Discovery and Purity in Archimedes

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Discovery and Purity in Archimedes

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Abstract

Philosophers of mathematics wonder about the applicability of mathematics to scientific explanations of the physical world. My inquiry is in the opposite direction: why and how can the study of physical phenomena be helpful to the advancement of mathematics? The value of purity has long driven changes and progress in mathematics. According to the ideal of purity, mathematics should be purged of ideas of an extraneous source, because they do not amount to true explanations and can be misleading. Meanwhile, mathematicians, including those who underscore purity, acknowledge the fruitfulness of borrowing foreign ideas to help with mathematical discovery. This dissertation studies Archimedes' *Method*, a work that highlights the fruitfulness of geometric discovery through mechanical imaginations. I argue that Archimedes brings out the heuristic potential of mechanics in two ways: one is to develop new methods that incorporate non-rigorous techniques inspired by the study of the physical world into rigorous mathematical demonstrations, the other is to envisage an art of discovery through mechanics, of which his *Method* provides starting points. In this dissertation I show that a dialogue between Archimedes' vision with regard to discovery and the ideal of mathematical purity can shed light on both the thought of Archimedes and the study of the history of mathematics.

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1. Uncovering Archimedes' approach of discovery

Critique of ancient geometry was a common theme among mathematicians and philosophers in the early modern period. An argument for this critique is the intransparency of the process of discovery in a certain type of demonstrations in ancient Greek geometry that employs the so-called method of exhaustion. While there were speculations about the informal technique of the ancients, Archimedes' *Method*, a work that emphasizes discovery *vis-à-vis* demonstration and showcases the approach used by Archimedes in his discovery of many ground-breaking results, was unknown to the modern world until the end of the nineteenth century. In what follows I will give a general introduction to the method of exhaustion (section 1), discuss several specimens of the critical and creative interpretations of ancient Greek geometry in the early modern period (section 2), and briefly go through the (re)discovery of Archimedes' *Method*, solely transmitted through the Archimedes Palimpsest, and explain how the heuristic approach showcased in the *Method* is related to Archimedes' rigorous demonstrations (section 3).

1.1 The method of exhaustion: terminology and examples

The standard practice of ancient Greek mathematics to prove the measurement of the dimension of a curvilinear figure is through the so-called method of exhaustion. The method of exhaustion approaches the curvilinear figure through an inexhaustible process of approximation by inscribing and/or circumscribing rectilinear figures. By using *reductio ad absurdum*, this method rigorously proves, albeit in a formulaic manner, that the limit of this approximation process is equal to the measurement of the curvilinear figure.

The term "exhaustion" incurs suspicion, as nothing is exhausted through the method of exhaustion. Grégoire de Saint-Vincent was the first to use the Latin verb *exhaurire* "to exhaust" to refer to proofs through the above-mentioned approach. In his 1647 work *Opus geometricum quadraturae circuli et sectionum coni*, Saint-Vincent gives the following scholium to a theorem proved through the method of exhaustion:

> Theorema iam demonstratum universalissimum est, extenditque sese ad sequentes propositiones fere omnes. Ne igitur idem discursus in singulis propositionibus labore inutile, et cum molestia lectoris repetendus esset, placuit totum exhaustionis negotium hoc loco terminis universalibus proponere et demonstrare. (*Dudctum plani in planum*, part 3 prop. 46; Saint-Vincent 1647: 740)
>
> The theorem just proved is most universal and can be extended to all subsequent propositions. Thus, to avoid the pointless labor of repeating the process in every single proposition to the reader's annoyance, I thought it best to set forth and prove here the entire business of exhaustion in universal terms.

This comment reflects the *Zeitgeist* of seventeenth-century mathematics: a proof through the method of exhaustion is cumbersome to write and unpleasant to read, and if it cannot be totally dispensed with, one would limit it to the most universal case possible so that the work is done once and for all. The phrase *exhaustionis negotium* suggests use of the word *exhaustio* in informal complaints about the ancient style of proof already before Saint-Vincent's publication. The sixth-century philosopher Simplicius already used the Greek verb δαπανᾶν "to exhaust" in his *Commentary to Aristotle's* Physics to describe Antiphon's circle quadrature:

> τοῦτο ἀεὶ ποιῶν ὥστε ποτὲ δαπανωμένου τοῦ ἐπιπέδου ἐγγραφήσεσθαί τι πολύγωνον τούτῳ τῷ τρόπῳ ἐν τῷ κύκλῳ, οὗ αἱ πλευραὶ διὰ σμικρότητα ἐφαρμόσουσι τῇ τοῦ κύκλου περιφερείᾳ. (9.55.6-8)
>
> Continually doing this [process of inscribing new polygons by duplicating the number of sides of the last one], with the result that at some point, while the area is being exhausted, a certain polygon will be inscribed in this way in the circle, whose sides, by smallness, coincide with the circumference of the circle.

This explanation of Antiphon's circle quadrature is a gloss to Aristotle's claim that "it is the geometer's job to refute the squaring of the circle by segments, but not to refute that of Antiphon" (*Phys*. I.2 185a16-17), which in turn is an example to illustrate the point that refutation of false statements should only concern those reached by arguing from the relevant principles (*Phys*. I.2 185a14-15). Simplicius, agreeing with Eudemus, says that the principle violated in Antiphon's quadrature is that "magnitudes are infinitely divisible" (ἐπ' ἄπειρον εἶναι τὰ μεγέθη διαιρετά, 9.55.22-23). But there is no serious conflict between infinite divisibility and Antiphon's idea. Antiphon could well admit that a continuum cannot be exhausted through finite divisions, and say at the same time that *if* it should be exhausted (perhaps through infinitely many divisions), the circle would coincide with a polygon.

The Euclidean circle quadrature continues with Antiphon's intuition but circumvents the quarrels caused by infinity by way of the so-called method of exhaustion. Euclid's *Elements* XII 2 squares the circle through the same construction proposed by Antiphon: inscribe in the circle a square, then an octagon, then a sixteen-sided polygon, then a thirty-two-sided, and so on. Such a construction will not exhaust the circle within finite steps, but the remainder between the polygon and the circle grows smaller and smaller without a positive lower bound. If we suppose towards contradiction that circles to one another are not as the squares on their diameters, then there is a certain area less than one of the two circles so that the ratio of this area to the other circle is the same as that of the squares to one another. But this will lead to contradiction with the fact that the remainder between the inscribed polygon and the circle can be smaller than any positive quantity as the number of the sides of the polygon duplicates itself.

Below is a sequence of inscribed polygons in a circle. At each step, a_i denotes what is left between the polygon and the circle, b_i denotes the area newly added to the previous polygon.

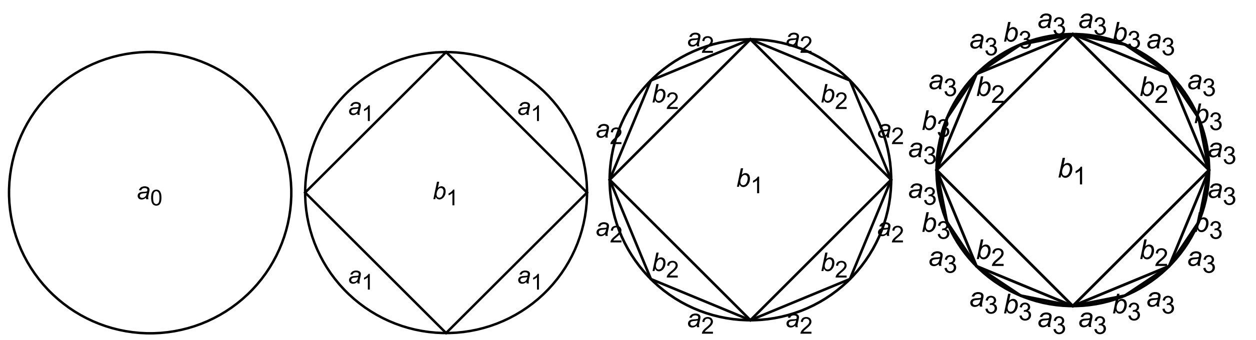

Figure 1-1 a circle, a square, an octagon, and a sixteen-sided polygon inscribed in the circle

At each step, the newly added area b_i is greater than the remainder a_i. In other words, more than half of the remainder of the circle is subtracted when a new polygon is inscribed. According to *Elements* X 1, if a magnitude is continually subtracted by at least a half of the remainder, then for any given magnitude, there is a finite step at which the remainder is less than the given magnitude. Now if we suppose towards contradiction that for two circles of radii R and r, their corresponding area A_0 and a_0 are not to each other as R^2 to r^2, one of the two ratios would be greater than the other. If $A_0 : a_0$ is less than $R^2 : r^2$, there is a certain magnitude $S < a_0$ such that $A_0 : S :: R^2 : r^2$. Let $\varepsilon = a_0 - S$. According to *Elements* X 1, there exists a finite number k such that at the k-th step, $a_k < \varepsilon = a_0 - S$, which means $S < a_0 - a_k$ = the area of the inscribed 2^{k+1} -sided polygon. Inscribe a similar polygon in the other circle, by proportion its area will be greater than the circle's area A_0, which is absurd. If, on the other hand, $A_0 : a_0$ is greater than $R^2 : r^2$, there is likewise a certain magnitude $S < A_0$ such that $S : a_0 :: R^2 : r^2$, and contradiction will result in the same way. Thus $A_0 : a_0 :: R^2 : r^2$.

The Euclidean circle quadrature is a classic example of the method of exhaustion in measuring the dimension of a figure: a curvilinear figure is approximated through a converging sequence of rectilinear ones, and a double *reductio ad absurdum* proof shows that the quantity under investigation has to be equal to that proposed in the enunciation of the theorem—double because one should prove that the quantity is neither greater nor less. The method of exhaustion is conventionally associated with the name Eudoxus, who was a younger contemporary of Plato and is said to have visited the latter's academy. In the preface letter to *On Sphere and Cylinder* I, Archimedes praises Eudoxus as the first to grasp[1] that a pyramid is a third of the prism of the same base and height and that a cone is a third of the cylinder of the same base and height (*SC* I H1.4.5-9), and in the preface to the *Method* Archimedes mentions these theorems again, while making a distinction between demonstration and discovery—while Eudoxus was first to prove rigorously these two theorems, it was Democritus who first made the assertion without proof (*Method* H2.430.1-9 = 43r col. 2[2]). This cubature of the cone is also included in the *Elements*, starting right after the circle quadrature and culminating in XII 9. The proof of the cubature is a three-dimensional version of that of the circle quadrature given by XII 2, following the same method of inscribing polygons in the circular base, doubling the sides, and giving a double *reductio*. The key step in the cubature of the cone is based on the cubature of the pyramid, i.e., a pyramid is a third of the triangular prism of the same base and height. This theorem, known as XII 7 is in turn based on the conclusion of XII 5, namely pyramids of the same height are to one another as their bases. The proof of XII 5 is different from the other exhaustion proofs in the *Elements* for slicing the figure in question in a parallel rather than polar fashion, which, as we will see, plays a prominent role in Archimedes' works. The exhaustion proofs in the *Elements* may have preserved traces of non-rigorous techniques, such as those to be extracted from

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[1] *SC* I H1.4.11-13: πολλῶν πρὸ Εὐδόξου γεγενημένων ἀξίων λόγου γεωμετρῶν συνέβαινεν ὑπὸ πάντων ἀγνοεῖσθαι μηδ' ὑφ' ἑνὸς κατανοηθῆναι. "Though many geometers before Eudoxus were noteworthy, it happened that these theorems were unknown to all, nor were they realized by anyone."

[2] The Archimedean texts in this paper are based on Heiberg's 1910-15 critical edition, though sometimes I may choose the reading in Netz and Wilson's transcription in Netz, Noel, Tchernetska, and Wilson 2011, vol. 2. When I cite texts from the *Method*, I show both the volume, page, and line numbers in Heiberg's edition and where they are in the Archimedes Palimpsest, e.g. H2.428.27-29 = 46v col. 2.11-15.

Antiphon's argument or Democritus' cone dilemma[3], but the definitive form of rigorous proofs as presented in the *Elements* is often attributed to Eudoxus.[4]

Archimedes pushed the centuries-long inquiry of circle quadrature to a more specific conclusion: the area of a circle is equal to the area of the triangle whose height is equal to the radius of the circle and base the circumference. He arrived at this conclusion through a modified form of the Eudoxean method of exhaustion. The modification lies in that not one, but two sequences of converging polygons, the inscribed and the circumscribed, are constructed to approximate the circle. If the area of the circle is greater than that of the triangle, then there will be an inscribed polygon whose area is greater than that of the triangle. But the circumference of the polygon is less than that of the circle, and the inradius less than the circle's radius. Consequently, the area of the polygon is less than that of the triangle, which contradicts the hypothesis. On the other hand, if the area of the circle is less than that of the triangle, then there will be a circumscribed polygon whose area is less than that of the triangle. But the circumference of the polygon is greater than that of the circle, and the inradius equal to the circle's radius, which means that the area of the polygon is less than that of the triangle, which, in turn, leads to contradiction.

As noted by scholars, the received text of Archimedes' circle quadrature, viz. the *Dimensio Circuli*, betrays multiple signs of heavy modification and excerption by later hands, including the absence of the Doric dialect (which was Archimedes' native tongue), inaccurate terminology, the

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[3] Knorr 1996 shows that *Elements* XII 5 "had an indivisibilist preliminary" (75), but "[t]he scheme of parallel sectioning of plane or solid figures is not readily convertible to a convergence construction of the one-sided 'approximative' type, adopted by Euclid. It does lead itself to the two-sided ('compression') method, however, the manner favored by Archimedes, its apparent inventor." (79)
[4] See e.g., Archimedes' preface to the *Quadrature of the Parabola* and Hero's preface to his *Metrica*, which attributes both circle quadrature and cone cubature to Eudoxus. See Knorr 1986: 78-79 for accounts of ancient geometry about the Eudoxean method.

circular use of prop. 3 in prop. 2, and discrepancies with late antique commentators.[5] It is questionable whether Archimedes followed *Elements* XII 2 so closely as the received text does in the construction of inscribed polygons. In late antique commentators who cite Archimedes' circle quadrature, details of the construction are either missing (as in Theon) or very likely interpolated by the commentator by following *Elements* XII 2 closely (such as Pappus' account).[6] Perhaps Archimedes did not construct polygons by doubling the sides, but the addition of a sequence of circumscribed polygons is unmistakably authentic, both because such a sequence is necessary for the argument and because parallel texts are found in Pappus' and Theon's commentaries.

Using two sequences of figures—one inscribed, the other circumscribed—to approximate a curvilinear figure proves to be a customary technique throughout Archimedes' work. A further development consists in the way in which the curvilinear figure under investigation is divided. The measurements of circle, cone, and sphere in *Elements* XII are carried out through a polar division from the center of a circle, whereas in Archimedes, curvilinear figures are usually divided into parallel slices on which rectilinear figures are inscribed and circumscribed. For example, in one of the proofs in Archimedes' *Quadrature of the Parabola*, a parabolic segment is divided by lines parallel to its axis, and each subdivided segment is bounded by two trapezia, one inscribed, the other circumscribed. For any given quantity, there exist a certain number of lines that divide the parabolic segment into enough many slices, so that the difference between the inscribed and the circumscribed trapezia is smaller than the given quantity. This proof will be further elaborated later in this chapter. For now, it is enough to mention that Archimedes widely applied this technique to the measurement of dimension and the determination of the center of

[5] For a detailed account see Knorr 1989: 375. See also Dijksterhuis 2014: 222; Heiberg's note in *Archimedis opera omnia*, vol. 1 p. 233n.: *omnino in toto hoc opusculo genus dicendi et exponendi brevitate tam neglegenti laborat, ut manum excerptoris pot ius quam Archimedis agnoscas*.
[6] Knorr 1989: 381-384.

gravity of various geometric objects ranging from triangle to spiral, from conic sections to solids generated through rotation.

1.2 Early modern reception of Archimedes: discontent and appropriation

Unlike the *Elements*, advanced works in ancient Greek mathematics mostly fell into oblivion in the Middle Ages. Despite the circulation of Arabo-Latin and Greco-Latin translations, the figure of Archimedes had long been conceived of as an engineer, a designer of war machines, a practitioner of the physical sciences, a man undisturbed by war or death, …, anything but a mathematician of the first rank. After the fifteenth-century reintroduction of Plutarch's *Lives* into the West, in which the *Life of Marcellus* gives the fullest surviving account of Archimedes' life and achievements, humanists and high artisans, such as Leon Battista Alberti, Leonardo da Vinci, Regiomontanus, and Giorgio Valla, showed an interest in Archimedes in combination with knowledge of his works. The sixteenth century witnessed the return of Archimedes *qua* mathematician. The *editio princeps* of the Archimedean corpus in the original Greek language came out in 1544, and selections of Moerbeke's Greco-Latin translation were printed and reprinted several times throughout this century. Also notable in the sixteenth century was the reception of Archimedes' works on quadrature, cubature, and the study of the center of gravity, as seen in Commandino's revised translation and commentaries and Maurolico's writings in the same fields that once engaged Archimedes.[7]

Simultaneous with the keen acknowledgement and reception of Archimedes was a critical assessment of the style of reasoning and the form of writing of ancient Greek mathematics. Sixteenth- and seventeenth-century mathematicians criticize the method of exhaustion for being

[7] See Høyrup 2022: 182-188; Laird 1991: 629-638, esp. 633ff.

“indirect” due to two kinds of reasoning. One strand is directed at the discipline of mathematics in general. Arguing from an Aristotelian position, some hold that mathematics is not a *causal* science, viz. science that reveals the true cause of the conclusion (Piccolomini 1547 and Pereyra 1562), or more radically, that there is no causal science at all (Gassendi 1658).[8] A basis for such reasoning is the extensive use of *reductio* in mathematics, on the grounds that proofs through *reductio* do not proceed from the true cause of the conclusion, but on the very opposite, by assuming wrong premises. Attacks on mathematics were met with efforts to defend the status of mathematics as a respectable science, and if possible, a causal one. Such a debate, known as the *Quaestio de certitudine mathematicarum*, was topical among sixteenth- and seventeenth-century intellectuals.[9] Strikingly, defenders of mathematics also hold that *reductio* proofs cannot be causal. For example, Barozzi 1560 agrees with Proclus that a converse theorem “is not proved through the true causes” (*non per veras causas demonstraretur*, 26r, 27r-v), but “through a deduction leading to impossibility” (*deductione ad impossibile*, 27r); Biancani 1615 defends the causal nature of mathematics but excludes *reductio* proofs from “demonstrations from the cause” (*demonstrationibus a causa*, 10).

Another strand of criticism accuses the ancients of concealing their true path of discovery. The trend of valuing *ars inveniendi* above *ars demonstrandi* in the sixteenth and seventeenth centuries was closely related to the emergence of algebra and analytic geometry.[10] Following Commandino’s 1588 Latin translation of Pappus’ *Collectio*, the concept of analysis became well-known and continually pondered over. In the beginning of *Collectio* VII, Pappus tells

[8] See, e.g., Malink 2020 for a discussion of Aristotle’s grounds for the inferiority of *reductio* to direct demonstrations and (mis)interpretations of this claim.
[9] Mancosu 1996: 8-33.
[10] See Rashed 2014 for an introduction of tenth-century Arabic mathematicians’ discussions of mathematical discovery and the role of analysis and synthesis in it.

Hermodorus that the domain of analysis equips one with the faculty of discovery (δύναμιν εὑρετικήν) of problems more advanced than those in the *Elements*. Pappus then goes on to give synopses of works relevant to analysis and introduce preparatory theorems for the study of those works. The vagueness in Pappus' description and the loss of many works mentioned by him made the idea of analysis all the more mysterious and tempting: analysis was commonly held to be the ancients' secret art of discovery, and such an opinion stimulated efforts to reimagine the lost art of discovery or to surpass it by making new discoveries.[11]

Against this backdrop, the method of exhaustion, like other styles of proof, e.g., *reductio* proofs in general, was considered a concealment rather than an exposition of discovery. Leibniz complains about the construction of converging rectilinear figures as some sort of rituals and values his indivisibles approach over the traditional rigorous style of proof for the former's simplicity. In his *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis*, a work written in 1676 but not published until 1993, Leibniz makes the following comment in a scholium to a proposition proved through constructing converging step-shaped (*gradiforme*) figures:

> Hac propositione supersedissem lubens, cum nihil sit magis alienum ab ingenio meo quam scrupulosae quorundam minutiae in quibus plus ostentationis est quam fructus, nam et tempus quibusdam velut caeremoniis consumunt, et plus laboris quam ingenii habent, et inventorum originem caeca nocte involvunt, quae mihi plerumque ipsis inventis videtur praestantior. Quoniam tamen non nego interesse Geometriae ut ipsae methodi ac principia inventorum tum vero theoremata quaedam praestantiora severe demonstrata habeantur, receptis opinionibus aliquid dandum esse putavi.
>
> I would have happily omitted this proposition, since nothing is more at odds with my talent than the gruesome details of certain matters, in which there is more display than outcome, for they consume time, so to speak, with some ceremonies, involve more work than talent, and conceal the origin of discoveries in blind night, which I think is generally more important than the discoveries themselves. But since I do not deny that it is important for Geometry that its methods, principles of discoveries, and particularly certain important

[11] See e.g. Descartes 1908: 376-377; Torricelli 1644: 56; Wallis 1685: 1, 3-4; Walter Charleton 1657: 63-65.

> theorems are considered demonstrated rigorously, I thought I should make concessions to the established opinion.

The work in which this scholium is written aims to lay a most rigorous foundation for the theory of indivisibles. It is for this reason that we see both a distaste for and a concession to the classical way of demonstration in this scholium, which is reminiscent of Saint-Vincent's idea of finishing the *totum exhaustio negotium* in the most general case possible. As we will see later, Archimedes also uses indivisibles in his *Method*, but in a way different from Leibniz's. Archimedes' have no extension in the dimension in which they compose a figure, whereas Leibniz's indivisibles have an infinitesimal extension—a quantity is infinitesimal if it is smaller than any given quantity. The distinction was already made by seventeenth-century mathematicians, in the language of homogeneous and heterogeneous, though in practice the two were often confused. If a continuum is composed of infinitesimal indivisibles, the indivisibles are homogeneous with what they compose, as they are of the same kind of quantity; but if a continuum is composed of indivisibles that have no extension in the dimension in which they compose the continuum, the indivisibles are heterogeneous, as they are of a lower dimension than what they compose.

Among many various indivisible techniques, I find Kepler's most interesting, partly because his technique stems from a close reading of Archimedes' circle quadrature, partly because his reading of Archimedes is particularly creative and imaginative. Kepler' 1615 work on *Nova stereometria doliorum vinariorum* is comprised of two parts, *Stereometria Archimedeae*, "Archimedes' stereometry," and *Stereometria dolii Austriaci*, "the stereometry of Austrian wine jar". The first part is further divided into a commentary on a collection of Archimedes' theorems and a *Supplementum ad archimedea*, "supplement to Archimedes." In the commentary to Archimedes' *Dimensio Circuli* prop. 2, Kepler criticizes the use of *reductio* (7v) and constructs a direct proof that aims to reveal Archimdes' true intention in the following way (*mihi sensus hic* *videtur*, 7r). He starts with an interesting assumption: the circumference of a circle has as many parts (*partes*) as it has points (*puncta*); let them be infinite.[12] Any part can be the base of an infinitesimal isosceles triangle whose two other sides are radii of the circle. Now let the pinnacles of all the triangles be fixed at the center of the circle and unroll and pull straight the circumference. The circle is thus transformed into a triangle composed of infinitely many triangles of which the height is equal to the radius and the bases together equal to the circumference of the circle.[13] Kepler concludes that "this is what the Archimedean deduction to impossibility intends" (*Hoc sibi vult illa Archimedea ad impossibile deductio*, 7v).

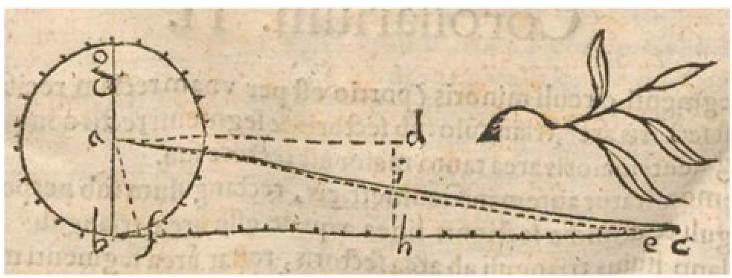

Figure 1-2 Part 1, theorem 2, schema II in the 1615 edition of *Nova stereometria*, 7r

In the *Supplementum ad Archimedea*, Kepler applies the same technique of unrolling to a solid that he calls "apple". An "apple" (labelled III below in Figure 1-3) is generated through the rotation of a circle around a chord in the circle. If we fix the axis, which is the common edge (*communis acies*, 22v) of all the circles, and pull straight the circumference of the apple, the apple will be unrolled into a cylinder segment (see below in Figure 1-4). This, Kepler says, follows the very principles by which Archimedes unrolls the circle into a triangle.[14]

[12] Kepler 1615: 7r. *Circuli BG circumferentia partes habet totidem, quod puncta, puta infinitas*.
[13] Kepler 1615: 7r-7v.
[14] Kepler 1615: 22r: "Explicetur corpus Mali iisdem legibus in Cylindricum segmentum, quibus Archimedes Theorem. I 1 explicavit circuli aream in triangulum rectangulum."

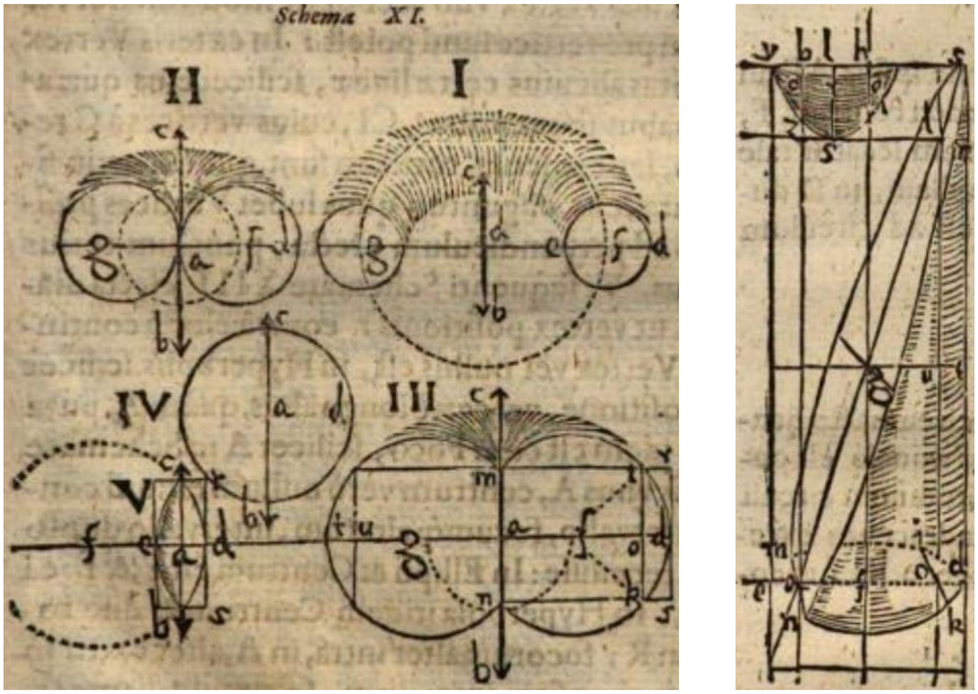

Figure 1-3 left: schema XI, part 1, in the 1615 edition of *Nova stereometria*, 18r; right: schema XIV, part 1, in the 1615 edition of *Nova stereometria*, 22v

This seems more of an appropriation than a plain reception of Archimedes. One year after the publication of *Nova stereometria*, Alexander Anderson published a defense of Archimedes titled *Vindiciae Archimedis sive, elenchus cyclometrae novae*, in which he says Archimedes was badly received by Kepler (*a M. Ioanne Keplero male acceptum*, 1616: 3) and asks, “What mind could grasp such a transformation?” (*quae mens capiat huiusmodi Metamorphoses*? 1616: 3) However, Kepler was not the first to read Archimedes’ circle quadrature as an outcome of unrolling the circumference. Theon’s *Commentary on Ptolemy’s* Almagest mentions Archimedes’ circle quadrature with the expression of unrolling. Theon says that Archimedes has proved that the rectangle contained by the diameter and the circumference, when “the circumference of the circle is unrolled into a straight line” (τὸ ὑπὸ τῆς διαμέτρου καὶ τῆς τοῦ κύκλου περιφερείας εἰς εὐθεῖαν ἐξαπλουμένης περιεχόμενον ὀρθογώνιον τετραπλάσιόν), is twice the area of the circle.[15] The same idea of unrolling is also echoed by Hero in his *Metrica*. Hero says that if we imagine (νοήσωμεν) that the surface of a cylinder "is unrolled, i.e. stretched out into a plane surface" (ἀνηπλωμένην, τουτέστιν ἐκτεταμένην εἰς ἐπίπεδον), it will be a rectangle (I 36); and if we imagine in the same way that the surface of a cone "is unrolled and stretched out into a plane surface" (<ἀν>ηπλωμένην καὶ εἰς ἐπίπεδον ἐκτεταμένην), it will be a circular sector (I 37). It was possible for Kepler to have access to this reading of Archimedes' circle quadrature as unrolling a circular figure into a straight one. Though Hero's *Metrica* was lost until 1896, Theon's *Commentary on Ptolemy's* Almagest was known throughout the Middle Ages, and the *editio princeps* of the Greek text was published in 1538. Still, Kepler's application of the unrolling process to solids is innovative.

Kepler's *Nova stereometria* turns out to provide a comparison to Archimedes' *Method* on a number of important questions: the composition of the continuum (recall the ambiguity when Kepler shifts between indivisible *points* and infinitesimal *parts* of a circle); imagined versus "real" figures (Kepler describes the unrolling of the circumference of a circle as *erunt bases imaginatae omnes in una recta*, "all the bases will be imagined on one straight line", 7 verso); the degree to which an account of discovery furnishes a proof; and finally, the cylinder segment, which Archimedes measures in a completely different way in the *Method*. Such a comparison could have hardly suggested itself to sixteenth- and seventeenth-century mathematicians, as they did not know the *Method*, a work long lost until its rediscovery in 1906.

[15] See Theon's commentary to *Almagest* I 4. For the word ἐξαπλουμένης meaning "unrolling", the Basel edition and Halma's critical edition has ἐξαπλουμένης (Basel edition 1538: 23, Halma 1821: 63), but Rome's critical edition has ἑξαπλουμένης, "multiplied by six" (Rome 395).

1.3 Three ways to measure the parabola: *Method* 1 and the *Quadrature of the Parabola*

The Ottoman Greek scholar Athanasios Papadopoulos-Kerameus was born in Thessaly in 1856. He was a self-taught Hellenist and devoted paleographer.[16] In 1899 he published a catalogue of the 447 manuscripts in the Metochion of the Holy Sepulchre in Constantinople. Manuscript 355, as Papadopoulos-Kerameus notes, is a prayer book made from parchments of older books. Such a recycled book is called a palimpsest, from πάλιν ψάω "I scrape over and over" in Greek, though the actual process of removing the old text consists in soaking in milk rather than scraping.[17] Through the milk bath, the old text fades but often leaves visible traces. For this reason, the new text is often written vertical to the old one to minimize confusion, which is also the case for ms. 355. Judging from the script, Papadopoulos-Kerameus estimates that the old text of ms. 355, also called the lower text, was written in the tenth century and the new text, or the upper text, was written in the thirteenth or fourteenth century. He noticed that the lower text was full of geometric figures and hard to read (τὸ μὲν παλαιὸν κείμενον, πλῆρες ὂν γεωμετρικῶν σχημάτων, ἔτι δὲ καὶ λίαν εὐανάγνωστον πολλαχοῦ, 329). As he could not identify the text, Papadopoulos-Kerameus transcribed a few lines of the lower text in the catalogue.[18]

Papadopoulos-Kerameus was not the first to notice this manuscript. When he catalogued the manuscript, a leaf was missing from it. The missing leaf was bought by the Cambridge University Library in 1876 from the executors of Constantin von Tischendorf, the person who found the Codex Sinaiticus, the oldest extant manuscript of the Bible. According to Tischendorf, during his search for the oldest Bible in the 1840s, he visited the patriarch of Constantinople,

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[16] Scanty information about his life and scholarly occupation can been gathered from a memorial note in Reinach 1913: 278-279, and n. 53 in Joassart 2010: 408.
[17] Agati 2017: 70.
[18] Papadopoulos-Kerameus 1899: 329.

who allowed him “to make any use of the manuscripts [he] found”[19] in the library. Among the 30 manuscripts there, Tischendorf found nothing interesting, “with the exception of a palimpsest upon mathematics”[20]. That is very likely the palimpsest Papadopoulos-Kerameus catalogued in 1899 and the origin of the single leaf torn off and kept by Tischendorf that now lies in the collection of the Cambridge University Library.

Upon reading Papadopoulos-Kerameus’ transcription, the Danish philologist Johan Ludvig Heiberg realized that the text was from Archimedes’ *On Sphere and Cylinder*. In the summer of 1906, Heiberg went to Constantinople to consult the palimpsest. It turns out that the lower text of the palimpsest is composed mostly of Archimedes’ works, after whom it is called the Archimedes Palimpsest. Among the seven Archimedean works in the lower text are the only extant Greek text of *On Floating Bodies* and two works thought to have been lost, the *Stomachion* and the *Method*. What awaits the manuscript afterwards, however, was the fate of theft, addition of forged gold leaf portraits, deterioration by mold, and finally, reappearance for auction in 1998 and digitization and preservation after that.[21]

After Heiberg’s identification and transcription of the *Method*, Archimedes’ way of discovery is no longer a subject of speculation. In the prefatory letter to the *Method*, Archimedes tells Eratosthenes, the recipient of the letter, that he investigated (θεωρεῖν) and discovered (εὑρίσκειν) theorems in a way different than how he proved (ἀποδεικνύναι) them. The way in which Archimedes made discoveries is showcased by the theorems in the *Method*: first, he assumes that a geometric object is “filled up with” (συμπληροῦσθαι) indivisibles of a lower dimension, as a plane is filled up with lines, a solid with planes, though sometimes he also uses the language of

[19] Tischendorf 1851: 274.
[20] Tischendorf 1851: 274.
[21] Easton and Noel 2010: 51-2.

“composed of” (συνεστάναι, συγκεῖσθαι); he then measures the area or volume by “weighing” the indivisibles on an imagined balance. If the indivisibles at one end of the balance are in equilibrium with the indivisibles on the other end, then by the law of the lever, the figure filled up with one group of indivisibles is knowable when both lever arms and the figure filled up with the other group of indivisibles are known.

The most appropriate word to characterize Archimedes’ way of discovery is not “method”, the word in the title, but “approach”. In the Archimedes Palimpsest, the full title of the *Method* was written in Alexandrian majuscule, a style of script that resembles upper case Greek letters in modern orthography, as opposed to the miniscule, or roughly speaking, “lower case letters” of the main text. The title reads, ἈΡΧΙΜΗΔΟΥΣ ΠΕΡΙ ΤΩ͂Ν ΜΗΧΑΝΙΚΩ͂Ν ΘΕΩΡΗΜΆΤΩΝ ΠΡ(ΟΣ) ΕΡΑΤΟΣΘΈΝΗΝ· ἜΦΟΔΟΣ. This means, literally, “Of Archimedes Concerning the Mechanical Theorems [Sent] to Eratosthenes, An Approach.” See below for the digitized image of the palimpsest.

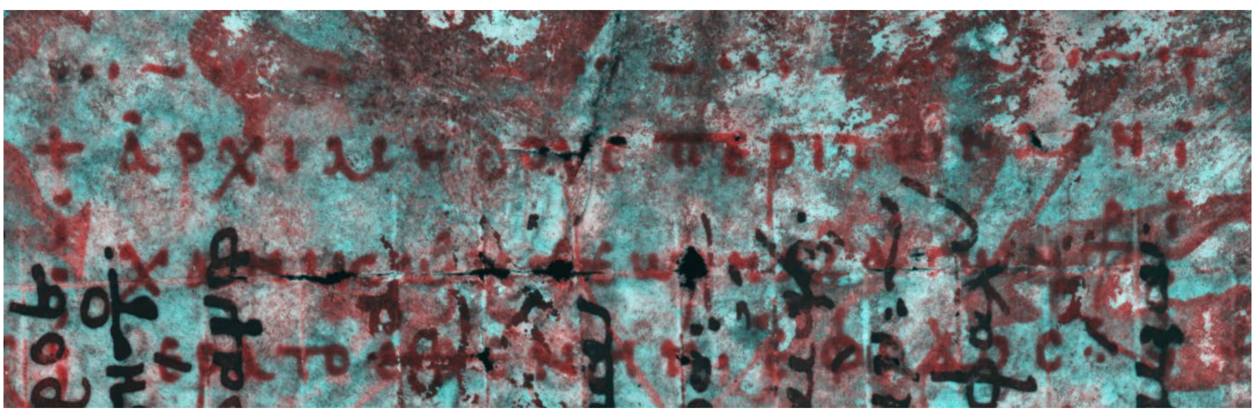

Figure 1-4 Digitized image of the Archimedes Palimpsest 46r vol. 2 lines 1-3 (046r-043v-Arch15r_Sinar_pesudo_no-veil)

While there is a middle point before the last word ἔφοδος, the degree to which this point marks a separation is unclear. Punctuation in medieval manuscripts was by no means standardized, and a middle point could denote a pause like that of a comma, or a weaker one like breathing pause, or it could be more of a visual or mental function like marking out a grammatical structure, etc.[22] It is possible that the words before the middle point are scribal intervention and the word ἔφοδος after it is the original title, though the presence of the middle point itself is not decisive evidence.[23]

In the 1913 Teubner edition, the word ἔφοδος in the title of the Archimedean treatise was translated as *methodus* in Latin by the editor and translator Heiberg. In 1907 Heiberg published the Greek text with a German translation by Zeuthen, and the latter used the word *Methodenlehre* to render ἔφοδος in the title[24]. The 1907 French translation by Théodore Reinach, the 1912 English translation by Sir Thomas Heath, and the 1927 Italian translation by Enrico Rufini followed Heiberg and Zeuthen in choosing the word *méthode*/method/*metodo*.[25] However, the word ἔφοδος means more accurately "approach" than "method", as the latter in modern scientific and philosophical context means a set of norms, procedures, and systematic views that facilitate or are expected to facilitate scientific discourse. The Greek ἔφοδος, on the other hand, is used more loosely by Greek authors, and its precise semantic connotations in Archimedes is difficult to decipher, as the word only appears once in the extant Archimedean corpus, viz. in the title under discussion. In the main text, however, the original word translated by Heiberg as *methodus* is τρόπος, which is used more generally and loosely than ἔφοδος, meaning "way, manner". Thus, the conventional rendering of ἔφοδος and τρόπος as method is not unquestioned. For example, Eberhard Knobloch thinks that the more accurate rendering of ἔφοδος would be "access, approach"[26] in English, and in Latin "aditus"[27], and he further observes that "[w]henever [Archimedes] spoke of the method we nowadays call 'mechanical method' in this treatise, he

[22] Parkes 2016 introduction. Cf. Thompson 1966: 60.
[23] Netz forthcoming.
[24] Heiberg and Zeuthen 1907.
[25] See Reinach 1907; Heath 1912; Rufini 1926.
[26] Knobloch 2000: 83.
[27] Knobloch 2000: 83.

used the word τρόπος, not ἔφοδος, without ever adding the attribute μηχανικός, mechanical."[28] It is true that Archimedes is remarkably reluctant to qualify his τρόπος. Whenever he refers to his τρόπος, he uses a demonstrative article, e.g., "through this way" (διὰ τούτου τοῦ τρόπου), a definite article, e.g., "through the way", "after writing down the way" (διὰ τοῦ τρόπου, τὸν τρόπον ἀναγράψας), or a phrase like "through the way shown" (διὰ τοῦ ἀποδειχθέντος τρόπου), never defining or describing the τρόπος of his investigation. Even the very first appearance of the word τρόπος is strikingly indefinite, when Archimedes says that he decided to write down and set forth the "peculiarity of *a certain* τρόπος" (τρόπου τινὸς ἰδιότητα H2.428.22 = 46v col. 2.3; my italic). These all suggest that Archimedes took his τρόπος as a way of thinking rather than a systematic method. Although for the title of the treatise I choose to follow the conventional translation "method" to render the Greek ἔφοδος for the sake of convenience, in my discussion of this work I opt for the word "approach" as a translation of Archimedes' τρόπος.

Archimedes' strategy of conveying his approach to the reader is not by definition and description, but through examples. The first example he gives, *Method* 1, is the quadrature of a parabolic segment. This example is uniquely important compared to other examples in the *Method*. According to the preface to the *Method*, the quadrature of a parabolic segment is a seminal finding in Archimedes' career:

> Γράφομεν οὖν πρῶτον τὸ καὶ πρῶτον φανὲν διὰ τῶν μηχανικῶν, ὅτι πᾶν τμῆμα ὀρθογωνίου κώνου τομῆς ἐπίτριτόν ἐστιν τριγώνου τοῦ βάσιν ἔχοντος τὴν αὐτὴν καὶ ὕψος ἴσον, μετὰ δὲ τοῦτο ἕκαστον τῶν διὰ τοῦ αὐτοῦ τρόπου θεωρηθέντων· (H2.430.19-23 = 57r col. 1.10-18)
>
> I therefore write down, first, that which is also the first finding that appeared to me through mechanics, namely every parabolic segment is 4/3 the triangle of the same base and height, and after this, each of the findings investigated in the same way.

[28] Knobloch 2000: 83.

These words lend an auto-biographical color to the *Method*. The quadrature of a parabolic segment is above all the start of a series of findings discovered in the same way. And the *Method* provides a collection of these findings. For some of these findings, we find rigorous counterparts in Archimedes' other works such as *On Sphere and Cylinder* and *On Conoids and Spheroids*, while rigorous proofs of other findings, viz., theorems about the measurement of the center of gravity of solids, cannot be found in his extant works, but are possibly included in his lost works on mechanics. What is unique about *Method* 1 is that its rigorous counterpart is not one, but two demonstrations, which compose the two halves of Archimedes' *Quadrature of the Parabola*. The first, by preserving the mechanical element of an imagined balance and the way of inscribing and circumscribing polygons, is a rigorized and generalizable development from *Method* 1, whereas the second, though strictly geometric, is uninformative about the process of discovery, and its approach is also case-specific.

A parabolic segment is the finite figure bounded by a parabola and a straight line that intersects the parabola in two different points. As the figure is bounded by two lines, let the straight one be called its base, the curved one its parabolic bound, the points at which the two intersect its two ends, and the point on the parabolic bound that has the greatest distance from the base its vertex. The greatest triangle that can be inscribed in the segment has the vertex and the two ends as its three vertices. Archimedes proves that the area of a parabolic segment is 4/3 that of the greatest inscribed triangle.

Certain properties of the parabola are assumed as known in both the *Method* and the *QP*. Below I will go through these properties while introducing the geometric configuration of *Method* 1. A notable trait of the ἔκθεσις "setting-out" part of ancient Greek mathematical writing is naming figures with points whose determination depends upon the figures they name. For instance, two lines ABC and DBE may be named first and their intersection B later, though B is already used to define the two lines. Rather than considering B as part of the name of the two lines, I think the name ABC is iconic in showing that B is between A and C, and the position of B is indefinite until further configurations are introduced.[29] I will keep this feature as I introduce the figure below.

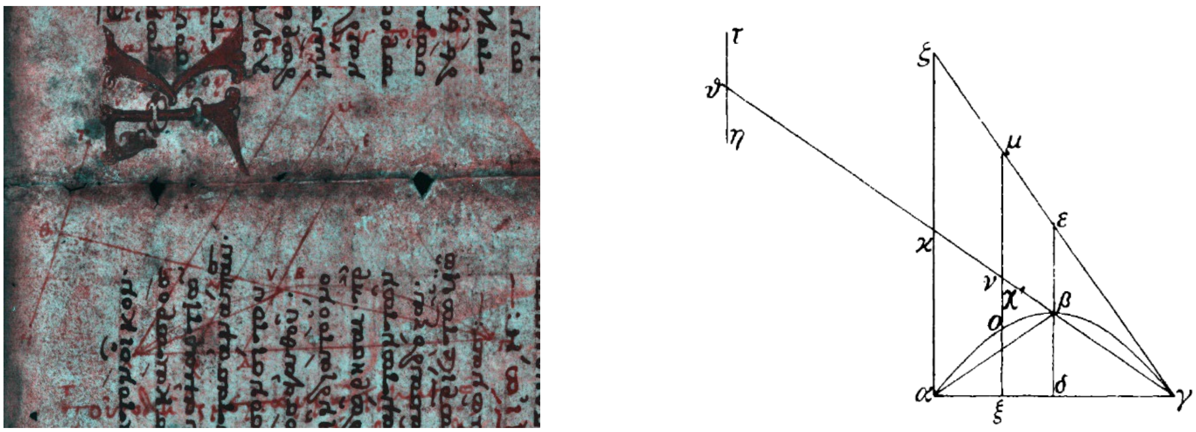

Figure 1-5 left: Digitized image of the diagram for *Method* 1 in the Archimedes Palimpsest (66r-71v_Arch17r_Sinar_pseudo_no-veil); right: Diagram for *Method* 1 in Heiberg's edition (vol. 2 p.435).

Let $\alpha\beta\gamma$ be a parabolic segment bounded by the straight line $\alpha\gamma$ and let β be the vertex of the segment. Draw straight lines $\delta\beta\varepsilon$ through the vertex β and $\alpha\kappa\zeta$ through one of its ends α parallel to the axis of the parabola, and let the tangent to the parabola at the other end γ intersect $\delta\beta\varepsilon$ at ε and $\alpha\kappa\zeta$ at ζ. Extend $\gamma\beta$ to meet $\alpha\zeta$ at κ. The first three propositions of *QP* are proved in works on the elements of conics. The first two propositions say that δ, the intersection of $\delta\beta\varepsilon$ and $\alpha\gamma$, is the midpoint of $\alpha\gamma$, and that β is the midpoint of $\delta\beta\varepsilon$. This means that $\gamma\kappa$ is the midline of triangle $\alpha\gamma\zeta$ and triangle $\alpha\beta\gamma$ is a quarter of triangle $\alpha\gamma\zeta$. Thus, it amounts to prove that the parabolic segment $\alpha\beta\gamma$ is a third of triangle $\alpha\gamma\zeta$. Now through any point ξ on $\alpha\gamma$ draw a line $\mu\xi$ parallel to $\varepsilon\delta$ and let it intersect $\gamma\zeta$, $\gamma\alpha$, $\gamma\kappa$, and the parabola $\alpha\beta\gamma$ at μ, ξ, ν, and o. Archimedes proves that $o\xi : \mu\xi :: \alpha\xi :$

[29] Cf. Acerbi 2020: 48.

$\alpha\gamma :: \kappa\nu : \gamma\kappa$. So far, we have gone through all the properties and relations needed to prove the theorem.

In *Method* 1, Archimedes proves the theorem in the following way. First, he extends $\gamma\kappa$ to θ so that $\gamma\theta$ is twice $\gamma\kappa$. The line $\gamma\theta$ serves as an imagined balance. As introduced above, Archimedes has shown that for any line $\mu\xi$ parallel to $\varepsilon\delta$ and intersecting $\gamma\zeta$, $\gamma\alpha$, $\gamma\kappa$, and the parabola $\alpha\beta\gamma$ at μ, ξ, ν, $o\xi : \mu\xi :: \alpha\xi : \alpha\gamma :: \kappa\nu : \kappa\theta$. This means if $o\xi$ is moved to θ, it will be in equilibrium with $\mu\xi$ around point κ. If we move to point θ all the line segments that make up (συνέστηκε) the parabolic segment $\alpha\beta\gamma$, they will balance all the line segments that make up the triangle $\alpha\gamma\zeta$. Since the triangle's center of gravity χ lies at one-third of $\kappa\gamma$ from κ, the parabolic segment $\alpha\beta\gamma$ is a third of the triangle $\alpha\gamma\zeta$, and therefore 4/3 of the triangle $\alpha\beta\gamma$.

The two most interesting features of this approach are also those which make it deviate from a standard geometric demonstration: the introduction of the balance and the relation between lines and the plane figure in which the lines are. Both topics will recur throughout this dissertation. Below I will focus on the language with which Archimedes introduces the imagined balance into geometry. In *Method* 1, the balance is introduced in the following words:

> ἐκβεβλής<θω ἡ ΓΒ ἐπὶ τὸ Κ, καὶ κείσθω τῇ ΓΚ ἴση ἡ ΚΘ>. Νοείσθω ζυγὸς ὁ ΓΘ καὶ μέσον αὐτοῦ τὸ Κ καὶ τῇ ΕΔ παράλληλος τυχοῦσα ἡ ΜΞ.
>
> Let line ΓΒ be extended to Κ and let line ΚΘ equal to ΓΚ be placed [along ΓΚ]. Let a balance ΓΘ be imagined and its fulcrum Κ [be imagined] and a random line ΜΞ parallel to ΕΔ [be imagined].

Several words and phrases in my translation need explanations. First, the passive imperative verb νοείσθω, from νοέω, to see with one's νόος—the verb's cognate noun meaning "mind", does *not* have a copulative force here. That is to say, it does *not* mean "let X be imagined as Y" but means "let something be imagined". The thing imagined is a balance, and it is specified as ΓΘ. This is seen in the definite article ὁ in the expression ὁ ΓΘ. Ancient Greek mathematical writings specify objects by a cluster of definite article + letter(s), often without a noun. And the definite article, in combination with the geometric configuration, denotes what the object is, e.g., ἡ AB denotes the line AB, as the Greek word for line γραμμή is feminine; and ὁ ΑΒΓ—if in the configuration there is a circle through the three points A, B, Γ—denotes the circle ΑΒΓ, as the Greek word for circle κύκλος is masculine. In *Method* 1, the masculine article ὁ in ὁ ΓΘ can only refer to ζυγὸς. Despite the definite article, the words ζυγὸς ὁ ΓΘ go together as an indefinite grammatical subject of the sentence, "a balance ΓΘ".[30] A contrast can be made with some other theorems in the *Method*. In *Method* 6, 9, and 12, the language used to introduce the balance is slightly different. For example, in *Method* 6, the words are νοείσθω ζυγὸς ἡ ΘΓ εὐθεῖα, "let the straight [line] ΘΓ be imagined as a balance". There the grammatical subject is ἡ ΘΓ εὐθεῖα, "the straight [line] ΘΓ", ζυγὸς is the predicate, and the verb νοείσθω has a copulative force, "let X be imagined as Y". The stake of this linguistic analysis is an ontological one. In ancient Greek geometry, when geometric objects are introduced, the default language is ἔστω "let there be"; and in the *Method*, when Archimedes introduces a balance to help with geometry, the balance is always imagined: either the balance is imagined to be, or that which is primarily a geometric object, namely a line, is taken as a balance through imagination.[31]

Another expression that needs explanation is μέσον αὐτοῦ τὸ Κ, to which the noun σημεῖον "point", agreeing with the neuter adjective μέσον, is to be supplied. With the main verb νοείσθω supplied, this expression literally means "let point K be imagined as its midpoint". But according to the construction, K *is* the midpoint of ΓΘ and need not to be imagined as such. The expression μέσον αὐτοῦ τὸ Κ is rather an abbreviated version of μέσον αὐτοῦ ἔστω τὸ Κ καὶ κρεμάσθω κατὰ τὸ Κ, "let K be the midpoint of the balance and let the balance be suspended at K", which

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[30] See Acerbi 2020 on the indefiniteness of such expressions.
[31] See Chen 2023 for a discussion of the diagrams for *Method* 12.

amounts to say that K is the fulcrum. The full expression is used in proposition 7 of the *QP*: μέσον αὐτοῦ ἔστω τὸ Β καὶ κρεμάσθω κατὰ τὸ Β, "let B be its midpoint and let it [= the balance] be suspended at B." After a few repetitions of this expression, Archimedes shifts to the shorter expression μέσον δὲ αὐτοῦ τὸ Β to indicate that B is the fulcrum. In the *Method*, he also uses the expression of midpoint to mean fulcrum of a balance.

As mentioned earlier, the *QP* contains two approaches to measure the parabolic segment. The first seventeen propositions comprise the first, mechanical one [*QP* 1], and the eighteenth to twenty-fourth propositions comprise the second, geometric one [*QP* 2]. In *QP* 1, the balance is introduced in a more empirical language than that used in *Method* 1:

> Νοείσθω δὲ τὸ [ὅτε ἐστὶν τὸ ἐν τᾷ θεωρίᾳ] προκείμενον [ὁρώμενον] ἐπίπεδον ὀρθὸν ποτὶ τὸν ὁρίζοντα, καὶ τᾶς ΑΒ γραμμᾶς [ἔπειτα] τὰ μὲν ἐπὶ τὰ αὐτὰ τῷ Δ κάτω νοείσθω, τὰ δὲ ἐπὶ θάτερα ἄνω, τὸ δὲ ΒΔΓ τρίγωνον ἔστω ὀρθογώνιον ὀρθὰν ἔχον τὰν ποτὶ τῷ Β γωνίαν καὶ τὰν ΒΓ πλευρὰν ἴσαν τᾷ ἡμισείᾳ τοῦ ζυγοῦ [δηλονότι ἴσης οὔσης τᾶς ΑΒ τῇ ΒΓ], κρεμάσθω δὲ τὸ τρίγωνον ἐκ τῶν Β, Γ σαμείων, κρεμάσθω δὲ καὶ ἄλλο χωρίον τὸ Ζ ἐκ τοῦ ἑτέρου μέρεος τοῦ ζυγοῦ κατὰ τὸ Α, καὶ ἰσορροπείτω τὸ Ζ χωρίον κατὰ τὸ Α κρεμάμενον τῷ ΒΔΓ τριγώνῳ οὕτως ἔχοντι, ὡς νῦν κεῖται. Φαμὶ δὴ τὸ Ζ χωρίον τοῦ ΒΔΓ τριγώνου μέρος τρίτον εἶμεν. (H2.272.11-23)

> Let a plane be imagined perpendicular to the horizon, and let the part of line AB that is on the same side with Δ be imagined to be below, and the other side above, and let there be a right triangle BΔΓ of which angle B is right and the side BΓ is equal to a half of the balance, and let the triangle be suspended from points B, Γ, and let another area Z be suspended from another part of the balance at point A, and let the area Z suspended at A be in equilibrium with the triangle BΔΓ in the way as it is now placed. I say, then, the area Z is a third of the triangle BΔΓ.

This is the πρότασις "enunciation" part of proposition 6 of the *QP*. The language is similar to *Method* 1 in its use of the word νοείσθω, "let it be imagined". Apart from that, the language is much more redundant than that of *Method* 1 and relies on a more empirical conception of balance and space. Several features are notable. First, Archimedes bothers to define "above" and "below" by mapping topological relations of the geometric configuration into those of the physical space, rather than just assuming that the reader will read the direction from the diagram.

He seems also to restrict himself to a horizontal position of the balance as an indicator of equilibrium. Most relevant to my previous discussions is the implicit exposition of the balance. It is to be inferred from the configuration and the conclusion that the balance is made up of two equal parts, line AB on the one hand and a side ΒΓ of the triangle ΒΔΓ on the other. Point B is also to be inferred to be the fulcrum. Interestingly, the side ΒΓ is said to be *equal* to a half of the balance, i.e. line AB, as if there is another line, the line of the balance, and the side ΒΓ of the triangle ΒΔΓ, by equality, coincides with a half of it.

After this introduction of the balance, the subsequent theorems in the *QP* use a language very close to that of *Method* 1. The enunciation of proposition 7 says

> Ἔστω πάλιν ζυγὸς ἁ ΑΓ γραμμά, μέσον δὲ αὐτᾶς ἔστω τὸ Β, καὶ κρεμάσθω κατὰ τὸ Β.
>
> Again, let line ΑΓ be a balance, and let point B be its midpoint, and let the balance be suspended at B.

The expression of midpoint, as explained earlier, becomes the way in which Archimedes refers to the fulcrum of the balance. A glaring divergence from *Method* 1 is the imperative ἔστω. It seems that Archimedes shifts back to the default language of Greek mathematical writings, once the distinction between geometric objects and the imagined balance had been made in the preceding proposition. In the *Method*, the word νοείσθω is used in a way syntactically equivalent to ἔστω: νοείσθω is usually the first word of the sentence and states the existence, albeit a mental one, of certain objects, which are the grammatical subjects of the sentence, each followed by a specific designation comprised of a definite article and one or more letters. From a syntactical perspective, νοείσθω functions as a mental version of ἔστω in Archimedes.

The approach of *QP* 1 is in a way strikingly similar to that of *Method* 1. *QP* 1 shares with *Method* 1 the same mechanical element of the imagined balance, replaces the parallel line in *Method* 1 with indefinitely thin parallel slices, and by using the method of exhaustion, avoids the controversy about the idea that a plane figure is composed of parallel lines. Below I give a detailed account of *QP* 1, with the hope that its essential similarity to *Method* 1 could be evident through such an account. To show the similarity more straightforwardly, I adapt the diagram of *Method* 1 into ones that suit the argument of *QP* 1. Manuscript diagrams of *QP* 1 will also be included later in this chapter.

Archimedes divides the parabolic segment $\alpha\beta\gamma$ and the triangle $\alpha\gamma\zeta$ each into an indefinite number *of* slices of equal width. The number of slices can be greater than any given number, so the parts can be smaller than any given quantity. The division is made in this way: divide $\alpha\zeta$ into n-many equal segments at $\pi_1, \pi_2, \ldots, \pi_{n-1}$. For each i, the line $\gamma\pi_i$ intersects the parabolic segment at o_i. Through o_i draw $\mu_i\xi_i$ parallel to the axis of the parabola intersecting $\alpha\gamma$ at ξ_i, $\gamma\zeta$ at μ_i, $\gamma\theta$ at ν_i. Let ρ_i be the intersection of $\mu_i\xi_i$ and $\gamma\pi_{i+1}$. By proposition 5 of the *QP*, $\xi_1, \xi_2, \ldots, \xi_{n-1}$ are also the n-division points of $\alpha\gamma$. Hence the lines $\mu_i\xi_i$ divide the triangle $\alpha\gamma\zeta$ into n-many trapezia of equal width and the parabolic segment $\alpha\beta\gamma$ into n-many parts of equal width.

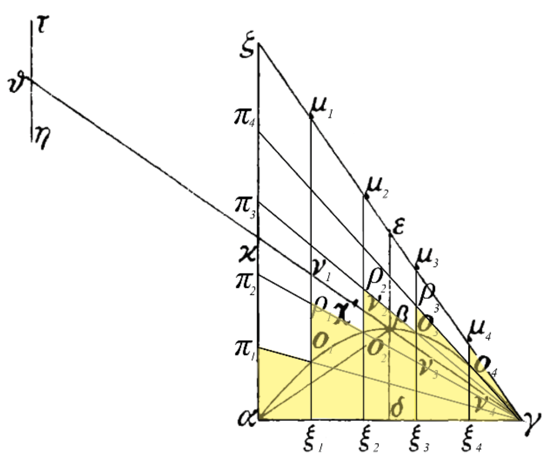

Figure 1-6

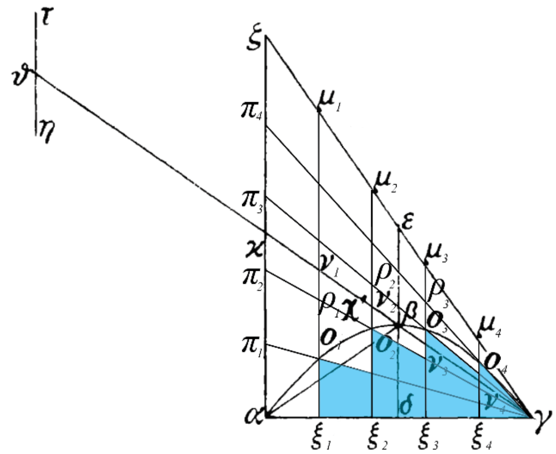

Figure 1-7

Consider the trapezia $\pi_1\xi_1$, $\rho_1\xi_2$, $\rho_2\xi_3$, …, $\rho_{n-2}\xi_{n-1}$, and the triangle $\mu_{n-1}\xi_{n-1}\gamma$ circumscribed about the parabolic segment $\alpha\beta\gamma$ (the yellow ones in Figure 1-7). In *Method* 1, Archimedes shows that $o_i\xi_i : \mu_i\xi_i :: \alpha\xi_i : \alpha\gamma$, and this is used again in *QP* 1. Taking this together with the fact that the ratio of trapezium $\rho_i\xi_{i+1}$ to trapezium $\mu_i\xi_{i+1}$ is the same as $\rho_i\xi_i : \mu_i\xi_i$, which is again the same as $o_{i+1}\xi_{i+1} : \mu_{i+1}\xi_{i+1}$, we have the following: if we move the trapezium $\rho_i\xi_{i+1}$ so that its center of gravity is at point θ, it will balance the corresponding trapezium $\mu_i\xi_{i+1}$ in triangle $\alpha\gamma\zeta$ when the center of gravity of trapezium $\mu_i\xi_{i+1}$ is moved to the midpoint of $\mu_{i+1}\xi_{i+1}$, that is, *further away from* the fulcrum κ than its original position (let α be ξ_0 and π_1 be ρ_0 for the sake of generality). Hence, if we move all the circumscribed trapezia so that their center of gravity is at point θ, they will balance triangle $\alpha\gamma\zeta$ when it is placed further away from the fulcrum κ. Since the center of gravity χ of triangle $\alpha\gamma\zeta$ lies on the one-third point of $\gamma\kappa$, the sum of all the circumscribed trapezia is greater than a third of the triangle $\alpha\gamma\zeta$. Similarly, the sum of all the inscribed trapezia (the blue ones in Figure 1-8) is less than a third of the triangle $\alpha\gamma\zeta$.

Archimedes then uses the method of exhaustion to prove that the area of the parabolic segment is neither smaller nor greater than a third of the triangle *αγζ*. The difference between the circumscribed and the inscribed trapezia are shown by the yellow trapezia in Figure 1-9 below, the sum of which is an *n-th* of the triangle *αγζ*. Since the number *n* is indefinite, the difference can be indefinitely small, in the sense that for any given quantity *Q* there is an *n* such that the difference is smaller than *Q*.

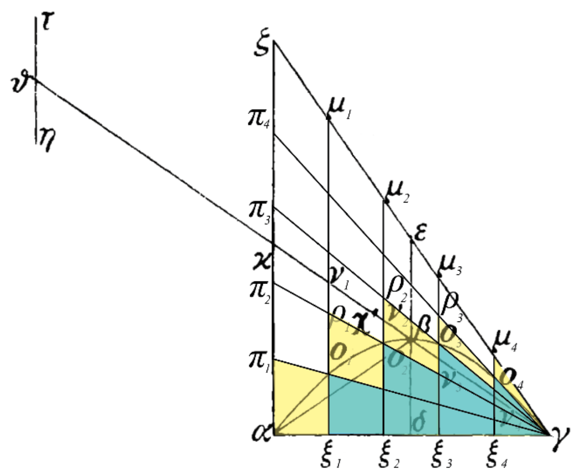

Figure 1-8 Showing both the circumscribed and the inscribed trapezia in one diagram

If the parabolic segment is smaller than a third of triangle *αγζ*, then there is an *n* such that the difference *δ* between the inscribed and the circumscribed *n*-many trapezia is less than the difference *Δ* between the parabolic segment and one-third of triangle *αγζ*. Since the sum of all the circumscribed trapezia is greater than a third of triangle *αγζ*, this means the parabolic segment (= 1/3 S_{Δ} *αγζ* - *Δ*) is less than the inscribed trapezia (= circumscribed trapezia - *δ*), which is a contradiction. If, on the other hand, the parabolic segment is greater than one-third of triangle *αγζ*, then there is an *n* such that the difference between the inscribed and the circumscribed *n*-many trapezia is smaller than the difference between the parabolic segment and one-third of triangle *αγζ*. Since the sum of all the inscribed trapezia is less than a third of triangle *αγζ*, this means the parabolic segment (= 1/3 S_{Δ} *αγζ* + *Δ*) is greater than the circumscribed trapezia (=

inscribed trapezia + δ), which is again a contradiction. Therefore, the parabolic segment has to be a third of the triangle $\alpha\gamma\zeta$.

I hope through my explication of *QP* 1 the following has become clear: a proof in the style of *Method* 1 is convertible into another in the style of *QP* 1, and vice versa. The difference between the two lies in the choice between indefinitely thin slices and the limit of such slices, i.e., when the slices have no width. If one opts for the former, then a double *reductio* is to be given; and if one opts for the latter, the proof will benefit from its conciseness but incur controversy. By contrast, *QP* 2 does not follow the same train of thought as *QP* 1 and *Method* 1 do. In *QP* 2, the construction is close to that of the Euclidean circle quadrature, viz. by inscribing polygons and doubling the number of sides. At each step, the newly added area is a quarter of the area added at the last step. Through such a construction, *QP* 2 not only arrives at the same conclusion as *QP* 1, but also shows what is arithmetically equivalent to 1+1/4+1/16+… = 4/3. However, *QP* 2 falls short of *QP* 1 in two aspects. *QP* 2 contains no clue as to how Archimedes first discovered the result, viz. through the way of *Method* 1, but *QP* 1 preserves from *Method* 1the geometric configuration, parallel slicing, and the idea of using equilibrium to assist geometry, giving the reader a great transparency of the process of discovery. Another aspect in which *QP* 1 prevails is the generality of the approach. All the theorems in the *Method* can be converted into proofs in the style of *QP* 1, whereas *QP* 2 is case-specific. Appended to this chapter is an example that converts *Method* 2 into a proof in the style of *QP* 1.

Although *QP* 1 is essentially similar to *Method* 1, one might say, the extent to which *QP* 1 is suggestive of the original process of discovery is speculative. In light of this consideration, a striking case provided by Torricelli furnishes a vivid example of how a reader of Archimedes managed to recover *Method* 1 from *QP* 1. *De dimensione parabolae*, a part of Torricelli's 1644 book *Opera geometrica*, deals with the quadrature of the parabola in twenty different ways. The latter half of *De dimensione parabolae* has a separate title, *Quadratura parabolae per novam indivisibilium geometriam pluribus modis absoluta*, meaning *Quadrature of the Parabola Through the New Geometry of Indivisibles Solved in Many Ways*. As mentioned in section two, the new geometry of indivisibles was practiced and advocated by many early modern thinkers, in ways different from one another. The approach Torricelli adopts is that of Cavalieri's, which, in very rough terms, compares objects by slicing them into parallel indivisibles of a lower dimension, an approach very close to that showcased in Archimedes' *Method*, with a major difference in the absence of the imagined balance. Most notably, proposition XX of Torricelli's *De dimensione parabola* squares a parabolic segment in the same way as *Method* 1 does.[32] I will not recapitulate how Torricelli does it but rather present the diagram in the 1644 edition of his book for a comparison with a diagram for Archimedes' *QP* in Codex H (Codex Parisiensis 2361).

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[32] See Boyer 1959: 125, "Among the eleven demonstrations by the geometry of indivisibles, he included one which is—oddly enough—almost identical with the mechanical quadrature given by Archimedes in the Method, a work not known in the seventeenth century. This coincidence shows how closely Cavalieri's geometry of indivisibles resembled the mathematical atomism upon which Archimedes' method was probably based."

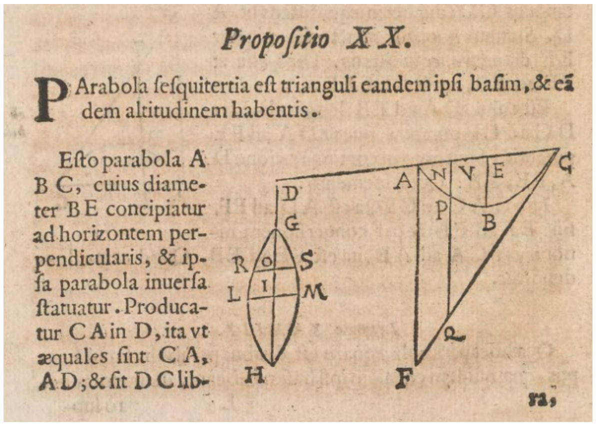

Figure 1-9 Diagram for prop. XX in the 1644 edition of Torricelli's *Opera geometrca*, p. 82

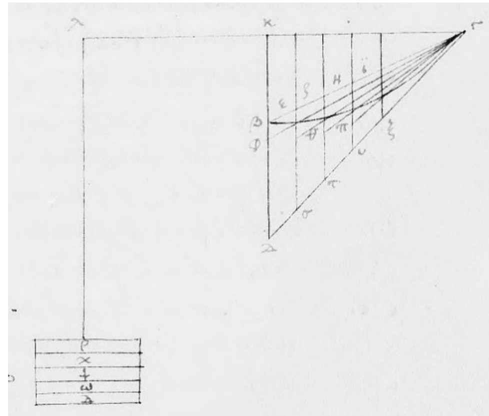

Figure 1-10 Diagram for prop. 15 in Codex Parisiensis 2361, p. 297

In both diagrams the parabola lies inverted and is positioned in a particular way such that the axis is vertical. In Torricelli's proof, an instruction that the geometric configuration should be constructed thus is meticulously made. Not only the argument, but also the structure of the balance, the position of the parabolic segment, and the spatial relations between different parts all follow *QP* 1 closely, with a change in replacing indefinitely thin slices with indivisibles. The program of Torricelli's *Opera geometrica*, just like many works of his time, is to tackle problems which Archimedes once tackled, in ways different from Archimedes', and to expand the scope to encompass problems not treated by Archimedes. As a part of such endeavor, however, proposition XX happens to reveal how Archimedes first discovered the quadrature of the parabola.

A complexity about Torricelli is that while promoting the indivisibles of Cavalieri, Torricelli does not commit himself to the view that the geometry of indivisibles is genuinely new. He says in the preface to the *Quadratura parabolae per novam indivisibilium geometriam pluribus modis absoluta* that

> Quod autem haec Indivisibilium Geometria novum penitus inventum sit, equidem non ausim affirmare. Crediderim potius veteres Geometras hac metodo usos in inventione Theorematum difficillimorum, quamquam in demonstrationibus aliam viam magis probaverint, sive ad occultandum artis arcanum, sive ne ulla invidis detractoribus proferretur occasio contradicendi. Quicquid est, certum est hanc Geometriam mirum esse pro inventione compendium, et innumera quasi imperscrutabilia Theoremata, brevibus, directis, affirmativisque demonstrationibus confirmare; quod per doctrinam antiquorum fieri minime potest. Haec enim est in Mathematicis spinetis via vere Regia, quam primus omnium aperuit, et ad publicum bonum complanavit mirabilium inventorum machinator Cavalerius. (Torricelli 1644: 56)

> However, I for my part would not dare to declare that this Geometry of Indivisibles is completely new. I would rather believe that the ancient Geometers used this method in the discovery of the most difficult Theorems, although in demonstrations they would rather endorse another way, either to conceal the secret of their art, or to avoid any opportunity for refutation to be offered to envious detractors. Whichever it is, this Geometry certainly is a marvelous shortcut for discovery and confirms countless impenetrable Theorems with brief, direct, and affirmative demonstrations; and this is hardly possible to happen through the teaching of the ancients. Indeed, this is truly the Royal way in the Mathematical thickets of thorns, which Cavalieri, the inventor of wonderful discoveries, was the first among all to open and make even for the common good.

Torricelli's pronounced preference for the new geometry over the ancients' teaching suggests a contrast with Archimedes. Right after *Method* 1, Archimedes says that *Method* 1 does not demonstrate the conclusion but only "creates a certain appearance that the conclusion is true" (ἔμφασιν δέ τινα πεποίηκε τὸ συμπέρασμα ἀληθὲς εἶναι H2.438.17-18 = 71v col. 1.20-22). It seems logical to contrast the two: while seventeenth-century mathematicians ventured to promote the new geometry at the expense of the old ideal of rigor, Archimedes, though anticipating the new geometry by more than a millennium, seemed to discredit his approach in favor of demonstration. But I suggest that Archimedes has a more complicated attitude toward the dynamics between demonstration and discovery. Archimedes' optimism in his way of discovery is no less firm than his early modern colleagues. In the preface to the *Method*, Archimedes reveals the intent of sharing his method: "for I suppose that some among the present or future generations will discover, through the approach shown, still other theorems which have not yet occurred to me" (ὑπολαμβάνω γάρ τινας ἢ τῶν ὄντων ἢ ἐπιγινομένων διὰ τοῦ ἀποδειχθέντος τρόπου καὶ ἄλλα θεωρήματα οὔπω ἡμῖν συνπαραπεπτωκότα εὑρήσειν H2.430.15-18 = 57r col. 1.6-10). I think this reflects the overarching goal of Archimedes' *Method*, and his concerns with the rigor of demonstration should be read in correlation with this goal.

Appendix: converting *Method* 2 to a demonstration in the style of *QP* 1

One can see from the above comparison that one can convert the theorems in the *Method* into demonstrations like *QP* 1 through two simple steps, (1) replacing the indivisibles with indefinitely thin slices and (2) adding a double *reductio* in the end. Let us try this formula with *Method* 2.

Figure 1-11 Digitized diagram for Method 2 in the Archimedes Palimpsest (65r-72v_Arch18r_Sinar_pseudo_no-veil)

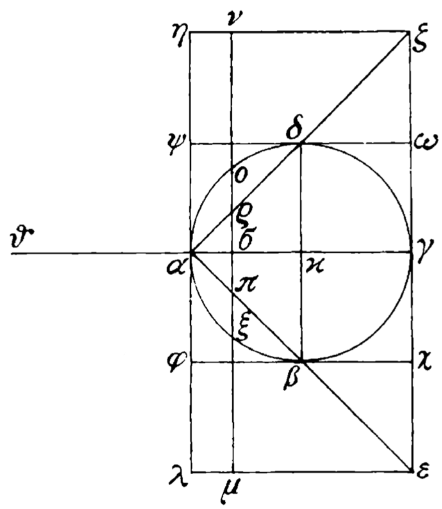

Figure 1-12 Diagram for *Method* 2 in Heiberg's edition (vol. 2 p.441)

Let κ be the center of the sphere $\alpha\beta\gamma\delta$ and $\alpha\gamma$ a diameter of the sphere. Let the plane perpendicular to $\alpha\gamma$ at κ intersect the sphere and produce a circle with a center κ and a diameter $\beta\delta$. Take α as the pinnacle and circle κ as the base, inscribe a cone $\alpha\beta\delta$ in the sphere $\alpha\beta\gamma\delta$. The aim of *Method* 2 is to show that the volume of the sphere $\alpha\beta\gamma\delta$ is four times the volume of the cone $\alpha\beta\delta$.

Archimedes draws an auxiliary cone $\alpha\varepsilon\zeta$ by extending the cone $\alpha\beta\delta$ to the plane tangent to the sphere $\alpha\beta\gamma\delta$ at γ, and a cylinder $\varepsilon\zeta\eta\lambda$ sharing the same base and axis as the cone $\alpha\varepsilon\zeta$. Since he has proved elsewhere that the cylinder $\varepsilon\zeta\eta\lambda$ is three times the cone $\alpha\varepsilon\zeta$, which is eight times the cone *αβδ*, what is needed is to prove that the cylinder *εζηλ* is two times the sum of the sphere *αβγδ* and the cone *αεζ*.

Extend *γα* to *θ* so that *αγ* = *αθ* and take *γθ* as a balance with the fulcrum at *α*. Let a random plane perpendicular to *αγ* at *σ* intersect the plane *αβγδ* at line *νμ*. Line *νμ* intersects the circle *αβγδ*, the triangle *αεζ*, and the rectangle *εζηλ* respectively at *ο* and *ξ*, *ρ* and *π*, and *ν* and *μ*. The intersections of the plane with the cone *αεζ*, the sphere *αβγδ*, and the cylinder *εζηλ* are three concentric circles on the same plane of radii *ρσ*, *οσ*, and *νσ*, and they are considered corresponding to each other. Archimedes shows that $\nu\sigma^2 : (\rho\sigma^2 + o\sigma^2) :: \alpha\gamma : \alpha\sigma$. This means circles *οσ* and *ρς* at *θ* balance circle *νσ* at its original position. If we move all the circles that make up the sphere *αβγδ* and the cone *αεζ* to point *θ*, they balance all the circles that "fill up" (συμπληρωθέντος 2.442.23) cylinder *εζηλ* at their original position. Since the center of gravity of the cylinder is the midpoint *κ* of *αγ*, sphere *αβγδ* plus cone *αεζ* is half of the cylinder *εζηλ*.

This proof can be adapted into a demonstration like *QP* 1 through the following steps.

Step 1 Replace the indivisibles with indefinitely thin slices. This is done through the following three steps.

Step 1.1 Divide $\alpha\gamma$ into n-many equal segments at σ_1, σ_2, …, σ_{n-1}. At each division point σ_i let a plane perpendicular to $\alpha\gamma$ intersect the plane $\alpha\beta\gamma\delta$ at line $\nu_i\mu_i$. Line $\nu_i\mu_i$ intersects the sphere $\alpha\beta\gamma\delta$ at o_i and ξ_i and the cone $\alpha\varepsilon\zeta$ at ρ_i and π_i. Through every o_i let there be a line $\psi_i\omega_i$ parallel to $\alpha\gamma$, through every ξ_i a line $\varphi_i\chi_i$, through every ρ_i a line $\kappa_i\lambda_i$, and through every π_i a line $\tau_i\upsilon_i$.

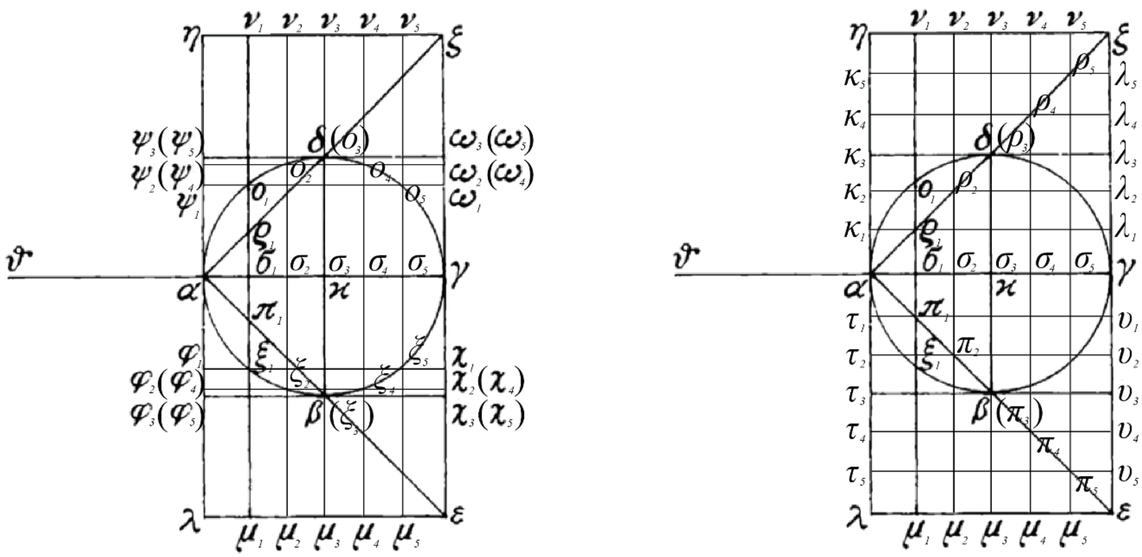

Figure 1-13

Step 1.2 Circumscribe about the sphere $\alpha\beta\gamma\delta$ cylinders with $\alpha\sigma_1$, $\sigma_1\sigma_2$, …, $\sigma_{n-1}\gamma$ as axes (the yellow ones) and inscribe in the sphere $\alpha\beta\gamma\delta$ cylinders with $\sigma_1\sigma_2$, …, $\sigma_{n-2}\sigma_{n-1}$ as axes (the blue ones). Circumscribe about the cone $\alpha\varepsilon\zeta$ cylinders with $\alpha\sigma_1$, $\sigma_1\sigma_2$, …, $\sigma_{n-1}\gamma$ as axes (the yellows ones) and inscribe in the cone $\alpha\varepsilon\zeta$ cylinders with $\sigma_1\sigma_2$, …, $\sigma_{n-2}\sigma_{n-1}$ as axes (the blue ones).

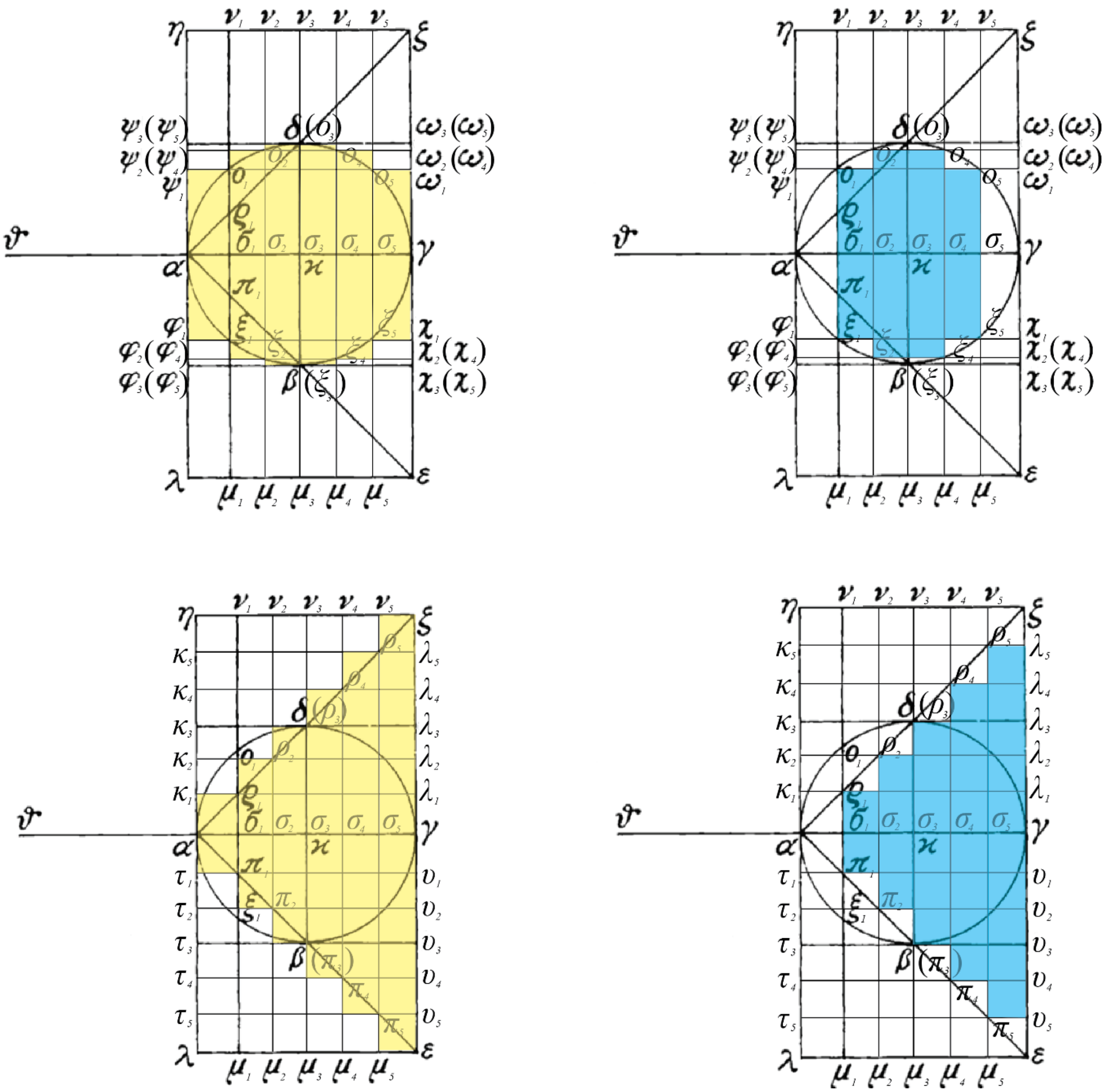

Figure 1-14

Step 1.3 Prove that the sum of all the circumscribed cylinders is greater than a half of cylinder $\varepsilon\zeta\eta\lambda$ and that the sum of all the inscribed cylinders is less than a half of cylinder $\varepsilon\zeta\eta\lambda$. In

Method 2 Archimedes shows that $\nu_i\sigma_i^2 : (\rho_i\sigma_i^2 + o_i\sigma_i^2) :: \alpha\gamma : \alpha\sigma_i$. This means if we move the cylinder with axis $\sigma_i\sigma_{i+1}$ and radius $o_{i+1}\sigma_{i+1}$ circumscribed about the sphere and the cylinder with axis $\sigma_i\sigma_{i+1}$ and radius $\rho_{i+1}\sigma_{i+1}$ circumscribed about the cone so that their center of gravity is at θ, they will balance the corresponding cylinder $\nu_i\nu_{i+1}\mu_{i+1}\mu_i$ in the cylinder $\varepsilon\zeta\eta\lambda$ when the center of gravity of cylinder $\nu_i\nu_{i+1}\mu_{i+1}\mu_i$ is moved from the midpoint of $\sigma_i\sigma_{i+1}$ to σ_{i+1}, that is, *further away from* the fulcrum α than its original position (for the sake of generality, let α be σ_0, γ be σ_n, η be ν_0, ζ be ν_n, λ be μ_0, and ε be μ_n). Hence, if we move all the circumscribed cylinders so that their center of gravity is at θ, they will balance cylinder $\varepsilon\zeta\eta\lambda$ when it is placed further away from the fulcrum α. Since the center of gravity of cylinder $\varepsilon\zeta\eta\lambda$ is the midpoint κ of $\alpha\gamma$, the sum of all the circumscribed cylinders is greater than a half of cylinder $\varepsilon\zeta\eta\lambda$. Similarly, the sum of all the inscribed cylinders is less than a half of cylinder $\varepsilon\zeta\eta\lambda$.

Step 1.4 Prove that the difference between the sum of all the circumscribed cylinders and the sum of all the inscribed cylinders is indefinitely small. The difference between the circumscribed and inscribed cylinders are the yellow cylinders and rings above. In the case of sphere $\alpha\beta\gamma\delta$, every ring of difference is equal to a parallel ring in either cylinder $\eta\nu\mu_1\lambda_1$ or cylinder $\nu_{n-1}\zeta\varepsilon\mu_{n-1}$,

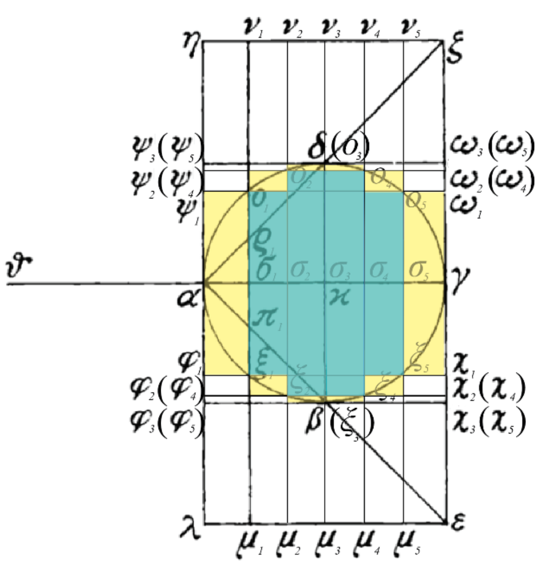

Figure 1-15

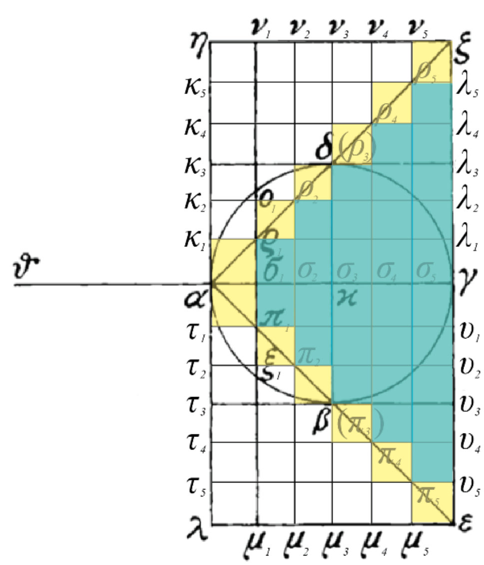

Figure 1-16

so that the sum of the difference is equal to (when *n* is an even number) or smaller than (when *n* is an odd number) two cylinders with *βδ* as the base and $\alpha\sigma_1$ as the axis, the sum of which is a half of an *n*-th of cylinder *εζηλ*. In the case of cone *αεζ*, every ring of difference is equal to a ring in cylinder $\eta\nu\mu_1\lambda_1$, so that the sum of the difference is equal to cylinder $\eta\nu\mu_1\lambda_1$, which is an *n*-th of cylinder *εζηλ*. Thus, the difference between the sum of all the circumscribed cylinders and the sum of all the inscribed cylinders is one and a half of an *n*-th of cylinder *εζηλ*, which can be made smaller than any given quantity by choosing the appropriate *n*.

Step 2 Add a double *reductio*:

If the sum of sphere *αβγδ* and cone *αεζ* is smaller than a half of cylinder *εζηλ*, then there is an *n* such that the difference *δ* between the inscribed and the circumscribed *n*-many cylinders is less than the difference *Δ* between the sum of sphere *αβγδ* and cone *αεζ* and a half of cylinder *εζηλ*. Since the the sum of all the circumscribed cylinders is greater than a half of cylinder *εζηλ*, this means the sum of sphere *αβγδ* and cone *αεζ* (= 1/2 *V εζηλ* - *Δ*) is less than the inscribed cylinders (= circumscribed cylinders - *δ*). If the sum of sphere *αβγδ* and cone *αεζ* is greater than a half of cylinder *εζηλ*, then there is an *n* such that the difference between the inscribed and the circumscribed *n*-many cylinders is smaller than the difference between the sum of sphere *αβγδ* and cone *αεζ* and a half of cylinder *εζηλ*. Since the sum of all the inscribed cylinders is less than a half of cylinder *εζηλ*, this means the sum of sphere *αβγδ* and cone *αεζ* (= 1/2 *V εζηλ* + *Δ*) is greater than the circumscribed cylinders (= inscribed cylinders + *δ*). Therefore, the sum of the sphere *αβγδ* and the cone *αεζ* has to be a half of the cylinder *εζηλ*.

Other proofs in the *Method* can be converted into proofs like *QP* 1 in the same way. What is unique in each proof is the ratio between indivisible lines or planar sections that can be visualized through a balance, yet all these lines and plane figures can be replaced with indefinitely thin slices in such a way that the original ratio is preserved in the relation between corresponding slices. Among the other proofs in the *Method*, the last four theorems about the cylinder hoof (12-15) and the four on the measurement of the center of gravity (5, 6, 9, 10) are not published elsewhere, and the rest (2, 3, 4, 7, 8, 11) are cubature problems, the results of which are demonstrated in Archimedes' other works through the method of exhaustion (*On the Sphere and Cylinder* I 21-32, *On Conoids and Spheroids* 27, 21, *On the Sphere and Cylinder* II 2, *On Conoids and Spheroids* 29 -32, and 25 respectively).

2. Archimedes and the ideal of mathematical purity

"Certain things, which first became clear to me in a mechanical way, were later demonstrated geometrically, because an investigation through this approach is separate from demonstration." Thus speaks Archimedes in the preface to his *Method*. According to existing scholarship, Archimedes acknowledges in saying so that his approach of discovery does not furnish a demonstration in the strict sense. Right after the first example proof *of* his approach, Archimedes comments that the investigation through his approach is not a demonstration but only creates "a certain indication that the conclusion is right", and that a demonstration is needed for the conclusion to be considered fully valid. In contrast to the traditional scholarly opinion, I contend that Archimedes' distinction of discovery and demonstration does not imply the deficiency of his heuristic approach; rather, with such statements, Archimedes distinguishes discovery from demonstration as an independent activity that aims at and leads to breakthroughs. I hope to show that Archimedes considers the introduction of mechanical elements instrumental and fruitful in making new discoveries in mathematics, by examining Archimedes' own discussions of his approach (section 1) and by showing that Archimedes' emphasis on discovery and the discourse about purity in modern mathematics can enlighten each other (section 2).

2.1 Archimedes' distinction between discovery and demonstration

The very first introducing phrase of Archimedes' approach is τρόπου τινὸς ἰδιότητα "the peculiarity of a certain approach", and Archimedes says that by traveling along it one will be able to grasp the *starting points* of the capacity to investigate mathematics through mechanics (καθ' ὅν ἐπιπορευόμενον ἔσται λαμβάνειν ἀφορμὰς εἰς τὸ δύνασθαί τινα τῶν ἐν τοῖς μαθήμασι θεωρεῖν διὰ τῶν μηχανικῶν; my emphasis). The word ἀφορμὰς "starting points" strikes me as envisaging a sufficiently developed art of discovery that starts with and goes beyond the "peculiarity of a certain approach" presented in the Method. The realization of such a vision is nonexistent in the extant corpus of Archimedes or ancient Greek mathematics at large, but the visionary nature of Archimedes' words is to be taken account of and makes a difference to our understanding of the purpose and nature of the *Method*. Below I contend that Archimedes' distinction between discovery and demonstration is not to be interpreted as an acknowledgment of the inferiority of his approach of discovery, but should be understood as what it is *prima facie*, i.e., a distinction between two different forms of intellectual endeavor. This reading is in alignment with the general goal of the *Method*, which is to bring attention to discovery, which has always been understudied in ancient and modern times alike.

There is an intertextuality between Archimedes' *Method* and *QP*. In the *Method* he makes the distinction that the same conclusions *became clear* to him in a *mechanical* way and *were demonstrated* in a *geometric* way. Two very similar, almost verbatim versions of this statement appear in his preface to the *QP*. In the *QP* he gives two ways to approach the quadrature of the parabolic segment, *QP* 1 and *QP* 2, both of which he calls demonstrations (ἀποδείξιας). In the preface to the *QP*, he also makes the distinction between discovery through mechanics and demonstration through geometry, referring to *QP* 1 and *QP* 2 respectively. Given the parallelism between *Method* 1 and *QP* 1, Dijksterhuis concludes that it is their difference, the indivisibles, not the balance, that makes the theorems in the *Method* considered by Archimedes to fail to be a demonstration. As Dijksterhuis argues, *Method* 1 is proved alternatively "by means of statical considerations, but this time without indivisibles"[33] in the *QP*, a work that "constitutes an official publication satisfying all requirements of exactness"[34].

[33] Dijksterhuis 2014: 319.
[34] Dijksterhuis 2014: 319.

This interpretation does not offer a satisfactory account of the distinction between discovery and demonstration, which Archimedes stresses repeatedly and eloquently. As shown in chapter one, *Method* 1 is convertible into *QP* 1 through merely formal modifications, so are other theorems in the *Method*. *Method* 1 and *QP* 1 are essentially the same and the indivisibles are no more than a short cut. Knorr 1996 argues against Dijksterhuis that the indivisibles are "merely a secondary aspect" (73) of Archimedes' approach, since Archimedes refers to his approach twice "by the phrase *dia tôn mêchanikôn*, never by a phrase denoting indivisibles" (73). Knorr also stresses the distinction between geometry and mechanics made in both the Method and the QP and makes the following inference:

> If the mechanical treatment were formally acceptable, there would be no need for a second "geometric" demonstration. By attaching it, Archimedes apparently wishes to forestall possible objections to the assumption of mechanical properties, like weight and equilibrium, in demonstrations dealing exclusively with geometric properties of figures. (1996: 73)

I agree with Knorr's emphasis on the distinction between discovery through mechanics and demonstration through geometry. But I do not think *QP* 2 is added due to formal considerations as Knorr suggests. As shown in chapter one, *QP* 2 and *QP* 1 use essentially different arguments, and *QP* 2 continues the practice of inscribing polygons inherited from circle quadrature. This suggests at least two aspects in which *QP* 2 is valuable in itself: a different argument for the same conclusion and a benevolent interaction with the tradition before Archimedes. A stronger disagreement I hold against Knorr and existing scholarly opinions in general is the implicit assumption that what is at stake is acceptability of a mathematical argument. Acceptability, whether formally or not, is only a marginal concern to Archimedes. Instead, it is the distinction of discovery *from* demonstration that is at stake.

Archimedes makes the distinction of discovery from demonstration three times in the *QP* and the *Method*. In Archimedes' own expressions, the verbs associated with mechanics are θεωρεῖν “to contemplate” and εὑρίσκειν “to discover”, and those with geometry are ἀποδεικνύναι and ἐπιδεικνύναι, which can both mean “to show”. It turns out to be a pitfall that ἀπόδειξις, the noun cognate of ἀποδεικνύναι, is the standard term for demonstration in mathematics, but δεικνύναι, not ἀποδεικνύναι, is the standard verb that means to demonstrate in Classical and Hellenistic mathematics. The word ἀποδεικνύναι is by far a secondary choice for the expression of the action of demonstrating, appearing in some of Archimedes’ prefatory letters and sporadically in his demonstrations. There are in addition two usages of the verb ἐπιδεικνύναι, “to display”, in the Archimedean corpus, once in *On the Spirals* to introduce what he is going to prove, and the other time in his distinction of discovery from demonstration, used interchangeably with ἀποδεικνύναι. An unequivocal usage of ἀποδεικνύναι in the sense of “to show”, without argumentative connotations, is used in reference to Archimedes’ own approach of discovery: in the preface to the *Method* he says that mathematicians will investigate theorems not yet discovered, through the approach shown (διὰ τοῦ ἀποδειχθέντος τρόπου). As pointed out in chapter one, Archimedes’ approach is not defined or described through words, but is shown through the example theorems in the *Method*.

Below we will go through the three occasions on which Archimedes makes the distinction of discovery from demonstration. In the prefatory letter to the *QP* sent to the recipient Dositheus, Archimedes says that

> T2.1 ἐπροχειριξάμεθα δὲ ἀποστεῖλαί τοι γράψαντες, ὡς Κόνωνι γράφειν ἐγνωκότες ἦμες, γεωμετρικῶν θεωρημάτων, ὃ πρότερον μὲν οὐκ ἦν τεθεωρημένον, νῦν δὲ ὑφ' ἁμῶν τεθεώρηται, πρότερον μὲν διὰ μηχανικῶν εὑρεθέν, ἔπειτα δὲ καὶ διὰ τῶν γεωμετρικῶν ἐπιδειχθέν. (H2.262.8-13)
>
> And I decided to write and send you, as I had been determined to write to Konon[35], a certain one of the geometric theorems, one that had never been investigated before and now has

[35] There is no extant letter from Archimedes to Konon. We know from the preface to the *QP* that Konon had been Archimedes’ frequent contact for mathematical exchange before Konon’s death. The preface to the *QP* implies that Archimedes did not know Dositheus in person at the time of writing the *QP*, and the

> been thoroughly investigated by me, first discovered through mechanics, and then shown through geometry.

I take the perfect passive verb τεθεώρηται as conveying the enduring effect of the completed action of θεωρεῖν, "to contemplate", which can also mean "to investigate" in this context. That is to say, τεθεώρηται encompasses discovery and possibly justification of what has been found through investigation. I render it accordingly as "has been thoroughly investigated" in my translation. The verb and its cognates also occur frequently in the *Method* as well, such as θεωρία, the activity of investigation, and θεώρημα, the result of investigation, i.e. theorem.

Another word to be glossed is the passive participle ἐπιδειχθέν from ἐπιδεικνύναι, "to display". The uncommon usage here of ἐπιδεικνύναι is noted by Heiberg, who shows in his apparatus that ἐπιδειχθέν is the reading of Valla's manuscript, which is the main basis for Heiberg's critical edition, and that Moerbeke's Latin translation has *demonstratis*, which could suggest ἀποδειχθέν as the original word in the text Moerbeke used, or could just be a sensible though less common rendering of ἐπιδειχθέν. The only other place where Archimedes uses the verb ἐπιδεικνύναι is in *On Spirals*, where he gives an overview of what he is going to *prove* in the next step (λοιπὸν δὲ ἐπιδειξοῦμες ὅτι, "in what follows I will prove…"). Since these two appearances of ἐπιδεικνύναι are the *lectio difficilior*, i.e. the more difficult reading which a scribe would be more likely to "correct" than introduce, they should not be explained away as scribal interventions. It is striking that both usages of ἐπιδεικνύναι are interchangeable with

QP is the very first letter sent to Dositheus for the purpose of connection-building and re-establishment of mathematical correspondence with colleagues. According to Archimedes, Dositheus was chosen as his new contact because Dositheus knew Konon and was competent in geometry. The expression of grieve (ἐλυπήθημες) and the praise of the dead (ἐν τοῖς μαθημάτεσσι θαυμαστοῦ) in the preface to the *QP* are not *pro forma*. In a sense, the *QP* functions as a letter of condolence and a commemoration of Konon as well.

ἀποδεικνύναι. The interchangeability between the two verbs is especially pronounced when Archimedes reiterates in the same work the distinction of discovery from demonstration:

> T2.2 ἀναγράψαντες οὖν αὐτοῦ τὰς ἀποδείξιας ἀποστέλλομες πρῶτον μὲν ὡς διὰ τῶν μηχανικῶν ἐθεωρήθη, μετὰ ταῦτα δὲ καὶ ὡς διὰ τῶν γεωμετρουμένων ἀποδείκνυται. (H2.264.26-266.2)
>
> So I wrote down the proofs and am dispatching them first in the way in which it was investigated through mechanics, and after this, also in the way in which it is demonstrated through the practice of geometry.

The contrast is again between two activities through mechanics and through geometry, while the expressions for the activities have changed. In T2.1, the verb associated with mechanics is εὑρίσκειν "to discover" and that with geometry is ἐπιδεικνύναι "to show", and here in T2.2 they are θεωρεῖν "to contemplate" and ἀποδεικνύναι "to prove" respectively. We will see later that in the *Method* the expression of θεωρεῖν is often interchangeable and juxtaposed with that of discovery, in opposition to ἀποδεικνύναι. Though in other places, such as the preface to *SC*, Archimedes also uses the expression of θεωρεῖν in the sense of discovery plus demonstration.[36] In the *QP*, the change of expression from discovery into investigation does not imply inconsistency. For when investigation is contrasted with demonstration, the emphasis is on the heuristic part of the process of the investigation. Such a change of expression is indeed very natural if one accepts the assumption that Archimedes does not always use words in their technical sense, and that under certain circumstances he might prefer expressions of strong rhetorical effects, such as in the current case he uses rhetorical antithesis and *variatio* of

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[36] Archimedes considers Eudoxus' proofs of the volume of the pyramid and that of the cone as the most outstanding findings among what have been the outcome of θεωρεῖν in geometry: διόπερ οὐκ ἂν ὀκνήσαιμι ἀντιπαραβαλεῖν αὐτὰ πρός τε **τὰ τοῖς ἄλλοις γεωμέτραις τεθεωρημένα** καὶ πρὸς τὰ δόξαντα πολὺ ὑπερέχειν τῶν ὑπὸ Εὐδόξου περὶ τὰ στερεὰ θεωρηθέντων , ὅτι πᾶσα πυραμὶς τρίτον ἐστὶ μέρος πρίσματος τοῦ βάσιν ἔχοντος τὴν αὐτὴν τῇ πυραμίδι καὶ ὕψος ἴσον, καὶ ὅτι πᾶς κῶνος τρίτον μέρος ἐστὶν τοῦ κυλίνδρου τοῦ βάσιν ἔχοντος τὴν αὐτὴν τῷ κώνῳ καὶ ὕψος ἴσον· (my emphasis)

expressions to highlight the distinction between mechanics and geometry. This distinction comprises a part of Archimedes' claim that he is breaking new ground in multiple respects: investigating a problem not investigated before (see T2.1), making a new finding, and exposing his approach of discovery (a contribution made explicit in the *Method*).

The lexical variations between T2.1 and T2.2 has an ostensible rhetorical effect, but what calls for explanation is the fact that ἐπιδεικνύναι is used as a lexical variant of ἀποδεικνύναι. The two words are more often contrasted than equated in the intellectual context that is historically close to Archimedes. The noun forms of the two verbs are distinguished in philosophical and scientific works in the Classical period. The noun ἐπίδειξις denotes an ostentatious display that impresses and moves the audience without imparting knowledge: it is a stock phrase of Plato's criticism of the rhetorical showpieces of the sophists (in the *Gorgias*, *Protagoras*, *Euthydemus*, etc.). Aristotle's category of epideictic oratory. On the other hand, ἀπόδειξις is a reliable and estimable form of argumentation. In the strict sense it is the proper form of scientific argumentation, as Aristotle defines in the *APo* that a demonstration (ἀπόδειξις) is a syllogistic argument that brings about knowledge. In a looser sense ἀπόδειξις can refer to a valid, or sometimes just convincing argument in general. Aristotle uses the word ἀπόδειξις to denote rhetorical persuasion in *Rhet*. I 1.11, "πίστις is also a kind of ἀπόδειξις." Plato contrasts ἀπόδειξις with τό εἰκός "argument of probability" several times. The strategy of τό εἰκός, which persuades through plausible assumptions and likely speculations, was popular in Athenian rhetoric and oratory. Plato takes τό εἰκός as a poor comparandum to ἀπόδειξις. In the *Phaedo*, Simmias distinguishes between arguments through ἀπόδειξις and those with probability (μετὰ εἰκότος τινὸς, 92d1), but he also mixes the two and says that **arguments** making **proofs** through **probability** are pretentious (τοῖς διὰ τῶν **εἰκότων** τὰς **ἀποδείξεις** ποιουμένοις **λόγοις** … οὖσιν ἀλαζόσιν, 92d3-4; my emphasis) and will deceive an unalert audience in all matters, and above all in the field of *geometry* (καὶ ἐν γεωμετρίᾳ καὶ ἐν τοῖς ἄλλοις ἅπασιν, 92d5). Plato stresses again in the *Theaetetus* that it is improper to use τό εἰκός instead of ἀπόδειξις in geometry. At 162e, Socrates imagines a Protagorean speaker criticizing an argument that is not an ἀπόδειξις but uses τό εἰκός instead. Socrates then has the Protagorean speaker exclude τό εἰκός from geometry: "if Theodorus or any other geometer does geometry by using argument of probability, he would be of utterly no worth" (ᾧ εἰ ἐθέλοι Θεόδωρος ἢ ἄλλος τις τῶν γεωμετρῶν χρώμενος γεωμετρεῖν, ἄξιος οὐδ' ἑνὸς μόνου ἂν εἴη, 162e6-7). The contrast between the scientific nature of ἀπόδειξις and the rhetorical nature of ἐπίδειξις also plays a role in several Hippocratic works. A very striking example is provided by the Hippocratic *On the Art*. The author of this work criticizes those who consider themselves to be making "a display of their own inquiry" (ἱστορίης οἰκείης **ἐπίδειξιν**; my emphasis) to be actually slandering the arts (1.1-3). In contrast, what the author will do is to make an **ἀπόδειξις** about the art of medicine (3.2-3). Interestingly, at the end of this work, the author places his own argument among the **ἐπιδείξιες** of those who know the arts, "which they display more happily from deeds than from words, being untrained in speaking" (ἐπιδείξιες … ἃς ἐκ τῶν ἔργων ἥδιον ἢ ἐκ τῶν λόγων, οὐ τὸ λέγειν καταμελετήσαντες, 13.4-5). This final use of ἐπίδειξις and the verb ἐπιδεικνύναι is obviously not in the sense of rhetorical display, but the author seems to be redefining and redirecting a good use of ἐπίδειξις.

As noted earlier, in the field of mathematics, the most common expression for demonstration is the verb δεικνύναι. In Euclid, the verb is used in demonstrations as a formulaic expression to introduce what is to be proved and to conclude that what was required to demonstrate has been demonstrated. And this is also the practice of other major mathematical authors, namely Archimedes and Apollonius. The noun ἀπόδειξις is the standard term for the noun demonstration, but it is used much less frequently than δεικνύναι. Its verb form ἀποδεικνύναι is even less common, and not always in the sense of demonstrating. As mentioned earlier, Archimedes uses the verb ἀποδεικνύναι in the sense of showing, in the context of showing his approach through the example theorems in the *Method*. The interchangeability between ἐπιδεικνύναι in T2.1 and ἀποδεικνύναι in T2.2 furnishes another example where ἀποδεικνύναι means showing and displaying. The contrast in both texts is between discovering through mechanics and presenting the findings through geometry, and demonstration is the proper form of such presentations in the field of mathematics.

Besides the lexical variation between ἀποδεικνύναι and ἐπιδεικνύναι, what is at stake in T2.2 in scholarship is the use of ἀποδείξιας, the plural accusative of ἀπόδειξις "demonstration" in the Doric form, to refer to both *QP* 1 and *QP* 2. This provides a comparison to Archimedes' word in the preface to the *Method*, where he stresses the distinction of discovery from demonstration for the third time:

> T2.3 Καὶ γάρ τινα τῶν πρότερόν μοι φανέντων μηχανικῶς ὕστερον γεωμετρικῶς ἀπεδείχθη διὰ τὸ χωρὶς ἀποδείξεως εἶναι τὴν διὰ τούτου τοῦ τρόπου θεωρίαν· (H2.428.26-29 = 46v col. 2.11-15)
>
> And indeed, certain findings which had first become clear to me mechanically were later demonstrated geometrically, since the investigation through this approach is separate from demonstration.

This brings us back to where we started: scholars have unanimously taken χωρὶς ἀποδείξεως to mean "without demonstration"[37], and *Method* 1 is said to be χωρὶς ἀποδείξεως "without demonstration" in the *Method*, while *QP* 1, which is essentially the same as *Method* 1, is called an ἀπόδειξις "demonstration" in the *QP*. This creates a question that needs to be addressed, namely on account of what is *Method* 1 considered by Archimedes to be deficient. But the

[37] See e.g., Heiberg 1910-15, Heath 1912, Dijksterhuis 2014, Knorr 1996, Netz forthcoming.

questioning might be happening too fast. The word χωρὶς can also mean "separate from, apart from" when used as a preposition, and the expression χωρὶς ἀποδείξεως might simply indicate that discovery through mechanics is separate from demonstration through geometry. There is another place where Archimedes uses the phrase χωρὶς ἀποδείξεως. A few lines later in the preface to the *Method*, Archimedes compares Eudoxus and Democritus: the former was the first to find out a demonstration of the theorem which the latter was the first to assert χωρὶς ἀποδείξεως "apart from demonstration", and Archimedes thinks one should attribute to the latter no small share of the finding. The phrase χωρὶς ἀποδείξεως refers to the fact that Democritus did not provide a demonstration, but that does not bear the negative connotation of deficiency; instead it emphasizes that discovery, albeit being χωρὶς ἀποδείξεως "apart from demonstration", is of great importance to mathematical inquiry. As a matter of fact, Archimedes never says that an investigation or discovery through his approach lacks demonstration. What he repeatedly says is that his approach of discovery is not a demonstration.

What immediately follows T2.3 in the original text supports the reading that it is a distinction, not deficiency, that Archimedes is making regarding the relationship between discovery and demonstration:

> T2.4 ἑτοιμότερον γάρ ἐστι προλαβόντα διὰ τοῦ τρόπου γνῶσίν τινα τῶν ζητημάτων πορίσασθαι τὴν ἀπόδειξιν μᾶλλον ἢ μηδενὸς ἐγνωσμένου ζητεῖν. (H2.428.29-430.1 = 46v col. 2.15-19)
>
> For it is easier to furnish the demonstration by pre-assuming a certain cognition through this approach than to seek with nothing known.

The article γάρ indicates that this sentence is explanatory of the preceding one, which is T2.3. And what it explains is *not* why a discovery through Archimedes' approach fails to be a demonstration, but in what way such a discovery facilitates demonstration. It is tempting to read

Archimedes' words about discovery in a negative light, given that demonstration has long been considered of supreme importance in mathematical and scientific inquiries. But that does not mean a different form of reasoning in mathematics and the sciences is necessarily inferior to demonstration. At least for Archimedes, discovery is not deficient or inferior to demonstration, but is distinct from and facilitates demonstration.

The overarching goal of the *Method* is to promote the status of discovery without impairing the established norms of rigorous demonstration. This can be seen from both the preface to and the content of the *Method*. As noted in chapter one, the *Method* is comprised of a series of theorems as examples to show Archimedes' approach of discovery. The preface to the *Method* makes this point clear, namely that discovery is valuable in itself and an art of it is of great help for the demonstration of what has been discovered. After giving an enunciation of the new findings, Archimedes tells Eratosthenes, the recipient of the letter, that he includes his own approach of discovery in addition to the proofs of the new findings.

> T2.5 ἐδοκίμασα γράψαι σοι καὶ εἰς τὸ αὐτὸ βιβλίον ἐξορίσαι τρόπου τινὸς ἰδιότητα, καθ' ὅν ἐπιπορευόμενον[38] ἔσται λαμβάνειν ἀφορμὰς εἰς τὸ δύνασθαί τινα τῶν ἐν τοῖς μαθήμασι θεωρεῖν διὰ τῶν μηχανικῶν. Τοῦτο δὲ πέπεισμαι χρήσιμον εἶναι οὐδὲν ἧσσον καὶ εἰς τὴν ἀπόδειξιν αὐτῶν τῶν θεωρημάτων. (H2.428.20-26 = 46v col. 2.1-11)
>
> I decided to write to you and set forth in the same book a peculiarity of a certain approach, by traveling along which it will be possible to grasp the starting points of the capacity to investigate certain matters in mathematics through mechanics. I am fully convinced that this is also no less useful for the demonstration of the findings of the investigation themselves.

The last word of this text, θεώρημα, usually means theorem in a mathematical context. But I render it here as the finding of an investigation, since the noun means the outcome of the action of θεωρεῖν, "to contemplate, investigate", and the core of this preface to the *Method* is the distinction of investigation and discovery (θεωρεῖν and εὑρίσκειν) from demonstration. The

[38] Following Netz and Wilson' reading.

comparative of T2.5, οὐδὲν ἧσσον "no less useful", has an implicit *comparandum*: the approach is no less useful for demonstration than for investigation and discovery *per se*. By using the comparative, the last sentence stresses that Archimedes' approach of discovery both cultivates the ability of discovery and is conducive to demonstration of what has been discovered through such an investigation.

T2.5 immediately precedes the key text we have discussed, namely T2.3. Archimedes makes the distinction of discovery from demonstration in T2.3 and explains the importance of discovery in T2.4. Following T2.4, the next legible sentence is a comparison between Eudoxus and Democritus in the following words:

> T2.6 <... Διόπερ καὶ τῶν θεωρη>μάτων τούτων, ὧν Εὔδοξος ἐξηύρηκεν πρῶτος τὴν ἀπόδειξιν, περὶ τοῦ κώνου καὶ τῆς πυραμίδος, ὅτι τρίτον μέρος ὁ μὲν κῶνος τοῦ κυλίνδρου, ἡ δὲ πυραμὶς τοῦ πρίσματος, τῶν βάσιν ἐχόντων τὴν αὐτὴν καὶ ὕψος ἴσον, οὐ μικρὰν ἀπονείμαι ἄν τις Δημοκρίτῳ μερίδα πρώτῳ τὴν ἀπόφασιν τὴν περὶ τοῦ εἰρημένου σχήματος χωρὶς ἀποδείξεως ἀποφηναμένῳ. (H2.430.1-9 = 43r col. 2.20-32)
>
> It is also for this reason that of the theorems which Eudoxus was the first to have found out the demonstration, regarding the cone and the pyramid, that a cone is a third of the cylinder and the pyramid is a third of the prism that share the same base and height, one would attribute no small share to Democritus, who was the first to have given the assertion about the aforementioned figure apart from demonstration.

This comparison is, again, not stressing the inferiority of discovery to demonstration, but is doing quite the opposite. It emphasizes that making an assertion without demonstration is a great contribution in itself. The phrase χωρὶς ἀποδείξεως in this text, as I noted above, is not an expression of deficiency, but of differentiation; and this is also the case when Archimedes says that an investigation through his approach is χωρὶς ἀποδείξεως, "separate from demonstration."

2.2 Discovery and the ideal of purity

Archimedes' distinction between investigating mechanically and demonstrating geometrically insinuates an idea about what a proper method of demonstration is, or is not. This makes him a fitting interlocutor with advocates of topical purity in the history of mathematics. A topically pure proof draws upon entities and principles that are "suggested by" or "inherent in" the content of the theorem.[39] I argue that a comparison between Archimedes and advocates of purity is enlightening in that it helps elucidate Archimedes' unique contribution to the meta-discourse about how to do mathematics on the one hand and accentuates discovery as a neglected and yet fruitful topic for the discussions of foundations of and progress in mathematics.

2.2.1 Purity in Aristotle and ancient Greek mathematics

Heinrich Scholz's 1930 seminal paper on "Der Axiomatik der Alten" presents a systematic view of Aristotle's theory of science, which has developed into a framework for interpreting the ideals and developments of science and mathematics.[40] One of the stipulations in this framework is Aristotle's prohibition of kind-crossing. Echoes of the prohibition of kind-crossing can be found in the discussions of purity in modern mathematics. While there are various ways to interpret Aristotle's prohibition of kind-crossing, in this section I introduce the argument Aristotle gives in support of his prohibition as an intellectual background against which the modern conception of topical purity arises. In addition, I include an ancient counterpart of mathematical purity, as shown in Pappus' critique of Apollonius and Archimedes according to the ancient classification of geometric problems.

[39] See e.g. Hilbert's characterization of topical purity cited below.

[40] Scholz 1930. See de Jong and Betti 2008 for an account of the historiographical development of the so-called Classical Model of Science.

In *Posterior Analytics* 1.7 Aristotle makes the following statement about crossing between different fields in proofs:

> Οὐκ ἄρα ἔστιν ἐξ ἄλλου γένους μεταβάντα δεῖξαι, οἷον τὸ γεωμετρικὸν ἀριθμητικῇ. (*APo* 75a38-39)
>
> Therefore, one cannot prove [a thing] by crossing from another kind, such as proving something geometric by arithmetic.

The reason for this is given by Aristotle as follows:

> ἐξ ὧν μὲν οὖν ἡ ἀπόδειξις, ἐνδέχεται τὰ αὐτὰ εἶναι· ὧν δὲ τὸ γένος ἕτερον, ὥσπερ ἀριθμητικῆς καὶ γεωμετρίας, οὐκ ἔστι τὴν ἀριθμητικὴν ἀπόδειξιν ἐφαρμόσαι ἐπὶ τὰ τοῖς μεγέθεσι συμβεβηκότα, εἰ μὴ τὰ μεγέθη ἀριθμοί εἰσι· τοῦτο δ' ὡς ἐνδέχεται ἐπί τινων, ὕστερον λεχθήσεται. ἡ δ' ἀριθμητικὴ ἀπόδειξις ἀεὶ ἔχει τὸ γένος περὶ ὃ ἡ ἀπόδειξις, καὶ αἱ ἄλλαι ὁμοίως. ὥστ' ἢ ἁπλῶς ἀνάγκη τὸ αὐτὸ εἶναι γένος ἢ πῇ, εἰ μέλλει ἡ ἀπόδειξις μεταβαίνειν. ἄλλως δ' ὅτι ἀδύνατον, δῆλον· ἐκ γὰρ τοῦ αὐτοῦ γένους ἀνάγκη τὰ ἄκρα καὶ τὰ μέσα εἶναι. εἰ γὰρ μὴ καθ' αὑτά, συμβεβηκότα ἔσται. (*APo* 75b2-12)
>
> Those from which the demonstration proceeds can be the same, but of those whose kinds are different, such as arithmetic and geometry, it is impossible to fit an arithmetic demonstration onto what are incidental to magnitudes, unless magnitudes are numbers—in what way this is possible with some cases will be explained later. An arithmetic demonstration always has the kind which the demonstration is about, and the same for other demonstrations as well. So that the kind must be the same either unqualifiedly or in some way, if a demonstration is to cross [from one kind to another]. That it is impossible to be otherwise is clear: for the extreme and the middle terms must be of the same kind. For if they do not belong to the subject in themselves, they will be incidentals.

This argument is based on Aristotle's theory of scientific knowledge. One way to acquire scientific knowledge is through demonstration, which is a type of syllogism that is true and produces knowledge. A syllogism draws a conclusion from two premises, e.g., all cats are blue, all blue things chase their tail, therefore all cats chase their tail. In this case, the term shared by both premises, "blue", is the middle term, and the other two terms "cats" and "chase their tail" are respectively the minor and major terms. Despite being a valid syllogism, this example of universal blue cats chasing their tail is however not a demonstration. In a demonstration, the premises should be true and explanatory of the conclusion. The condition of being explanatory requires them to be ontologically prior to and better known than the conclusion, by being more universal and further removed from perception. Ultimately the premises should also be indemonstrable, serving as the starting point of demonstrable knowledge. Through such a demonstration, one knows the conclusion *p* by knowing *q*, that *q* is the explanation of *p*, and that the conclusion cannot be otherwise.

Then why must the extreme and middle terms belong to the subject in themselves and not incidentally? For Aristotle, if they belong to the subject only incidentally, the conclusion is not necessary, which means they can be otherwise. And this makes the syllogism no longer capable of producing scientific knowledge. Thus the transgression from one kind to another in a demonstration is to be prohibited. [41]

In the practice of ancient Greek mathematics, there is a concern about what can be called the ancient counterpart of purity. Like the modern discourse about purity, ancient mathematicians developed standards of the proper method of proof and systematized mathematical knowledge following a formal model. But their considerations of purity are different both from Aristotle's prohibition of kind-crossing and from the modern idea of topical purity. To begin with, the ancient classification of geometric problems is not based on the content, but on the most elementary possible solution of the problem, "elementary" referring to the level of sophisticatedness of the curves involved in the solution. In the fourth book of his *Collectio*, Pappus criticizes geometers for using techniques alien to the problem itself:

> δοκεῖ δέ πως ἁμάρτημα τὸ τοιοῦτον οὐ μικρὸν εἶναι τοῖς γεωμέτραις, ὅταν ἐπίπεδον πρόβλημα διὰ τῶν κωνικῶν ἢ τῶν γραμμικῶν ὑπό τινος εὑρίσκηται, καὶ τὸ σύνολον ὅταν ἐξ ἀνοικείου λύηται γένους, οἷόν ἐστιν τὸ ἐν τῷ πέμπτῳ τῶν Ἀπολλωνίου κωνικῶν ἐπὶ τῆς παραβολῆς πρόβλημα καὶ ἡ ἐν τῷ περὶ τῆς ἕλικος ὑπὸ Ἀρχιμήδους λαμβανομένη στερεοῦ νεῦσις ἐπὶ κύκλον· (*Collectio* iv 270.28-272.3)

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[41] For interpretations of Aristotle's prohibition of kind-crossing see e.g. Scholz 1930, Barnes 1993, Detel 1993, Diestelzweig 2013, Hintikka 1972, McKirahan 1992, Hankinson 2005, Steinkrüger 2018.

> It seems that such an error is not small for geometers, when a plane problem is solved by somebody through conics or other curves and in general when a problem is solved through an alien kind, such as the problem about the parabola in the fifth book of Apollonius' *Conics* and the solid neusis used by Archimedes on a circle in his *On Spirals*.

Pappus' criticism of Apollonius and Archimedes is based on his classification of geometric problems. According to Pappus, ancient geometers classified problems into plane, solid, and linear. Problems that can be *solved* through straight line and circular arc are called plane; those that can be *solved* when one or more conic sections are used to discover the solution are called solid; the last type is called linear because these problems are solved by lines that "have a more complicated and forced generation" (ποικιλωτέραν καὶ βεβιασμένην ἔχουσαι τὴν γένεσιν, *Collectio* iii 54.18-19), e.g. helix, quadratrix, conchoids, and cisoids, and these lines have many paradoxical and incidental properties about them (πολλὰ καὶ παράδοξα περὶ αὐτὰς ἔχουσαι συμπτώματα, *Collectio* iii 54.21-22). The category of linear problems is not entirely clear and does not have a uniform standard for determination. Pappus' characterization of "having a more complicated and forced generation" and "having many paradoxical and incidental properties" is more of a complaint than a definition. Indeed, he is not content with the mechanical element in the generation of the more complicated lines. For instance, he gives "a geometric way" (γεωμετρικῶς) to construct the quadratrix, by projecting a cylindrical helix onto a plane, in contrast to the traditional "more mechanical" (μηχανικωτέρα) construction attributed to Hippias of Elis, which is the path of the intersecting point of (i) the motion of one side of a square moving parallelly and uniformly to the opposite side and (ii) the synchronous motion of another side, perpendicular to the first, rotating uniformly around a vertex. It is hard to tell which of the two ways is "purer", since Pappus' resorts to solid considerations to construct a plane curve. A likely reason why Hippias' is less preferable lies in the dependence on a μηχανή "device". Arguably all lines are constructed through motion, but Hippias' quadratrix requires an accurate control of speed, which in turn requires design and operation of a device, making it μηχανικός, "mechanical" accordingly.

Along this line of thinking, if a problem involves only plane figures but can only be solved by solid curves, then it is a solid problem, not a plane problem. For example, the trisection of a random angle is not a plane problem according to Pappus' classification, because it cannot be solved through plane curves. Thus, the type of purity suggested by Pappus' classification is different from Aristotle's prohibition of kind-crossing, unless Aristotle's kind were to be understood as the set of concepts and axioms necessary to prove a theorem. However, the question of what is or is not necessary to prove a theorem, in combination with the avoidance of kind-crossing, plays an active role in the history of mathematics. In particular, this motivates the search for purer proofs, or otherwise the proof of impossibility, i.e., to show that something is provably improvable or inconstructable in certain ways or without qualification. For examples, the impossibility of trisecting an angle through plane curves was not proved until 1837 by Pierre Wantzel; attempts to prove the parallel axiom persevered from antiquity to the modern era, until non-Euclidean geometries were developed without yielding contradictions in the first half of the nineteenth century.

2.2.2 Purity in modern mathematics

The prime number theorem was first proved in the late nineteenth century with recourse to methods of complex analysis. But the search for an "elementary" proof of the theorem did not stop until such proofs were developed by Erdös and Selberg independently in 1948. Selberg was awarded the Fields Medal in 1950 for his elementary proof. According to Rota 1997, the idea that the distribution of the primes should be proved "on the basis of an analysis of the concept of prime without appealing to extraneous techniques" (115) contributes to the search for an elementary proof of the theorem.

Detlefsen 2008 gives a useful overview of the broadly speaking Aristotelian concern for purity among contemporary mathematicians. He cites mathematicians from various fields highlighting their achievements of using "only *definitions* of terms contained in the theorem" (Detlefsen 2008: 191) and stressing that what motivates their research is the notion that a result is only properly proved when the proof does not resort to extraneous techniques.[42] Some mathematicians also explain why such a proof is desirable: Edmonds 1986 and Stanton and Zelberger 1989 both comment on their topically pure proofs that such proofs can provide those working in the same field with better insight into the subject. If possible, a particularly strict type of purity may be pursued. The first paragraph of Woo 1971 highlights that his proof of the Lebesgue decomposition theorem (a theorem from measure theory) "uses nothing from measure theory beyond the definitions needed to state the theorem" (Woo 1971: 183).

From a historical point of view, an important advocate of topical purity in the modern era is Bolzano. The best-known example of Bolzano's work with regard to purity is arguably his pure proof of the intermediate value theorem. The intermediate theorem says that if there is a function f continuous on the interval $[a,b]$, then for any value t between $f(a)$ and $f(b)$, there exists a certain $u \in [a,b]$ such that $f(u) = t$. The theorem seems intuitively correct. Simply imagine a continuous curve that extends from one side of a line to the other side, there must be a point where the curve intersects with the line. But this is to Bolzano an unsatisfactory justification

[42] See, e.g., the abstract of Formanek 1973: "the main interest is that the proof is very elementary and uses little more than the definition of 'Noetherian'." Stanton and Zelberger 1989 starts off with the following comment: "To a true combinatorialist, a combinatorial result is not properly proved until it receives a direct combinatorial proof… However to non-combinatorialists, a direct combinatorial proof is 'just another proof.'"

because it proves a theorem about quantity in general through considerations of a special type of quantity, i.e., spatial quantity (assuming that quantity is applicable to space). According to Bolzano,

> The most common kind of proof depends on a truth borrowed from *geometry*, namely, *that every continuous line of simple curvature of which the ordinates are first positive and then negative* (or conversely) *must necessarily intersect the x-axis somewhere at a point that lies in between those ordinates*. There is certainly no objection against the *correctness*, nor indeed against the *obviousness* of this geometrical proposition. But it is equally clear that it is an intolerable offence against *correct method* to derive truths of *pure* (or general) mathematics (i.e. arithmetic, algebra, analysis) from considerations which belong to a merely *applied* (or special) part, namely *geometry*. Indeed, have we not felt and recognized for a long time the incongruity of such μετάβασις εἰς ἄλλο γένος? Have we not already avoided this whenever possible in hundreds of other cases, and regarded this avoidance as a merit? So if we wish to be consistent must we not try and do the same here?—For in fact, if one considers that the proofs of the science should not merely be *confirmations* [*Gewissmachungen*], but rather *justifications* [*Begründungen*], i.e. presentations of the objective reason for the truth concerned, then it is self-evident that the strictly scientific proof, or the objective reason, of a truth which holds equally for *all* quantities, whether in space or not, cannot possibly lie in a truth which holds merely for quantities which are in *space*. (Bolzano 1817: 4-5, trans. Russ 1980)

In this comment, Bolzano identifies Aristotelian epistemology as the source of the ideal of purity in the history of mathematics. The phrase μετάβασις εἰς ἄλλο γένος, "crossing into another kind", is a clear reference to the Aristotelian passage discussed above (*APo* 1.7). The distinction between *Gewissmachungen* and *Begründungen* is also reminiscent of Aristotle's distinction between demonstration "of the fact that" (τὸ ὅτι) and demonstration "of the reason why" (τὸ διότι), of which the former only convinces that the conclusion is right while the latter gives the true cause of the conclusion.[43]

Bolzano also criticizes geometry on the basis of his understanding of Aristotle's prohibition of kind-crossing:

> I must point out that I believed I could never be satisfied with a completely strict proof *if it were not derived from the same concepts* which the thesis to be proved contained, but rather made use of some fortuitous alien, *intermediate concept* [*zufälligen, fremdartigen*

[43] See Detlefsen 2008 for further discussions of Bolzano's ideal of purity.

> *Mittelbegriffes*], which is always an erroneous μετάβασις εἰς ἄλλο γένος. In this respect I considered it an error in geometry that all propositions about angles and ratios of straight lines to one another (in triangles) are proved by means of *considerations of the plane* for which there is no cause [*Veranlassung*] in the theses to be proved. (Bolzano 1804 introduction para. 4, trans. Russ, in Ewald 1999: 173)

Bolzano's standard for purity is based on a very strict understanding of Aristotle's kind (cf. Woo 1971 cited above). Instead of the discipline to which the theorem belongs to, Bolzano seems to take the particular genus of the entities involved in the statement of the theorem to be the boundary within which the proof should confine itself. For Bolzano, a proof about triangle should confine itself to components of the triangle alone, i.e. point, line, and angle, and should not resort to considerations of the plane.

Bolzano's ideal of purity is a strict and radical one that not everybody would agree with or find feasible. In fact, what is to be considered the proper content of a theorem is open to interpretation. On the looser side of the spectrum is the proposal that the proper content of a theorem is what is needed to understand the theorem. Depending on how one construes "understanding", this proposal may amount to defining the content of a theorem as what is needed to explain or prove the theorem, or the "hidden higher-order content"[44]. Take Hilbert's foundational work on geometry as an example. Hilbert notes that the plane version of Desargues' theorem can only be proved using either solid or metrical considerations, whereas the theorem is plane and projective. Commenting on the impurity, or impossibility of a pure proof, Hilbert characterizes topical purity in the following way:

> We are therefore for the first time in a position to make *a critique of means of proof*. In modern mathematics such criticism is very often made, where the aim is to preserve *the purity of method* [*die Reinheit der Methode*], i.e. when proving theorems, to use (if possible) only those means that are suggested [nahe gelegt] by the content of the theorem. (Hallet and Majer 2004: 315–6)

[44] See, e.g., Hallet 2008 and Isaacson 1996.

Depending on what is to be considered close to the content of the theorem, one may conclude that the theorem cannot have a pure proof, i.e., a plane and projective proof, or argue that the deeper content of the theorem is solid or metrical, as opposed to the apparent plane and projective nature of the statement.[45] Hilbert's concluding remark at the end of his 1899 *Festschrift* (the first edition of his *Grundlagen der Geometrie*) to some extent undermines the force of topical purity that he mentioned earlier:

> In this investigation the ground rule was to discuss every question that arises in such a way so as to find out at the same time whether it can be answered in a specified way with certain limited means.... The impossibility of certain solutions and problems thus plays a prominent role in modern mathematics, and the drive to answer questions of this type was oftentimes the cause for the discovery of new and fruitful areas of investigation…. The ground rule according to which the principles of the possibility of a proof should be discussed at all is very intimately connected with the requirement for the "purity" of the methods of proof which has been championed by many mathematicians with great emphasis. This requirement is basically none other than a subjective form of the ground rule followed here. (Hilbert 1899: 89-90, trans. L. Unger, in Hilbert 1971: 106-107)

In the same spirit Hilbert also makes the following comment in the notes for his 1898-99 winter course on "Grundlage der Euklidischen Geometrie":

> Of course, the recourse to different kinds of means often has a *deeper* and *justified* ground, and beautiful and *fruitful relations* are uncovered, e.g. the prime number problem and the $\zeta(x)$ function, potential theory and analytic functions, etc. In any case, one should never pass by such an *occurrence* of the interaction of different areas carelessly. (Hallett and Majer 2004: 236)

For Hilbert, the value of topical purity lies in that it opens up investigations of undiscovered relations between different fields. Impurity, rather than purity, is heuristically most helpful in this light.

A further example drawn from Dedekind is given below. According to Dedekind, a pure and rigorous of definition of the continuum should not appeal to geometric intuition. But I argue that in forming such a definition, Dedekind relies on geometric intuition as a heuristic tool.

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[45] Hallet, cf. Arana and Mancosu 2012.

Dedekind's 1872 work on *Stetigkeit und irrationale Zahlen* aims to lay "a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis". For Dedekind, the conception of the rational numbers as points on a straight line is an analogy that, though helpful, is extraneous to the study of numbers:

> For our immediate purpose, however, another property of the system *R* [viz., the system of all rational numbers] is still more important; it may be expressed by saying that the system *R* forms a well-arranged domain of one dimension extending to infinity on two opposite sides. What is meant by this is sufficiently indicated by my use of expressions borrowed from geometric ideas; but just for this reason it will be necessary to bring out clearly the corresponding purely arithmetic properties in order to avoid even the appearance as if arithmetic were in need of ideas foreign to it. (Dedekind 1872: 6, trans. Dedekind 1901[46])

Already in the preface to this work, Dedekind expresses his thoughts about geometric intuition (*geometrische Anschauung*), which he regards as "exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time" (Dedekind 1872: 1), but "can make no claim to being scientific" (Dedekind 1872: 1). In the comment above he again shows a similarly mixed attitude: geometric ideas are sufficient in conveying certain abstract properties of arithmetic, but they are foreign to the field and should be purged for the sake of rigor and purity. For this reason, Dedekind also holds that the approach of introducing irrational numbers as additional points on the line whose distances from the origin are not commensuarble with the unit length is not the proper way to define irrational numbers. The approach he takes instead is to "define irrational numbers by means of the rational numbers alone" (Dedekind 1872: 10). Dedekind's remark below reveals that purity is a major factor that motivates his foundational program.

> For, the way in which the irrational numbers are usually introduced is based directly upon the conception of extensive magnitudes — which itself is nowhere carefully defined — and

[46] All translations of Dedekind's work on *Stetigkeit und irrationale Zahlen* cited in this dissertation are from Dedekind 1901.

> explains number as the result of measuring such a magnitude by another of the same kind. Instead of this I demand that arithmetic shall be developed out of itself. (Dedekind 1872: 10)

In the meantime, the geometric idea of a line proves to be a very fruitful heuristic tool for Dedekind's work on a rigorous definition of continuity. Dedekind takes continuity to be completeness: to put it the other way round, the system of all rational numbers is discontinuous because it is incomplete, or containing gaps. For example, $\sqrt{2}$ is a gap because it seperates the ordered "line" of rational numbers into two halves and does not belong to either half. In his work Dedekind explicates how he progresses from the assumed continuity of a geometric line to the construction of continuum[47], which leads to a rigorously arithmetic definition of continuity that is purged of geometric intuition.

It is easy to see how Archimedes stands apart from the mainstream attitude that values purity over untraditional, boundary-crossing approaches. In the same time, Archimedes shares with advocates of purity the idea of proper methods of mathematics, as seen in his rigorous proofs of the results discovered mechanically. There is also a consensus about the fruitfulness of introducing foreign ideas to the study of mathematics shared by both Archimedes and foundationalists like Dedekind and Hilbert. Archimedes' unique contribution consists in his emphasis on discovery as an independent intellectual activity that is not merely problem-solving or a preparatory stage of demonstration. As demonstration has its instituted methods, so should discovery, though no method had been instituted by the time of Archimedes. Archimedes does

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[47] "The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in the line. If space has at all a real existence it is not necessary for it to be continuous; many of its properties would remain the same even were it discontinuous. And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desired, from filling up its gaps, in thought, and thus making it continuous ; this filling up would consist in a creation of new pointindividuals and would have to be effected in accordance with the above principle."12

not claim to be the inventor of his approach[48], but his contribution consists in envisaging the institution of a proper method of discovery and taking a substantive step towards that vision. This is also a reaffirmation that it is most appropriate to render Archimedes' τρόπος as approach rather than method, i.e., a nascent form of the visionary method of discovery.

[48] See Knorr 1996 on the tradition of indivisibilist techniques in antiquity before and after Archimedes.

3. *Method* 14: an effort of rigorization

While the previous chapters focus on the theme of discovery in Archimedes' *Method*, this chapter shifts to a related and complementary topic, namely rigorization. I argue that *Method* 14, a special proof in the *Method*, showcases an intermediate form between discovery through non-rigorous means and rigorization of the key finding through the discovery. *Method* 14 contains an arithmetic element, which replaces the use of the imagined balance and is similar to the arithmetic method in Archimedes' rigorous works such as *On Conoids and Spheroids* [*CS*]. This shows that Archimedes' use of the balance in the *Method* is not only fruitful regarding discovery but also helpful to the development of new techniques in the context of rigorous demonstration.

3.1 The use of Lemma 11 in *Method* 14

At the beginning of the *Method*, Archimedes include eleven προλαμβανόμενα, "things assumed beforehand" (without proof), rendered by Heiberg as *Lemmata*. The last of these *Lemmata*, Lemma 11, is proved in the *CS* as the first lemma of that work. In the *Method*, Archimedes gives only the conclusion of the lemma:

> **T3.1 = Lemma 11** χρησόμεθα δὲ καὶ [ἐν τῶι προγεγραμμένωι Κωνοειδῶν] τῶιδε θεωρήματι· ἐὰν ὁποσαοῦν μεγέθη ἄλλοις μεγέθεσιν ἴσοις τὸ πλῆθος **(i)** κατὰ δύο τὸν αὐτὸν ἔχηι λόγον τὰ ὁμοίως τεταγμένα, **(ii)** ἧι δὲ τὰ πρῶτα μεγέθη πρὸς ἄλλα μεγέθη ἐν λόγοις[49] ὁποιοισοῦν, ἢ τὰ πάντα ἤ τινα αὐτῶν, [καὶ τ]ὰ ὕστερα μεγέθη πρὸς τὰ ὁμό[λ]ογα ἐν τοῖς αὐτοῖς λόγοις ἧ, **(iii)** πάντα τὰ πρῶτα μεγέθη πρὸς πάντα τὰ λεγόμενα τὸν αὐτὸν ἕξει λόγον, ὃν ἔχει πάντα τὰ ὕστερον πρὸς πάντα τὰ λεγόμενα. (H2.434.3-12 = 64v col. 2.31-57r col. 1.8)
>
> I will also use the following theorem in the preface to the *Conoids*: if a collection of magnitudes of any number have **(i)** the same ratio two by two as another collection of magnitudes, equal in multitude and arranged in the same manner, and **(ii)** if whatever ratio the first collection of magnitudes is to another collection of magnitudes [the third

[49] πρὸς ἄλλα μεγέθη ἐν λόγοις Heiberg ἐν τόποις Netz and Wilson. The digitized image of this line is hardly legible.

> collection], either all or some of them, the same ratio is the second collection of magnitudes to another series of corresponding magnitudes [the fourth collection], then **(iii)** the ratio of all of the first collection to all of the third is the same as the ratio of all of the second to all of the fourth.

This lemma is frequently used in the *CS*, usually in combination with the method of exhaustion. After evenly slicing the object under investigation into parallel slices, Archimedes compares four series: series 1: $a_0, a_1, \ldots, a_n$; series 2: $b_0, b_1, \ldots, b_n$; series 3: $c_0, c_1, \ldots c_m$; series 4: $d_0, d_1, \ldots, d_m$ (*m* is no greater than *n*). Archimedes shows that (i) for any *i*, *j* no greater than *n*, $a_i : a_j :: b_i : b_j$, and (ii) for any *i* no greater than *m*, $a_i : c_i :: b_i : d_i$. Then according to Lemma 11, we have (iii) $\sum_{i=0}^{n} a_i : \sum_{i=0}^{m} c_i :: \sum_{i=0}^{n} b_i : \sum_{i=0}^{m} d_i$.

While Lemma 11 is included in the *Method*, none of the theorems in the *Method* makes explicit use of it. In Heiberg's 1906 transcription, a crucial part of the text of *Method* 14 is a lacuna, since that part of the palimpsest was not legible to the naked eye. But Lemma 11 is very likely to have been used there to fill in the logical gap of the proof. The new transcription by Netz and Wilson based on digitized images of the palimpsest reveals what is contained in that lacuna. Below is the text in *Method* 14 that applies Lemma 11 transcribed by Netz and Wilson[50]. The conventions used for the transcriptions are as follows: ụ̣ṇ̣c̣ẹṛṭạị̣ṇ̣ ṛẹạḍịṇg̣ [restored by the editors] (scribal abbreviation)

> **T3.2** 110v col. 1 line 21 ἔστα̣ι̣ 22 **τινὰ μεγέθ̣η ἴ̣σ̣α ἀλλήλο[ις**, τὰ] τ[ρί]- 23 γων̣α τὰ ἐν τῶι πρίσμα̣τ̣ι, (καὶ) **ἕ-**24 **τερα μεγέθη**, αἵ εἰσιν εὐθεῖα̣ι̣ [ἐν] 25 τῶ ΔΗ παραλλη[λογ]ράμμ[ω] πα-26 ράλληλοι οὖσαι τ̣[ῆ]ι̣ ΖΚ **ἴ[σ]αι̣ ἀλ̣-**27 **λήλ̣ο̣ι̣ς̣** [κα]ὶ ἔτι **τ̣ῷ̣ι πλήθει ἴσα** 28 τοῖ̣ς ἐν τῶ πρίσματι τριγων[(οις)]· 29 ἔσται δὲ καὶ **ἕτ̣ερα τρ[ί]γωνα** τὰ 30 γενόμενα ἐν τῶι ἀποτμηθέν-31 τ̣ι̣ **ἴσα τῶι πλήθει** τοῖ̣ς̣ γενομέ̣ 32 νοις ἐν τῶι πρίσματι τρι-33 γώνοι[ς]· καὶ **αἱ ἕτε[ραι εὐθ]ε̣ῖ̣[αι]** 34 ἀπολαμβανόμεναι ἀ̣[πὸ] τῶν 110v col. 2 line 1 ἀγομένων π(αρὰ) τὴν Κ[Ζ μετ]αξυ 2 τῆς τοῦ ὀρθογωνίο̣[υ κώ]νου 3 τομῆς καὶ̣ τῆς ΕΗ **ἴσ̣[αι] τ̣ῶ̣ι** 4 **πλήθει** ταῖς ἐν τῶι ΔΗ π[αραλ]-5 ληλογράμμωι ἠγμέναις π[(αρὰ)] 6 τὴν ΚΖ, καὶ ἔσται πάντα τὰ 7 τρίγωνα τὰ ἐν τῶι πρίσματι 8 πρὸς πάντα τὰ τρίγωνα τὰ 9 ἐν τῶι ἀποτμηθέντι τῶι ἀπὸ 10 τοῦ κυρίνδρου

[50] For digitized images of the palimpsest and Netz and Wilson' transcription see https://archimedespalimpsest.net/

ἀφηρημένα, 11 οὕτως πᾶσαι αἱ εὐθεῖαι αἱ ἐν 12 τῶι ΔΗ παραλληλογράμμωι (πρὸς) 13 πάσας τὰς εὐθείας τὰς μετα-14 ξυ τῆς τοῦ ὀρθογωνίου κώνου 15 τομῆς καὶ τῆς ΕΗ εὐθείας. (καὶ) 16 **(Composition Statement)** ἐκ μὲν τῶν ἐν τῶι πρίσματι τρι-17 γώνων σύγκειται τὸ πρίσμα, ἐκ 18 δὲ τῶν ἐν τῶι ἀποτμήματι τῶι 19 ἀπ̣ὸ τοῦ κυ[λ]ί̣ν[δ]ρου τὸ ἀ[π]ό̣τμη̣-20 μα, ἐκ δὲ τῶν εὐθειῶν τῶν ἐν 21 τῶι ΔΗ παραλληλογράμμωι (τῶν) 22 παρὰ τὴν ΚΖ τὸ ΔΗ παραλλη-23 λόγραμμον, ἐκ δὲ τῶν [εὐ]θειῶν 24 μεταξὺ τῆς τοῦ ὀρθογωνίου κώ-25 νου τομῆς καὶ τῆς ΕΗ τὸ τμῆ-26 μ̣α τῆς παραβολῆς· ὡς (ἄρα) τὸ πρ(ίσ)-27 μα (πρὸς) τὸ ἀπότμημα το(ῦ) ἀπὸ τ(οῦ) 28 κ̣υ̣λίνδρου, οὕτως τὸ ΔΗ πα[ρ]αλ-29 ληλόγραμμον (πρὸς) τὸ ΕΖΗ τμῆμα 30 τὸ περιεχόμενον ὑπὸ τῆς τοῦ 31 ὀρθογωνίου κώνο̣υ τομῆς καὶ 32 τῆς ΕΗ̣ εὐθ̣ε̣ί̣ας. (my emphasis)

There will be **certain magnitudes equal to one another**, namely the triangles in the prism, and **other magnitudes**, which are the lines in the parallelogram ΔΗ, parallel to ΖΚ and **equal to one another**, and moreover, **equal in multitude** to the triangles in the prism. Again, there will be yet **other triangles** that come to be in the segment cut away from the cylinder, **equal in multitude** to the triangles that come to be in the prism. And there will still be **other lines** taken from the lines drawn parallel to ΚΖ and between the orthotome segment and ΕΗ, **equal in multitude** to the lines drawn parallel to ΚΖ in the parallelogram ΔΗ; and it will be the case that as all the triangles in the prism are to all the triangles taken in the segment cut away from the cylinder, so all the lines in the parallelogram ΔΗ are to all the lines between the orthotome segment and ΕΗ. Also, **(Composition Statement)** the prism is composed of the triangles in the prism, the segment cut away from the cylinder the triangles in it, the parallelogram ΔΗ the lines parallel to ΚΖ in it, and the parabolic segment the lines between the orthotome segment and ΕΗ. Therefore, as the prism is to the segment cut away from the cylinder, so the parallelogram ΔΗ is to the parabolic segment ΕΖΗ bounded by the orthotome segment and ΕΗ.

Archimedes does not use any expression like “by Lemma 11” to signal the application of Lemma 11 in this text, but rather shows that the magnitudes under examination satisfy the condition listed in Lemma 11, which I highlight in bold, and draws the inference according to Lemma 11, which I underline. Through a comparison with Lemma 11, the application should be obvious.

Unlike the proofs in the *CS*, the entities in *Method* 14 are not finitely many, nor are they countable at all. This makes the application of Lemma 11 to *Method* 14 a dubious move: in the finite case, the phrasing “all” refers to the sum of the series, but clearly the interpretation of “sum” does not work in the case of a continuum. The key point is that using Lemma 11 is an alternative way to do the job of the imagined balance. To see this, let us go through the train of thoughts of *Method* 14 and see what role Lemma 11 plays in it.

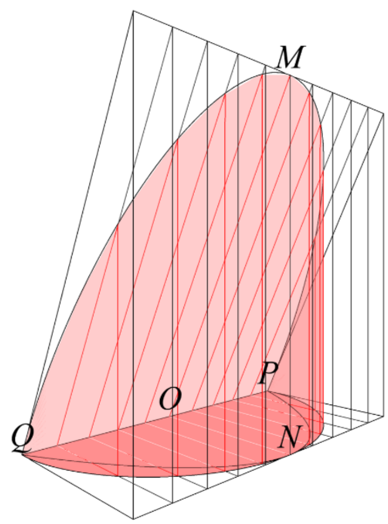

Figure 3-1

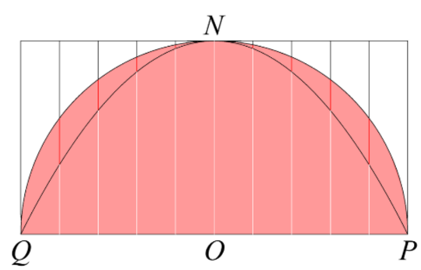

Figure 3-2

Let point *O* be the center of the lower base of a cylinder, line *PQ* a diameter of the lower base, and plane *OMN* perpendicular to *PQ*. The part contained by plane *PQM* and the original cylinder is the segment the volume of which we are measuring (see Figure 3-1). In the semicircle *PNQ*, inscribe a parabolic segment through *P*, *N*, and *Q*. Consider a random plane parallel to *OMN* intersecting the cylinder segment at more than one point. This plane produces a series of intersections with the geometric objects constructed. In the cylinder segment it produces a triangle, and in the triangular prism circumscribed around the cylinder segment a greater triangle similar to the one in the cylinder segment. On the plane *PNQ*, the intersections are two collinear lines, the shorter one in the parabolic segment, and the longer one in the rectangle circumscribed about the parabolic segment (see Figure 3-2). These four magnitudes are considered corresponding to each other.

Archimedes proves that the four corresponding magnitudes are proportional: the ratio of the triangular intersection in the cylinder segment to its corresponding intersection in the triangular prism is the same as the ratio of its corresponding line segment in the parabola to its corresponding line segment in the rectangle. By Lemma 11,[51] the conclusion follows that the ratio of the cylinder segment to the triangular prism is the same as the ratio of the parabolic segment to the rectangle, which is 2:3. Since the entire rectangular prism circumscribed about the cylinder is four times the triangular prism, the cylinder segment is a sixth of the rectangular prism circumscribed about the original cylinder.

Method 14 is one of the three different approaches to prove this result. The mechanical proof, comprised of *Method* 12 and 13, uses the balance along with indivisibles and is developed entirely differently from *Method* 14. Judging from what is extant and legible, *Method* 15, a rigorous proof of the result through the method of exhaustion, uses the same geometric configuration as *Method* 14, with the lines in *Method* 14 being replaced with indefinitely thin slices. Thus it is possible that Lemma 11 is also used in *Method* 15. *Method* 14 is a unique proof in the *Method* in that, while using the informal technique of indivisibles, it replaces the imagined balance with Lemma 11. The use of Lemma 11 in *Method* 14 shows an effort to prove the result on an arithmetic basis.[52]

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[51] On a modern interpretation, Lemma 11 assumes that multiple integrals are associative: $\iiint f(x,y,z)dxdydz = \int(\iint f(x,y,z)dxdy)dz$.

[52] The parallelism between Archimedes' indivisibles (esp. without the balance) and Cavalieri's has been widely observed, see, e.g., Mancosu 1996: 34-35, Malet 1996: 13, Knorr 1996: 80.

3.2 Infinite collections as "equal in multitude"

A new finding in Netz and Wilson's new transcription is the expression ἴσα/ἴσαι τῷ πλήθει "equal in multitude". This is said of the four collections of triangles or lines compared in T3.1. What is striking about this text is that neither the triangles nor the lines are finite. While Archimedes avoids any expressions of infinity in this text, but vaguely describes the triangles and the lines as τινὰ μεγέθη "some magnitudes", the expression is still a statement of equality of multitude between collections of items that are not finitely many, for the key reason that Archimedes' approach avoids the cumbersome converging construction and double *reductio* of the method of exhaustion consists in that the former's division of figure is not finite. Also, by construction, each item in one collection finds three corresponding items in the other three collections respectively (the four corresponding items are the intersections of the same plane with each of the two solids and two plane figures). This amounts to three one-to-one mappings between a collection and each of the other three collections. It is tempting to see a connection between equality in multitude and one-to-one mapping, as both are present in *Method* 14.[53] But nowhere in *Method* 14 or other works does Archimedes *use* one-to-one mapping as the criterion by which he judges that two collections of infinitely many items are equal in multitude. It may well be the case that Archimedes thinks that no infinite plurality is greater than or less than another, but all infinite pluralities are equal. Then all infinite collections are equal in multitude simply because they are infinite, and Archimedes does not need any criterion for equality or inequality between infinite pluralities.

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[53] See Netz, Saito, and Tchernetska 2001.

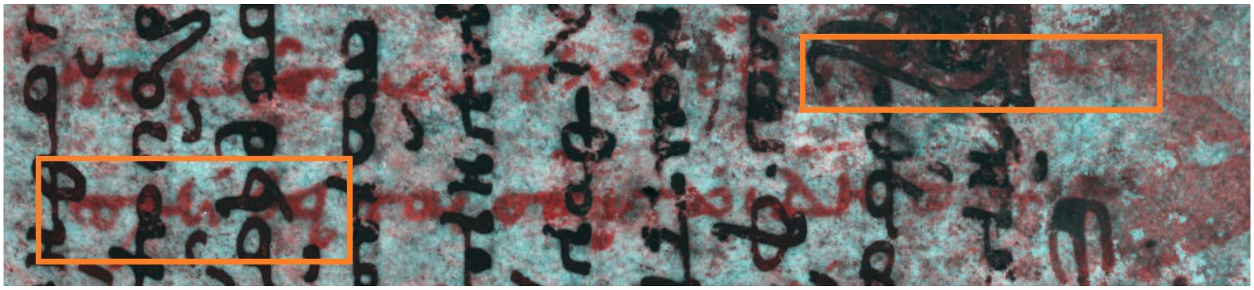

Figure 3-3 110v-105r_Arch27v_Sinar_pseudo_no_veil, col. 2, lines 3-4

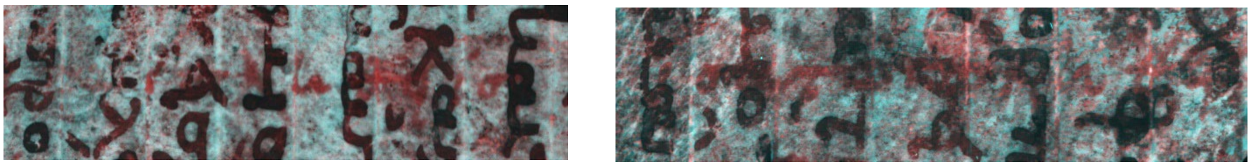

Figure 3-4 left: 110v-105r_Arch27v_Sinar_pseudo_no_veil, col. 1, line 27, τ̣ῶ̣ι πλήθει ἴσα; right: 110v-105r_Arch27v_Sinar_pseudo_no_veil, col. 1, line 31, ἴσα τῶι πλήθει

The idea that all infinite pluralities are equal is compatible with the intellectual milieu of Greek antiquity. The assumption that an infinity plurality is not less than anything is used by Alexander and Philoponus in their commentaries on Aristotle to argue against infinite plurality. Their arguments against infinite plurality are made by constructing a collection of which a given infinite plurality is a proper part. This will lead to absurdity on the premises that (i) the whole is greater than its proper part and that (ii) an infinite plurality is not less than anything.[54] Plutarch's *De communibus notiis* gives an argument that no infinity is greater than or less than another. This comes closer than Alexander's and Philoponus' premise (ii) to the view of *Method* 14 that some infinite collections are equal in multitude. To ridicule the Stoic doctrine of infinite divisibility, Plutarch says that the Stoics contradict the common notion that the whole has more component parts than any of its proper part does, but

> **T3.3 (Part-Whole Equinumerosity)** γενόμενοι δὲ Στωικοὶ τἀναντία λέγουσι καὶ δοξάζουσιν, ὡς οὐκ ἔστιν ἐκ πλειόνων μορίων ὁ ἄνθρωπος ἢ ὁ δάκτυλος οὐδ' ὁ κόσμος ἢ ὁ ἄνθρωπος. **(No-Infinity-Is-Greater Argument)** ἐπ' ἄπειρον γὰρ ἡ τομὴ προάγει τὰ σώματα, τῶν δ' ἀπείρων οὐθέν ἐστι πλέον οὐδ' ἔλαττον οὐδ' ὅλως ὑπερβάλλον πλῆθος, ἢ

54 Rosen 2020: 486-88.

παύσεται τὰ μέρη τοῦ ὑπολειπομένου μεριζόμενα καὶ παρέχοντα πλῆθος ἐξ αὑτῶν. (1079A8-B4)

Once people have become Stoics, they say and think the opposite, that a man is not composed of more parts than a finger, nor the universe than a man. For division drives bodies to infinity, and of infinities none is more or less or exceeds [another] in multitude at all, or else the parts of that which falls short will cease being divided and providing multitude from themselves.

The latter part of this text gives the No-Part-Is-Greater Argument that no infinity is greater or less than another as an explication of the former part that holds the view of Part-Whole Equinumerosity that the whole does not have more parts than its proper part does. The No-Part-Is-Greater Argument in this text is closer to the infinity view of *Method* 14, because it speaks of different infinities as if they are comparable in multitude and says that no infinity is greater or less than another. A difference from *Method* 14 lies in the omission of the possibility of equality between two infinite pluralities. In addition, a complexity of this presentation of the Stoic doctrine of infinite divisibility lies in Plutarch's rhetorical goal to attack the Stoics, rather than to faithfully present them. While the Part-Whole Equinumerosity is attributed to the Stoics by Plutarch (λέγουσι καὶ δοξάζουσιν, "they say and think", 1079A8), it is unclear from T3.3 whether the explication of this view through the No-Part-Is-Greater Argument is Stoic or Plutarch's.

The No-Infinity-Is-Greater Argument in Plutarch is compatible with Alexander's and Philoponus' premise (ii) that an infinite plurality is not less than anything. Also, I cannot find any argument of ancient origin that some infinities are greater than others. Instead, such a logical inference had always been forceful evidence of the absurdity of the infinite in antiquity. It is likely that Archimedes, sharing the same intellectual milieu, developed the idea that the collection of all the triangles and that of all the lines are equal in multitude simply because they are infinite.

3.3 Applicability of Lemma 11

Lemma 11 is applicable to the measurement of various geometric objects beyond the cylinder segment: it is applicable to the measurement theorems in the *Method* except *Method* 1, and its finite version, *CS* 1, is frequently used in the *CS* in the measurement of the volume of different solids. In what follows I argue that Lemma 11 embodies an effort of rigorization and arithmetization of the use of the imagined balance. My argument starts with the observation that many theorems of the *Method* can be proved in the style of *Method* 14, i.e. removing the balance and using Lemma 11, if applicable. Then I give a survey of the application of *CS* 1, the finite version of Lemma 11, in the measurement of different solids in the *CS*. This will make clear Archimedes' contributions to the development of the technique of the method of exhaustion: He introduces a generalizable method of parallel slicing and deducing proportional relationships between corresponding slices, which, unlike the Euclidean proofs of circle quadrature, is not dependent on the shape of the figure under examination.

We have discussed *Method* 1 in the first chapter and *Method* 14 earlier in this chapter. *Method* 12 and 13 comprise an alternative proof of *Method* 14, using again the balance and relying on a significantly different argument than that of *Method* 14. Among the other theorems, *Method* 5, 6, 9, and 10 determine the center of gravity of solids, *Method* 1, 2, 3, 4, 7, and 8 are about the measurement of geometric objects, and *Method* 11 is concerned with both the volume and the center of gravity of any right segment of an amblyconoid. Except for *Method* 1, Lemma 11 can be applied to all the other measurement theorems, namely *Method* 2, 3, 4, 7, 8, and the measurement part of 11. Among these, *Method* 8, 10, and 11 only give conclusions and omit the proof, which Archimedes says can be made in the way (τρόπος) already indicated in the proofs of the previous theorems. For those measurement theorems that are given a proof in the *Method*, which are *Method* 2, 3, 4, and 7, their proofs can be converted into ones in the style of *Method* 14, by removing the balance and using Lemma 11 instead. For example, *Method* 2 can be rewritten in the following way:

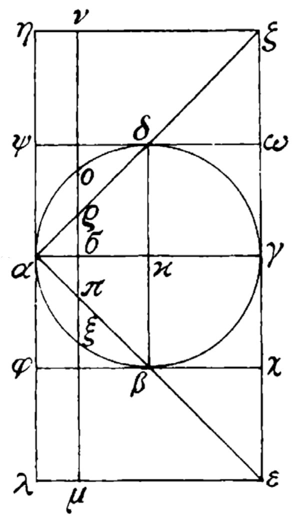

Figure 3-5

As shown in Figure 3-6, a cone $\alpha\varepsilon\zeta$ is inscribed in a cylinder $\varepsilon\zeta\eta\lambda$. Take their common axis $\alpha\gamma$ as diameter and construct a sphere $\alpha\beta\gamma\delta$. Through any point σ on $\alpha\gamma$ draw a plane perpendicular to $\alpha\gamma$, intersecting the cone $\alpha\varepsilon\zeta$, the sphere $\alpha\beta\gamma\delta$, and the cylinder $\varepsilon\zeta\eta\lambda$ and producing three circles with radius $\rho\sigma$, $o\sigma$ and $\nu\sigma$ respectively. It is proved in *Method* 2 that for any point σ, $\nu\sigma^2 : (\rho\sigma^2 + o\sigma^2) :: \alpha\gamma : \alpha\sigma$. Because $\alpha\sigma = \rho\sigma$ and $\alpha\gamma = \nu\sigma$, we have $\nu\sigma^2 : (\rho\sigma^2 + o\sigma^2) :: \nu\sigma : \rho\sigma$. Let the circle with radius $\nu\sigma$ in the cylinder $\varepsilon\zeta\eta\lambda$ be a_i, the line segment $\nu\mu$ in the square $\alpha\gamma\zeta\eta$ be b_i, the sum of circle with radius $\rho\sigma$ in the cone $\alpha\varepsilon\zeta$ and the circle with radius $o\sigma$ in the sphere $\alpha\beta\gamma\delta$ be c_i, and the line segment $\rho\sigma$ in the triangle $\alpha\gamma\delta$ be d_i. We have for **(a)** any i and j, $a_i : a_j :: b_i : b_j$, because all the *A*s are equal and all the *B*s are equal, and **(b)** for any i, $a_i : c_i :: b_i : d_i$, since $\nu\sigma^2 : (\rho\sigma^2 + o\sigma^2) :: \nu\sigma : \rho\sigma$. From this we conclude that **(c)** the ratio of the cylinder $\varepsilon\zeta\eta\lambda$ to the sum of the sphere $\alpha\beta\gamma\delta$ and the cone $\alpha\varepsilon\zeta$ is the same as the ratio of the square $\alpha\gamma\zeta\eta$ to the triangle $\alpha\gamma\delta$, which is 2:1. Since the cone $\alpha\varepsilon\zeta$ is a third of the cylinder of the same base and height, the sphere $\alpha\beta\gamma\delta$ is a half of the cone $\alpha\varepsilon\zeta$.

Lemma 11 is also applicable to the other theorems on measurement, namely *Method* 3, 4, 7, 8, and 11. Thus we can prove these theorems in the style of *Method* 14 just like what we have done with *Method* 2. These theorems of measurement, including *Method* 2, have rigorous counterparts through the method of exhaustion in Archimedes' other works. Below I give a chart of the rigorous proofs of each of these theorems.

Table 3-1

	Rigorous proof	Content
Method 2	*SC* I 34	Volume of a sphere
Method 3	*CS* 27	Volume of a spheroid
Method 4	*CS* 21	Volume of a right segment of an orthoconoid
Method 7	*SC* II 2	Volume of a spheric segment
Method 8 (statement only)	*CS* 29 and 31	Volume of a spheroid segment
Method 11 (statement only)	*CS* 25	Volume of a right segment of an amblyconoid

Some of the rigorous proofs, *CS* 21, 25, and 27, which correspond to *Method* 4, 11, and 3, use the finite version of Lemma 11. Most notably, *CS* 21 uses the same argument as its counterpart *Method* 4. The theorem is about the measurement of a right segment of an orthoconoid, and an orthoconoid is the solid generated by a parabola rotating around its axis. The commonality between *Method* 4 and *CS* 21 lies in two aspects. Both slice the orthoconoid, the cylinder circumscribed about it, and the cone inscribed in it into corresponding parallel slices. And both use the same proportionality between the corresponding slices, which fulfils condition (b) of Lemma 11 (see section 3.1). If one converts *Method* 4 into a proof in the style of *Method*

14, which is easily doable by following the way in which *Method* 2 is converted, then the new proof—let it be called *Method* 4-Lemma 11—will be even closer to *CS* 21 than *Method* 14, as they both use Lemma 11. The difference between *Method* 4-Lemma 11 and *CS* 21 will only lie in the latter's use of the method of exhaustion.

Apart from similarity, differences between the theorems in the *Method* and their rigorous counterparts are also informative. Comparing *Method* 2 to the rigorous proof of *SC* I 34 and the sphere cubature in the *Elements*, the ways in which the three divide the sphere are significantly different. The Euclidean sphere cubature divides the sphere through a polar division and inscribes in each part a pyramid, *Method* 2 divides the sphere into parallel slices and uses an auxiliary cone, and *SC* I 34 inscribes solids in the sphere through a mixed division, in which the parallel division is predominant—it first inscribes in a greater circle of the sphere an even-sided polygon, then rotates the polygon around one of its diameter, and this results in a solid made up of cones and conic frusta whose bases are parallel. The innovation Archimedes brings to the measurement of a sphere lies in the parallel slicing he uses in both proofs, and this way of division does not depend on the particular shape of the figure to be measured but is generally applicable. For example, *Method* 3, the cubature of a spheroid, is not easily approached through a polar division. Instead, *Method* 3 follows the paradigm of *Method* 2 very closely in the same use of an auxiliary cone and the same proportionality between the corresponding slices. The rigorous proof of *Method* 3, which is *CS* 27, uses a purely parallel division along the axis of the spheroid. The process of parallel slicing, approximation through inscribing and circumscribing cylinders in each slice, and applying Lemma 11 has been established as a fixed approach shared by many proofs in the *CS*. This approach is a fruitful result of the attempt to formalize the intuition in the use of the balance in the *Method*, and Lemma 11 plays the role of arithmetization in this process.

3.4 Limitations of Lemma 11

In this section I show that there are two limitations of Lemma 11. First, Lemma 11 in its own right is not a reliable way of measurement. It is rather the combination of Lemma 11 with parallel slicing that is widely applicable and effective. This limitation of reliability is shown below through an example. The second limitation is about the generality of the approach of using Lemma 11 in combination with parallel slicing. Such an approach only partly captures the intuition in Archimedes' use of the imagined balance. As evidence for my argument, Lemma 11 is not applicable to *Method* 1, but the argument of *Method* 14 relies on the conclusion of *Method* 1, viz. the area of a parabolic segment. Thus, the idea of an imagined balance and the concept of the center of gravity remain irreducible and fundamental to Archimedes' measurement theorems. Instead of considering this a failure to fully arithmetize and rigorize Archimedes' approach of discovery, I think this provides alternative perspectives to the discussions about the foundation of mathematics and the relation between mathematical truths and physical phenomena.

Below I give an example where all the conditions of Lemma 11 are satisfied but the conclusion is wrong. I believe the error of this example is caused by the polar division.

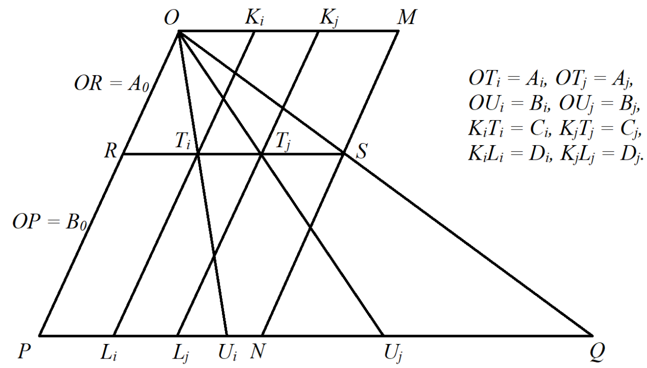

Figure 3-6

In the triangle *OPQ*, *RS* is parallel to *PQ*, intersecting *OP* and *OQ* at *R* and *S* respectively. Draw from the vertex *O* a random line intersecting *RS* and *PQ* at T_i and U_i. Let $A_i = OT_i$ and $B_i = OU_i$. It is easy to prove that for any *i* and *j*, triangle OT_iT_j is similar to triangle OU_iU_j. From this we have **(i)** for any *i* and *j*, $A_i : A_j :: B_i : B_j$. Also, for any A_i, there is a unique B_i, i.e. the whole in which A_i is a part.

Now draw a parallelogram *OMNP* with *OM* equal and parallel to *RS*. For any *i*, draw through T_i a line parallel to *OP*, intersecting *OM* and *PN* at K_i and L_i. Let $C_i = K_iL_i$ and $D_i = K_iT_i$. It is easy to show that C_i is equal to *OP* (= A_0) and D_i is equal to *OR* (= B_0). Again, for any C_i, there is a unique D_i, i.e. the whole in which C_i is a part.

Also, for every T_i there is a unique K_i, which means that for every A_i there exists a unique C_i. Whether we believe all infinites are equal or take one-to-one mapping as the criterion of cardinality, we have all of $\{A_i\}$, all of $\{B_i\}$, all of $\{C_i\}$, and all of $\{D_i\}$ are equinumerous. Again, triangle OT_iK_i is similar to triangle $U_iT_iL_i$, and triangle OT_iR is similar to triangle OU_iP. This means **(ii)** for any *i*, $A_i : C_i :: A_i : A_0 :: B_i : B_0 :: B_i : D_i$.

Thus, by Lemma 11, we have **(iii)** all of $\{A_i\}$ (= triangle *ORS*) : all of $\{C_i\}$ (= parallelogram *OMSR*) :: all of $\{B_i\}$ (= triangle *OPQ*) : all of $\{D_i\}$ (= parallelogram *OMNP*). But this is wrong. Triangle *ORS* is a half of parallelogram *OMSR*, but triangle *OPQ* is greater than a half of parallelogram *OMNP* (which would be triangle *OPN*).

The motion and division of a circle has been a classic locus for discussions of infinity and continuity. For example, the wheel paradox in the pseudo-Aristotelian *Mechanica* has stimulated solutions and debates across a wide time range from antiquity to the early modern times and finds echoes in Bolzano's discussions of the one-to-one mapping between the points on two concentric arcs in his *Paradoxien des Unendlichen*, a work influential to Cantor.[55] My example with Lemma 11 follows this tradition by using a mixed division—I divide the triangle through a polar division and the parallelogram through a parallel division. This example shows that a *reliable* generalization of Archimedes' approach of discovery assumes a parallel division, which, though never made explicit, is a consistent practice in both the *Method* and many of Archimedes' exhaustion proofs such as *QP* 1, some in *EP* I and II, and all the major findings of *SC* I and *CS*.

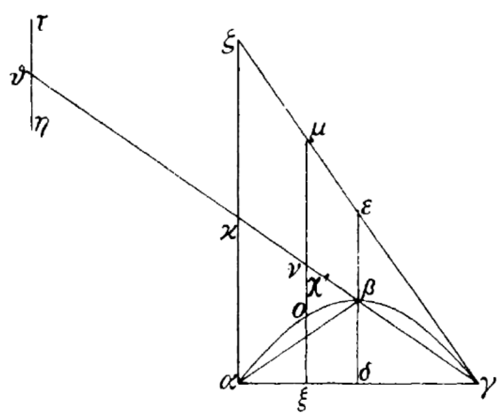

Figure 3-7 Diagram for *Method* 1 in Heiberg's edition (vol. 2 p.435)

[55] For the history of the wheel paradox, see Drabkin 1950.

The second limitation of Lemma 11 is about its generality. For some cases, Lemma 11 is not applicable, but the mechanical element of the balance remains useful. This is the case for Method 1. And what further undermines the generality of Lemma 11 is the fact that Method 14, which is done by usind Lemma 11 and not the balance, ultimately relies on the conculison of Method 1. In other words, Lemma 11 can fully replace the balance in the cases of *Method* 2, 3, 4, 7, 8, 11, but in the only case where it is actually used by Archimedes independetly without the balance[56], i.e. *Method* 14, Lemma 11 does not replace the balance on a fundamental level.

It is easy to show why Lemma 11 is not applicable to *Method* 1. Below I use Heiberg's diagram again to illustrate this point. In a parabolic segment $\alpha\beta\gamma$, draw through the vertex β lines $\delta\beta\varepsilon$ and $\alpha\kappa\zeta$ parallel to the axis of the parabola, and let the tangent to the parabola at γ intersect $\delta\beta\varepsilon$ at ε and $\alpha\kappa\zeta$ at ζ. Through any point ξ on $\alpha\gamma$ draw a line $\mu\xi$ parallel to $\varepsilon\beta\delta$ intersecting $\gamma\zeta$, $\gamma\alpha$, $\gamma\kappa$, and the parabola $\alpha\beta\gamma$ at μ, ξ, ν, and o.

The inapplicability of Lemma 11 is straightforward: Let $\mu\xi$ be A_i, $\alpha\gamma$ be B_i, $o\xi$ be C_i, and $\alpha\xi$ be D_i. It is proved in *Method* 1 that (b) $\mu\xi : o\xi :: \alpha\gamma : \alpha\xi$. But (a), namely $A_i : A_j :: B_i : B_j$ for any i and j, does not hold, because for any i, $B_i = \alpha\gamma$ is constant but $A_i = \mu\xi$ is not. This makes Lemma 11 inapplicable to *Method* 1.

On the other hand, the conclusion of *Method* 1 is used as a basis for *Method* 14. By Lemma 11, we have **(c)** the triangular prism : the cylinder hoof :: the parallelogram : the parabolic segment. It is from *Method* 1 that we know that the parallelogram : the parabolic segment :: 3:2. Given Lemma 11's inapplicability to *Method* 1, the absence of the balance in *Method* 14 is not true, but the arithmetization of the balance through Lemma 11 is only partially successful.

[56] *Method* 12 uses both Lemma 11 and the balance.

Conclusion

The method of parallel slicing and using an arithmetic tool, i.e., Lemma 11, is a critical development of the technique of the method of exhaustion. In the *Elements*, the exhaustion proofs of the measurement of circle, cone, and sphere all use a polar division that depends on the circularity of the objects, and the measurement of the pyramid uses a parallel division, which is also convenient for its rectilinear shape. On the other hand, Archimedes' demonstrations in the *CS* do not depend on the shape of the objects but follow a generalizable approach of slicing the geometric object under question into parallel slices and looking for proportionality between corresponding slices.[57]

This way of applying Lemma 11 is a fruitful product of the rigorization and arithmetization of Archimedes' innovative use of the balance in the *Method*. Still, Lemma 11 does not provide a thorough and universal foundation for what Archimedes finds illuminating in the use of the balance. Despite the wide applicability of Lemma 11, *Method* 1 cannot be adapted into a proof in the style of *Method* 14, because Lemma 11 is not applicable to *Method* 1. On the contrary, *Method* 14 relies on *Method* 1 for using its conclusion, i.e., the area of a parabolic segment. Therefore, the use of the balance in *Method* 1 is not reducible to a theory of proportion but remains a basis for more advanced theorems like that of *Method* 14.

There remains the question of what exactly is being rigorized in Archimedes' new technique of combining parallel slicing with Lemma 11. So far I have referred to this as "the key finding" or "what Archimedes finds illuminating", which may or may not seem to a reader to be an alternative way of saying "intuition". I do not find it helpful to repel intuition from mathematical

[57] Cf. Knorr 1996: 79: "Indeed, we can derive from indivisibilist precedents, like the sectioning of the cone and pyramid, a possible incentive for Archimedes' development of the alternative convergence technique." On Archimedes two-sided compression technique see Knorr 1993: 161-63.

investigations or scientific activities in general, but there is the need to explicate in each case what the intuition is specifically. It is true that Archimedes studies the lever law and the center of gravity in his works *On the Equilibrium of Planes* I and II and applies some of the results he achieves in that work to the *Method*. But the application is not done by attributing physical properties, such as weight and density, to geometric objects, but is done the other way round: Archimedes develops a mathematical model for the study a set of physical phenomena that pertains to equilibrium. This model is worth studying in its own right because it has the potential to push forward the development of mathematics itself. The example proofs in the *Method* and the rigorous demonstrations in the *CS* are efforts to bring out that potential in two different directions. The one is to make the mechanical intuition the starting point of the visionary goal of instituting an art of discovery, the other is to use arithmetic tools to incorporate non-rigorous techniques inspired by mechanical intuition into mathematics.

Sir Michael Atiyah 1995 lists four ways in which mathematicians interact with ideas and techniques that arise from physics. The first is to "[t]ake the heuristic results 'discovered' by physicists and try to give rigorous proofs by other methods" (Atiyah 1995: 1), which corresponds to the exhaustion proofs such as *QP* 2 and the proofs in *SC* I and II; the second is to "try to understand the physics involved and enter into a dialogue with physicists concerned" (Atiyah 1995: 1); the third is to "try to develop the physics on a rigorous basis so as to give a formal justification to the conclusions" (Atiyah 1995: 2); and the fourth is to "try to understand the deeper meanings of the physics-mathematics connection". Archimedes' new technique of exhaustion proofs through parallel slicing and applying Lemma 11 comes close to the third approach, and his study of the level law and the center of gravity can be interpreted as a form of the second approach, with a possible interest in the fourth approach. On the other hand, the strategy employed in the *Method* falls out of all these four categories. While it is helpful to incorporate non-rigorous techniques borrowed from the physical sciences into mathematical *demonstrations*, there is also the possible strategy that leaves the process of discovery as open and transparent as possible in order to facilitate mathematical *discoveries*. I hope this insight of Archimedes is of help to those who, like Archimedes, endeavor to contribute to the mathematical enterprise and enjoy and value the process of discovery *per se*.

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