Stable Directions for Excited States of Nonlinear Schrödinger Equations

STABLE DIRECTIONS FOR EXCITED STATES OF NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract

We consider nonlinear Schrödinger equations in $\mathbb{R}^3$. Assume that the linear Hamiltonians have two bound states. For certain finite codimension subset in the space of initial data, we construct solutions converging to the excited states in both non-resonant and resonant cases. In the resonant case, the linearized operators around the excited states are non-self adjoint perturbations to some linear Hamiltonians with embedded eigenvalues. Although self-adjoint perturbation turns embedded eigenvalues into resonances, this class of non-self adjoint perturbations turn an embedded eigenvalue into two eigenvalues with the distance to the continuous spectrum given to the leading order by the Fermi golden rule.

1 Introduction

Consider the nonlinear Schrödinger equation

$$i\partial_t \psi = (-\Delta + V)\psi + \lambda|\psi|^2\psi, \qquad \psi(t=0) = \psi_0 \tag{1.1}$$

where V is a smooth localized potential, λ is an order one parameter and $\psi = \psi(t,x) : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{C}$ is a wave function. The goal of this paper is to study the asymptotic dynamics of the solution for initial data ψ_0 near some *nonlinear excited state*.

For any solution $\psi(t) \in H^1(\mathbb{R}^3)$ the L^2-norm and the Hamiltonian

$$\mathcal{H}[\psi] = \int \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}V|\psi|^2 + \frac{1}{4}\lambda|\psi|^4 \, dx \ , \tag{1.2}$$

are constants for all t. The global well-posedness for small solutions in $H^1(\mathbb{R}^3)$ can be proved using these conserved quantities and a continuity argument.

We assume that the linear Hamiltonian $H_0 := -\Delta + V$ has two simple eigenvalues $e_0 < e_1 < 0$ with normalized eigen-functions ϕ_0, ϕ_1. The nonlinear bound states to the Schrödinger equation (1.1) are solutions to the equation

$$(-\Delta + V)Q + \lambda |Q|^2 Q = EQ \ . \tag{1.3}$$

They are critical points to the Hamiltonian $\mathcal{H}[\psi]$ defined in (1.2) subject to the constraint that the L^2-norm of ψ is fixed.

We may obtain two families of such bound states by standard bifurcation theory, corresponding to the two eigenvalues of the linear Hamiltonian. For any E sufficiently close to e_0 so that $E - e_0$ and λ have the same sign, there is a unique positive solution $Q = Q_E$ to (1.3) which decays exponentially as $x \to \infty$. See Lemma 2.1 of [21]. We call this family the *nonlinear ground states* and we refer to it as $\{Q_E\}_E$. Similarly, there is a *nonlinear excited state* family $\{Q_{1,E_1}\}_{E_1}$. We will abbreviate them as Q and Q_1. From the same Lemma 2.1 of [21], these solutions are small and we have $\|Q_E\| \sim |E - e_0|^{1/2}$ and $\|Q_{1,E_1}\| \sim |E_1 - e_1|^{1/2}$.

It is well-known that the family of nonlinear ground states is stable in the sense that if

$$\inf_{\Theta, E} \left\| \psi(t) - Q_E \, e^{i\Theta} \right\|_{L^2}$$

is small for $t = 0$, it remains so for all t, see [13]. Let $\|\cdot\|_{L^2_{\text{loc}}}$ denote a local L^2 norm, for example the L^2-norm in a ball with large radius. One expects that this difference actually approaches zero in local L^2 norm, i.e.,

$$\lim_{t \to \infty} \inf_{\Theta, E} \left\| \psi(t) - Q_E \, e^{i\Theta} \right\|_{L^2_{\text{loc}}} = 0 \ . \tag{1.4}$$

If $-\Delta + V$ has only one bound state, it is proved in [17] [10] that the evolution will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. Suppose now that $-\Delta + V$ has two bound states: a ground state ϕ_0 with eigenvalue e_0 and an excited state ϕ_1 with eigenvalue e_1. It is proved in [20] that the evolution with initial data ψ_0 near some Q_E will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. See also [3] for the one dimensional case, [18] for nonlinear Klein-Gorden equations with one unstable bound state.

If the initial data is not restricted to near the ground states, the problem becomes much more delicate due to the presence of the excited states. On physical ground, quantum mechanics tells us that excited states are unstable and all perturbations should result in a release of radiation and the relaxation of the excited states to the ground states. Since bound states are periodic orbits, this picture differs from the classical one where periodic orbits are in general stable.

There were extensive linear analysis for bound states of nonlinear Schrödinger and wave equations, see, e.g., [15, 16, 14, 5, 6, 22, 23]. A special case of Theorem 3.5 of [6], page 330, states that

Theorem A *Let* $H_1 = -\Delta + V - E_1$*. The matrix operator*

$$JH_1 = \begin{bmatrix} 0 & H_1 \\ -H_1 & 0 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

is structurally stable if and only $e_0 > 2e_1$*.*

The precise meaning of structural stability was given in [6]. Roughly speaking, it means that the operator remains stable under small perturbations. Theorem A will not be directly used in this paper.

As we will see later, the linearized operator around an excited state is a perturbation of JH_1. Thus, two different situations occurs:

1. Non-resonant case: $e_0 > 2e_1$. $\quad (e_{01} < |e_1|)$.
2. Resonant case: $e_0 < 2e_1$. $\quad (e_{01} > |e_1|)$.

Here $e_{01} = e_1 - e_0 > 0$. In the resonant case, Theorem A says the linearized operator is in general unstable, which agrees with the physical picture. In the non-resonant case, however, the linearized operator becomes stable. The difference here is closely related to the fact that $2e_1 - e_0$ lies in the continuum spectrum of H_0 only in the resonant case.

In the resonant case, the unstable picture is confirmed for most data near excited states in our work [21]. We prove that, as long as the ground state component in $\psi_0 - Q_1$ is larger than $\|\psi_0\|^2$ times the size of the dispersive part corresponding the continuous spectrum, the solution will move away from the excited states and relax and stabilize to ground states locally. Since $\|\psi_0\|^2$ is small, this assumption allows the dispersive part to be much larger than the ground state component.

There is a small set of data where [21] does not apply, namely, those data with ground state component in $\psi_0 - Q_1$ smaller than $\|\psi_0\|^2$ times the size of the dispersive part. The aim of this paper is to show that this restriction is almost optimal: we will construct within this small set of initial data a "hypersurface" whose corresponding solutions converge to *excited states*.

This does not contradict with the physical intuition since this hypersurface in certain sense has zero measure and can not be observed in experiments. These solutions, however, show that linear instability does not imply all solutions to be unstable. In the language of dynamical systems, *the excited states are one parameter family of hyperbolic fixed points and this hypersurface is contained in the stable manifold of the fixed points.* We believe that this surface is the whole stable manifold.

We will also construct solutions converging to excited states in the non-resonant case, where it is expected since the linearized operator is stable. We now state our assumptions on the potential V:

Assumption A0: $H_0 := -\Delta + V$ acting on $L^2(\mathbb{R}^3)$ has two simple eigenvalues $e_0 < e_1 < 0$, with normalized eigenvectors ϕ_0 and ϕ_1.

Assumption A1: The bottom of the continuous spectrum to $-\Delta + V$, 0, is not a generalized eigenvalue, i.e., not an eigenvalue nor a resonance. There is a small $\sigma > 0$ such that

$$|\nabla^\alpha V(x)| \le C \langle x \rangle^{-5-\sigma}, \qquad \text{for } |\alpha| \le 2 .$$

Also, the functions $(x \cdot \nabla)^k V$, for $k = 0, 1, 2, 3$, are $-\Delta$ bounded with a $-\Delta$-bound < 1:

$$\left\|(x \cdot \nabla)^k V\phi\right\|_2 \le \sigma_0 \left\|-\Delta\phi\right\|_2 + C \left\|\phi\right\|_2, \qquad \sigma_0 < 1, \quad k = 0, 1, 2, 3 .$$

Assumption A1 contains some standard conditions to assure that most tools in linear Schrödinger operators apply. In particular, it satisfies the assumptions of [24] so that the wave operator $W_H = \lim_{t\to\infty} e^{itH_0} e^{it\Delta}$ satisfies the $W^{k,p}$ estimates for $k \le 2$. These conditions are certainly not optimal.

Let $e_{01} = e_1 - e_0$ be the spectral gap of the ground state. In the resonant case $2e_{01} > |e_0|$ so that $2e_1 - e_0$ lies in the continuum spectrum of H_0, we further assume

Assumption A2: For some $s_0 > 0$,

$$\gamma_0 \equiv \inf_{|s|<s_0} \lim_{\sigma\to 0+} \operatorname{Im}\left(\phi_0\phi_1^2,\ \frac{1}{H_0+e_0-2e_1+s-\sigma i}\,\mathbf{P}_\mathrm{c}{}^{H_0}\phi_0\phi_1^2\right) > 0. \qquad (1.5)$$

Note that $\gamma_0 \ge 0$ since the expression above is quadratic. This assumption is generically true.

Let $Q_1 = Q_{1,E_1}$ be a nonlinear excited state with $\|Q_{1,E_1}\|_2$ small. Since (Q_1, E_1) satisfies (1.3), the function $\psi(t,x) = Q_1(x)e^{-iE_1 t}$ is an exact solution of (1.1). If we consider solutions $\psi(t,x)$ of (1.1) of the form

$$\psi(t,x) = [Q_1(x) + h(t,x)]\ e^{-iE_1 t}$$

with $h(t,x)$ small in a suitable sense, then $h(t,x)$ satisfies

$$\partial_t h = \mathcal{L}_1 h + \text{nonlinear terms}$$

where $\mathcal{L}_1$, the *linearized operator around the nonlinear excited state solution* $Q_1(x)e^{-iE_1 t}$, is defined by

$$\mathcal{L}_1 h = -i\left\{(-\Delta + V - E_1 + 2\lambda Q_1^2)\,h + \lambda Q_1^2\,\overline{h}\,\right\}. \qquad (1.6)$$

Theorem 1.1 *Suppose $H_0 = -\Delta + V$ satisfies assumptions A0–A1. Suppose either*

(NR) $e_0 > 2e_1$, or

(R) $e_0 < 2e_1$, and the assumption A2 for γ_0 holds.

Then there are $n_0 > 0$ and $\varepsilon_0(n) > 0$ defined for $n \in (0, n_0]$ such that the following holds. Let $Q_1 := Q_{1,E_1}$ be a nonlinear excited state with $\|Q_1\|_{L^2} = n \le n_0$, and let $\mathcal{L}_1$ be the corresponding linearized operator. For any $\xi_\infty \in \mathbf{H}_\mathrm{c}(\mathcal{L}_1) \cap (W^{2,1} \cap H^2)(\mathbb{R}^3)$ with $\|\xi_\infty\|_{W^{2,1}\cap H^2} = \varepsilon$, $0 < \varepsilon \le \varepsilon_0(n)$, there is a solution $\psi(t,x)$ of (1.1) and a real function $\theta(t) = O(t^{-1})$ for $t > 0$ so that

$$\|\psi(t) - \psi_\mathrm{as}(t)\|_{H^2} \le C\varepsilon^2(1+t)^{-7/4},$$

where $C = C(n)$ and

$$\psi_\mathrm{as}(t) = Q_1\, e^{-iE_1 t + i\theta(t)} + e^{-iE_1 t} e^{t\mathcal{L}_1}\xi_\infty.$$

To prove this theorem, a detailed spectral analysis of the linearized operator $\mathcal{L}_1$ is required. We shall classify the spectrum of $\mathcal{L}_1$ completely in both non-resonant and resonant cases, see Theorems 2.1 and 2.2. It is well-known that the continuous spectrum Σ_c of $\mathcal{L}_1$ is the same as that of JH_1, i.e., $\Sigma_c = \{si : s \in \mathbb{R}, |s| \geq |E_1|\}$. The point spectrum of $\mathcal{L}_1$ is more subtle. By definition, $H_1\phi_1 = -(E_1 - e_1)\phi_1$ and $H_1\phi_0 = -(E_1 - e_0)\phi_0$, and thus the matrix operator JH_1 has 4 eigenvalues $\pm i(E_1 - e_1)$ and $\pm i(E_1 - e_0)$. In the non-resonant case, the eigenvalues of $\mathcal{L}_1$ are purely imaginary and are small perturbations of these eigenvalues. In the resonant case, the eigenvalues $\pm i(E_1 - e_0)$ are embedded inside the continuum spectrum Σ_c. In general perturbation theory for embedded eigenvalues, they turn into resonances under self-adjoint perturbations. The operator $\mathcal{L}_1$ is however not a self-adjoint perturbation of H_1. In this case, we shall prove that *the embedded eigenvalues $\pm i(E_1 - e_0)$ split into four eigenvalues $\pm\omega_*$ and $\pm\bar{\omega}_*$ with the real part given approximately by the Fermi golden rule* (see [12] Chap.XII.6):

$$n^4 \operatorname{Im}\left(\lambda\phi_0\phi_1^2, \frac{1}{-\Delta + V + e_0 - 2e_1 - 0i}\, \mathbf{P}_c \lambda\phi_1^2\phi_0\right).$$

Here n is the size of Q_1, see (2.41). In particular, $e^{t\mathcal{L}_1}$ is *exponentially unstable* with the decay rate (or the blow-up rate) given approximately by the Fermi golden rule. In other words, *although self-adjoint perturbation turns embedded eigenvalues into resonances, the non-self adjoint perturbations given by $\mathcal{L}_1$ turns an embedded eigenvalue into two eigenvalues with the shifts in the real axis given to the leading order by the Fermi golden rule.* The dynamics of self-adjoint perturbation of embedded eigenvalues were studied in [19].

In the appendix we will prove the existence of solutions vanishing locally as $t \to \infty$, independent of the number of bound states in H_0. Although some weaker versions of this proposition are expected, it has never been proved in current form and we include it for completeness.

Proposition 1.2 *Suppose $H_0 = -\Delta + V$ satisfies assumption A1. There is an $\varepsilon_0 > 0$ such that the following holds. For any $\xi_\infty \in \mathbf{H}_c(H_0) \cap (W^{2,1} \cap H^2)(\mathbb{R}^3)$ with $0 < \|\xi_\infty\|_{W^{2,1}\cap H^2} = \varepsilon \leq \varepsilon_0$, there is a solution $\psi(t,x)$ of* (1.1) *of the form*

$$\psi(t) = e^{-itH_0}\xi_\infty + g(t), \qquad (t \geq 0),$$

with $\|g(t)\|_{H^2} \leq C\varepsilon^2(1+t)^{-2}$.

2 Linear analysis for excited states

As mentioned in §1, there is a family $\{Q_{1,E_1}\}_{E_1}$ of nonlinear excited states with the frequency E_1 as the parameter. They satisfy

$$(-\Delta + V)Q_1 + \lambda |Q_1|^2 Q_1 = E_1 Q_1 \ . \tag{2.1}$$

Let $Q_1 = Q_{1,E_1}$ be a fixed nonlinear excited state with $n = \|Q_{1,E_1}\|_2 \le n_0 \ll 1$. The linearized operator around the nonlinear bound state solution $Q_1(x)e^{-iE_1 t}$ is defined in (1.6)

$$\mathcal{L}_1 h = -i \left\{ (-\Delta + V - E_1 + 2\lambda Q_1^2)\, h + \lambda Q_1^2 \, \overline{h} \, \right\}.$$

We will study the spectral properties of $\mathcal{L}_1$ in this section. It properties are best understood in the complexification of $L^2(\mathbb{R}^3, \mathbb{C})$.

Definition 2.1 *Identify $\mathbb{C}$ with $\mathbb{R}^2$ and $L^2 = L^2(\mathbb{R}^3, \mathbb{C})$ with $L^2(\mathbb{R}^3, \mathbb{R}^2)$. Denote by $\mathbb{C}L^2 = L^2(\mathbb{R}^3, \mathbb{C}^2)$ the complexification of $L^2(\mathbb{R}^3, \mathbb{R}^2)$. $\mathbb{C}L^2$ consists of 2-dimensional vectors whose components are in L^2. We have the natural embedding*

$$f \in L^2 \longrightarrow \begin{bmatrix} \mathrm{Re}\, f \\ \mathrm{Im}\, f \end{bmatrix} \in \mathbb{C}L^2.$$

We equip $\mathbb{C}L^2$ with the natural inner product: For $f, g \in \mathbb{C}L^2$, $f = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$, $g = \left[\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right]$, we define

$$(f, g) = \int \bar{f} \cdot g \, dx = \int (\bar{f}_1 g_1 + \bar{f}_2 g_2) \, dx. \tag{2.2}$$

Denote by $\mathbf{RE}$ *the operator first taking the real part of functions in $\mathbb{C}L^2$ and then pulling back to L^2:*

$$\mathbf{RE} : \mathbb{C}L^2 \to L^2, \quad \mathbf{RE} \begin{bmatrix} f \\ g \end{bmatrix} = (\mathrm{Re}\, f) + i(\mathrm{Re}\, g).$$

Recall the matrix operator JH_1 defined in Theorem A. Since $H_1\phi_1 = -(E_1 - e_1)\phi_1$ and $H_1\phi_0 = -(E_1 - e_0)\phi_0$, the matrix operator JH_1 has 4 eigenvalues $\pm i(E_1 - e_1)$ and $\pm i(E_1 - e_0)$ with corresponding eigenvectors

$$\begin{bmatrix} \phi_1 \\ -i\phi_1 \end{bmatrix}, \quad \begin{bmatrix} \phi_1 \\ i\phi_1 \end{bmatrix}, \quad \begin{bmatrix} \phi_0 \\ -i\phi_0 \end{bmatrix}, \quad \begin{bmatrix} \phi_0 \\ i\phi_0 \end{bmatrix}. \tag{2.3}$$

Notice that

$$E_1 - e_1 = O(n^2), \qquad E_1 - e_0 = e_{01} + O(n^2). \tag{2.4}$$

The continuous spectrum of JH_1 is

$$\Sigma_c = \{si : \ s \in \mathbb{R}, |s| \geq |E_1|\}\,, \tag{2.5}$$

which consists of two rays on the imaginary axis.

The operator $\mathcal{L}_1$ in its matrix form

$$\begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}, \qquad \text{with} \begin{cases} L_- = -\Delta + V - E_1 + \lambda Q_1^2 \\ L_+ = -\Delta + V - E_1 + 3\lambda Q_1^2 \end{cases} \tag{2.6}$$

is a perturbation of JH_1. By Weyl's lemma, the continuous spectrum of $\mathcal{L}_1$ is also Σ_c. The eigenvalues are more complicated. In both cases ($e_{01} < |e_1|$ and $e_{01} > |e_1|$) they are near 0 and $\pm ie_{01}$. As we shall see, in both cases 0 is an eigenvalue of $\mathcal{L}_1$. The main difference between the two cases are the eigenvalues near ie_{01} and $-ie_{01}$. If $e_{01} < |e_1|$, then ie_{01} lies outside the continuous spectrum and $\mathcal{L}_1$ has an eigenvalue near ie_{01} which is purely imaginary. On the other hand, if $e_{01} > |e_1|$, then ie_{01} lies inside the continuous spectrum. Generically it splits under perturbation and the eigenvalues of $\mathcal{L}_1$ near $\pm ie_{01}$ have non-zero real parts.

We shall show that $L^2(\mathbb{R}^3, \mathbb{C})$, as a real vector space, can be decomposed as the direct sum of three invariant subspaces

$$L^2(\mathbb{R}^3, \mathbb{C}) = S(\mathcal{L}_1) \oplus \mathbf{E}_1(\mathcal{L}_1) \oplus \mathbf{H}_c(\mathcal{L}_1) \tag{2.7}$$

Here $S(\mathcal{L}_1)$ is the generalized null space, $\mathbf{E}_1(\mathcal{L}_1)$ is a generalized eigenspaces and $\mathbf{H}_c(\mathcal{L}_1)$ corresponds to the continuous spectrum. Both $S(\mathcal{L}_1)$ and $\mathbf{E}_1(\mathcal{L}_1)$ are finite dimensional.

Recall the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They are self-adjoint and

$$\sigma_1 \mathcal{L}_1 = \mathcal{L}_1^* \sigma_1, \qquad \sigma_3 \mathcal{L}_1 = -\mathcal{L}_1 \sigma_3, \tag{2.8}$$

where $\mathcal{L}_1^* = \left[\begin{smallmatrix} 0 & -L_+ \\ L_- & 0 \end{smallmatrix}\right]$.

Let $R_1 = \partial_{E_1} Q_{1,E_1}$. Direct differentiation of (2.1) with respect to E_1 gives $L_+ R_1 = Q_1$. Since $L_- Q_1 = 0$ and $L_+ R_1 = Q_1$, we have $\mathcal{L}_1 \left[\begin{smallmatrix} 0 \\ Q_1 \end{smallmatrix}\right] = 0$ and $\mathcal{L}_1 \left[\begin{smallmatrix} R_1 \\ 0 \end{smallmatrix}\right] = -\left[\begin{smallmatrix} 0 \\ Q_1 \end{smallmatrix}\right]$. We will show $\dim_{\mathbb{R}} S(\mathcal{L}_1) = 2$, hence

$$S(\mathcal{L}_1) = \operatorname*{span}_{\mathbb{R}} \left\{ \begin{bmatrix} 0 \\ Q_1 \end{bmatrix}, \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \right\}. \tag{2.9}$$

$\mathbf{H}_c(\mathcal{L}_1)$ can be characterized as

$$\mathbf{H}_c(\mathcal{L}_1) = \left\{ \psi \in \mathbb{C}L^2 : (\sigma_1 \psi, f) = 0, \ \forall f \in S(\mathcal{L}_1) \oplus \mathbf{E}_1(\mathcal{L}_1) \right\}. \tag{2.10}$$

We will use (2.10) as a working definition of $\mathbf{H}_c(\mathcal{L}_1)$. After we have proved the spectrum of $\mathcal{L}_1$ and the resolvent estimates, we will use the wave operator of $\mathcal{L}_1$ (see [4, 24, 25]) to show that (2.10) agrees with the usual definition of the continuous spectrum subspace. See §2.5.

The space $\mathbf{E}_1(\mathcal{L}_1)$, however, has very different properties in the two cases, due to whether $\pm i(E_1 - e_0)$ are embedded eigenvalues of JH_1. We will consider $\mathbf{E}_1 = \mathbf{E}_1(\mathcal{L}_1)$ as a subspace of $L^2(\mathbb{R}^3, \mathbb{R}^2)$ and denote by $\mathbb{C}\mathbf{E}_1 \subset \mathbb{C}L^2$ the complexification of $\mathbf{E}_1$. We will show that $\mathbb{C}\mathbf{E}_1$ is a direct sum of eigenspaces of $\mathcal{L}_1$ in $\mathbb{C}L^2$. We also have

$$(\sigma_1 f, g) = 0, \quad \forall f \in S(\mathcal{L}_1), \ \forall g \in \mathbf{E}_1(\mathcal{L}_1). \tag{2.11}$$

We have the following two theorems for the two cases.

Remark The case $e_0 = 2e_1$: The spectral property of $\mathcal{L}_1$ is not clear.

Theorem 2.1 (Non-resonant case) *Suppose $e_0 > 2e_1$, and the assumptions A0-A1 hold. Let $Q_1 = Q_{1,E_1}$ be a nonlinear excited state with sufficiently small L^2-norm, and let $\mathcal{L}_1$ be defined as in* (1.6).

(1) The eigenvalues of $\mathcal{L}_1$ are 0 and $\pm\omega_*$. The multiplicity of 0 is two. The other eigenvalues are simple. Here $\omega_* = i\kappa$, κ is real, $\kappa = e_{01} + O(n^2)$. There is no embedded eigenvalue. The bottoms of the continuous spectrum are not eigenvalue nor resonance.

(2) The space $L^2 = L^2(\mathbb{R}^3, \mathbb{C})$, as a real vector space, can be decomposed as in (2.7). *Here $S(\mathcal{L}_1)$ and $\mathbf{H}_c(\mathcal{L}_1)$ are given in* (2.9) *and* (2.10), *respectively; $\mathbf{E}_1(\mathcal{L}_1)$ is the space corresponding to the perturbation of the eigenvalues $\pm i(E_1 - e_0)$ of JH_1. We have the orthogonality relation* (2.11).

(3) Let $\mathbb{C}\mathbf{E}_1$ denotes the complexification of $\mathbf{E}_1 = \mathbf{E}_1(\mathcal{L}_1)$. $\mathbb{C}\mathbf{E}_1$ is 2-complex-dimensional. $\mathbf{E}_1$ is 2-real-dimensional. We have

$$\begin{aligned} \mathbb{C}\mathbf{E}_1 &= \operatorname*{span}_{\mathbb{C}} \left\{ \Phi, \overline{\Phi} \right\}, \\ \mathbf{E}_1 &= \operatorname*{span}_{\mathbb{R}} \left\{ \left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right] \right\}. \end{aligned} \tag{2.12}$$

Here $\Phi = \left[\begin{smallmatrix} u \\ -iv \end{smallmatrix}\right]$ is an eigenfunction of $\mathcal{L}_1$ with eigenvalue ω_*. u and v are real-valued L^2-functions satisfying $L_+ u = -\kappa v$, $L_- v = -\kappa u$ and $(u, v) = 1$.

u and v are perturbations of ϕ_0. $\bar\Phi = \left[\begin{smallmatrix} u \\ iv \end{smallmatrix}\right]$ is another eigenfunction with eigenvalue $-\omega_*$. We have $\mathcal{L}_1\Phi = \omega_*\Phi$, $\mathcal{L}_1\bar\Phi = -\omega_*\bar\Phi$.

(4) For any function $\zeta \in \mathbf{E}_1(\mathcal{L}_1)$, there is a unique $\alpha \in \mathbb{C}$ so that

$$\zeta = \textbf{RE } \alpha\Phi,$$

and we have $\mathcal{L}_1\zeta = \textbf{RE } \omega_*\alpha\Phi$, $e^{t\mathcal{L}_1}\zeta = \textbf{RE } e^{t\omega_*}\alpha\Phi$.

(5) We have the orthogonality relations in (2.10) and (2.11). Hence any $\psi \in L^2$ can be decomposed as (see (2.7))

$$\psi = a\begin{bmatrix} R_1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ Q_1 \end{bmatrix} + c\begin{bmatrix} u \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ v \end{bmatrix} + \eta \tag{2.13}$$

with $\eta \in \mathbf{H}_\text{c}(\mathcal{L}_1)$,

$$\begin{aligned} a &= (Q_1, R_1)^{-1}(Q_1, \operatorname{Re}\psi), \qquad & c &= (u,v)^{-1}(v, \operatorname{Re}\psi), \\ b &= (Q_1, R_1)^{-1}(R_1, \operatorname{Im}\psi), \qquad & d &= (u,v)^{-1}(u, \operatorname{Im}\psi). \end{aligned} \tag{2.14}$$

(6) Let $M_1 \equiv \mathbf{E}_1(\mathcal{L}_1) \oplus \mathbf{H}_\text{c}(\mathcal{L}_1)$. We have

$$M_1 \equiv \mathbf{E}_1(\mathcal{L}_1) \oplus \mathbf{H}_\text{c}(\mathcal{L}_1) = \begin{bmatrix} Q_1^\perp \\ R_1^\perp \end{bmatrix}. \tag{2.15}$$

There is a constant $C_2 > 1$ such that, for all $\phi \in M_1$ and all $t \in \mathbb{R}$, we have

$$C_2^{-1}\|\phi\|_{H^k} \le \left\|e^{t\mathcal{L}_1}\phi\right\|_{H^k} \le C_2\|\phi\|_{H^k}\ , \qquad (k = 1,2). \tag{2.16}$$

(7) Decay estimates: For all $\eta \in \mathbf{H}_\text{c}(\mathcal{L}_1)$, for all $p \in [2,\infty]$, one has

$$\left\|e^{t\mathcal{L}_1}\eta\right\|_{L^p} \le C|t|^{-3(\frac{1}{2}-\frac{1}{p})}\|\eta\|_{L^{p'}}.$$

Theorem 2.2 (Resonant case) *Suppose $e_0 < 2e_1$, and the assumptions A0-A2 hold. Let $Q_1 = Q_{1,E_1}$ be a nonlinear excited state with sufficiently small L^2-norm, and let $\mathcal{L}_1$ be defined as in (1.6).*

(1) The eigenvalues of $\mathcal{L}_1$ are 0, $\pm\omega_*$ and $\pm\bar\omega_*$. The multiplicity of 0 is two. The other eigenvalues are simple. Here $\omega_* = i\kappa + \gamma$, $\kappa, \gamma > 0$, $\kappa = e_{01} + O(n^2)$, and $\frac{3}{4}\lambda^2\gamma_0 n^4 \le \gamma \le Cn^4$. ($\gamma_0$ is given in (1.5)). There is no embedded eigenvalue. The bottoms of the continuous spectrum are not eigenvalue nor resonance.

There is an ω_*-eigenvector Φ, $\mathcal{L}_1\Phi = \omega_*\Phi$, which is of order one in L^2 and $\Phi - \left[\begin{smallmatrix} \phi_0 \\ -i\phi_0 \end{smallmatrix}\right]$ is locally small in the sense that

$$\left|(\phi, \Phi - \left[\begin{smallmatrix} \phi_0 \\ -i\phi_0 \end{smallmatrix}\right])\right| \le Cn^2\|\langle x\rangle^r\phi\|_{L^2}, \tag{2.17}$$

for any $r > 3$. However, Φ is not a perturbation of $\left[\begin{smallmatrix} \phi_0 \\ -i\phi_0 \end{smallmatrix}\right]$ in $\mathbb{C}L^2$. In fact, $\Phi = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ with $u - \phi_0$ and $v + i\phi_0$ of order one in L^2,

$$u = \phi_0 - \frac{1}{-\Delta + V - E_1 - \kappa + \gamma i}\, \mathbf{P}_c(H_0)\lambda\phi_0 Q_1^2 + O(n^2) \quad \textit{in } L^2,$$

and $v = -L_+ u/\omega_*$. Note $-E_1 - \kappa = e_0 - 2e_1 + O(n^2)$.

(2) The space $L^2 = L^2(\mathbb{R}^3, \mathbb{C})$, as a real vector space, can be decomposed as in (2.7). Here $S(\mathcal{L}_1)$ and $\mathbf{H}_c(\mathcal{L}_1)$ are given in (2.9) and (2.10), respectively; $\mathbf{E}_1(\mathcal{L}_1)$ is the space corresponding to the perturbation of the eigenvalues $\pm i(E_1 - e_0)$ of JH_1. We have the orthogonality relation (2.11).

(3) Let $\mathbb{C}\mathbf{E}_1$ denotes the complexification of $\mathbf{E}_1 = \mathbf{E}_1(\mathcal{L}_1)$. $\mathbb{C}\mathbf{E}_1$ is 4-complex-dimensional. $\mathbf{E}_1$ is 4-real-dimensional. If we write $\Phi = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] = \begin{bmatrix} u_1 + u_2 i \\ v_1 + v_2 i \end{bmatrix}$ with u_1, u_2, v_1, v_2 real-valued L^2 functions, we have

$$\begin{aligned} \mathbb{C}\mathbf{E}_1 &= \operatorname*{span}_{\mathbb{C}} \left\{\Phi, \bar\Phi, \sigma_3\Phi, \sigma_3\bar\Phi\right\}, \\ \mathbf{E}_1 &= \operatorname*{span}_{\mathbb{R}} \left\{ \begin{bmatrix} u_1 \\ 0 \end{bmatrix}, \begin{bmatrix} u_2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_1 \end{bmatrix}, \begin{bmatrix} 0 \\ v_2 \end{bmatrix} \right\}. \end{aligned} \tag{2.18}$$

Recall $\sigma_3 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. The other eigenvectors are $\bar\Phi$, $\sigma_3\Phi$ and $\sigma_3\bar\Phi$,

$$\mathcal{L}_1\Phi = \omega_*\Phi, \quad \mathcal{L}_1\bar\Phi = \bar\omega_*\bar\Phi, \quad \mathcal{L}_1\sigma_3\Phi = -\omega_*(\sigma_3\Phi), \quad \mathcal{L}_1\sigma_3\bar\Phi = -\bar\omega_*(\sigma_3\bar\Phi). \tag{2.19}$$

(4) For any function $\zeta \in \mathbf{E}_1(\mathcal{L}_1)$, there is a unique pair $(\alpha, \beta) \in \mathbb{C}^2$ so that

$$\zeta = \mathbf{RE}\left\{\alpha\Phi + \beta\sigma_3\Phi\right\}, \tag{2.20}$$

and we have $\mathcal{L}_1\zeta = \mathbf{RE}\ \{\omega_*\alpha\Phi - \omega_*\beta\sigma_3\Phi\}$, $e^{t\mathcal{L}_1}\zeta = \mathbf{RE}\ \{e^{t\omega_*}\alpha\Phi + e^{-t\omega_*}\beta\sigma_3\Phi\}$.

(5) We have the orthogonality relations in (2.10) and (2.11). Moreover, $\sigma_1\bar\Phi \perp \left\{\bar\Phi, \sigma_3\Phi, \sigma_3\bar\Phi\right\}$, $\sigma_1\Phi \perp \left\{\Phi, \sigma_3\Phi, \sigma_3\bar\Phi\right\}$, and $\int \bar u v dx = 0$, etc. For any function $\psi \in \mathbb{C}L^2$, if we decompose

$$\psi = a\begin{bmatrix} R_1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ Q_1 \end{bmatrix} + \alpha_1\Phi + \alpha_2\bar\Phi + \beta_1\sigma_3\Phi + \beta_2\sigma_3\bar\Phi + \eta \tag{2.21}$$

where $a, b, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ and $\eta \in \mathbf{H}_c(\mathcal{L}_1)$, then we have

$$\begin{aligned} a &= c_1(\sigma_1 \left[\begin{smallmatrix} 0 \\ Q_1 \end{smallmatrix}\right], \psi), \qquad & b &= c_1(\sigma_1 \left[\begin{smallmatrix} R_1 \\ 0 \end{smallmatrix}\right], \psi), \\ \alpha_1 &= c_2(\sigma_1\bar\Phi, \psi), \qquad & \alpha_2 &= \bar c_2(\sigma_1\Phi, \psi), \\ \beta_1 &= -c_2(\sigma_1\sigma_3\bar\Phi, \psi), \qquad & \beta_2 &= -\bar c_2(\sigma_1\sigma_3\Phi, \psi), \end{aligned} \tag{2.22}$$

where $c_1^{-1} = (Q_1, R_1)$ *and* $c_2^{-1} = (\sigma_1\bar{\Phi}, \Phi) = \int 2uvdx$. *(Note* $c_1\lambda > 0$.*) The statement that* $\psi \in \mathbf{E}_1$ *is equivalent to that* $a, b \in \mathbb{R}$, $\alpha_1 = \alpha_2 = \alpha/2$, $\beta_1 = \beta_2 = \beta/2$ *and* η *is real. In this case,*

$$\psi = a \begin{bmatrix} R_1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ Q_1 \end{bmatrix} + \mathbf{RE}\{\alpha\Phi + \beta\sigma_3\Phi\} + \eta, \tag{2.23}$$

with $a, b \in \mathbb{R}$, $\eta \in \mathbf{H}_c(\mathcal{L}_1)$ *real,* $\alpha, \beta \in \mathbb{C}$, *and*

$$\alpha = P_\alpha(\psi) \equiv 2c_2(\sigma_1\bar{\Phi}, \psi), \quad \beta = P_\beta(\psi) \equiv -2c_2(\sigma_1\sigma_3\bar{\Phi}, \psi). \tag{2.24}$$

P_α *and* P_β *are maps from* L^2 *to* $\mathbb{C}$.

(6) There is a constant $C > 1$ *such that, for all* $\eta \in \mathbf{H}_c(\mathcal{L}_1)$ *and all* $t \in \mathbb{R}$, *we have*

$$C^{-1}\|\eta\|_{H^k} \le \left\|e^{t\mathcal{L}_1}\eta\right\|_{H^k} \le C\|\eta\|_{H^k}, \qquad (k = 1, 2).$$

(7) Decay estimates: For all $\eta \in \mathbf{H}_c(\mathcal{L}_1)$, *for all* $p \in [2, \infty]$, *one has*

$$\left\|e^{t\mathcal{L}_1}\eta\right\|_{L^p} \le C|t|^{-3(\frac{1}{2}-\frac{1}{p})}\|\eta\|_{L^{p'}},$$

where $C = C(n, p)$ *depends on* n.

Remark (i). In (6), we restrict ourselves to $\mathbf{H}_c(\mathcal{L}_1)$, not M_1 as in Theorem 2.1. (ii). In (3), Φ is not a perturbaiton of $\left[\begin{smallmatrix} \phi_0 \\ -i\phi_0 \end{smallmatrix}\right]$. Also, the L^2 functions u_1 and u_2 are independent of each other. So are v_1 and v_2. (iii) In (7) the constant depends on n since there are eigenvalues which are very close to the continuous spectrum.

Since the proof of Theorem 2.1 is easier, we postpone it to the last subsection §2.8. We will focus on proving Theorem 2.2 in the following subsections.

2.1 Perturbation of embedded eigenvalues and their eigenvectors

In this subsection we study the eigenvalues of $\mathcal{L}_1$ near ie_{01}. By symmetry we get also the information near $-ie_{01}$.

For our fixed nonlinear excited state $Q_1 = Q_{1,E_1}$, let $H = -\Delta + V - E_1 + \lambda Q_1^2$. ($H$ is L_1 in (2.6).) Let $\widetilde{\phi}_0$ denote a positive normalized ground state of H, with ground state energy $-\rho$ which is very close to $-e_{01}$. Hence the bottom of the continuous spectrum of H, which is close to $|e_1|$, is less than ρ. We have

$$H Q_1 = 0, \qquad H\widetilde{\phi}_0 = -\rho\widetilde{\phi}_0.$$

$$Q_1 = n\phi_1 + O(n^3), \qquad \widetilde{\phi}_0 = \phi_0 + O(n^2). \tag{2.25}$$

We want to solve the eigenvalue problem $\mathcal{L}_1 \Phi = \omega_* \Phi$ with ω_* near ie_{01}. Write $\Phi = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$. The problem has the form

$$\begin{bmatrix} 0 & H \\ -(H + 2\lambda Q_1^2) & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \omega_* \begin{bmatrix} u \\ v \end{bmatrix}$$

for some ω_* near ie_{01} and for some complex L^2-functions u, v. We have

$$Hv = \omega_* u, \qquad (H + 2\lambda Q_1^2)u = -\omega_* v.$$

Thus $H(H + 2\lambda Q_1^2)u = -\omega_*^2 u$. Suppose $\omega_* = i\kappa + \gamma$ with $\gamma > 0$. Since $\operatorname{Im}(-\omega_*^2) < 0$ and H is real, it is more convenient to solve $H(H + 2\lambda Q_1^2)\bar{u} = -\bar{\omega}_*^2 \bar{u}$ instead. If we decompose $\bar{u} = a\widetilde{\phi}_0 + bQ_1 + h$ with $h \in \mathbf{H}_c(H)$, we find $b = 0$ since $\bar{u} \in \operatorname{Image} H$. Since $a \neq 0$, we may assume $\bar{u} = \widetilde{\phi}_0 + h$. Let $A = H 2\lambda Q_1^2$ and $z = -\bar{\omega}_*^2 \sim \rho^2$. (A small $\operatorname{Re}\omega_* > 0$ corresponds to a small $\operatorname{Im} z > 0$.) We have

$$(H^2 + A)(\widetilde{\phi}_0 + h) = z(\widetilde{\phi}_0 + h),$$

i.e.,

$$z\widetilde{\phi}_0 + zh = \rho^2\widetilde{\phi}_0 + A\widetilde{\phi}_0 + (H^2 + A)h. \tag{2.26}$$

Taking projection $\mathbf{P}_c = \mathbf{P}_c(H)$, we get

$$zh = \mathbf{P}_c A\widetilde{\phi}_0 + (H^2 + \mathbf{P}_c A\,\mathbf{P}_c)h\ .$$

If $\operatorname{Im} z \neq 0$,

$$h = -(H^2 + \mathbf{P}_c A\,\mathbf{P}_c - z)^{-1}\,\mathbf{P}_c A\widetilde{\phi}_0. \tag{2.27}$$

On the other hand, if $\operatorname{Im} z = 0$, then h is generically not in L^2. We will assume $\operatorname{Im} z \neq 0$ in this subsection. The non-existence of eigenvalues with $\operatorname{Im} z = 0$ will be proved in next subsection.

Taking inner product of (2.26) with $\widetilde{\phi}_0$, we get

$$z = \rho^2 + (\widetilde{\phi}_0, A\widetilde{\phi}_0) + \left(\widetilde{\phi}_0, Ah\right).$$

Substituting (2.27), we get

$$z = \rho^2 + (\widetilde{\phi}_0, A\widetilde{\phi}_0) - \left(\widetilde{\phi}_0, A(H^2 + \mathbf{P}_{\rm c} A\, \mathbf{P}_{\rm c} - z)^{-1}\, \mathbf{P}_{\rm c} A\widetilde{\phi}_0\right). \tag{2.28}$$

If A is self-adjoint, then the signs of the imaginary parts of the two sides of the equation are different. Thus z is real and generically h is not in L^2. In our case, $A = H2\lambda Q_1^2$ is not self-adjoint. Recall $H\widetilde{\phi}_0 = -\rho\widetilde{\phi}_0$. Equation (2.28) becomes the following fixed point problem

$$z = f(z) \tag{2.29}$$

where

$$\begin{aligned} f(z) &= \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widetilde{\phi}_0) \\ &\quad + \rho\left(\widetilde{\phi}_0 2\lambda Q_1^2,\ (H^2 + H\,\mathbf{P}_{\rm c} 2\lambda Q_1^2\,\mathbf{P}_{\rm c} - z)^{-1} H\,\mathbf{P}_{\rm c} 2\lambda Q_1^2 \widetilde{\phi}_0\right). \end{aligned} \tag{2.30}$$

Let

$$R(z) = (H^2 - z)^{-1} H = \frac{1}{2(H - \sqrt{z})} + \frac{1}{2(H + \sqrt{z})}, \tag{2.31}$$

where $\sqrt{z}$ takes the branch $\operatorname{Im}\sqrt{z} > 0$ if $\operatorname{Im} z > 0$. We can expand $f(z)$ as

$$f(z) = \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widetilde{\phi}_0) + \sum_{k=1}^{\infty} \rho 2\lambda \left(\widetilde{\phi}_0 Q_1, [2\lambda Q_1\,\mathbf{P}_{\rm c} R(z)\,\mathbf{P}_{\rm c} Q_1]^k\, Q_1 \widetilde{\phi}_0\right). \tag{2.32}$$

Let

$$\begin{aligned} z_0 &= \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widetilde{\phi}_0), \\ z_1 &= z_0 + 4\rho\lambda^2 \left(\widetilde{\phi}_0 Q_1^2,\ R(z_0 + 0i)\,\mathbf{P}_{\rm c} Q_1^2 \widetilde{\phi}_0\right). \end{aligned}$$

We have $|z_1 - z_0| \le Cn^4$ from its explicit form, (cf. (2.35) of Lemma 2.3 below). We also have, by (2.25) and (1.5)

$$\operatorname{Im} z_1 = \operatorname{Im} 4\rho\lambda^2 \left(\widetilde{\phi}_0 Q_1^2,\ \frac{1}{2(H - 0i)}\ \mathbf{P}_{\rm c} Q_1^2 \widetilde{\phi}_0\right) \ge \tfrac{7}{4}\, e_{01} \lambda^2 n^4\, \gamma_0 + O(n^6) > 0.$$

Let $r_0 = \frac{1}{4}((e_{01})^2 - |e_1|^2)$ be a length of order 1. Denote the regions

$$G = \left\{x + iy : |x - \rho^2| < r_0,\ 0 < y < r_0\right\}, \tag{2.33}$$

$$D = B(z_1, n^5) = \left\{ z : |z - z_1| \le n^5 \right\}. \tag{2.34}$$

Clearly $z_0 \in G$, $z_1 \in D \subset G$. Also observe that the real part of all points in G are greater than $|E_1|^2$. We will solve the fixed point problem (2.29) in D. We need the following two lemmas.

Lemma 2.3 *Fix $r > 3$. There is a constant $C_1 > 0$ such that, for all $z \in G$,*

$$\left\| \langle x \rangle^{-r} \, \mathbf{P}_c R(z) \, \mathbf{P}_c \, \langle x \rangle^{-r} \right\|_{(L^2, L^2)} \le C_1, \tag{2.35}$$

$$\left\| \langle x \rangle^{-r} \, \mathbf{P}_c \frac{d}{dz} R(z) \, \mathbf{P}_c \, \langle x \rangle^{-r} \right\|_{(L^2, L^2)} \le C_1 (\operatorname{Im} z)^{-1/2}. \tag{2.36}$$

Here $\mathbf{P}_c = \mathbf{P}_c(H)$. Moreover, for $w_1, w_2 \in G$,

$$\begin{aligned} &\left\| \langle x \rangle^{-r} \, \mathbf{P}_c [R(w_1) - R(w_2)] \, \mathbf{P}_c \, \langle x \rangle^{-r} \right\|_{(L^2, L^2)} \\ &\qquad \le C_1 (\max(\operatorname{Im} w_1, \operatorname{Im} w_2))^{-1/2} |w_1 - w_2|. \end{aligned} \tag{2.37}$$

Proof. We have

$$R(z) = (H^2 - z)^{-1} H = \frac{1}{2(H - \sqrt{z})} + \frac{1}{2(H + \sqrt{z})} \tag{2.38}$$

Since $\frac{1}{2(H+\sqrt{z})}$ is regular in a neighborhood of $\bar{G}$, it is sufficient to prove the lemma with $R(z)$ replaced by $R_1(z) := (H - \sqrt{z})^{-1}$.

That $\left\| \langle x \rangle^{-r} \, \mathbf{P}_c R_1(z) \, \mathbf{P}_c \, \langle x \rangle^{-r} \right\|_{(L^2, L^2)} \le C_1$ is well-known, see e.g. [1], [7]. The estimate (2.36) will follow from (2.37) by taking limit. We now show (2.37) for $R_1(z)$. For any $w_1, w_2 \in G$, we have $|\sqrt{w_1} - \sqrt{w_2}| \le |w_1 - w_2|$. Write $\sqrt{w_1} = a_1 + ib_1$ and $\sqrt{w_2} = a_2 + ib_2$. We may assume $0 < b_1 < b_2$. Let $w_3 \in G$ be the unique number such that $\sqrt{w_3} = a_1 + ib_2$.

For any $u, v \in L^2$ with $\|u\|_2 = \|v\|_2 = 1$, let $u_1 = \mathbf{P}_c \langle x \rangle^{-r} u$, $v_1 = \mathbf{P}_c \langle x \rangle^{-r} v$. We have $u_1, v_1 \in L^1 \cap L^2(\mathbb{R}^3)$ and

$$\begin{aligned} &\left| \left(u, \langle x \rangle^{-r} \, \mathbf{P}_c \, [R_1(w_1) - R_1(w_3)] \, \mathbf{P}_c \, \langle x \rangle^{-r} \, v \right) \right| \\ &= \left| \int_0^\infty (u_1, e^{-it(H - a_1)} v_1) \, (e^{-b_1 t} - e^{-b_2 t}) \, dt \right| \\ &\le \int_0^\infty (1 + t)^{-3/2} (e^{-b_1 t} - e^{-b_2 t}) \, dt \le C (b_2^{1/2} - b_1^{1/2}) \le b_2^{-1/2} (b_2 - b_1). \end{aligned}$$

Here we have used the decay estimate for e^{-itH} with $H = -\Delta + V - E_1 - \lambda Q_1^2$, namely,

$$\left\| e^{-itH} \, \mathbf{P}_c \phi \right\|_{L^\infty} \le C |t|^{-3/2} \, \|\phi\|_{L^1} \tag{2.39}$$

under our Assumption A1. See [7, 8, 24].

We also have

$$\begin{aligned}
&\left|\left(u, \langle x\rangle^{-r}\ \mathbf{P}_{\rm c}\left[R_1(w_3)-R_1(w_2)\right]\ \mathbf{P}_{\rm c}\ \langle x\rangle^{-r}\ v\right)\right| \\
&= \left|\int_0^\infty (u_1, e^{-it(H-s_2-ib_2)}v_1)\,(e^{i(a_1-a_2)t}-1)\,dt\right| \\
&\le \int_0^\infty (1+t)^{-3/2}e^{-b_2 t}\,|e^{i(a_1-a_2)t}-1|\,dt \le Cb_2^{-1/2}|a_1-a_2|.
\end{aligned}$$

Since $|a_1-a_2|+|b_1-b_2| \sim |\sqrt{w_1}-\sqrt{w_2}| \le |w_1-w_2|$, we conclude

$$\left|\left(u, \langle x\rangle^{-r}\ \mathbf{P}_{\rm c}\left[R_1(w_1)-R_1(w_2)\right]\ \mathbf{P}_{\rm c}\ \langle x\rangle^{-r}\ v\right)\right| \le Cb_2^{-1/2}|w_1-w_2|.$$

Hence we have (2.37). **Q.E.D.**

Lemma 2.4 *Recall the regions G and D are defined in* (2.33)–(2.34)*.*

(1) $f(z)$ defined by (2.30) *is well-defined and analytic in G.*

(2) $|f'(z)| \le Cn^4(\operatorname{Im} z)^{-1/2}$ in G and $|f'(z)| \le 1/2$ in D.

(3) for $w_1, w_2 \in G$,

$$|f(w_1)-f(w_2)| \le Cn^4(\max(\operatorname{Im} w_1, \operatorname{Im} w_2))^{-1/2}|w_1-w_2|.$$

(4) $f(z)$ maps D into D.

Proof. By (2.35), the expansion (2.32) can be bounded by

$$|f(z)| \le C + CC_1n^4 + CC_1^2n^6 + \cdots$$

and thus converges. Since every term in (2.32) is analytic, $f(z)$ is well-defined and analytic. We also get the estimates in (2). To prove (3), let $b = \max(\operatorname{Im} w_1, \operatorname{Im} w_2)$. Then from (2.35)–(2.37),

$$|f(w_1)-f(w_2)| \le \sum_{k=1}^\infty CkC_1^k n^{2k+2}b^{-1/2}|w_1-w_2| \le Cn^4b^{-1/2}|w_1-w_2|.$$

It remains to show (4). We first estimate $|f(z_1)-z_1|$. Write $z_1 = z_0+a+bi$. Recall that $|a| < Cn^4$ and $\frac{1}{4}\lambda^2\gamma_0 n^4 < |b| < Cn^4$. Using (2.37) and (2.35) we have

$$\begin{aligned}
|f(z_1)-z_1| &= \left(\widetilde{\phi}_0 Q_1^2, \left[R(z_1)-R(z_0+0i)\right]\ \mathbf{P}_{\rm c}Q_1^2\widetilde{\phi}_0\right) \\
&\quad + \sum_{k=2}^\infty \left(\widetilde{\phi}_0 Q_1, \left[Q_1\,\mathbf{P}_{\rm c}R(z_1)\,\mathbf{P}_{\rm c}Q_1\right]^k Q_1\widetilde{\phi}_0\right) \\
&\le Cn^4b^{-1/2}(|a|+|b|) + CC_1^2n^6 + CC_1^3n^8 + \cdots \le Cn^6
\end{aligned}$$

Hence $|f(z_1)-z_1| \le Cn^6$. For any $z \in D$, we have

$$|f(z)-z_1| \le |f(z)-f(z_1)| + |f(z_1)-z_1| \le \frac{1}{2}|z-z_1| + Cn^6 \le |z-z_1|.$$

Hence $f(z) \in D$. This proves (4). **Q.E.D.**

We are ready to solve (2.29) in G. By Lemma 2.4 (1), (2) and (4), the map $f \to f(z)$ is a contraction mapping in D and hence has a unique fixed point z_* in D. By (3), for any $z \in G$ we have $|f(z)-f(z_*)| \le Cn^4 (\operatorname{Im} z_*)^{-1/2}|z-z_*| \le \frac{1}{2}|z-z_*|$. Hence there is no other fixed point of $f(z)$ in G.

By symmetry, there is another unique fixed point with negative imaginary part. Moreover, they have the size indicated in Theorem 2.2. We will prove in Subsections 2.2 and 2.3 that ω_* does not admit generalized eigenvectors and that there is no purely imaginary eigenvalue near ie_{01}, i.e., there is no embedded eigenvalue. Hence ω_*, and $-\bar{\omega}_*$ are simple and are the only eigenvalues near ie_{01}.

We now look more carefully on z_* and u_*, where u_* denotes the unique solution of $H(H+2\lambda Q_1^2)u_* = -\omega_*^2 u_*$ with the form $u_* = \widetilde{\phi}_0 + \bar{h}_*$.

Recall $|z_1 - z_*| \le n^5$ and

$$z_1 = \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widetilde{\phi}_0) + 4\rho\lambda^2 \left(\widetilde{\phi}_0 Q_1^2,\ R(z_0+0i)\,\mathbf{P}_{\rm c} Q_1^2 \widetilde{\phi}_0\right),$$

where $z_0 = \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widetilde{\phi}_0)$. Hence

$$\begin{aligned}\sqrt{z_*} &= \sqrt{z_1} + O(n^5) \\ &= \rho - (\widetilde{\phi}_0 \lambda Q_1^2 \widetilde{\phi}_0) + 2\lambda^2 \left(\widetilde{\phi}_0 Q_1^2,\ R(z_0+0i)\,\mathbf{P}_{\rm c} Q_1^2 \widetilde{\phi}_0\right) \\ &\quad + \frac{1}{4\rho}(\widetilde{\phi}_0 \lambda Q_1^2 \widetilde{\phi}_0)^2 + O(n^5).\end{aligned}$$

Since $z_* = -\bar{\omega}_*^2$, we have $\bar{\omega}_* = i\sqrt{z_*}$. Thus if we write $\omega_* = i\kappa + \gamma$, then

$$\begin{aligned}\kappa &= \rho - (\widetilde{\phi}_0 \lambda Q_1^2 \widetilde{\phi}_0) + \frac{1}{4\rho}(\widetilde{\phi}_0 \lambda Q_1^2 \widetilde{\phi}_0)^2 \\ &\quad + \operatorname{Re} 2\lambda^2 \left(\widetilde{\phi}_0 Q_1^2,\ R(z_0+0i)\,\mathbf{P}_{\rm c} Q_1^2 \widetilde{\phi}_0\right) + O(n^5), \qquad (2.40) \\ \gamma &= -\operatorname{Im} 2\lambda^2 \left(\widetilde{\phi}_0 Q_1^2,\ R(z_0+0i)\,\mathbf{P}_{\rm c} Q_1^2 \widetilde{\phi}_0\right) + O(n^5).\end{aligned}$$

By (2.31), (2.25), and expansion into series,

$$\begin{aligned}\gamma &= \operatorname{Im} \lambda^2 \left(\widetilde{\phi}_0 Q_1^2,\ (H - \sqrt{z_0} - 0i)\,\mathbf{P}_{\rm c} Q_1^2 \widetilde{\phi}_0\right) + O(n^5) \\ &= \operatorname{Im} \lambda^2 n^4 \left(\phi_0 \phi_1^2,\ \frac{1}{-\Delta + V - E_1 - \sqrt{z_0} - 0i}\ \mathbf{P}_{\rm c} \phi_1^2 \phi_0\right) + O(n^5). \qquad (2.41)\end{aligned}$$

By (1.5), $\gamma \geq \lambda^2 n^4 \gamma_0 + O(n^5)$.

By (2.27) and $A = H2\lambda Q_1^2$, we have

$$h_* = -(H^2 + \mathbf{P}_c H2\lambda Q_1^2 \mathbf{P}_c - z_*)^{-1} H \mathbf{P}_c 2\lambda Q_1^2 \widetilde{\phi}_0, \tag{2.42}$$

where $\mathbf{P}_c = \mathbf{P}_c(H)$. We now expand the resolvent on the right side as in (2.32). Then by Lemma 2.3, we can derive $|(\phi, h)| \leq Cn^2 \|\langle x \rangle^r \phi\|_2$, for any $r > 3$.

We now show that h_* is bounded in L^2 with a bound uniform in n. Recall $\sqrt{z_*} = \kappa + i\gamma$ with $\kappa \sim e_{01}$, $\gamma > \frac{1}{2}\lambda^2\gamma_0 n^4$. Since $Q_1 = n\phi_1 + O(n^3)$, by expansion and (2.25) we have

$$\begin{aligned} h_* &= -(H^2 - z_*)^{-1} H \mathbf{P}_c(H) 2\lambda\phi_0 Q_1^2 + O(n^2) \\ &= -(H - \sqrt{z_*})^{-1} \mathbf{P}_c(H) \lambda\phi_0 Q_1^2 + O(n^2) \\ &= -\frac{1}{-\Delta + V + s - \gamma i} \mathbf{P}_c(H_0) \lambda\phi_0 Q_1^2 + O(n^2), \end{aligned} \tag{2.43}$$

where $s = -E_1 - \kappa = e_0 - 2e_1 + O(n^2)$. Here we have used the fact that

$$\mathbf{P}_c(H)\phi = \mathbf{P}_c(H_0)\phi + n^2 \sum_{k=1}^{N} (\psi_k^*, \phi)\psi_k$$

for some local functions ϕ_k, ϕ_k^* of order one. We will show that the leading term on the right of (2.43) is of order one in L^2. It follows from the same proof that $O(n^2)$ on the right is also in L^2 sense.

Observe that, for $f(p) \in L^2$ with $\|f\|_2 \leq 1$,

$$\begin{aligned} &\int \frac{1}{p^2 - s + \gamma i} f(p) \cdot \frac{1}{p^2 - s - \gamma i} \bar{f}(p)\, dp \\ &= \int |f(p)|^2 \frac{1}{(p^2 - s)^2 + \gamma^2}\, dp \\ &\leq C + C \int_{\sqrt{s}/2}^{3\sqrt{s}/2} \frac{1}{(|r - \sqrt{s}| + \gamma)^2}\, dr \\ &= C + 2C \int_0^{\sqrt{s}/2} \frac{1}{(r + \gamma)^2}\, dr \leq C + C/\gamma. \end{aligned}$$

Using wave operator for $-\Delta + V$, we have similar estimates if p^2 is replaced by $-\Delta + V$. Since $\lambda\phi_0 Q_1^2 = O(n^2)$,

$$(h_*, h_*) \leq Cn^2\gamma^{-1}n^2 \leq C,$$

where C is independent of n. Since $u = \widetilde{\phi}_0 + \bar{h} = \phi_0 + \bar{h} + O(n^2)$, we have obtained the u part of the estimates $\|\Phi\|_{L^2} \leq C$ and (2.17). The corresponding estimate for v can be proved using $v = (-L_+)u/\omega_*$.

2.2 Resolvent estimates

In this subsection we study the resolvent $R(w) = (w - \mathcal{L}_1)^{-1}$. Note that $R(w)$ had a different meaning in the previous subsection. We will prove resolvent estimates along the continuous spectrum Σ_c and determine all eigenvalues outside of Σ_c.

Let L^2_r denote the weighted L^2 spaces for $r \in \mathbb{R}$:

$$L^2_r = \left\{ f : (1+x^2)^{r/2} f(x) \in L^2(\mathbb{R}^3) \right\}.$$

We will prove the following lemma.

Lemma 2.5 *Let $R(w) = (w - \mathcal{L}_1)^{-1}$ be the resolvent of $\mathcal{L}_1$. Let $\mathbf{B} = B(L^2_r, L^2_{-r})$, the space of bounded operators from L^2_r to L^2_{-r} with $r > 3$. Recall $\omega_* = i\kappa + \gamma$. For $\tau \geq |E_1|$ we have*

$$\|R(i\tau \pm 0)\|_{\mathbf{B}} + \|R(-i\tau \pm 0)\|_{\mathbf{B}} \leq C(1+\tau)^{-1/2} + C(|\tau - \kappa| + n^4)^{-1}. \tag{2.44}$$

The constant C is independent of n. We also have

$$\left\|R^{(k)}(i\tau \pm 0)\right\|_{\mathbf{B}} + \left\|R^{(k)}(-i\tau \pm 0)\right\|_{\mathbf{B}} \leq C(1+\tau)^{-(1+k)/2} + C(|\tau - \kappa| + n^4)^{-1}. \tag{2.45}$$

for derivatives, where $k = 1, 2$.

We first consider $R_0(w) = (w - JH_1)^{-1}$. Recall $H_1 = -\Delta + V - E_1$. Since

$$\begin{aligned}(w - JH_1)^{-1} &= \begin{bmatrix} w & -H_1 \\ H_1 & w \end{bmatrix}^{-1} = \frac{1}{H_1^2 + w^2} \begin{bmatrix} w & H_1 \\ -H_1 & w \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} (H_1 - iw)^{-1} + \frac{1}{2} \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} (H_1 + iw)^{-1},\end{aligned} \tag{2.46}$$

the estimates of $R_0(w)$ can be derived from that of $(H_1 - iw)^{-1}$ and $(H_1 + iw)^{-1}$. By assumption, the bottom of the continuous spectrum of H_1, $-E_1$, is not an eigenvalue nor a resonance of H_1. Hence $(H_1 - z)^{-1}$ is uniformly bounded in $\mathbf{B}$ for z away from $e_0 - E_1$ and $e_1 - E_1$, see [7]. By (2.46) and (2.4), $R_0(w)$ is uniformly bounded in $\mathbf{B}$ for w with $\operatorname{dist}(w, \Sigma_p) \geq n$, where $\Sigma_p = \{0, ie_{01}, -ie_{01}\}$.

Write

$$\mathcal{L}_1 = JH_1 + W, \qquad W = \begin{bmatrix} 0 & \lambda Q_1^2 \\ -3\lambda Q_1^2 & 0 \end{bmatrix}.$$

For $R(w) = (w - \mathcal{L}_1)^{-1}$ we have

$$R(w) = (1 - R_0(w)W)^{-1} R_0(w) = \sum_{k=0}^{\infty} [R_0(w)W]^k R_0(w). \tag{2.47}$$

Since $R_0(w)$ is uniformly bounded in $\mathbf{B}$ for w with $\text{dist}(w, \Sigma_p) > n$, and W is localized and small, (2.47) converges and $(w - \mathcal{L}_1)^{-1}$ is uniformly bounded in $\mathbf{B}$ for w with $\text{dist}(w, \Sigma_p) > n$ and we have

$$\|R(w)\|_B \le C \, \text{dist}(w, \Sigma_p)^{-1}, \qquad (n \le \text{dist}(w, \Sigma_p) \le 1). \tag{2.48}$$

Recall $\Sigma_c = \{is : |s| \ge |E_1|\}$ is the continuous spectrum of JH_1 and $\mathcal{L}_1$. For w in the region

$$\{w : \text{dist}(w, \Sigma_p) \ge n, w \notin \Sigma_c\}, \tag{2.49}$$

we have

$$\|R_0(w)\|_{(L^2, L^2)} \le C \, \text{dist}(w, \Sigma_c)^{-1}.$$

By (2.47), and because W is localized and small,

$$\begin{aligned} \|R(w)\|_{(L^2,L^2)} &\le \|R_0(w)\|_{(L^2,L^2)} + \\ &+ \sum_{k=1}^{\infty} C \|R_0(w)\|_{(L^2,L^2)} \left\{ Cn^2 \|R_0(w)\|_{\mathbf{B}} \right\}^{k-1} \|R_0(w)\|_{(L^2,L^2)} \\ &\le C \, \text{dist}(w, \Sigma_c)^{-1} + C \, \text{dist}(w, \Sigma_c)^{-2}. \end{aligned}$$

Hence $R(w)$ is uniformly bounded in (L^2, L^2) in a neighborhood of w. In particular, there is no eigenvalue of $\mathcal{L}_1$ in the above region (2.49). Note that this region includes a neighborhood of the bottom of the continuous spectrum Σ_c, $\pm iE_1$, except those in Σ_c. Hence the eigenvalues can occur only in $\{w : \text{dist}(w, \Sigma_p) < n\}$ or Σ_c.

The circle $\{w : |w| = \sqrt{n}\}$ is in the resolvent set of $\mathcal{L}_1$. By [12] Theorem XII.6, the Cauchy integral

$$P = \frac{1}{2\pi i} \oint_{|w| = \sqrt{n}} (w - \mathcal{L}_1)^{-1} \, dw$$

gives the L^2-projection onto the generalized eigenspaces with eigenvalues inside the disk $\{w : |w| < \sqrt{n}\}$. Moreover, the dimension of P is an upper bound for the sum of the dimensions of those eigenspaces. However, since the projection $P_0 = (2\pi i)^{-1} \oint_{|w|=\sqrt{n}} R_0(w)\, dw$ has dimension 2 (see (2.3)–(2.4)), and

$$P - P_0 = \frac{1}{2\pi i} \oint_{|w|=\sqrt{n}} \sum_{k=1}^{\infty} [R_0(w)W]^k\, R_0(w)\, dw$$

is convergent and bounded in (L^2, L^2) by

$$\begin{aligned}&\le C \left\| R_0(w) \right\|_{(L^2,L^2)} n^2 \sum_{k=0}^{\infty} \left(Cn^2 \left\| R_0(w) \right\|_{\mathbf{B}} \right)^k \left\| R_0(w) \right\|_{(L^2,L^2)} \\ &\le Cn^{-1/2} Cn^2 n^{-1/2} = Cn,\end{aligned}$$

(here we have used (2.48)), the dimension of P is also two. Since we already have two generalized eigenvectors $\left[\begin{smallmatrix} 0 \\ Q_1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} R_1 \\ 0 \end{smallmatrix}\right]$ with eigenvalue 0, we have obtained all generalized eigenvectors with eigenvalues in the disk $|w| < \sqrt{n}$. Together with the results in §2.1, we have obtained all eigenvalues outside of Σ_c: 0, $\pm\omega_*$ and $\pm\overline{\omega_*}$.

We next study $R(w) = (w - \mathcal{L}_1)^{-1}$ for w near $\pm ie_{01}$: $|w - ie_{01}| < n$ or $|w + ie_{01}| < n$. Let us assume $w = i\tau - \varepsilon$ with $\tau, \varepsilon > 0$, thus $-w^2$ lies in G (defined in (2.33)). The other cases are similar. Let $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathbb{C}L^2$. We want to solve the equation

$$(w - \mathcal{L}_1) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{2.50}$$

We have

$$wu - Hv = f, \qquad wv + (H + 2\lambda Q_1^2)u = g.$$

Cancelling v, we get (recall $A = H2\lambda Q_1^2$)

$$w^2 u + (H^2 + A)u = F, \qquad F = wf + Hg.$$

Write $u = \alpha\widetilde{\phi}_0 + \beta\widehat{Q}_1 + \eta$ with $\eta \in \mathbf{H}_{\rm c}(H)$ and $\widehat{Q}_1 = Q_1/\left\| Q_1 \right\|_2$. Also denote $\zeta = \alpha\widetilde{\phi}_0 + \beta\widehat{Q}_1$. We have

$$\begin{aligned}\left(w^2 + H^2 + \mathbf{P}_{\rm c} A\right)\eta &= \mathbf{P}_{\rm c} F - \mathbf{P}_{\rm c} A\zeta, \\ \left(w^2 + H^2 + \mathbf{P}^{\perp} A\right)\zeta &= \mathbf{P}^{\perp} F - \mathbf{P}^{\perp} A\eta.\end{aligned}$$

Here $\mathbf{P}_{\rm c} = \mathbf{P}_{\rm c}(H)$ and $\mathbf{P}^{\perp} = 1 - \mathbf{P}_{\rm c}$. Solving η in terms of ζ, we get

$$\eta = \left(w^2 + H^2 + \mathbf{P}_{\rm c} A\, \mathbf{P}_{\rm c}\right)^{-1} (\mathbf{P}_{\rm c} F - \mathbf{P}_{\rm c} A\zeta). \tag{2.51}$$

Substituting the above into the ζ equation we get

$$\left(w^2 + H^2 + \mathbf{P}^{\perp} A - \mathbf{P}^{\perp} A \left(w^2 + H^2 + \mathbf{P}_c A \mathbf{P}_c\right)^{-1} \mathbf{P}_c A\right) \zeta = \widetilde{F}, \tag{2.52}$$

$$\widetilde{F} = \mathbf{P}^{\perp} F - \mathbf{P}^{\perp} A \left(w^2 + H^2 + \mathbf{P}_c A \mathbf{P}_c\right)^{-1} \mathbf{P}_c F.$$

Using $\widetilde{\phi}_0$ and $\widehat{Q}_1$ as basis, we can put (2.52) into matrix form

$$\begin{bmatrix} a & b \\ 0 & w^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} (\widetilde{\phi}_0, \widetilde{F}) \\ (\widehat{Q}_1, \widetilde{F}) \end{bmatrix}, \tag{2.53}$$

where (recall $H\widetilde{\phi}_0 = -\rho\widetilde{\phi}_0$, $H\widehat{Q}_1 = 0$)

$$a = w^2 + \rho^2 - \rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widetilde{\phi}_0) + \rho(\widetilde{\phi}_0 2\lambda Q^2, \left(w^2 + H^2 + \mathbf{P}_c A \mathbf{P}_c\right)^{-1} H \mathbf{P}_c 2\lambda Q_1^2 \widetilde{\phi}_0),$$

$$b = -\rho(\widetilde{\phi}_0 2\lambda Q_1^2 \widehat{Q}_1) + \rho(\widetilde{\phi}_0 2\lambda Q^2, \left(w^2 + H^2 + \mathbf{P}_c A \mathbf{P}_c\right)^{-1} H \mathbf{P}_c 2\lambda Q_1^2 \widehat{Q}_1).$$

Thus

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1/a & -b/(aw^2) \\ 0 & w^{-2} \end{bmatrix} \begin{bmatrix} (\widetilde{\phi}_0, \widetilde{F}) \\ (\widehat{Q}_1, \widetilde{F}) \end{bmatrix}. \tag{2.54}$$

We now consider the case when w is near the continuous spectrum Σ_c. We will assume $w = i\tau - \varepsilon$ with $|\tau - e_{01}| < n$ and $\varepsilon > 0$ much smaller. The case $w = i\tau + \varepsilon$ follows similarly.

Let $z = -w^2$. It follows that $z \in G$ and $\operatorname{Re} z > 0$ is small. The idea of what follows is to compare z with z_*, the fixed point found in §2.1.

We have $a = -z + f(z) = (z_* - z) + (-f(z_*) + f(z))$. Using Lemma 2.4 (3) with $w_1 = z$ and $w_2 = z_*$, we have

$$|a| \geq |z - z_*| - |-f(z_*) + f(z)| \geq \tfrac{1}{2}|z - z_*| = \tfrac{1}{2}|w^2 - \bar{\omega}_*^2| \geq C|w - \bar{\omega}_*|.$$

Recall $\omega_* = i\kappa + \gamma$ with $\gamma \sim n^4$. Hence $|a| \geq C(|\tau - \kappa| + n^4)$. Thus

$$|\alpha| + |\beta| \leq C(1 + |a|^{-1})\|\widetilde{F}\|_{L^2} \leq C(|\tau - \kappa| + n^4)^{-1} \left(\|f\|_{L^2} + \|g\|_{L^2}\right).$$

By (2.51) and $F = wf + Hg$,

$$\eta = \Omega w \mathbf{P}_c f + \Omega H \mathbf{P}_c g - \Omega w \mathbf{P}_c A \zeta,$$

where $\Omega = \left(w^2 + H^2 + \mathbf{P}_{\rm c} A\, \mathbf{P}_{\rm c}\right)^{-1}$. Substituting the above into (2.51), we can solve η and we have

$$\|\eta\|_{L^2} \le C\left(\|f\|_{L^2} + \|g\|_{L^2}\right) + Cn^2(|\tau-\kappa| + n^4)^{-1}\left(\|f\|_{L^2} + \|g\|_{L^2}\right).$$

We conclude, for $u = \alpha\widetilde{\phi}_0 + \beta\widehat{Q}_1 + \eta$,

$$\|u\|_{L^2} \le \left(C + C(|\tau-\kappa| + n^4)^{-1}\right)\left(\|f\|_{L^2} + \|g\|_{L^2}\right).$$

We can estimate v similarly. Thus, for $\tau \in (e_{01} - n, e_{01} + n)$,

$$\|R(i\tau \pm 0)\|_{\mathbf{B}} \le C + C(|\tau-\kappa| + n^4)^{-1}.$$

For $\tau > e_{01} + n$ and $w = i\tau + 0$, using $R(w) = (1 + R_0(w)W)^{-1}R_0(w)$ and the fact that $\|R_0(w)\|_{\mathbf{B}} \le C(1+\tau)^{-1/2}$, (see [7] Theorem 9.2), we have $\|R(i\tau + 0)\|_{\mathbf{B}} \le C\tau^{-1/2}$.

For $\tau \in [|E_1|, e_{01} - n]$, by the same argument we have $\|R(i\tau + 0)\|_{\mathbf{B}} \le C$.

The derivative estimates for the resolvent is obtained by induction argument and by differentiating the relation $R(1 + WR_0) = R_0$ and using the relations $(1+WR_0)^{-1} = 1 - WR$ and $(1 + R_0W)^{-1} = 1 - RW$. See the proof of [7] Theorem 9.2. We have proved Lemma 2.5.

2.3 Nonexistence of generalized ω_*-eigenvector

We now show that ω_* is simple and Φ is the only generalized ω_*-eigenvector, i.e., there is no vectors ϕ with $(\mathcal{L}_1 - \omega_*)\phi \ne 0$ but $(\mathcal{L}_1 - \omega_*)^k\phi = 0$ for some $k \ge 2$. Suppose the contrary, then we may find a vector $\begin{bmatrix} u \\ v \end{bmatrix}$ with $(\omega_* - \mathcal{L}_1)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_* \\ v_* \end{bmatrix}$. That is, $w = \omega_*$ and $\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} u_* \\ v_* \end{bmatrix}$ in the system (2.50). We have $F = wu_* + Hv_* = 2\omega_* u_*$. Since $u_* = \widetilde{\phi}_0 + \bar{h}_*$ with $\bar{h}_* \in \mathbf{H}_{\rm c}(H)$, we have $(\widehat{Q}_1, \widetilde{F}) = (\widehat{Q}_1, F) = (\widehat{Q}_1, 2\omega_* u_*) = 0$. Hence $\beta = 0$. Also

$$\begin{aligned}(\widetilde{\phi}_0, \widetilde{F}) &= (\widetilde{\phi}_0, F) - (\widetilde{\phi}_0 H 2\lambda Q_1^2\left(w^2 + H^2 + \mathbf{P}_{\rm c} A\, \mathbf{P}_{\rm c}\right)^{-1}\mathbf{P}_{\rm c} F)\\ &= 2\omega_* + \rho(\widetilde{\phi}_0 2\lambda Q_1^2\left(w^2 + H^2 + \mathbf{P}_{\rm c} A\, \mathbf{P}_{\rm c}\right)^{-1} 2\omega_* \bar{h}_*)\\ &= 2\omega_*\left[1 + \rho(\Phi, \Omega\overline{\Omega} H\Phi)\right],\end{aligned}$$

where $\Omega = \left(w^2 + H^2 + \mathbf{P}_{\rm c} A\, \mathbf{P}_{\rm c}\right)^{-1}$ and $\Phi = \mathbf{P}_{\rm c}\widetilde{\phi}_0 2\lambda Q_1^2$. Since the main term in $(\Phi, \Omega\overline{\Omega}H\Phi)$,

$$(\Phi, (w^2 + H^2)^{-1}(\bar{w}^2 + H^2)^{-1}H\Phi),$$

is positive, $(\widetilde{\phi}_0, \widetilde{F})$ is not zero.

On the other hand, $a = \omega_*^2 + f(-\omega_*^2) = -\bar{z}_* + f(\bar{z}_*) = 0$. Hence there is no solution for α. This shows ω_* is simple (and so are $-\omega_*, \pm\bar{\omega}_*$).

Once we have an eigenvector Φ with $\mathcal{L}_1\Phi = \omega_*\Phi$ and ω_* complex, then we have three other eigenvalues and eigenvectors as given in (2.19). Hence we have found all eigenvalues and eigenvectors of $\mathcal{L}_1$. $\mathbb{C}\mathbf{E}_1$ is the combined eigenspace of $\pm\omega_*$ and $\pm\bar{\omega}_*$. It is easy to check that $\mathbf{RE}\,\mathbb{C}\mathbf{E}_1 = \mathbf{E}_1$. We have proved parts (1)–(3) of Theorem 2.2 in §2.1 to §2.3.

2.4 Nonexistence of embedded eigenvalues

In this subsection we prove that there is no embedded eigenvalue $i\tau$ with $|\tau| > |E_1|$. Suppose the contrary, we may assume $\tau > -E_1 > 0$ and $\mathcal{L}_1\psi = i\tau\psi$ for some $\psi \in \mathbb{C}L^2$. We will derive a contradiction.

Let $H_* = -\Delta - E_1$. We can decompose

$$\mathcal{L}_1 = JH_* + A, \qquad A = \begin{bmatrix} 0 & V + \lambda Q_1^2 \\ -V - 3\lambda Q_1^2 & 0 \end{bmatrix}. \tag{2.55}$$

Hence $(i\tau - JH_*)\psi = A\psi$. By the same computation of (2.46) we have

$$(w - JH_*)^{-1} = (H_* - iw)^{-1}M_+ + (H_* + iw)^{-1}M_-,$$

where

$$M_+ = \frac{1}{2}\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}, \qquad M_- = \frac{1}{2}\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}.$$

Thus, with $w = i\tau$, we have

$$\psi = (i\tau - JH_*)^{-1}A\psi = (H_* + \tau)^{-1}\phi_+ + (H_* - \tau)^{-1}\phi_-, \tag{2.56}$$

where $\phi_+ = M_+A\psi$ and $\phi_- = M_-A\psi$. By Assumption A1 on the decay of V and that $\psi \in L^2$, both $\phi_+, \phi_- \in L^2_{5+\sigma}$ with $\sigma > 0$. Since $-\tau$ is outside the spectrum of H_*, we have $(H_* + \tau)^{-1}\phi_+ \in L^2_{5+\sigma}$. Let $s = E_1 + \tau > 0$. We have $H_* - \tau = -\Delta - s$. By assumption $\psi \in \mathbb{C}L^2$, hence so is $(H_* - \tau)^{-1}\phi_-$. Therefore $(p^2 - s)^{-1}\widehat{\phi_-}(p) \in L^2$. Since $\phi_- \in L^2_{5+\sigma}$, ϕ_- is continuous and we can conclude

$$\widehat{\phi_-}(p)\big|_{|p|=\sqrt{s}} = 0. \tag{2.57}$$

We now recall [11] page 82, Theorem IX.41: Suppose $f \in L^2_r$ with $r > 1/2$ and let $B_s f = \left((p^2 - s)^{-1} \widehat{f}\right)^{\vee}$. Suppose $\hat{f}(p)\big|_{|p|=\sqrt{s}} = 0$. Then for any $\varepsilon > 0$, one has $B_s f \in L^2_{r-1-2\varepsilon}$ and $\|B_s f\|_{L^2_{r-1-2\varepsilon}} \le C_{r,\varepsilon,s} \|f\|_{L^2_r}$ for some constant $C_{r,\varepsilon,s}$.

In our case, we have $f = \phi_-$, $\varepsilon = \sigma/2$ and $r = 5 + \sigma$. We conclude $(H_* - \tau)^{-1}\phi_- = B_s f \in L^2_4$. Thus $\psi \in L^2_4$.

However, since $(z - \mathcal{L}_1)\psi = (z - i\tau)\psi$, we have $R(z)\psi = (z - i\tau)^{-1}\psi$. Thus we have

$$\left\|(z - i\tau)^{-1}\psi\right\|_{L^2_{-r}} \le C \|\psi\|_{L^2_4},$$

where the constant C remains bounded as $z \to i\tau$ by Lemma 2.5. This is clearly a contradiction. Thus ψ does not exist.

2.5 Absence of eigenvector and resonance at bottom of continuous spectrum

In this subsection we show that there is no eigenvector and resonance at $\pm iE_1$. We want to show that, for $n = \|Q_{1,E_1}\|_{L^2}$ sufficiently small, the null space of $\mathcal{L}_1 \pm iE_1$ in $X = L^2_{-r}$, $r > 1/2$, is zero. Let us consider the case at $i|E_1|$. Suppose otherwise, we have a sequence $Q_{1,E_1(k)} \to 0$ and ψ_k so that

$$\left(\mathcal{L}_{1,E_1(k)} + iE_1(k)\right)\psi_k = 0, \quad \|\psi_k\|_X = 1.$$

As in the previous subsection, we write $\mathcal{L}_{1,E_1(k)} = JH_* + A_k$, where $H_* = -\Delta - E_1(k)$ and $A_k = JV + \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} \lambda Q^2_{1,E_1(k)}$. By (2.56) with $\tau = |E_1(k)|$ we have

$$\psi_k = (i\tau - JH_*)^{-1}A_k\psi_k = (-\Delta + 2\tau)^{-1}M_+A_k\psi_k + (-\Delta)^{-1}M_-A_k\psi_k$$

in X. Note that $(-\Delta+2\tau)^{-1}M_+A_k$ and $(-\Delta)^{-1}M_-A_k$ are compact operators in X, with a bound uniform in k. Since X is a reflexive Banach space, we can find a subsequence, which we still denote by ψ_k, converging weakly to some $\psi_* \in X$. Thus $\tau \to |e_1|$, $(-\Delta+2\tau)^{-1}M_+A_k\psi_k \to (-\Delta-2e_1)^{-1}M_+JV\psi_*$ and $(-\Delta)^{-1}M_-A_k\psi_k \to (-\Delta)^{-1}M_+JV\psi_*$ strongly in X. Thus

$$\psi_* = (-\Delta - 2e_1)^{-1}M_+JV\psi_* + (-\Delta)^{-1}M_+JV\psi_*$$

and $\psi_k \to \psi_*$ strongly. Hence $\|\psi_*\|_X = \lim \|\psi_k\|_X = 1$ and $(JH_1 + ie_1)\psi_* = 0$ by (2.56) again. This contradiction to our assumption shows our claim.

Another proof:

We will use the resolvent estimates Lemma 2.5 to give a proof, without using that $\mathcal{L}_1$ is a perturbation of JH_1. Suppose the contrary that we have $\psi \in X$ which satisfies $\psi \neq 0$, $\mathcal{L}_1\psi = i\tau\psi$, with $\tau = |E_1|$. Write $\mathcal{L}_1 = JH_* + A$ as before and let $R(z) = (z - \mathcal{L}_1)^{-1}$ and $R_0(z) = (z - JH_*)^{-1}$. We have $(i\tau - JH_*)\psi = A\psi$, hence

$$\psi = R_0(i\tau)A\psi \qquad \text{in } X. \tag{2.58}$$

Let $w = \sigma_1 A\psi = A^*\sigma_1\psi$. (Note $A^* = \sigma_1 A \sigma_1$). We have that $Vw \in L^2_r$. By Lemma 2.5 the L^2_{-r}-norm of $R(z)Vw$ is uniformly bounded as $z \to i\tau$. We will derive a contradiction.

Recall the resolvent identity $R(z)A = -1 - (1 - R_0(z)A)^{-1}$. Hence $\rho(z) \equiv (1 - R_0(z)A)^{-1}w = R(z)Aw + w$ is also uniformly bounded in L^2_{-r} as $z \to i\tau$.

Recall [7] Lemma 2.3 that $(-\Delta - z)^{-1} = (-\Delta)^{-1} + O(\sqrt{|z|})$ in $\mathcal{B}\left(H^{-1}_s, H^1_{-s}\right)$ for $s > 1$. Hence for z near $i\tau$ we have by (2.56)

$$R_0(z) = R_0(i\tau) + O(\sqrt{|z - i\tau|})$$

in $\mathcal{B}\left(H^{-1}_s, H^1_{-s}\right)$. Therefore

$$\begin{aligned}(w, w) &= (A^*\sigma_1\psi,\ (1 - R_0(z)A)\rho(z)) \\ &= (A^*\sigma_1\psi,\ [1 - R_0(i\tau)A]\rho(z)) + (A^*\sigma_1\psi, O(\sqrt{|z - i\tau|})A\rho(z)).\end{aligned}$$

Since $\sigma_1 J = -J\sigma_1$, $\sigma_1(z - JH_*) = (z + JH_*)\sigma_1 = (z - JH_*)^*\sigma_1$. Hence $\sigma_1 R_0(i\tau) = R_0(i\tau)^*\sigma_1$ and we have by (2.58)

$$\sigma_1\psi = \sigma_1 R_0(i\tau)A\psi = R_0(i\tau)^*\sigma_1 A\psi = R_0(i\tau)^* A^*\sigma_1\psi.$$

Hence

$$(A^*\sigma_1\psi,\ R_0(i\tau)A\rho(z)) = (A^* R_0(i\tau)^* A^*\sigma_1\psi,\ \rho(z)) = (A^*\sigma_1\psi,\ \rho(z)).$$

Hence $(A^*\sigma_1\psi,\ [1 - R_0(i\tau)A]\rho(z)) = 0$. Since $\rho(z)$ is uniformly bounded in L^2_r, we have

$$(w, w) = (A^*\sigma_1\psi, O(\sqrt{|z - i\tau|})A\rho(z)) = O(\sqrt{|z - i\tau|})$$

as $z \to i\tau$. Thus $w = 0$. Hence $(i\tau - JH_*)\psi = 0$. If we write $\psi = \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, then

$$i\tau u - (-\Delta + \tau)v = 0, \quad (-\Delta + \tau)u + i\tau v = 0.$$

One gets $\Delta(u - iv) = 0$ immediately. Since $u, v \in X$, we conclude $u = iv$ and $(-\Delta + 2\tau)u = 0$. Hence $u, v = 0$. This finishes the proof.

2.6 Proof of Theorem 2.2 (4)–(6)

We first show the orthogonality conditions. Recall $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It is self-adjoint in $\mathbb{C}L^2$. Let $\mathcal{L}_1^*$ be the adjoint of $\mathcal{L}_1$ in $\mathbb{C}L^2$. We have $\mathcal{L}_1^* = \begin{bmatrix} 0 & -L_+ \\ L_- & 0 \end{bmatrix}$ and $\mathcal{L}_1^* = \sigma_1 \mathcal{L}_1 \sigma_1$.

Suppose $\mathcal{L}_1 f = \omega_1 f$ and $\mathcal{L}_1 g = \omega_2 g$ with $\bar{\omega}_1 \neq \omega_2$. We have $\mathcal{L}_1^* \sigma_1 f = \sigma_1 \mathcal{L}_1 f = \omega_1 \sigma_1 f$ and hence

$$\omega_2(\sigma_1 f, g) = (\sigma_1 f, \omega_2 g) = (\sigma_1 f, \mathcal{L}_1 g) = (\mathcal{L}_1^* \sigma_1 f, g) = (\omega_1 \sigma_1 f, g) = \bar{\omega}_1(\sigma_1 f, g).$$

Hence we must have $(\sigma_1 f, g) = 0$. Therefore we have $\sigma_1 \bar{\Phi} \perp \bar{\Phi}, \sigma_3 \Phi, \sigma_3 \bar{\Phi}$, $\sigma_1 \Phi \perp \Phi, \sigma_3 \Phi, \sigma_3 \bar{\Phi}$, etc. If we write $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Phi = \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right]$, then we have

$$\int \bar{u} v \, dx = 0. \tag{2.59}$$

In other words, $(u_1, v_1) + (u_2, v_2) = 0$ and $(u_1, v_2) = (u_2, v_1)$.

If $f \in S(\mathcal{L}_1)$ and $\mathcal{L}_1 g = \omega_2 g$ with $\omega_2 \neq 0$. We have $(\mathcal{L}_1^*)^2 \sigma_1 f = 0$, hence

$$(\sigma_1 f, \omega_2^2 g) = (\sigma_1 f, \mathcal{L}_1^2 g) = ((\mathcal{L}_1^*)^2 \sigma_1 f, g) = (0, g).$$

Hence $(\sigma_1 f, g) = 0$. In terms of components, we get $(Q_1, u_1) = (Q_1, u_2) = 0$, $(R_1, v_1) = (R_1, v_2) = 0$. The above shows (2.22). The rest of (4) and (5) follows directly.

To prove (6), we first prove the following spectral gap

$$L_+|_{\{Q_1, v_1, v_2\}^\perp} > \tfrac{1}{2}|e_1| \ , \qquad L_-|_{\{R_1, u_1, u_2\}^\perp} > \tfrac{1}{2}|e_1| \ . \tag{2.60}$$

We will show the first assertion. Note that by (2.17) we have

$$v_1 = \mathbf{P}_c(L_-) v_1 + O(n^2), \qquad v_2 = -\phi_0 + \mathbf{P}_c(H_1) v_2 + O(n^2)$$

in L^2. In particular $\|v_2\|_{L^2} \geq 1/2$, and $(v_1, L_- v_1) \geq (v_1, L_- \mathbf{P}_c(L_-) v_1) - Cn^2 \geq -Cn^2$. By (2.59)

$$(v_1, L_- v_1) + (v_2, L_- v_2) = (v, L_- v) = (v, \omega u) = 0.$$

Hence $(v_2, L_+ v_2) = (v_2, L_- v_2) + O(n^2) \leq Cn^2$. We also have $(Q_1, L_+ Q_1) = (Q_1, L_- Q_1) + O(n^4) = 0 + O(n^4)$. Let $Q_1' = Q_1 - (Q_1, v_2) v_2 / \|v_2\|_2^2$. We have $Q_1' \perp v_j$ and $Q_1' = Q_1 + O(n^3)$ by (2.17) again. Hence $(Q_1', L_+ Q_1') =$

$(Q_1, L_+Q_1)+O(n^4) = O(n^4) \le Cn^2(Q_1', Q_1')$. We conclude that $L_+|_{\text{span}\{Q_1,v_2\}} \le Cn^2$. Since L_+ is a perturbation of H_1, it has exactly two eigenvalues below $\frac{1}{2}|e_1|$. By minimax principle we have $L_+|_{\{Q_1,v_2\}^\perp} > \frac{1}{2}|e_1|$. This shows the first assertion of (2.60). The second assertion is proved similarly.

Let $\mathbf{Q}(\psi)$ denote the quadratic form: (see e.g. [22, 23])

$$\mathbf{Q}(\psi) = (f, L_+f) + (g, L_-g) \ , \qquad \text{if } \psi = f + ig \ . \tag{2.61}$$

One can show for any $\psi \in L^2$

$$\mathbf{Q}(e^{t\mathcal{L}_1}\psi) = \mathbf{Q}(\psi) \ , \qquad \text{for all } t, \tag{2.62}$$

by direct differentiation in t. By (2.60) one has

$$\mathbf{Q}(\eta) \sim \|\eta\|_{H^1}^2 \ , \qquad \text{for any } \eta \in \mathbf{H}_{\text{c}}(\mathcal{L}_1).$$

Thus

$$\left\|e^{t\mathcal{L}_1}\eta\right\|_{H^1}^2 \sim \mathbf{Q}(e^{t\mathcal{L}_1}\eta) = \mathbf{Q}(\eta) \sim \|\eta\|_{H^1}^2 \, .$$

Similarly, we have by (2.60) and the above relation

$$\|\eta\|_{H^3}^2 \sim \|\mathcal{L}_1\eta\|_{H^1}^2 \sim \mathbf{Q}(\mathcal{L}_1\eta).$$

Since $\mathbf{Q}(\mathcal{L}_1\eta) = \mathbf{Q}(e^{t\mathcal{L}_1}\mathcal{L}_1\eta)$, we have $\|\eta\|_{H^3} \sim \left\|e^{t\mathcal{L}_1}\eta\right\|_{H^3}$. By interpolation we have $\|\eta\|_{H^2} \sim \left\|e^{t\mathcal{L}_1}\eta\right\|_{H^2}$. We have proven (6).

2.7 Wave operator and decay estimate

It remains to prove the decay estimate (7). We will use the wave operator. We will compare $\mathcal{L}_1$ with JH_*, where $H_* = -\Delta - E_1$. Recall we write $\mathcal{L}_1 = JH_* + A$ in §2.4, (2.55). Keep in mind that H_* has no bound states and A is local. Define $W_+ = \lim_{t\to+\infty} e^{-t\mathcal{L}_1}e^{tJH_*}$. Let $R(z) = (z - \mathcal{L}_1)^{-1}$ and $R_*(z) = (z - JH_*)^{-1}$. We have

$$\begin{aligned}
&W_+f - f \\
&= \lim_{\varepsilon\to0+} \int_{|E_1|}^{+\infty} R(i\tau + \varepsilon)A\left[R_*(i\tau - \varepsilon) - R_*(i\tau + \varepsilon)\right] f \, d\tau \\
&- \lim_{\varepsilon\to0+} \int_{|E_1|}^{+\infty} R(-i\tau + \varepsilon)A\left[R_*(-i\tau - \varepsilon) - R_*(-i\tau + \varepsilon)\right] f \, d\tau.
\end{aligned}$$

Yajima [24, 25] was the first to give a general method for proving the $(W^{k,p}, W^{k,p})$ estimates for the wave operators of self-adjoint operators. This method was extended by Cuccagna [4] to non-selfadjoint operators in the form we are considering. (He also used idea from Kato [9]). One key ingredient in this approach is the resolvent estimates near the continuous spectrum, which in many cases can be obtained by the Jensen-Kato [7] method. (See [24] Lemmas 3.1 and 3.2 and [4] Lemmas 3.9 and 3.10). In our current setting, this estimate is provided by the Lemma 2.5. We can thus follow the proof of [4] to obtain that W_+ is an operator from $\mathbb{C}L^2$ onto $\mathbf{H}_c(\mathcal{L}_1)$. Furthermore, W_+ and its inverse (restricted to $\mathbf{H}_c(\mathcal{L}_1)$) are bounded in (L^p, L^p)-norm for any $p \in [1, \infty]$. (Note this bound depends on n since our bound on $R(w)$ depends on n.) By the intertwining property of the wave operator we have

$$e^{t\mathcal{L}_1}\, \mathbf{P}_c = W_+ e^{tJH_*} (W_+)^* \, \mathbf{P}_c.$$

The decay estimate in (7) follows from the decay estimate of e^{tJH_*}.

The proof of Theorem 2.2 is complete.

2.8 Proof of Theorem 2.1

By the same Cauchy integral argument as in Subsection 2.2, the only eigenvalues of $\mathcal{L}_1$ are inside the disks $\{w : |w| < \sqrt{n}\}$, $\{w : |w - ie_{01}| < \sqrt{n}\}$ and $\{w : |w + ie_{01}| < \sqrt{n}\}$. Moreover, their dimensions are 2, 1 and 1, respectively, the same as that of JH_1. It counts the dimension of (generalized) eigenspaces of $\mathcal{L}_1$ in $\mathbb{C}L^2$. It also counts the dimensions of the restriction of these spaces in $L^2 = L^2(\mathbb{R}^3, \mathbb{R}^2)$ as a real-valued vector space.

By (2.9), we already have two generalized eigenvectors near 0. Hence we have everything near 0.

Since the dimension is 1 near ie_{01}, there is only a simple eigenvalue ω_* near ie_{01}. We have $\omega_* = ie_{01} + O(n^2)$ since the difference between $\mathcal{L}_1$ and JH_1 is of order $O(n^2)$. ω_* has to be purely imaginary, otherwise $-\bar{\omega}_*$ is another eigenvalue near ie_{01}, cf. (2.19), and the dimension can not be 1. (This also follows from the Theorem of Grillakis.)

By the same arguments in §2.2-2.4 we can prove resolvent estimates and the non-existence of embedded eigenvalues. Also, the bottoms of the continuous spectrum are not eigenvalue nor resonance.

Let Φ be an eigenvector corresponding to ω_*. Since $\mathcal{L}_1\Phi = \omega_*\Phi$ and $\bar{\omega}_* = -\omega_*$, we have $\mathcal{L}_1\bar{\Phi} = -\omega_*\bar{\Phi}$. Hence the (unique) eigenvalue near $-ie_{01}$ is $-\omega_*$ with eigenvector $\bar{\Phi}$. Write $\Phi = \left[\begin{smallmatrix} u \\ -iv \end{smallmatrix}\right]$. We may assume u is real. Writing out $\mathcal{L}_1\Phi = i\kappa\Phi$ we get $L_-v = -\kappa u$ and $L_+u = -\kappa v$. Hence v is also real. We can normalize u so that $(u, v) = 1$ or -1. Since Φ is a perturbation of $\left[\begin{smallmatrix} \phi_0 \\ -i\phi_0 \end{smallmatrix}\right]$, we have $(u, v) = 1$.

With this choice of u, v, let $\mathbb{C}\mathbf{E}_1$ and $\mathbf{E}_1$ be defined as in (2.12). $\mathbb{C}\mathbf{E}_1$ is the combined eigenspace corresponding to $\pm\omega_*$. Clearly $\mathbf{RE}\ \mathbb{C}\mathbf{E}_1 \subset \mathbf{E}_1$. Since

$$a\begin{bmatrix} u \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ v \end{bmatrix} = \mathbf{RE}\ \alpha\Phi, \qquad \alpha = a + bi,$$

we have $\mathbf{RE}\ \mathbb{C}\mathbf{E}_1 = \mathbf{E}_1$. That the choice of α is unique can be checked directly. The statement that if $\zeta = \mathbf{RE}\ \alpha\Phi$ then $\mathcal{L}_1\zeta = \mathbf{RE}\ \omega_*\alpha\Phi$ and $e^{t\mathcal{L}_1}\zeta = \mathbf{RE}\ e^{t\omega_*}\alpha\Phi$ is clear. We have proved (3) and (4).

Clearly, $S(\mathcal{L}_1)$, $\mathbf{E}_1(\mathcal{L}_1)$ and $\mathbf{H}_c(\mathcal{L}_1)$ defined as in (2.9), (2.12) and (2.10) are invariant subspaces of L^2 under $\mathcal{L}_1$, and we have the decomposition (2.7). This is (2).

For (5), note that (2.10) is by definition. For (2.11), we have

$$(Q_1, u) = (Q_1, (-\kappa)^{-1}L_-v) = (L_-Q_1, (-\kappa)^{-1}v) = 0,$$
$$(R_1, v) = (R_1, (-\kappa)^{-1}L_+u) = (-\kappa)^{-1}(L_+R_1, u) = (-\kappa)^{-1}(Q_1, u) = 0.$$

(2.14) comes from the orthogonal relations directly.

The first statement of (6) is because of (5). For the rest of (6), We first prove the following spectral gap

$$L_+|_{\{Q_1,v\}^\perp} > \tfrac{1}{2}|e_1|\ , \qquad L_-|_{\{R_1,u\}^\perp} > \tfrac{1}{2}|e_1|\ . \tag{2.63}$$

Since L_+ is a perturbation of H_1, it has exactly two eigenvalues below $\frac{1}{2}|e_1|$. Notice that $(Q_1, L_+Q_1) = (Q_1L_-Q_1) + O(n^4) = O(n^4)$ and $(v, L_+v) = (v, -\kappa u) = -\kappa$. Since $Q_1 = n\phi_1 + O(n^3)$ and $v = \phi_0 + O(n^2)$, one has $(Q_1, v) = O(n^3)$. Thus one can show $L_+|_{\mathrm{span}\{Q_1,v\}} \le Cn^2$. If there is a $\phi \perp Q_1, v$ with $(\phi, L_+\phi) \le \frac{1}{2}|e_1|(\phi, \phi)$, then we have $L_+|_{\mathrm{span}\{Q_1,v,\phi\}} \le \frac{1}{2}|e_1|$, which contradicts with the fact that L_+ has exactly two eigenvalues below $\frac{1}{2}|e_1|$ by minimax principle. This shows the first part of (2.63). The second part is proved similarly.

Recall the quadratic form $\mathbf{Q}(\psi)$ defined in (2.61) in §2.6. Also recall (2.62) that $\mathbf{Q}(e^{t\mathcal{L}_1}\psi) = \mathbf{Q}(\psi)$ for all t and all $\psi \in L^2$. By the spectral gap (2.63) one has

$$\mathbf{Q}(\eta) \sim \|\eta\|_{H^1}^2 , \quad \mathbf{Q}(\mathcal{L}_1\eta) \sim \|\eta\|_{H^3}^2 , \qquad \text{for any } \eta \in \mathbf{H}_{\rm c}(\mathcal{L}_1) . \tag{2.64}$$

For $\psi \in M_1$, we can write $\psi = \zeta + \eta$, where $\zeta = \mathbf{RE}\ \alpha\Phi$, $\alpha \in \mathbb{C}$ and $\eta \in \mathbf{H}_{\rm c}(\mathcal{L}_1)$. Notice that, by orthogonality in (2.10),

$$\mathbf{Q}(\psi) = -|\alpha|^2\kappa(u,v) + \mathbf{Q}(\eta) ,$$

which is not positive definite, (recall $(u,v) = 1$). However,

$$\|\psi\|_{H^1}^2 \sim |\alpha|^2 + \|\eta\|_{H^1}^2 . \tag{2.65}$$

To see it, one first notes that $\|\psi\|_{H^1}^2$ is clearly bounded by the right side. Because of (2.14), one has $|\alpha|^2 \le C\,\|\psi\|_{H^1}^2$. One also has $\|\eta\|_{H^1}^2 \le C\,\|\phi\|_{H^1}^2 + C|\alpha|^2$. Hence (2.65) is true.

Therefore for $\psi = (\,\mathbf{RE}\ \alpha\Phi) + \eta$ we have

$$\begin{aligned}
\left\|e^{t\mathcal{L}_1}\psi\right\|_{H^1}^2 &\sim \left\|e^{t\mathcal{L}_1}\ \mathbf{RE}\ \alpha\Phi\right\|_{H^1}^2 + \left\|e^{t\mathcal{L}_1}\eta\right\|_{H^1}^2 && \text{(by (2.65))}\\
&\sim |e^{-it\omega_*}\alpha|^2 + \mathbf{Q}(e^{t\mathcal{L}_1}\eta) && \text{(by (4), (2.64))}\\
&\sim |\alpha|^2 + \mathbf{Q}(\eta) && \text{(by (2.62))} .
\end{aligned}$$

Hence we have $\left\|e^{t\mathcal{L}_1}\psi\right\|_{H^1}^2 \sim \|\psi\|_{H^1}^2$ for all t. By an argument similar to that in §2.6, we have $\left\|e^{t\mathcal{L}_1}\psi\right\|_{H^k} \sim \|\psi\|_{H^k}$ for $k = 3, 2$. We have shown (6).

The decay estimate in (7) is obtained as in Theorem 2.2 (7). The constant C, however, is independent of n in the non-resonant case. The proof of Theorem 2.1 is complete.

3 Solutions converging to excited states

In this section we prove Theorem 1.1 using Theorems 2.1 and 2.2. Since the proof for the non-resonant case is easier, we will first prove the resonant case and then sketch the non-resonant case. Note that we could follow the approach of Theorem 1.5 of [20] if we had the transform $\mathcal{L}_1\,\mathbf{P}_{\rm c}^{\mathcal{L}_1} = -U^{-1}iAU\,\mathbf{P}_{\rm c}^{\mathcal{L}_1}$ as in [20]. However, it is not easy to define A and U for $\mathcal{L}_1$ and hence we choose another approach. This approach also gives another proof for Theorem 1.5 of [20].

Fix E_1 and $Q_1 = Q_{1,E_1}$. Let $\mathcal{L}_1$ be the corresponding linearized operator, and $\mathbf{P}_{M_1}$, $\mathbf{P}_{\mathbf{E}_1}$ and $\mathbf{P}_c^{\mathcal{L}_1}$ the corresponding projections with respect to $\mathcal{L}_1$. For any $\xi_\infty \in \mathbf{H}_c(\mathcal{L}_1)$ with small $H^2 \cap W^{2,1}$ norm, we want to construct a solution $\psi(t)$ of the nonlinear Schrödinger equation (1.1) with the form

$$\psi(t) = [Q_1 + a(t)R_1 + h(t)]\, e^{-iE_1 t + i\theta(t)} \ ,$$

where $a(t), \theta(t) \in \mathbb{R}$ and $h(t) \in M_1 = \mathbf{E}_1 \oplus \mathbf{H}_c(\mathcal{L}_1)$. Substituting the above ansatz into (1.1) and using $\mathcal{L}_1 iQ_1 = 0$ and $\mathcal{L}_1 R_1 = -iQ_1$, we get

$$\partial_t h = \mathcal{L}_1 h + i^{-1}F(aR_1 + h) - i\dot\theta(Q_1 + aR_1 + h) - aiQ_1 - \dot a R_1,$$

where

$$F(k) = \lambda Q_1(2|k|^2 + k^2) + \lambda |k|^2 k, \qquad k = aR_1 + h. \tag{3.1}$$

The condition $h(t) \in M_1$ can be satisfied by requiring that $h(0) \in M_1$ and

$$\dot a = (c_1 Q_1, \, \mathrm{Im}(F + \dot\theta h)), \tag{3.2}$$

$$\dot\theta = -[a + (c_1 R_1, \, \mathrm{Re}\, F)] \cdot [1 + (c_1 R_1, R_1)a + (c_1 R_1, \mathrm{Re}\, h)]^{-1}, \tag{3.3}$$

where $c_1 = (Q_1, R_1)^{-1}$ and $F = F(aR_1 + h)$. The equation for h becomes

$$\partial_t h = \mathcal{L}_1 h + P_M F_{\mathrm{all}}, \qquad F_{\mathrm{all}} = i^{-1}(F + \dot\theta(aR_1 + h)).$$

The proofs of the two cases diverge here. For the resonant case we decompose, using the decomposition of M_1 and (2.20) of Theorem 2.2,

$$h(t) = \zeta(t) + \eta(t), \qquad \zeta(t) = \mathbf{RE}\,\{\alpha(t)\Phi + \beta(t)\sigma_3\Phi\},$$

where $\alpha(t), \beta(t) \in \mathbb{C}$ and $\eta(t) \in \mathbf{H}_c(\mathcal{L}_1)$. Note

$$\mathcal{L}_1 \zeta = \mathbf{RE}\,\{\omega_* \alpha \Phi - \omega_* \beta \sigma_3 \Phi\}.$$

Recall $\omega_* = i\kappa + \gamma$ with $\kappa, \gamma > 0$. Taking the projections P_α and P_β defined in (2.24) of Theorem 2.2 of the h-equation, we have

$$\dot\alpha = \omega_* \alpha + P_\alpha F_{\mathrm{all}}, \tag{3.4}$$

$$\dot\beta = -\omega_* \beta + P_\beta F_{\mathrm{all}}. \tag{3.5}$$

Taking projection $\mathbf{P}_c^{\mathcal{L}_1}$ we get the equation for η,

$$\partial_t \eta = \mathcal{L}_1 \eta + \mathbf{P}_c^{\mathcal{L}_1} i^{-1}\dot\theta \eta + \mathbf{P}_c^{\mathcal{L}_1} \widetilde F, \qquad \widetilde F = i^{-1}(F + \dot\theta(aR_1 + \zeta)).$$

We single out $\mathbf{P}_c^{\mathcal{L}_1} i^{-1}\dot{\theta}\eta$ since it is a global linear term in η and cannot be treated as error. Let

$$\widetilde{\eta} = \mathbf{P}_c^{\mathcal{L}_1} e^{i\theta}\eta.$$

Note $\eta = \widetilde{\eta} + \mathbf{P}_c^{\mathcal{L}_1}(1-e^{i\theta})\eta$ and $\mathbf{P}_c(1-e^{i\theta})$ is a bounded map from $\mathbf{H}_c(\mathcal{L}_1)\cap H^2$ into itself with its norm bounded by $C|\theta|$. Hence if θ is sufficiently small, we can solve η in terms of $\widetilde{\eta}$ by expansion:

$$\eta = U_\theta \widetilde{\eta}, \qquad U_\theta \equiv \sum_{j=0}^{\infty} [\, \mathbf{P}_c(1-e^{i\theta})]^j. \tag{3.6}$$

The equation for $\widetilde{\eta}$ is

$$\begin{aligned}\partial_t \widetilde{\eta} &= \mathbf{P}_c^{\mathcal{L}_1} e^{i\theta}(i\dot{\theta}\eta + \partial_t \eta) \\ &= \mathcal{L}_1 \widetilde{\eta} + \left\{ \mathbf{P}_c^{\mathcal{L}_1} e^{i\theta}\mathcal{L}_1 - \mathcal{L}_1 \mathbf{P}_c^{\mathcal{L}_1} e^{i\theta} \right\} \eta \\ &\quad + \mathbf{P}_c^{\mathcal{L}_1} e^{i\theta}\left\{ i\dot{\theta}\eta - \dot{\theta}\, \mathbf{P}_c^{\mathcal{L}_1} i\eta + \mathbf{P}_c^{\mathcal{L}_1}\widetilde{F} \right\}\end{aligned}$$

Note that $i\dot{\theta}\eta - \dot{\theta}\,\mathbf{P}_c^{\mathcal{L}_1} i\eta = (1-\mathbf{P}_c^{\mathcal{L}_1}) i\dot{\theta}\eta$ and

$$\begin{aligned}\left\{ \mathbf{P}_c^{\mathcal{L}_1} e^{i\theta}\mathcal{L}_1 - \mathcal{L}_1 \mathbf{P}_c^{\mathcal{L}_1} e^{i\theta} \right\} \eta &= \mathbf{P}_c^{\mathcal{L}_1}[e^{i\theta}, \mathcal{L}_1]\eta = \mathbf{P}_c^{\mathcal{L}_1} \sin\theta [i, \mathcal{L}_1]\eta \\ &= \mathbf{P}_c^{\mathcal{L}_1} \sin\theta 2\lambda Q_1^2 \bar{\eta}.\end{aligned}$$

Hence we have

$$\partial_t \widetilde{\eta} = \mathcal{L}_1 \widetilde{\eta} + \mathbf{P}_c^{\mathcal{L}_1}\left\{ \sin\theta 2\lambda Q_1^2 \bar{\eta} + e^{i\theta}(1-\mathbf{P}_c^{\mathcal{L}_1}) i\dot{\theta}\eta + e^{i\theta}\,\mathbf{P}_c^{\mathcal{L}_1}\widetilde{F} \right\}$$

For a given profile ξ_∞, let

$$\widetilde{\eta}(t) = e^{t\mathcal{L}_1}\xi_\infty + g(t). \tag{3.7}$$

We have the equation

$$\partial_t g = \mathcal{L}_1 g + \mathbf{P}_c^{\mathcal{L}_1}\left\{ \sin\theta 2\lambda Q_1^2 \bar{\eta} + e^{i\theta}(1-\mathbf{P}_c^{\mathcal{L}_1}) i\dot{\theta}\eta + e^{i\theta}\,\mathbf{P}_c^{\mathcal{L}_1}\widetilde{F} \right\}. \tag{3.8}$$

We want $g(t) \to 0$ as $t \to \infty$ in some sense.

Summarizing, we write the solution $\psi(t)$ in the form

$$\begin{aligned}\psi(t) = \Big\{ &Q_1 + a(t)R_1 + \mathbf{RE}\,\{\alpha(t)\Phi + \beta(t)\sigma_3\Phi\} \\ &+ U_{\theta(t)}(e^{t\mathcal{L}_1}\xi_\infty + g(t)) \Big\}\, e^{-iE_1 t + i\theta(t)}\ ,\end{aligned} \tag{3.9}$$

with $a(t)$, $\theta(t)$, $\alpha(t)$, $\beta(t)$ and $g(t)$ satisfying (3.2), (3.3), (3.4), (3.5), and (3.8), respectively.

The main term of F is

$$F_0 = \lambda Q_1 \left(2|\xi|^2 + \xi^2\right) + \lambda|\xi|^2\xi, \qquad \xi(t) = U_{\theta(t)} e^{t\mathcal{L}_1}\xi_\infty.$$

Notice that, if $\|\xi_\infty\|_{H^2\cap W^{2,1}} \le \varepsilon \ll 1$, then $\xi(t)$ satisfies

$$\|\xi(t)\|_{H^2} \le C(n)\varepsilon, \quad \|\xi(t)\|_{W^{2,\infty}} \le C(n)\varepsilon|t|^{-3/2}, \quad \left\||\xi|^2\xi(t)\right\|_{H^2} \le C(n)\varepsilon^3 \langle t\rangle^{-3}.$$

Here we have used the boundedness and decay estimates for $e^{t\mathcal{L}_1}\,\mathbf{P}_c^{\mathcal{L}_1}$ in Theorem 2.2 (6)–(7). Since Q_1 is fixed, it does not matter that the constant depends on n. The main term of F_0 is quadratic in ξ. Hence

$$\|F_0(t)\|_{H^2} \le C\varepsilon^2 \langle t\rangle^{-3}.$$

As it will become clear, we have the freedom to choose ξ_∞ and $\beta_0 = \beta(0)$. We require that $\xi_\infty \in \mathbf{H}_c(\mathcal{L}_1)$ and

$$\|\xi_\infty\|_{H^2\cap W^{2,1}} \le \varepsilon, \qquad |\beta_0| \le \varepsilon^2/4, \tag{3.10}$$

with ε sufficiently small. With given ξ_∞ and β_0, we will define a contraction mapping Ω in the following space

$$\begin{aligned}\mathcal{A} =\big\{&(a,\theta,\alpha,\beta,g) : [0,\infty) \to \mathbb{R}\times\mathbb{R}\times\mathbb{C}\times\mathbb{C}\times(\mathbf{H}_c(\mathcal{L}_1)\cap H^2),\\ &|a(t)|, |\alpha(t)|, |\beta(t)|, \le \varepsilon^{7/4}(1+t)^{-2},\\ &\|g(t)\|_{H^2} \le \varepsilon^{7/4}(1+t)^{-7/4}, \quad |\theta(t)| \le 2\varepsilon^{7/4}(1+t)^{-1} \;\big\}\end{aligned}$$

For convenience, we introduce a variable $b = \dot\theta$. Our map Ω is defined by

$$\Omega : (a,\theta,\alpha,\beta,\eta) \longrightarrow (a^\triangle, \theta^\triangle, \alpha^\triangle, \beta^\triangle, \eta^\triangle)$$

where, with $c_1 = (Q_1, R_1)^{-1}$ and $F = F(aR + h)$ defined in (3.1),

$$
\begin{aligned}
h(t) &= \zeta(t) + \eta(t) \\
\zeta(t) &= \mathbf{RE}\, \{\alpha(t)\Phi + \beta(t)\sigma_3\Phi\}, \qquad \eta(t) = U_{\theta(t)}(e^{t\mathcal{L}_1}\xi_\infty + g(t)) \\
b(t) &= -[a + (c_1R_1,\, \operatorname{Re} F)] \cdot [1 + (c_1R_1, R_1)a + (c_1R_1, \operatorname{Re} h)]^{-1} \\
a^{\triangle}(t) &= \int_\infty^t (c_1Q_1,\, \operatorname{Re}(F + bh))\, ds \\
\theta^{\triangle}(t) &= \int_\infty^t b(s)\, ds \\
\alpha^{\triangle}(t) &= \int_\infty^t e^{\omega_*(t-s)} P_\alpha i^{-1}(F + b(aR + h))\, ds \\
\beta^{\triangle}(t) &= e^{-\omega_* t}\beta_0 + \int_0^t e^{-\omega_*(t-s)} P_\beta i^{-1}(F + b(aR + h))\, ds \\
g^{\triangle}(t) &= \int_\infty^t e^{\mathcal{L}_1(t-s)}\, \mathbf{P}_c^{\mathcal{L}_1} \Big\{ \sin\theta 2\lambda Q_1^2 \bar{\eta} + e^{i\theta}(1 - \mathbf{P}_c^{\mathcal{L}_1}) ib\eta + \\
&\qquad\qquad + e^{i\theta}\, \mathbf{P}_c^{\mathcal{L}_1} i^{-1}(F + b(aR + \zeta)) \Big\}\, ds.
\end{aligned}
$$

We will use Strichartz estimate for the term $\sin\theta 2\lambda Q_1^2\bar{\eta}$ in the g-integral:

$$
\left\| \int_\infty^t e^{\mathcal{L}_1(t-s)}\, \mathbf{P}_c^{\mathcal{L}_1} f(s,\cdot)\, ds \right\|_{L_x^2} \le C(n) \left\{ \int_\infty^t \|f(s,\cdot)\|_{L_x^{r'}}^{q'}\, ds \right\}^{1/q'} \tag{3.11}
$$

for $\frac{3}{r} + \frac{2}{q} = \frac{3}{2}$, $2 < q \le \infty$. Here $'$ means the usual conjugate exponent. Eq. (3.11) can be proved by either using wave operator to map $e^{t\mathcal{L}_1}$ to $e^{-it(-\Delta - E_1)}$, or by using the decay estimate Theorem 2.2 (7) and repeating the usual proof for Strichartz estimate. We will also use

$$
\|\phi\|_{H^2} \sim \|\mathcal{L}_1\phi\|_{L^2} \qquad \text{for } \phi \in \mathbf{H}_c(\mathcal{L}_1),
$$

which follows from the spectral gap (2.60). Since $\sin\theta 2\lambda Q_1^2\bar{\eta}$ is local and bounded by $C(n)\varepsilon^2(1+t)^{-1} \cdot \varepsilon(1+t)^{-3/2}$, by choosing q large we have

$$
\begin{aligned}
&\left\| \int_\infty^t e^{\mathcal{L}_1(t-s)}\, \mathbf{P}_c^{\mathcal{L}_1} \sin\theta 2\lambda Q_1^2\bar{\eta}\, ds \right\|_{H^2} \\
&\le C(n) \left\| \int_\infty^t e^{\mathcal{L}_1(t-s)}\, \mathbf{P}_c^{\mathcal{L}_1} \mathcal{L}_1 \sin\theta 2\lambda Q_1^2\bar{\eta}\, ds \right\|_{L_x^2} \\
&\le C(n) \left\{ \int_\infty^t \varepsilon^3(1+s)^{-(5/2)q'}\, ds \right\}^{1/q'} = C(n)\varepsilon^3(1+t)^{-5/2+1/q'}.
\end{aligned}
$$

In particular, we get $C\varepsilon^3(1+t)^{-7/4}$ by choosing $q = 4$.

Note $|b(t)| \le 2|a(t)|$. Since $t - s < 0$ in the integrand of α, $\operatorname{Re}\omega_*(t-s) < 0$ and the α-integral converges. Similarly $\operatorname{Re}\omega_*(t-s) > 0$ in the integrand of

β and hence the β-integration converges. Observe that we have the freedom of choosing β_0 and ξ_∞. Since $e^{-\omega_* t}\beta_0$ decays exponentially, the main term of $\beta(t)$ when t large is given by F_0, not $e^{-\omega_* t}\beta_0$. Direct estimates show that

$$|\alpha(t)| \le C(n)\varepsilon^2(1+t)^{-3}, \quad |\beta(t)| \le \varepsilon^2 e^{-\gamma t}/4 + C(n)\varepsilon^2(1+t)^{-3},$$

$$|a(t)|, |b(t)| \le C(n)\varepsilon^2(1+t)^{-2}, \quad |\theta(t)| \le C(n)\varepsilon^2(1+t)^{-1},$$

$$\|g(t)\|_{H^2} \le C(n)\varepsilon^2(1+t)^{-7/4}.$$

It is easy to check that the map Ω is a contraction if ε is sufficiently small. Thus we have a fixed point in $\mathcal{A}$, which gives a solution to the system (3.2), (3.3), (3.4), (3.5), and (3.8). Since it lies in $\mathcal{A}$, we also have the desired estimates. We obtain $\alpha(0)$, $a(0)$ and $\theta(0)$ as functions of ξ_∞ and β_0.

Recall $\psi_{\rm as}(t) = Q_1 e^{-iE_1 t + i\theta(t)} + e^{-iE_1 t} e^{t\mathcal{L}_1}\xi_\infty$ and we have

$$\psi(t) = [Q_1 + U_{\theta(t)} e^{t\mathcal{L}_1}\xi_\infty]\, e^{-iE_1 t + i\theta(t)} + O(t^{-7/4}) \qquad \text{in } H^2.$$

Since $\mathbf{P}_c^{\mathcal{L}_1}(1-e^{i\theta}) = O(\theta(t)) = O(t^{-1})$, by the definition (3.6) of U_θ,

$$\begin{aligned} U_{\theta(t)} e^{t\mathcal{L}_1}\xi_\infty &= [1 + \mathbf{P}_c^{\mathcal{L}_1}(1-e^{i\theta})] e^{t\mathcal{L}_1}\xi_\infty + O(t^{-2}) \\ &= (2-e^{i\theta}) e^{t\mathcal{L}_1}\xi_\infty + (1-\mathbf{P}_c^{\mathcal{L}_1})(1-e^{i\theta}) e^{t\mathcal{L}_1}\xi_\infty + O(t^{-2}) \end{aligned}$$

in H^2. Since $(1-\mathbf{P}_c^{\mathcal{L}_1})$ is a local operator, $(1-\mathbf{P}_c^{\mathcal{L}_1})(1-e^{i\theta})e^{t\mathcal{L}_1}\xi_\infty = O(t^{-1}\cdot t^{-3/2})$. Also, $e^{i\theta}(2-e^{i\theta}) = 1 + O(\theta^2) = 1 + O(t^{-2})$. Hence we have $\psi(t) - \psi_{\rm as}(t) = O(t^{-7/4})$ in H^2. We have proven Theorem 1.1 under assumption (R).

We now sketch the proof for the non-resonant case. The only difference is that we define $\zeta(t)$ as $\mathbf{RE}\ \alpha(t)\Phi$ and write $\psi(t)$ in the form

$$\psi(t) = \left\{Q_1 + a(t)R_1 + \mathbf{RE}\ (\alpha(t)\Phi) + U_{\theta(t)}(e^{t\mathcal{L}_1}\xi_\infty + g(t))\right\}\, e^{-iE_1 t + i\theta(t)}.$$

The function $\alpha(t)$ still satisfies (3.4) but with a purely imaginary eigenvalue ω_*. The previous proof will go through if we remove all terms related to β.

4 Appendix: Vanishing solutions

In this appendix we prove Proposition 1.2. Recall $H_0 = -\Delta + V$. The propagator e^{-iH_0t} is bounded in H^s, $s \ge 0$, and satisfies the decay estimate of the form (2.39):

$$\left\| e^{-itH_0} \mathbf{P_c}^{H_0} \phi \right\|_{L^\infty} \le C|t|^{-3/2} \left\| \phi \right\|_{L^1} \tag{4.1}$$

under our assumption A1. See [7, 8, 24].

For any $\xi_\infty \in \mathbf{H}_c(H_0)$ with small $H^2 \cap W^{2,1}$ norm, we want to construct a solution $\psi(t)$ of (1.1) with the form

$$\psi(t) = e^{-iH_0t}\xi_\infty + g(t), \qquad g(t) = \text{error}. \tag{4.2}$$

Let $\xi(t) = e^{-iH_0t}\xi_\infty$. Suppose $\|\xi_\infty\|_{H^2 \cap W^{2,1}} = \varepsilon$, $0 < \varepsilon \le \varepsilon_0$, we have by (4.1),

$$\|\xi(t)\|_{H^2} \le C_1\varepsilon, \|\xi(t)\|_{W^{2,\infty}} \le C_1\varepsilon|t|^{-3/2}, \quad \left\| |\xi|^2\xi(t) \right\|_{H^2} \le C_1\varepsilon^3(1+t)^{-3}$$

for some constant C_1.

The error term $g(t)$ satisfies

$$\partial_t g = -iH_0 g + F$$

with $g(t) \to 0$ as $t \to \infty$ in certain sense, and

$$F(t) = -i\lambda|\psi|^2\psi = -i\left|\xi(t) + g(t)\right|^2 \left(\xi(t) + g(t)\right), \quad \xi(t) = e^{-iH_0t}\xi_\infty. \tag{4.3}$$

We define a solution by (4.3) and

$$g(t) = \int_\infty^t e^{-iH_0(t-s)} F(s)\, ds\ . \tag{4.4}$$

Note that our $g(t)$ belongs to L^2 and is not restricted to the continuous spectrum component of H_0. Also note that the main term in F is $|\xi|^2\xi(t)$, which is of order t^{-3} in H^2. Hence $g(t) \sim t^{-2}$.

We define a contraction mapping in the following class

$$\mathcal{A} = \left\{ g(t) : [0, \infty) \to H^2(\mathbb{R}^3),\ \|h(t)\|_{H^2} \le C_1\varepsilon^3(1+t)^{-2} \right\}\ .$$

This class is not empty since it contains the zero function. We also define the norm

$$\|g\|_{\mathcal{A}} := \sup_{t>0} (1+t)^2 \left\| g(t) \right\|_{H^2}.$$

For $g(t) \in \mathcal{A}$ we define

$$\Omega : g(t) \to g^{\triangle}(t) = -i\lambda \int_{\infty}^{t} e^{-iH_0(t-s)} \left(|\xi + g|^2 (\xi + g) \right)(s)\, ds \ .$$

It is easy to check that

$$\begin{aligned} \left\| g^{\triangle}(t) \right\|_{H^2} &\le \int_t^{\infty} \|F(t)\|_{H^2}\, ds \\ &\le \int_t^{\infty} C_1 \varepsilon^3 \langle s \rangle^{-3} + C\varepsilon^5 \langle s \rangle^{-7/2}\, ds \le C_1 \varepsilon^3 \langle t \rangle^{-2} \ , \end{aligned}$$

if ε_0 is sufficiently small. This shows that the map Ω maps $\mathcal{A}$ into itself. Similarly one can show $\|\Omega g_1 - \Omega g_2\|_{\mathcal{A}} \le \|g_1 - g_2\|_{\mathcal{A}}$, if $g_1, g_2 \in \mathcal{A}$. Therefore our map Ω is a contraction mapping and we have a fixed point. Hence we have a solution $\psi(t)$ of the form (4.2) with $e^{-itH_0}\xi_\infty$ as the main profile.

Remark. The above existence result holds no matter how many bound states H_0 has. The situation is different if we linearize around a nonlinear excited state. In that case, the propagator $e^{t\mathcal{L}_1}$, ($\mathcal{L}_1$ is the linearized operator), may not be bounded in whole L^2.

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