



Nearby Cycles of Whittaker Sheaves

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Nearby cycles of Whittaker sheaves

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by

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Abstract

In this thesis we study the nearby cycles of a Whittaker sheaf as it degenerates to an object of the principal series category. In the case of a finite-dimensional flag variety, we prove that the nearby cycles sheaf corresponds under Beilinson-Bernstein localization to the big projective object of the BGG category \mathcal{O} . We also introduce a resolution of singularities for Drinfeld's compactification, using it to prove a key local acyclicity statement for the intersection cohomology sheaf. This acyclicity is then applied in the study of nearby cycles of Whittaker sheaves on Drinfeld's compactification. Namely, we express the strata restrictions of the nearby cycles sheaf in Langlands dual terms, and also describe the subquotients of the monodromy filtration using the Picard-Lefschetz oscillators introduced by Schieder.

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Introduction

1 Overview

1.1 Whittaker functions originated in number theory, in the context of harmonic analysis for reductive groups over local and global fields. The idea is to restrict automorphic functions to a space on which the associated p -adic or adelic group acts, and then to extract their Fourier coefficients with respect to the unipotent radical of a Borel.

If we work over a function field, meaning a local or global field of positive characteristic, then many of the coset spaces appearing in the theory of automorphic representations have a natural geometric structure over the finite ground field. In this setting, the *fonctions-faisceaux* correspondence allows us to pass from an ℓ -adic sheaf on a scheme (or a more general algebro-geometric object) to a function on its set of points over the ground field. Many functions of interest can be geometrized in this way, and Whittaker functions are no exception.

In this thesis all geometry takes place in characteristic zero, with algebraic D-modules serving as a sheaf theory. However, the main results hold *mutatis mutandis* for ℓ -adic sheaves in characteristic $p \neq \ell$.

1.2 As Whittaker functions in number theory are of interest largely due to their relevance to the Langlands program, so Whittaker sheaves play an important role in the geometric Langlands program. In [FGV], the authors constructed a local geometric Whittaker model and used it to give a geometric version of the Casselman-Shalika formula. Their results imply that Whittaker sheaves on the affine Grassmannian are equivalent to representations of the Langlands dual group.

Due to the inherent difficulties of constructing Whittaker sheaves in the local setting (for

instance, their supports are always infinite-dimensional), the authors of *loc. cit.* used Drinfeld’s compactification, a global object, as an approximation. Although intrinsically local methods are now available, for our purposes their approach offers a technical simplification, and moreover Whittaker sheaves on Drinfeld’s compactification play a role in the global Langlands theory.

1.3 The purpose of this thesis is to study the degeneration of a Whittaker sheaf as its Fourier coefficients tend to zero. This degeneration is encoded in a perverse sheaf¹ via the operation of nearby cycles. The property of having vanishing Fourier coefficients means that the nearby cycles sheaf belongs to the principal series category, which geometrizes the universal principal series representation in automorphic representation theory.

We attempt to describe this nearby cycles sheaf, with its associated monodromy endomorphism, as explicitly as possible. This enhances our understanding of gluing in the extended Whittaker category, an important ingredient in Gaitsgory’s program to prove the global unramified geometric Langlands conjecture. Namely, in [G2] Gaitsgory conjectured that the functor of extended Whittaker coefficient embeds the category of automorphic sheaves into the extended Whittaker category. This has been proved for a general linear group in [Bera], but remains open for a general reductive group, and is perhaps the most substantial obstacle to the completion of Gaitsgory’s program.

The extended Whittaker category is stratified by partial Whittaker categories, which interpolate between the nondegenerate Whittaker category and the principal series category. It can be recovered by a formal procedure from the partial Whittaker categories and the “gluing functors” between them, given by $!$ -extension followed by $!$ -restriction. These gluing functors can be recovered from corresponding nearby cycles functors as derived invariants of monodromy. Although the gluing functors can be studied directly as in [AG2], nearby cycles has the advantage of being t -exact, so that it sends perverse sheaves to perverse sheaves.

¹For us “perverse sheaf” will be synonymous with “holonomic D-module.” We do not require regular singularities, and in fact Whittaker sheaves are always irregular.

1.4 In Part I we study nearby cycles of Whittaker D-modules on a finite-dimensional flag variety. This can be viewed as a toy model for the situation on Drinfeld’s compactification. Moreover, Theorem 2.3.1, which says that the nearby cycles sheaf corresponds to the big projective module under Beilinson-Bernstein localization, may be of independent interest.

Part II is an interlude in which we introduce a resolution of singularities for Drinfeld’s compactification. This resolution is a certain moduli stack of stable maps, similar to the “Kontsevich compactification” introduced in the appendix to [FFKM] for the space of quasimaps. We use this resolution to prove a local acyclicity property of the intersection cohomology sheaf on Drinfeld’s compactification, to be applied in Part III. This acyclicity was asserted without proof as Proposition 7.9 in Gaitsgory’s work [G1] on the quantum geometric Langlands correspondence, where it was used to construct an equivalence between representations of the quantum group at a generic parameter and a category of twisted Whittaker sheaves. Lysenko has also applied our local acyclicity theorem in [L] to establish results analogous to Gaitsgory’s in the metaplectic Langlands program.

In Part III we study the degeneration of a Whittaker sheaf on Drinfeld’s compactification to an object of the principal series category. In [AG2] this degeneration is implemented by a gluing functor, meaning $!$ -extension from the general locus followed by $!$ -restriction to the special fiber. This operation produces a complex of sheaves, which the authors of *loc. cit.* link to the constant term of Poincaré series. We instead consider the closely related nearby cycles sheaf, computing its strata restrictions in terms of the Langlands dual Lie algebra, and describing the subquotients of the monodromy filtration in terms of the Picard-Lefschetz oscillators introduced in [Sch1].

2 Survey of the main results

2.1 *Notation and conventions:* Let k be an algebraically closed field of characteristic zero and G a connected reductive group over k . Fix a Borel subgroup $B \subset G$ with unipotent

radical N and put $T := B/N$. Fix a splitting $T \rightarrow B$, so that there is a well-defined opposite Borel B^- with unipotent radical N^- . The set of vertices of the Dynkin diagram of G will be denoted by I .

For Y a scheme or stack locally of finite type, we write $D(Y)$ for the unbounded derived category of D -modules on Y , and $D_{\text{hol}}(Y)$ for the subcategory of bounded complexes with holonomic cohomology sheaves. For our purposes these can be viewed as triangulated categories, with the exception of Section 9.2, where the language of DG categories is used.

If $f : Y \rightarrow Z$ is a smooth morphism of relative dimension d , we sometimes write

$$f^\Delta := f^![-d] : D(Z) \rightarrow D(Y)$$

for the cohomologically normalized inverse image functor.

2.2 This main body of this thesis is divided into three parts. They can be read independently, and are logically independent except for the application of Corollary 8.1.1.1 in the second proof of Theorem 2.7.1.

In Part I we work with the finite-dimensional flag variety G/B . In this case, the nearby cycles sheaf belongs to $(D_{\text{hol}}(G/B)^N)^\heartsuit$, the abelian category of N -equivariant holonomic D -modules on the flag variety G/B , which is equivalent to the principal block of the BGG category \mathcal{O} under Beilinson-Bernstein localization (see [BB]).

The N -orbits in G/B are labeled by the Weyl group W . For $w \in W$ we denote by $C_w = NwB/B$ the corresponding Schubert cell, and $X_w = \overline{C_w}$ the Schubert variety. The big cell C_{w_0} (here $w_0 \in W$ is the longest element) is an N -torsor, and the opposite Borel B^- determines a trivialization $N \cong C_{w_0}$.

Fix a homomorphism $\psi : N \rightarrow \mathbb{G}_a$ which is nondegenerate, meaning ψ is nontrivial on each of the simple root subgroups. Let \exp denote the exponential D -module on \mathbb{G}_a . In this context the Whittaker sheaf \mathscr{W}_1 is defined to be the direct image of $\psi^\Delta \exp$ along the open embedding $N \cong C_{w_0} \rightarrow G/B$ onto the big cell. Both direct image functors give the

same result, for the following reason: the nondegeneracy assumption on ψ implies that it is nontrivial when restricted to the stabilizer of any point of G/B in the complement of the big cell. Thus $\psi^\Delta \exp$ extends cleanly to G/B , i.e. the canonical map from the $!$ -extension to the $*$ -extension is an isomorphism.

The torus T acts on G/B by left translations. Fixing a dominant regular coweight $\gamma : \mathbb{G}_m \rightarrow T$, we obtain an action of \mathbb{G}_m on G/B . Denote by $a_\gamma : \mathbb{G}_m \times G/B \rightarrow G/B$ the action map. Define $\mathscr{W} := \alpha_\gamma^\Delta \mathscr{W}_1$. Thus we may consider the nearby cycles sheaf $\Psi(\mathscr{W})$, where nearby cycles is taken with respect to the projection $\mathbb{A}^1 \times G/B \rightarrow \mathbb{A}^1$. By construction $\Psi(\mathscr{W})$ is a perverse sheaf on G/B .

2.3 The abelian category $(D_{\text{hol}}(G/B)^N)^\heartsuit$ has finitely many simple objects, labeled by elements of the Weyl group W , which we will denote by \mathcal{L}_w for $w \in W$. Recall that $\mathcal{L}_w = \text{IC}_{X_w}$, the intersection cohomology sheaf on X_w , or rather its direct image in G/B . Denote by \mathcal{P}_w a projective cover of \mathcal{L}_w , so \mathcal{P}_w is characterized up to noncanonical isomorphism as an indecomposable projective object which maps onto \mathcal{L}_w .

We denote by $e \in W$ the identity element. Observe that \mathcal{L}_e is the delta sheaf at the closed Schubert cell, which is a point. Its projective cover \mathcal{P}_e is the longest indecomposable projective object in $(D_{\text{hol}}(G/B)^N)^\heartsuit$; it is often referred to as the *big projective*.

Theorem 2.3.1. *The sheaf $\Psi(\mathscr{W})$ is a projective cover of \mathcal{L}_e , i.e. we have $\Psi(\mathscr{W}) \cong \mathcal{P}_e$.*

It is well-known that \mathcal{P}_e can also be characterized as the unique tilting extension of the constant perverse sheaf on the big cell. Here *tilting* means that its $!$ - and $*$ -restrictions to the Schubert cells are perverse. In Part III we will show that the nearby cycles of a Whittaker sheaf on Drinfeld's compactification is also tilting with respect to a suitable stratification.

Theorem 2.3.1 gives a canonical geometric construction of \mathcal{P}_e . Another such construction, also involving a Whittaker sheaf, appears in Lemma 4.4.11 of [BY], as well as Proposition 14.3.1 of [FG] (see also the related [BBM]). Namely, letting \mathscr{W}^- be the Whittaker sheaf on G/B attached to a nondegenerate $\psi^- : N^- \rightarrow \mathbb{G}_a$, the N -average of \mathscr{W}^- is

isomorphic to \mathcal{P}_e . The $!$ - and $*$ -averaging functors give the same result thanks to the Verdier self-duality of \mathcal{P}_e , although the canonical morphism from the $!$ -average to the $*$ -average is not an isomorphism.

2.4 Let X be a connected smooth projective curve over k . For any algebraic group H we write Bun_H for the moduli stack of H -bundles on X . We will assume for simplicity that the derived subgroup of G is simply-connected.

The purpose of Part II is to introduce a resolution of singularities for Drinfeld's relative compactification $\overline{\mathrm{Bun}}_B \rightarrow \mathrm{Bun}_G$ (see Section 7.1 for a definition sketch). In the appendix to [FFKM] such a resolution is constructed, when X has genus zero, for the space of quasimaps, which is the fiber of Drinfeld's compactification over the trivial G -bundle. Their resolution parameterizes stable maps (in the sense of Kontsevich, see [Kon]) into the flag variety. When the G -bundle is allowed to vary, we obtain a relative compactification $\overline{\mathrm{Bun}}_B^K \rightarrow \mathrm{Bun}_G$ mapping to $\overline{\mathrm{Bun}}_B$ over Bun_G , which we call Kontsevich's compactification. The stack $\overline{\mathrm{Bun}}_B^K$, unlike the space of quasimaps, is smooth for X of arbitrary genus (see Proposition 6.4.1 below). Moreover, our Proposition 8.4.1 implies that the complement of Bun_B in $\overline{\mathrm{Bun}}_B^K$ is a normal crossings divisor.

In the case $G = \mathrm{GL}_n$ Laumon's compactification resolves the singularities of $\overline{\mathrm{Bun}}_B$ (see e.g. [Kuz]), but it is unclear how to generalize this construction to arbitrary reductive G . The approach via stable maps looks completely different: whereas Laumon's compactification consists of quasimaps into a smooth stack, Kontsevich's compactification allows the curve to vary and to acquire nodes.

2.5 We use Kontsevich's compactification to prove Theorem 8.2.1, which says that the (possibly twisted) IC sheaf of $\overline{\mathrm{Bun}}_B$ is universally locally acyclic over Bun_G at points sufficiently antidominant relative to their defect. This follows from Theorem 8.1.1, which asserts that the projection $\overline{\mathrm{Bun}}_B^K \rightarrow \mathrm{Bun}_G$ is smooth at points sufficiently antidominant relative to their defect. In the untwisted case, this immediately implies that the IC sheaf of $\overline{\mathrm{Bun}}_B^K$, a shif-

ted constant sheaf, is universally locally acyclic over Bun_G in the relevant domain. Some additional care is required in the twisted case, with the normal crossings property of the boundary playing a crucial role. Then we apply the decomposition theorem for the proper map $\overline{\text{Bun}}_B^K \rightarrow \overline{\text{Bun}}_B$, which implies that the twisted IC sheaf of $\overline{\text{Bun}}_B$ is a direct summand of the pushforward of the twisted IC sheaf of $\overline{\text{Bun}}_B^K$. Since universal local acyclicity is preserved under proper direct image and taking direct summands, Theorem 8.2.1 follows.

2.6 We now summarize the main results of this thesis, which appear in Part III.

Fix a square root $\omega_X^{\otimes \frac{1}{2}}$ of the canonical bundle on X , which gives rise to a T -bundle $\rho(\omega) := 2\rho(\omega_X^{\otimes \frac{1}{2}})$. Denote by $\overline{\text{Bun}}_{N^\omega}$ the fiber of $\overline{\text{Bun}}_B \rightarrow \text{Bun}_T$ over $\rho(\omega)$, which contains the dense open substack $\text{Bun}_{N^\omega} \subset \overline{\text{Bun}}_{N^\omega}$. The choice of nondegenerate $\psi : N \rightarrow \mathbb{G}_a$ gives rise to a map

$$\text{ev} : \text{Bun}_{N^\omega} \longrightarrow \mathbb{G}_a,$$

constructed for example in Section 4.1 of [FGV].

As in *loc. cit.*, one shows that $\text{ev}^\Delta \exp$ extends cleanly to a perverse sheaf \mathscr{W}_1 on $\overline{\text{Bun}}_{N^\omega}$. Although we use the same notation for the Whittaker sheaf as in Part I, it should not cause any confusion, since G/B will not appear in Part III.

By construction, there is a canonical action of T on $\overline{\text{Bun}}_{N^\omega}$. Denote by

$$a_\gamma : \mathbb{G}_m \times \overline{\text{Bun}}_{N^\omega} \rightarrow \overline{\text{Bun}}_{N^\omega}$$

the action map induced by the fixed dominant regular coweight $\gamma : \mathbb{G}_m \rightarrow T$. We put $\mathscr{W} := a_\gamma^\Delta \mathscr{W}_1$, a perverse sheaf on $\mathbb{G}_m \times \overline{\text{Bun}}_{N^\omega}$. The main object of study in this thesis is the perverse sheaf $\Psi(\mathscr{W})$ on $\overline{\text{Bun}}_{N^\omega}$.

2.7 The stack $\overline{\text{Bun}}_{N^\omega}$ has a stratification by defect of the generalized N -bundle. Our first main theorem is a description of the restrictions of $\Psi(\mathscr{W})$ to the strata. In particular we will prove that this sheaf is tilting with respect to the defect stratification, meaning its $!$ - and

*-restrictions to the strata are perverse (although generally not lisse).

Write Λ for the lattice of cocharacters of T , and denote by $\Lambda^{\text{pos}} \subset \Lambda$ the positive coweights with respect to B . The defect stratification of $\overline{\text{Bun}}_{N^\omega}$ is indexed by Λ^{pos} , and for each $\mu \in \Lambda^{\text{pos}}$ we denote the locally closed stratum embedding by

$$j_{=\mu} : \overline{\text{Bun}}_{N^\omega, =\mu} \longrightarrow \overline{\text{Bun}}_{N^\omega}.$$

Denote by $X^{(n)}$ the n -fold symmetrized power of X . If $\mu = \sum_{i \in I} n_i \alpha_i$ for some nonnegative integers n_i (here α_i is the simple coroot corresponding to i) then the corresponding configuration space of points in X is defined by

$$X^\mu := \prod_{i \in I} X^{(n_i)},$$

and there is a smooth surjection

$$m_\mu : \overline{\text{Bun}}_{N^\omega, =\mu} \longrightarrow X^\mu.$$

In [BG2] the authors introduced certain factorizable perverse sheaves Ω^μ on the configuration spaces X^μ . The $!$ -fiber of Ω^μ at a point $\sum_i \mu_i x_i$ is

$$\bigotimes_i C^\bullet(\check{\mathfrak{n}})^{\mu_i},$$

where $\check{\mathfrak{n}}$ is the unipotent radical of a Borel subalgebra in the Langlands dual Lie algebra $\check{\mathfrak{g}}$, and $C^\bullet(\check{\mathfrak{n}})$ is its \check{T} -graded cohomological Chevalley complex.

Theorem 2.7.1. *For any $\mu \in \Lambda^{\text{pos}}$ there is an isomorphism $j_{=\mu}^! \Psi(\mathscr{W}) \xrightarrow{\sim} m_\mu^\Delta \Omega^\mu$.*

In fact, Ω^μ is indecomposable and m_μ has contractible fibers, so the isomorphism in the theorem is automatically unique up to scaling. Since $\Psi(\mathscr{W})$ is Verdier self-dual, Theorem 2.7.1 implies that it is tilting with respect to the defect stratification.

2.8 Recall that the *monodromy filtration* on $\Psi(\mathcal{W})$ is the unique filtration by perverse sheaves

$$F_{-m} \subset \cdots \subset F_{m-1} \subset F_m = \Psi(\mathcal{W})$$

such that for all $1 \leq i \leq n$, the i^{th} power of the monodromy endomorphism induces an isomorphism $F_i/F_{i-1} \xrightarrow{\sim} F_{-i}/F_{-i-1}$. The associated graded sheaf $\text{gr } \Psi(\mathcal{W})$ has an action of the so-called Lefschetz \mathfrak{sl}_2 such that the lowering operator is induced by the monodromy endomorphism and F_i/F_{i-1} has weight i for the Cartan operator.

We now formulate a description of $\text{gr } \Psi(\mathcal{W})$ in terms of certain factorizable perverse sheaves with \mathfrak{sl}_2 -action, called the *Picard-Lefschetz oscillators* after [Sch1] and [Sch2] (see also Section 3.2 of *loc. cit.*).

First let us define the Picard-Lefschetz oscillators on $X^{(n)}$. Let std denote the standard 2-dimensional representation of the Lefschetz \mathfrak{sl}_2 , and write sgn for the sign character of the symmetric group Σ_n . The $\Sigma_n \times \mathfrak{sl}_2$ -representation $\text{sgn} \otimes \text{std}^{\otimes n}$ (here Σ_n also permutes the std factors) gives rise to a local system with \mathfrak{sl}_2 -action on the disjoint locus $X_{\text{disj}}^{(n)}$, by applying the associated bundle construction to the canonical Σ_n -torsor $X_{\text{disj}}^n \rightarrow X_{\text{disj}}^{(n)}$. Then \mathcal{P}_n is defined as the intermediate extension to $X^{(n)}$ of the (perverse cohomological shift of the) local system attached to $\text{sgn} \otimes \text{std}^{\otimes n}$. The perverse sheaf \mathcal{P}_n carries an \mathfrak{sl}_2 -action by functoriality, and is evidently semisimple.

Recall that a *Kostant partition* of $\mu \in \Lambda^{\text{pos}}$ is an expression of the form $\mu = \sum_{\beta \in R^+} n_\beta \beta$, where the n_β are nonnegative integers and R^+ denotes the set of positive coroots. Denote by $\text{Kost}(\mu)$ the set of Kostant partitions of μ . To any $\mathfrak{k} \in \text{Kost}(\mu)$ given by $\mu = \sum_{\beta \in R^+} n_\beta \beta$ we attach a partially symmetrized power

$$X^{\mathfrak{k}} := \prod_{\beta \in R^+} X^{(n_\beta)},$$

which is equipped with a canonical finite map $\iota^{\mathfrak{k}} : X^{\mathfrak{k}} \rightarrow X^\mu$.

The *Picard-Lefschetz oscillator* on X^μ is defined by the formula

$$\mathcal{P}^\mu := \bigoplus_{\mathfrak{k} \in \text{Kost}(\mu)} i_*^{\mathfrak{k}} \left(\boxtimes_{\beta \in R^+} \mathcal{P}_{n_\beta} \right).$$

In particular, we have $\mathcal{P}^{n\alpha} = \mathcal{P}_n$ for α a simple coroot. By construction, \mathcal{P}^μ is a semisimple perverse sheaf with \mathfrak{sl}_2 -action.

Theorem 2.8.1. *There is an \mathfrak{sl}_2 -equivariant isomorphism*

$$\text{gr } \Psi(\mathcal{W}) \xrightarrow{\sim} \bigoplus_{\mu \in \Lambda^{\text{pos}}} j_{=\mu, !*} m_\mu^\Delta \mathcal{P}^\mu. \quad (2.8.1)$$

In particular $\text{gr } \Psi(\mathcal{W})$ is semisimple. Using the theorem we can compute the kernel of the monodromy operator, whose simple subquotients are the lowest weight sheaves for the Lefschetz \mathfrak{sl}_2 -action.

Corollary 2.8.1.1. *The canonical morphism $j_{=0, !} \text{IC}_{\text{Bun}_{N^\omega}} \rightarrow \Psi(\mathcal{W})$ is an isomorphism onto the kernel of the monodromy operator.*

3 Further directions

3.1 We now try to place Theorem 2.3.1 in context, and to suggest a possible generalization, namely a geometric interpretation of translation out of the wall.

As explained in Remark 14 of [W], Theorem 4.4.1 of [BY] implies that if we allow a general $\psi : N \rightarrow \mathbb{G}_a$, the resulting category of possibly degenerate Whittaker sheaves is equivalent to a possibly singular block of the category \mathcal{O} : the more degenerate ψ is, the less singular is the central character defining the block. At one extreme we can identify the category of nondegenerate Whittaker sheaves on G/B with the most singular block of \mathcal{O} , since they are both equivalent to finite-dimensional vector spaces. This equivalence can be normalized to send the generator of the singular block to \mathcal{W}_1 . At the other extreme we have

the aforementioned equivalence between N -equivariant D-modules on G/B and the regular block of the category \mathcal{O} .

An interesting problem is to study the nearby cycles functors between degenerate Whittaker categories, and to determine which functors between blocks of \mathcal{O} they correspond to. Translation from the most singular block to the regular block sends the generator of the singular block to the big projective. Therefore, in that case, the nearby cycles functor identifies with translation out of the wall. In future work the author hopes to investigate whether this holds more generally.

3.2 Now we transition back to the case of Drinfeld's compactification. A key step in the proof of Theorem 2.8.1 is establishing that $\text{gr } \Psi(\mathcal{W})$ is semisimple. Below we will give a standard weight-theoretic argument, justified by Mochizuki's theory of weights for holonomic D-modules developed in [M]. We also present a more direct proof of Theorem 2.8.1 which uses the remarkable action, constructed in [FFKM], of \mathfrak{g} on the intersection cohomology of quasimap spaces. As we now explain, this action can be obtained via a deformation-theoretic argument from the global unramified geometric Langlands conjecture, together with its expected compatibility with geometric Eisenstein series. This leads us to a natural generalization of the action from [FFKM], which the author intends to return to in future work.

Write $\bar{\mathfrak{p}} : \overline{\text{Bun}}_B \rightarrow \text{Bun}_G$ and $\bar{\mathfrak{q}} : \overline{\text{Bun}}_B \rightarrow \text{Bun}_T$ for the canonical morphisms. The functor of *compactified Eisenstein series* $\text{Eis}_{!*} : D(\text{Bun}_T) \rightarrow D(\text{Bun}_G)$, introduced in [BG1], is defined by

$$\text{Eis}_{!*}(-) = \bar{\mathfrak{p}}_*(\text{IC}_{\overline{\text{Bun}}_B} \overset{!}{\otimes} \bar{\mathfrak{q}}^!(-)).$$

For any algebraic group H , we denote by LS_H the DG algebraic stack of de Rham H -local systems on X . For any DG stack \mathcal{Y} we denote by $\text{QCoh}(\mathcal{Y})$ the unbounded derived category of quasicoherent sheaves on \mathcal{Y} .

The form of the geometric Langlands conjecture which we now state is a coarse version

of the one formulated in [AG1], but at this level of precision it goes back to Beilinson and Drinfeld. It is expected to enjoy various compatibilities, but here we only describe the conjectural interaction with Eisenstein series. Let

$$\mathfrak{t}^{\text{spec}} : \text{LS}_{\check{T}} \longrightarrow \text{LS}_{\check{G}}$$

be the map induced by the inclusion $\check{T} \rightarrow \check{G}$.

Conjecture 3.2.1 (Global unramified geometric Langlands). *There is a fully faithful functor*

$$\mathbb{L}_G : \text{QCoh}(\text{LS}_{\check{G}}) \longrightarrow D(\text{Bun}_G)$$

which makes the following square commute up to natural isomorphism:

$$\begin{array}{ccc} \text{QCoh}(\text{LS}_{\check{T}}) & \xrightarrow{\rho(\omega) \circ \mathbb{L}_T} & D(\text{Bun}_T) \\ \downarrow \mathfrak{t}_*^{\text{spec}} & & \downarrow \text{Eis}_{!*} \\ \text{QCoh}(\text{LS}_{\check{G}}) & \xrightarrow{\mathbb{L}_G} & D(\text{Bun}_G). \end{array}$$

Here $\rho(\omega)$ denotes the automorphism of $D(\text{Bun}_T)$ given by translation by the same-named T -bundle.

3.3 Kodaira-Spencer theory tells us that the tangent space to $\text{LS}_{\check{G}}$ at a \check{G} -local system $E_{\check{G}}$ is identified with $H_{\text{dR}}^\bullet(X, \check{\mathfrak{g}}_{E_{\check{G}}})[1]$, where $\check{\mathfrak{g}}_{E_{\check{G}}}$ denotes the local system attached to $E_{\check{G}}$ and the adjoint representation $\check{\mathfrak{g}}$. Moreover, the (derived) endomorphism algebra of the skyscraper sheaf $\delta_{E_{\check{G}}}$ in $\text{QCoh}(\text{LS}_{\check{G}})$ is the enveloping algebra of the DG Lie algebra $H_{\text{dR}}^\bullet(X, \check{\mathfrak{g}}_{E_{\check{G}}})$, and in particular $H_{\text{dR}}^\bullet(X, \check{\mathfrak{g}}_{E_{\check{G}}})$ acts on $\delta_{E_{\check{G}}}$.

Applying these principles in the case $E_{\check{G}} = \mathfrak{t}^{\text{spec}}(E_{\check{T}})$ for a \check{T} -local system $E_{\check{T}}$, we derive the following concrete consequence of Conjecture 3.2.1. According to geometric class field theory $\mathbb{L}_T(\delta_{E_{\check{T}}})$ is a multiplicative flat line bundle, and in particular is invariant under translation by any T -bundle.

Conjecture 3.3.1. *The DG Lie algebra $H_{\mathrm{dR}}^\bullet(X, \check{\mathfrak{g}}_{E_{\check{T}}})$ acts on $\mathrm{Eis}_{!*}(\mathbb{L}_T(\delta_{E_{\check{T}}}))$.*

Moreover, Conjecture 3.2.1 predicts that the full endomorphism algebra of $\mathrm{Eis}_{!*}(\mathbb{L}_T(\delta_{E_{\check{T}}}))$ is the enveloping algebra of $H_{\mathrm{dR}}^\bullet(X, \check{\mathfrak{g}}_{E_{\check{T}}})$. We also remark that $\mathbb{L}_T(\delta_{E_{\check{T}}})$ has a non-conjectural description: it is the multiplicative line bundle with flat connection on Bun_T corresponding to $E_{\check{T}}$ under geometric class field theory. This means that it is characterized (among multiplicative flat line bundles) by the property that for any coweight $\lambda : \check{T} \rightarrow \mathbb{G}_m$, its inverse image along the map $X \rightarrow \mathrm{Bun}_T$ sending $x \mapsto \mathcal{P}_T^{\mathrm{triv}}(\lambda \cdot x)$ is $\lambda(E_{\check{T}})$.

When $E_{\check{T}}$ is trivial, which is the case we will apply below, Conjecture 3.3.1 says that $\check{\mathfrak{g}} \otimes H_{\mathrm{dR}}^\bullet(X)$ acts on $\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$. In Section 14.3 we specify the action of certain generators of $\check{\mathfrak{g}} \otimes H_{\mathrm{dR}}^\bullet(X)$, namely the factors in the triangular decomposition. In [FFKM] the authors verify the necessary relations for $\check{\mathfrak{g}} \otimes H^0(X)$, which suffices for our application. This approach seems intractable when trying to prove Conjecture 3.3.1 in full generality: the derivedness of the Lie algebra $\check{\mathfrak{g}} \otimes H_{\mathrm{dR}}^\bullet(X)$, or more generally $H_{\mathrm{dR}}^\bullet(X, \check{\mathfrak{g}}_{E_{\check{T}}})$, makes the checking of relations difficult or impossible to do “by hand.” In future work the author hopes to prove Conjecture 3.3.1 by a local-to-global argument, constructing the action from a more tractable local version which can be deduced from work of Raskin [R2].

I The case of a finite-dimensional flag variety²

4 Nearby cycles

4.1 Let Y be a scheme locally of finite type over k equipped with a morphism $f : Y \rightarrow \mathbb{A}^1$. Write $Y_0 := f^{-1}(0)$ and $\mathring{Y} := f^{-1}(\mathbb{G}_m)$. We temporarily denote the embeddings by

$$Y_0 \xrightarrow{i} Y \xleftarrow{j} \mathring{Y}.$$

In [Bei1] Beilinson gave a construction of the *unipotent nearby cycles functor*

$$\Psi : D_{\text{hol}}(\mathring{Y}) \longrightarrow D_{\text{hol}}(Y_0),$$

which has the following properties:

1. Ψ is t-exact,
2. Ψ commutes with inverse image along smooth maps and direct image along proper maps,
3. if \mathcal{F} in $D(Y)$ is lisse, then $\Psi(\mathcal{F}|_{\mathring{Y}}) = i^! \mathcal{F}[1]$,
4. Ψ comes equipped with an endomorphism such that for any perverse \mathcal{F} in $D_{\text{hol}}(\mathring{Y})$, the resulting endomorphism of $\Psi(\mathcal{F})$ is nilpotent.

This endomorphism of Ψ is called the *monodromy*. For the purposes of Part I we can treat Ψ as a black box, but in Section 9.3 below we review Beilinson's construction and give an interpretation which can serve as an alternative definition of Ψ .

²Part I has a substantial overlap with the article [C1].

5 Proof of Theorem 2.3.1

5.1 Now we return to the situation of Sections 2.2-2.3. Before proving the theorem, we make some observations.

Remark 5.1.1. Theorem 2.3.1 implies that up to isomorphism the D-module $\Psi(\mathcal{W})$ does not depend on γ , but in fact γ determines the monodromy endomorphism. Namely, note that \mathcal{P}_e is weakly T -equivariant and so $\mathfrak{h} := \text{Lie } T$ acts on \mathcal{P}_e by endomorphisms. If we fix an isomorphism $\varphi : \Psi(\mathcal{W}) \xrightarrow{\sim} \mathcal{P}_e$, then the action of $-d\gamma \in \mathfrak{h}$ on \mathcal{P}_e corresponds to the monodromy endomorphism of $\Psi(\mathcal{W})$ by Claim 2 in the appendix to [AB] (this makes sense independently of φ because $\text{End}(\mathcal{P}_e)$ is commutative, see e.g. [Bern]).

Example 5.1.2. Consider the case $G = \text{SL}_2$, so that $G/B = \mathbb{P}^1$. Since $N = \mathbb{G}_a$ we can take $\psi = \text{id}$, so that $\mathcal{W}_1 = j_* \exp$ where $j : \mathbb{A}^1 \rightarrow \mathbb{P}^1$ is the inclusion. Moreover $T = \mathbb{G}_m$ and γ is the n^{th} power map on \mathbb{G}_m for some $n > 0$.

Let $s = w_0 \in W = S_2$ be the nontrivial (and longest) element of the Weyl group. Then \mathcal{L}_s is the IC sheaf on \mathbb{P}^1 and \mathcal{L}_e is the delta sheaf at ∞ . The sheaf \mathcal{P}_e has a composition series

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 = \mathcal{P}_e$$

with $\mathcal{F}_1 \cong \mathcal{L}_e$, $\mathcal{F}_2/\mathcal{F}_1 \cong \mathcal{L}_s$, and $\mathcal{P}_e/\mathcal{F}_2 \cong \mathcal{L}_e$. The algebra of endomorphisms of \mathcal{P}_e is canonically identified with $k[\epsilon]/(\epsilon^2)$, and the monodromy endomorphism of $\Psi(\mathcal{W})$ corresponds to $-n\epsilon$.

Proof of Theorem 2.3.1. The proof proceeds in two steps: we will first show that for all $w \neq e$ we have $\text{Ext}^i(\Psi(\mathcal{W}), \mathcal{L}_w) = 0$ for all i , then prove that $\text{Hom}(\Psi(\mathcal{W}), \mathcal{L}_e) = k$ and $\text{Ext}^i(\Psi(\mathcal{W}), \mathcal{L}_e) = 0$ for $i \neq 0$.

First, we claim that for any $w \neq e$ there exists a simple parabolic subgroup $P \subset G$ (in particular, $P \neq B$) such that $\mathcal{L}_w \cong \pi^\Delta \mathcal{F}$ for some object $\mathcal{F} \in D(G/P)^N$, where $\pi : G/B \rightarrow G/P$ is the projection. Indeed, if α is a simple root such that $\ell(ws_\alpha) = \ell(w) - 1$,

then \overline{NwB} is stable under the right action of $P = B \sqcup Bs_\alpha B$. The claim follows because \mathcal{L}_w is pushed forward from the IC sheaf on X_w .

We will now show that $\pi_! \Psi(\mathcal{W}) = 0$, so that

$$\mathrm{Ext}^i(\Psi(\mathcal{W}), \mathcal{L}_w) \cong \mathrm{Ext}^i(\pi_! \Psi(\mathcal{W}), \mathcal{F}[-1]) = 0$$

for all i . Since Ψ commutes with pushforward along proper morphisms, we need only prove that $\tilde{\pi}_! \mathcal{W} = 0$, where $\tilde{\pi} := \pi \times \mathrm{id}_{\mathbb{G}_m}$. Because \mathcal{W} is cleanly extended from $C_{w_0} \times \mathbb{G}_m$, the sheaf $\tilde{\pi}_! \mathcal{W}$ is $*$ -extended from the $N \times \mathbb{G}_m$ -orbit $(Nw_0P/P) \times \mathbb{G}_m$. Thus by the ψ^Δ exp-equivariance it suffices to check that $i_{w_0}^! \tilde{\pi}_! \mathcal{W} = 0$, where $i_{w_0} : \mathbb{G}_m \rightarrow (G/P) \times \mathbb{G}_m$ is the constant section w_0P . One has $w_0Pw_0^{-1} \cap N = N_{-w_0\alpha}$, so the desired vanishing follows from base change. This is because for any $t \neq 0$, the character $\mathrm{ad}_{\gamma(t)}(\psi)$ is nontrivial when restricted to $N_{-w_0\alpha}$, and a nontrivial exponential D-module has vanishing de Rham cohomology.

Finally, we show that $\mathrm{Ext}^i(\Psi(\mathcal{W}), \mathcal{L}_e) = 0$ if $i \neq 0$ and $\mathrm{Hom}(\Psi(\mathcal{W}), \mathcal{L}_e) = k$. Recall that there is a surjection $j_{w_0,*}(\mathrm{IC}_{C_{w_0}}) \rightarrow \mathcal{L}_e$ whose kernel does not have \mathcal{L}_e as a subquotient, where $j_{w_0} : C_{w_0} \rightarrow G/B$ is the inclusion. This implies that

$$\mathrm{Ext}^i(\Psi(\mathcal{W}), j_{w_0,*}(\mathrm{IC}_{C_{w_0}})) \xrightarrow{\sim} \mathrm{Ext}^i(\Psi(\mathcal{W}), \mathcal{L}_e)$$

for all i . But $\mathcal{W}|_{C_{w_0} \times \mathbb{G}_m}$ extends to a lisse sheaf on $C_{w_0} \times \mathbb{A}^1$, so that properties 2 and 3 of Ψ from Section 4 imply that $\Psi(\mathcal{W})|_{C_{w_0}} = \mathcal{W}|_{C_{w_0} \times \{0\}}[1]$. The latter is the IC sheaf on C_{w_0} , whence the claim. □

II A resolution of singularities for Drinfeld's compactification³

6 The Kontsevich compactification

6.1 Recall that a map $f : C \rightarrow Y$ from a nodal curve to a variety Y is called *stable* if C admits no first-order infinitesimal automorphisms along which f is constant. Said differently, the automorphism group of f in the moduli of curves mapping to Y is finite. Concretely, this means that if $C_0 \subset C$ is a rational component on which f is constant, then C_0 contains at least three nodes of C , with a similar condition on genus one components which will not be relevant.

6.2 In what follows, the letter S will always denote an affine scheme.

Definition 6.2.1. The *Kontsevich compactification* $\overline{\text{Bun}}_B^K$ is defined as follows. A point $S \rightarrow \overline{\text{Bun}}_B^K$ is a commutative square

$$\begin{array}{ccc} C & \longrightarrow & \text{pt}/B \\ \downarrow p & & \downarrow \\ X \times S & \xrightarrow{\mathcal{P}_G} & \text{pt}/G, \end{array}$$

where $C \rightarrow S$ is a flat family of connected nodal projective curves of arithmetic genus g , the map p has degree 1, and the induced map $C \rightarrow (G/B)_{\mathcal{P}_G}$ is stable over every geometric point of S .

For a geometric point $\text{Spec } k \rightarrow \overline{\text{Bun}}_B^K$, the projection $p : C \rightarrow X$ restricts to an isomorphism on some component, i.e. it admits a canonical section $X \rightarrow C$. Such a section does not exist for a general family. Let $\overset{\circ}{X}$ be the intersection of the smooth locus of C with X .

³Part II has a substantial overlap with the article [C2].

Observe that the connected components of $C \setminus \overset{\circ}{X}$ have arithmetic genus zero and hence are simply connected. In particular, the dual graph of C is a tree, and all irreducible components not equal to X are isomorphic to \mathbb{P}^1 .

6.3 The open locus in $\overline{\text{Bun}}_B^K$ where p is an isomorphism (equivalently, where C is smooth over S) identifies with Bun_B , and the projection $\overline{\mathfrak{p}}^K : \overline{\text{Bun}}_B^K \rightarrow \text{Bun}_G$ extends $\text{Bun}_B \rightarrow \text{Bun}_G$. The degree of a point in $\overline{\text{Bun}}_B^K$ is defined to be the degree of the B -bundle, and we denote the component of degree $\lambda \in \Lambda$ by $\overline{\text{Bun}}_B^{K,\lambda}$.

Proposition 6.3.1. *The stack $\overline{\text{Bun}}_B^K$ is Artin and locally of finite type, and for any $\lambda \in \Lambda$ the projection*

$$\overline{\mathfrak{p}}^{K,\lambda} : \overline{\text{Bun}}_B^{K,\lambda} \rightarrow \text{Bun}_G$$

is proper (but not schematic in general).

Proof. For any point $S \rightarrow \text{Bun}_G$ the fiber product $\overline{\text{Bun}}_B^K \times_{\text{Bun}_G} S$ is the open (and closed) locus in the stack $\overline{\mathcal{M}}_g((G/B)_{\mathcal{P}_G})$ of stable maps defined on arithmetic genus g curves where the composition with $(G/B)_{\mathcal{P}_G} \rightarrow X \times S$ has degree one. It is well-known that $\overline{\mathcal{M}}_g((G/B)_{\mathcal{P}_G})$ is representable by a Deligne-Mumford stack locally of finite type (see [AO]), from which it follows that $\overline{\text{Bun}}_B^K$ is an Artin stack locally of finite type. Moreover, in *loc. cit.* it is shown that $\overline{\mathcal{M}}_g((G/B)_{\mathcal{P}_G}) \rightarrow S$ is proper when the degree is fixed. □

6.4 Denote by \mathcal{M}_X the moduli stack of connected nodal curves C equipped with a proper degree one map $p : C \rightarrow X$ satisfying $R^1p_*\mathcal{O}_C = 0$. Note that in this definition X need not be proper. If X is proper, then \mathcal{M}_X can also be described as the moduli stack of proper connected nodal curves of arithmetic genus g equipped with a degree one map to X .

Proposition 6.4.1. *Both \mathcal{M}_X and the projection $\pi : \overline{\text{Bun}}_B^K \rightarrow \mathcal{M}_X$ are smooth, so $\overline{\text{Bun}}_B^K$ is smooth.*

Proof. First we prove that \mathcal{M}_X is smooth. Observe that

$$T_p(\mathcal{M}_X) = R\Gamma(C, T(C/X))[1],$$

so we must prove that $R^i\Gamma(C, T(C/X)) = 0$ for $i > 1$. A simple local calculation shows that $T(C)$ is concentrated in degrees 0 and 1, and that $H^1(T(C))$ is the sum of the delta sheaves at the nodes. It follows that $R^i\Gamma(C, T(C/X)) = 0$ for $i > 2$, so it remains to check that

$$R^1\Gamma(C, T(C)) \longrightarrow R^1\Gamma(C, p^*T(X))$$

is surjective.

Applying Serre duality, we must show that

$$R^0\Gamma(C, \omega_C \otimes p^*T^*(X)) \longrightarrow R^0\Gamma(C, \omega_C \otimes T^*(C))$$

is injective, where here $\omega_C = \det T^*(C)$ is the dualizing line bundle. We claim that the composition

$$R^0\Gamma(C, \omega_C \otimes p^*T^*(X)) \longrightarrow R^0\Gamma(C, \omega_C \otimes T^*(C)) \longrightarrow R^0(\overset{\circ}{X}, T^*(\overset{\circ}{X})^{\otimes 2})$$

is injective, where the second map is restriction to $\overset{\circ}{X}$, the intersection of X with the smooth locus of C . Observe that for any irreducible component C_0 other than X we have

$$(\omega_C \otimes p^*T^*(X))|_{C_0} \cong \omega_{C_0}.$$

Since ω_{C_0} has no global sections, any global section of the line bundle $\omega_C \otimes p^*T^*(X)$ is supported on X , which implies the desired injectivity.

To prove smoothness of π , fix a geometric point $\xi : \text{Spec } k \rightarrow \overline{\text{Bun}}_B^K$ over $p : C \rightarrow X$.

The fiber $\pi^{-1}(p)$ maps to $\text{Map}(C, \text{pt}/B)$, which is smooth because C is a curve. The square

$$\begin{array}{ccc} \pi^{-1}(p) & \longrightarrow & \text{Bun}_G \\ \downarrow & & \downarrow \\ \text{Map}(C, \text{pt}/B) & \longrightarrow & \text{Map}(C, \text{pt}/G), \end{array}$$

is not Cartesian, but $\pi^{-1}(p)$ is the open locus in the fiber product defined by the stability condition. It therefore suffices to prove that the map $p^* : \text{Bun}_G \rightarrow \text{Map}(C, \text{pt}/G)$ is smooth at \mathcal{P}_G . Since both domain and target are smooth stacks, we need only show that

$$R^1\Gamma(X, \mathfrak{g}_{\mathcal{P}_G}) \longrightarrow R^1\Gamma(C, p^*(\mathfrak{g}_{\mathcal{P}_G}))$$

is surjective.

This is equivalent to the claim that

$$R^0\Gamma(C, \omega_C \otimes p^*(\mathfrak{g}_{\mathcal{P}_G}^*)) \longrightarrow R^0\Gamma(X, \omega_X \otimes \mathfrak{g}_{\mathcal{P}_G}^*) \quad (6.4.1)$$

is injective. Recall that C is Gorenstein, i.e. ω_C is a line bundle. A global section of $\omega_C \otimes p^*(\mathfrak{g}_{\mathcal{P}_G}^*)$ vanishes on any irreducible component $C_0 \neq X$, since the restriction of the vector bundle there is isomorphic to $\mathfrak{g}^* \otimes \omega_{C_0}$ and $C_0 \cong \mathbb{P}^1$. Thus both sides of (6.4.1) inject into $R^0\Gamma(\mathring{X}, \omega_X \otimes \mathfrak{g}_{\mathcal{P}_G}^*)$, which finishes the proof. □

7 The morphism to Drinfeld's compactification

7.1 We recall the definition of Drinfeld's compactification $\overline{\text{Bun}}_B$. We assume for simplicity that the derived group of G is simply connected. Let $\check{\Lambda}$ be the weights of G and denote the dominant weights by $\check{\Lambda}^+$. A map $S \rightarrow \overline{\text{Bun}}_B$ is the following data: a G -bundle \mathcal{P}_G and a

T -bundle \mathcal{P}_T on $X \times S$, and for every $\check{\lambda} \in \check{\Lambda}^+$ a monomorphism of coherent sheaves

$$\kappa^{\check{\lambda}} : \check{\lambda}(\mathcal{P}_T) \longrightarrow V_{\mathcal{P}_G}^{\check{\lambda}},$$

satisfying Plücker-type relations (see [BG1] for details).

Constructing the morphism $\overline{\text{Bun}}^K \rightarrow \overline{\text{Bun}}_B$ requires some preliminary observations.

Lemma 7.1.1. *The stack Bun_B is dense in $\overline{\text{Bun}}_B^K$. In particular, if $S \rightarrow \overline{\text{Bun}}_B^K$ is smooth then the resulting map $p : C \rightarrow X \times S$ is an isomorphism over a locus of codimension at least 2 in $X \times S$.*

Proof. Density follows from the fact that the complement of Bun_B in $\overline{\text{Bun}}_B^K$ is a normal crossings divisor: combine Proposition 6.4.1 and Proposition 8.4.1 below.

Notice that the locus $D \subset X \times S$ over which p is not an isomorphism is finite over S . If $S \rightarrow \overline{\text{Bun}}_B^K$ is smooth then the image of D in S is a proper subset, whence the claim. □

7.2 Fix a morphism $\xi : S \rightarrow \overline{\text{Bun}}_B^K$ with S an affine scheme. To define the desired map $S \rightarrow \overline{\text{Bun}}_B$ we use the G -bundle \mathcal{P}_G on $X \times S$ obtained from ξ , so we still need to specify a T -bundle \mathcal{P}_T and, for every $\check{\lambda} \in \check{\Lambda}^+$, a monomorphism of coherent sheaves

$$\kappa^{\check{\lambda}} : \check{\lambda}(\mathcal{P}_T) \longrightarrow V_{\mathcal{P}_G}^{\check{\lambda}}$$

satisfying the Plücker relations.

The point ξ also includes a family of curves $C \rightarrow S$ with a degree one map $p : C \rightarrow X \times S$ and a B -reduction

$$G \times^B \mathcal{P}_B \xrightarrow{\sim} p^* \mathcal{P}_G,$$

which can be viewed as a collection of vector bundle embeddings

$$\iota^{\check{\lambda}} : \check{\lambda}(\mathcal{P}_B) \longrightarrow p^*V_{\mathcal{P}_G}^{\check{\lambda}}$$

for $\check{\lambda} \in \check{\Lambda}^+$.

Proposition 7.2.1. *For every $\check{\lambda} \in \check{\Lambda}$ the direct image $p_*\check{\lambda}(\mathcal{P}_B)$ is a perfect complex on $X \times S$, so that its determinant line bundle $\det(p_*\check{\lambda}(\mathcal{P}_B))$ is well-defined. There exists a unique T -bundle \mathcal{P}_T on $X \times S$ such that $\check{\lambda}(\mathcal{P}_T) = \det(p_*\check{\lambda}(\mathcal{P}_B))$ for all $\check{\lambda} \in \check{\Lambda}$, and for all $\check{\lambda} \in \check{\Lambda}^+$ the embeddings $\iota^{\check{\lambda}}$ canonically induce monomorphisms of coherent sheaves*

$$\kappa^{\check{\lambda}} : \check{\lambda}(\mathcal{P}_T) \longrightarrow V_{\mathcal{P}_G}^{\check{\lambda}}$$

satisfying the Plücker relations.

Proof. By a standard descent argument we reduce to the case that $\xi : S \rightarrow \overline{\text{Bun}}_B^K$ is smooth. Then S is smooth by Proposition 6.4.1, so $p_*\check{\lambda}(\mathcal{P}_B)$ is a perfect complex because p is proper and $X \times S$ is smooth. To show that \mathcal{P}_T exists, we claim that the isomorphisms

$$(\check{\lambda} + \check{\mu})(\mathcal{P}_B) \xrightarrow{\sim} \check{\lambda}(\mathcal{P}_B) \otimes \check{\mu}(\mathcal{P}_B)$$

on C canonically give rise to isomorphisms

$$\det(p_*(\check{\lambda} + \check{\mu})(\mathcal{P}_T)) \xrightarrow{\sim} \det(p_*\check{\lambda}(\mathcal{P}_T)) \otimes \det(p_*\check{\mu}(\mathcal{P}_T))$$

on $X \times S$. By Lemma 7.1.1, these isomorphisms are defined away from a locus of codimension at least 2, whence they extend uniquely to $X \times S$. Similarly, apply Lemma 7.1.1 to obtain the $\kappa^{\check{\lambda}}$ from the $\iota^{\check{\lambda}}$. The Plücker relations on $X \times S$ are inherited from those on C by construction. □

We summarize the construction of the map as follows.

Proposition 7.2.2. *The construction in Proposition 7.2.1 defines a proper morphism $\overline{\text{Bun}}_B^K \rightarrow \overline{\text{Bun}}_B$ which restricts to the identity on Bun_B and commutes with the projections to Bun_G .*

Proof. Since \det commutes with inverse image, the construction defines a morphism of stacks by base change for quasi-coherent sheaves. It clearly restricts to the identity on Bun_B and commutes with the projections to Bun_G . Since $\overline{\text{Bun}}_B^{K,\lambda} \rightarrow \text{Bun}_G$ is proper, so is $\overline{\text{Bun}}_B^{K,\lambda} \rightarrow \overline{\text{Bun}}_B$. Proposition 7.3.1 below shows that $\overline{\text{Bun}}_B^K \rightarrow \overline{\text{Bun}}_B$ preserves degree and hence is proper. □

7.3 The defect of a geometric point $\xi : \text{Spec } k \rightarrow \overline{\text{Bun}}_B^K$ is the Λ^{pos} -weighted effective divisor defined as follows. Let C_1, \dots, C_n be the connected components of $C \setminus \overset{\circ}{X}$ and $\{x_i\} = C_i \cap X$. Then the defect of ξ is

$$- \sum_{i=1}^n \deg(\mathcal{P}_B|_{C_i}) \cdot x_i.$$

For any $\mu \in \Lambda^{\text{pos}}$ we denote by $\overline{\text{Bun}}_{B, \leq \mu}^K$ the open locus where

$$- \sum_{i=1}^n \deg(\mathcal{P}_B|_{C_i}) \leq \mu.$$

Proposition 7.3.1. *The image of $\xi : \text{Spec } k \rightarrow \overline{\text{Bun}}_B^K$ under $\overline{\text{Bun}}_B^K \rightarrow \overline{\text{Bun}}_B$ has the same defect as ξ , and its saturation is obtained by restricting the B -reduction on C along the section $s : X \rightarrow C$. In particular $\overline{\text{Bun}}_B^K \rightarrow \overline{\text{Bun}}_B$ preserves degree.*

Proof. Suppose that $n = 1$: the general case will follow by a straightforward induction. Fix $\check{\lambda} \in \check{\Lambda}^+$ and put $\mathcal{L} = \check{\lambda}(\mathcal{P}_B)$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow s_* s^*(\mathcal{L}) \oplus i_* i^*(\mathcal{L}) \longrightarrow \delta_{x_1} \longrightarrow 0$$

on C , where $i : C_1 \rightarrow C$ is the inclusion. Pushing forward to X , we obtain the exact triangle

$$p_*\mathcal{L} \longrightarrow s^*(\mathcal{L}) \oplus (\delta_{x_1} \otimes R\Gamma(C_1, i^*\mathcal{L})) \longrightarrow \delta_{x_1}.$$

The claim is that $\det(Rp_*\mathcal{L}) \xrightarrow{\sim} s^*(\mathcal{L})(d \cdot x_1)$, where $d := \deg(i^*\mathcal{L}) \leq 0$.

Suppose that $i^*\mathcal{L}$ is nontrivial. Since $s^*\mathcal{L} \rightarrow \delta_{x_1}$ is an epimorphism, we get a short exact sequence

$$0 \longrightarrow R^0p_*\mathcal{L} \longrightarrow s^*\mathcal{L} \longrightarrow \delta_{x_1} \longrightarrow 0$$

and an isomorphism

$$R^1p_*\mathcal{L} \xrightarrow{\sim} \delta_{x_1} \otimes R^1\Gamma(C_1, i^*\mathcal{L}).$$

Thus $R^0p_*\mathcal{L} = s^*(\mathcal{L})(-x_1)$ and $\det(R^1p_*\mathcal{L}) = \mathcal{O}_X((-d-1) \cdot x_1)$. The monomorphism of coherent sheaves associated with $\text{Spec } k \rightarrow \overline{\text{Bun}}_B^K \rightarrow \overline{\text{Bun}}_B$ and $\check{\lambda}$ is the composition

$$\det(p_*\mathcal{L}) \xrightarrow{\sim} s^*(\mathcal{L})(d \cdot x) \longrightarrow s^*\mathcal{L} \longrightarrow s^*p^*V_{\mathcal{P}_G}^{\check{\lambda}} = V_{\mathcal{P}_G}^{\check{\lambda}},$$

and since the last map is a vector bundle embedding the claim follows in this case.

If $i^*\mathcal{L}$ is trivial (this can only occur if the semisimple rank of G is at least 2), then we have instead a short exact sequence

$$0 \longrightarrow R^0p_*\mathcal{L} \longrightarrow s^*(\mathcal{L}) \oplus \delta_{x_1} \longrightarrow \delta_{x_1} \longrightarrow 0$$

and $R^1p_*\mathcal{L} = 0$. Therefore $R^0p_*\mathcal{L} \xrightarrow{\sim} s^*\mathcal{L}$, so the claim is immediate in this case. □

8 Relative smoothness and local acyclicity

8.1 Now we show that the Kontsevich compactification is smooth over Bun_G at points sufficiently antidominant relative to their defect.

Proposition 8.1.1. *Fix $\mu \in \Lambda^{\text{pos}}$. If $\lambda \in \Lambda$ satisfies $\langle \check{\alpha}, \lambda - \mu' \rangle > 2g - 2$ for all negative roots $\check{\alpha}$ and all $0 \leq \mu' \leq \mu$, then the morphism*

$$\overline{\text{Bun}}_{B, \leq \mu}^{K, \lambda} \longrightarrow \mathcal{M}_X \times \text{Bun}_G$$

is smooth. In particular, the composition with $\mathcal{M}_X \times \text{Bun}_G \rightarrow \text{Bun}_G$ is smooth.

Proof. The fiber over $p : C \rightarrow X$ and \mathcal{P}_G is the open locus in $\text{Map}_X(C, (G/B)_{\mathcal{P}_G})$ consisting of stable maps. Fixing a point $f : C \rightarrow (G/B)_{\mathcal{P}_G}$ in the fiber, we have

$$T_f(\text{Map}_X(C, (G/B)_{\mathcal{P}_G})) = R\Gamma(C, f^*T((G/B)_{\mathcal{P}_G}/X)).$$

It therefore suffices to prove that

$$R^1\Gamma(C, f^*T((G/B)_{\mathcal{P}_G}/X)) = 0$$

when the degree λ of f (i.e. the degree of the resulting B -bundle \mathcal{P}_B on C) satisfies the given hypothesis.

One can check that

$$f^*T((G/B)_{\mathcal{P}_G}/X) \xrightarrow{\sim} (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}.$$

The latter vector bundle can be filtered by subbundles so that the associated graded is $(\mathfrak{n}^-)_{\mathcal{P}_T}$, where \mathcal{P}_T is the T -bundle induced by \mathcal{P}_B . Let $\lambda_X = \text{deg}(\mathcal{P}_B|_X)$ and note that $\lambda - \lambda_X \leq \mu$. Thus the hypothesis on λ implies that

$$\langle \lambda_X, \check{\alpha} \rangle = \langle \lambda, \check{\alpha} \rangle - \langle \lambda - \lambda_X, \check{\alpha} \rangle > 2g - 2$$

for any negative root $\check{\alpha}$, whence $R^1\Gamma(X, \check{\alpha}(\mathcal{P}_T)) = 0$.

Now we induct on the number of components in C . Fix a leaf C_0 of the tree C , i.e. an irreducible component other than X which contains a single node c , and put $C' :=$

$C \setminus (C_0 \setminus \{c\})$. Then we have a short exact sequence

$$0 \longrightarrow (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}|_{C_0}(-c) \longrightarrow (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B} \longrightarrow (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}|_{C'} \longrightarrow 0.$$

By the inductive hypothesis it suffices to prove that $R^1\Gamma(C_0, (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}|_{C_0}(-c)) = 0$. This follows from the observation that $(\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}|_{C_0}$ is generated by global sections.

□

The following corollary will be applied in Part III to prove Theorem 2.7.1.

Corollary 8.1.1.1. *For $\lambda, \mu \in \Lambda$ as in Proposition 8.1.1, $\mathrm{IC}_{\overline{\mathrm{Bun}}_B, \leq \mu}^\lambda$ is universally locally acyclic over Bun_G .*

Proof. The pushforward of $\mathrm{IC}_{\overline{\mathrm{Bun}}_B^K}$ along $\overline{\mathrm{Bun}}_B^K \rightarrow \overline{\mathrm{Bun}}_B$ is semisimple by Theorem 8.3.2.4 of [Sun], and since this morphism is birational it follows that the pushforward contains $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ as a summand. The proposition implies that $\mathrm{IC}_{\overline{\mathrm{Bun}}_B, \leq \mu}^{\kappa, \lambda}$ is ULA over Bun_G for λ, μ as above, and now the claim follows because the ULA condition is preserved under proper pushforward and taking summands.

□

8.2 In the remainder of this section we show that Corollary 8.1.1.1 still holds after we introduce a twist. For us the twisting parameter is a multiplicative local system on \mathbb{G}_m , so twistings are in bijection with k/\mathbb{Z} .

Let $\mathcal{L}_{\mathrm{Bun}_G}$ be the determinant line bundle on Bun_G , and let $\mathcal{L}_{\mathrm{Bun}_T}$ be the line bundle on Bun_T whose fiber at \mathcal{P}_T is

$$\det R\Gamma(X, \mathfrak{h} \otimes \mathcal{O}_X) \otimes \bigotimes_{\check{\alpha}} \det R\Gamma(X, \check{\alpha}(\mathcal{P}_T)),$$

where the tensor product runs over the roots of G . The twisting line bundle on $\overline{\mathrm{Bun}}_B$, which

appears naturally in the theory of Eisenstein series, is

$$\mathcal{L}_{\overline{\text{Bun}}_B} := \overline{\mathfrak{p}}^*(\mathcal{L}_{\text{Bun}_G})^{-1} \otimes \overline{\mathfrak{q}}^*(\mathcal{L}_{\text{Bun}_T}),$$

where $\overline{\mathfrak{p}} : \overline{\text{Bun}}_B \rightarrow \text{Bun}_G$ and $\overline{\mathfrak{q}} : \overline{\text{Bun}}_B \rightarrow \text{Bun}_T$ are the natural maps.

Observe that $\mathcal{L}_{\overline{\text{Bun}}_B}$ trivializes canonically over Bun_B . This is because for a fixed \mathcal{P}_B with induced G -bundle \mathcal{P}_G and T -bundle \mathcal{P}_T , the adjoint bundle $\mathfrak{g}_{\mathcal{P}_G}$ admits a filtration whose subquotients are the line bundles $\check{\alpha}(\mathcal{P}_T)$ and the trivial vector bundle $\mathfrak{h} \otimes \mathcal{O}_X$.

For any twisting c , the corresponding category of twisted D-modules $D^c(\overline{\text{Bun}}_B)$ is defined as the category of D-modules on the total space of $\mathcal{L}_{\overline{\text{Bun}}_B}$ which are \mathbb{G}_m -equivariant against the character sheaf c . Since $\mathcal{L}_{\overline{\text{Bun}}_B}$ trivializes canonically over Bun_B , we obtain an equivalence

$$D^c(\overline{\text{Bun}}_B) \xrightarrow{\sim} D(\text{Bun}_B).$$

Under this equivalence we can view IC_{Bun_B} as an object of $D^c(\overline{\text{Bun}}_B)$, and we denote by $\text{IC}_{\overline{\text{Bun}}_B}^c$ its intermediate extension in $D^c(\overline{\text{Bun}}_B)$.

Theorem 8.2.1. *For $\lambda, \mu \in \Lambda$ as in Proposition 8.1.1, $\text{IC}_{\overline{\text{Bun}}_B, \leq \mu}^c$ is universally locally acyclic over Bun_G .*

The proof will occupy the rest of this section.

8.3 We will deduce Theorem 8.2.1 from the following key lemma. Let $f : \mathcal{Y} \rightarrow \mathcal{Z}$ be a morphism of smooth Artin stacks, and let $D \subset \mathcal{Y}$ be a normal crossings divisor. Suppose we are given a line bundle of the form

$$\mathcal{O}_{\mathcal{Y}}(\sum_i r_i D_i), \tag{8.3.1}$$

where the D_i are the irreducible components of D and $r_i \in \mathbb{Z}$. There is a corresponding category of twisted sheaves $D^c(\mathcal{Y})$. Since the line bundle trivializes canonically over $\mathcal{Y} \setminus D$,

we can consider the twisted IC sheaf $\mathrm{IC}_{\mathcal{Y}}^c$.

The normal crossings divisor D gives rise to a stratification on \mathcal{Y} . Namely, for $n \geq 0$ let $\mathcal{Y}_n \subset \mathcal{Y}$ be the codimension n stratum, which is the locus where D is locally isomorphic to an intersection of n coordinate hyperplanes in affine space.

Lemma 8.3.1. *If $\mathcal{Y}_n \rightarrow \mathcal{Z}$ is smooth for all $n \geq 0$, then $\mathrm{IC}_{\mathcal{Y}}^c$ is universally locally acyclic over \mathcal{Z} .*

Proof. Since the ULA property is local in the smooth topology on the source and target, the assertion reduces to the case that $Y = \mathcal{Y}$ and $Z = \mathcal{Z}$ are schemes. Let $C \subset T^*(Y)$ be the singular support of IC_Y^c . By the characterization of singular support introduced in [Bei2], it suffices to prove that for any point $y \in Y(k)$, we have $(df_y)^{-1}(C_y) \setminus \{0\} = \emptyset$.

We may assume that the multiplicities r_i in (8.3.1) are all nonzero. If c is trivial, then C is the zero section. Since the hypothesis implies that f is smooth, the claim follows in this case. If c is nontrivial, then for $y \in Y_n$ the singular support $C_y \subset T_y^*(Y)$ is the union of the n lines orthogonal to the n hyperplanes in $T_y(Y)$ determined by the normal crossings property of D . The span of C_y is precisely the kernel of $T_y^*(Y) \rightarrow T_y^*(Y_n)$, so the injectivity of $T_{f(y)}^*(Z) \rightarrow T_y^*(Y_n)$ implies the claim in this case also.

□

8.4 In order to apply Lemma 8.3.1, we must show that the boundary of the Kontsevich compactification has normal crossings. According to Proposition 6.4.1, we need only prove the corresponding claim for \mathcal{M}_X .

If $f : X \rightarrow Y$ is an étale map of smooth curves, there is a corresponding morphism $\mathcal{M}_Y \rightarrow \mathcal{M}_X$ which sends $C \rightarrow Y \times S$ to $X \times_Y C \rightarrow X \times S$. It is étale at a k -point $C \rightarrow Y$ if and only if for all nodes y_1, \dots, y_n of C contained in Y , the fiber $f^{-1}(y_i)$ is a single point.

Proposition 8.4.1. *The complement $\partial\mathcal{M}_X$ of the open point in \mathcal{M}_X given by X is a normal crossings divisor.*

Proof. Fix a k -point $p : C \rightarrow X$ of $\partial\mathcal{M}_X$. The claim reduces to the case that $X = \mathbb{A}^1$ as follows. Choose an open $U \subset X$ which contains all the nodes x_1, \dots, x_n of C contained in X , and such that there exists an étale function $f : U \rightarrow \mathbb{A}^1$ with the property that $f^{-1}(f(x_i)) = \{x_i\}$ for all $1 \leq i \leq n$. Then we obtain maps $\mathcal{M}_X \rightarrow \mathcal{M}_U \leftarrow \mathcal{M}_{\mathbb{A}^1}$, and the image of p in \mathcal{M}_U lifts to \tilde{p} in $\mathcal{M}_{\mathbb{A}^1}$. Moreover, $\mathcal{M}_X \rightarrow \mathcal{M}_U$ is étale and $\mathcal{M}_{\mathbb{A}^1} \rightarrow \mathcal{M}_U$ is étale at \tilde{p} .

For $X = \mathbb{A}^1$ we will explicitly construct a smooth chart around p . There is a canonical stratification

$$X = C_0 \subset C_1 \subset \dots \subset C_m = C$$

defined as follows: the subgraph of the dual graph corresponding to C_i contains the vertices which are connected to the vertex X by a path consisting of at most i edges. We will use the notation $\mathbb{A}_{\text{disj}}^r$ to denote the open locus in \mathbb{A}^r where no two coordinates are equal.

Define $S_1 := \mathbb{A}_{\text{disj}}^n \times \mathbb{A}^n$. For any $1 \leq j \leq n$ let

$$Z_{1j} := \{(z, (z_i), (t_i)) \in \mathbb{A}^1 \times S_1 \mid z = z_j, t_j = 0\}.$$

Note that the Z_{1j} are pairwise disjoint and let Y_1 be the blow-up of $\mathbb{A}^1 \times S_1$ along their union. The family $Y_1 \rightarrow S_1$ defines a smooth chart $S_1 \rightarrow \mathcal{M}_{\mathbb{A}^1}$ around C_1 , since $H^0T_{C_1}(\mathcal{M}_{\mathbb{A}^1})$ is spanned by the n deformations which move the nodes in X , along with the n deformations which remove the components of $C_1 \setminus C_0$.

Since Z_{1j} has a Zariski neighborhood isomorphic to a linear embedding $\mathbb{A}^{r-2} \cap V \rightarrow V$ for an open set $V \subset \mathbb{A}^r$, the normal bundle $N(Z_{1j}/\mathbb{A}^1 \times S_1)$ is trivial. In particular we can choose a coordinate $\pi_{1j} : E_{1j} \rightarrow \mathbb{P}^1$ on the exceptional divisor $E_{1j} = \mathbb{P}(N(Z_{1j}/\mathbb{A}^1 \times S_1))$ with the property that $\pi_{1j}^{-1}(\infty)$ intersected with each fiber over S_1 is contained in the singular locus. Let n_2 be the number of connected components of $C_2 \setminus C_1$ and put $S_2 = S_1 \times \mathbb{A}_{\text{disj}}^{n_2} \times \mathbb{A}^{n_2}$. Let

$$\sigma_2 : \{1, \dots, n_2\} \rightarrow \{1, \dots, n\}$$

be a function such that $\sigma_2^{-1}(i)$ is the number of components of $C_2 \setminus C_1$ whose closure intersects the i^{th} component of $C_1 \setminus C_0$. For each $1 \leq j \leq n_2$ define

$$Z_{2j} := \{(y, (z_i), (t_i)) \in Y_1 \times \mathbb{A}_{\text{disj}}^{n_2} \times \mathbb{A}^{n_2} \mid y \in E_{1\sigma_2(j)}, \pi_{1\sigma_2(j)}(y) = z_j, t_j = 0\},$$

and let Y_2 be the blow-up of $Y_1 \times \mathbb{A}_{\text{disj}}^{n_2} \times \mathbb{A}^{n_2}$ along the union of the Z_{2j} . Then $Y_2 \rightarrow \mathbb{A}^1 \times S_2$ defines a smooth chart $S_2 \rightarrow \mathcal{M}_{\mathbb{A}^1}$ around C_2 .

Continue in this way to obtain a smooth chart $S_m \rightarrow \mathcal{M}_{\mathbb{A}^1}$ around C . At the inductive step, the center Z_{ij} of each blow-up has a neighborhood of the form $\mathbb{A}^{r-2} \cap V \rightarrow V$ for an open $V \subset \mathbb{A}^r$, so that a suitable coordinate on the resulting exceptional divisor can be chosen. The preimage of $\partial \mathcal{M}_{\mathbb{A}^1}$ in S_m equals the preimage of the union of the coordinate hyperplanes under the smooth projection $S_m \rightarrow \mathbb{A}^n$.

□

The theorem now follows readily.

Proof of Theorem 8.2.1. We apply Lemma 8.3.1 with $\mathcal{Y} = \overline{\text{Bun}}_{B, \leq \mu}^{K, \lambda}$ and $\mathcal{Z} = \text{Bun}_G$. The boundary in $\overline{\text{Bun}}_B^K$ is the preimage of the boundary in \mathcal{M}_X under the canonical projection, so the boundary in the former is a normal crossings divisor by Propositions 6.4.1 and 8.4.1. Let $\mathcal{L}_{\overline{\text{Bun}}_B^K}$ be the inverse image of $\mathcal{L}_{\text{Bun}_B}$. Since $\mathcal{L}_{\overline{\text{Bun}}_B^K}$ trivializes canonically over Bun_B , it has the form (8.3.1). Moreover, Proposition 8.1.1 implies that the strata in the boundary of $\overline{\text{Bun}}_{B, \leq \mu}^{K, \lambda}$ determined by the normal crossings structure are smooth over Bun_G , i.e. the hypotheses of Lemma 8.3.1 are satisfied and hence $\text{IC}_{\overline{\text{Bun}}_{B, \leq \mu}^{K, \lambda}}^c$ is ULA over Bun_G . Now apply the argument from the proof of Corollary 8.1.1.1 to deduce the same for $\text{IC}_{\overline{\text{Bun}}_{B, \leq \mu}^\lambda}^c$.

□

III Nearby cycles of Whittaker sheaves on Drinfeld's compactification⁴

9 Alternative construction of nearby cycles

9.1 We now reinterpret Beilinson's construction of nearby cycles from [Bei1], which will help to streamline our first proof of Theorem 2.7.1. In this section we use the language of DG categories in the sense of Lurie, meaning k -linear stable ∞ -categories: see Chapter I.1 in [GR] for a summary of the theory.

Our notation is as in Section 4. The projection $\mathring{Y} \rightarrow \mathbb{G}_m$ gives rise to an action of $D(\mathbb{G}_m)$, viewed as a symmetric monoidal DG category under $!$ -tensor product, on the DG category $D(\mathring{Y})$. We claim that this naturally induces an action of the DG algebra $H^\bullet(\mathbb{G}_m)$ on the identity functor of $D(\mathring{Y})$. Namely, we view the action of $D(\mathbb{G}_m)$ on $D(\mathring{Y})$ as a monoidal functor

$$D(\mathbb{G}_m) \longrightarrow \mathrm{Fun}(D(\mathring{Y}), D(\mathring{Y})).$$

Since $H^\bullet(\mathbb{G}_m)$ is the algebra of endomorphisms of the monoidal unit in $D(\mathbb{G}_m)$, this monoidal functor induces the desired action.

In particular, for any holonomic D-module \mathcal{F} on \mathring{Y} , we have an action of $H^\bullet(\mathbb{G}_m)$ on $i^!j_!\mathcal{F}$ by functoriality. The point $1 \in \mathbb{G}_m$ induces an augmentation $H^\bullet(\mathbb{G}_m) \rightarrow k$, and since $H^\bullet(\mathbb{G}_m)$ is generated by a single element in cohomological degree 1, the endomorphism algebra $\mathrm{End}_{H^\bullet(\mathbb{G}_m)\text{-mod}}(k)$ is canonically isomorphic to the power series ring $k[[t]]$. Here $H^\bullet(\mathbb{G}_m)\text{-mod}$ is the DG category of modules over the DG algebra $H^\bullet(\mathbb{G}_m)$.

⁴Part III has a substantial overlap with the article [C3].

9.2 Our construction of unipotent nearby cycles is as follows: it is the functor

$$\tilde{\Psi} : D_{\text{hol}}(\mathring{Y}) \longrightarrow D(Y_0)$$

given by the formula

$$\tilde{\Psi}(-) = k \otimes_{H^\bullet(\mathbb{G}_m)} i^! j_!(-).$$

In particular $\text{End}_{H^\bullet(\mathbb{G}_m)\text{-mod}}(k) = k[[t]]$ acts on $\tilde{\Psi}$. By construction $\tilde{\Psi}$ takes values in ind-holonomic D-modules, which implies that the generator in $k[[t]]$ acts locally nilpotently.

More precisely, the action of $H^\bullet(\mathbb{G}_m)$ on the identity functor of $D(\mathring{Y})$ defines a lift of the latter to a functor

$$D(\mathring{Y}) \longrightarrow H^\bullet(\mathbb{G}_m)\text{-mod}(D(\mathring{Y})),$$

and likewise on the holonomic subcategory. Then $\tilde{\Psi}$ is by definition the composition

$$D_{\text{hol}}(\mathring{Y}) \longrightarrow H^\bullet(\mathbb{G}_m)\text{-mod}(D_{\text{hol}}(\mathring{Y})) \xrightarrow{i^! j_!} H^\bullet(\mathbb{G}_m)\text{-mod}(D(Y_0)) \longrightarrow D(Y_0),$$

where the last functor is tensor product with the augmentation $H^\bullet(\mathbb{G}_m)$ -module. Since $H^\bullet(\mathbb{G}_m)$ is isomorphic to $\text{Sym}(k[-1])$, an action of $H^\bullet(\mathbb{G}_m)$ is the same as an action of the abelian DG Lie algebra $k[-1]$. Tensor product with the augmentation $H^\bullet(\mathbb{G}_m)$ -module corresponds to (homotopy) $k[-1]$ -coinvariants.

9.3 The following lemma will be used to compare our construction with Beilinson's. It says that the homotopy coinvariants of $H^1(\mathbb{G}_m)[-1]$ acting on the constant sheaf $k_{\mathbb{G}_m}$ is the "infinite Jordan block."

For any $a \geq 1$ let L_a be the shifted D-module on \mathbb{G}_m corresponding to the local system whose monodromy is a unipotent Jordan block of rank a . There are canonical injections

$L_a \rightarrow L_{a+1}$ (in the heart of the constructible t-structure), and we put

$$L_\infty := \operatorname{colim}_{a \geq 1} L_a.$$

Observe that L_∞ has a canonical injective “shift” endomorphism with cokernel $k_{\mathbb{G}_m}$.

Lemma 9.3.1. *There is a canonical isomorphism*

$$k \otimes_{H^\bullet(\mathbb{G}_m)} k_{\mathbb{G}_m} \xrightarrow{\sim} L_\infty$$

in $D(\mathbb{G}_m)$, which identifies the action of the generator in $k[[t]] = \operatorname{End}_{H^\bullet(\mathbb{G}_m)\text{-mod}}(k)$ with the canonical endomorphism of L_∞ .

Proof. First, observe that since any self-extension of L_∞ splits, the action of $H^1(\mathbb{G}_m)[-1]$ on L_∞ is trivial. Thus the canonical inclusion $k_{\mathbb{G}_m} = L_1 \rightarrow L_\infty$ factors through a map

$$k \otimes_{H^\bullet(\mathbb{G}_m)} k_{\mathbb{G}_m} \longrightarrow L_\infty. \tag{9.3.1}$$

To prove that this map is an isomorphism, we use a Koszul-type resolution of the augmentation $H^\bullet(\mathbb{G}_m)$ -module. Namely, put $M_1 := H^\bullet(\mathbb{G}_m)$, and let

$$M_2 := \operatorname{cofib}(H^\bullet(\mathbb{G}_m)[-1] \rightarrow H^\bullet(\mathbb{G}_m))$$

be the cofiber (i.e. mapping cone) of the action of a generator of $H^1(\mathbb{G}_m)$. Proceeding inductively, we define

$$M_{a+1} := \operatorname{cofib}(H^\bullet(\mathbb{G}_m)[-a] \rightarrow M_a),$$

so there are canonical maps $M_a \rightarrow M_{a+1}$. We define M_∞ to be the colimit of the M_a . Since the module M_a has cohomology $k \oplus k[-a]$, the canonical map $M_\infty \rightarrow k$ to the augmentation $H^\bullet(\mathbb{G}_m)$ -module is an isomorphism. In particular $k[t]$ acts on M_∞ , and it is not hard to see that the generator acts by the canonical “shift” map $M_\infty \rightarrow M_\infty$ with cofiber M_1 .

Thus we may replace the left hand side of (9.3.1) with $M_\infty \otimes_{H^\bullet(\mathbb{G}_m)} k_{\mathbb{G}_m}$. A straightforward inductive argument shows that $M_a \otimes_{H^\bullet(\mathbb{G}_m)} k_{\mathbb{G}_m}$ maps isomorphically onto L_a , which proves that (9.3.1) is an isomorphism which moreover preserves the filtrations on both sides. The action of $k[[t]]$ on both sides agrees by inspection: both endomorphisms shift the filtration by 1.

□

Proposition 9.3.2. *The functor $\tilde{\Psi}$ is isomorphic to the unipotent nearby cycles functor Ψ from [Bei1], and in particular preserves holonomicity. Moreover, if \mathcal{F} is \mathbb{G}_m -equivariant, then $\Psi(\mathcal{F})$ is unipotently \mathbb{G}_m -monodromic, and the monodromy endomorphism agrees with the obstruction to \mathbb{G}_m -equivariance.*

Proof. Let \mathcal{F} be a holonomic D-module on Y° . Then Beilinson's nearby cycles is given by the formula

$$\Psi(\mathcal{F}) := \operatorname{colim}_{a \geq 1} i^! j_! (\mathcal{F} \otimes^* f^* L_a).$$

Moreover, we have $\Psi(\mathcal{F}) = H^0 i^! j_! (\mathcal{F} \otimes^* f^* L_a)$ for large a , which is evidently holonomic.

The functor

$$D_{\text{hol}}(\mathbb{G}_m) \longrightarrow D(Y_0)$$

given on objects by $\mathcal{M} \mapsto i^! j_! (\mathcal{F} \otimes^* f^* \mathcal{M})$ admits a unique colimit-preserving extension to the category of ind-holonomic sheaves on \mathbb{G}_m , so one can write

$$\Psi(\mathcal{F}) = i^! j_! (\mathcal{F} \otimes^* f^* L_\infty).$$

The functor Ψ carries a canonical endomorphism induced by the shift endomorphism $L_\infty \rightarrow L_\infty$. Now it follows from Lemma 9.3.1 that $\tilde{\Psi}(\mathcal{F}) \xrightarrow{\sim} \Psi(\mathcal{F})$, preserving the monodromy endomorphisms on both sides.

For the second statement, note that the \mathbb{G}_m -equivariance of $i^! j_! \mathcal{F}$ implies that $\Psi(\mathcal{F}) = H^0 i^! j_! (\mathcal{F} \otimes^* f^* L_a)$ is \mathbb{G}_m -monodromic (here a is large). By construction, the monodromy

endomorphism is induced by the canonical endomorphism of L_a with one-dimensional kernel and cokernel. But the latter is precisely the obstruction to \mathbb{G}_m -equivariance for L_a , so the claim follows from the functoriality of this obstruction. □

It follows from Proposition 9.3.2 that $\tilde{\Psi}$ enjoys the standard properties of the unipotent nearby cycles functor: it is t-exact, commutes with Verdier duality, and commutes with proper direct image and smooth inverse image.

10 Drinfeld compactifications and Zastava spaces

10.1 Recall that $\overline{\text{Bun}}_{N^\omega}$ is defined to be the fiber product

$$\begin{array}{ccc} \overline{\text{Bun}}_{N^\omega} & \longrightarrow & \overline{\text{Bun}}_B \\ \downarrow & & \downarrow \\ \text{Spec } k & \xrightarrow{\rho(\omega)} & \text{Bun}_T. \end{array}$$

Since T acts on N by group automorphisms, the T -bundle $\rho(\omega)$ gives rise to a group scheme N^ω over X , and the open stratum $\text{Bun}_{N^\omega} \rightarrow \overline{\text{Bun}}_{N^\omega}$ is identified with the moduli stack of N^ω -bundles, as the notation suggests. By construction, T acts on $\overline{\text{Bun}}_{N^\omega}$ in such a way that $\overline{\text{Bun}}_{N^\omega} \rightarrow \overline{\text{Bun}}_B$ factors through a closed embedding

$$\overline{\text{Bun}}_{N^\omega}/T \longrightarrow \overline{\text{Bun}}_B.$$

For each $\mu \in \Lambda^{\text{pos}}$ the corresponding stratum $\overline{\text{Bun}}_{N^\omega, =\mu}$ fits into a fiber square

$$\begin{array}{ccc} \overline{\text{Bun}}_{N^\omega, =\mu} & \longrightarrow & \text{Bun}_B \\ \downarrow & & \downarrow \\ X^\mu & \longrightarrow & \text{Bun}_T, \end{array}$$

where the lower horizontal morphism is the twisted Abel-Jacobi map $D \mapsto \rho(\omega)(D)$. We

write

$$\overline{\text{Bun}}_{N^\omega, \leq \mu} := \bigcup_{\mu' \leq \mu} \overline{\text{Bun}}_{N^\omega, = \mu'}$$

and $j_{\leq \mu}$ for the corresponding open embedding.

The embedding $j_{=\mu}$ of the stratum, which is known to be affine, extends to a finite map

$$j_{\geq \mu} : \overline{\text{Bun}}_{N^\omega, \geq \mu} \longrightarrow \overline{\text{Bun}}_{N^\omega},$$

where $\overline{\text{Bun}}_{N^\omega, \geq \mu} := X^\mu \times_{\text{Bun}_T} \overline{\text{Bun}}_B$.

10.2 Now we introduce the Zastava spaces, which are factorizable local models for $\overline{\text{Bun}}_{N^\omega}$. There are several versions of Zastava space, and notations vary significantly within the literature (ours is similar to [AG2]).

Define the Zastava space Z to be the open locus in $\overline{\text{Bun}}_{N^\omega} \times_{\text{Bun}_G} \text{Bun}_{B^-}$ where the generalized N -reduction and B^- -reduction are transverse generically on X . It is well-known that Z is a scheme, with connected components

$$Z^\lambda := Z \cap (\overline{\text{Bun}}_{N^\omega} \times_{\text{Bun}_G} \text{Bun}_{B^-}^{\lambda + \deg \rho(\omega)})$$

indexed by $\lambda \in \Lambda^{\text{pos}}$.

The map $\mathfrak{p}^- : \text{Bun}_{B^-} \rightarrow \text{Bun}_G$ gives rise to $'\mathfrak{p}^- : Z \rightarrow \overline{\text{Bun}}_{N^\omega}$. It is shown in [BG1] that $\mathfrak{p}^{-, \lambda}$ and therefore $'\mathfrak{p}^{-, \lambda}$ are smooth for λ sufficiently dominant. Moreover, given a quasicompact open $U \subset \overline{\text{Bun}}_{N^\omega}$, for λ sufficiently dominant the image of $Z^\lambda \rightarrow \overline{\text{Bun}}_{N^\omega}$ contains U .

For each $0 \leq \mu \leq \lambda$ we have the corresponding stratum $Z_{=\mu}^\lambda := Z^\lambda \times_{\overline{\text{Bun}}_{N^\omega}} \overline{\text{Bun}}_{N^\omega, = \mu}$ with locally closed embedding

$$'j_{=\mu}^\lambda : Z_{=\mu}^\lambda \longrightarrow Z^\lambda.$$

Define $\overset{\circ}{Z}^\lambda := Z_{=0}^\lambda$. Similarly, we have $Z_{\leq\mu}^\lambda := Z^\lambda \times_{\overline{\text{Bun}}_{N^\omega}} \overline{\text{Bun}}_{N^\omega, \leq\mu}$ with the open embedding

$$j_{\leq\mu}^\lambda : Z_{\leq\mu}^\lambda \longrightarrow Z^\lambda.$$

Put $Z_{\geq\mu}^\lambda := Z \times_{\overline{\text{Bun}}_{N^\omega}} \overline{\text{Bun}}_{N^\omega, \geq\mu}$, so that $j_{\geq\mu}^\lambda$ extends to the finite map

$$j_{\geq\mu}^\lambda : Z_{\geq\mu}^\lambda \longrightarrow Z^\lambda.$$

10.3 Let us recall the factorization structure on Z . For every $\lambda \in \Lambda^{\text{pos}}$ there is a canonical map $\pi^\lambda : Z^\lambda \rightarrow X^\lambda$, which is well-known to be affine. We write $(X^{\lambda_1} \times X^{\lambda_2})_{\text{disj}} \subset X^{\lambda_1} \times X^{\lambda_2}$ for the open locus where the two divisors are disjoint. Similarly, put

$$(Z^{\lambda_1} \times Z^{\lambda_2})_{\text{disj}} := (Z^{\lambda_1} \times Z^{\lambda_2}) \times_{X^{\lambda_1} \times X^{\lambda_2}} (X^{\lambda_1} \times X^{\lambda_2})_{\text{disj}}.$$

The factorization structure is a canonical morphism $(Z^{\lambda_1} \times Z^{\lambda_2})_{\text{disj}} \rightarrow Z^{\lambda_1 + \lambda_2}$ which fits into a fiber square

$$\begin{array}{ccc} (Z^{\lambda_1} \times Z^{\lambda_2})_{\text{disj}} & \longrightarrow & Z^{\lambda_1 + \lambda_2} \\ \downarrow & & \downarrow \\ (X^{\lambda_1} \times X^{\lambda_2})_{\text{disj}} & \longrightarrow & X^{\lambda_1 + \lambda_2}. \end{array}$$

The factorization structure is compatible with the defect stratification in the following sense. The factorization structure on the strata consists of, for each decomposition $\mu_1 + \mu_2 = \mu$ satisfying $0 \leq \mu_1 \leq \lambda_1$ and $0 \leq \mu_2 \leq \lambda_2$, a morphism $(Z_{=\mu_1}^{\lambda_1} \times Z_{=\mu_2}^{\lambda_2})_{\text{disj}} \rightarrow Z_{=\mu}^{\lambda_1 + \lambda_2}$ which fits into a fiber square

$$\begin{array}{ccc} \coprod_{\mu_1 + \mu_2 = \mu} (Z_{=\mu_1}^{\lambda_1} \times Z_{=\mu_2}^{\lambda_2})_{\text{disj}} & \longrightarrow & Z_{=\mu}^{\lambda_1 + \lambda_2} \\ \downarrow & & \downarrow \\ (X^{\lambda_1} \times X^{\lambda_2})_{\text{disj}} & \longrightarrow & X^{\lambda_1 + \lambda_2}. \end{array}$$

One has similar factorization structures on $Z_{\leq\mu}^\lambda$ and $Z_{\geq\mu}^\lambda$. Moreover, these factorization structures are compatible with $j_{=\mu}^\lambda$, $j_{\geq\mu}^\lambda$, etc.

10.4 We will also need the compactified Zastava space \overline{Z} , which is the open locus in $\overline{\text{Bun}}_{N^\omega} \times_{\text{Bun}_G} \overline{\text{Bun}}_{B^-}$ where the generalized N - and B^- -reductions are generically transverse. In particular there is an open embedding $j^- : Z \rightarrow \overline{Z}$ obtained from $j^- : \text{Bun}_{B^-} \rightarrow \overline{\text{Bun}}_{B^-}$ by base change.

For any $\nu \in \Lambda^{\text{pos}}$ we put

$${}_{=\nu}\overline{Z} := \overline{Z} \times_{\overline{\text{Bun}}_{B^-}} \overline{\text{Bun}}_{B^-, =\nu},$$

and similarly for ${}_{\leq\nu}\overline{Z}$ and ${}_{\geq\nu}\overline{Z}$.

The projections π^λ extend to proper morphisms $\overline{\pi}^\lambda : \overline{Z}^\lambda \rightarrow X^\lambda$. The factorization structure on Z extends to \overline{Z} in a way compatible with both defect stratifications on \overline{Z} .

11 Adelic invariance

11.1 Before proving Theorem 2.7.1, we will show that $j_{=\mu}^! \Psi(\mathcal{W})$ is pulled back from X^μ for any $\mu \in \Lambda^{\text{pos}}$. This property is equivalent to invariance under the ‘‘adelic N^ω ,’’ as we now explain.

For any $x \in X$, we define the open substack $\overline{\text{Bun}}_{N^\omega}^x \subset \overline{\text{Bun}}_{N^\omega}$ to consist of those generalized N^ω -bundles whose defect is disjoint from x . A point of the ind-algebraic stack $\mathcal{H}_{N^\omega}^x$ consists of two points of $\overline{\text{Bun}}_{N^\omega}^x$ together with an identification over $X \setminus \{x\}$. Note that $\mathcal{H}_{N^\omega}^x$ has the structure of a groupoid acting on $\overline{\text{Bun}}_{N^\omega}^x$. The fibers of $\mathcal{H}_{N^\omega}^x$ over $\overline{\text{Bun}}_{N^\omega}^x \times \overline{\text{Bun}}_{N^\omega}^x$ are isomorphic to ind-affine space $\text{colim}_n \mathbb{A}^n$, which implies that the functor which forgets $\mathcal{H}_{N^\omega}^x$ -equivariance is fully faithful, i.e. $\mathcal{H}_{N^\omega}^x$ -equivariance is a property.

We say that an object of $D(\overline{\text{Bun}}_{N^\omega})$ is $N^\omega(\mathbb{A})$ -equivariant if, for every $x \in X$, its restriction to $\overline{\text{Bun}}_{N^\omega}^x$ is $\mathcal{H}_{N^\omega}^x$ -equivariant.

Proposition 11.1.1. *An object \mathcal{F} of $D(\overline{\text{Bun}}_{N^\omega})$ is $N^\omega(\mathbb{A})$ -equivariant if and only if, for*

every $\mu \in \Lambda^{\text{pos}}$, the canonical morphism

$$\mathfrak{m}_\mu^* \mathfrak{m}_{\mu,*} \mathfrak{j}_{=\mu}^! \mathcal{F} \longrightarrow \mathfrak{j}_{=\mu}^! \mathcal{F}$$

is an isomorphism.

11.2 For each $x \in X$, denote by \mathcal{O}_x the completed local ring of X at x , with fraction field K_x . If R is a k -algebra, we denote by $R \hat{\otimes} \mathcal{O}_x$ and $R \hat{\otimes} K_x$ the respective completed tensor products.

The local Hecke stack $\mathcal{H}_{N^\omega}^{\text{loc},x}$ is defined as follows: a $\text{Spec } R$ -point of $\mathcal{H}_{N^\omega}^{\text{loc},x}$ consists of two N^ω -bundles over $\text{Spec}(R \hat{\otimes} \mathcal{O}_x)$ equipped with an isomorphism over $\text{Spec}(R \hat{\otimes} K_x)$. There is a natural restriction map $\text{res}^x : \mathcal{H}_{N^\omega}^x \rightarrow \mathcal{H}_{N^\omega}^{\text{loc},x}$.

Our choice of nondegenerate $\psi : N \rightarrow \mathbb{G}_a$ induces a map $\mathcal{H}_{N^\omega}^{\text{loc},x} \rightarrow \mathbb{G}_a$ in the following way. The projection $N \rightarrow N/[N, N]$ induces a morphism

$$\mathcal{H}_{N^\omega}^{\text{loc},x} \longrightarrow \prod_I \mathcal{H}_{\mathbb{G}_a^\omega}^{\text{loc},x},$$

where \mathbb{G}_a^ω is the additive group scheme over X attached to the canonical line bundle ω_X . Note that there is a canonical isomorphism of groupoids

$$\mathcal{H}_{\mathbb{G}_a^\omega}^{\text{loc},x} \xrightarrow{\sim} \mathbb{G}_a^\omega(\mathcal{O}_x) \backslash \mathbb{G}_a^\omega(K_x) / \mathbb{G}_a^\omega(\mathcal{O}_x),$$

so taking residues defines a morphism $\text{rsd}^x : \mathcal{H}_{\mathbb{G}_a^\omega}^{\text{loc},x} \rightarrow \mathbb{G}_a$. The composition

$$\text{rsd}_\psi^x : \mathcal{H}_{N^\omega}^x \xrightarrow{\text{res}^x} \mathcal{H}_{N^\omega}^{\text{loc},x} \longrightarrow \prod_I \mathcal{H}_{\mathbb{G}_a^\omega}^{\text{loc},x} \xrightarrow{\prod \text{rsd}^x} \prod_I \mathbb{G}_a \xrightarrow{\text{add}} \mathbb{G}_a$$

is an additive character, meaning it is a morphism of groupoids.

It follows that $\tilde{\chi}^x := \text{rsd}_\psi^{x,1} \chi$ is a character sheaf on $\mathcal{H}_{N^\omega}^x$, i.e. it is multiplicative for the groupoid structure. In particular we can speak of $(\mathcal{H}_{N^\omega}^x, \tilde{\chi}^x)$ -equivariant sheaves on $\overline{\text{Bun}}_{N^\omega}^x$,

which form a full subcategory of $D(\overline{\text{Bun}}_{N^\omega}^x)$. Likewise, if a sheaf on $\overline{\text{Bun}}_{N^\omega}$ is $(\mathcal{H}_{N^\omega}^x, \tilde{\chi}^x)$ -equivariant for all $x \in X$ we say that it is $(N^\omega(\mathbb{A}), \tilde{\chi})$ -equivariant. Although we will not use this fact, the category of $(N^\omega(\mathbb{A}), \tilde{\chi})$ -equivariant sheaves on $\overline{\text{Bun}}_{N^\omega}$ is equivalent to the derived category of vector spaces, being generated by the object \mathcal{W}_1 introduced above. Moreover $(\mathcal{H}_{N^\omega}^x, \tilde{\chi}^x)$ -equivariance for a single $x \in X$ implies $(N^\omega(\mathbb{A}), \tilde{\chi})$ -equivariance.

Observe that there is a natural T -action on $\mathcal{H}_{N^\omega}^{\text{loc},x}$. Using the chosen dominant regular cocharacter $\gamma : \mathbb{G}_m \rightarrow T$, the resulting \mathbb{G}_m -action on $\mathcal{H}_{N^\omega}^{\text{loc},x}$ is contracting. In particular, it extends to an action $\mathbb{A}^1 \times \mathcal{H}_{N^\omega}^{\text{loc},x} \rightarrow \mathcal{H}_{N^\omega}^{\text{loc},x}$ of the multiplicative monoid \mathbb{A}^1 . The $!$ -pullback of χ along the composition

$$\mathbb{A}^1 \times \mathcal{H}_{N^\omega}^x \xrightarrow{\text{id}_{\mathbb{A}^1} \times \text{res}^x} \mathbb{A}^1 \times \mathcal{H}_{N^\omega}^{\text{loc},x} \longrightarrow \mathcal{H}_{N^\omega}^{\text{loc},x} \longrightarrow \mathbb{G}_a$$

defines an \mathbb{A}^1 -family $\tilde{\chi}_{\text{ext}}^x$ of character sheaves on $\mathcal{H}_{N^\omega}^x$. Its $!$ -restriction to $\{1\} \times \mathcal{H}_{N^\omega}^x$ is $\tilde{\chi}_x$, and it is trivial along $\{0\} \times \mathcal{H}_{N^\omega}^x$.

Lemma 11.2.1. *The sheaf $\Psi(\mathcal{W})$ is $N^\omega(\mathbb{A})$ -equivariant.*

Proof. Fix $x \in X$; we omit restriction to $\overline{\text{Bun}}_{N^\omega}^x$ from the notation in what follows. By construction \mathcal{W} is $\tilde{\chi}_{\text{ext}}^x|_{\mathbb{G}_m \times \mathcal{H}_{N^\omega}^x}$ -equivariant. Since $\tilde{\chi}_{\text{ext}}^x|_{\{0\} \times \mathcal{H}_{N^\omega}^x}^!$ is the trivial character sheaf, it follows from our construction of Ψ (or equivalently, Beilinson's) that $\Psi(\mathcal{W})$ is $\mathcal{H}_{N^\omega}^x$ -equivariant as desired. □

12 Restriction to the strata

12.1 Now we give the first proof of Theorem 2.7.1 by deducing it from Theorem 1.3.6 in [AG2], which describes the restrictions to the strata of $i^!j_!\mathcal{W}$ in terms of the perverse sheaf Ω . Since we work with a fixed dominant regular coweight γ rather than the entire torus T , it will be necessary to prove a slightly different formulation of the latter theorem.

As in Section 7.2 of [AG2], the inclusion

$$D(\overline{\text{Bun}}_{N^\omega})^{N^\omega(\mathbb{A})} \longrightarrow D(\overline{\text{Bun}}_{N^\omega})$$

of $N^\omega(\mathbb{A})$ -equivariant sheaves on $\overline{\text{Bun}}_{N^\omega}$ admits a right adjoint, which we denote by

$$\text{Av}_*^{N^\omega(\mathbb{A})} : D(\overline{\text{Bun}}_{N^\omega}) \longrightarrow D(\overline{\text{Bun}}_{N^\omega})^{N^\omega(\mathbb{A})}.$$

Let i and j be as in Section 2.6, and write $p : \mathbb{A}^1 \times \overline{\text{Bun}}_{N^\omega} \rightarrow \overline{\text{Bun}}_{N^\omega}$ for the projection. We will also abusively denote $p \circ j$ by p .

Proposition 12.1.1. *There is a canonical isomorphism $i^! j_! \mathcal{W} \xrightarrow{\sim} \text{Av}_*^{N^\omega(\mathbb{A})} p_! \mathcal{W}$.*

Proof. See Section 10.3 of [AG2], where the claim is proved for the action of the entire torus T . The same proof applies *mutatis mutandi* to our claim, which involves only the \mathbb{G}_m -action. □

First proof of Theorem 2.7.1. Since $j_{=\mu}^!$ preserves (homotopy) colimits, we see that

$$j_{=\mu}^! \Psi(\mathcal{W}) \xrightarrow{\sim} k_{H^\bullet(\mathbb{G}_m)} \otimes j_{=\mu}^! i^! j_! \mathcal{W}.$$

By Proposition 12.1.1 we have

$$j_{=\mu}^! i^! j_! \mathcal{W} \xrightarrow{\sim} j_{=\mu}^! \text{Av}_*^{N^\omega(\mathbb{A})} p_! \mathcal{W},$$

and Proposition 11.1.1 implies that

$$j_{=\mu}^! \text{Av}_*^{N^\omega(\mathbb{A})} p_! \mathcal{W} \xrightarrow{\sim} \mathfrak{m}_\mu^* \mathfrak{m}_{\mu,*} j_{=\mu}^! p_! \mathcal{W}.$$

Now Theorem 1.3.6 of [AG2] yields

$$\mathbf{m}_\mu^* \mathbf{m}_{\mu,*} \mathbf{j}_{=\mu}^! \mathcal{W} \xrightarrow{\sim} \mathbf{m}_\mu^\Delta \Omega^\mu \otimes H_c^\bullet(\mathbb{G}_m)[1]. \quad (12.1.1)$$

It remains to show that under the composed isomorphism

$$\mathbf{j}_{=\mu}^! i^! \mathbf{j}_! \mathcal{W} \xrightarrow{\sim} \mathbf{m}_\mu^\Delta \Omega^\mu \otimes H_c^\bullet(\mathbb{G}_m)[1],$$

the action of $H^\bullet(\mathbb{G}_m)$ on the left hand side corresponds to the natural action on $H_c^\bullet(\mathbb{G}_m)$ on the right hand side. Since $H_c^\bullet(\mathbb{G}_m) = H^\bullet(\mathbb{G}_m)[-1]$ as $H^\bullet(\mathbb{G}_m)$ -modules, this will finish the proof.

It is clear that

$$\mathbf{j}_{=\mu}^! i^! \mathbf{j}_! \mathcal{W} \xrightarrow{\sim} \mathbf{m}_\mu^* \mathbf{m}_{\mu,*} \mathbf{j}_{=\mu}^! \mathcal{W}$$

intertwines the actions of $H^\bullet(\mathbb{G}_m)$, since it is obtained by evaluating a morphism of functors on \mathcal{W} . Tracing through the proof of Theorem 1.3.6 in [AG2], we see that the isomorphism (12.1.1) is also obtained by evaluating a morphism of functors on \mathcal{W} , with the appearance of $H_c^\bullet(\mathbb{G}_m)$ accounted for by the isomorphism

$$a_{\gamma,!} \mathcal{W} \xrightarrow{\sim} \mathcal{W}_1 \otimes H_\bullet(\mathbb{G}_m)[-1] \xrightarrow{\sim} \mathcal{W}_1 \otimes H_c^\bullet(\mathbb{G}_m)[1].$$

The latter isomorphism intertwines the actions of $H^\bullet(\mathbb{G}_m)$ as needed. □

12.2 The rest of this subsection is devoted to the second proof of Theorem 2.7.1. This proof applies Theorem 4.6.1 in [R1], which says that Ω can be realized as the twisted cohomology of Zastava space. Accordingly, we must formulate the analogue of Theorem 2.7.1 on Zastava

space. First, the Whittaker sheaf: we claim that

$$\mathscr{W}_{Z^\lambda} := (\mathrm{id}_{\mathbb{G}_m} \times {}'p^{-,\lambda})^! \mathscr{W}[\dim \overline{\mathrm{Bun}}_{N^\omega} - \dim Z^\lambda]$$

is perverse for any λ . For λ sufficiently dominant ${}'p^{-,\lambda}$ is smooth, so that \mathscr{W}_{Z^λ} is the cohomologically normalized inverse image of the perverse sheaf \mathscr{W} . If $\lambda' \leq \lambda$ then we can pull back \mathscr{W}_{Z^λ} along $\mathrm{id}_{\mathbb{G}_m}$ times the factorization morphism

$$(Z^{\lambda'} \times Z^{\lambda-\lambda'})_{\mathrm{disj}} \longrightarrow Z^\lambda,$$

and it is not hard to see that we obtain the restriction of $\mathscr{W}_{Z^{\lambda'}} \boxtimes \mathscr{W}_{Z^{\lambda-\lambda'}}$. Since the factorization map is étale, this implies that $\mathscr{W}_{Z^{\lambda'}}$ is perverse as desired.

Since the map $\overline{\mathrm{Bun}}_{N^\omega} \rightarrow \mathrm{Bun}_G$ is T -equivariant for the trivial action of T on Bun_G , we obtain an action of T on Z which makes ${}'p^-$ a T -equivariant map. In particular our fixed dominant regular cocharacter $\gamma : \mathbb{G}_m \rightarrow T$ induces an action of \mathbb{G}_m on Z , and since \mathscr{W} was \mathbb{G}_m -equivariant, so is \mathscr{W}_Z .

Theorem 12.2.1. *For any $\mu \in \Lambda^{\mathrm{pos}}$ there is an isomorphism*

$${}'j_{=\mu}^! \Psi(\mathscr{W}_Z) \xrightarrow{\sim} {}'m_\mu^\Delta \Omega^\mu.$$

We will need to use the factorization structure on Z in the following way. First, observe that \mathscr{W}_Z admits a natural factorization structure. Thus $\Psi(\mathscr{W}_Z)$ admits a factorization structure by the Künneth formula for nearby cycles. Although the Künneth formula holds for the total nearby cycles functor, in this case the total nearby cycles equals the unipotent nearby cycles because \mathscr{W}_Z is \mathbb{G}_m -equivariant.

12.3 In the second proof, Theorems 2.7.1 and 12.2.1 will be proved simultaneously by an inductive argument. The argument uses the following key lemma.

Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of Artin stacks with \mathcal{Y} smooth, and suppose we are given a function $\mathcal{Y} \rightarrow \mathbb{A}^1$. Let $g : S \rightarrow \mathcal{Y}$ be a morphism where S is an affine scheme and consider the cartesian square

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} S & \xrightarrow{g} & \mathcal{X} \\ \downarrow f & & \downarrow f \\ S & \xrightarrow{g} & \mathcal{Y}. \end{array}$$

Write $i : S_0 \rightarrow S$ for the inclusion of the vanishing locus of the function $S \rightarrow \mathcal{Y} \rightarrow \mathbb{A}^1$.

Lemma 12.3.1. *For any $\mathcal{F} \in D(\mathcal{X})$ which is universally locally acyclic (ULA) over \mathcal{Y} and any $\mathcal{G} \in D(S)$, there is a canonical isomorphism*

$$\Psi(g^!(\mathcal{F}) \otimes^{\mathbb{L}} f^!(\mathcal{G})) \xrightarrow{\sim} g^!(\mathcal{F}) \otimes^{\mathbb{L}} f^!(i^! \Psi(\mathcal{G})).$$

Fix $\nu \in \Lambda^{\text{pos}}$. We will apply the lemma in the case $\mathcal{X} = \mathbb{A}^1 \times \overline{\text{Bun}}_{B^-, \leq \nu}^\lambda$, $\mathcal{F} = \text{IC}_{\mathbb{A}^1} \boxtimes j_!^-(\text{IC}_{\text{Bun}_{B^-, \leq \nu}^\lambda})|_{\overline{\text{Bun}}_{B^-, \leq \nu}^\lambda}$, and $\mathcal{Y} = \mathbb{A}^1 \times \text{Bun}_G$. Let us check that the ULA property holds when λ is sufficiently dominant relative to ν .

According to Corollary 4.5 in [BG2], the perverse sheaf $j_!^-(\text{IC}_{\text{Bun}_{B^-, \leq \nu}^\lambda})|_{\overline{\text{Bun}}_{B^-, \leq \nu}^\lambda}$ has a filtration by $\eta \leq \nu$, with its subquotients being the perverse sheaves $j_{\geq \eta, !}^-(\Omega^\eta \boxtimes \text{IC}_{\text{Bun}_{B^-, \leq \nu - \eta}^{\lambda - \eta}})$. Here Ω^η is attached to $\check{\mathfrak{n}}^-$ rather than $\check{\mathfrak{n}}$. Since the ULA property is stable under extensions and exterior products, it suffices to show that these subquotients are ULA over Bun_G .

Observe that the diagram

$$\begin{array}{ccc} X^\eta \times \overline{\text{Bun}}_{B^-, \leq \nu - \eta}^{\lambda - \eta} & \xrightarrow{j_{\geq \eta, !}^-} & \overline{\text{Bun}}_{B^-, \leq \nu - \eta}^{\lambda - \eta} \\ \downarrow & & \downarrow \\ \overline{\text{Bun}}_{B^-, \leq \nu - \eta}^{\lambda - \eta} & \longrightarrow & \text{Bun}_G, \end{array}$$

commutes, where the left vertical arrow is projection onto the second factor. Since $j_{\geq \eta, !}^-$ is proper, the functor $j_{\geq \eta, !}^-$ preserves the ULA property. Thus it suffices to prove that, for λ sufficiently dominant, $\text{IC}_{\text{Bun}_{B^-, \leq \nu - \eta}^{\lambda - \eta}}$ is ULA over Bun_G for all $\eta \leq \nu$. This follows immediately from Corollary 4.1.1.1 in [C2].

Second proof of Theorems 2.7.1 and 12.2.1. Observe that Theorem 2.7.1 is trivial on the open stratum, since

$$\Psi(\mathcal{W})|_{\text{Bun}_{N^\omega}} = \Psi(\mathcal{W}|_{\mathbb{G}_m \times \text{Bun}_{N^\omega}}) = \Psi(\text{IC}_{\mathbb{G}_m} \boxtimes \text{IC}_{\text{Bun}_{N^\omega}}) = \text{IC}_{\text{Bun}_{N^\omega}},$$

and similarly for Theorem 12.2.1 on \mathring{Z} .

We begin by proving Theorem 12.2.1 for the deepest strata, i.e. the closed embeddings

$$'j_{=\mu}^\mu : X^\mu \longrightarrow Z^\mu.$$

Recall that $\Psi(\mathcal{W}_Z)$ is \mathbb{G}_m -monodromic by construction, so the contraction principle says that

$$'j_{=\mu}^{\mu,!} \Psi(\mathcal{W}_Z) = \pi_!^\mu \Psi(\mathcal{W}_Z).$$

Write $\mathring{\pi}^\mu := \pi^\mu \circ 'j_{=0}^\mu$. Theorem 4.6.1 in [R1] implies that there is an isomorphism

$$(\text{id}_{\mathbb{G}_m} \times \mathring{\pi})_! (\mathcal{W}_Z|_{\mathbb{G}_m \times \mathring{Z}}) \xrightarrow{\sim} \text{IC}_{\mathbb{G}_m} \boxtimes \Omega,$$

compatible with the factorization structures. Since \mathcal{W}_Z is $!$ -extended from $\mathbb{G}_m \times \mathring{Z}$, we obtain

$$(\text{id}_{\mathbb{G}_m} \times \pi)_! \mathcal{W}_Z \xrightarrow{\sim} \text{IC}_{\mathbb{G}_m} \boxtimes \Omega.$$

Since $\pi = \bar{\pi} \circ 'j^-$ and Ψ commutes with proper pushforwards, we have

$$\bar{\pi}_! \Psi((\text{id}_{\mathbb{G}_m} \times 'j^-)_! \mathcal{W}_Z) \xrightarrow{\sim} \Omega.$$

Therefore it suffices to prove that the canonical morphism

$$'j_!^- \Psi(\mathcal{W}_Z) \longrightarrow \Psi((\text{id}_{\mathbb{G}_m} \times 'j^-)_! \mathcal{W}_Z)$$

is an isomorphism. Since Ψ commutes with Verdier duality we can replace the $!$ -pushforwards with $*$ -pushforwards.

Fix $S \rightarrow \overline{\text{Bun}}_{N^\omega}$ with S an affine scheme and apply Lemma 12.3.1 with $f = \text{id}_{\mathbb{A}^1} \times \bar{\mathfrak{p}}^-$,

$$g : \mathbb{A}^1 \times S \longrightarrow \mathbb{A}^1 \times \overline{\text{Bun}}_{N^\omega} \longrightarrow \mathbb{A}^1 \times \text{Bun}_G,$$

$\mathcal{F} = \text{IC}_{\mathbb{A}^1} \boxtimes \text{j}_*^-(\omega_{\text{Bun}_{B^-}})|_{\overline{\text{Bun}}_{B^-, \leq \nu}^\lambda}$, and $\mathcal{G} = \mathcal{W}|_{\mathbb{A}^1 \times S}^!$. Then the lemma yields an isomorphism

$$\Psi((\text{id}_{\mathbb{G}_m} \times \text{j}^-)_* \mathcal{W}_Z)|_{\leq \nu \overline{Z}^\lambda} \xrightarrow{\sim} \text{j}_*^-(\Psi(\mathcal{W}_Z))|_{\leq \nu \overline{Z}^\lambda}$$

for λ sufficiently dominant. Changing λ if necessary so that $\lambda \geq \mu$, we can restrict this isomorphism along the map

$$(\leq \nu \overline{Z}^\mu \times \overset{\circ}{Z}^{\lambda-\mu})_{\text{disj}} \longrightarrow \leq \nu \overline{Z}^\lambda.$$

By factorizability we obtain the desired isomorphism on $\leq \nu \overline{Z}^\mu$. Since ν was arbitrary, Theorem 12.2.1 is proved for the deepest strata.

Now we prove Theorem 2.7.1. Fix $\mu \in \Lambda^{\text{pos}}$ and choose $\lambda \geq \mu$ dominant enough that $Z_{=\mu}^\lambda$ surjects smoothly onto $X^\mu \times_{\text{Bun}_T} \text{Bun}_B$. Note that $(X^\mu \times \overset{\circ}{Z}^{\lambda-\mu})_{\text{disj}}$ is one of the connected components of the fiber product

$$(X^\mu \times X^{\lambda-\mu})_{\text{disj}} \times_{X^\lambda} Z_{=\mu}^\lambda,$$

and that the former surjects onto $\overline{\text{Bun}}_{N^\omega, =\mu}$. Theorem 12.2.1 for the deepest and open strata implies that the cohomologically normalized pullback of $\text{j}_{=\mu}^! \Psi(\mathcal{W})$ to $(X^\mu \times \overset{\circ}{Z}^{\lambda-\mu})_{\text{disj}}$ is $\Omega^\mu \boxtimes \text{IC}_{\overset{\circ}{Z}^{\lambda-\mu}}$. Theorem 2.7.1 follows once we observe that the composition

$$(X^\mu \times \overset{\circ}{Z}^{\lambda-\mu})_{\text{disj}} \longrightarrow \overline{\text{Bun}}_{N^\omega, =\mu} \xrightarrow{\mathfrak{m}_\mu} X^\mu$$

is the projection onto the first factor and apply Lemma 11.2.1.

The previous paragraph implies Theorem 12.2.1 holds on the stratum $Z_{=\mu}^\lambda$. Let $\lambda' \geq \mu$ and change λ if necessary so that $\lambda \geq \lambda'$. By restricting along the morphism

$$(Z_{=\mu}^{\lambda'} \times \mathring{Z}^{\lambda-\lambda'})_{\text{disj}} \longrightarrow Z_{=\mu}^\lambda$$

and invoking factorization, we obtain Theorem 12.2.1.

The remainder of Theorems 2.7.1 and 12.2.1 follows as in the first proof. □

13 First proof of Theorem 2.8.1

13.1 Like Theorem 2.7.1, we formulate the analogue of Theorem 2.8.1 on Zastava space.

Theorem 13.1.1. *For any $\lambda \in \Lambda^{\text{pos}}$, there is an \mathfrak{sl}_2 -equivariant isomorphism of factorizable sheaves*

$$\text{gr } \Psi(\mathcal{W}_{Z^\lambda}) \xrightarrow{\sim} \bigoplus_{0 \leq \mu \leq \lambda} {}'j_{=\mu, !*}^{\lambda} {}'m_{\mu}^{\lambda, \Delta} \mathcal{P}^{\mu}. \quad (13.1.1)$$

Now we work out three of the simplest cases of Theorem 13.1.1. For brevity, we will write $\Psi := \Psi(\mathcal{W}_{Z^\lambda})$.

Example 13.1.2. Let α be a simple coroot. There is an isomorphism $Z^\alpha \cong X \times \mathbb{A}^1$ under which $\mathring{Z}^\alpha \cong X \times \mathbb{G}_m$. The canonical map $\mathring{Z}^\alpha \rightarrow \mathbb{A}^1$ is given in these terms by $(x, t) \mapsto \frac{1}{t}$. It follows from Example 4.3 of [C1] that Ψ is the cohomologically normalized pullback of the unique indecomposable tilting sheaf on \mathbb{A}^1 which extends $\text{IC}_{\mathbb{G}_m}$. Moreover, the monodromy filtration

$$F_{-1} \subset F_0 \subset F_1 = \Psi$$

satisfies $F_{-1} \cong \text{IC}_{Z_{=\alpha}^\alpha}$, $F_0/F_{-1} \cong \text{IC}_{Z^\alpha}$, and $F_1/F_0 \cong \text{IC}_{Z_{\neq\alpha}^\alpha}$, where $Z_{=\alpha}^\alpha = X \times \{0\} \subset X \times \mathbb{A}^1 = Z^\alpha$. This confirms Theorem 13.1.1 in the case $\lambda = \alpha$.

Example 13.1.3. Now consider the case $\lambda = 2\alpha$. By Theorem 12.2.1 for $\mu = 2\alpha$, we have a short exact sequence

$$\Omega^{2\alpha} \longrightarrow \Psi \longrightarrow {}'j_{\leq\alpha,*}^{2\alpha} {}'j_{\leq\alpha}^{2\alpha,*} \Psi$$

(recall that $\Omega^{2\alpha}$ is the clean extension of the sign local system on $X_{\text{disj}}^{(2)}$). Similarly, applying Theorem 12.2.1 for $\mu = \alpha$ and 0 we obtain an exact triangle

$${}'j_{=\alpha,*}^{2\alpha} \text{IC}_{Z_{\geq\alpha}^{2\alpha}} \longrightarrow {}'j_{\leq\alpha,*}^{2\alpha} {}'j_{\leq\alpha}^{2\alpha,*} \Psi \longrightarrow {}'j_{=0,*}^{2\alpha} \text{IC}_{Z^{2\alpha}},$$

where we used the fact that $\Omega^\alpha \cong \text{IC}_X$. Applying Verdier duality to the equation in Corollary 4.5 of [BG2] (or rather the analogous equation on Zastava space), we have

$$[{}'j_{=0,*}^{2\alpha} \text{IC}_{Z^{2\alpha}}] = [\text{IC}_{Z^{2\alpha}}] + [\text{IC}_{Z_{\geq\alpha}^{2\alpha}}] + [\Omega^{2\alpha}],$$

where we identified $\Omega^{2\alpha}$ with its Verdier dual. Finally, one computes the simple constituents of ${}'j_{=\alpha,*}^{2\alpha} \text{IC}_{Z_{\geq\alpha}^{2\alpha}}$ as follows: first, consider the short exact sequence

$$\text{IC}_{Z_{\geq\alpha}^{2\alpha}} \longrightarrow {}'j_{=\alpha,*}^{2\alpha} \text{IC}_{Z_{\geq\alpha}^{2\alpha}} \longrightarrow {}'j_{=2\alpha}^{2\alpha,!} \text{IC}_{Z_{\geq\alpha}^{2\alpha}}[1].$$

To compute the third term, observe that there is a Cartesian square

$$\begin{array}{ccc} X^2 & \longrightarrow & Z_{\geq\alpha}^{2\alpha} \\ \downarrow & & \downarrow {}'j_{\geq\alpha}^{2\alpha} \\ X^{(2)} & \xrightarrow{{}'j_{=2\alpha}^{2\alpha}} & Z^{2\alpha}, \end{array}$$

and that the !-restriction of $\text{IC}_{Z_{\geq\alpha}^{2\alpha}}$ along the top horizontal morphism is $\text{IC}_{X^2}[-1]$. Since ${}'j_{\geq\alpha}^{2\alpha}$ is finite and birational onto its image, we can use base change to compute

$${}'j_{=2\alpha}^{2\alpha,!} \text{IC}_{Z_{\geq\alpha}^{2\alpha}}[1] = {}'j_{=2\alpha}^{2\alpha,!} {}'j_{\geq\alpha,*}^{2\alpha} \text{IC}_{Z_{\geq\alpha}^{2\alpha}}[1] = \text{IC}_{Z_{\geq\alpha}^{2\alpha}} \oplus \Omega^{2\alpha}.$$

Summarizing, we have

$$[\Psi] = [\mathrm{IC}_{Z^{2\alpha}}] + 2[\mathrm{IC}_{\overline{Z^{2\alpha}}}] + [\mathrm{IC}_{Z_{=2\alpha}^{2\alpha}}] + 3[\Omega^{2\alpha}].$$

Now we will determine which graded component of $\mathrm{gr} \Psi$ each simple subquotient lies in. In what follows, “weight” refers to an eigenvalue of the Lefschetz Cartan operator. Since the monodromy filtration is compatible with the factorization structure, when we pull back $\mathrm{gr} \Psi$ along the factorization map

$$(Z^\alpha \times Z^\alpha)_{\mathrm{disj}} \longrightarrow Z^{2\alpha}$$

we get $(\mathrm{gr} \Psi(\mathcal{W}_{Z^\alpha}))^{\boxtimes 2}$ restricted to $(Z^\alpha \times Z^\alpha)_{\mathrm{disj}}$. By the previous example, the latter sheaf with \mathfrak{sl}_2 -action is isomorphic to

$$(\mathrm{std}^{\otimes 2} \otimes \mathrm{IC}_{X^2}) \oplus (\mathrm{std} \otimes \mathrm{IC}_{X \times Z^\alpha}) \oplus (\mathrm{std} \otimes \mathrm{IC}_{Z^\alpha \times X}) \oplus (\mathrm{triv} \otimes \mathrm{IC}_{Z^\alpha \times Z^\alpha}). \quad (13.1.2)$$

It follows immediately that $\mathrm{IC}_{Z^{2\alpha}}$ has weight 0 and that the two copies of $\mathrm{IC}_{\overline{Z^{2\alpha}}}$ have weights ± 1 . Since $\mathrm{std}^{\otimes 2} \cong V_2 \oplus \mathrm{triv}$, the three copies of $\Omega^{2\alpha}$ have weights $-2, 0,$ and 2 , and $\mathrm{IC}_{Z_{=2\alpha}^{2\alpha}}$ has weight 0.

In terms of the monodromy filtration

$$F_{-2} \subset F_{-1} \subset F_0 \subset F_1 \subset F_2 = \Psi,$$

we have $F_{-2} \cong \Omega^{2\alpha} \cong F_2/F_0$, $F_{-1}/F_{-2} \cong \mathrm{IC}_{\overline{Z^{2\alpha}}} \cong F_1/F_0$, and F_0/F_{-1} has simple constituents $\mathrm{IC}_{Z^{2\alpha}}$, $\mathrm{IC}_{Z_{=2\alpha}^{2\alpha}}$, and $\Omega^{2\alpha}$. So in order to prove Theorem 13.1.1 in the case $\lambda = 2\alpha$, it remains to show that F_0/F_{-1} is semisimple. Its pullback along the factorization map is semisimple, and since semisimplicity is étale local, the restriction of F_0/F_{-1} to $Z^{2\alpha} \setminus \pi^{-1}(X)$ is semisimple. But Ψ has no simple subquotients supported on $\pi^{-1}(X)$, so F_0/F_{-1} is the intermediate

extension of its restriction to $Z^{2\alpha} \setminus \pi^{-1}(X)$ and therefore semisimple.

Example 13.1.4. Suppose $\lambda = \alpha + \beta$ is a coroot, where α and β are distinct simple coroots. Applying Theorem 12.2.1 for $\mu = \lambda = \alpha + \beta$, we obtain the short exact sequence

$$\Omega^\lambda \longrightarrow \Psi \longrightarrow {}'j_{<\lambda,*}^\lambda {}'j_{<\lambda}^{\lambda,*} \Psi.$$

Similar considerations yield the short exact sequence

$${}'j_{=\alpha,*}^\lambda \mathrm{IC}_{Z_{=\alpha}^\lambda} \oplus {}'j_{=\beta,*}^\lambda \mathrm{IC}_{Z_{=\beta}^\lambda} \longrightarrow {}'j_{<\lambda,*}^\lambda {}'j_{<\lambda}^{\lambda,*} \Psi \longrightarrow {}'j_{=0,*}^\lambda \mathrm{IC}_{Z^\lambda}.$$

Applying Corollary 4.5 of [BG2], we have

$$[{}'j_{=0,*}^\lambda \mathrm{IC}_{Z^\lambda}] = [\mathrm{IC}_{Z^\lambda}] + [\mathrm{IC}_{\overline{Z_{=\alpha}^\lambda}}] + [\mathrm{IC}_{\overline{Z_{=\beta}^\lambda}}] + [\Upsilon^\lambda],$$

where Υ^λ is the Verdier dual of Ω^λ . According to Section 1.3.2 in [AG2], in this case Ω^λ is the $*$ -extension of $\mathrm{IC}_{X_{\mathrm{disj}}^2}$ to $X^\lambda = X^2$, whence Υ^λ is the $!$ -extension. In particular $[\Omega^\lambda] = [\Upsilon^\lambda] = [\mathrm{IC}_{X^2}] + [\mathrm{IC}_X]$. As for the remaining simple constituents, consider the short exact sequence

$$\mathrm{IC}_{\overline{Z_{=\alpha}^\lambda}} \longrightarrow {}'j_{=\alpha,*}^\lambda \mathrm{IC}_{Z_{=\alpha}^\lambda} \longrightarrow {}'j_{=\alpha}^{\lambda,!} \mathrm{IC}_{\overline{Z_{=\alpha}^\lambda}}[1],$$

and similarly for β . The third term is IC_{X^2} , so finally we see that

$$[\Psi] = [\mathrm{IC}_{Z^\lambda}] + 2[\mathrm{IC}_{\overline{Z_{=\alpha}^\lambda}}] + 2[\mathrm{IC}_{\overline{Z_{=\beta}^\lambda}}] + 4[\mathrm{IC}_{X^2}] + 2[\mathrm{IC}_X].$$

Now we compute the weights of the simple subquotients of Ψ . We have the factorization morphism

$$(Z^\alpha \times Z^\beta)_{\mathrm{disj}} \longrightarrow Z^{\alpha+\beta},$$

and after pulling back $\mathrm{gr} \Psi$ the result is (13.1.2), up to relabeling β as α . As in Example

13.1.3, it follows that $\mathrm{IC}_{Z^{\alpha+\beta}}$ has weight 0, the two copies of $\mathrm{IC}_{\overline{Z^{\lambda}_{\alpha}}}$ have weights ± 1 and likewise for $\mathrm{IC}_{\overline{Z^{\lambda}_{\beta}}}$, and the four copies of IC_{X^2} have weights $-2, 0, 0,$ and 2 . We will see below that in any case where λ is a coroot, there are two simple subquotients of Ψ isomorphic to IC_X , with weights ± 1 . In terms of the monodromy filtration, we have $F_{-2} \cong \mathrm{IC}_{X^2} \cong F_2/F_1$, F_{-1}/F_{-2} and F_1/F_0 each have simple subquotients $\mathrm{IC}_{\overline{Z^{\lambda}_{\alpha}}}$, $\mathrm{IC}_{\overline{Z^{\lambda}_{\beta}}}$, and IC_X , and F_0/F_{-1} has simple subquotients $\mathrm{IC}_{Z^{\lambda}}$ and IC_{X^2} , the latter with multiplicity two. As in Example 13.1.3, one uses factorization to show that F_0/F_{-1} is semisimple. To prove the semisimplicity of F_{-1}/F_{-2} and F_1/F_0 , it is enough to show that there are no extensions between IC_X and $\mathrm{IC}_{\overline{Z^{\lambda}_{\alpha}}}$. By Verdier duality, it suffices to prove that

$$\mathrm{Ext}_{D(Z^{\lambda})}^1(\mathrm{IC}_X, \mathrm{IC}_{\overline{Z^{\lambda}_{\alpha}}}) = 0.$$

We have $j_{=\lambda}^{\lambda,!} \mathrm{IC}_{\overline{Z^{\lambda}_{\alpha}}} = \mathrm{IC}_{X^2}[-1]$, whence $\Delta^! \mathrm{IC}_{\overline{Z^{\lambda}_{\alpha}}} = \mathrm{IC}_X[-2]$. Thus

$$\mathrm{RHom}_{D(Z^{\lambda})}(\mathrm{IC}_X, \mathrm{IC}_{\overline{Z^{\lambda}_{\alpha}}}) = H^{\bullet}(X)[-2],$$

and in particular Ext^1 vanishes.

13.2 The following lemma will be used in both proofs of Theorems 2.8.1 and 13.1.1. For any $\mu \in \Lambda^{\mathrm{pos}}$ write

$$\Delta^{\mu} : X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B \longrightarrow \overline{\mathrm{Bun}}_{N^{\omega}}$$

for the finite map defined as the composition of $j_{\geq \mu}$ and the embedding

$$X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B = X \times_{X^{\mu}} \overline{\mathrm{Bun}}_{N^{\omega}, \geq \mu} \longrightarrow \overline{\mathrm{Bun}}_{N^{\omega}, \geq \mu}$$

induced by the diagonal map $X \rightarrow X^{\mu}$.

Lemma 13.2.1. *If μ is a coroot, then there is an indecomposable subquotient \mathcal{M} of $\Psi(\mathcal{W})$*

with a filtration

$$\mathcal{M}_{-1} \subset \mathcal{M}_0 \subset \mathcal{M}_1 = \mathcal{M}$$

such that $\mathcal{M}_{-1} \cong \Delta_*^\mu \mathrm{IC}_{X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B}$, $\mathcal{M}_0 / \mathcal{M}_{-1} \cong \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$, and $\mathcal{M} / \mathcal{M}_0 \cong \Delta_*^\mu \mathrm{IC}_{X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B}$.

Proof. First, we claim the subsheaf $\mathbf{j}_{=0,!} \mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ of $\Psi(\mathcal{W})$ has a quotient \mathcal{M}_0 of the form described above. Recall that $\mathbf{j}_{=0,!} \mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ has a descending filtration with subquotients $\mathbf{j}_{=\nu,!} \mathbf{r}_\nu^\Delta \Omega^\nu$, and in particular has $\mathbf{j}_{=\mu,!} \Omega^\mu$ as a subquotient and $\mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$ as a quotient. The former sheaf has $\Delta_*^\mu \mathrm{IC}_{X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B}$ as a quotient because μ is a coroot, so it suffices to show that for any $0 < \nu < \mu$ we have

$$\mathrm{Ext}^1(\mathbf{j}_{=\nu,!} \mathbf{r}_\nu^\Delta \Omega^\nu, \Delta_*^\mu \mathrm{IC}_{X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B}) = 0,$$

or dually

$$\mathrm{Ext}^1(\Delta_*^\mu \mathrm{IC}_{X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B}, \mathbf{j}_{=\nu,!} \mathbf{r}_\nu^\Delta \Upsilon^\nu) = 0$$

where Υ^ν is the Verdier dual of Ω^ν . One computes using base change that

$$\Delta^{\mu,!} \mathbf{j}_{=\nu,!} \mathbf{r}_\nu^\Delta \Upsilon^\nu = \Delta^! \Upsilon^\nu \overset{!}{\otimes} \Delta^{\mu,!} \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}.$$

Since $\Delta^{\mu,!} \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$ is concentrated in cohomological degrees ≥ 1 , and both $\Delta^! \Upsilon^\nu$ and $\Delta^{\mu,!} \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$ have lisse (actually constant) cohomology sheaves, their $!$ -tensor product is concentrated in cohomological degrees ≥ 2 . It follows that the Ext^1 above vanishes.

We have shown that $\mathbf{j}_{=0,!} \mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ has a quotient \mathcal{M}_0 which fits into a short exact sequence

$$\Delta_*^\mu \mathrm{IC}_{X \times_{\mathrm{Bun}_T} \overline{\mathrm{Bun}}_B} \longrightarrow \mathcal{M}_0 \longrightarrow \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}.$$

This sequence does not split because \mathcal{M}_0 is the quotient of the indecomposable sheaf $\mathbf{j}_{=0,!} \mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ with simple cosocle $\mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}}$. Dually, we obtain a subsheaf $\mathcal{M} / \mathcal{M}_{-1}$ of $\mathbf{j}_{=0,*} \mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ of the desired form, from which follows the existence of \mathcal{M} .

□

We will first give a proof of Theorem 2.8.1 under the assumption that $\text{gr } \Psi(\mathscr{W})$ is semisimple. The semisimplicity can be proved via Mochizuki's theory of weights for holonomic D -modules, since \mathscr{W} is pure and (up to shift) the monodromy filtration on nearby cycles of a pure sheaf coincides with the weight filtration (see Corollary 9.1.10 in [M]).

First proof of Theorems 2.8.1 and 13.1.1. Both sides of the isomorphism (2.8.1) restrict to $\text{IC}_{\text{Bun}_{N^\omega}}$. Suppose that we have constructed the isomorphism over $\overline{\text{Bun}}_{N^\omega, < \mu}$. Then for $\lambda \geq \mu$ sufficiently dominant, pulling back along $'\mathfrak{p}^-$ yields the isomorphism (13.1.1) over $Z_{< \mu}^\lambda$. One obtains (13.1.1) on $Z_{< \mu}^\mu$ by pullback along the factorization map

$$(Z_{< \mu}^\mu \times \overset{\circ}{Z}^{\lambda - \mu})_{\text{disj}} \longrightarrow Z_{< \mu}^\lambda,$$

since the inverse images of both sides of the isomorphism factorize and are constant along the second component. The same argument yields (13.1.1) on $Z_{< \mu}^{\mu'} = Z^{\mu'}$ for $\mu' < \mu$.

On the other hand, one can use factorization to obtain (13.1.1) on $Z^\mu \setminus \pi^{-1}(X)$. Namely, for $\mu_1 + \mu_2 = \mu$, $\mu_1, \mu_2 < \mu$, the pullback of both sides of the isomorphism along

$$(Z^{\mu_1} \times Z^{\mu_2})_{\text{disj}} \longrightarrow Z^\mu$$

are identified. Since this factorization map is étale but not necessarily an embedding, we must argue that the isomorphism descends to its image. This immediately reduces to the case that $\mu = n \cdot \alpha$ for some simple root α . In this case, both sides of (13.1.1) are the intermediate extension of their restriction to $\pi^{-1}(X_{\text{disj}}^\mu)$, so it suffices to show that the isomorphism over $(Z^\alpha)_{\text{disj}}^n$ descends to $\pi^{-1}(X_{\text{disj}}^\mu) \subset Z^\mu$. Since $(Z^\alpha)_{\text{disj}}^n$ is a Σ_n -torsor over $\pi^{-1}(X_{\text{disj}}^\mu)$ and Σ_n is generated by transpositions, the claim reduces to the case $n = 2$. But this was already done in Example 13.1.3.

Note that $Z_{< \mu}^\mu \cup (Z^\mu \setminus \pi^{-1}(X)) = Z^\mu \setminus \Delta(X)$. The isomorphisms of the previous two

paragraphs clearly agree on $Z_{<\mu}^\mu \cap (Z^\mu \setminus \pi^{-1}(X))$, hence glue to an isomorphism away from the main diagonal.

If μ is not a coroot, then we claim that $\Psi(\mathscr{W}_{Z^\mu})$ has no simple subquotients supported on the main diagonal, whence $\text{gr } \Psi(\mathscr{W}_{Z^\mu})$ is the intermediate extension of its restriction to $Z^\mu \setminus \Delta(X)$. This is true for the right hand side of (13.1.1) by construction, so the claim implies that the isomorphism extends to Z^μ in this case. By Theorem 12.2.1 there is a filtration of $\Psi(\mathscr{W}_{Z^\mu})$ by the sheaves $'j_{=\nu,*}^\mu 'm_\nu^{\mu,\Delta} \Omega^\nu$ for $0 \leq \nu \leq \mu$. Using Corollary 4.5 of [BG2], one can show that $'j_{=\nu,*}^\mu 'm_\nu^{\mu,\Delta} \Omega^\nu$ surjects onto $'j_{=\mu,*}^\mu \text{add}_*(\Omega^\nu \boxtimes \Upsilon^{\mu-\nu})$, and that no subquotient of the kernel is supported on X^μ . Now the claim follows, because out of the latter sheaves only Ω^μ and Υ^μ could have subquotients supported on the diagonal, and by Section 3.3 of *loc. cit.* this occurs if and only if μ is a coroot.

Suppose that μ is a coroot. Then $\Delta_* \text{IC}_X$ appears as a subquotient of Ω^μ and of Υ^μ with multiplicity one. By the analysis in the previous paragraph $\Delta_* \text{IC}_X$ appears as a summand of $\text{gr } \Psi(\mathscr{W}_{Z^\mu})$ with multiplicity two, and there are no other subquotients supported on the main diagonal. Thus the isomorphism (13.1.1) extends to Z^μ , and it remains to show that \mathfrak{sl}_2 acts on the summand $\text{IC}_X^{\oplus 2}$ of $\text{gr } \Psi(\mathscr{W}_{Z^\mu})$ as the standard representation.

The only other possibility is that \mathfrak{sl}_2 acts on $\text{IC}_X^{\oplus 2}$ trivially, which would imply that the subquotient \mathscr{M} from Lemma 13.2.1 is a subquotient of F_0/F_{-1} . But \mathscr{M} is indecomposable and F_0/F_{-1} is semisimple, so this is impossible.

Having constructed the isomorphism of Theorem 13.1.1 over Z^μ , we can complete the inductive step of Theorem 2.8.1 by extending the isomorphism from $\overline{\text{Bun}}_{N^\omega, <\mu}$ to $\overline{\text{Bun}}_{N^\omega, \leq \mu}$. Choose $\lambda \geq \mu$ dominant enough that $Z_{\leq \mu}^\lambda$ surjects smoothly onto $\overline{\text{Bun}}_{N^\omega, \leq \mu}$. As in the proof of Theorem 2.7.1, note that $(Z^\mu \times \mathring{Z}^{\lambda-\mu})_{\text{disj}}$ is one of the connected components of the fiber product

$$(X^\mu \times X^{\lambda-\mu})_{\text{disj}} \times_{X^\lambda} Z_{\leq \mu}^\lambda,$$

and that the former surjects onto $\overline{\text{Bun}}_{N^\omega, \leq \mu}$. By factorization, the cohomologically normalized pullback of $\text{gr } \Psi(\mathscr{W})$ to $(Z^\mu \times \mathring{Z}^{\lambda-\mu})_{\text{disj}}$ is $(\text{gr } \Psi(\mathscr{W}_{Z^\mu})) \boxtimes \text{IC}_{\mathring{Z}^{\lambda-\mu}}$.

A factorization argument as in the proof of Theorem 12.2.1 allows us to construct the isomorphism (13.1.1) over $Z_{\leq \mu}^\lambda$ for arbitrary $\lambda \geq \mu$, which completes the proof of Theorem 13.1.1.

□

Proof of Corollary 2.8.1.1. It suffices to prove the corresponding claim on Z^λ for any $\lambda \in \Lambda^{\text{pos}}$. The morphism

$${}'j_{=0,!}^\lambda \text{IC}_{Z^\lambda} \longrightarrow \Psi(\mathcal{W}_{Z^\lambda})$$

is injective because $\Psi(\mathcal{W}_{Z^\lambda})$ is tilting, so it suffices to show that ${}'j_{=0,!}^\lambda \text{IC}_{Z^\lambda}$ and the kernel of monodromy have the same class in the Grothendieck group. By factorization and induction this holds away from the main diagonal. Arguing as in the above proof of Theorem 13.1.1, if λ is not a coroot then we are done. If λ is a coroot then Corollary 4.5 of [BG2] implies that the only subquotient of ${}'j_{=0,!}^\lambda \text{IC}_{Z^\lambda}$ supported on the main diagonal is $\Delta_* \text{IC}_X$ with multiplicity one. Theorem 13.1.1 implies that the same is true for the kernel of the monodromy operator on $\Psi(\mathcal{W}_{Z^\lambda})$.

□

14 Second proof of Theorem 2.8.1

14.1 In this section we will give a proof of Theorems 2.8.1 and 13.1.1 which does not use weights for irregular holonomic D -modules to prove the semisimplicity of $\text{gr } \Psi(\mathcal{W})$, but instead depends on Conjecture 3.3.1 (but only the part proved in [FFKM]). First we make the statement of the conjecture more precise in the case of a trivial \check{T} -local system by specifying the action of generators of $\check{\mathfrak{g}} \otimes H^\bullet(X)$ on $\bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B}$.

We construct the action of $\check{\mathfrak{h}} \otimes H^\bullet(X)$ as follows. Pullback along the evaluation map

$$X \times \text{Bun}_T \longrightarrow \text{pt}/T$$

defines a homomorphism

$$\mathrm{Sym}(\mathfrak{h}^*[-2]) = H^\bullet(\mathrm{pt}/T) \longrightarrow H^\bullet(X) \otimes H^\bullet(\mathrm{Bun}_T).$$

By adjunction we obtain a morphism $\mathfrak{h}^* \otimes H_\bullet(X)[-2] \longrightarrow H^\bullet(\mathrm{Bun}_T)$. Identifying $\mathfrak{h}^* \cong \check{\mathfrak{h}}$ and $H_\bullet(X)[-2] \cong H^\bullet(X)$, the latter using Poincaré duality, we obtain a morphism

$$\check{\mathfrak{h}} \otimes H^\bullet(X) \longrightarrow H^\bullet(\mathrm{Bun}_T).$$

Then the action of $H^\bullet(\mathrm{Bun}_T)$ on ω_{Bun_T} induces by functoriality the desired action of $\check{\mathfrak{h}} \otimes H^\bullet(X)$ on $\mathrm{Eis}_{!*} \omega_{\mathrm{Bun}_T} = \bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}_B}}$.

Next we construct the action of $\check{\mathfrak{n}} \otimes H^\bullet(X)$. Denote by $\mathcal{U}(\check{\mathfrak{n}})$ the factorization algebra whose fiber at $\sum_i \mu_i x_i \in X^\mu$ is

$$\bigotimes_i U(\check{\mathfrak{n}})^{\mu_i},$$

where the superscript μ_i indicates the corresponding \check{T} -graded component. The following is part of Theorem 5.6 in [BG2], which itself is a restatement of results from [FFKM]: for any $\mu \in \Lambda^{\mathrm{pos}}$ there is a canonical morphism

$$\mathfrak{j}_{\geq \mu, !}(\mathcal{U}(\check{\mathfrak{n}})^\mu \boxtimes \mathrm{IC}_{\overline{\mathrm{Bun}_B}}) \longrightarrow \mathrm{IC}_{\overline{\mathrm{Bun}_B}},$$

which induces an isomorphism

$$\mathcal{U}(\check{\mathfrak{n}})^\mu \boxtimes \mathrm{IC}_{\mathrm{Bun}_B} \xrightarrow{\sim} \mathfrak{j}_{=\mu}^! \mathrm{IC}_{\overline{\mathrm{Bun}_B}}.$$

Here we abuse notation slightly by denoting the maps $\mathfrak{j}_{\geq \mu} : X^\mu \times \overline{\mathrm{Bun}_B} \rightarrow \overline{\mathrm{Bun}_B}$ and $\mathfrak{j}_{=\mu} : X^\mu \times \mathrm{Bun}_B \rightarrow \overline{\mathrm{Bun}_B}$ by the same symbols we used in the case of $\overline{\mathrm{Bun}_{N^\omega}}$. The latter are obtained from the former by base change along $\overline{\mathrm{Bun}_{N^\omega}} \rightarrow \overline{\mathrm{Bun}_B}$.

Pushing forward to Bun_G , we obtain a morphism

$$H^\bullet(X^\mu, \mathcal{U}(\check{\mathfrak{n}})^\mu) \otimes \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B} \longrightarrow \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B}.$$

The object $\mathcal{U}(\check{\mathfrak{n}})^\mu$ is concentrated in (perverse) cohomological degrees ≥ 1 , and if μ is a coroot then we have $H^1(\mathcal{U}(\check{\mathfrak{n}})^\mu) = \text{IC}_X$. The resulting morphism $k_X \rightarrow \mathcal{U}(\check{\mathfrak{n}})^\mu$ induces

$$H^\bullet(X) \otimes \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B} \longrightarrow H^\bullet(X^\mu, \mathcal{U}(\check{\mathfrak{n}})^\mu) \otimes \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B} \longrightarrow \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B},$$

which defines the action of $\check{\mathfrak{n}}_\mu \otimes H^\bullet(X) \cong H^\bullet(X)$ on $\bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B}$.

Dually, for any $\mu \in \Lambda^{\text{pos}}$ there is a canonical morphism

$$\text{IC}_{\overline{\text{Bun}}_B} \longrightarrow \mathfrak{j}_{\geq \mu, *}(\mathcal{U}^\vee(\check{\mathfrak{n}}^-)^\mu \boxtimes \text{IC}_{\overline{\text{Bun}}_B}),$$

which induces an isomorphism

$$\mathfrak{j}_{=\mu}^* \text{IC}_{\overline{\text{Bun}}_B} \xrightarrow{\sim} \mathcal{U}^\vee(\check{\mathfrak{n}}^-)^\mu \boxtimes \text{IC}_{\overline{\text{Bun}}_B}.$$

Here $\mathcal{U}^\vee(\check{\mathfrak{n}}^-)$ is by definition the Verdier dual of $\mathcal{U}(\check{\mathfrak{n}}^-)$.

Thus we obtain a morphism

$$\bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B} \longrightarrow H^\bullet(X^\mu, \mathcal{U}(\check{\mathfrak{n}}^-)^\mu)^\vee \otimes \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B},$$

or by adjunction

$$H^\bullet(X^\mu, \mathcal{U}(\check{\mathfrak{n}}^-)^\mu) \otimes \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B} \longrightarrow \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B}.$$

If μ is a coroot, then as before we have a morphism $H^\bullet(X) \rightarrow H^\bullet(X^\mu, \mathcal{U}(\check{\mathfrak{n}}^-)^\mu)$, which defines the action of $\check{\mathfrak{n}}_{-\mu} \otimes H^\bullet(X) \cong H^\bullet(X)$ on $\bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B}$.

14.2 Fix a coroot μ . Recall the subquotient \mathcal{M} of $\Psi(\mathcal{W})$ from Lemma 13.2.1. The action of $\check{\mathfrak{g}}$ on $\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ relates to our problem through the following key lemma, whose proof will occupy this subsection.

Lemma 14.2.1. *The sheaf \mathcal{M} does not descend to $\overline{\mathrm{Bun}}_{N^\omega}/\mathbb{G}_m$.*

First, observe that \mathcal{M}_0 and $\mathcal{M}/\mathcal{M}_{-1}$ descend to $\overline{\mathrm{Bun}}_{N^\omega}/T$ and hence to $\overline{\mathrm{Bun}}_{N^\omega}/\mathbb{G}_m$, being subquotients of $\mathfrak{j}_{=0,!} \mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ and $\mathfrak{j}_{=0,*} \mathrm{IC}_{\mathrm{Bun}_{N^\omega}}$ respectively. The obstruction to descent of \mathcal{M} to $\overline{\mathrm{Bun}}_{N^\omega}/\mathbb{G}_m$ is the resulting composition

$$\Delta_*^\mu \mathrm{IC}_{(X \times_{\mathrm{Bun}} T \overline{\mathrm{Bun}}_B)/\mathbb{G}_m} \longrightarrow \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}/\mathbb{G}_m}[1] \longrightarrow \Delta_*^\mu \mathrm{IC}_{(X \times_{\mathrm{Bun}} T \overline{\mathrm{Bun}}_B)/\mathbb{G}_m}[2]. \quad (14.2.1)$$

Similarly, the obstruction to its descent to $\overline{\mathrm{Bun}}_{N^\omega}/T$ is the composition

$$\Delta_*^\mu \mathrm{IC}_{(X \times_{\mathrm{Bun}} T \overline{\mathrm{Bun}}_B)/T} \longrightarrow \mathrm{IC}_{\overline{\mathrm{Bun}}_{N^\omega}/T}[1] \longrightarrow \Delta_*^\mu \mathrm{IC}_{(X \times_{\mathrm{Bun}} T \overline{\mathrm{Bun}}_B)/T}[2]. \quad (14.2.2)$$

Denote by $\mathring{\Delta}^\mu : X \times_{\mathrm{Bun}} T \mathrm{Bun}_B \rightarrow \overline{\mathrm{Bun}}_{N^\omega}$ the locally closed embedding given by composing Δ^μ with the open embedding $X \times_{\mathrm{Bun}} T \mathrm{Bun}_B \rightarrow X \times_{\mathrm{Bun}} T \overline{\mathrm{Bun}}_B$. Composition with the canonical morphisms

$$\mathring{\Delta}_!^\mu \mathrm{IC}_{(X \times_{\mathrm{Bun}} T \mathrm{Bun}_B)/T} \rightarrow \Delta_*^\mu \mathrm{IC}_{(X \times_{\mathrm{Bun}} T \overline{\mathrm{Bun}}_B)/T}$$

and

$$\Delta_*^\mu \mathrm{IC}_{(X \times_{\mathrm{Bun}} T \overline{\mathrm{Bun}}_B)/T} \rightarrow \mathring{\Delta}_*^\mu \mathrm{IC}_{(X \times_{\mathrm{Bun}} T \mathrm{Bun}_B)/T}$$

gives

$$\mathrm{End}(\Delta_*^\mu \mathrm{IC}_{(X \times_{\mathrm{Bun}} T \overline{\mathrm{Bun}}_B)/T}) \longrightarrow H^\bullet(X \times \mathrm{pt}/T), \quad (14.2.3)$$

since the map $(X \times_{\mathrm{Bun}} T \mathrm{Bun}_B)/T \rightarrow X \times \mathrm{pt}/T$ induces an isomorphism on cohomology.

Lemma 14.2.2. *The image of the endomorphism (14.2.2) under (14.2.3) is*

$$-1 \otimes h_\mu \in H^0(X) \otimes \mathfrak{h}^* \subset H^2(X \times \text{pt}/T).$$

Proof. Theorem 5.1.5 in [BG1] says that $\text{IC}_{\overline{\text{Bun}}_B}$ is ULA over Bun_T , which implies that the $!$ -restriction of $\text{IC}_{\overline{\text{Bun}}_B}$ to $\overline{\text{Bun}}_{N^\omega}/T$ is $\text{IC}_{\overline{\text{Bun}}_{N^\omega}/T}[\dim T - \dim \text{Bun}_T]$. It follows that the $!$ -restriction of $\Delta_*^\mu \text{IC}_{X \times \overline{\text{Bun}}_B}$ to $\overline{\text{Bun}}_{N^\omega}/T$ is a shift of $\Delta_*^\mu \text{IC}_{(X \times \text{Bun}_T, \overline{\text{Bun}}_B)/T}$, where we abusively write $\Delta^\mu : X \times \overline{\text{Bun}}_B \rightarrow \overline{\text{Bun}}_B$ for the similarly-defined finite map. This gives rise to a commutative square

$$\begin{array}{ccc} \text{End}(\Delta_*^\mu \text{IC}_{X \times \overline{\text{Bun}}_B}) & \longrightarrow & H^\bullet(X \times \text{Bun}_T) \\ \downarrow & & \downarrow \\ \text{End}(\Delta_*^\mu \text{IC}_{(X \times \text{Bun}_T, \overline{\text{Bun}}_B)/T}) & \longrightarrow & H^\bullet(X \times \text{pt}/T), \end{array}$$

where the upper horizontal arrow is defined similarly to (14.2.3) and the right vertical arrow is id_X times restriction along $\rho(\omega) : \text{pt}/T \rightarrow \text{Bun}_T$. The previous subsection implies that \mathcal{M}_0 and $\mathcal{M}/\mathcal{M}_{-1}$ extend to $\overline{\text{Bun}}_B$, giving rise to a morphism

$$\Delta_*^\mu \text{IC}_{X \times \overline{\text{Bun}}_B} \longrightarrow \text{IC}_{\overline{\text{Bun}}_B}[1] \longrightarrow \Delta_*^\mu \text{IC}_{X \times \overline{\text{Bun}}_B}[2] \quad (14.2.4)$$

which restricts to (14.2.2) on $\overline{\text{Bun}}_{N^\omega}/T$. Thus it suffices to show that the image of (14.2.4) in $H^2(X \times \text{Bun}_T)$ restricts to $-1 \otimes h_\mu$ along $\text{id}_X \times \rho(\omega)$.

Observe that

$$\text{End}(\Delta_*^\mu \text{IC}_{X \times \overline{\text{Bun}}_B}) = \text{Hom}(\Delta_*^\mu(k_X \boxtimes \text{IC}_{\overline{\text{Bun}}_B}), \Delta_*^\mu(\omega_X \boxtimes \text{IC}_{\overline{\text{Bun}}_B}))[-2],$$

so $\bar{\mathfrak{p}}_*$ induces a morphism

$$\text{End}(\Delta_*^\mu \text{IC}_{X \times \overline{\text{Bun}}_B}) \longrightarrow \text{Hom}(H^\bullet(X) \otimes \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B}, H_\bullet(X) \otimes \bar{\mathfrak{p}}_* \text{IC}_{\overline{\text{Bun}}_B})[-2].$$

Composing with the canonical map $k \rightarrow H^\bullet(X)$ and its dual $H_\bullet(X) \rightarrow k$, we obtain

$$\mathrm{End}(\Delta_*^\mu \mathrm{IC}_{X \times \overline{\mathrm{Bun}}_B}) \longrightarrow \mathrm{End}(\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B})[-2]. \quad (14.2.5)$$

By construction, the image of (14.2.2) under (14.2.5) coincides with the action of $f_\mu e_\mu \in U(\check{\mathfrak{g}})$.

By composing with the morphisms $\mathfrak{p}! \mathrm{IC}_{\mathrm{Bun}_B} \rightarrow \bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ and $\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B} \rightarrow \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B}$, we obtain

$$\mathrm{End}(\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}) \longrightarrow \mathrm{Hom}(\mathfrak{p}! \mathrm{IC}_{\mathrm{Bun}_B}, \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B}). \quad (14.2.6)$$

Note that (14.2.6) annihilates the endomorphism of $\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ given by the action of $e_\mu f_\mu$, since it factors through a sheaf supported on the boundary. Proposition 4.9 in [FFKM] says that the relation $[e_\mu, f_\mu] = h_\mu$ holds in $\mathrm{End}(\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B})$, which implies that the images of $f_\mu e_\mu$ and $-h_\mu$ under (14.2.6) coincide.

Now consider the commutative square

$$\begin{array}{ccc} \mathrm{End}(\Delta_*^\mu \mathrm{IC}_{X \times \overline{\mathrm{Bun}}_B}) & \xrightarrow{(14.2.3)} & H^\bullet(X \times \mathrm{Bun}_T) \\ (14.2.5) \downarrow & & \downarrow \\ \mathrm{End}(\bar{\mathfrak{p}}_* \mathrm{IC}_{\overline{\mathrm{Bun}}_B})[-2] & \xrightarrow{(14.2.6)} & \mathrm{Hom}(\mathfrak{p}! \mathrm{IC}_{\mathrm{Bun}_B}, \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B})[-2], \end{array}$$

where the right vertical morphism is the composition

$$\begin{aligned} H^\bullet(X \times \mathrm{Bun}_T) &= \mathrm{Hom}(\mathring{\Delta}_!^\mu \mathrm{IC}_{X \times \mathrm{Bun}_B}, \mathring{\Delta}_*^\mu \mathrm{IC}_{X \times \mathrm{Bun}_B}) \\ &\longrightarrow \mathrm{Hom}(H^\bullet(X) \otimes \mathfrak{p}! \mathrm{IC}_{\mathrm{Bun}_B}, H_\bullet(X) \otimes \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B})[-2] \\ &\longrightarrow \mathrm{Hom}(\mathfrak{p}! \mathrm{IC}_{\mathrm{Bun}_B}, \mathfrak{p}_* \mathrm{IC}_{\mathrm{Bun}_B})[-2]. \end{aligned}$$

Note that $\mathrm{id}_X \times \rho(\omega) : X \times \mathrm{pt}/T \rightarrow X \times \mathrm{Bun}_T$ admits a canonical retraction, given by the projection $X \times \mathrm{Bun}_T \rightarrow X$ and the evaluation map $X \times \mathrm{Bun}_T \rightarrow \mathrm{pt}/T$. So far we have

shown that the image of (14.2.2) under the resulting composition

$$R^2 \text{End}(\Delta_*^\mu \text{IC}_{(X \times \text{Bun}_T \overline{\text{Bun}_B})/T}) \longrightarrow H^2(X \times \text{pt}/T) \longrightarrow H^2(X \times \text{Bun}_T) \longrightarrow R^0 \text{Hom}(\mathfrak{p}_! \text{IC}_{\text{Bun}_B}, \mathfrak{p}_* \text{IC}_{\text{Bun}_B})$$

agrees with the image of $-1 \otimes h_\mu$ under

$$H^2(X \times \text{pt}/T) \longrightarrow H^2(X \times \text{Bun}_T) \longrightarrow R^0 \text{Hom}(\mathfrak{p}_! \text{IC}_{\text{Bun}_B}, \mathfrak{p}_* \text{IC}_{\text{Bun}_B}),$$

so it suffices to show that the latter composition is injective.

We have $H^2(X \times \text{pt}/T) = \check{\mathfrak{h}} \oplus H^2(X)$, and we have already used the fact that for each $\lambda \in \Lambda$, an element $h \in \check{\mathfrak{h}}$ maps to $\langle h, \lambda \rangle$ times the canonical morphism $\mathfrak{p}_! \text{IC}_{\text{Bun}_B^\lambda} \rightarrow \mathfrak{p}_* \text{IC}_{\text{Bun}_B^\lambda}$. One checks that the canonical generator of $H^2(X)$ maps to canonical map $\mathfrak{p}_! \text{IC}_{\text{Bun}_B} \rightarrow \mathfrak{p}_* \text{IC}_{\text{Bun}_B}$ itself, which proves the desired injectivity. □

Note that Lemma 14.2.2 already implies that \mathcal{M} does not descend to $\overline{\text{Bun}}_{N^\omega}/T$, since $h_\mu \neq 0$.

Proof of Lemma 14.2.1. The morphism (14.2.1) induces an element of $H^2(X \times \text{pt}/\mathbb{G}_m)$ in the same way that (14.2.2) gives rise to $-1 \otimes h_\mu \in H^2(X \times \text{pt}/T)$. Moreover, these constructions fit into a commutative square

$$\begin{array}{ccc} \text{End}(\Delta_*^\mu \text{IC}_{(X \times \text{Bun}_T \overline{\text{Bun}_B})/T}) & \longrightarrow & H^\bullet(X \times \text{pt}/T) \\ \downarrow & & \downarrow \\ \text{End}(\Delta_*^\mu \text{IC}_{(X \times \text{Bun}_T \overline{\text{Bun}_B})/\mathbb{G}_m}) & \longrightarrow & H^\bullet(X \times \text{pt}/\mathbb{G}_m) \end{array}$$

where the vertical morphisms are induced by γ , and in particular (14.2.2) maps to (14.2.1) along the left vertical morphism. The image of h_μ under $H^2(\text{pt}/T) \rightarrow H^2(\text{pt}/\mathbb{G}_m) = k$ is the positive integer $\langle h_\mu, \gamma \rangle$, so the lemma follows. □

14.3 We need another, more elementary lemma. Fix $\mu \in \Lambda^{\text{pos}}$ and $\mathfrak{k} \in \text{Kost}(\mu)$ given by $\mu = \sum n_\beta \beta$.

Lemma 14.3.1. *If \mathcal{L} is a nonconstant simple summand of $\boxtimes_{\beta \in R^+} \mathcal{P}_{n_\beta}$, then*

$$\Delta^! l_*^{\mathfrak{k}} \mathcal{L} = 0 = \Delta^* l_*^{\mathfrak{k}} \mathcal{L}.$$

Proof. This follows from the fact that the local system on $X_{\text{disj}}^{(n)}$ associated to a nontrivial irreducible Σ_n -representation extends cleanly over the main diagonal. □

Second proof of Theorems 2.8.1 and 13.1.1. We proceed as in the first proof, until we have constructed the isomorphism (13.1.1) over $Z^\mu \setminus \Delta(X)$ and reduced to the case that μ is a coroot. We showed that in this case $\Psi(\mathcal{W}_{Z^\mu})$ contains $\Delta_* \text{IC}_X$ with multiplicity two but has no other subquotients supported on the main diagonal (in particular, the two sides of (13.1.1) agree in the Grothendieck group). Either \mathfrak{sl}_2 acts trivially on both copies of $\Delta_* \text{IC}_X$ or they have weights 1 and -1 . In order to show that (13.1.1) extends to Z^μ we need to rule out the first case, and then show that $\text{gr } \Psi(\mathcal{W}_{Z^\mu})$ is semisimple.

We must rule out the possibility that \mathfrak{sl}_2 acts trivially on the subquotient \mathcal{M} from Lemma 13.2.1. Since the monodromy endomorphism of $\Psi(\mathcal{W}_{Z^\mu})$ is the obstruction to \mathbb{G}_m -equivariance, this would imply that \mathcal{M} is \mathbb{G}_m -equivariant, contradicting Lemma 13.2.1.

Now we finish the proof that $\text{gr } \Psi(\mathcal{W}_{Z^\mu})$ is semisimple. Using the \mathfrak{sl}_2 -action, it decomposes into the direct sum of its isotypic components, indexed by the irreducible \mathfrak{sl}_2 -representations. The previous paragraph says that the two copies of $\Delta_* \text{IC}_X$ are subquotients of the standard isotypic component. Thus the other isotypic components have no subquotients supported on the main diagonal, so they are the same as the corresponding isotypic components on the right hand side of (13.1.1) and in particular are semisimple. We will show that for any

simple subquotient $\mathcal{L} \neq \Delta_* \mathrm{IC}_X$ of the std-isotypic component we have

$$\mathrm{Ext}^1(\Delta_* \mathrm{IC}_X, \mathcal{L}) = 0 = \mathrm{Ext}^1(\mathcal{L}, \Delta_* \mathrm{IC}_X),$$

from which it follows that $\mathrm{IC}_X^{\oplus 2}$ is a direct summand of the std-isotypic component. Since the other summand has no subquotients supported on the main diagonal, it is semisimple by the induction hypothesis.

We will show that $H^i(\Delta^! \mathcal{L}) = 0$ for $i \leq 1$, which implies that $\mathrm{Ext}^1(\Delta_* \mathrm{IC}_X, \mathcal{L}) = 0$. The other vanishing follows by applying Verdier duality: since $\mathrm{gr} \Psi(\mathcal{W}_{Z^\mu})$ is Verdier self-dual, so is its std-isotypic component. Observe that \mathcal{L} has the form $'j_{=\nu, !}^\mu 'm_\nu^{\mu, \Delta} \mathcal{F}$ for some $0 < \nu \leq \mu$ and a simple summand \mathcal{F} of \mathcal{P}^ν . The case $\nu = 0$ is excluded because then $\mathcal{L} = \mathrm{IC}_{Z^\mu}$ has weight 0 for the Cartan operator in \mathfrak{sl}_2 .

First suppose that $\nu < \mu$. Then we have

$$\Delta^! j_{=\nu, !}^\mu 'm_\nu^{\mu, \Delta} \mathcal{F} \xrightarrow{\sim} \Delta^! \mathcal{F} \otimes^! \Delta^! \mathrm{IC}_{Z^{\mu-\nu}}.$$

Since $\Delta^! \mathrm{IC}_{Z^{\mu-\nu}}$ is concentrated in cohomological degrees ≥ 1 (see [BG2]) and $\Delta^! \mathcal{F}$ is concentrated in degrees ≥ 0 , and both complexes have lisse cohomology sheaves, their $!$ -tensor product is concentrated in degrees ≥ 2 as desired.

Finally, we address the case $\nu = \mu$, where $\mathcal{F} = \mathcal{L}$. By Lemma 14.3.1, we can assume \mathcal{L} is a summand of $\mathrm{add}_* \mathrm{IC}_{\prod X^{(n_\beta)}}$ for some Kostant partition $\mu = \sum n_\beta \beta$. If $\sum n_\beta \geq 3$ then the claim follows by base change. By assumption $\mathcal{L} \neq \Delta_* \mathrm{IC}_X$, so $\sum n_\beta > 1$. This leaves only the case $\sum n_\beta = 2$, and since μ is a coroot the only possibility is that $\mu = \beta_1 + \beta_2$ is a sum of two distinct coroots. As shown in Example 13.1.4, in this case the std-isotypic component of $\mathrm{gr} \Psi(\mathcal{W}_{Z^\mu})$ is just $\mathrm{IC}_X^{\oplus 2}$.

□

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