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Extended Supersymmetric Moduli Space and a SUSY/Non-SUSY Duality

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We study $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theories coupled to an adjoint chiral field with superpotential. We consider the full supersymmetric moduli space of these theories obtained by adding all allowed chiral operators. These include higher-dimensional operators that introduce a field-dependence for the gauge coupling. We show how Feynman diagram/matrix model/string theoretic techniques can all be used to compute the IR glueball superpotential. Moreover, in the limit of turning off the superpotential, this leads to a deformation of $\mathcal{N} = 2$ Seiberg-Witten theory. In the case where the superpotential drives the squared gauge coupling to a negative value, we find that supersymmetry is spontaneously broken, which can be viewed as a novel mechanism for breaking supersymmetry. We propose a new duality between a class of $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theories with field-dependent gauge couplings and a class of $U(N)$ gauge theories where supersymmetry is softly broken by nonzero expectation values for auxiliary components of spurion superfields.

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1. Introduction

The last decade has seen great progress in our understanding of the dynamics of $\mathcal{N} = 1$ supersymmetric gauge theories, with string theory playing a large role in these developments thanks to its rich web of dualities. In particular, motivated by string theoretic considerations [1], a perturbative approach was proposed for the computation of glueball superpotentials in certain $\mathcal{N} = 1$ supersymmetric gauge theories using matrix models [2], which leads to non-perturbatively exact information for these theories at strong coupling. Further evidence for this proposal was provided through direct computations [3], as well as from consideration of $\mathcal{N} = 1$ chiral rings [4].

The simplest class of gauge theories considered in [1] involve an $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory with an adjoint superfield Φ together with a superpotential

$$\text{Tr}W(\Phi) = \sum_k a_k \text{Tr}\Phi^k.$$

In this paper, we consider further deforming this theory by the most general set of single-trace chiral operators. This is accomplished by the introduction of superpotential terms

$$\int d^4x d^2\theta \text{Tr}[\alpha(\Phi)\mathcal{W}_\alpha\mathcal{W}^\alpha],$$

where \mathcal{W}_α is the field strength superfield. In string theory, these theories are constructed by wrapping D5 branes on vanishing cycles in local Calabi-Yau three-folds, where the addition of a background B -field which depends holomorphically on one complex coordinate of the three-fold leads to the above deformation, with

$$\alpha(\Phi) = B(\Phi) = \sum_k t_k \Phi^k.$$

We show how the strongly coupled IR dynamics of these theories can be understood using both string theoretic techniques (large N duality via a geometric transition) and a direct field theory computation as in [3]. Moreover, following [5], we can consider the limit where $W(\Phi)$ is set to zero, in which case we recover an $\mathcal{N} = 2$ supersymmetric theory with Lagrangian given by

$$\mathcal{L} = \int d^4x d^4\theta \mathcal{F}(\Phi).$$

The prepotential $\mathcal{F}(\Phi)$ is related to $\alpha(\Phi)$ by

$$\mathcal{F}''(\Phi) = \alpha(\Phi), \tag{1.1}$$

where Φ is an adjoint-valued $\mathcal{N} = 2$ chiral multiplet. In this limit, our solution reduces to that of the extended Seiberg-Witten theory with general prepotential (1.1). Our results are in complete agreement with the beautiful earlier work of [6], which uses Konishi anomaly [4] and instanton techniques [7] to study these same supersymmetric gauge theories.¹

The stringy perspective which we develop, however, sheds light on nonsupersymmetric phases of these theories, which will be our main focus. In particular, it turns out that if $\alpha(\Phi)$ is chosen appropriately, there are vacua where supersymmetry is broken. The idea is that a suitable choice of higher-dimensional operators can lead to negative values of g_{YM}^2 for certain factors of the gauge group. Motivated by string theory considerations, we will show that strong coupling effects can make sense of the negative value for g_{YM}^2 , and at the same time lead to supersymmetry breaking. In the string theory construction, this arises from the presence of antibranes in a holomorphic B -field background. When g_{YM}^2 is negative in all the gauge group factors, we propose a complete UV field theory description of these vacua. This is another $U(N)$ gauge theory, already studied in [9,10,11], with an adjoint field $\tilde{\Phi}$ and superpotential

$$\int d^2\theta \text{Tr}[\tilde{t}_0 \tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha + \tilde{W}(\tilde{\Phi})], \quad (1.2)$$

where

$$\tilde{W}(\tilde{\Phi}) = \sum_k (a_k + 2it_k \theta\theta) \tilde{\Phi}^k.$$

Note that since the spurion auxiliary fields have nonzero vevs t_k , this theory breaks supersymmetry.

A duality between supersymmetric and nonsupersymmetric theories may appear contradictory. The way this arises is as follows (see figure 1). We have an IR effective $\mathcal{N} = 1$ theory which is valid below a cutoff scale Λ_0 . The IR theory is formulated in terms of chiral fields which we collectively denote by χ (for us, these are glueball fields). The theory depends on some couplings t , and for each value of t we find two sets of vacua – one which is supersymmetric, and one which is not. However, for any given values of t , only one of these vacua is physical, in that the expectation value of the chiral fields is below the cutoff scale $|\langle\chi\rangle| < \Lambda_0$. The other solution falls outside of this region of validity. In particular, in one regime of parameter space, only the supersymmetric solution is acceptable. As we change t , the supersymmetric solution leaves the allowed region of field space, and at the

¹ A special case of these theories with a particular choice of $W(\Phi)$ was also studied in [8].

same time the nonsupersymmetric solution enters the allowed region. We obtain in this way a duality between a supersymmetric and a nonsupersymmetric theory. Moreover, we are able to identify two dual UV theories. However, unlike the effective IR theory, which is valid for the entire parameter space, each UV theory is valid only for part of the full parameter space. The supersymmetric IR solution matches onto a supersymmetric UV theory, and the nonsupersymmetric IR solution matches onto another UV theory where supersymmetry is broken softly by spurions.

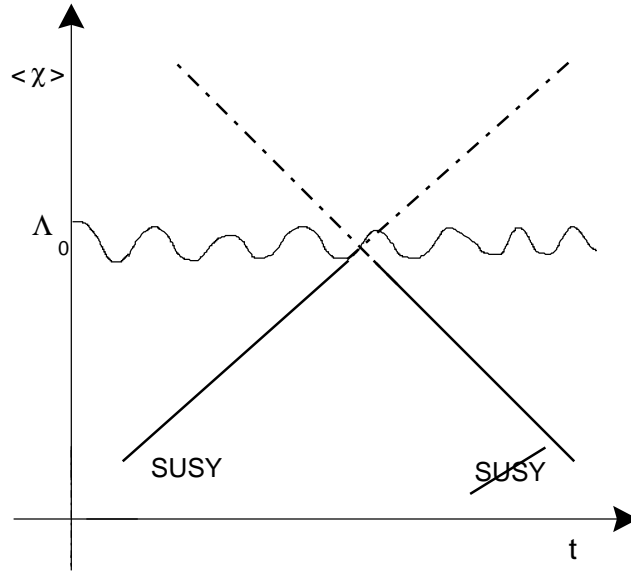


Fig. 1. A phase diagram for the supersymmetric/nonsupersymmetric duality. The horizontal axis represents the full parameter space, and the vertical axis represents field vevs. A wavy line at the cutoff Λ_0 is the region where we begin to lose validity of a given solution – we can trust solutions only below this scale.

The organization of this paper is as follows: In section 2 we establish the basic field theories which will be studied. In section 3 we show how these field theories can be realized in type IIB string theory on local Calabi-Yau three-folds. In section 4 we show how this string theory construction leads to a solution for the IR dynamics of the theory. In section 5 we derive the same result directly from field theory considerations. In section 6 we specialize to the $\mathcal{N} = 2$ case. In section 7 we consider these field theories when some of the gauge couplings g_{YM}^2 become negative. We explain why this leads to supersymmetry-breaking and propose a dual description. Some aspects of the effective superpotential computation are presented in an appendix.

2. Field Theory

Consider an $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory with no hypermultiplets. Classically, this theory is described by a holomorphic prepotential $\mathcal{F}(\Phi)$ which appears in the $\mathcal{N} = 2$ Lagrangian,

$$\mathcal{L} = \int d^4x d^4\theta \mathcal{F}(\Phi) \tag{2.1}$$

where Φ is an adjoint-valued $\mathcal{N} = 2$ chiral multiplet, and

$$\mathcal{F}(\Phi) = \frac{t_0}{2} \text{Tr} \Phi^2. \tag{2.2}$$

Above, t_0 determines the classical gauge coupling and θ angle

$$t_0 = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}, \tag{2.3}$$

and the integral in (2.1) is over a chiral half of the $\mathcal{N} = 2$ superspace. The low energy dynamics of this theory were studied in [12], where it was shown that the theory admits a solution in terms of an auxiliary Riemann surface and one-form.

This theory admits a natural extension via the introduction of higher-dimensional single-trace chiral operators,

$$\mathcal{F}(\Phi) = \sum_{k=0} \frac{t_k}{(k+1)(k+2)} \text{Tr} \Phi^{k+2}, \tag{2.4}$$

which deform the theory in the ultraviolet. One effect of these new terms is that the effective gauge coupling at a given point in moduli space now depends explicitly on the expectation value of the scalar component ϕ of the superfield Φ ,

$$t_0 \rightarrow \mathcal{F}''(\phi) = \sum_{k=0} t_k \phi^k.$$

We therefore define

$$\alpha(\Phi) \equiv \mathcal{F}''(\Phi).$$

In this paper, we will solve for the low energy dynamics of this extended Seiberg-Witten theory.

We will also study deformations of the theory (2.1) to an $\mathcal{N} = 1$ supersymmetric theory by the addition of a superpotential,

$$\mathrm{Tr}W(\Phi) = \sum_{k=0}^{n+1} a_k \mathrm{Tr}\Phi^k, \quad (2.5)$$

for the $\mathcal{N} = 1$ chiral multiplet Φ that sits inside $\mathbf{\Phi}$. In $\mathcal{N} = 1$ language, the full superpotential of the theory then becomes

$$\int d^2\theta \left(\mathrm{Tr} [\alpha(\Phi) \mathcal{W}_\alpha \mathcal{W}^\alpha] - \mathrm{Tr}W(\Phi) \right), \quad (2.6)$$

where \mathcal{W}_α is the gaugino superfield.

Classically, the superpotential $W(\Phi)$ freezes the eigenvalues of ϕ at points in the moduli space where

$$W'(\phi) = 0. \quad (2.7)$$

For generic superpotential, we can write

$$W'(x) = g \prod_{i=1}^n (x - e_i), \quad (2.8)$$

with e_i all distinct, so the critical points are isolated and the choice of a vacuum breaks the gauge symmetry as

$$U(N) \rightarrow \prod_{k=1}^n U(N_k) \quad (2.9)$$

for the vacuum with N_i of the eigenvalues of ϕ placed at each critical point $x = e_i$.

As long as the effective gauge couplings of the low-energy theory are positive, i.e.

$$\mathrm{Im}[\alpha(e_i)] = \left(\frac{4\pi}{g_{\mathrm{YM}}^2} \right)_i > 0, \quad i = 1, \dots, n \quad (2.10)$$

the general aspects of the low energy dynamics of this theory are readily apparent. In the vacuum (2.9), at sufficiently low energies, the theory is pure $\mathcal{N} = 1$ super-Yang-Mills, which is expected to exhibit confinement and gaugino condensation.

When the original $\mathcal{N} = 2$ theory has canonical prepotential (2.2), the condition (2.10) is satisfied trivially, and in this case the problem of computing the vacuum expectation values of gaugino condensates in the $\mathcal{N} = 1$ theory,

$$S_k = \text{Tr} \mathcal{W}_{\alpha,k} \mathcal{W}_k^\alpha, \quad (2.11)$$

has been studied extensively from both string theory [1] and gauge theory [3,4] perspectives. The question can be posed in terms of the computation of an effective glueball superpotential [1],

$$\mathcal{W}_{\text{eff}}(S_i),$$

whose critical points give the supersymmetric vacua of the theory. In this paper, we will show how to compute \mathcal{W}_{eff} for the $\mathcal{N} = 1$ theory with the more general prepotential (2.4). Note that physically inequivalent choices of $\alpha(\Phi)$ correspond to polynomials in Φ of degree at most $n - 1$. This is because, for the supersymmetric theory, any operator of the form

$$\text{Tr} [\Phi^k W'(\Phi) \mathcal{W}_\alpha \mathcal{W}^\alpha] \sim 0$$

is trivial in the chiral ring [4].

In section 7, we will ask what happens when (2.10) is not satisfied and it appears that some of the gauge couplings of (2.9) become negative in the vacuum. We will show that in this case, the theory (2.6) generically breaks supersymmetry. Moreover, the supersymmetry-breaking vacua still exhibit gaugino condensation and confinement, and we will be able to compute the corresponding expectation values (2.11) as critical points of a certain effective *scalar* potential $V_{\text{eff}}(S_i)$.

3. The String Theory Construction

In this section we give the string theory realization of the above gauge theory. To begin with, we consider type IIB string theory compactified on an A_1 singularity,

$$uv = y^2, \quad (3.1)$$

which is fibered over the complex x -plane. This has a singularity for all x at $u, v, y = 0$, which can be resolved by blowing up a finite \mathbf{P}^1 . Wrapping N D5 branes on the \mathbf{P}^1 gives a $d = 4$ $U(N)$ $\mathcal{N} = 2$ gauge theory at sufficiently low energies. The adjoint scalar ϕ of the gauge theory corresponds to motion of the branes in the x -plane.

In the microscopic $\mathcal{N} = 2$ gauge theory we also have a choice of prepotential $\mathcal{F}(\Phi)$. What does this correspond to geometrically? To answer this, note that the microscopic prepotential determines the bare 4d gauge coupling, which arises in the geometry from the presence of nonzero B -fields,

$$\frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} = \int_{\mathbf{P}^1} \left(B_{\text{RR}} + \frac{i}{g_s} B_{\text{NS}} \right). \quad (3.2)$$

In the undeformed theory with the prepotential (2.2), the gauge coupling was a constant t_0 . This translates to the statement that, classically, as the ALE space is fibered over the x -plane, the Kähler modulus of the \mathbf{P}^1 (in particular the B -fields in (3.2)) does *not* vary with x . In the extended Seiberg-Witten theory, the complexified gauge coupling becomes ϕ -dependent. Since the adjoint scalar ϕ parameterizes the positions of the D5 branes in the x plane, making the gauge coupling ϕ -dependent should correspond to letting the background B -fields in (3.2) be x -dependent,

$$B(x) = \int_{S_x^2} \left(B_{\text{RR}} + \frac{i}{g_s} B_{\text{NS}} \right), \quad (3.3)$$

where the integral on the right hand side is over the S^2 at a point in the x plane. In order to reproduce the gauge theory, we require

$$B(x) \rightarrow B_0(x) = \alpha(x) = \sum_{k=0}^{n-1} t_k x^k. \quad (3.4)$$

To summarize, the gauge theory in section 2 is realized as the low-energy limit of N D5 branes wrapped on an $A_1 \times \mathbf{C}$ singularity with H -flux turned on,

$$\int_{S_x^2} H_0 = dB_0(x) \neq 0. \quad (3.5)$$

It may seem surprising that turning on H -flux does not break supersymmetry down to $\mathcal{N} = 1$.² In the case at hand, the flux we are turning on is due to a B -field that varies *holomorphically* over the complex x -plane. It is known that if the B -field varies holomorphically, the full $\mathcal{N} = 2$ supersymmetry is preserved [13].

As was explained in [1], turning on a superpotential $\text{Tr}W(\Phi)$ for the adjoint chiral superfield, as in (2.5), corresponds in the geometry to fibering the ALE space over complex x -plane nontrivially,

$$uv = y^2 - W'(x)^2, \quad (3.6)$$

where

$$W(x) = \sum_{k=1}^{n+1} a_k x^k.$$

The resulting manifold is a Calabi-Yau three-fold and supersymmetry is broken to $\mathcal{N} = 1$. After turning on $W(x)$, the minimal S^2 's (the holomorphic \mathbf{P}^1 's) are isolated at n points in the x -plane, $x = e_i$, which are critical points of the superpotential,

$$W'(x) = g \prod_{i=1}^n (x - e_i).$$

At each of these points, the geometry develops a conifold singularity, which is resolved by a minimal \mathbf{P}^1 . The gauge theory vacuum where the gauge symmetry is broken as in (2.9) corresponds to choosing N_i of the D5 branes to wrap the i 'th \mathbf{P}^1 . In particular, the tree-level gauge coupling for the branes wrapping the \mathbf{P}^1 at $x = e_i$ is given by

$$\int_{\mathbf{P}^1_i} B_0 = \left(\frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} \right)_i = \alpha(e_i), \quad (3.7)$$

which agrees with the classical values in the gauge theory.

In summary, we can engineer the $\mathcal{N} = 1$ theory obtained from the extended $\mathcal{N} = 2$ theory by the addition of a superpotential $W(\Phi)$ with N D5 branes wrapping the S^2 in the Calabi-Yau (3.6), with background flux H_0 . In the next section, we will study the closed-string dual of this theory.

² The fact that it preserves at least $\mathcal{N} = 1$ supersymmetry is clear for a holomorphic B -field, since the variation of the superpotential $W = \int H \wedge \Omega$ with respect to variations of Ω vanishes if H is holomorphic.

4. The Closed String Dual

The open-string theory on the D5 branes has a dual description in terms of pure geometry with fluxes. The gauge theory on the D5 branes which wrap the \mathbf{P}^1 's develops a mass gap as it confines in the IR. The confinement of the open-string degrees of freedom can be thought of as leading to the disappearance of the D5 branes themselves. This has a beautiful geometric realization [1] which we review presently.

In flowing to the IR, the D5 branes deform the geometry around them so that the \mathbf{P}^1 's they wrap get filled in, and the S^3 's surrounding the branes get finite sizes. This is a conifold transition for each minimal S^2 , after which the geometry is deformed from (3.6) to

$$uv = y^2 - W'(x)^2 + f_{n-1}(x), \quad (4.1)$$

where $f_{n-1}(x)$ is a polynomial in x of degree $n - 1$. This has n coefficients which govern the sizes of the n resulting S^3 's.

In addition, there is H -flux generated in the dual geometry,

$$H = H_{\text{RR}} + \frac{i}{g_s} H_{\text{NS}}.$$

Before the transition, the S^3 's were contractible and had RR fluxes through them due to the enclosed brane charge. After the transition, they are no longer contractible, but the fluxes must remain. In other words we expect the disappearance of the branes to induce (log-)normalizable RR flux, localized near the branes' previous locations, which we denote by H_{RR} . If we denote the S^3 that replaces the k 'th S^2 by A_k -cycles, then

$$\oint_{A_k} H_{\text{RR}} = N_k. \quad (4.2)$$

It is also natural to expect that there will be no H_{RR} flux through the B_k -cycles, as there were no branes to generate it. In other words

$$\int_{B_k} H_{\text{RR}} = 0. \quad (4.3)$$

In addition to the induced flux H_{RR} , we have a background flux H_0 due to the variation of the background B_0 field, which was present even when there

were no branes, and which we denote by $H_0 = dB_0$. Thus we expect the total flux after the transition to be given by

$$H = H_{\text{RR}} + dB_0.$$

Note that before the transition, there are no compact 3-cycles, and so there is no compact flux associated with dB_0 . It is then natural to postulate that after the transition, dB_0 will have no net flux through any of the compact 3-cycles. Moreover, far from the branes, we expect B_0 to be given by its value before the transition. For the noncompact 3-cycles in the dual geometry, denoted by B_k , we can then explicitly evaluate the periods of H_0 ,

$$\int_{B_k} H_0 = \int_{B_k} dB_0 = \oint_{S^2_{\Lambda_0}} B_0 = B(\Lambda_0). \quad (4.4)$$

Because these cycles are noncompact, the integral is regulated by the introduction of a long distance cutoff Λ_0 in the geometry. As usual, we identify this scale with the UV cutoff in the gauge theory.

To summarize, the total flux $H = H_{\text{RR}} + dB_0$ after the transition should be determined by the following facts: H_{RR} is (log-)normalizable, with only nonzero A_k periods (given by N_k), and far from the branes, B_0 is given by its background value (3.4), i.e.,

$$dB_0 \sim d\alpha(x) = \sum_{k=1}^{n-1} kt_k x^{k-1}.$$

The fact that the deformed background flux is given by an exact form dB_0 emphasizes the fact that it is cohomologically trivial and has no nonzero periods around compact 3-cycles.

The striking aspect of the duality is that in the dual geometry, the gaugino superpotential \mathcal{W}_{eff} becomes purely classical. We will turn to its computation in the next subsection.

4.1. The effective superpotential

The effective superpotential is classical in the dual geometry and is generated by fluxes,

$$\mathcal{W}_{\text{eff}} = \int_{CY} (H_{\text{RR}} + H_0) \wedge \Omega,$$

where Ω is a holomorphic three-form on the Calabi-Yau,

$$\Omega = \frac{dx \wedge dy \wedge dz}{z}.$$

This has a simpler description as an integral over the Riemann surface Σ which is obtained from (4.1) by setting the $u, v = 0$:

$$0 = y^2 - W'(x)^2 + f_{n-1}(x). \quad (4.5)$$

The Riemann surface Σ is a double cover of the complex x -plane, branched over n cuts. The 3-cycles A_k and B_k of Calabi-Yau three-fold descend to one-cycles on the Riemann surface Σ , with A_k cycles running around the cuts and B_k cycles running from the branch points to the cutoff (see figure 2). In addition, H_{RR} descends to a one-form on Σ with periods (4.2), (4.3). Moreover, Ω descends to a one form on Σ , given by

$$y dx,$$

where y solves (4.5). The effective superpotential then reduces to an integral over the Riemann surface,

$$\mathcal{W}_{\text{eff}} = \int_{CY} (H_{\text{RR}} + H_0) \wedge \Omega = \int_{\Sigma} (H_{\text{RR}} + dB_0) \wedge y dx. \quad (4.6)$$

The one-form H_{RR} is defined by its periods

$$\oint_{A_i} H_{\text{RR}} = N_i, \quad \int_{B_i} H_{\text{RR}} = 0,$$

and the asymptotic behavior of B_0 is determined by

$$dB_0(x) \sim \pm d\alpha(x),$$

where \pm correspond to the values of the one-form on the top and bottom sheets of Σ .

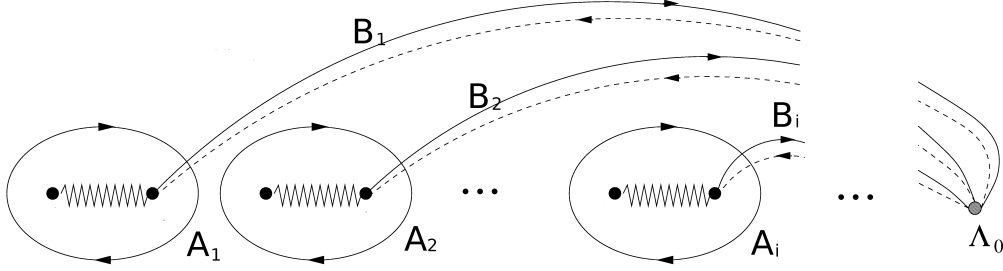


Fig. 2. The Calabi-Yau three-fold (4.1) projects to the x -plane by setting $u = v = 0$. This can be described as a multi-cut Riemann surface Σ , where the nontrivial three-cycles of the Calabi-Yau reduce to one-cycles as drawn.

The evaluation of the superpotential is now straightforward. Using the Riemann bilinear identities, we can evaluate the first term,

$$\int_{\Sigma} H_{\text{RR}} \wedge y dx = \sum_{k=1}^n \oint_{A_k} H_{\text{RR}} \int_{B_k} y dx - \oint_{A_k} y dx \int_{B_k} H_{\text{RR}} = \sum_{k=1}^n N^k \frac{\partial \mathcal{F}_0}{\partial S_k}$$

where

$$\oint_{A_k} y dx = S_k, \quad \int_{B_k} y dx = \frac{\partial \mathcal{F}_0}{\partial S_k},$$

and \mathcal{F}_0 is the genus 0 prepotential of the Calabi-Yau. The background contribution to the superpotential is also straightforward to evaluate, since there are no internal periods for the flux,

$$\int_{\Sigma} dB_0 \wedge y dx = \oint_P B_0(x) y dx \sim \pm \sum_{k=1}^n \oint_{A_k} \alpha(x) y dx,$$

where the last equality follows from the fact that $B_0(x) = \alpha(x)$ for large x by Cauchy's theorem (since the cycle around P is homologous to the sum of all the A_k -cycles).

Thus, the full effective superpotential is

$$\mathcal{W}_{\text{eff}} = \sum_{k=1}^n N_k \frac{\partial}{\partial S_k} \mathcal{F}_0 + \oint_{A_k} \alpha(x) y dx. \quad (4.7)$$

This expression is in line with our intuition from the open-string description. Namely, to the leading order we have

$$\oint_{A_k} \alpha(x) y dx \sim \alpha(e_k) S_k + \dots$$

where the omitted terms are higher order in S_i . To this approximation, the superpotential is given by

$$\mathcal{W}_{\text{eff}} \sim \sum_k \alpha(e_k) S_k + N_k \frac{\partial \mathcal{F}_0}{\partial S_k} + \dots$$

Note that the first term above comes from the classical superpotential of the gauge theory, since the A_i -cycle periods S_i in the geometry are identified with glueball superfields in the gauge theory. The coefficient of S_i in the effective superpotential is the microscopic gauge coupling of the $U(N_i)$ gauge group factor in the low energy effective field theory. This is precisely equal to the B -field on the S^2 wrapped by the branes (3.7).

However, this cannot be the whole story. After the deformation, the location of the \mathbf{P}^1 is no longer well defined, as the \mathbf{P}^1 at the point $x = e_k$ has disappeared and been replaced by an S^3 which is a branch cut on the x -plane. The geometry has been deformed around the branes and the two sheets of the Riemann surface connect through a smooth throat. We need to specify *where* the gauge coupling is to be evaluated, and since the point in the x -plane has been replaced by a throat, the most natural guess is that we smear the B -field over the cuts. This is precisely what (4.7) does! In the appendix, we provide more details for the derivation of (4.7) based on the use of the Riemann bilinear identities.

In the next section, we will show that the same effective superpotential follows from a direct gauge theory computation. Moreover, we will relate the gauge theory computation to an effective matrix model. We will also give a more explicit expression for \mathcal{W}_{eff} ,

$$\mathcal{W}_{\text{eff}} = \sum_{k=0}^{n-1} t_k \frac{\partial}{\partial a_k} \mathcal{F}_0 + N_k \frac{\partial}{\partial S_k} \mathcal{F}_0, \quad (4.8)$$

which arises from the following nontrivial identity that we prove in section 5 using the formulation of the topological string in terms of matrix models [14]:

$$\oint_P \alpha(x) y dx = \sum_{k=0}^{n-1} t_k \frac{\partial}{\partial a_k} \mathcal{F}_0.$$

Equations (4.7) and (4.8) agree with the results of [6,8].

The form of the superpotential (4.8) suggests a dual role played by (a_k, S_k) and (t_k, N_k) – indeed it suggests a formulation in terms of fluxes [15] (see also [10]). We can think of the fluxes N_k as turning on auxiliary fields for the S_k superfields in the $\mathcal{N} = 2$ effective theory, where S_k is the lowest component of the superfield,

$$S_k \rightarrow S_k + \cdots + 2iN_k\theta_2\theta_2 + \cdots$$

The $\mathcal{N} = 1$ superpotential arises by the integration over half of the chiral $\mathcal{N} = 2$ superspace

$$\int d^4\theta \mathcal{F}_0(S_k) = \int d^2\theta N_k \frac{\partial \mathcal{F}_0}{\partial S_k} + \cdots$$

Similarly, we can view the background parameters a_k as scalar components of non-normalizable superfields, and the t_k as the corresponding fluxes leading to vevs for their associated auxiliary fields,

$$a_k \rightarrow a_k + \cdots + 2it_k\theta_2\theta_2 + \cdots$$

Thus the full superpotential can be obtained from the $\mathcal{N} = 2$ formulation simply by giving vevs (t_k, N_k) to the auxiliary fields of (a_k, S_k) .

4.2. Extrema of the superpotential

With the closed-string dual of our gauge theory identified, we turn to the extremization of the flux superpotential. We wish to solve

$$\frac{\partial \mathcal{W}_{\text{eff}}}{\partial S_k} = \int_{\Sigma} (H_{\text{RR}} + H_0) \wedge \frac{\partial}{\partial S_k} y dx = 0. \quad (4.9)$$

From (4.8) this can be written as

$$\sum_{i=0}^{n-1} t_i \eta_{ik} = \sum_{i=1}^n N_i \tau_{ik} \quad (4.10)$$

where η is an $n \times n$ matrix,

$$\eta_{ik} = \frac{\partial^2 \mathcal{F}_0}{\partial a_i \partial S_k} \quad (4.11)$$

and τ_{ik} is the usual period matrix,

$$\tau_{ik} = \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_k}. \quad (4.12)$$

Note that for a fixed choice of Higgs branch, specified by N_i , the number of parameters specifying the choice of $B_0(x)$ and the number of parameters determining the normalizable deformations of the geometry, given by $f_{n-1}(x)$, are both equal to n . Therefore we would expect to generically have a one-to-one map. This allows us to invert the problem. Instead of asking how B_0 determines f_{n-1} , i.e.,

$$B_0 \rightarrow f_{n-1},$$

we can instead ask for which choice of $B_0(x)$ we obtain a given deformed geometry, $f_{n-1}(x)$, i.e.,

$$B_0 \leftarrow f_{n-1}.$$

In this formulation, the extremization problem has a simple solution. We choose a set of complex structure moduli for the Riemann surface,

$$y^2 = (W'(x; a))^2 - f_{n-1}(x; a, S),$$

by picking values for the S_i (or equivalently for the coefficients of f_{n-1}). This completely determines the matrices τ_{ij} and η_{ij} through (4.11) and (4.12). The equations (4.9), (4.10) can then be thought of as n linear equations for the n coupling constants $\{t_i\}_{i=0}^{n-1}$, thus determining $B_0(x)$.

The equations (4.9) determine the explicit form of the flux $H_{\text{RR}} + H_0$ on the solution. Recall that, off-shell, $H_{\text{RR}} + H_0$ was defined by its compact periods,

$$\oint_{A_i} H_{\text{RR}} + H_0 = N_i \quad \int_{B_i} H_{\text{RR}} + H_0 = \alpha(\Lambda_0), \quad (4.13)$$

and asymptotic behavior for large x ,

$$H_{\text{RR}} + H_0(x) \sim \pm dB(x).$$

The equations of motions (4.9) then imply that the one-form $H_{\text{RR}} + H_0$ is holomorphic on the punctured Riemann surface $\Sigma - \{P, Q\}$, and given by

$$H_{\text{RR}} + H_0 = \sum_{k=1}^n N_k \frac{\partial}{\partial S_k} y dx - \sum_{k=0}^{n-1} t_k \frac{\partial}{\partial a_k} y dx. \quad (4.14)$$

Above, P and Q correspond to points at infinity of the top and the bottom sheet of the Riemann surface, and

$$\frac{\partial}{\partial S_k} y dx$$

are linear combinations of the $n - 1$ holomorphic differentials on Σ ,

$$\frac{x^k dx}{y}, \quad k = 0, \dots, n - 2,$$

together with $x^{n-1} dx/y$, which has a pole at infinity.

To derive this, we note that (4.9) implies that $H_{\text{RR}} + H_0$ is orthogonal to the complete set of holomorphic differentials in the interior. This implies that $H_{\text{RR}} + H_0$ is holomorphic away from the punctures. We can also show that (4.14) has the correct periods and asymptotic behavior. Consider the periods of $\omega_i = \frac{\partial}{\partial S_i} y dx$,

$$\oint_{A_k} \omega_i = \delta_i^k, \quad \int_{B_k} \omega_i = \tau_{ik} \quad (4.15)$$

and the periods of $\rho_i = \frac{\partial}{\partial a_i} y dx$,

$$\oint_{A_k} \rho_i = 0, \quad \int_{B_k} \rho_i = \eta_{ik} + \Lambda_0^i. \quad (4.16)$$

The reason for the Λ_0^i term in (4.16) is that $\frac{\partial \mathcal{F}_0}{\partial S_i}$ is the B_i -period with boundary term subtracted. The A_k periods also match – this is because the $\frac{\partial}{\partial a_k}$ derivative is taken at fixed S_k , per definition. Using these periods and (4.10), we immediately see that (4.14) has the correct periods (4.13). It is also clear that the large x behavior is dominated by ρ_i and this yields $d\alpha(x)$ for the large x behavior of $H_{\text{RR}} + H_0$ as required.

5. Gauge Theory Derivation

In this section we will sketch the derivation of the effective glueball superpotential directly in the gauge theory language, and show that this exactly reproduces the results of the string theoretic derivation. In [3] the effective superpotential for the glueball superfields was computed by explicitly integrating out the chiral superfield Φ . This is possible as long as we are only interested in

the chiral $\int d^2\theta$ terms in the effective action. In the absence of the deformation (2.6), computation of the relevant gauge theory Feynman graphs with Φ running around loops directly translates into the computation of planar diagrams in a certain auxiliary matrix integral. We will see that this is the case even after the deformation, albeit with a novel deformation of the relevant matrix integral.

Let us review the results of [3]. For simplicity, consider the vacuum where the $U(N)$ gauge symmetry is unbroken. The propagators for Φ can be written in the Schwinger parameterization as

$$\int ds_i \exp[-s_i(p_i^2 + \mathcal{W}^\alpha \pi_\alpha + m)],$$

where s_i are the Schwinger times, p_i are the bosonic momenta, and π_α the fermionic momenta. The mass parameter m is given by $m = W''(\phi_0)$. These propagators have the property that each Φ loop brings down two insertions of the glueball superfield \mathcal{W}_α . Using the chiral ring relation

$$\{\mathcal{W}_\alpha, \mathcal{W}_\beta\} \sim 0, \tag{5.1}$$

only those operators of the form

$$S^k = (\text{Tr} \mathcal{W}_\alpha \mathcal{W}^\alpha)^k$$

are nontrivial as F-terms. In particular, there must be *at most* two insertions of \mathcal{W}_α per index loop. This implies that only planar Φ -diagrams contribute to the superpotential – nonplanar graphs have fewer index loops than momentum loops.

The integration over bosonic and fermionic loop momenta in a planar diagram with h holes gives a constant factor,

$$NhS^{h-1}, \tag{5.2}$$

independent of the details of the diagram. The planar graphs have one more index loop (hole) than momentum loop, and there is one insertion of S per momentum loop, with h choices of which index loop goes unoccupied. At the same time, the index summation for the unoccupied loop leads to the factor of N .

The rest of the computation, namely combinatorial factors, contributions of vertices, and an additional factor of $1/m^{h-1}$ from the propagators, is captured by a zero-dimensional, auxiliary holomorphic matrix theory with path integral

$$Z_M = \frac{1}{\text{Vol}(U(M))} \int d\Phi \exp(-\text{Tr}W(\Phi)/g_{\text{top}}), \quad (5.3)$$

where Φ is an $M \times M$ matrix, and $W(\Phi)$ is the same superpotential as in (2.5). The coefficient

$$\mathcal{F}_{0,h}$$

of (5.2) is computed by summing over the planar graphs of Z_M with h holes and extracting the coefficient of $M^h g_{\text{top}}^{h-2}$. In other words, by rewriting the sum

$$\mathcal{F}_0(S) = \sum_h \mathcal{F}_{0,h} S^h$$

where

$$Z_M \sim \exp(-\mathcal{F}_0/g_{\text{top}}^2).$$

In the semiclassical approximation, the effective superpotential of the undeformed theory is simply

$$\mathcal{W}_{\text{eff}} = t_0 S + N \partial_S \mathcal{F}_0(S).$$

In the full answer, \mathcal{F}_0 contains a $\frac{1}{2} S^2 \log S$ piece which, in the matrix model, comes from the volume of the gauge group in (5.3).

5.1. The deformed matrix model

Now consider the gauge theory with the more general tree-level superpotential (2.6) (for a special form of the superpotential, this theory was studied in [8]). In this case, the propagators of the theory are unchanged, but there are now additional vertices coming from the first term in (2.6). What is the effect of this? Clearly, it is still only the planar graphs that can contribute to the amplitude, since nonplanar graphs still have too few index loops to absorb the \mathcal{W}_α insertions. This, together with (5.1), implies that the extra vertices from $\text{Tr}[\alpha(\Phi)\mathcal{W}_\alpha\mathcal{W}^\alpha]$ can only be brought down once for each planar graph, where

they are inserted on the sole index loop that would have otherwise been unoccupied. The prescription for extracting the contributions of these new graphs from the matrix model is now clear. Consider the deformed matrix model

$$Z_M = \frac{1}{\text{Vol}(U(M))} \int d\Phi \exp(-\text{Tr}W(\Phi)/g_{\text{top}} + \text{Tr} \Lambda \alpha(\Phi)/g_{\text{top}}), \quad (5.4)$$

where the matrix Λ stands for $\mathcal{W}_\alpha \mathcal{W}^\alpha$ insertions that do not come from the propagators. Summing over planar graphs, the matrix integral now has the form

$$Z_M \sim \exp(-\mathcal{F}_0/g_{\text{top}}^2 - \text{Tr} \Lambda \mathcal{G}_0/g_{\text{top}} + \dots)$$

where the omitted terms contain higher powers of traces of Λ that will not play any role. The effective superpotential, including the contribution of the new vertices from $\text{Tr}[\alpha(\Phi)\mathcal{W}_\alpha\mathcal{W}^\alpha]$, is now

$$\mathcal{W}_{\text{eff}} = S\mathcal{G}_0(S) + N\partial_S\mathcal{F}_0(S).$$

Note that it is manifest in the matrix model that the effective superpotential is invariant under the addition to $\alpha(\Phi)$ of terms the form $\Phi^k W'(\Phi)$, as mentioned in section 2. These terms can be removed by a shift in Φ

$$\Phi \rightarrow \Phi + \Lambda \Phi^k,$$

and as such they do not affect the matrix integral.

It is easy to generalize this to vacua of the gauge theory where the gauge group is broken as in (2.9). The superpotential in these vacua is computed by the same matrix model, but where one now considers the perturbative expansion about the more general vacuum, where the gauge symmetry of the matrix model is broken to $\prod_{k=1}^n U(M_k)$ [16]. The contributions of insertions of

$$\text{Tr}[\alpha(\Phi_k)\mathcal{W}_{\alpha,k}\mathcal{W}_k^\alpha]$$

are now captured by deforming the matrix model to

$$Z_M = \frac{1}{\prod_k \text{Vol}(U(M_k))} \int \prod_k d\Phi_k \dots \exp\left(-\frac{1}{g_{\text{top}}} \sum_k (\text{Tr}W(\Phi_k) + \text{Tr} \Lambda_k \alpha(\Phi_k))\right)$$

where the omitted terms ... are gauge fixing terms [16] corresponding to the choice for Φ to be block diagonal, and breaking the gauge symmetry to $\prod_k U(M_k)$. Summing over the planar graphs returns

$$Z_M \sim \exp \left(-\mathcal{F}_0/g_{\text{top}}^2 - \sum_k \text{Tr} \Lambda_k \mathcal{G}_{0,k}/g_{\text{top}} - \dots \right)$$

where \mathcal{F}_0 and $\mathcal{G}_{0,k}$ are functions of the matrix model 't Hooft couplings $g_{\text{top}} M_k$. These are identified with the glueballs S_i in the physical theory. The effective superpotential is now given by

$$\mathcal{W}_{\text{eff}} = \sum_k S_k \mathcal{G}_{0,k} + N_k \partial_{S_k} \mathcal{F}_0,$$

and all that remains is to compute the new terms in $\mathcal{G}_{0,k}$.

5.2. Matrix model computation

Now let us compute the relevant correction from the matrix model. Since we are only interested in the planar graphs linear in $\text{Tr} \Lambda_k$, the contribution of interest can be extracted from the special case where we choose

$$\Lambda_k = \lambda_k \mathbf{1}_{M_k \times M_k}$$

The matrix model partition function then becomes

$$Z_M = \int \dots \exp \left(- \sum_k \lambda_k \text{Tr} \alpha(\Phi_k)/g_{\text{top}} \right) \sim \exp \left(-\mathcal{F}_0/g_{\text{top}}^2 - \sum_k M_k \lambda_k \mathcal{G}_{0,k}/g_{\text{top}} \right)$$

which implies

$$\mathcal{G}_{0,k} = \langle \text{Tr}[\alpha(\Phi_k)] \rangle / M_k,$$

where the expectation value is evaluated in the planar limit of the $\prod_k U(M_k)$ vacuum of the undeformed matrix model. These can be computed using well known large M matrix model saddle point techniques [2]. The answer can be formulated in terms of a Riemann surface,

$$y^2 - (W'(x))^2 + f_{n-1}(x) = 0,$$

with a one-form ydx , where the coefficients of f_{n-1} are chosen so that

$$M_k g_{\text{top}} = \oint_{A_k} ydx.$$

Namely, the result is that

$$\langle \text{Tr } \alpha(\Phi_k) \rangle = \frac{1}{g_{\text{top}}} \oint_{A_k} \alpha(x) y dx.$$

Since the glueballs S_k are identified with $M_k g_{\text{top}}$ in the matrix model, we can write the corresponding contribution to the effective superpotential

$$\delta \mathcal{W}_{\text{eff}} = \sum_k S_k \mathcal{G}_{0,k}$$

simply as

$$\delta \mathcal{W}_{\text{eff}} = \sum_k \oint_{A_k} \alpha(x) y dx.$$

A look back at (4.7) shows that this agrees with the result of our string theoretic analysis. Moreover, this is consistent with the results of [4] for the expectation values of the corresponding chiral ring elements.

In the next subsection, we will use matrix model technology to derive the identity (4.8) for expressing $\delta \mathcal{W}_{\text{eff}}$, as a function of S_k .

5.3. Evaluation of $\delta \mathcal{W}_{\text{eff}}$

To begin with, note that $\delta \mathcal{W}_{\text{eff}}$ can be rewritten as

$$\delta \mathcal{W}_{\text{eff}} = \sum_k \frac{S_k \langle \text{Tr } \alpha(\Phi_k) \rangle}{M_k} = g_{\text{top}} \sum_k \langle \text{Tr } \alpha(\Phi_k) \rangle = g_{\text{top}} \langle \text{Tr } \alpha(\Phi) \rangle$$

where the trace is over the $M \times M$ matrix Φ .³ The expectation value is now straightforward to compute. The problem amounts to the computation of

$$\langle \text{Tr } \Phi^k \rangle, \quad k = 0, \dots, n-1$$

³ This leads to the same expression (4.7) for the large M average using $y(x) = W'(x) + g_{\text{top}} \langle \frac{1}{x-\Phi} \rangle$, and the fact that the sum over the A_k -cycles is homologous to the cycle around infinity in x -plane.

in the matrix model. Recall that

$$W(\Phi) = \sum_{k=0}^{n+1} a_k \Phi^k,$$

which implies that, for $k = 0, \dots, n-1$

$$\langle \text{Tr} \Phi^k \rangle = -\frac{g_{\text{top}}}{Z_M} \frac{\partial Z_M}{\partial a_k}$$

with Z_M as defined in (5.3). In particular, since

$$Z_M \sim \exp\left(-\frac{1}{g_{\text{top}}^2} \mathcal{F}_0(S, a)\right),$$

it follows that

$$\langle \text{Tr} \Phi^k \rangle = \frac{1}{g_{\text{top}}} \frac{\partial \mathcal{F}_0}{\partial a_k}.$$

Thus we have derived (4.8),

$$\delta \mathcal{W}_{\text{eff}} = \sum_{k=0}^{n-1} t_k \frac{\partial \mathcal{F}_0}{\partial a_k}.$$

6. The $\mathcal{N} = 2$ Gauge Theory

6.1. Extended Seiberg-Witten theory

With the results of the previous section in hand, we are now in position to recover the solution to the extended $\mathcal{N} = 2$ theory with classical prepotential

$$\mathcal{F}(\Phi) = \sum_{k=0} \frac{t_k}{(k+1)(k+2)} \text{Tr} \Phi^{k+2}. \quad (6.1)$$

The analysis of this section closely mirrors the approach taken in [5], and the results also follow from [6].

To begin with, consider a special case of the $\mathcal{N} = 1$ theories studied in the previous section. We deform the extended $U(N)$ $\mathcal{N} = 2$ theory (6.1) to $\mathcal{N} = 1$ by the addition of a degree $N+1$ superpotential,

$$W(\Phi) = \sum_{k=0}^{N+1} a_k x^k \quad (6.2)$$

with

$$W'(\Phi) = g \prod_{k=1}^N (x - e_k). \quad (6.3)$$

In particular, we now study a generic vacuum on the Coulomb branch of the theory, where the gauge symmetry is broken as

$$U(N) \rightarrow U(1)^N.$$

This is important, because if we now take the limit of vanishing superpotential (6.2) while keeping the expectation value of the adjoint fixed,

$$g \rightarrow 0, \quad e_k = \text{const},$$

we expect to recover the $\mathcal{N} = 2$ vacuum at the same point in moduli space. As discussed in section 3, this corresponds in string theory language to reverting to studying N D5 branes on the \mathbf{P}^1 in the A_1 ALE space, but with a holomorphically varying B -field turned on. The nontrivial B -field background corresponds in the low energy theory on the branes to turning on the higher dimensional terms in the classical prepotential (6.1).

We found in section 4 that the critical point of this theory corresponds to a Riemann surface

$$y^2 = (W'(x; a))^2 - f_{N-1}(x; S, a) \quad (6.4)$$

where the N parameters t_k in (6.1) are determined in terms of the complex structure moduli S_i of (6.4) by extremizing the superpotential (4.9). Moreover, at the critical point, the net flux $H_{\text{RR}} + H_0$ is given by a holomorphic one-form on the Riemann surface (6.4),

$$H_{\text{RR}} + H_0 = \sum_{k=1}^N \frac{\partial}{\partial S_k} y dx - \sum_{k=0}^{N-2} t_k \frac{\partial}{\partial a_k} y dx, \quad (6.5)$$

with periods

$$\begin{aligned} \oint_{A_i} H_{\text{RR}} + H_0 &= 1 \\ \int_{B_i} H_{\text{RR}} + H_0 &= \alpha(\Lambda_0) \\ \oint_P x^{-k} (H_{\text{RR}} + H_0) &= k t_k, \quad k = 1, \dots, N-2. \end{aligned}$$

It turns out that all of the holomorphic information about the $\mathcal{N} = 2$ theory in the infrared can be recovered from calculations in the $\mathcal{N} = 1$ theory, just as in [5]. To observe this, we note that if we extract an overall factor of g from y in (6.4) and use new g -independent functions $\widetilde{W} \equiv \frac{1}{g}W$ and $\widetilde{f}_{N-1} \equiv \frac{1}{g^2}f_{N-1}$, then

$$y = g\sqrt{\widetilde{W}(x)^2 + \widetilde{f}_{N-1}(x)},$$

and the periods of y have a trivial g -dependence. In particular,

$$\frac{1}{g}S_i, \quad \frac{1}{g}\frac{\partial\mathcal{F}_0}{\partial S_i},$$

are independent of g . Consequently, the period matrix

$$\tau_{ij} = \frac{\partial^2\mathcal{F}_0}{\partial S_i\partial S_j} = \frac{\partial}{\partial(S_i/g)}\left(\frac{1}{g}\frac{\partial\mathcal{F}_0}{\partial S_j}\right)$$

is independent of g . This fact can be made more manifest by considering the geometry in question,

$$\frac{y^2}{g^2} = \widetilde{W}(x)^2 + \widetilde{f}_{N-1}(x).$$

It is clear that the variation of g can just be absorbed into a rescaling of the coordinate y .

It is also crucial that in the process of sending $g \rightarrow 0$, the values of t_k for which the Riemann surface in question satisfies the equations of motion remain fixed. The superpotential

$$\mathcal{W}_{\text{eff}} = \int_{\Sigma} (H_{\text{RR}} + H_0) \wedge y dx$$

is simply proportional to g , and hence its critical points are g -independent.

Lastly, we note that the Seiberg-Witten one-form on the Riemann surface can be recovered from the $\mathcal{N} = 1$ analysis as well. First note that the H -flux $H_{\text{RR}} + H_0$ at the critical point of the superpotential is given by a g -independent holomorphic one-form (6.5). Just as in [5], it follows that the Seiberg-Witten one-form on the Riemann surface is given by

$$\lambda_{\text{SW}} = x(H_{\text{RR}} + H_0) \tag{6.6}$$

which we can read off from the $\mathcal{N} = 1$ theory. This can be seen as follows. Periods of λ_{SW} compute the masses of dyons in the $\mathcal{N} = 2$ theory. However,

these dyons can be identified with D3 branes wrapping Lagrangian 3-cycles in the Calabi-Yau, or one-cycles on the Riemann surface, and their mass can be derived from string theory to be given by periods of the one-form (6.6).

In summary, we can obtain the full $\mathcal{N} = 2$ curve and the Seiberg-Witten one-form λ_{SW} that capture the low energy dynamics of the extended $\mathcal{N} = 2$ theory (6.1). These results are consistent with those obtained recently in [17] using very different techniques. There, the authors formulate the solution of the $\mathcal{N} = 2$ theory in terms of a hyperelliptic curve of genus $N - 1$

$$y^2 = \prod_{i=1}^N (x - a_{i,+})(x - a_{i,-}), \quad (6.7)$$

and a holomorphic one-form $d\Phi$ with the properties that

$$\begin{aligned} \oint_{A_i} d\Phi &= 1 \\ \int_{B_i} d\Phi &= 0 \\ \oint_P x^{-k} d\Phi &= kt_k, \quad k = 1, \dots, N - 2 \end{aligned}$$

and which is related to the Seiberg-Witten one-form by

$$\lambda_{\text{SW}} = x d\Phi.$$

Comparing with our results, it is clear that $d\Phi$ should be identified with $H_{\text{RR}} + H_0$.

The agreement is almost complete, apart from two points. First, our Seiberg-Witten curve (6.4) is not a generic genus N hyperelliptic curve like (6.7), but rather is one where all the dependence on the parameters t_k is in the polynomial $f_{N-1}(x)$ of degree $N - 1$. More precisely, note that the defining equation of the hyperelliptic curve has $2N$ parameters and generally all such parameters appear. However, half the parameters correspond to the choice of the point on the Coulomb branch e_i , while the other half define the quantum deformation which depends on the choice of the $\alpha(x)$. In our formulation, there is a natural way to separate how these parameters appear in the defining equation of the Seiberg-Witten curve. Secondly, there is an apparent discrepancy in that in the current solution, the B_i periods of $H_{\text{RR}} + H_0$ do not all vanish, but are instead equal to $\alpha(\Lambda_0)$. It is possible that in the definition of the B_k integrals (6.1) of [17], there is a hidden subtraction of the value of the integral at infinity, which would account for the vanishing B_k periods and resolve this discrepancy.

7. Duality and Supersymmetry Breaking

In this section we study the phase structure of the $\mathcal{N} = 1$ models under consideration. We find that there is a region in the parameter space where supersymmetry is broken. This leads to a novel and calculable mechanism for breaking supersymmetry. Even though this method for supersymmetry breaking is motivated by string theoretic considerations, we will see that it can also be phrased entirely in terms of the underlying $\mathcal{N} = 1$ supersymmetric gauge theory.

The organization of this section is as follows. We first discuss some general features of the phase structure for these theories, and point out a region where classical considerations are not sufficient to provide a reasonable picture. We next turn to focus on the meaning of this new phase and show how string dualities can shed light on its meaning. Furthermore, we show that, generically, supersymmetry is spontaneously broken in the new phase. We propose UV dual field theory descriptions for some of these phases which turn out to be $\mathcal{N} = 1$ supersymmetric gauge theories with supersymmetry broken softly by nonzero expectation values for the auxiliary components of spurion superfields.

7.1. Parameter space with $g_{\text{YM}}^2 < 0$

Consider the $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory studied in the previous sections, with adjoint field Φ together with superpotential $W(\Phi)$, and gauge kinetic term in Lagrangian is captured by $\alpha(\Phi)$ as below

$$\int d^4x d^2\theta \text{Tr} [\alpha(\Phi) \mathcal{W}_\alpha \mathcal{W}^\alpha].$$

As already discussed, the classical vacua correspond to all the ways of distributing the eigenvalues of ϕ among the critical points of $W'(\phi) = 0$. For concreteness, let

$$W'(\Phi) = g \prod_{i=1}^n (\Phi - e_i),$$

and consider the classical vacuum with N_i eigenvalues of Φ equal to e_i . For generic superpotential, Φ will be massive, and at sufficiently low energies the light degrees of freedom describe pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory

with gauge symmetry $\prod_i U(N_i)$. The coupling constant of each of the $U(N_i)$ in the UV is given by

$$\alpha_i = \alpha(e_i).$$

As long as the gauge coupling for each factor of the gauge group is positive, i.e.,

$$\text{Im}[\alpha_i] = \frac{4\pi}{(g_{\text{YM}})_i^2} > 0 \tag{7.1}$$

for all i with $N_i \neq 0$, we expect a supersymmetric theory in the IR to which the analysis of the previous section applies. This suggests the question: *What is the meaning of the phase where (7.1) is not satisfied for some i ?* It is to this question which we now turn our attention.

One may be inclined to consider such cases as pathological, as one is not able to give a meaning to such a theory in the UV. However, we also know from various examples that the appearance of a negative g_{YM}^2 is often the smoking gun for the existence of a dual description. Thus all we can conclude is that when $\text{Im}[\alpha(e_i)]$ do not have the correct sign, the original UV picture is not appropriate, and we should look for an alternative description.

Generically⁴, for an arbitrary choice of $W(\Phi)$ and $\alpha(\Phi)$, $\text{Im} \alpha(e_i)$ will not have the same sign for all the critical points, and thus some vacua will have gauge group factors with $g_{\text{YM}}^2 < 0$. We have a practical way to analyze the IR theory in these vacua directly from the field theory approach. We can start with parameters such that the UV theory makes sense, and then compute the effective IR action in terms of the glueball superfields, as discussed in the previous sections. We then change the parameters so that the UV theory would formally develop a negative value of g_{YM}^2 for some of the gauge group factors. However, the effective IR theory *still* makes sense when we do this, so we can simply study the IR action, without worrying about the dual UV description. As we will show, in the IR theory this change of parameters leads to supersymmetry-breaking.

We are thus naturally led to ask: *What is the corresponding UV theory in such cases?* When only some of the gauge couplings are negative, we will

⁴ Generic in the sense of generic functions $\alpha(x)$ and $W(x)$. From a field theory perspective, it is natural for the nonrenormalizable operators in $\alpha(\Phi)$ to be suppressed by large mass scales, in which case the phenomenon discussed in this section will be unusual.

argue that supersymmetry is broken, but we will not have a full field theory description in the UV. However, if they are *all* negative, we can formulate a complete UV field theory description for which supersymmetry is manifestly broken. In all cases, the UV description provided by string theory exists, and we will argue that it involves both branes and antibranes.

In the general, these theories have two sources of supersymmetry breaking. One, which comes from any of the gauge factors with negative $\text{Im } \alpha(e_i)$, corresponds to giving a nonzero vev to spurion auxiliary fields. The other effect comes from the fact that when both signs of $\text{Im } \alpha(e_i)$ are present, the interaction between the gauge group factors are not supersymmetric, as each factor tries to preserve a different supersymmetry.

We first study the situation of the first kind – all $\text{Im } \alpha(e_i)$ negative – where the internal dynamics of the gauge theory softly break supersymmetry. For this case, we quantify the supersymmetry-breaking effect in terms of a dimensionless parameter which measures fractional mass splittings in the supermultiplets. Moreover, we motivate and provide strong evidence for the existence of a dual nonsupersymmetric UV theory. We motivate this from field theory as well as describing its natural explanation in the context of string theory.

We then move to the multi-sign case and show that when some $\text{Im } \alpha(e_i)$ have different signs, there is an additional effect which breaks supersymmetry. Essentially, this arises from each factor of the gauge group trying to preserve a different half of a background $\mathcal{N} = 2$ supersymmetry, and charged bifundamental matter communicates supersymmetry breaking. For this case, we only have a stringy dual description in the UV.

7.2. Negative gauge couplings and duality

We now discuss, from both string theory and field theory perspectives, how a gauge coupling squared becoming negative can be sensibly understood in terms of the dual description. The simple example which we review, where both the original and the dual theories are supersymmetric, has already been studied in [18].

Consider N D5 branes on the resolved conifold geometry with a single \mathbf{P}^1 . As in section 3, we can view this geometry as obtained by fibering an A_1 ALE singularity over the x -plane as

$$uv = y^2 - W'(x)^2 \quad (7.2)$$

where

$$W(x) = \frac{1}{2}mx^2. \quad (7.3)$$

We turn on a *constant* B -field through the S^2 at the tip of the ALE space,

$$\alpha = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2} = \int_{S_x^2} \left(B_{\text{RR}} + \frac{i}{g_s} B_{\text{NS}} \right).$$

In the language of section 2, this means that the gauge coupling is independent of ϕ . More generally, the effective gauge coupling of the $4d$ $U(N)$ theory is given by $4\pi/g_{\text{YM}}^2 = \sqrt{r^2 + B_{\text{NS}}^2}/g_s$, where r is the physical volume of the \mathbf{P}^1 . This is usually written in terms of single complex variable t , the complexified Kähler class, given by $t = B_{\text{NS}} + ir$, as $4\pi/g_{\text{YM}}^2 = |t|/g_s$. In the present paper we have permanently set $r = 0$, so $t = B_{\text{NS}}$.

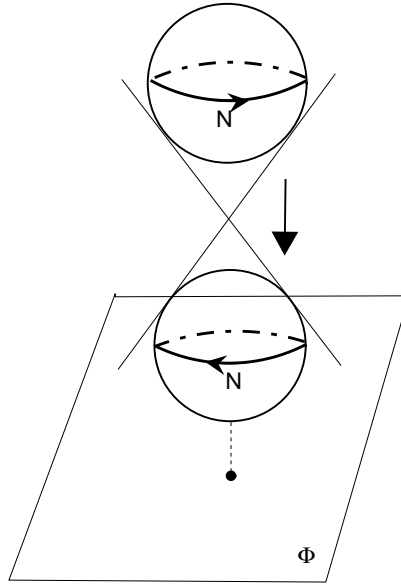


Fig. 3. By changing the B -field, an S^2 undergoes a flop, and N branes on the S^2 become N antibranes on the flopped S^2 . If the B -field is constant on x -plane, then the antibrane system preserves an $\mathcal{N} = 1$ supersymmetry *opposite* to that of the brane system. If the B -field varies holomorphically, then the B -field and antibranes preserve orthogonal $\mathcal{N} = 1$ supersymmetries, leading to a stable $\mathcal{N} = 0$ vacuum.

Now consider the same geometry, but with the complexified Kähler class varied so that it undergoes a flop (see figure 3), corresponding to $t \rightarrow -t$. We now get a *new* \mathbf{P}^1 . Moreover, the charge of the wrapped D5 branes on this flopped \mathbf{P}^1 is opposite to what it was before the flop. Therefore, in order to conserve D5 brane charge across the flop, we will end up with anti-D5 branes on the new \mathbf{P}^1 . In the case of constant B -field, we again obtain a $U(N)$ gauge theory with $\mathcal{N} = 1$ supersymmetry at low energies. However, the $\mathcal{N} = 1$ supersymmetry that the theory preserves after the flop has to be *orthogonal* to the original one, since branes and antibranes preserve different supersymmetries.

This stringy duality is directly manifested in field theory. It turns out, as we now review, that this situation has a simple and elegant realization in terms of the glueball superfields which emerge as the IR degrees of freedom. Consider first the situation before the flop. In the IR, we have a deformed conifold geometry where S , the modulus of the deformation, is identified with the glueball superfield, $S = \text{Tr} \mathcal{W}_\alpha \mathcal{W}^\alpha$. The Veneziano-Yankielowicz superpotential, which can be derived in either the field theory or the dual string theory, is given by

$$\mathcal{W}(S) = -\alpha S + N \partial_S \mathcal{F}_0 = -\alpha S + \frac{1}{2\pi i} N S \left(\log \left(\frac{S}{m \Lambda_0^2} \right) - 1 \right).$$

As was already reviewed in previous sections, in the gravitational dual picture, the two terms above correspond to flux contributions to the superpotential. One should note that this effective description is only valid for field values where $|S/m| \ll |\Lambda_0^2|$.

Extremizing \mathcal{W} with respect to S gives

$$\partial_S \mathcal{W} = 0 \rightarrow S^N = (m \Lambda_0^2)^N \exp(2\pi i \alpha). \tag{7.4}$$

As long as the bare UV gauge coupling satisfies

$$\text{Im}[\alpha] = \frac{4\pi}{g_{\text{YM}}^2} \gg 0,$$

this is an acceptable solution in the sense that S is within the allowed region of field space. Note that in addition to the chiral superfield, the theory in the IR still has a $U(1)$ vector multiplet, because only the $SU(N) \subset U(N)$ is confined. In the string theory construct, the extra $U(1)$ is identified with the reduction of

the 4-form IIB gauge potential on the deformed S^3 . In other words, this theory describes a massive chiral multiplet consisting of S and its fermionic partner ψ , as well as a massless photon A and its partner λ ,

$$(S, \psi), \quad (A, \lambda). \quad (7.5)$$

Together these would form an $\mathcal{N} = 2$ chiral multiplet before the supersymmetry is broken to $\mathcal{N} = 1$ by fluxes.

Now consider the same theory, but in the limit where

$$\text{Im}(\alpha) \ll 0,$$

which *would have corresponded* to $1/g_{\text{YM}}^2 \ll 0$. Then the above solution (7.4) is not valid anymore, since $|S/m| \gg |\Lambda_0^2|$ lies outside the regime of validity of the effective theory. Thus the original supersymmetry is broken, since we cannot set $\partial_S \mathcal{W}$ to zero. Even so, as was shown in [18], there are still physical vacua which minimize an effective scalar potential V_{eff} . Moreover, the theory in these minima is exactly the same as one would expect for the IR limit of an $\mathcal{N} = 1$ supersymmetric $U(N)$ theory, with a positive squared gauge coupling. In fact, a new supersymmetry does re-emerge! It turns out that ψ becomes the massless goldstino of the original supersymmetry which is broken, whereas λ picks up a mass and becomes the superpartner of S under the new supersymmetry, giving realigned supermultiplets

$$(S, \lambda), \quad (A, \psi). \quad (7.6)$$

This beautifully reflects the physics of the string theory construction. After the flop, the D5 branes are replaced by anti-D5 branes, which still give rise to a $U(N)$ gauge theory with $\mathcal{N} = 1$ supersymmetry, albeit a *different* supersymmetry than the original one, explaining the above realignment.

Let us review in more detail how the flop is manifested in the IR field theory of [18]. When $\text{Im}(\alpha) \ll 0$, we must look for critical points of the physical potential

$$V_{\text{eff}} = g^{S\bar{S}} |\partial_S \mathcal{W}|^2. \quad (7.7)$$

At leading order, the theory spontaneously breaks an underlying $\mathcal{N} = 2$ supersymmetry, so the tree-level Kähler metric should be determined by special

geometry. While we do not expect this to be an exact statement, we nevertheless make the assumption for the remainder of this section that the Kähler metric is that of the $\mathcal{N} = 2$ theory⁵. Thus the action for the IR dual is given by

$$\int d^4x d^2\theta d^2\bar{\theta} \Lambda^{-4} [\bar{S}_i \partial_i \mathcal{F}_0 - c.c.] + \left[\int d^4x d^2\theta W(S_i) + c.c. \right] \quad (7.8)$$

where Λ^4 gets identified with M_{string}^4 in the string context. This leads to the Kähler metric

$$\mathcal{G}_{S\bar{S}} = \text{Im}(\tau) \cdot \Lambda^{-4},$$

where

$$\tau(S) = \partial_S^2 \mathcal{F}_0 = \frac{1}{2\pi i} \log \left(\frac{S}{m\Lambda_0^2} \right).$$

The effective potential can then be made explicit,

$$V_{\text{eff}} = \frac{2i}{(\tau - \bar{\tau})} |\alpha - N\tau|^2,$$

and the critical points, $\partial_S V_{\text{eff}} = 0$, are located at the solutions to

$$\frac{2i}{(\tau - \bar{\tau})^2} \partial_S^3 \mathcal{F}_0 (\bar{\alpha} - N\bar{\tau}) (\alpha - N\bar{\tau}) = 0.$$

This can be satisfied through either

$$\alpha - N\tau = 0 \quad \text{or} \quad \alpha - N\bar{\tau} = 0. \quad (7.9)$$

The first solution preserves the manifest $\mathcal{N} = 1$ supersymmetry, and corresponds to the solution of $\partial_S \mathcal{W} = 0$. The second solution does not preserve the original supersymmetry as $\partial_S \mathcal{W} \neq 0$. Only *one* of these two solutions is valid at a given point in parameter space if S is to be within the field theory cut-off of $|S| \ll |m\Lambda_0^2|$. For $\text{Im}(\alpha) > 0$ the first solution is physical, and this is the supersymmetric solution we discussed above. However, for $\text{Im}(\alpha) = 1/g_{\text{YM}}^2 < 0$, it is the second solution which is physical, and we obtain

$$S^N = (m\Lambda_0^2)^N \exp(2\pi i \bar{\alpha}). \quad (7.10)$$

This solution looks very much like the solution (7.4) for the original $U(N)$ confining theory, except that $\alpha \rightarrow \bar{\alpha}$. This is what one would expect if we were discussing the theory of N antibranes on the flopped geometry. In fact, as discussed in detail in [18] one can show that this theory is indeed supersymmetric, with supermultiplets aligned as in (7.6).

⁵ See [19] for a discussion of stringy corrections to the Kähler metric.

7.3. Supersymmetry breaking by background fluxes

Now consider the same geometry as in the previous subsection, but with a holomorphically varying B -field introduced. If we wrap branes on the conifold, this gives rise to the supersymmetric theories considered in sections 3-4. However, in the case of antibranes, we will see that supersymmetry is in fact broken. This is due to the fact that, while branes preserve the same half of the background $\mathcal{N} = 2$ supersymmetry as the B -field, antibranes preserve an opposite half.

As in the previous section, we will consider branes and antibranes on the conifold geometry (7.2) with superpotential given by (7.3), but now with the holomorphically varying B -field given by⁶

$$B(\Phi) = t_0 + t_2\Phi^2. \quad (7.11)$$

We will study this from the perspective of the IR effective field theory of the glueball superfield S . Because of the underlying $\mathcal{N} = 2$ structure of this theory, we will have a good description regardless of whether it is branes or antibranes which are present. In the next subsection, we will provide UV *field* theories describing both situations.

The superpotential in the dual geometry is given by (4.7), which we repeat here for convenience

$$\mathcal{W}(S) = - \oint_A B(x)ydx + N \frac{\partial \mathcal{F}_0}{\partial S}. \quad (7.12)$$

An explicit computation in the geometry yields an exact expression for the first term,

$$\oint_A B(x)ydx = t_0 S + t_2 \frac{S^2}{m}.$$

The scalar potential is again given by (7.7) with the same metric and prepotential \mathcal{F}_0 , but now with superpotential (7.12). There are two vacua which extremize the potential, $\partial_S V_{\text{eff}} = 0$,

$$\begin{aligned} - \left(t_0 + 2t_2 \frac{S}{m} \right) + N\tau &= 0, \\ - \left(t_0 + 2t_2 \frac{S}{m} \right) + N\bar{\tau} + 4\pi i(\tau - \bar{\tau})t_2 \frac{S}{m} &= 0. \end{aligned} \quad (7.13)$$

⁶ We could have also added a term linear in Φ , but this has no effect due to the symmetry of the problem.

The first solution satisfies $\partial\mathcal{W} = 0$. This has solutions in the case where branes are present, with

$$\text{Im}[\alpha] \gg 0.$$

Here α is defined as $\alpha = t_0 + 2t_2 \frac{S}{m}$, and large positive values of $\text{Im}[\alpha]$ give $|S/m| \ll |\Lambda_0^2|$ within the allowed region. This vacuum is manifestly supersymmetric, and we have studied it in sections 3-4.

We can instead study antibranes by allowing the geometry to undergo a flop, so

$$\text{Im}[\alpha] \ll 0.$$

Then the supersymmetric solution is unphysical, and we instead study solutions to the second equation in (7.13). We already know that the manifest supersymmetry is entirely broken in this vacuum, because $\partial\mathcal{W} \neq 0$. Moreover the fact that the second equation in (7.13) is not holomorphic in S suggests that no accidental supersymmetry emerges here, unlike the cases in previous subsection and [18]. We can directly observe the fact that supersymmetry is broken in this vacuum by computing the tree-level masses of the bosons and fermions in the theory, and showing that there is a nonzero mass splitting.

From the $\mathcal{N} = 1$ Lagrangian, we can read off the fermion masses,

$$\begin{aligned} \Lambda^{-4} m_\psi &= \frac{1}{2i(\text{Im}\tau)^2} \frac{1}{2\pi i S} \left(t_0 + N\bar{\tau} + 2t_2 \frac{S}{m} \right) + \frac{1}{\text{Im}\tau} \frac{2t_2}{m} \\ \Lambda^{-4} m_\lambda &= \frac{1}{2i(\text{Im}\tau)^2} \frac{1}{2\pi i S} \overline{\left(t_0 + N\tau + 2t_2 \frac{S}{m} \right)}, \end{aligned}$$

while the bosonic masses are given by

$$\Lambda^{-4} m_{b,\pm}^2 = \frac{1}{\text{Im}\tau} (\partial\bar{\partial}V_{\text{eff}} \pm |\partial\partial V_{\text{eff}}|).$$

Evaluating the masses in the brane vacuum, we see that λ is massless and acts as a partner of the gauge field A , while ψ is a superpartner to S . In other words, supersymmetry pairs up the bosons and fermions as in (7.5).

Evaluating the masses in the antibrane vacuum, ψ becomes the massless goldstino. However, there is no longer a bose/fermi degeneracy like where the background B -field was constant. Instead,

$$m_{b,\pm}^2 = |m_\lambda|^2 \pm 4\pi\Lambda^4 |m_\lambda \partial\alpha|. \tag{7.14}$$

This mass splitting shows quite explicitly that all supersymmetries are broken in this vacuum. We can capture the strength of this breaking with a dimensionless quantity,

$$\epsilon = \frac{\Delta m_b^2}{m_b^2} \sim 2\pi\Lambda^4 \left| \frac{2t_2/m}{m_\lambda} \right| \sim \frac{t_2 S \log |S|}{m N}.$$

We can get a heuristic understanding of this measure of supersymmetry breaking as follows. The reason supersymmetry is broken in this phase is that B -field varies in a way incompatible with the normalizable fluxes/branes. Thus its variation over the cut in the IR geometry is a natural way to quantify supersymmetry breaking. More precisely, we expect that measuring

$$\epsilon = \Delta B$$

across the cut should give a quantification of the supersymmetry breaking by a dimensionless number. Evaluating this explicitly yields $\epsilon = t_2 S/m$, which is in rough agreement (up to a factor of order $\log |S|/N$) with the dimensionless quantity coming from the mass splittings.

7.4. A susy/non-susy duality

Motivated by the considerations of the previous example, we now formulate a duality between two field theories – one which is manifestly supersymmetric, and the other in which supersymmetry is broken softly by spurions. Consider an $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theory with an adjoint field Φ and superpotential terms

$$\int d^2\theta_1 \text{Tr} [B(\Phi)\mathcal{W}_\alpha\mathcal{W}^\alpha + W(\Phi)] \quad (7.15)$$

where, as before,

$$B(\Phi) = \sum_{k=0}^{n-1} t_k \Phi^k, \quad W(\Phi) = \sum_{k=0}^{n+1} a_k \Phi^k.$$

Consider a choice of parameters (a_k, t_k) such that

$$\text{Im}B(e_k) < 0 \quad (7.16)$$

for all e_k with $W'(e_k) = 0$. Then this theory is not sensible in this regime as it has no unitary vacuum. However, we propose that this theory is dual to

another $U(N)$ gauge theory already studied in [11], with an adjoint field $\tilde{\Phi}$ and superpotential term

$$\int d^2\theta_2 \text{Tr}[\bar{t}_0 \tilde{\mathcal{W}}_\alpha \tilde{\mathcal{W}}^\alpha + \tilde{W}(\tilde{\Phi})] \quad (7.17)$$

where

$$\tilde{W}(\tilde{\Phi}) = \sum_{k=1}^{n+1} (a_k + 2it_k \theta_2 \theta_2) \tilde{\Phi}^k.$$

Note that since the auxiliary field in the spurion supermultiplets have vevs t_k , this theory breaks supersymmetry. Also, the fermionic parts of the superspaces for these two actions are not related in any way. Indeed, they are orthogonal subspaces of an underlying $\mathcal{N} = 2$ superspace. This is indicated by the first theory being formulated in terms of coordinates θ_1^α , and the second in terms of θ_2^α – two *different* $\mathcal{N} = 1$ superspaces.

Note that this is natural from the string theory perspective. In the regime of parameters where (7.16) holds, one should describe the physics in terms of the flopped geometry, and ask how the antibrane theory perceives the geometry. Since the background B -field is holomorphic, it breaks supersymmetry. Indeed the tension of the antibranes will vary as they change position in the x -plane (and we do not expect a canceling term as would be the case for branes). We thus expect the potential to depend on x through a term proportional to the B -field,

$$V_{\text{eff}} \sim \text{Im}B(x). \quad (7.18)$$

Indeed, the soft supersymmetry-breaking term in (7.17) gives precisely this contribution when we identify the eigenvalues of Φ with positions in the x -plane. Moreover, note that in going from (7.15) to (7.17) we have flipped the sign of $\text{Im}(t_0) \sim 1/g_{\text{YM}}^2$, which is consistent with the fact that (7.17) describes the same physics from the antibrane perspective. As an aside, note that in this section (unlike in much of the rest of the paper), t_0 and $t_{k>0}$ enter on different footings.

We now provide evidence for this duality. We will show that both theories (7.15) and (7.17) have the *same* IR description in terms of glueball fields. The effective superpotential for the supersymmetric theory we have already discussed, and is given by

$$\int d^2\theta_1 \mathcal{W}_{\text{eff}}(S_i, a_k) \quad (7.19)$$

where

$$\mathcal{W}_{\text{eff}} = \sum_i t_0 S_i + \sum_{k>0} t_k \frac{\partial \mathcal{F}_0}{\partial a_k} + \sum_i N_i \frac{\partial \mathcal{F}_0}{\partial S_i}.$$

The effective glueball theory for the nonsupersymmetric theory, in which auxiliary spurion fields have nonzero vevs, has been studied in [10,11]. As shown in [11], turning on soft supersymmetry-breaking terms that give spurionic F-terms to the a_i in the UV theory has the expected effect in the IR of simply giving spurionic F-terms to $a_{k>0}$ in that theory,

$$\int d^2\theta_2 \widetilde{\mathcal{W}}_{\text{eff}}(S_i, a_k + 2it_k\theta_2\theta_2) \quad (7.20)$$

where

$$\widetilde{\mathcal{W}}_{\text{eff}} = \bar{t}_0 S_i + \sum_i N_i \frac{\partial \mathcal{F}_0}{\partial S_i}.$$

We will see that the two effective glueball theories are in fact identical!

As we reviewed in section 4, one way to arrive at the effective IR theory is via a dual gravity theory. Both theories (7.19) and (7.20) originate from the same Calabi-Yau after the transition, and so have the *same* underlying $\mathcal{N} = 2$ theory with prepotential $\mathcal{F}_0(S, a)$ at low energies,⁷

$$\text{Im}\left(\int d^2\theta_1 d^2\theta_2 \mathcal{F}_0(S_i, a_k)\right),$$

with appropriate fluxes or auxiliary spurion fields turned on. In fact, it was shown in [20,21] that turning on fluxes is also equivalent to giving vevs to auxiliary fields, so both (7.19) and (7.20) can be thought of as originating from the $\mathcal{N} = 2$ theory with prepotential $\mathcal{F}_0(S, a)$, where auxiliary fields for the glueball fields S_i and the background fields a_k are subsequently given vevs. This breaks supersymmetry explicitly to $\mathcal{N} = 1$ in the case of (7.19), and to $\mathcal{N} = 0$ in case of (7.20).

To be more precise, (7.19) can be obtained by shifting the auxiliary fields of the $\mathcal{N} = 2$ multiplets containing S and a according to

$$S_i \rightarrow S_i + 2iN_i\theta_2\theta_2, \quad a_k \rightarrow a_k + 2it_k\theta_2\theta_2, \quad k > 0,$$

⁷ More precisely, the Lagrangian also contains the $\mathcal{N} = 2$ FI terms $t_0 F_{11}^i + \bar{t}_0 F_{22}^i$ where F^i 's are the auxiliary fields discussed in the text.

and integrating over θ_2 . Meanwhile, (7.20) arises from instead shifting

$$S_i \rightarrow S_i + 2iN_i\theta_1\theta_1 \quad a_k \rightarrow a_k + 2it_k\theta_2\theta_2, \quad k > 0,$$

and integrating over θ_1 .

These two situations differ in how they shift the auxiliary fields F_{11}^i and $F_{22}^i = \overline{F}_{11}^i$ which lie in the $\mathcal{N} = 2$ chiral multiplet containing S_i ,

$$\mathcal{S}_i = S_i + \dots + \theta_1\theta_1 F_{11}^i + \theta_2\theta_2 F_{22}^i.$$

Shifts of fields alone cannot affect any aspect of the physics if the shift can be undone by an allowed field redefinition. Indeed, the difference between the shifts of (7.19) and (7.20) is an allowed auxiliary field redefinition, so these theories are equivalent! Put another way, in integrating out the auxiliary fields, we end up summing over all of their values, so any difference between the two theories will disappear. Note that, if F_{11}^i and F_{22}^i were independently fluctuating degrees of freedom, we could use this argument to say that both theories were equivalent to the original $\mathcal{N} = 2$ theory. They are not, however, since the auxiliary field shifts we made cannot be undone by a field redefinition obeying $F_{22}^i = \overline{F}_{11}^i$, which the fluctuating part of the auxiliary fields must satisfy.

To make this duality more explicit, we will show that both theories give rise to the same IR effective potential, $V_{\text{eff}}(S_i)$. For the supersymmetric theory (7.19), the superpotential (7.15) is

$$\mathcal{W}_{\text{eff}} = t_k \frac{\partial \mathcal{F}_0}{\partial a_k} + N_i \frac{\partial \mathcal{F}_0}{\partial S_i},$$

which leads to an effective potential

$$V_{\text{eff}} = \mathcal{G}^{i\bar{j}} (N^k \tau_{ki} + t_0 + t^k \eta_{ki}) \overline{(N^r \tau_{rj} + t_0 + t^r \eta_{rj})},$$

where in the summation $t^k \eta_{ki}$, we have removed the $m = 0$ term and written it explicitly. This will be convenient for the manipulations below, where we will continue to use this summation convention. We can rewrite V_{eff} grouped by order in t^k ,

$$\begin{aligned} V_{\text{eff}} &= \mathcal{G}^{i\bar{j}} N^k \tau_{kj} \overline{N^r \tau_{rj}} + \mathcal{G}^{i\bar{j}} (t_0 + t^k \eta_{ki}) \overline{(t_0 + t^r \eta_{rj})} \\ &\quad + \mathcal{G}^{i\bar{j}} N^k \tau_{ki} \overline{(t_0 + t^r \eta_{rj})} + \mathcal{G}^{i\bar{j}} (t_0 + t^k \eta_{ki}) \overline{N^r \tau_{rj}}. \end{aligned} \tag{7.21}$$

Now we will show that the effective potential of the nonsupersymmetric theory (7.17) agrees with (7.19). The Lagrangian can be written in $\mathcal{N} = 1$ superspace,

$$\begin{aligned} \mathcal{L} = \text{Im} \left(\int d^2\theta d^2\bar{\theta} \bar{S}_i \frac{\partial \mathcal{F}_0}{\partial S_i} \right) + \text{Im} \left(\int d^2\theta \frac{1}{2} \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j} \mathcal{W}_{\alpha,i} \mathcal{W}_j^\alpha \right) \\ + \left(\int d^2\theta \widetilde{\mathcal{W}}_{\text{eff}}(S) + c.c. \right), \end{aligned} \quad (7.22)$$

and the superpotential of this nonsupersymmetric theory is simply

$$\widetilde{\mathcal{W}}_{eff} = \bar{t}_0 S_i + N_i \frac{\partial \mathcal{F}_0}{\partial S_i}.$$

Let F_i be the auxiliary field in the S_i superfield. Performing the $d^2\theta$ integral for the superpotential term (the last terms of (7.22)) and picking out the spurion contribution (note that $\frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial a_k} = \eta_{ik}$), gives

$$\int d^2\theta \widetilde{\mathcal{W}}_{\text{eff}}(S) = (\bar{t}_0 + N_i \tau_{ij}) F_j + 2i N_i \eta_{ik} t_k.$$

The remaining terms come from the Kähler potential term (the first term of (7.22)). This gives $\mathcal{G}_{i\bar{j}} F_i \bar{F}_j$ before spurion deformation, while the spurions produce additional terms, giving a total contribution

$$\text{Im} \left(\int d^2\theta d^2\bar{\theta} S_i \frac{\partial \overline{\mathcal{F}_0}}{\partial S_i} \right) = \mathcal{G}_{i\bar{j}} F_i \bar{F}_j + F_i \bar{\eta}_{ik} \bar{t}_k + \bar{F}_i \eta_{ik} t_k + \dots$$

With the full F-term Lagrangian, it is now easy to check that integrating out the auxiliary fields F_i , produces precisely the effective potential (7.21), which arose from the supersymmetric theory (7.19).

We have seen that the tree-level effective potentials for the supersymmetric theory (7.15) and the nonsupersymmetric theory (7.17) agree exactly, corroborating the proposed duality between the two theories.

7.5. Multi-cut geometries and supersymmetry breaking

In the previous subsections we have focused on the case where all gauge couplings have the same sign, positive or negative. We now shift to consider the more general case in which both signs are present. For simplicity, we will focus on the case where the superpotential has two critical points, with a brief

discussion of the generalization to an arbitrary number of critical points reserved for the end of this subsection.

In particular, we now consider the UV theory where the superpotential appearing in the geometry (7.2) is given by

$$W(\Phi) = g\text{Tr} \left(\frac{1}{3}\Phi^3 - m^2\Phi \right)$$

and the holomorphic variation of the B -field gives rise to an effective field-dependent gauge coupling

$$\alpha(\Phi) = t_0 + t_1\Phi.$$

The two critical points of the superpotential are given by $\Phi = \pm m$, at which points the gauge coupling takes values

$$\alpha_{\pm} \equiv \alpha(\pm m) = t_0 \pm mt_1.$$

We wish to study the case where the imaginary parts of gauge couplings have opposite signs (see figure 4). Without loss of generality, we then consider

$$\text{Im}(\alpha_-) \ll 0 \ll \text{Im}(\alpha_+). \tag{7.23}$$

We will consider the vacuum where the $U(N)$ gauge group is broken to $U(N_1) \times U(N_2)$ with N_i both nonzero. It is clear from the discussion in section 7.2 that this theory is that of N_1 branes wrapping the S^2 at e_1 and N_2 antibranes wrapping the flopped S^2 at e_2 .

There are now two sources of supersymmetry breaking present. First, for the N_2 antibranes (even if $N_1 = 0$), supersymmetry is broken due to the holomorphic variation of the B -field, as discussed in section 7.3. However, this effect is secondary to that which arises from the fact that branes and antibranes are both present and preserve disparate halves of the background supersymmetry. This more dominant source of supersymmetry breaking was studied in a slightly simpler context in [18,22,23].

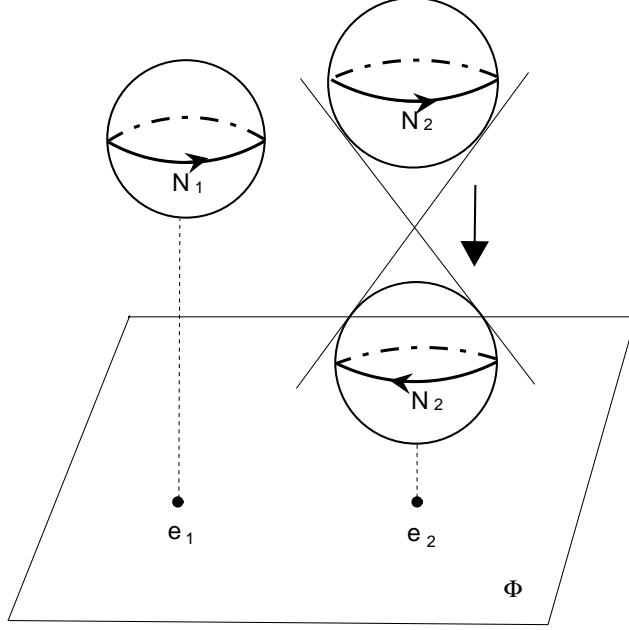


Fig. 4. By changing the parameters of the B -field continuously, it can be arranged for *only* the second S^2 to undergo a flop, with the N_2 branes replaced by N_2 antibranes on the flopped S^2 at e_2 . This configuration clearly breaks supersymmetry, as branes and antibranes preserve orthogonal supersymmetries.

We now show that this stringy UV picture is borne out in the dual IR theory. The superpotential for the closed-string dual geometry is given by (4.8)

$$\mathcal{W}(S_1, S_2) = t_0(S_1 + S_2) + t_1 \frac{\partial \mathcal{F}_0}{\partial a_1} + N_k \frac{\partial \mathcal{F}_0}{\partial S_k}.$$

In the large N_i limit, it is a sufficient approximation to work to 1-loop order in the associated matrix model. For the geometry in question, the superpotential then takes the form

$$\mathcal{W}(S_1, S_2) = \alpha_+ S_1 + \alpha_- S_2 + N_1 \frac{\partial \mathcal{F}_0}{\partial S_1} + N_2 \frac{\partial \mathcal{F}_0}{\partial S_2},$$

where $a_1 = -m^2 g$ and \mathcal{F}_0 was computed in [1],

$$\begin{aligned} \partial_{S_1} \mathcal{F}_0 &\approx \frac{1}{2\pi i} \left(-W(e_1) + S_1 \left(\log \frac{S_1}{8gm^3} - 1 \right) - 2(S_1 + S_2) \log \left(\frac{\Lambda_0}{2m} \right) \right), \\ \partial_{S_2} \mathcal{F}_0 &\approx \frac{1}{2\pi i} \left(-W(e_2) + S_2 \left(\log \frac{S_2}{8gm^3} - 1 \right) - 2(S_1 + S_2) \log \left(\frac{\Lambda_0}{2m} \right) \right). \end{aligned} \tag{7.24}$$

Note that at this order, the effect of the varying B -field is just to change the effective coupling constants in the superpotential from α_0 to α_{\pm} . As a result,

the only supersymmetry-breaking effects which appear are due to the presence of antibranes.

This theory has no physical supersymmetric vacua, so in order to study its low energy dynamics, we minimize the physical scalar potential,

$$\Lambda^{-4} V_{\text{eff}} = \mathcal{G}^{i\bar{j}} \partial_i \mathcal{W} \overline{\partial_{\bar{j}} \mathcal{W}},$$

where again the Kähler metric is determined by $\mathcal{N} = 2$ supersymmetry,

$$\mathcal{G}_{i\bar{j}} = \text{Im}(\tau_{ij}) = \text{Im} \left(\frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j} \right).$$

The critical points are given by solutions to

$$\mathcal{G}^{i\bar{a}} \mathcal{G}^{b\bar{j}} \mathcal{F}_{abk} (\alpha_i - N^l \bar{\tau}_{li}) \overline{(\alpha_j - N^r \tau_{rj})} = 0.$$

At one-loop order in the matrix model, \mathcal{F}_{ijk} only has nonzero diagonal elements, in which case the vacuum equations simplify. In particular, for the case at hand they simplify to

$$\begin{aligned} N_1 \tau_{11} &= \alpha_+ - N_2 \bar{\tau}_{12}, \\ N_2 \bar{\tau}_{22} &= \alpha_- - N_1 \tau_{12}, \end{aligned}$$

and using the expression for the Kähler metric arising from (7.24), we obtain following explicit solutions

$$\begin{aligned} (S_1)^{N_1} &= (2gm\Lambda_0^2)^{N_1} \exp(2\pi i \alpha_+) \left(\frac{\Lambda_0^2}{4m^2} \right)^{-N_2} \\ (-S_2)^{N_2} &= (2gm\Lambda_0^2)^{N_2} \exp(2\pi i \bar{\alpha}_-) \left(\frac{\Lambda_0^2}{4m^2} \right)^{-N_1}. \end{aligned}$$

In addition, we can compute the vacuum energy, and find it to be

$$\frac{V_{\text{eff}*}}{\Lambda^4} = 4N_2 |\text{Im} \alpha_-| + \frac{4}{\pi} N_1 N_2 \log \left| \frac{\Lambda_0}{2m} \right|.$$

The first term we identify as the brane tension due to antibranes on the flopped \mathbf{P}^1 , which agrees with (7.18), while the second term suggests a Coulomb *repulsion* between brane stacks preserving opposite supersymmetries. A similar

expression for the potential energy between branes and antibranes can be found in [18,22,23].

We can further study the masses of the bosonic and fermionic excitations about the nonsupersymmetric vacua. At the current order of approximation, most of the expressions from [18] still hold. We obtain four distinct bosonic masses, given by [18]

$$(m_{\pm,c})^2 = \frac{(a^2 + b^2 + 2abcv) \pm \sqrt{(a^2 + b^2 + 2abcv)^2 - 4a^2b^2(1-v)^2}}{2(1-v)^2} \quad (7.25)$$

where $c = \pm 1$,

$$a \equiv \Lambda^4 \left| \frac{N_1}{2\pi S_1 \text{Im}\tau_{11}} \right|, \quad b \equiv \Lambda^4 \left| \frac{N_2}{2\pi S_2 \text{Im}\tau_{22}} \right|$$

$$v \equiv \frac{(\text{Im}\tau_{12})^2}{\text{Im}\tau_{11} \text{Im}\tau_{22}},$$

and Λ is a mass scale in the action (7.8). The tree-level fermionic masses can also be computed from the off-shell $\mathcal{N} = 1$ Lagrangian. As in [18], they are given by⁸

$$m_{\psi_i} = \left(\frac{a}{1-v}, 0 \right), \quad m_{\lambda_i} = \left(0, \frac{b}{1-v} \right). \quad (7.26)$$

The presence of two massless fermions can be thought of as representing two goldstinos due to the breaking of off-shell $\mathcal{N} = 2$ supersymmetry. Alternatively, this fermion spectrum can be viewed as the natural result of breaking supersymmetry collectively with branes and antibranes. There is a light gaugino localized on both the branes and the antibranes. However, since these preserve different supersymmetries, we see the gauginos as arising one from the gaugino sector and one from the sfermion sector with respect to a given $\mathcal{N} = 1$ superspace.

For a generic choice of parameters, supersymmetry breaking is not small, and there is no natural way to pair up bosons and fermions in order to write a

⁸ Note that the relation [24]

$$\sum_{\text{boson}} m^2 - \sum_{\text{fermion}} m^2 = \text{Tr}(-)^F m^2 = 0$$

holds for our system, as well as for (7.14).

mass splitting as a measure of how badly supersymmetry is broken. However from the mass formula we have given, it is clear that in the limit $v \rightarrow 0$, the spectrum becomes supersymmetric, and there does emerge a natural pairing of bosonic and fermionic excitations. In this limit, v becomes a good dimensionless measure of the mass splitting, and we can write it in terms of parameters $(\Lambda_0, \alpha_{\pm}, m, N_i)$ as

$$v = \frac{N_1 N_2 \left(\log \left| \frac{\Lambda_0}{2m} \right| \right)^2}{\left(\pi |\text{Im}(\alpha_+)| + \Delta N \log \left| \frac{\Lambda_0}{2m} \right| \right) \left(\pi |\text{Im}(\alpha_-)| - \Delta N \log \left| \frac{\Lambda_0}{2m} \right| \right)}.$$

where $\Delta N = N_1 - N_2$. For $\Delta N = 0$, this further simplifies to

$$v = \frac{N^2 \left(\log \left| \frac{\Lambda_0}{2m} \right| \right)^2}{\pi^2 \left(|\text{Im}(\alpha_+)| \right) \left(|\text{Im}(\alpha_-)| \right)}.$$

This vanishes and supersymmetry is restored for large separation $mt_1 \gg 1$, corresponding to the extreme weak-coupling limit. One can also consider another extreme where $N_1 \gg N_2$. In this limit we again expect supersymmetry to be restored. Indeed, in this limit $v \propto N_2/N_1$, and so vanishes.

It should be noted that, unlike the case where all gauge couplings are negative and the background flux is small, in this case the dimensionless parameter v does depend explicitly on the cutoff Λ_0 . This may be related to the fact that, in this case, there is no field theory description in the UV. Namely, even though we know that this system should be described by branes and antibranes, these brane configurations do not admit a good field theory limit. Nevertheless the arguments of the previous section can be used to show that *below* the scale of gauge symmetry breaking, there is an effective field theory description in terms of a $\prod_i U(N_i)$ gauge theory which breaks supersymmetry and captures the same IR physics. In this theory, the gauge group factors with positive gauge couplings have an effective field dependent gauge coupling, while those with negative gauge couplings have supersymmetry softly broken by spurions. However, this is not a satisfactory description for the full dual UV theory.

Before concluding this section, let us briefly consider the generalization of the previous discussion to the n -cut geometry. Here, the superpotential in (7.2) is given by

$$W'(\Phi) = g \prod_{i=1}^n (\Phi - e_i).$$

Starting with D5 branes wrapped on n shrinking \mathbf{P}^1 's at $x = e_i$, we perform a geometric transition and study the dual closed-string geometry with n finite S^3 's. The distance between critical points are

$$\Delta_{ij} \equiv e_i - e_j.$$

From the period expansion of [1] we have following expressions in a semiclassical regime

$$2\pi i \tau_{ii} = 2\pi i \frac{\partial^2 \mathcal{F}_0}{\partial S_i^2} \approx \log \left(\frac{S_1}{W''(e_i) \Lambda_0^2} \right) + \mathcal{O}(S)$$

$$2\pi i \tau_{ij} = 2\pi i \frac{\partial^2 \mathcal{F}_0}{\partial S_i \partial S_j} \approx -\log \left(\frac{\Lambda_0^2}{\Delta_{ij}^2} \right) + \mathcal{O}(S)$$

Generalizing the vacuum condition from the two cut geometry, the physical minima of effective potential are then determined by

$$0 = -\text{Re}(\alpha_i) + \sum_j \text{Re}(\tau)_{ij} N_j,$$

$$0 = -\text{Im}(\alpha_i) + \sum_j \text{Im}(\tau)_{ij} N_j \delta_j$$

where $\delta_i \equiv \text{sign}[\text{Im} \alpha_i]$. The expectation values of S_i are expressed explicitly below,

$$\langle S_i \rangle^{N_i} = (W''(e_i) \Lambda_0^2)^{N_i} \prod_{j \neq i}^{\delta_i \delta_j > 0} \left(\frac{\Lambda_0}{\Delta_{ij}} \right)^{2N_j} \prod_{k \neq i}^{\delta_i \delta_k < 0} \left(\frac{\Lambda_0}{\Delta_{ik}} \right)^{-2N_k} \exp(2\pi i \alpha_i), \quad \delta_i > 0$$

$$\langle S_i \rangle^{N_i} = (W''(e_i) \Lambda_0^2)^{N_i} \prod_{j \neq i}^{\delta_i \delta_j > 0} \left(\frac{\Lambda_0}{\Delta_{ij}} \right)^{2N_j} \prod_{k \neq i}^{\delta_i \delta_k < 0} \left(\frac{\Lambda_0}{\Delta_{ik}} \right)^{-2N_k} \exp(2\pi i \bar{\alpha}_i), \quad \delta_i < 0.$$

The vacuum energy density formula is now given by

$$\frac{V_{\text{eff}*}}{\Lambda^4} = 2 \sum_i N_i (|\text{Im} \alpha_i| - \text{Im} \alpha_i) + \left(\sum_{i,j}^{\delta_i > 0, \delta_j < 0} \frac{2}{\pi} N_i N_j \log \left| \frac{\Lambda_0}{\Delta_{ij}} \right| \right), \quad (7.27)$$

where the first term is the brane tension contribution from each flopped \mathbf{P}^1 with negative g_{YM}^2 (matching with (7.18)), and the second term suggests that opposite brane types interact to contribute a *repulsive* Coulomb potential energy (as in the cases considered in [23])

7.6. Decay mechanism for nonsupersymmetric systems

It is straightforward to see how the nonsupersymmetric systems studied in this section can decay. This is particularly clear in the UV picture. If the gauge coupling constants are all negative, the branes want to sit at the critical point which minimizes $|\text{Im}B(e_i)|$, as this will give the smallest vacuum energy according to (7.27). Thus we expect that in this case the system will decay to one which is the $U(N)$ theory of antibranes in a holomorphic B -field background. This still breaks supersymmetry, but it is completely stable. Considering that RR charge has to be conserved, no further decay is possible.

If there are some critical points where $\text{Im}B(e_i)$ is positive, there is no unique stable vacuum. Instead, there are as many as there are ways of distributing N branes amongst the critical points $x = e_i$ where $\text{Im}B(e_i) > 0$. Thus, we find numerous supersymmetric vacua which could be the end point of the decay process, each one minimizing the potential energy to zero. As in [18], these decays can be reformulated in the closed-string dual in terms of Euclidean D5 brane instantons, which effectively transfer branes/flux from one cut to another.

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Appendix A. Computation of \mathcal{W}_{eff}

Here we provide more detail on the derivation of (4.7) using the Riemann bilinear identity and its extension to a noncompact Riemann surface Σ . In particular, we wish to compute the integral

$$\int_{\Sigma} \chi \wedge \lambda \tag{A.1}$$

for closed one-forms χ and λ which are now allowed to have arbitrarily bad divergences at infinity. We need to be extra careful due to this worse-than-usual behavior at infinity. In particular, the contribution of the interior of the Riemann surface will be exactly the same as the usual case, with the only difference coming from a careful treatment of contributions coming from the boundary at infinity.

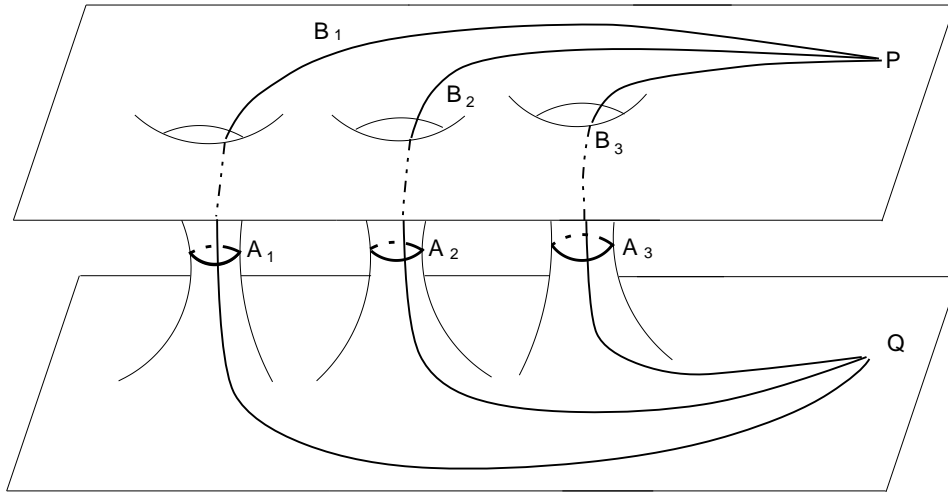


Fig. 5. A noncompact Riemann surface represented as a compact Riemann surface Σ with two points P and Q at infinity removed.

We can represent the noncompact Riemann surface Σ as a compact Riemann surface of genus n with two points representing the points at infinity on the top and bottom sheet (labeled by P and Q , respectively) removed. The derivation of the Riemann bilinear identity on the surface then goes through as usual, by cutting the Riemann surface open into a disk, except that we get an additional contribution from the boundary piece connecting the points P and Q (see figure 5). In particular, the contributions of the $n - 1$ compact

B -cycles $B_i - B_{i+1}$ and the dual $n - 1$ compact A -cycles are the usual ones. The contribution from the boundary at infinity is given by

$$\oint_P f\lambda + \oint_Q f\lambda - \oint_P \chi \int_Q^P \lambda \quad (\text{A.2})$$

where $\chi = df$ and f is a function defined on the simply connected domain which represents the cut-open surface Σ .⁹ Evaluating this for our case of interest, with

$$\begin{aligned} \lambda &= ydx \\ \chi &= H_{\text{RR}} + H_0, \end{aligned}$$

(A.2) gives a contribution

$$\oint_P B(x)ydx - \oint_P (H_{\text{RR}} + H_0) \int_Q^P ydx$$

where we have used the fact that $\oint_P = -\oint_Q$ and that $H_{\text{RR}} + H_0 \sim dB(x)$ for large x (and so at the contour around P). Combining all contributions, the superpotential can indeed be rewritten as

$$\mathcal{W}_{\text{eff}} = \sum_{i=1}^n \oint_{A_i} B(x)ydx - \sum_{i=1}^n N_i \partial_{S_i} \mathcal{F}_0$$

⁹ Note that when f has at worst a logarithmic divergence at P and Q , and λ has at worst a simple pole, then we can write

$$\oint_P f\lambda + \oint_Q f\lambda = (f(P) - f(Q)) \oint_P \lambda = \int_Q^P \chi \oint_P \lambda$$

which returns the standard form for the integral (A.2). However, in the case where f has poles at P and Q , the resulting equations are modified.

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