{w_i w_j\}_{1 \leq i \leq j \leq n}$ of $\mathrm{Sym}^2 \mathbb{C}^n$, the coordinates of $m' f(v_1, v_2, v_3) + n' g(v_1, v_2, v_3) + p' h(v_1, v_2, v_3)$ must be expressible as polynomials in the coordinates of $mv_1 + nv_2 + pv_3$, and the coordinates of $m' f(v_1, v_2, v_3)^2 + n' g(v_2, v_2, v_3)^2 + p' h(v_1, v_2, v_3)^2$ must be expressible as polynomials in the coordinates of $mv_1 + nv_2 + pv_3$ and $mv_1^2 + nv_2^2 + pv_3^2$. In particular, the coordinates of

$$\begin{aligned} &(m' + n' + p') p_2(f(v_1, v_2, v_3), g(v_1, v_2, v_3), h(v_1, v_2, v_3)) \\ &- p_1(f(v_1, v_2, v_3), g(v_1, v_2, v_3), h(v_1, v_2, v_3))^2 = \\ m_1' m_2' (L(v_1) - L(v_2))^2 &+ m_1' m_3' (L(v_1) - L(v_3))^2 + m_2' m_3' (L(v_2) - L(v_3))^2 \end{aligned}$$

must be expressible as polynomials in the coordinates of $mv_1 + nv_2 + pv_3$ and $mv_1^2 + nv_2^2 + pv_3^2$. Setting all $r_{i,j}$ to be zero except for $x = r_{1,1}$, $y = r_{2,1}$, $z = r_{3,1}$ (or considering the coefficient of w_i^2), we see that there must exist λ_1, λ_2 such that

$$m'n'(x - y)^2 + m'p'(x - z)^2 + n'p'(y - z)^2 = \lambda_1 (mx + ny + pz)^2 + \lambda_2 (mx^2 + ny^2 + pz^2).$$

Setting $x = y = z$ we see that $\lambda_2 = -(m + n + p)\lambda_1$, and therefore

$$m'n'(x-y)^2 + m'p'(x-z)^2 + n'p'(y-z)^2 = -\lambda_1(mn(x-y)^2 + np(x-z)^2 + np(y-z)^2).$$

But this is only possible if $m'n', m'p', n'p'$ are proportional to mn, mp, np, which implies that (m', n', p') is a multiple of (m, n, p), a contradiction. □

We now show Conjecture 5.1.11 for $k \leq 3$ and show $N_4((1, 2, 4, 8)) \geq 15$.

Proof of Theorem 5.1.12. These results were found by Magma computations [10]. To show $N_3((1, 2, 4)) \geq 7$ we show that there are no linear relations between degree 7 products of polynomials of the form $x^i + 2y^i + 4z^i$, and to show $N_4 \geq 15$ we show that there are no linear relations between degree 15 products of polynomials of the form $x^i + 2y^i + 4z^i + 8w^i$. To show $N_3 = 7$, we used Magma to find all linear relations over $\mathbb{C}(m_1, m_2, m_3)$ between degree 15 products of polynomials of the form $p_i(m_1, m_2, m_3, x_1, x_2, x_3)$ (the degree of p_i being i). There is in fact a unique linear relation up to scaling, and after clearing the denominators the expression was of the form

$$(m_1 + m_2 + m_3)(m_1 + m_2)(m_1 + m_3)(m_2 + m_3)m_1m_2m_3p_8 = g(m_1, m_2, m_3, p_1, \ldots, p_7)$$

for some polynomial g. As $(m_1 + m_2 + m_3)(m_1 + m_2)(m_1 + m_3)(m_2 + m_3)m_1m_2m_3$ is invertible we have a relation showing $R_7 = R_8$, and hence Proposition 5.5.2 implies $N_3 \geq 7$. Thus $N_3 = 7$. □

5.6 Explicit recursions for weighted power sums

In this section, we sketch a construction of an explicit polynomial expressing p_N in terms of the previous p_i (which by Lemma 5.3.6 is equivalent to relating e_N to the previous e_i). By Proposition 5.5.2 we also get explicit recursions for $N' \geq N$.

First, we will construct expressions realizing some power of x_1 to be inside the ideal $(p_1, \ldots,) \subset \mathbb{C}[m_1, \ldots, m_k, x_1, \ldots, x_k][\{\frac{1}{\sum_{i \in S} m_i}\}_S]$ by induction on k. Supposing we have an expression for k variables

$$x_1^{w_k} - g_1(\{m_i, x_i\}_{i=1}^k)p_1(\{m_i, x_i\}_{i=1}^k) + \ldots + g_{w_k}(\{m_i, x_i\}_{i=1}^k)p_{w_k}(\{m_i, x_i\}_{i=1}^k) = 0,$$

we show how to construct a relation in $k+1$-variables

$$x_1^{w_{k+1}} - g'_1(\{m_i, x_i\}_{i=1}^{k+1})p_1(\{m_i, x_i\}_{i=1}^{k+1}) + \ldots + g'_{w_{k+1}}(\{m_i, x_i\}_{i=1}^{k+1})p_{w_{k+1}}(\{m_i, x_i\}_{i=1}^{k+1}) = 0.$$

From the relation (denoting p_i for $p_i(\{m_i, x_i\}_{i=1}^{k+1})$)

$$\prod_{1 \leq i < j \leq k+1} (x_i - x_j)^2 = \frac{1}{m_1 \ldots m_{k+1}} \begin{bmatrix} p_0 & p_1 & \ldots & p_k \\ p_1 & p_2 & \ldots & p_{k+1} \\ \ldots & \ldots & \ldots & \ldots \\ p_k & p_{k+1} & \ldots & p_{2k} \end{bmatrix},$$

we see that such a relation in $k+1$ variables is equivalent to producing an expression

$$x_1^{w_{k+1}} - g'_1(\{m_i, x_i\}_{i=1}^{k+1})p_1(\{m_i, x_i\}_{i=1}^{k+1}) + \ldots + g'_{w_{k+1}}(\{m_i, x_i\}_{i=1}^{k+1})p_{w_{k+1}}(\{m_i, x_i\}_{i=1}^{k+1})$$

divisible by $\prod_{1 \leq i < j \leq k+1}(x_i - x_j)^2$. To do this it suffices to find for every $1 \leq i < j \leq k+1$ such an expression divisible by $x_i - x_j$ as then we can multiply together the squares of all such expressions. So suppose we fix $1 \leq i < j \leq k+1$. We obtain such an expression for $k+1$ variables from the k-variable relation by replacing

$p_\ell(\{m_i, x_i\}_{i=1}^{k})$ with $p_\ell(\{m_i, x_i\}_{i=1}^{k+1})$, replacing in g_ℓ the variables $m_1, \ldots, m_k$ with

$$m_1, \ldots, m_{i-1}, m_i + m_j, m_{i+1}, \ldots, m_{j-1}, m_{j+1}, \ldots, m_{k+1}$$

and replacing the variables $x_1, \ldots, x_k$ in g_ℓ with

$$x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k+1}.$$

Indeed, if we let $x_i = x_j$ in such an expression, we obtain the k-variable expression with the m_ℓ and x_ℓ replaced with the above sets of variables. Thus the induction is complete. Permuting the variables, we obtain similar expressions for $x_2^{w_k}, \ldots, x_{k+1}^{w_k}$.

By the above relations, we may now set up w_k^k linear equations expressing x_1 times a monomial $x_1^{i_1} \ldots x_k^{i_k}$ with $0 \leq i_1, \ldots, i_k < w_k$ as a linear combination of other such monomials with each coefficient a multiple of some p_i. Expressing this system of equations as a matrix, the Cayley-Hamilton theorem [45, Theorem 10.2] then says that x_1 satisfies the characteristic polynomial of this matrix, yielding a monic polynomial satisfied by x_1 with coefficients in the algebra

$$R = \mathbb{C}[m_1, \ldots, m_k][\{\frac{1}{\sum_{i \in S} m_i}\}_S][p_1, p_2, \ldots].$$

We similarly obtain such polynomials for $x_2, \ldots, x_k$, and multiplying these k polynomials together yields a single monic polynomial h with coefficients in R satisfied by $x_1, \ldots, x_k$. The expression $m_1 x_1 h(x_1) + m_2 x_2 h(x_2) + \ldots + m_k x_k h(x_k)$ is then manifestly an expression for $p_{k w_k^k + 1}$ in terms of the previous p_i.

This construction yields a polynomial of degree roughly $k!^{2k}$. If we had used the effective Nullstellensatz [40, Theorem 1.3] at the first stage to bound w_k it would have given a bound of approximately $k!^k$ for $N_k(P)$ and $2^{k^2} k!^k$ for N_k, though the construction would be non-explicit.

5.7 The singularities of colored incidence strata in smooth curves

In this section, we prove Theorem 5.1.4.

Proof of Theorem 5.1.4. Let $\overline{P} = (\overline{m_1}, \ldots \overline{m_k})$, and denote

$$\overline{m_1} + \ldots + \overline{m_k} = (d_1, \ldots, d_r).$$

First we prove the result for $C = \mathbb{A}^1$.

Let $\overline{n_1}, \ldots, \overline{n_t}$ denote the distinct $\overline{m_i}$, and suppose $\overline{n_i}$ appears ℓ_i times in $\overline{P}$. Then the normalization of $\Delta^{k,r}_{\overline{P}}(\mathbb{A}^1)$ is the morphism

$$\begin{aligned} \Phi_{\overline{P}} : \mathrm{Sym}^{\ell_1}\mathbb{A}^1 \times \ldots \times \mathrm{Sym}^{\ell_t}\mathbb{A}^1 \to (\mathbb{A}^\infty)^r \\ (F_1, \ldots, F_t) \mapsto (\prod_{s=1}^{t} F_s^{(\overline{n_s})_1}, \ldots, \prod_{s=1}^{t} F_s^{(\overline{n_s})_r}). \end{aligned}$$

Indeed, by Lemma 5.3.7, the inclusion $\Gamma(\Delta^{k,r}_{\overline{P}}(\mathbb{A}^1)) \subset \mathbb{C}[x_1, \ldots, x_k]$ is finite, and the map $\Phi_{\overline{P}}$ corresponds to the inclusion

$$\Gamma(\Delta^{k,r}_{\overline{P}}(\mathbb{A}^1)) \subset \mathbb{C}[x_1, \ldots, x_k]^{S_{\ell_1} \times \ldots \times S_{\ell_t}},$$

which is thus also finite. It is furthermore birational as it is generically injective, and the domain is normal, so it is the normalization as desired.

Branches at $\overline{X}$ are in bijection with points of $\Phi^{-1}_{\overline{P}}(\overline{X})$, so the description of the branches immediately follows. Now we check when $\phi_{\overline{P}}$ is an immersion at $\overline{Y}$, i.e.

when $(d\Phi_P)_{\overline{Y}}$ is not of full rank.

Suppose

$$Y = (F_1(z), \ldots, F_t(z)) \in \mathrm{Sym}^{\ell_1}\mathbb{A}^1 \times \ldots \times \mathrm{Sym}^{\ell_t}\mathbb{A}^1,$$

where $F_i = z^{\ell_i} + w_{i,1}z^{\ell_i - 1} + \ldots + w_{i,\ell_i}$ is a degree ℓ_i monic polynomial in z. By taking the derivative with respect to each of the coefficients of each of the F_i separately, the Jacobian of $\Phi_{\overline{P}}$ is seen to be given by the block matrix $\begin{bmatrix} M_{i,j} \end{bmatrix}^T$ with $1 \leq i \leq t$, $1 \leq j \leq r$, where $M_{i,j}$ is the $\ell_i \times \infty$ matrix below (whose columns are indexed by powers of z)[2]. We will by abuse of notation use power series and infinite row vectors interchangeably, where the correspondence in one direction is given by listing the coefficients.

$$M_{i,j} = \begin{bmatrix} \partial/\partial w_{i,1} \prod_{s=1}^t F_s^{(\overline{n_s})_j} \\ \ldots \\ \partial/\partial w_{i,\ell_i} \prod_{s=1}^t F_s^{(\overline{n_s})_j} \end{bmatrix} = (\overline{n_i})_j \begin{bmatrix} z^{\ell_i - 1}\left(F_i^{(\overline{n_i})_j - 1} \prod_{s \neq i} F_s^{(\overline{n_s})_j}\right) \\ z^{\ell_i - 2}\left(F_i^{(\overline{n_i})_j - 1} \prod_{s \neq i} F_s^{(\overline{n_s})_j}\right) \\ \ldots \\ 1 \cdot \left(F_i^{(\overline{n_i})_j - 1} \prod_{s \neq i} F_s^{(\overline{n_s})_j}\right) \end{bmatrix}$$

$$= (\overline{n_i})_j \begin{bmatrix} z^{\ell_i - 1}(z^{d_i - \ell_i} + a_{i,1}z^{d_i - \ell_i - 1} + \ldots) \\ z^{\ell_i - 2}(z^{d_i - \ell_i} + a_{i,1}z^{d_i - \ell_i - 1} + \ldots) \\ \ldots \\ 1 \cdot (z^{d_i - \ell_i} + a_{i,1}z^{d_i - \ell_i - 1} + \ldots) \end{bmatrix}$$

$$= (\overline{n_i})_j \begin{bmatrix} z^{d_i - 1} & a_{i,1}z^{d_i - 2} & a_{i,2}z^{d_i - 3} & \ldots & a_{i,\ell_i - 1}z^{d_i - \ell_i} & a_{i,\ell_i}z^{d_i - \ell_i - 1} & \ldots \\ 0 & z^{d_i - 2} & a_{i,1}z^{d_i - 3} & \ldots & a_{i,\ell_i - 2}z^{d_i - \ell_i} & a_{i,\ell_i - 1}z^{d_i - \ell_i - 1} & \ldots \\ 0 & 0 & z^{d_i - 3} & \ldots & a_{i,\ell_i - 3}z^{d_i - \ell_i} & a_{i,\ell_i - 2}z^{d_i - \ell_i - 1} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots & z^{d_i - \ell_i} & a_{i,1}z^{d_i - \ell_i - 1} & \ldots \end{bmatrix}.$$

[2] Here we use the fact that taking the derivative with respect to a coefficient of an F_i can be computed formally using the chain rule rather than expanding the series and computing the derivative of each coefficient of the resulting series separately.

The rank of $d\Phi_{\overline{P}}$ is the same as the row rank of $[M_{i,j}]$. Multiplying each row of $M_{i,j}$ by $\left(\prod_{s=1}^{t} F_s^{(\overline{n_s})_j}\right)^{-1}$ is equivalent to right multiplication by an invertible (and upper triangular) $\infty \times \infty$ matrix (independent of i). Doing so does not change the $\mathbb{C}$-linear dependencies between the rows of the block matrix $[M_{i,j}]$. We see that

$$\left(\prod_{s=1}^{t} F_s^{(\overline{n_s})_j}\right)^{-1} M_{i,j} = (\overline{n_i})_j \begin{bmatrix} z^{\ell_i-1}/F_i \\ z^{\ell_i-2}/F_i \\ \dots \\ 1/F_i \end{bmatrix}$$

$$= (\overline{n_i})_j \begin{bmatrix} z^{\ell_i-1}(z^{-\ell_i} + b_{i,1}z^{-\ell_i-1} + \dots) \\ z^{\ell_i-2}(z^{-\ell_i} + b_{i,1}z^{-\ell_i-1} + \dots) \\ \dots \\ 1 \cdot (z^{-\ell_i} + b_{i,1}z^{-\ell_i-1} + \dots) \end{bmatrix}$$

$$= (\overline{n_i})_j \begin{bmatrix} z^{-1} & b_{i,1}z^{-2} & b_{i,2}z^{-3} & \dots & b_{i,\ell_i-1}z^{-\ell_i} & b_{i,\ell_i}z^{-\ell_i-1} & \dots \\ 0 & z^{-2} & b_{i,1}z^{-3} & \dots & b_{i,\ell_i-2}z^{-\ell_i} & b_{i,\ell_i-1}z^{-\ell_i-1} & \dots \\ 0 & 0 & z^{d_i-3} & \dots & b_{i,\ell_i-3}z^{-\ell_i} & b_{i,\ell_i-2}z^{-\ell_i-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & z^{-\ell_i} & b_{i,1}z^{-\ell_i-1} & \dots \end{bmatrix}$$

Hence the rows of the block matrix $\begin{bmatrix} M_{i,j} \end{bmatrix}$ are linearly dependent if and only if there is a $\mathbb{C}$-linear dependence between the vectors

$$\frac{z^{\ell_1-1}}{F_1}\overline{n_1}, \frac{z^{\ell_1-2}}{F_1}\overline{n_1}, \dots, \frac{1}{F_1}\overline{n_1}, \dots, \frac{z^{\ell_t-1}}{F_t}\overline{n_t}, \frac{z^{\ell_t-2}}{F_t}\overline{n_t}, \dots, \frac{1}{F_t}\overline{n_t},$$

i.e. there exist polynomials G_i with $\deg(G_i) < \deg(F_i)$ and $\sum \frac{G_i}{F_i}\overline{n_i} = 0$. Denote by

T the set of roots of all F_i. By partial fraction decomposition, we can write

$$\{\frac{G_i}{F_i} \mid \deg(G_i) < \deg(F_i)\} = \bigoplus_{\substack{\alpha \in T, j \geq 1 \\ (z-\alpha)^j | f_i}} \mathbb{C}\frac{1}{(z-\alpha)^j},$$

and hence there is a linear dependency of this form if and only if there exists an $\alpha \in T$ and $j \geq 1$ such that $\{\overline{n_i} \mid (z-\alpha)^j \text{ divides } f_i\}$ has a linear dependency. Obviously if this occurs then it occurs when $j = 1$ for some $\alpha \in T$, and the result now follows for $\Delta^{k,r}_{\overline{P}}(\mathbb{A}^1)$.

Now, let C be an arbitrary smooth affine curve. By passing to an open subset we may assume that there is an étale morphism

$$\Psi : C \to \mathbb{A}^1,$$

inducing a map

$$\widetilde{\Psi} : \Delta^{k,r}_{\overline{P}}(C) \to \Delta^{k,r}_{\overline{P}}(\mathbb{A}^1).$$

We first show the result for $\overline{m_i} \in \mathbb{N}^k$. To do this it suffices to show that the formal completion of $\overline{X} \in \Delta^{k,r}_{\overline{P}}(C)$ maps isomorphically to the formal completion of $\widetilde{\Psi}(\overline{X})$. Indeed, we may represent $\Delta^{k,r}_{\overline{P}}(C)$ as the quotient of a certain ordered colored incidence strata by $S_{d_1} \times \ldots \times S_{d_r}$. Let

$$\Delta^{ord}_{\overline{P}}(C) \subset C^{d_1} \times \ldots \times C^{d_r}$$

be all ordered incidences of d_i distinguishable points of color i for each i which correspond to an element of $\Delta^{k,r}_{\overline{P}}(C)$, then $\Delta^{k,r}_{\overline{P}}(C) = \Delta^{ord}_{\overline{P}}(C)/S_{d_1} \times \ldots \times S_{d_r}$. Note that $\Delta^{ord}_{\overline{P}}(C)$ is simply the union of many diagonally embedded copies of C^k encoding which distinguished points are incident. It is also clear that the map

$$\Delta^{ord}_{\overline{P}}(C) \to \Delta^{ord}_{\overline{P}}(\mathbb{A}^1)$$

induces an $S_{d_1} \times \ldots \times S_{d_r}$-equivariant isomorphism at the completion of any point in $\Delta^{ord}_{\overline{P}}(C)$. Let I_C be the ideal of the $S_{d_1} \times \ldots \times S_{d_r}$-orbit corresponding to $\overline{X}$ in $\Delta^{ord}_{\overline{P}}(C)$, and define the ideal $I_{\mathbb{A}^1}$ similarly for $\Delta^{ord}_{\overline{P}}(\mathbb{A}^1)$. By [47, Lemma 2], completion at the ideal of a $S_{d_1} \times \ldots \times S_{d_r}$-orbit commutes with quotient by a finite group, so applying this to I_C and $I_{\mathbb{A}^1}$ the result follows.

Now, we consider the general case. Embed $C \subset \mathbb{A}^n$ for some n. Then we have identically to the $\mathbb{A}^1$ case that the normalization of $\Delta^{k,r}_{\overline{P}}$ is

$$\mathrm{Sym}^{\ell_1} C \times \ldots \times \mathrm{Sym}^{\ell_t} C \to \Delta^{k,r}_{\overline{P}}(C),$$

which under the Chow embeddings is represented by the map

$$(F_1(z), \ldots, F_t(z)) \mapsto (\prod_{s=1}^{t} F_s^{(\overline{n_s})_1}(z), \ldots, \prod_{s=1}^{t} F_s^{(\overline{n_s})_r}(z))$$

where F_i is a polynomial of the form $(z - \overline{v_1}) \ldots (z - \overline{v_{\ell_i}})$, with $\overline{v_i} \in C \subset \mathbb{A}^n$, treated as an element of

$$\mathbb{C}^n \times \mathrm{Sym}^2 \mathbb{C}^n \times \ldots \times \mathrm{Sym}^{\ell_i} \mathbb{C}^n.$$

If we vary the $\overline{n_i}$ over distinct vectors in $\mathbb{C}^r$ such that

$$(\underbrace{\overline{n_1}, \ldots, \overline{n_1}}_{\ell_1}, \ldots, \underbrace{\overline{n_t}, \ldots, \overline{n_t}}_{\ell_t}) \in ((\mathbb{C}^r)^k)^\circ,$$

we obtain a family of maps from $\prod_{i=1}^t \mathrm{Sym}^{\ell_i} C$ to affine space. For fixed $\overline{Y} = (F_1, \ldots, F_t)$, the fiber-wise Jacobian matrix varies polynomially in the $\overline{n_i}$.

Let now $\overline{X} \in \Delta_P^{k,r}(C)$ be a colored configuration and let $\overline{Y} \in \prod_{i=1}^t \mathrm{Sym}^{\ell_i} C$ map to $\overline{X}$ under the normalization map. Suppose the configuration $\overline{Y}$ is supported at the points $c_1, \ldots, c_u \in C$, such that the set of labels of points colliding at c_i, excluding repetition, is $\{\overline{n_j}\}_{j \in A_i}$ and $A_1 \sqcup \ldots \sqcup A_u = \{1, \ldots, t\}$. We show now that if for some fixed i, $\{\overline{n_j}\}_{j \in A_i}$ are linearly dependent, then the normalization map at $\overline{Y}$ is not an immersion. Indeed, as for fixed $F_1, \ldots, F_t$ the Jacobian depends polynomially on the $\overline{n_j}$, as the Jacobian drops rank whenever $\overline{n_i} \in \mathbb{N}^r$ satisfies $(\overline{n_j})_{j \in A_i}$ are linearly dependent, by Claim 5.7.1 this is also true when the $\overline{n_i}$ are arbitrary complex vectors.

Claim 5.7.1. *Let* $\mathrm{Mat}_{a,b}$ *be the affine space of* $a \times b$ *matrices, where* a, b *are positive integers. Let* $\mathrm{Mat}_{a,b}^k \subset \mathrm{Mat}_{a,b}$ *be the locus of matrices with rank at most* k*, where* k *is a positive integer. Then the set of points* $\mathrm{Mat}_{a,b}^k \cap \mathrm{Mat}_{a,b}(\mathbb{N})$ *is Zariski-dense in* $\mathrm{Mat}_{a,b}^k$.

Proof of Claim 5.7.1. Let Λ be the $a \times b$ matrix of rank k with k 1's on its diagonal and all other entries zero. Then the product $\mathrm{Mat}_{a,a} \cdot \Lambda \cdot \mathrm{Mat}_{b,b}$ is exactly $\mathrm{Mat}_{a,b}^k$ and since $\mathrm{Mat}_{a,a}(\mathbb{N}) \times \mathrm{Mat}_{b,b}(\mathbb{N})$ is Zariski-dense in $\mathrm{Mat}_{a,a} \times \mathrm{Mat}_{b,b}$, the claim follows. □

To conclude, we must show that there are no other points where the Jacobian drops rank. Because $C \to \mathbb{A}^1$ is étale, the induced map

$$\mathrm{Sym}^{\ell_1} C \times \ldots \times \mathrm{Sym}^{\ell_r} C \to \mathrm{Sym}^{\ell_1} \mathbb{A}^1 \times \ldots \times \mathrm{Sym}^{\ell_r} \mathbb{A}^1$$

is étale and hence induces isomorphisms at the level of tangent spaces. Consider the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{Sym}^{\ell_1} C \times \ldots \times \mathrm{Sym}^{\ell_t} C & \longrightarrow & \Delta_P^{k,r}(C) \\
\downarrow & & \downarrow \\
\mathrm{Sym}^{\ell_1} \mathbb{A}^1 \times \ldots \times \mathrm{Sym}^{\ell_t} \mathbb{A}^1 & \longrightarrow & \Delta_P^{k,r}(\mathbb{A}^1)
\end{array}$$

If the bottom horizontal map has injective Jacobian at some point then the top horizontal map must have injective Jacobian at the corresponding point, and the result follows. $\square$

5.8 Isomorphisms between moduli spaces

In this section, we apply Theorem 5.1.4 to show that two natural moduli spaces are not isomorphic, negatively answering a question of Farb and Wolfson [24].

Proof of Theorem 5.1.7. By [24], $Rat^*_{d,n}$ is isomorphic to the space of $(n+1)$-tuples of monic degree d polynomials $(f_0, \ldots, f_n)$ which share no common root. Thus $Rat^*_{d,n}$ and $Poly^{d(n+1),1}_{n+1}$ are open subsets of $\mathbb{A}^{d(n+1)}$ whose complements $(Rat^*_{d,n})^c$ and $(Poly^{d(n+1),1}_{n+1})^c$ are codimension n, which is at least 2 by assumption. By Hartog's Extension Theorem, an isomorphism $Rat^*_{d,n} \to Poly^{d(n+1),1}_{n+1}$ extends to a birational morphism $\mathbb{A}^{d(n+1)} \to \mathbb{A}^{d(n+1)}$, which must be an isomorphism since $\mathbb{A}^{d(n+1)}$ is integral and separated. Furthermore, the isomorphism $\mathbb{A}^{d(n+1)} \to \mathbb{A}^{d(n+1)}$ restricts to an isomorphism between their complements $(Rat^*_{d,n})^c \to (Poly^{d(n+1),1}_{n+1})^c$.

Let $\overline{P}$ be the partition of the vector $(d, \ldots, d) \in \mathbb{N}^{n+1}$ with parts

$$\overline{P} : (1, \ldots, 1), (1, 0, \ldots, 0)^{d-1}, (0, 1, 0, \ldots, 0)^{d-1}, \ldots, (0, 0, \ldots, 1)^{d-1}$$

and let Q be the partition of $(n+1)d$ with parts

$$Q : n+1, 1^{(d-1)(n+1)}$$

We now identify

$$(Rat^*_{d,n})^c = \Delta_{\overline{P}}^{(d-1)(n+1)+1,n+1}(\mathbb{A}^1)$$
$$(Poly_{n+1}^{d(n+1),1})^c = \Delta_{Q}^{(d-1)(n+1)+1}(\mathbb{A}^1).$$

Now, as $d \geq 2$, from Theorem 5.1.4 we immediately conclude that $\Delta_{Q}^{(d-1)(n+1)+1}(\mathbb{A}^1)$ is singular in codimension 1 when a point of multiplicity 1 and $n+1$ collide, whereas $\Delta_{\overline{P}}^{(d-1)(n+1)+1,n+1}(\mathbb{A}^1)$ is first singular in codimension n, along the sub-incidence stratum $\Delta_{\overline{R}}^{(d-2)(n+1)+2,n+1}(\mathbb{A}^1)$ where $\overline{R}$ is the vector partition

$$\overline{R} : (1, \dots, 1)^2, (1, 0, \dots, 0)^{d-2}, (0, 1, 0, \dots, 0)^{d-2}, \dots, (0, 0, \dots, 1)^{d-2}.$$

□

5.9 Equivalence of constructions and Deligne categories

In this section, we show that the constructions from Section 5.2 are equivalent, and in particular prove Theorem 5.1.9.

5.9.1 Equivalence of Construction 1 and Construction 2

In this subsection we show that Constructions 1 and 2 from Section 5.2 are equivalent.

First we show the equivalence for $\mathbb{A}^n$, i.e. when $A = \mathbb{C}[x_1, \dots, x_n]$. Construction 2 at the level of rings yields the subalgebra of $\Gamma((\mathbb{C}^k)^\circ) \otimes \mathbb{C}[\{x_{i,j}\}_{1 \leq i \leq k, 1 \leq j \leq n}]$ generated by expressions of the form

$$\sum_{i=1}^{k} m_i \prod_{j=1}^{n} x_{i,j}^{s_j}$$

where $s_1, \ldots, s_n \in \mathbb{Z}_{\geq 0}$.

Let $v_1, \ldots, v_n$ be the standard basis vectors of $\mathbb{C}^n$, and write $\overline{x_i} = \sum x_{i,j} v_j$ for the vector in $\mathbb{C}^n$ with entries $x_{i,j}$ with $1 \leq j \leq n$. By Lemma 5.3.6, at the level of rings, Construction 1 yields the subalgebra of $\Gamma((\mathbb{C}^k)^\circ) \otimes \mathbb{C}[\{x_{i,j}\}_{1\leq i\leq k, 1\leq j\leq n}]$ generated by the coefficients in the $\prod v_i^{\mu_i}$-basis of expressions of the form

$$m_1 \overline{x_1}^\ell + \ldots + m_k \overline{x_k}^\ell.$$

The subalgebra from Construction 1 is clearly contained in the subalgebra from Construction 2, and conversely the expression $\sum_{i=1}^k m_i \prod_{j=1}^n x_{i,j}^{s_j}$ appears as a multiple of the $v_1^{s_1} \ldots v_n^{s_n}$-coefficient of $m_1 \overline{x_1}^\ell + \ldots + m_k \overline{x_k}^\ell$ with $\ell = s_1 + \ldots + s_n$, so the two constructions in fact yield the same subalgebra.

Finally, suppose that $\mathrm{Spec}A \subset \mathbb{A}^n$ is a variety with $A = \mathbb{C}[x_1, \ldots, x_n]/I$. Then Construciton 1 yields the image of the composite

$$(\mathbb{C}^k)^\circ \times (\mathrm{Spec}A)^k \hookrightarrow (\mathbb{C}^k)^\circ \times (\mathbb{A}^n)^k \to (\mathbb{C}^k)^\circ \times \prod_{i=1}^{N_k} \mathrm{Sym}^i \mathbb{C}^n.$$

which by Lemma 5.3.6 corresponds to the subalgebra of $\Gamma((\mathbb{C}^k)^\circ) \otimes A^{\otimes k}$ generated by the coefficients of $m_1 \overline{x_1}^\ell + \ldots + m_k \overline{x_k}^\ell$. The same argument as above shows that this is the subalgebra generated by all expressions of the form $\sum_{i=1}^k m_i \prod_{j=1}^n x_{i,j}^{s_j}$, and as every element of A is a $\mathbb{C}$-linear combination of monomials $x_1^{s_1} \ldots x_n^{s_n}$, this subalgebra is equivalently described as the subalgebra generated by elements

$$\sum_{i=1}^k m_i (1^{\otimes i-1} \otimes a \otimes 1^{\otimes k-i}),$$

which is precisely Construction 2.

5.9.2 Equivalence of Construction 2 and Construction 3

In this subsection we prove Theorem 5.1.9.

To do this, we need to unwind the construction in the Deligne category. We fix complex $m_1, \ldots, m_k \in \mathbb{C} \backslash \mathbb{Z}_{\geq 0}$ summing to $d \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ and let $I = I(m_1, \ldots, m_k)$.

The Deligne category $\mathrm{Rep}(S_d)$ is semi-simple with irreducible objects X_λ that are parametrized by Young tableaux λ. We write $\mathbf{1} = X_\emptyset$, $\mathfrak{h}'$ for the X_λ corresponding to the standard representation, and let $\mathfrak{h} = \mathbf{1} \oplus \mathfrak{h}'$ be the object corresponding to the permutation representation. The following fact will allow us to work with elements in algebras.

Claim 5.9.1. *The subcategory of* $\mathrm{Rep}(S_d)$ *generated by* $\mathbf{1}$ *is equivalent to the category* Vect_k *of finite-dimensional vector spaces given by the functor* $\mathrm{Hom}(\mathbf{1}, -)$.

Proof. The inverse to $\mathrm{Hom}(\mathbf{1}, -)$ takes a vector space V to $V \otimes \mathbf{1}$, and the result follows since $\mathrm{Hom}(\mathbf{1}, \mathbf{1}) = \mathbb{C}$. □

Therefore, if we have an algebra inside $\mathrm{Ind}(\mathrm{Rep}(S_d))$ which as an object is a direct sum of $\mathbf{1}$'s, then we may treat it as an honest $\mathbb{C}$-algebra by applying $\mathrm{Hom}(\mathbf{1}, -)$.

Definition 5.9.2. *Given an object* B *in* $\mathrm{Rep}(S_d)$ *or* $\mathrm{Ind}(\mathrm{Rep}(S_d))$, *let* B^{S_d} *denote the* $\mathbf{1}$*-isotypic component of* B.

Using the fact that $\mathrm{Rep}(S_d)$ is semisimple, we see the functor $(-)^{S_d}$ has the property that if B is an algebra in $\mathrm{Rep}(S_d)$ or $\mathrm{Ind}(\mathrm{Rep}(S_d))$ and if J is an ideal in B, then J^{S_d} is an ideal in B^{S_d}, and $(B/J)^{S_d} \cong B^{S_d}/J^{S_d}$. Finally, we have $\mathrm{Hom}(\mathbf{1}, B) \cong \mathrm{Hom}(\mathbf{1}, B^{S_d})$ for any object B, and by Claim 5.9.1 we have

$$\mathrm{Hom}(\mathbf{1}, B/J) = \mathrm{Hom}(\mathbf{1}, B)/\mathrm{Hom}(\mathbf{1}, J) = \mathrm{Hom}(\mathbf{1}, B^{S_d})/\mathrm{Hom}(\mathbf{1}, J^{S_d}).$$

Proof of Theorem 5.1.9. We are interested in computing the algebra

$$\operatorname{Hom}(\mathbf{1}, A^{\otimes d}/I) = \operatorname{Hom}(\mathbf{1}, (A^{\otimes d})^{S_d}) / \operatorname{Hom}(\mathbf{1}, I^{S_d}).$$

We first characterize I^{S_d}.

Consider the diagram

$$\begin{array}{ccccc} A^{\otimes d} & \xmapsto{\text{Res}} & A^{\otimes m_1} \boxtimes \ldots \boxtimes A^{\otimes m_k} & \longleftarrow & (A^{\otimes m_1})^{S_{m_1}} \boxtimes \ldots \boxtimes (A^{\otimes m_k})^{S_{m_k}} \\ \uparrow & & \uparrow & \nearrow & \\ (A^{\otimes d})^{S_d} & \xmapsto{\text{Res}} & \operatorname{Res}((A^{\otimes d})^{S_d}) & & \end{array}$$

Given a subobject J' of a commuatative algebra B in $\operatorname{Rep}(S_d)$ or $\operatorname{Ind}(\operatorname{Rep}(S_d))$, the ideal J'' generated by J' is the image of the map $B \otimes J' \to B \otimes B \to B$.

Claim 5.9.3. *I^{S_d} is the largest ideal $J' \subset (A^{\otimes d})^{S_d}$ such that $\operatorname{Res}(J') \subset (J_{m_1})^{S_{m_1}} \boxtimes \ldots \boxtimes (J_{m_1})^{S_{m_k}}$.*

Proof. We clearly have $\operatorname{Res}(I^{S_d}) \subset \operatorname{Res}(I) \subset J_{m_1} \boxtimes \ldots \boxtimes J_{m_k}$, so taking invariants we see that $\operatorname{Res}(I^{S_d}) \subset (J_{m_1})^{S_{m_1}} \boxtimes \ldots \boxtimes (J_{m_1})^{S_{m_k}}$. We now show that I^{S_d} is the largest ideal in $(A^{\otimes d})^{S_d}$ with this property.

Let $J' \subset (A^{\otimes d})^{S_d}$ be an ideal such that $\operatorname{Res}(J')$ lies in $J_{m_1} \boxtimes \ldots \boxtimes J_{m_k}$, and let J'' be the $A^{\otimes d}$-ideal generated by J'. Since Res is an exact fuctor it commutes with taking images, so we have $\operatorname{Res}(J'')$ is the ideal generated by $\operatorname{Res}(J')$. As $\operatorname{Res}(J') \subset J_{m_1} \boxtimes \ldots \boxtimes J_{m_k}$, we deduce $\operatorname{Res}(J'') \subset J_{m_1} \boxtimes \ldots \boxtimes J_{m_k}$, so $J'' \subset I$ by definition of I. In particular, this implies $J' \subset I^{S_d}$. □

Now, Res is an equivalence of categories when restricted to the 1-isotypic part of $\operatorname{Rep}(S_d)$ and $\operatorname{Rep}(S_{m_1}) \boxtimes \ldots \boxtimes \operatorname{Rep}(S_{m_k})$, so

$$\operatorname{Hom}(\mathbf{1}, (A^{\otimes d})^{S_d}) = \operatorname{Hom}(\mathbf{1}, \operatorname{Res}((A^{\otimes d})^{S_d}))$$

and therefore the ideal $\mathrm{Hom}(\mathbf{1}, I^{S_d}) \subset \mathrm{Hom}(\mathbf{1}, (A^{\otimes d})^{S_d})$ is the kernel of the product of the multiplication maps

$$\begin{aligned}\mathrm{Hom}(\mathbf{1}, A^{\otimes d}) &= \mathrm{Hom}(\mathbf{1}, \mathrm{Res}(A^{\otimes d})) \\ &\to \mathrm{Hom}(\mathbf{1}, (A^{\otimes m_1})^{S_{m_1}}) \otimes \ldots \otimes \mathrm{Hom}(\mathbf{1}, (A^{\otimes m_k})^{S_{m_k}}) \to A \otimes \cdots \otimes A.\end{aligned}$$

By [22, Proposition 4.1], we have

$$\mathrm{Hom}(\mathbf{1}, A^{\otimes d}) \cong S(A)/(1_A = d),$$

where $S(A)$ is the symmetric algebra generated by A. The relation $(1_A = d)$ indicates the unit $1_A \in A$ is identified with $d \in A^0 \cong \mathbb{C}$. We first explicitly describe the map

$$S(A)/(1_A = d) \to S(A)/(1_A = m_1) \otimes \ldots \otimes S(A)/(1_A = m_k).$$

To do this, we note that $A^{\otimes d}$ is canonically the quotient of the tensor algebra $T(A \otimes \mathfrak{h})$ by certain relations [22, Proposition 4.3]. We claim that $(A^{\otimes d})^{S_d}$ is generated by the invariant generators arising from the composite

$$\overline{a} : \mathbf{1} \xrightarrow{a \otimes \mathrm{Id}} A \otimes \mathbf{1} \to A \otimes \mathfrak{h} \to T(A \otimes \mathfrak{h}) \to A^{\otimes d}.$$

Indeed, we know that $\mathrm{Hom}(\mathbf{1}, (A^{\otimes d})) \cong S(A)/(1_A = d)$, and under this isomorphism the elements of $A \subset S(A)/(1_A = d)$ correspond to precisely to the homorphisms described above.

Now, we have the diagram

$$\begin{array}{ccc} A^{\otimes d} & \xmapsto{Res} & A^{\otimes m_1} \boxtimes \ldots \boxtimes A^{\otimes m_k} \\ \uparrow & & \uparrow \\ A \otimes \mathfrak{h} & \xmapsto{Res} & A \otimes (\mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k) \\ \uparrow & & \uparrow \\ A \otimes \mathbf{1} & \xmapsto{Res} & A \otimes (\mathbf{1} \oplus \ldots \oplus \mathbf{1}) \end{array}$$

where by $\mathfrak{h}_i$ we really mean $\mathbf{1} \boxtimes \ldots \boxtimes \mathbf{1} \boxtimes \mathfrak{h}_i \boxtimes \mathbf{1} \boxtimes \ldots \boxtimes \mathbf{1}$ where $\mathfrak{h}_i$ is in the i'th place. Thus the map

$$S(A)/(1_A = d) \to S(A)/(1_A = m_1) \otimes \ldots \otimes S(A)/(1_A = m_k)$$

is given on generators by

$$a \mapsto (a \otimes 1 \otimes \ldots \otimes 1) + (1 \otimes a \ldots \otimes 1) + \ldots + (1 \otimes 1 \otimes \ldots \otimes a).$$

Now, we consider the multiplication map $A^{\otimes m_i} \to A \otimes \mathbf{1}$. By definition, this is given by factoring the map $T(A \otimes \mathfrak{h}) \to A \otimes \mathbf{1}$ through $A^{\otimes m_i}$, where the map is given by setting for each j

$$(A \otimes \mathfrak{h})^{\otimes j} \to A \otimes \mathbf{1}$$

induced by the maps $A^{\otimes j} \to A$ and $\mathfrak{h}^{\otimes j} \to \mathbf{1}$ given by the basis element in the partition algebra $\mathbb{C}P_{j,0}$ [18, Definition 2.11] corresponding to the finest partition of j.

The multiplication maps induce algebra maps $S(A)/(1_A = m_i) \to A$ which we want to determine at the level of generators. Given a generator $a \in A \subset S(A)$, we compute the map to A as arising from the composite $\mathbf{1} \to A \otimes \mathbf{1} \to A \otimes \mathfrak{h} \to A \otimes \mathbf{1}$ where the composite map is a on the first coordinate, and $\mathbf{1} \to \mathfrak{h} \to \mathbf{1}$ on the second coordinate, which is multiplication by m_i [18, Definition 2.11]. Therefore, the map of algebras, $S(A)/(1_A = m_i) \to A$ takes each generator $a \in A \subset S(A)$ to $m_i a \in A$.

Therefore, the composite map

$$S(A)/(1_A = d) \to S(A)/(1_A = m_1) \otimes \ldots \otimes S(A)/(1_A = m_k) \to \underbrace{A \otimes \ldots \otimes A}_{k}$$

takes

$$a \mapsto m_1(a \otimes 1 \otimes \ldots \otimes 1) + m_2(1 \otimes a \otimes \ldots \otimes 1) + \ldots + m_k(1 \otimes 1 \otimes \ldots \otimes a),$$

and the quotient $\operatorname{Hom}(\mathbf{1}, (A^{\otimes d})^{S_d}) / \operatorname{Hom}(\mathbf{1}, I^{S_d})$ is therefore isomorphic to the subalgebra of $A^{\otimes k}$ generated by such elements, which is precisely Construction 2. $\square$

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