Several Compactness Results in
Gauge Theory and Low Dimensional Topology

Several compactness results in gauge theory and low dimensional topology

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Abstract

This thesis studies several compactness problems in gauge theory and explores their applications is low dimensional topology.

The first chapter studies a connection between taut foliations and Seiberg-Witten theory. Let Y be a closed oriented 3-manifold and $\mathcal{F}$ a smooth oriented foliation on Y. Assume that $\mathcal{F}$ does not admit any transverse invariant measure. This chapter constructs an invariant $c(\mathcal{F})$ for $\mathcal{F}$ which takes value in $\widetilde{HM}_\bullet(Y)$. The invariant is well defined up to a sign. If two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are homotopic through foliations without transverse invariant measure, then $c(\mathcal{F}_1) = c(\mathcal{F}_2)$. The grading of $c(\mathcal{F})$ is represented by the homotopy class of the tangent plane field of $\mathcal{F}$, and its image in $HM_\bullet(Y)$ is nonzero.

The second chapter proves a deformation invariance for the parity of the number of Klein-bottle leaves in a smooth taut foliation. Given two smooth cooriented taut foliations, assume that every Klein-bottle leaf has non-trivial linear holonomy, and assume that the two foliations can be smoothly deformed to each other through taut foliations, then the parities of the number of Klein-bottle leaves are the same.

The third chapter is more analytic in nature. It proves that the zero locus of a $\mathbb{Z}/2$ harmonic spinor on a 4 dimensional manifold is 2-rectifiable and has locally finite Minkowski content.

Contents

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0. Introduction

Gauge theory is an important tool in low dimensional topology. By studying solutions to partial differential equations, such as the Yang-Mills equation or the Seiberg-Witten equations, it gives insight into the topology of three and four-dimensional manifolds beyond classical results.

To apply gauge theory to low dimensional topology, one studies the moduli space of solutions to certain gauge-theoretic PDEs, and use the properties of the moduli space to understand the background manifold. The moduli space itself usually depends on some geometric data such as a Riemannian metric or a perturbation, but in the case when the moduli space is compact or the boundary of the compactification is well-understood, one can extract information from the moduli space that is independent of the geometric data. One of the reasons that the Yang-Mills equation and the Seiberg-Witten equations are useful in topology is that in many cases, the boundaries of the compactifications of the relevant moduli spaces are either empty or can be understood.

This thesis studies several compactness problems in gauge theory and explores their applications is low dimensional topology.

The first chapter studies a connection between taut foliations and Seiberg-Witten theory. Let Y be a three-manifold endowed with a smooth taut foliation, then there is a canonical symplectization for $\mathbb{R} \times Y$. The symplectic structure gives rise to a version of perturbed Seiberg-Witten equations. Assume furthermore that the foliation has no transverse invariant measure, then the symplectic form can be made exact. The strategy of this chapter is similar to [21]. It proves that although the equation is defined on a *non-compact* manifold, the moduli space of solutions to the perturbed Seiberg-Witten equations is *compact*, therefore one can define an invariant for the foliation in the monopole Floer homology group of Y. The invariant does not change under smooth deformations of the foliation among foliations without transverse invariant measure. As an application, one can construct two such foliations that

are homotopic as plane fields but have different invariants, and it implies that the h-principle does not hold for smooth foliations without transverse invariant measure.

The second chapter studies the moduli space of closed J-holomorphic curves in the symplectization of a taut foliation. It turns out that this moduli space is closely related to the space of closed leaves. In general, the moduli space is not compact, but when the foliation has $\mathbb{Z}/2$ symmetry, one can extract a $\mathbb{Z}/2$-invariant from the moduli space that is invariant under deformations. This idea is used to study the Klein-bottle leaves in taut foliations, and one can prove that the parity of the number of Klein-bottle leaves is invariant under deformations, under certain non-degeneracy assumptions.

The third chapter is more analytic in nature. Recently, many efforts have been made to generalize the Yang-Mills theory and the Seiberg-Witten theory to other gauge-theoretic equations. One of the analytical difficulties is that the relevant moduli spaces are usually not compact, and the compactification is so far difficult to understand. It has been shown by Taubes [38, 40, 39, 42, 41], and Haydys-Walpuski [18] that in many cases, the boundary of the moduli space is described by $\mathbb{Z}/2$ harmonic spinors. The zero locus of a $\mathbb{Z}/2$ harmonic spinor plays an important role in the structure of the boundary of the compactified moduli space, but unfortunately, it can a priori be complicated. It was proved by Taubes [40] that the zero locus must have Hausdorff codimension 2. The third chapter of the thesis proves that it is rectifiable and has finite Minkowski content.

1. A monopole Floer invariant for foliations without transverse invariant measure

Let Y be a closed oriented 3-manifold and $\mathcal{F}$ a smooth oriented foliation on Y. Assume that $\mathcal{F}$ does not admit any transverse invariant measure. This chapter constructs an invariant $c(\mathcal{F})$ for $\mathcal{F}$ which takes value in $\widetilde{HM}_\bullet(Y)$. The invariant is well defined up to a sign. If two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are homotopic through foliations without transverse invariant measure, then $c(\mathcal{F}_1) = c(\mathcal{F}_2)$. The grading of $c(\mathcal{F})$ is represented by the homotopy class of the tangent plane field of $\mathcal{F}$, and its image in $HM_\bullet(Y)$ is nonzero.

1.1. **Introduction.** Let Y be a smooth oriented three manifold and $\mathcal{F}$ an oriented foliation on Y. The foliation $\mathcal{F}$ is called taut if for every point $p \in Y$ there exists an embedded circle in Y passing through p and transverse to $\mathcal{F}$. When $\mathcal{F}$ is smooth, it can be written as $\ker \lambda$ for some smooth 1-form λ, and the following theorem gives some geometric and analytic characterizations of the tautness of $\mathcal{F}$.

Theorem 1.1 (Rummler [31] , Sullivan [33]). *If $\mathcal{F} = \ker \lambda$ is a smooth oriented foliation on Y, then the following conditions are equivalent:*

(1) *For $\forall p \in Y$, there exists an embedded circle in Y passing through p and transverse to $\mathcal{F}$.*

(2) *There exists a closed 2-form ω on Y such that $\omega \wedge \lambda > 0$ everywhere.*

(3) *There exists a Riemannian metric on Y such that the leaves of $\mathcal{F}$ are minimal surfaces.*

(4) *There is no transverse invariant measure on $\mathcal{F}$ that defines an exact foliation cycle.*

Tautness can also be defined for higher dimensional manifolds, but this chapter will only consider foliations on smooth oriented three manifolds.

The existence problem of taut foliations on a given three manifold has been studied for decades. By Roussarie-Thurston theorem, if Y admits a taut foliation then every embedded sphere of Y is either nullhomotopic or isotopic to a leaf. Reeb's stability theorem then implies that if a reducible manifold supports a taut foliation then it has to be $S^2 \times S^1$ with the product foliation. Gabai [14] proved that every irreducible three manifold with $b_1 \geqslant 1$ supports a taut foliation. The existence problem for taut foliations on manifolds with $b_1 = 0$, namely on rational homology spheres, is not yet solved. It was proved in [23] that if Y is a rational homology sphere supporting a smooth taut foliation, then the reduced monopole Floer homology $HM_{\bullet}(Y)$ must be nonzero. This implies, for example, that lens spaces do not support any smooth taut foliations. The theorem was proved for general three manifolds in [22] and it was further generalized to C^0 taut foliations in [5].

The flexibility of taut foliations has also been studied for years. Eynard-Bontemps [13] proved that if two taut foliations can be homotoped to each other via plane fields then they can be homotoped to each other via foliations. On the other hand, Vogel [44] and Bowden [6] recently constructed examples of taut foliations that are homotopic as plane fields but cannot be homotoped to each other via taut foliations.

The proofs of the obstruction of existence of taut foliatons and of Vogel and Bowden's examples rely on the following theorem. The C^2 case of the theorem was due to Eliashberg and Thurston, and the C^0 case was proved by Bowden:

Theorem 1.2 (Eliashberg-Thurston [12], Bowden [5])**.** *Let $\mathcal{F}$ be an orientable C^0 foliation on Y which is not homeomorphic to the product foliation of $S^2 \times S^1$, then $\mathcal{F}$ can be C^0 approximated by a pair of positive and negative contact structures. If the foliaiton is taut, then the contact structures are weakly semi-fillable.*

The non-vanishing of $HM_{\bullet}(Y)$ for taut foliations then follows from the theorem that for a negative weakly semi-fillable contact structure on Y, the image of the contact element in $HM_{\bullet}(Y)$ is always nonzero [22]. For the examples in [44] and [6],

it was first proved that the perturbed contact structures are uniquely determined up to isotopy by the taut foliations, and then the claims were proved by studying the isotopy classes of the those contact structures.

This chapter proposes a method to study the interaction between $\mathcal{F}$ and the monopole Floer homology without invoking Eliashberg-Thurston theorem, when $\mathcal{F}$ is a smooth taut foliation. For technical reasons, one needs to assume that $\mathcal{F}$ does not have any transverse invariant measure. By entry 4 of theorem 1.1 this assumption is stronger than the tautness of $\mathcal{F}$. Let $\mathcal{F} = \ker \lambda$, Sullivan [33] proved that $\mathcal{F}$ has no transverse invariant measure if and only if there exists an exact 2-form ω on Y such that $\omega \wedge \lambda$ is everywhere positive. Therefore, on rational homology spheres a foliation $\mathcal{F}$ has no transverse invariant measure if and only if it is taut.

The idea of this chapter is inspired by [21]. For a taut foliation $\mathcal{F} = \ker \lambda$ on Y, a canonical symplectic structure on $Y \times \mathbb{R}$ will be constructed. There is a metric g on $Y \times \mathbb{R}$ which is compatible with this symplectic structure and having bounded geometry. Now consider a metric on $Y \times \mathbb{R}$ which equals g on $Y \times (-\infty, -1]$ and is cylindrical on $Y \times [0, \infty)$. It will be proved in theorem 1.46 that a suitable counting of solutions of the Seiberg-Witten equations gives an invariant $c(\mathcal{F}) \in \widetilde{HM}_\bullet(Y)$ for the foliation $\mathcal{F}$. The invariant is defined up to a sign, and it is independent of the choice of the 1-form λ and other parameters appeared in the construction. It is also invariant under deformations of the foliation through foliations without transverse measure.

The construction of $c(\mathcal{F})$ follows a similar structure as the definition of the monopole Floer contact element. For a positive contact structure ξ on Y the contact element takes value in $\widetilde{HM}_\bullet(-Y)$. However, unlike contact structures, the foliation $\mathcal{F}$ does not dictate a canonical orientation for the three manifold Y, hence the invariant $c(\mathcal{F})$ can take value in either $\widetilde{HM}_\bullet(Y)$ or $\widetilde{HM}_\bullet(-Y)$, depending on the choice of

orientations and conventions. To simplify notations, the invariant $c(\mathcal{F})$ is defined to be an element of $\widetilde{HM}_\bullet(Y)$ instead of $\widetilde{HM}_\bullet(-Y)$.

In section 1.6 it will be proved that the grading of $c(\mathcal{F})$ is given by the homotopy class of $\mathcal{F}$ as a plane field, and that $c(\mathcal{F})$ has a nonzero image in $HM_\bullet(Y)$. Therefore the construction of $c(\mathcal{F})$ implies the nonvanishing theorem of $HM_{[\mathcal{F}]}(Y)$ (theorem 41.4.1 of [22]) for foliations without transverse invariant measure. Since taut foliations on rational homology spheres do not have transverse invariant measure, this gives the same obstruction for the existence of smooth taut foliations on rational homology spheres as obtained in [23]. With some more effort, one can use the nonvanishing theorem for foliations without transverse invariant measure to prove the nonvanishing theorem of taut foliations when Y is an atoroidal manifold but not a surface bundle over S^1. The proof was explained to the author by Jonathan Bowden and it will be given in section 1.7.

The invariant $c(\mathcal{F})$ can also be used to study the flexibility of foliations without transverse invariant measure. Section 1.7 will give a construction of smooth foliations without transverse invariant measure that are homotopic as plane fields but have different foliation invariants in $\widetilde{HM}_\bullet(Y)$. Therefore, these foliations can not be deformed to each other via smooth foliations without transverse invariant measure.

Besides taut foliations on rational homology spheres, smooth foliations with no transverse invariant measure can also be constructed from Anosov flows. Let v be a vector field on Y that gnenerates an Anosov flow, then the normal bundle of v splits to the direct sum of a contracting subspace and a expanding subspace. The plane field generated by v and the expanding direction of the normal bundle is then a foliation without transverse invariant measure. As an example, let Σ is a compact hyperbolic surface, let Y be its unit tangent bundle, then $Y = \Gamma \backslash PSL_2(\mathbb{R})$ for some cocompact subgroup $\Gamma \subset PSL_2(\mathbb{R})$. Let e_1, e_2, and e_3 be left invariant vector fields

of $\Gamma\backslash PSL_2(\mathbb{R})$ which take value $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ respectively at the unit element. Then e_1 defines an Anosov flow on Y, and e_2 represents the expanding direction of its normal bundle, therefore $\ker(e^3) = \text{span}\{e_1, e_2\}$ is a foliaiton without transverse invariant measure. Let $\{e^1, e^2, e^3\}$ be the dual basis of $\{e_1, e_2, e_3\}$, then $de^1 = -e^2 \wedge e^3$, $de^2 = -2e^1 \wedge e^2$, and $de^3 = 2e^1 \wedge e^3$, thus de^3 is an exact form that is positive on this foliation.

It should be pointed out that for foliations without transverse invariant measure there is a canonical way to deform the foliation to contact structures by a linear deformation (theorem 2.1.2 of [12]), therefore the contact element of the deformed contact class also defines an invariant for the foliation. For now it is not clear to the author whether these two invariants are the same.

The chapter is organized as follows. Sections 1.2 and 1.3 write down the perturbed Seiberg-Witten equations that will be used to define the invariant $c(\mathcal{F})$. Section 1.4 provides the Fredholm theory necessary for the definition of moduli space. Section 1.5 proves a uniform exponential decay estimate for the solutions of the Seiberg-Witten equations. This is the only step that one needs to assume that $\mathcal{F}$ is a foliation without transverse invariant measure rather than just a taut foliation. A similar uniform exponential decay estimate was also used in the definition of contact elements in [23], which referred to the analytical reults in [21], but the analytical details were not completely given there. In [27], in the remark after lemma 2.2.7, the analysis in [21] was explained, but the argument was for compact 4-manifolds with contact boundary, not for manifolds with cylindrical ends which was needed in the definition of contact elements. The method developed in section 1.5 of this chapter can also be used to fill in the analytical details for the definition of contact elements. Section 1.6 uses the analytical results proved in the previous sections to establish the invariant $c(\mathcal{F})$. Finally, section 1.7 discusses its topological applications.

1.2. **Setting the stage.** From now on, let Y be a closed oriented 3-manifold and $\mathcal{F}$ a smooth oriented foliation on Y. The orientations of Y and $\mathcal{F}$ give a co-orientation of $\mathcal{F}$. Take a smooth non-zero 1-form λ such that $\mathcal{F} = \ker \lambda$ and λ is positive on the positive side of $\mathcal{F}$. By Frobenius theorem, $\lambda \wedge d\lambda = 0$. Since $\mathcal{F}$ has no transverse invariant measure, by Sullivan [33] there exists an exact 2-form ω such that $\omega \wedge \lambda > 0$ everywhere on Y. Take a smooth 1-form θ such that $d\theta = \omega$.

Consider the cylinder $Y \times \mathbb{R}$, let t be the coordinate of the $\mathbb{R}$-component. Use the same notations ω, λ, and θ to denote the pull back forms on $Y \times \mathbb{R}$ when there is no danger of confusion. Let $\Omega = \omega + d(t\lambda)$ be a 2-form on $Y \times \mathbb{R}$, then Ω is a symplectic form. Let $\Theta = \theta + t\lambda$, then $\Omega = d\Theta$.

Fix a metric g_0 on Y such that $|\lambda|_{g_0} = 1$ and $\lambda = *\omega$. Locally ω can be written as $\omega = e^1 \wedge e^2$ where e^1 and e^2 are orthonormal cotangent vector fields on Y. Since $\lambda \wedge d\lambda = 0$, there is a unique μ_1 such that $d\lambda = \mu_1 \wedge \lambda$ and $\langle \lambda, \mu_1 \rangle_{g_0} = 0$. Now $d\mu_1 \wedge \lambda = d(\mu_1 \wedge \lambda) = d^2 \lambda = 0$, hence there is a unique μ_2 such that $d\mu_1 = \mu_2 \wedge \lambda$ and $\langle \mu_2, \lambda \rangle_{g_0} = 0$.

Now define a Riemannian metric on $Y \times \mathbb{R}$ compatible with Ω as follows. Notice that $\Omega = e^1 \wedge e^2 + dt \wedge \lambda + t\mu_1 \wedge \lambda$. Take

$$(1.1) \qquad g = e^1 \otimes e^1 + e^2 \otimes e^2 + (1 + t^2)\lambda \otimes \lambda + \frac{1}{1 + t^2}(dt + t\mu_1) \otimes (dt + t\mu_1).$$

It is easy to verify that g does not depend on the choice of e^1 and e^2, and that it is compatible with Ω. (i.e. Ω is a self-dual 2-form of length $\sqrt{2}$ under the metric g.) Denote the Riemannian manifold $(Y \times \mathbb{R}, g)$ by X.

Lemma 1.3. *The symplectic manifold $(X, \Omega = d\Theta)$ has the following properties:*

(1) X is complete.

(2) The injectivity radius of X is bounded from below by a positive number.

(3) Let R be the curvature tensor of X, then $\nabla^k R$ is bounded for each k.

(4) The tensor $\nabla^k \Theta$ is bounded for each k.

Proof. Let $v = x \cdot \frac{\partial}{\partial t} + u$ be a tangent vector of X, where x is a real number and u is a vector tangent to the Y component of X. By the definition of g and Cauchy's inequality:

$$|v| \cdot \sqrt{t^2 |\mu_1|^2 + t^2 + 1}$$

$$\geq \sqrt{|u|^2 + \frac{1}{1+t^2} \left(x + t \cdot \mu_1(u) \right)^2} \cdot \sqrt{t^2 |\mu_1|^2 + (1+t^2)}$$

$$\geq |t| |\mu_1| |u| + |x + t \cdot \mu_1(u)|$$

$$\geq |x|.$$

Therefore $|v| \geq |x| / \sqrt{1 + z \cdot t^2}$, where $z = \sup |\mu_1|^2 + 1$. The length of each curve from the slice $t = -T$ to $t = T$ is thus at least

$$\int_{-T}^{T} 1/\sqrt{1 + z \cdot t^2} \, dt.$$

Since $\int_{-\infty}^{\infty} 1/\sqrt{1 + z \cdot t^2} \, dt = +\infty$, this proves the completeness of X.

For the boundedness of $\nabla^k R$ and $\nabla^k \Omega$, use the moving frame method. Take an arbitrary point q on Y, choose a contractible neighborhood U_q of q, and fix a choice

of e^1 and e^2 on U_q. Let

$$e^3 = \sqrt{1 + t^2} \cdot \lambda,$$

$$e^4 = \frac{1}{\sqrt{1 + t^2}}(dt + t \cdot \mu_1).$$

Then $\{e^1, e^2, e^3, e^4\}$ form an orthonormal basis of the cotangent bundle on $U_q \times \mathbb{R}$. There exist smooth functions ν_i on U_q $(i = 1, 2, ..., 10)$, such that

$$d\,e^1 = \nu_1 \, e^1 \wedge e^2 + \nu_2 \, e^1 \wedge \lambda + \nu_3 \, e^2 \wedge \lambda,$$

$$d\,e^2 = \nu_4 \, e^1 \wedge e^2 + \nu_5 \, e^1 \wedge \lambda + \nu_6 \, e^2 \wedge \lambda,$$

$$\mu_1 = \nu_7 \, e^1 + \nu_8 \, e^2,$$

$$\mu_2 = \nu_9 \, e^1 + \nu_{10} \, e^2.$$

By schrinking the neighborhood U_q if necessary, assume that $\nabla^k \nu_i$ is bounded for each k. A straight forward calculation shows:

$$(1.2) \quad \begin{cases} d\,e^1 & = \nu_1 \, e^1 \wedge e^2 + \frac{\nu_2}{\sqrt{1+t^2}} \, e^1 \wedge e^3 + \frac{\nu_3}{\sqrt{1+t^2}} \, e^2 \wedge e^3, \\[2mm] d\,e^2 & = \nu_4 \, e^1 \wedge e^2 + \frac{\nu_5}{\sqrt{1+t^2}} \, e^1 \wedge e^3 + \frac{\nu_6}{\sqrt{1+t^2}} \, e^2 \wedge e^3, \\[2mm] d\,e^3 & = \frac{t}{\sqrt{1+t^2}} \, e^4 \wedge e^3 + \frac{\nu_7}{1+t^2} \, e^1 \wedge e^3 + \frac{\nu_8}{1+t^2} \, e^2 \wedge e^3, \\[2mm] d\,e^4 & = \frac{1}{1+t^2} e^4 \wedge (\nu_7 \, e^1 + \nu_8 \, e^2) - \frac{t}{1+t^2} e^3 \wedge (\nu_9 \, e^1 + \nu_{10} \, e^2). \end{cases}$$

Write

$$de^i = \sum_{j \neq k} a^i_{jk} \, e^j \wedge e^k,$$

such that $a^i_{jk} = -a^i_{kj}$, then the equations above imply that $\nabla^m a^i_{jk}$ is bounded for each m.

Suppose $\nabla e^i = \omega^i_j \otimes e^j$, then the connection matrix $\{\omega^j_i\}$ can be calculated from $\{a^i_{jk}\}$ by the formula

$$\omega^i_j = (-a^k_{ji} + a^i_{kj} + a^j_{ik})e^k,$$

and the curvature matrix under the basis $\{e^i\}$ is given by $d\omega_i^j - \omega_i^k \wedge \omega_k^j$. Since a_{jk}^i and their exterior derivatives are bounded, it follows that under the basis $\{e^i\}$ the matrix of $\nabla^k R$ are bounded on $U_q \times \mathbb{R}$ for each k. This proves the boundedness of $\nabla^k R$ on $U_q \times \mathbb{R}$. Since Y is compact, it is covered by finitely many such U_q's, therefore $\nabla^k R$ are bounded on $X = Y \times \mathbb{R}$.

For the estimates on Θ, write θ as

$$\theta = \nu_{11}e^1 + \nu_{12}e^2 + \nu_{13}\lambda,$$

then

$$\Theta = \nu_{11}e^1 + \nu_{12}e^2 + \frac{t + \nu_{13}}{\sqrt{1 + t^2}}e^3,$$

and the same calculation proves the boundedness of $\nabla^k \Theta$.

For the lower bound on injectivity radius one needs the following theorem:

Theorem (Cheeger-Gromov-Taylor [9]). *Let (M, g) be a complete Riemannian manifold, let $K > 0$ be a constant such that the sectional curvature of M is bounded by K from above. Let $0 < r < \frac{\pi}{4\sqrt{K}}$. Then the injectivity radius at each point $p \in M$ satisfies the following inequality:*

$$(1.3) \qquad \mathrm{inj}(p) \geq r\frac{Vol(B_M(p, r))}{Vol(B_M(p, r)) + Vol_{T_pM}(B_{T_pM}(0, 2r))}.$$

Here $Vol_{T_pM}(B_{T_pM}(0, 2r))$ denotes the volume of the ball of radius $2r$ in T_pM, where both the volume and the distance function are defined using the pull-back of the metric g to T_pM via the exponential map.

Back to the proof of lemma 1.3. Let $K > 0$ be an upper bound of the sectional curvature. For each point $q \in Y$, let L_q be the leaf of $\mathcal{F}$ through q, the metric on L_q is taken to be the restriction from g_0. Let $\epsilon = \inf_{q \in Y} \mathrm{inj}(L_q)$. Since Y is compact, ϵ is positive. Take a positive constant r such that $0 < r < \min\{\frac{\pi}{4\sqrt{K}}, \epsilon, 1\}$. Let

$p = (q, t) \in X$. The following argument will show that $\mathrm{inj}(p)$ is bounded from below by a positive constant independent of p. Without loss of generality, assume $|t| > 1$.

By the volume comparison theorem, there is a positive constant z_1 depending only on K and r such that

$$(1.4) \qquad\qquad \mathrm{Vol}_{T_pX}(B_{T_pX}(0, 2r)) \leqslant z_1.$$

Let $D(q, r/3)$ be the open disk of radius $r/3$ on L_q centering at q, Let

$$U = \{x \in Y | \mathrm{dist}_{g_0}(x, D(q, r/3)) < \frac{r}{3\sqrt{1 + t^2}}\}.$$

Then the distance from each point in U to $D(q, r/3)$ under the metric $g|_{Y \times \{t\}}$ is less than $r/3$, thus the distance from each point of U to q is less than $2r/3$. Therefore,

$$B_{X,g}(p, r) \supseteq U \times (e^{-r/3} t, e^{r/3} t).$$

The volume of U under the metric g_0 is bounded from below by a constant multiple of $r/(3\sqrt{1 + t^2})$ where the constant depends only on g_0 and $\mathcal{F}$, thus the volume of $U \times (e^{-r/3} t, e^{r/3} t)$ under the product metric of $Y \times \mathbb{R}$ is bounded from below by a positive constant. Notice that the volume form of the product metric on $Y \times \mathbb{R}$ is the same as the volume form of g. Therefore

$$(1.5) \qquad\qquad \mathrm{Vol}(B_X(p, r)) > \frac{1}{z_2}$$

for some positive constant z_2 depending on $\mathcal{F}$, g_0 and r. The lower bound of injectivity radius of X follows immediately from (1.3), (1.4), and (1.5). $\qquad\square$

Remark 1.4. The fact that the injectivity radius of X is bounded from below could be counter intuitive because of the factor $\frac{1}{1+t^2}$ in the definition of g. In fact, by the proof of lemma 1.3 one can visualize the geometry of X as follows. First consider the three manifold Y with the metric g_0. For any $x \in Y$, $r, \epsilon > 0$, let L_x be the leaf

of $\mathcal{F}$ containing x with the induced metric from g_0, let D_r be the r-neighborhood of x in L_x, and let $D_r(\epsilon)$ be the ϵ neighborhood of D_r in Y. When r is fixed and ϵ is small, $D_r(\epsilon)$ looks like a thin slice near D_r. Now let $r_0 > 0$ be a lower bound of the injectivity radius, then a normal neighborhood of X centering at (q, t) with radius r_0 contains the set $D_{r_0/3}\big(\frac{r_0}{3\sqrt{1+t^2}}\big) \times (e^{-r_0/3}t, e^{r_0/3}t)$. When t is large, this looks like a thin slice near a portion of the leaf containing x times a long interval.

The rest of this section gives some technical definitions for later reference.

Definition 1.5. Let M be a noncompact Riemannian 4-manifold possibly with boundary. Then M is called a cylindrical end if M is isometric to $Y_{g_0} \times [0, +\infty)$ with the product metric.

Definition 1.6. Let M be a noncompact Riemannian 4-manifold possibly with boundary. Then M is called a symplectic end if the following conditions hold:

(1) ∂M is compact, each geodesic ray on M either intersects ∂M or can be extended infinitely.

(2) There is an open neighborhood Z of ∂M such that Z has compact closure and $\mathrm{inj}(M)$ is bounded away from zero on $M - Z$.

(3) Let R be the curvature tensor of M, then $\nabla^k R$ is bounded for each k.

(4) There exists an exact symplectic form $\Omega = d\Theta$ which is compatible with the metric on M, such that for each k, the tensor $\nabla^k \Theta$ is bounded.

Definition 1.7. Let M be a complete manifold, and $Z \subset M$ an open submanifold such that $\overline{Z}$ is a manifold with boundary. A component of $M - Z$ is said to be a symplectic end of M if it is a symplectic end as defined in definition 1.6, and the symplectic structure extends to a neighborhood of the component. The component is said to be a cylindrical end of M if it is a cylindrical end as defined in definition 1.5, and the product metric extends to a neighborhood of the component.

Whenever a manifold M is said to have symplectic or cylindrical ends, it will always be assumed that the subset Z has been specified in advance.

Remark 1.8. The definition of symplectic ends is a generalization of the AFAK structure discussed in [21].

By lemma 1.3, X is a manifold with symplectic ends.

1.3. **The Seiberg-Witten equations.**

1.3.1. *Spin^c Structures.* For any oriented Riemannian 4-manifold M, the positively oriented orthonormal frame bundle is a principal SO(4) bundle, denote it by P. A Spin^c structure $\mathfrak{s}$ on M is a lifting of P to a principal Spin^c(4) bundle. Denote the set of isomorphism classes of Spin^c structures on M by Spin$^c(M)$.

Although the definition of Spin^c structures depends on a choice of metric, the following well-known lemma shows that different choices of metrics give essentially the same set of Spin^c structures:

Lemma 1.9. *Let M_1, M_2 be two oriented Riemannian 4-manifolds, let $\varphi : M_1 \to M_2$ be an orientation-preserving diffeomorphism. Then the pull back map φ^* defines a bijection from $Spin^c(M_2)$ to $Spin^c(M_1)$.* $\qquad\square$

Once a Spin^c structure $\mathfrak{s}$ on the 4-manifold M is chosen there are some standard constructions. This subsection briefly summarizes them in order to fix notations. For details, see for example [26]. The notations here are mostly following [22]. There are two maps from Spin^c(4) to U(2), thus $\mathfrak{s}$ gives rise to two associated $\mathbb{C}^2$ bundles S^+ and S^-. The bundle $S = S^+ \oplus S^-$ is called the spinor bundle. There is a Clifford action $\rho : TM \to \mathrm{Hom}(S, S)$ induced by the Spin^c structure, which satisfies $\rho(X)^2 = -|X|^2$. The action ρ extends to T^*M and $\wedge^2 T^*M$ by the Riemannian metric. The bundles S^+ and S^- are labelled so that $\wedge_+^2 T^*M$ acts nontrivially on S^+.

A unitary connection A on S is called a Spin^c connection if $\nabla_A \rho = 0$, where the connection on TM is taken to be the Levi-Civita connection. Every Spin^c connection decomposes as a sum of unitary connections on S^+ and S^-, and it induces a connection on $\det(S^+)$. Conversely every unitary connection on $\det(S^+)$ is defined by a unique Spin^c connection on S. Use A^t to denote the connection on $\det(S^+)$ defined by A. If A is a Spin^c connection on S, use D_A to denote the Dirac operator defined by A.

There is a similar definition for Spin^c structures on 3-manifolds. Notice that $SO(3) \cong SU(2)/\{\pm 1\}$ and $U(2) \cong SU(2) \times U(1)/\{\pm 1\}$, hence there is a map from $U(2)$ to $SO(3)$ given by projection to the first component. A Spin^c structure on a oriented Riemannian 3 manifold Y is defined to be a lifting of the oriented orthonormal frame bundle to a $U(2)$ bundle. If $\mathfrak{t}$ is a Spin^c structure on a 3-manifold Y, its spinor bundle is $\mathfrak{t} \times_{U(2)} \mathbb{C}^2$ and there is a Clifford action $\rho : TM \to \mathrm{Hom}(S, S)$. A unitary connection B on the spinor bundle is called a Spin^c connection iff $\nabla_B \rho = 0$.

If M is a Riemannian 4-manifold with a compatible symplectic structure, then there is a canonically defined Spin^c structure on M such that

$$S^+ = T^{0,0} M \oplus T^{0,2} M,$$

$$S^- = T^{0,1} M.$$

On this canonical Spin^c structure, there is a canonical section of S^+ given by $1 \in \Gamma(M, T^{0,0} M)$, and there is a canonical Spin^c connection A_0 on S such that $D_{A_0} \Phi_0 = 0$ (cf. [19]).

1.3.2. *Configuration spaces.* If M is a manifold with each end being either symplectic or cylindrical, this subsection defines a suitable configuration space on M for the study of moduli spaces. First, following notations of [21], define a configuration space on compact manifolds:

Definition 1.10. If $\mathfrak{t}$ is a Spin^c structure on a compact 3-manifold N, let S be the spinor bundle. Define $\mathcal{C}_k(N, \mathfrak{t})$ to be the set of pairs (B, ψ) such that B is a locally L^2_k Spin^c-connection for $\mathfrak{t}$, and ψ is a locally L^2_k section of S. Define $\mathcal{C}(N, \mathfrak{t}) = \cap_{k \geqslant 1} \mathcal{C}_k(N, \mathfrak{t})$.

Definition 1.11. If $\mathfrak{s}$ is a Spin^c structure on a compact 4-manifold M, let S be the spinor bundle. Define $\mathcal{C}_k(M, \mathfrak{s})$ to be the set of pairs (A, ϕ) such that A is a locally L^2_k Spin^c-connection for $\mathfrak{s}$, and ϕ is a locally L^2_k section of S^+. Define $\mathcal{C}(M, \mathfrak{s}) = \cap_{k \geqslant 1} \mathcal{C}_k(M, \mathfrak{s})$.

Now suppose M is a Riemannian 4-manifold with each end being either symplectic or cylindrical. Let H_i be its cylindrical ends and G_j be the symplectic ends. Let $\mathfrak{s}$ be a Spin^c structure on M such that $\mathfrak{s}|_{G_j}$ is isomorphic to the canonical Spin^c structure on G_j. Let τ_j be an isomorphism from $\mathfrak{s}|_{G_j}$ to the canonical Spin^c structure on G_j, and assume that τ_j can be smoothly extended to a neighborhood of G_j. For later references, the following definition gives a name to such structures.

Definition 1.12. The structure $(\mathfrak{s}, \{\tau_j\})$ satisfying the assumptions above is called an admissible Spin^c structure on M.

Notice that M has the product metric on each H_i, therefore the restriction of $\mathfrak{s}$ to H_i is the product Spin^c structure given by a Spin^c structure $\mathfrak{t}_i$ on Y, as described in [22, p. 89]. The spinor bundle of $\mathfrak{t}_i$ is isomorphic to $S^+|_{H_i}$. If $(A, \phi) \in \mathcal{C}_k(H_i, \mathfrak{s}|H_i)$, then by restriction to each slice it defines a path $(B(t), \psi(t))$, where the $(B(t), \text{Spin}^c(t)) \in \mathcal{C}_{k-1}(Y, \mathfrak{t}_i)$.

The following space is the suitable configuration space for the purpose of this chapter:

Definition 1.13. Let M be a manifold with each end being either symplectic or cylindrical, let $(\mathfrak{s}, \{\tau_j\})$ be an admissible Spin^c structure, let A_j be the pull back of

the canonical connection by τ_j, let ϕ_j be the pull back of the canonical section by τ_j. Let $r > 0$ be a real number. Define $\mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ to be the set of pairs (A, ϕ) such that:

(1) A is a locally L^2_k Spin^c-connection for S, ϕ is a locally L^2_k section of S^+.

(2) On each symplectic end G_j,

$$A - A_j \in L^2_k(G_j, iT^*M),$$

$$\phi - \sqrt{r}\,\phi_j \in L^2_{k,A_j}(G_j, S^+).$$

(3) For each cylindrical end H_i, the restriction of (A, ϕ) on H_i gives a path in $\mathcal{C}_{k-1}(Y, \mathfrak{t}_i)$ which converges to some point in the configuration space of $(Y, \mathfrak{t}_i)$.

Define $\mathcal{C}(M, \mathfrak{s}, r, \{\tau_j\}) = \cap_{k \geqslant 1} \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$.

1.3.3. *Strongly tame perturbations.* In [22, p. 85], the Chern-Simons-Dirac functional $\mathcal{L}$ is defined on $\mathcal{C}(Y, \mathfrak{t})$ once a base connection B_0 is chosen. A class of tame perturbations of $\mathcal{L}$ is then studied and they played a crucial rule for the regularity of the moduli spaces. For the purpose of this chapter, however, a stronger version of tameness needs to be introduced. Later a version of Seiberg-Witten equations will be defined on manifolds with each end being either symplectic or cylindrical using strongly tamed perturbations.

Notice that if $\mathfrak{q}$ is a perturbation of the gradient of the Chern-Simons-Dirac functional, then it defines a perturbation $(\hat{\mathfrak{q}}^0, \hat{\mathfrak{q}}^1)$ on the cylinder $Y \times [0, 1]$. (See secion 10.1 of [22]).

Definition 1.14. A perturbation $\mathfrak{q}$ of $\mathcal{L}$ is called strongly tame if

(1) It is a tame perturbation as defined in definition 10.5.1 of [22].

(2) There is a constant m_0 such that

$$\|\hat{q}^0(A, \phi)\|_{C^0} \leqslant m_0(\|\phi\|_{C^0} + 1)$$

for all $(A, \phi) \in \mathcal{C}(Y \times [0,1], \mathfrak{t}')$, where $\mathfrak{t}'$ is the product Spin^c structure on $Y \times [0,1]$ defined by $\mathfrak{t}$.

(3) There is a constant m_1 such that

$$\|\hat{q}^1(A, \phi)\|_{C^0} \leqslant m_1$$

for all $(A, \phi) \in \mathcal{C}(Y \times [0,1], \mathfrak{t}')$.

Using the calculations in [22, p. 176] it is straight forward to verify that the cylindrical functions constructed in section 11.1 of [22] are strongly tame, therefore all the results about the class of tame perturbations proved in [22] works for strongly tame perturbations. In particular, a Banach space $\mathcal{P}$ of strongly tame perturbations can be defined so that the conclusions of theorem 11.6.1 of [22] is satisfied, and moreover for every $\mathfrak{q} \in \mathcal{P}$ the norm of $\mathfrak{q}$ satisfies

(1.6) $$\|\hat{q}^0(A, \phi)\|_{C^0} \leqslant \|\mathfrak{q}\| \left(\|\phi\|_{C^0} + 1\right),$$

(1.7) $$\|\hat{q}^1(A, \phi)\|_{C^0} \leqslant \|\mathfrak{q}\|,$$

for all $(A, \phi) \in \mathcal{C}(Y \times [0,1], \mathfrak{t}')$.

1.3.4. *Perturbed Seiberg-Witten equations.* Let M be a Riemannian 4-manifold with each end being either symplectic or cylindrical, let H_i be the cylindrical ends and G_j be the symplectic ends. Let $(\mathfrak{s}, \{\tau_j\})$ be an admissible Spin^c structure on M. Let $r > 0$ be a constant. This section defines a perturbed version of Seiberg-Witten equations on $\mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ when $k \geqslant 5$. The reason to take $k \geqslant 5$ is to ensure that the elements in $\mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ are in C^2.

For $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$, define the Seiberg-Witten map

$$\mathfrak{F}(A, \phi) = (\rho(F_{A^t}^+) - (\phi\phi^*)_0, D_A\phi).$$

By definition, $\mathfrak{F}(A, \phi)$ is a section of $i\mathfrak{su}(S^+) \oplus S^-$.

Recall that in the end of subsection 1.3.3, a Banach space $\mathcal{P}$ of strongly tame perturbations was defined, and the norm of $\mathcal{P}$ satisfies (1.6) and(1.7). Definition 22.1.1 of [22] defined a class of admissible perturbations as one of the ingredients in the definition of monopole Floer homology. A straight forward modification of theorem 15.1.1 and proposition 15.1.3 of [22] shows that admissible perturbations form a residual subset of $\mathcal{P}$. Now define a perturbation on the cylindrical ends as follows. Take an admissible perturbation $\mathfrak{q}_{\mathfrak{t}} \in \mathcal{P}$ for each Spin^c structure $\mathfrak{t}$ on Y. The choice of $\mathfrak{q}_{\mathfrak{t}}$ will be fixed for the rest of the chapter. Use $\hat{q}_{\mathfrak{t}}$ to denote the corresponding 4 dimensional perturbations on $Y \times \mathbb{R}$. For each cylindrical end H_i, take another element $\mathfrak{p}_i \in \mathcal{P}$ such that $\|\mathfrak{p}_i\| < 1$. The perturbations $\mathfrak{p}_i$ do not have to be admissible, later they will be chosen to obtain transversality. Let $\hat{p}_i$ be the 4 dimensional perturbations corresponding to $\mathfrak{p}_i$.

Next, define a perturbation on the symplectic ends. For each G_j, let Ω_j be the corresponding symplectic form, let ϕ_j, A_j be the pull back of the canonical section and the canonical Spin^c structure via τ_j. Define a perturbation $\hat{u}_j \in C^\infty(G_j, i\mathfrak{su}(S^+)\oplus S^-)$ such that

$$\hat{u}_j = (-r(\phi_j\phi_j^*)_0 + \rho(F_{A_j^t}^+), 0)$$

$$(1.8) \qquad = \big(-\frac{ir}{4}\rho(\Omega_j) + \rho(F_{A_j^t}^+), 0\big)$$

Finally, glue these perturbations together. For each end H_i, the perturbation $\hat{q}_i$ extends smoothly to a neighborhood H_i' of H_i. Similarly each $\hat{u}_j$ extends smoothly to a neighborhood G_j' of G_j. Let $\eta \geqslant 0$ be a smooth cutoff function on M such that $\text{supp}\,\eta \subset (\cup G_j') \cup (\cup H_i')$ and $\eta = 1$ on a neighborhood of $(\cup G_j) \cup (\cup H_i)$. Let $\rho \geqslant 0$ be a nonzero smooth function on $(0, +\infty)$ with compact support, and let $t_i : H_i \to [0, +\infty)$ be the projection, take $\eta_i = \rho \circ t_i$. Define

$$(1.9) \qquad \hat{\mu} = \eta \sum_i \hat{q}_i + \eta \sum_j \hat{u}_j + \sum_i \eta_i\, \hat{p}_i.$$

The Seiberg-Witten equation that will be considered in this chapter is then an equation on $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ which reads as:

(1.10) $$\mathfrak{F}(A, \phi) = \hat{\mu}(A, \phi).$$

Notice that any section $\phi \in \Gamma(G_j, S^+)$ can be decomposed as

(1.11) $$\phi = \sqrt{r} \left(\phi_j \cdot \tau_j^*(\alpha) + \tau_j^*(\beta) \right)$$

where $\alpha \in \Gamma(G_j, \mathbb{C})$ and $\beta \in \Gamma(G_j, T^{0,2} G_j)$. For any Spin^c connection A on G_j, define a unitary connection ∇'_A on $T^{0,2} G_j$ as:

$$\nabla'_A s = (\nabla_{\tau_*(A)} s)^{(0,2)}, \quad \forall s \in C^\infty(G_j, T^{0,2} G_j).$$

On the other hand, $a = A - A_j$ is an $i\mathbb{R}$ valued 1-form it gives a unitary connection on the trivial $\mathbb{C}$-bundle on G_j. Given $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$, define the energy density function E on the symplectic ends as:

$$E = |1 - |\alpha|^2 - |\beta|^2|^2 + |\beta|^2 + |\nabla_a \alpha|^2 + |\nabla'_A \beta|^2 + |F_a|^2.$$

If $k \geqslant 3$, then for each for each $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ the sections $(1 - \alpha)$, β, and a are in $L^2_{1,A_j} \cap C^0$, therefore

$$\int_{G_j} E < +\infty.$$

1.4. **Fredholm theory.** Let M be a manifold with *only* symplectic ends, and suppose $(\mathfrak{s}, \{\tau_j\})$ is an admissible Spin^c structure. This section studies the Fredholm theory for the operator $\mathfrak{F}$ on $\mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$. As before, assume that $k \geqslant 5$. The argument in this section is a slight modification of the arguments of Kronheimer and Mrowka in [21].

The space $\mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ is acted upon by the following gauge group

$$\mathcal{G}_{k+1} = \left\{ u : M \to \mathbb{C}^* \big| |u| = 1 \text{ and } 1 - u \in L^2_{k+1} \right\}.$$

Using standard techniques of gauge theory (see lemma 1.2 of [43], for example), it is straight forward to prove that if $\mathfrak{a}$, $\mathfrak{b} \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ and if there is a locally L^2 gauge transformation u such that $u(\mathfrak{a}) = \mathfrak{b}$, then $u \in \mathcal{G}_{k+1}$.

Fix a pair $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$, and let $l \leqslant k$. Let $\{G_j\}$ be the ends of M, and let A_0 be an extension of the pull back of the canonical connections on $\cup G_j$. Define

$$\delta_1 : L_{l+1}(i\mathbb{R}) \to L_l^2(iT^*M) \oplus L_{l,A_0}^2(S^+),$$

$$\delta_2 : L_{l+1}^2(iT^*M) \oplus L_{l+1,A_0}^2(S^+) \to L_l^2(i\mathfrak{su}(S^+)) \oplus L_{l,A_0}^2(S^-),$$

as

$$\delta_1(f) = (-df, f\phi),$$

$$\delta_2(a, \varphi) = \big(2\rho(d^+a) - (\phi\varphi^* + \varphi\phi^*)_0, D_A\varphi + \rho(a)\phi\big).$$

Then δ_1 and δ_2 are the tangent maps of the gauge transformation and the Seiberg-Witten map.

The Seiberg-Witten equations used in this chapter follow the conventions of [37]. It is different from the Seiberg-Witten equations used in [21] by a coefficient of 2. To apply the calculations in [21], define the isomorphisms

$$\sigma_1 : iT^*M \oplus S^+ \to T^*M \oplus S^+$$

$$(a, \varphi) \mapsto (a, \varphi/\sqrt{2}),$$

$$\sigma_2 : i\mathfrak{su}(S^+) \oplus S^- \to i\mathfrak{su}(S^+) \oplus S^-$$

$$(s, \varphi) \mapsto (s/2, \varphi/\sqrt{2}).$$

Let $\delta_1' = \sigma_1 \circ \delta_1^{-1}$, $\delta_2' = \sigma_2 \circ \delta_2 \circ \sigma_1^{-1}$. Then

$$\delta_1'(f) = \big(-df, f(\frac{\phi}{\sqrt{2}})\big),$$

$$\delta_2'(a, \varphi) = \left(\rho(d^+ a) - (\frac{\phi}{\sqrt{2}} \cdot \varphi^* + \varphi \cdot \frac{\phi^*}{\sqrt{2}})_0, D_A \varphi + \rho(a)\frac{\phi}{\sqrt{2}}\right).$$

Now δ_1' and δ_2' become the linearized Seiberg-Witten operators at $(A, \frac{\phi}{\sqrt{2}})$ as in the conventions of [21]. Let $(\delta_1')^*$ and $(\delta_2')^*$ be their formal dual. Let $\mathscr{D} = (\delta_1')^* + \delta_2'$ and $\mathscr{D}^* = \delta_1' + (\delta_2')^*$.

The goal of this section is to prove the following proposition.

Proposition 1.15. *Let M be a manifold with symplectic ends. There exists a constant r_0 depending on M, such that when $r > r_0$, the map*

$$\mathscr{D} : L_{l+1}^2(iT^*M) \oplus L_{l+1, A_0}^2(S^+) \to L_l^2(i\mathbb{R}) \oplus L_l^2(i\mathfrak{su}(S^+)) \oplus L_{l, A_0}^2(S^-)$$

is always a Fredholm operator for any $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ and $0 \leqslant l \leqslant k$. Let $e(M, \mathfrak{s}, \{\tau_j\})$ be the relative Euler number where the trivializations on the ends are the pull back of canonical sections via $\{\tau_j\}$, then the index of $\mathscr{D}$ equals $e(M, \mathfrak{s}, \{\tau_j\})$.

Definition 1.16. Let E, F be two vector boundles over M with smooth unitary connections A_E and A_F. Let k be a positive integer, let $r > 0$. An operator $D : C^\infty(E) \to L_{loc}^2(F)$ is called (k, r)-admissible if the following conditions hold:

(1) $D = \iota \circ \nabla_{A_E} + T$, where $\iota : T^*M \otimes E \to F$ and $T : E \to F$ are bundle maps with bounded norm.

(2) For $\forall 0 \leqslant l \leqslant k$, D extends continuously to a map

$$D : L_{l+1, A_E}^2(E) \to L_{l, A_F}^2(F).$$

(3) (Regularity) For $\forall 0 \leqslant l \leqslant k$, if $Ds = u$ in the sense of distributions, and if $s \in L^2(E)$, $u \in L_{l, A_F}^2(F)$, then $s \in L_{l+1, A_E}^2(E)$. Moreover, there is a constant C independent of s such that

(1.12)
$$\|s\|_{L_{l+1, A_E}^2} \leqslant C\left(\|Ds\|_{L_{l, A_F}^2} + \|s\|_{L^2}\right).$$

(4) There exists a bundle map $R \in \mathrm{Hom}(E, F)$, such that

(1.13)
$$\int_M |Ds|^2 = \int_M |\nabla_{A_E} s|^2 + r|s|^2 + \langle Rs, s \rangle$$

for every compactly supported $s \in L^2_{1,A_E}(E)$, and there exists a compact set $Z \subset M$ such that $|R| < r/2$ on $M - Z$.

Lemma 1.17. *There exists a constant $r_0 > 0$ such that if $r > r_0$, then the operators $\mathscr{D}$ and $\mathscr{D}^*$ are both $(k, r/2)$-admissible.*

Proof. Condition 1 of admissibility comes from definitions of $\mathscr{D}$ and $\mathscr{D}^*$, condition 2 is proved by Sobolev embeddings, condition 3 follows from standard elliptic regularity. Conditions 4 follow from Propositions 3.8 and 3.10 of [21]. $\qquad\square$

Proposition 1.18. *If an operator D and its formal dual D^* are both (k, r) admissible, then for $0 \leqslant l \leqslant k$, the map $D : L^2_{l+1,A_E}(E) \to L^2_{l,A_F}(F)$ is Fredholm, and $\mathrm{ind}\, D = \dim \ker D - \dim \ker D^*$.*

To prove the proposition one needs the following two lemmas.

Lemma 1.19. *Let D be a (k, r) admissible operator. If a sequence $\{s_i\} \subset L^2_{1,A_E}(E)$ satisfies*

$$\|s_i\|_{L^2} \leqslant 1,$$

$$Ds_i \to y \ in \ L^2(F).$$

Then there exists $s \in L^2(E)$ such that $s_i \to s$ in $L^2_{1,A_E}(E)$ and $Ds = y$.

Proof. By standard elliptic regularity, there exists a subsequence converging to some section $s \in L^2_{1,loc}$ in L^2_{loc}. Denote the subsequence by the same notation s_i. The limit s satisfies

$$\|s\|_{L^2} \leqslant \limsup_{i \to \infty} \|s_i\|_{L^2} \leqslant 1,$$

$$Ds = y \text{ as distributions.}$$

Therefore by the regularity of D, the section $s \in L^2_{1,A_E}$.

Let $u_i = s_i - s$, then

(1.14)
$$\|u_i\|_{L^2} \leqslant 2,$$

(1.15)
$$\lim_{i \to \infty} \|Du_i\|_{L^2} = 0,$$

and since $s_i \to s$ locally in L^2,

(1.16)
$$\lim_{i \to \infty} \int_J \|u_i\|_{L^2}^2 = 0$$

for any compact subset $J \subset M$. For any set $S \subset M$, denote the interior of S by S°. Let β_n be a sequence of compactly supported Lipschitz functions such that

$$1 \geqslant \beta_{n+1} \geqslant \beta_n \geqslant 0,$$

$$\operatorname{supp} \beta_n \subset \beta_{n+1}^{-1}(1)^\circ,$$

$$\bigcup_n \beta_n^{-1}(1)^\circ = M,$$

$$|\nabla \beta_n| \leqslant \frac{1}{n}.$$

Such a sequence β_n can be constructed for example as follows. Let x be any point on M, let d_x be the distance function to x, let $\eta : \mathbb{R}^{\geqslant 0} \to \mathbb{R}^{\geqslant 0}$ be a Lipschitz function such that $|\nabla \eta| \leqslant 1$, $\operatorname{supp} \eta \subset [0,3]$, and $\eta = 1$ on $[0,1]$. Then β_n can be defined by $\beta_n = \eta(d_x/n)$.

Let Z be the compact set in definition 1.16. Take a number m such that $Z \subset \beta_m^{-1}(1)^\circ$. For any $n \geqslant m$, equation (1.13) gives

$$r\|(\beta_n - \beta_m)u_i\|_{L^2}^2 + \|\nabla_{A_E}\big((\beta_n - \beta_m)u_i\big)\|_{L^2}^2$$

$$= \|D\big((\beta_n - \beta_m)u_i\big)\|_{L^2}^2 - \int_M (\beta_n - \beta_m)^2 \langle R(u_i), u_i \rangle$$

$$\leqslant \|Du_i\|_{L^2}^2 + \|\iota\big(d(\beta_n - \beta_m) \otimes u_i\big)\|_{L^2}^2 + \frac{r}{2}\int_M |\beta_n - \beta_m|^2 \, |u_i|^2.$$

Thus

$$\frac{r}{2}\|(\beta_n - \beta_m)u_i\|_{L^2}^2 \leqslant \|Du_i\|_{L^2}^2 + \frac{1}{m^2} \cdot \sup |\iota|^2 \cdot \|u_i\|_{L^2}^2.$$

Let $n \to \infty$, this gives

$$\|(1 - \beta_m)u_i\|_{L^2}^2 \leqslant \frac{2}{r}\|Du_i\|_{L^2}^2 + \frac{2}{m^2 r} \cdot \sup |\iota|^2 \cdot \|u_i\|_{L^2}^2.$$

Therefore

$$\|u_i\|_{L^2}^2 \leqslant 2\|\beta_m u_i\|_{L^2}^2 + 2\|(1 - \beta_m)u_i\|_{L^2}^2$$

$$\leqslant 2\|\beta_m u_i\|_{L^2}^2 + \frac{4}{r}\|Du_i\|_{L^2}^2 + \frac{4}{m^2 r} \cdot \sup |\iota|^2 \cdot \|u_i\|_{L^2}^2.$$

Take $m > \sqrt{\frac{8}{r}} \cdot \sup |\iota|$, then

$$\|u_i\|_{L^2}^2 \leqslant 4\|\beta_m u_i\|_{L^2}^2 + \frac{8}{r}\|Du_i\|_{L^2}^2.$$

By (1.15) and (1.16), this proves $\lim_{i \to \infty} \|u_i\|_{L^2} = 0$. Hence the regularity of D implies that $u_i \to 0$ in $L^2_{1,A_E}(E)$, therefore $s_i \to s$ in $L^2_{1,A_E}(E)$ and the lemma is proved. $\square$

Lemma 1.20. *If D is a (k, r) admissible operator, then the map*

$$D : L^2_{1,A_E}(E) \to L^2(F)$$

has finite dimensional kernel and closed range.

Proof. The lemma 1.19 implies that the unit ball of $\ker D$ is compact, therefore $\ker D$ has finite dimension.

Now let y_i be a convergent sequence in $\mathrm{Im}\, D$, let $y = \lim_{i \to \infty} y_i$. Take a sequence $\{s_i\}$ such that $Ds_i = y_i$ and $s_i \perp \ker D$ in the space $L^2_{1,A_E}(E)$.

If $\|s_i\|_{L^2}$ are bounded then lemma 1.19 asserts that $\{s_i\}_{i \geqslant 0}$ has a convergent subsequence in $L^2_{1,A_E}(E)$, hence $y \in \operatorname{Im} D$.

If $\|s_i\|_{L^2}$ are not bounded, without loss of generality assume that

$$\lim_{i \to \infty} \|s_i\|_{L^2} = \infty.$$

Take $u_i = s_i/\|s_i\|_{L^2}$, then $Du_i \to 0$, $\|u_i\|_{L^2} = 1$. Lemma 1.19 then asserts that a subsequence of u_i converges to an element of $\ker D$ in $L^2_{1,A_E}(E)$, contradicting the assumption that $s_i \perp \ker D$. $\qquad\square$

Proof of proposition 1.18. First prove the result for $l = 0$. In this case, every element u in the L^2 orthogonal complement of $\operatorname{Im} D$ satisfies $D^*u = 0$ in the sense of distributions. The regularity of the operator D thus imples $u \in L^2_{1,A_F}(F)$. On the other hand, every element of $\ker D^*$ is perpendicular to $\operatorname{Im} D$ in the L^2 norm. Hence $\operatorname{Im} D^\perp \cong \ker D^*$. Apply lemma 1.20 to D and D^*, this proves that D is Fredholm and $\dim \operatorname{coker} D = \dim \ker D^*$.

The result for $0 < l \leqslant k$ follows from the regularity of D. In fact, let R_l be the image of $D : L^2_{l+1,A_E}(E) \to L^2_{l,A_F}(F)$, then $R_l = R_0 \cap L^2_{l,A_F}(F)$ and the result for $l = 0$ implies that R_l is a closed subspace of $L^2_{l,A_F}(F)$ of finite codimension. Since R_l is dense in R_0 under the L^2 norm, the codimension of R_l in $L^2_{l,A_F}(F)$ equals the codimension of R_0 in $L^2(F)$. $\qquad\square$

Corollary 1.21. *Let r_0 be as in lemma 1.17 and assume $r > r_0$, then $\mathscr{D}$ and $\mathscr{D}^*$ are Fredholm operators.* $\qquad\square$

The rest of this section calculates the index of $\mathscr{D}$. The following lemma proves the index formula when the relative Euler number $e(M, \mathfrak{s}, \{\tau_j\})$ is zero.

Lemma 1.22. *Let r_0 be as in lemma 1.17 and let $r > r_0$. If*

$$e(M, \mathfrak{s}, \{\tau_j\}) = 0,$$

then ind $\mathscr{D} = 0$.

Proof. Since $e(M, \mathfrak{s}, \{\tau_j\}) = 0$, there exists an element

$$(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, 1, \{\tau_j\})$$

such that $|\phi| > 1/2$ everywhere. Define

$$(A, \Phi) = (A, \sqrt{r}\,\phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\}).$$

For sufficiently large r, (A, Φ) satisfies

$$(1.17) \quad |\Phi(x)|^2 \geqslant \frac{1}{2} \max_{x \in M} \Big(|Ric(x)| + |\mathscr{R}(x)| + \frac{1}{4}|s(x)| + \frac{1}{2}|F_A(x)|$$
$$+ |\nabla_A \Phi(x)| + |D_A \Phi(x)| \Big)$$

where $\mathscr{R}$ is the tensor defined from the Weyl curvature as in proposition 3.10 of [21]. By lemma 3.11 of [21], inequality (1.17) implies that both $\mathscr{D}$ and $\mathscr{D}^*$ have trivial kernel, hence by proposition 1.18, the operator $\mathscr{D}$ defined at (A, Φ) has ind $\mathscr{D} = 0$. Since the index is invariant under homotopy, this implies ind $\mathscr{D} = 0$ at every point of $\mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$. $\qquad\square$

To calculate the index for the general case, one needs the following lemma:

Lemma 1.23. *Let $(M_1, \mathfrak{s}_1)$ and $(M_2, \mathfrak{s}_2)$ be two 4-manifolds with $Spin^c$ structures. Then there exists a $Spin^c$ structure $\mathfrak{s}_1 \# \mathfrak{s}_2$ on $M_1 \# M_2$, such that*

$$\mathfrak{s}_1 \# \mathfrak{s}_2 |_{M_i - pt} \cong \mathfrak{s}_i |_{M_i - pt}, \quad i = 1, 2.$$

Proof. Define a new metric on $M_1 - \text{pt}$ and $M_2 - \text{pt}$ such that the ends of the two manifolds are cylinders $S^3 \times [0, \infty)$ with the standard product metric. By lemma 1.9, $\mathfrak{s}_1$ and $\mathfrak{s}_2$ pulls back to $Spin^c$ structures on $M_1 - \text{pt}$ and $M_2 - \text{pt}$ with the new metric. On the end of $M_i - \text{pt}$ the $Spin^c$ structure is a product $Spin^c$ structure given by a

Spin^c structure on S^3. Since $H^1(S^3) = H^2(S^3) = 0$, there is only one Spin^c structure $\mathfrak{t}$ on S^3 up to isomorphism, and any self-isomorphism of $\mathfrak{t}$ is homotopic to identity. Therefore up to homotopy there is a unique way to glue the Spin^c structures on the ends of $M_1 - \mathrm{pt}$ and $M_2 - \mathrm{pt}$ together and that gives the desired Spin^c structure $\mathfrak{s}_1 \# \mathfrak{s}_2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Lemma 1.24. *If M is a manifold with symplectic ends, and suppose that $(\mathfrak{s}, \{\tau_j\})$ are given, then there exists a closed 4-manifold with a Spin^c structure $(N, \mathfrak{s}_N)$ such that $e(M \# N, \mathfrak{s} \# \mathfrak{s}_N, \{\tau_j\}) = 0$*

Proof. By lemma 28.2.3 of [22], for any compact 4-manifold with a Spin^c structure,

$$\left(c_2(S^+) - \frac{1}{4}c_1(S^+)^2\right)[N] = -\frac{1}{4}\big(2\chi(N) + 3\sigma(N)\big).$$

Therefore for two given compact 4-manifolds with Spin^c structures $(N_1, \mathfrak{s}_1)$ and $(N_2, \mathfrak{s}_2)$ one has the following identity:

$$c_2(N_1 \# N_2, S^+)[N_1 \# N_2] = c_2(N_1, S_1^+)[N_1] + c_2(N_2, S_2^+)[N_2] + 1,$$

where S, S_1 and S_2 are the spinor bundles for $\mathfrak{s}_1 \# \mathfrak{s}_2$, $\mathfrak{s}_1$, and $\mathfrak{s}_2$. Since connected sum is a local construction this also proves

$$e(M \# N, \mathfrak{s} \# \mathfrak{s}_N, \{\tau_j\}) = e(M, \mathfrak{s}, \{\tau_j\}) + c_2(N, S^+)[N] + 1,$$

therefore one needs to find compact manifolds $(N, \mathfrak{s}_N)$ such that

$$(1.18) \qquad\qquad c_2(N, S^+)[N] = -e(M, \mathfrak{s}, \{\tau_j\}) - 1.$$

Notice that $c_1(S^+)$ can take any characteristic element in the intersection form. Therefore there is a Spin^c structure on $\mathbb{C}P^2$ such that $c_1(S^+)$ equals a generator of $H^2(\mathbb{C}P^2, \mathbb{Z})$, and for this Spin^c structure $c_2 = -2$. On the other hand the canonical Spin^c structure on $\mathbb{C}P^2$ has a nowhere vanishing section and therefore $c_2 = 0$. By

taking connected sum of these two Spin^c structures, one can easily construct a closed manifold $(N, \mathfrak{s}_N)$ satisfing (1.18). Hence the lemma is proved. $\qquad\square$

Proof of proposition 1.15. The Fredholm property of $\mathscr{D}$ is already contained in Corollary 1.21, the rest of the proof gives the index formula.

If $e(M, \mathfrak{s}, \{\tau_j\}) = 0$, the index formula follows from lemma 1.22. If $(M, \mathfrak{s})$ is a closed manifold with a Spin^c structure, it is well-known that $\operatorname{ind} \mathcal{D} = e(M, \mathfrak{s})$ (see for example lemma 27.1.1 and lemma 28.2.3 of [22]).

For the general case, let M be a manifold with symplectic ends, let $(\mathfrak{s}_M, \{\tau_j\})$ be the compatible Spin^c structure. By lemma 1.24, there exists a closed manifold N with a Spin^c structure $\mathfrak{s}_N$ such that $e(M \# N, \mathfrak{s} \# \mathfrak{s}_N, \{\tau_j\}) = 0$. Let $(M', \mathfrak{s}_{M'})$ be an arbitrary closed manifold with a Spin^c structure. For a manifold X with symplectic ends and a compatible Spin^c structure, use ind_X to denote the index of the corresponding operator $\mathcal{D}$. Then the excision property of index implies

$$\operatorname{ind}_M = \operatorname{ind}_{M'} - \operatorname{ind}_{M' \# N} + \operatorname{ind}_{M \# N}.$$

Therefore the index formula for the general case follows from lemma 1.22 and the index formula for closed manifolds. $\qquad\square$

1.5. **Exponential decay of solutions.** Let M be a manifold with each end being either symplectic or cylindrical with an admissible Spin^c structure $(\mathfrak{s}, \{\tau_j\})$. The purpose of this section is to establish an exponential decay estimate for solutions of equation (1.10) in $\mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$. As before, assume that $k \geqslant 5$, therefore on each symplectic end G_j,

$$(1.19) \qquad \int_{G_j} E < +\infty.$$

By definition, the compatible symplectic structure on each symplectic end G_j can be extended to a neighborhood of G_j. Let A_0 be a smooth extention of the canonical

connections from the symplectic ends to $(M, \mathfrak{s})$ such that on the cylindrical ends A_0 is in temporal gauge and is invariant under translations. For the rest of the section, the connection A_0 will be fixed. Let Φ_0 be the pull back of the canonical section on $\cup G_j$.

Let $\phi = \sqrt{r}\,(\Phi_0 \cdot \tau_j^* \alpha + \tau_j^* \beta)$ be the decomposition of ϕ on G_j as in (1.11). Let $a = A - A_0$.

Let $K > 0$ be an upper bound of the sectional curvature of M, let $\epsilon_j > 0$ be positive numbers such that the compatible symplectic structure on each symplectic end G_j can be extended to the ϵ_j-neighborhood of G_j. Let $\epsilon_0 = \min\{\mathrm{inj}(W), \pi/(2\sqrt{K}), \min_j \epsilon_j, 1\}$. For each $x \in M$, consider the Gaussian normal coordinates centered at x with radius ϵ_0. Let $\{g_{ij}^{(x)}\}_{1 \leqslant i,j \leqslant 4}$ be the metric matrix under this coordinate frame. Since M has bounded geometry, for each $k \geqslant 0$ the functions $\partial^k g_{ij}^{(x)}$ are uniformly bounded for any x.

To avoid excessively wordy explanations for the dependence of constants, the following conventions will be assumed for the rest of the chapter: when a constant is said to be depending on a manifold M with each end being either symplectic or cylindrical, it means the constant depends on the manifold M and the admissible Spin^c structure $(\mathfrak{s}, \{\tau_j\})$, the choice of the perturbation $\mathfrak{q}_t$, the cutoff functions in the definition of equation (1.10), and the choice of the connection A_0. The notations z_i will denote constants that only depend on M. It is always assumed that $r > r_0$, where r_0 is a positive constant sufficiently large so that the estimates will work. The value of r_0 may increase as the argument moves on, and r_0 depends only on M.

1.5.1. *Convergence of configurations.* This subsection defines a version of convergence up to gauge transformations, and provides a sufficient condition for the existence of convergent subsequences. The result might be obvious to experts, but a proof is given here for lack of a direct reference.

Definition 1.25. Suppose $\{(M_n, g_n)\}_{n \geqslant 1}$ is a sequence of complete oriented Riemann-ian 4-manifolds. For each n, let p_n be a point on M_n, let $\mathfrak{s}_n$ be a Spin^c structure on M_n and S_n be the corresponding spinor bundle, and $\rho_n : TM_n \to \mathrm{Hom}(S_n, S_n)$ the Clifford actions. Let A_n be a locally L_k^2 Spin^c-connection for $\mathfrak{s}_n$, let ϕ_n be a locally L_k^2 section of S_n^+.

The sequence $\{(M_n, g_n, p_n, \mathfrak{s}_n, A_n, \phi_n)\}_{n \geqslant 1}$ is said to be convergent to another con-figuration

$$(M, g, p, \mathfrak{s}, A, \phi)$$

up to gauge transformations, if there exists a sequence

$$\{(d_n, U_n, V_n, \varphi_n, \tilde{\varphi}_n, u_n)\}_{n \geqslant 1}$$

such that the following conditions hold:

(1) For any n, the element d_n is a positive number, and $\lim\limits_{n \to \infty} d_n = +\infty$. The element V_n is an open neighborhood of p_n, and U_n is an open neighborhood of p. The open sets V_n and U_n satisfy $B_{M_n}(p_n, d_n) \subset V_n$ and $B_M(p, d_n) \subset U_n$. The element φ_n is a diffeomorphism from U_n to V_n mapping p to p_n. Moreover, for each compact subset K of M,

$$\lim\limits_{n \to \infty} \|\varphi_n^*(g_n) - g\|_{C^m(U_n \cap K)} = 0, \quad \forall m \in \mathbb{N}.$$

(2) The element $\tilde{\varphi}_n$ is a map from $S_n|_{V_n}$ to $S|_{U_n}$, which is a smooth isomorphism of vector bundles lifting φ_n. Moreover, for each compact subset K of M,

$$\lim\limits_{n \to \infty} \|\tilde{\varphi}_n^*(\rho_n) - \rho\|_{C^m(U_n \cap K)} = 0, \quad \forall m \in \mathbb{N}$$

(3) The element u_n is a gauge transformation of $\mathfrak{s}_n$ on V_n. Moreover, for each compact subset K of M,

$$\lim\limits_{n \to \infty} \|\tilde{\varphi}_n^*(u_n(A_n, \phi_n)) - (A, \phi)\|_{C^m(U_n \cap K)} = 0, \quad \forall m \in \mathbb{N}$$

Definition 1.26. A sequence of pointed complete Riemannian manifolds

$$\{(M_n, g_n, p_n)\}_{n \geqslant 0}$$

is said to have uniformly bounded geometry if the following conditions hold:

(1) There exists $\epsilon_0 > 0$ such that for all n, the injectivity radius of M_n is greater than ϵ_0.

(2) For each integer k, there is a constant $N_k > 0$ such that the norm of the kth covariant derivative of the curvature tensor of M_n is bounded by N_k for every n.

The following result is essentially a properness property for the Seiberg-Witten map.

Proposition 1.27. *Let $\{(M_n, g_n, p_n)\}_{n \geqslant 1}$ be a sequence of pointed complete oriented Riemannian 4-manifolds with uniformly bounded geometry, let ϵ_0 be a positive lower bound of their injecticity radii. Let $\mathfrak{s}_n$ be a $Spin^c$ structure on M_n, let A_n be locally L_1^2 $Spin^c$-connections for $\mathfrak{s}_n$ and ϕ_n be locally L_1^2 sections of S_n^+. Assume that there exists a constant $C > 0$ such that for every point $x \in M_n$,*

(1.20) $$\int_{B_{M_n}(x,\epsilon_0)} |F_{A_n}|^2 < C,$$

(1.21) $$|\phi_n(x)| < C.$$

Moreover, assume that

(1.22) $$D_{A_n}(\phi_n) = 0$$

for each n, and assume that $\mathfrak{F}(A_n, \phi_n)$ is smooth for each n and there is a sequence of positive numbers C_k such that

(1.23) $$\|\mathfrak{F}(A_n, \phi_n)\|_{C^k} < C_k, \quad \forall k, n \geqslant 1.$$

Then there exists a subsequence of $\{(M_n, g_n, p_n, \mathfrak{s}_n, A_n, \phi_n)\}_{n \geqslant 1}$ which converges to some $(M, g, p, \mathfrak{s}, A, \phi)$ up to gauge transformations in the sense of definition 1.25.

Remark 1.28. The assumption (1.22) is not essential. However, if $D_{A_n}(\phi_n)$ is not 0, one needs to be more careful about the formulation of (1.23). Since this chapter will only use the result when (1.22) is satisfied, the more general statement will not be discussed here.

Proof. Since $\{(M_n, g_n, p_n)\}$ have uniformly bounded geometry, there is a pointed complete Riemannian manifold (M, g, p) such that a subsequence of $\{(M_n, g_n, p_n)\}$ converges to (M, g, p) in C^∞, in the sence as defined in [30]. Without loss of generality assume the subsequence is $\{(M_n, g_n, p_n)\}$ itself. Then by the definition there exists a sequence $\{(d_n, U_n, V_n, \phi_n)\}$ such that condition 1 of definition 1.25 is satisfied.

By (1.20), the L^2 norm of $\varphi_n^*(F_{A_n})$ is bounded on any compact subset of M. Take any fixed compact surface Σ in M, let $N(\Sigma)$ be a tubular neighborhood of Σ. Then the L^2 bound of $\varphi_n^*(F_{A_n})$ implies that the sequence $\int_{N(\Sigma)} |\varphi_n^*(F_{A_n})|$ is bounded. Let $P(\Sigma)$ be a closed 2-form that is supported in $N(\Sigma)$ and represents the Poincaré dual of Σ, then

$$\int_\Sigma c_1(\varphi_n^*(\mathfrak{s}_n)) = \frac{1}{2\pi i} \int_{N(\Sigma)} \varphi_n^*(F_{A_n}) \wedge P(\Sigma)$$

is uniformly bounded for any compact surface Σ in M. Therefore there exists a subsequence $\{n_i\}$ of $\{n\}$ such that for each compact subset K of M, the Chern class $c_1(\varphi_n^*(\mathfrak{s}_{n_i}))|_K$ is constant for large i. By taking a further subsequence one may assume that $\varphi_n^*(\mathfrak{s}_{i_n})|_K$ is a constant sequence up to isomorphisms when n is large. Without loss of generality, assume this subsequence is the original sequence. Then there is a Spin^c structure $\mathfrak{s}$ on M such that $\mathfrak{s}|U_n \cong \varphi_n^*(\mathfrak{s}_n|_{V_n})$. Now φ can be lifted to $\tilde{\varphi}$ so that condition 2 of definition 1.25 holds.

The only thing remaining to prove is the existence of the gauge transformations u_n satisfying condition 3 in definition 1.25. By theorem 5.2.1 of [22] one only needs

to prove the boundedness of the analytic energy of $\{(A_n, \phi_n)\}$ on $\varphi_n^{-1}(K)$ for any given compact subset $K \subset M$. (The theorem in [22] requires the manifold and the Spin^c structure to be fixed but the same argument works for a sequence of manifolds satisfying conditions 1 and 2 of definition 1.25.)

Without loss of generality, assume that $\overline{V_n}$ is compact for each n. Fix a positive integer m and let $n \to +\infty$, one only needs to prove:

$$\mathcal{E}^{an}|_{V_m}(A_n, \phi_n) < N_m, \quad \forall n > m$$

for some constant N_m depending on m.

Since the manifolds M_n have uniformly bounded geometry, without loss of generality (by taking a further subsequences if necessary) assume that $U_{n-1} \subset\subset U_n$, and that there are cut-off functions $\chi_n \geqslant 0$ defined on M such that $\operatorname{supp} \chi_n \subset U_n$, $\chi_n|_{U_{n-1}} = 1$, and the pull-back of χ_n to M_n satisfies $|\nabla \varphi_n^*(\chi_n)| \leqslant 1$ for all n.

For $n > m$, let $(A_n', \phi_n') = (A_n, \varphi^*(\chi_{m+1}) \cdot \phi_n)$. By assumptions (1.21) and (1.23),

$$(1.24) \qquad \|\mathfrak{F}(A_n', \phi_n')\|_{L^2} < C', \quad \forall n \geqslant m \geqslant 1$$

for some constant C' depending on m.

Since A_n' is compactly supported on V_m, the topological energy of (A_n', ϕ_n') on V_m (see definition 4.5.4 of [22]) is

$$(1.25) \qquad \mathcal{E}^{top}(A_n', \phi_n')|_{V_m} = \frac{1}{4} \int_{V_n} F_{(A_n')^t} \wedge F_{(A_n')^t},$$

and it is bounded because of (1.20). Since

$$\mathcal{E}^{an}(A_n', \phi_n') = \mathcal{E}^{top}(A_n', \phi_n') + \|\mathfrak{F}(A_n', \phi_n')\|_{L^2}^2,$$

(1.24) and (1.25) imply the boundedness of $\mathcal{E}^{an}(A_n', \phi_n')$ on V_{m+1}. Notice that one has $(A_n, \phi_n) = (A_n', \phi_n')$ on V_m, so this implies the boundedness of $\mathcal{E}^{an}(A_n, \phi_n)$ on V_m. $\qquad \square$

1.5.2. C^0 *bound.* Let M be a manifold with each end being either symplectic or cylindrical, and $(\mathfrak{s}, \{\tau_j\})$ be an admissible Spin^c structure. Let

$$(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$$

be a solution of equation (1.10). Let $a = A - A_0$. This subsection proves a C^0 estimate for ϕ and F^+. The main result is the following estimate:

Proposition 1.29. *There exists a constant z such that $|\phi| \leqslant z \cdot \sqrt{r}$ and $|F_a^+| \leqslant z \cdot r$ at every point of M.*

The proof starts with a C^0 estimate on symplectic ends. As before, use G_j to denote the symplectic ends of M. Let N_j be the ϵ_0 neighborhood of ∂G_j. The following lemma is inspired by lemma 3.23 of [21].

Lemma 1.30. *There exists a constant $z > 0$, such that for every locally L_k^2 configuration (A, ϕ) defined on the end G_j solving (1.10), the inequality*

$$|\phi| \leqslant z \cdot \sqrt{r}$$

holds on $G_j - N_j$.

Proof. By inequality (2.2) of [37], the following inequality holds on G_j:

$$\frac{1}{2} d^* d |\phi|^2 + |\nabla_A \phi|^2 + \frac{1}{4} |\phi|^2 (|\phi|^2 - r) - z_1 \cdot |\phi|^2 \leqslant 0.$$

Require $r_0 > 4z_1$, and throw away the term $|\nabla_A \phi|^2$, one obtains

$$\frac{1}{2} d^* d |\phi|^2 + \frac{1}{4} |\phi|^2 (|\phi|^2 - 2r) \leqslant 0.$$

For any $x \in G_j - N_j$, use B_x to denote the ball on M centered at x with radius ϵ_0. Let ρ be the distance function to x, let $f = 1/(\epsilon_0^2 - \rho^2)^2$ be a radial function on B_x. Take a normal coordinate of B_x centered at x and let g be the determinant of the

metric matrix, then on $B_x - \{x\}$,

$$d^* df = -\frac{1}{\rho^{n-1}\sqrt{g}}\frac{\partial}{\partial \rho}\Big(\rho^{n-1}\sqrt{g} \cdot \frac{df}{d\rho}(\rho)\Big).$$

Notice that M has bounded geometry, hence $\|g\|_{C^0}$ and $\|\nabla g\|_{C^0}$ are both bounded by constants independent of x, and g is bounded away from 0. A straight forward calculation shows that for some large constant $z_2 > 0$,

$$\frac{1}{2}d^*d\big((z_2)^2 rf\big) + \frac{1}{4}\big((z_2)^2 rf\big)\big((z_2)^2 rf - 2r\big) \geqslant 0.$$

By the maximum principle, this shows $|\phi|^2 \leqslant (z_2)^2 rf + 2r$ on B_x, hence $|\phi(x)| \leqslant z \cdot \sqrt{r}$ for some constant z. $\qquad\square$

The following lemma deals with the cylindrical ends:

Lemma 1.31. *Let $H_i \cong Y_i \times [0, +\infty)$ be a cylindrical end, use t to denote the projection from H_i to $[0, +\infty)$. Then there exists a constant z only depending on the perturbations $\mathfrak{q_t}$ such that*

$$\limsup_{t \to +\infty} |\phi| \leqslant z$$

Proof. Since $\mathfrak{q}$ is an admissible perturbation, there are only finitely many critical points of the corresponding perturbed Chern-Simons-Dirac functional $\mathcal{L}$ modulo gauge transformations. Let Σ be the set of critical points of $\mathcal{L}$, let

$$z = \max_{(B,\psi) \in \Sigma} \|\psi\|_{C^0}$$

Then the definition of $\mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ implies $\limsup\limits_{t \to +\infty} |\phi| \leqslant z$. $\qquad\square$

Now starts the prove of proposition 1.29.

Proof of proposition 1.29. There are two possibilities:

. *Case 1.* The supremum $\sup_M |\phi|$ is either not achieved on M or achieved in $G_j - N_j$ for some symplectic end G_j. In this case $\sup_M |\phi| \leqslant z \cdot \sqrt{r}$ by lemma 1.30 and lemma 1.31.

. *Case 2.* For some $x_0 \in M - \cup(G_j - N_j)$, $|\phi(x_0)| = \sup_M |\phi|$. Let B_{x_0} be the closed ball on M centered at x_0 with radius ϵ, where $\epsilon < \epsilon_0$ is a positive constant to be determined later. Then (ϕ, A) satisfies the following equations on B_{x_0}:

$$(1.26) \qquad \rho(F_{A^t}^+) = (\phi\,\phi^*)_0 + \hat{\mu}^0(A, \phi),$$

$$(1.27) \qquad D_A\phi = \hat{\mu}^1(A, \phi).$$

Since the perturbations $\mathfrak{p}_i$ and $\mathfrak{q}_{\mathfrak{t}}$ are strongly tame and $\|\mathfrak{p}_i\| < 1$, there exists a constant z' such that the following holds on B_{x_0}:

$$(1.28) \qquad \|\hat{\mu}^0(A, \phi)\|_{C^0} \leqslant z' \cdot (1 + |\phi(x_0)|),$$

$$(1.29) \qquad \|\hat{\mu}^1(A, \phi)\|_{C^0} \leqslant z'.$$

Apply D_A to both sides of (1.27),

$$D_A^2\phi = D_A(\hat{\mu}^1).$$

By the Weitzenböck formula, this implies

$$(1.30) \qquad \nabla_A^*\nabla_A\phi + \frac{1}{2}\rho(F_{A^t}^+)\phi + \frac{1}{4}s\phi = D_A(\hat{\mu}^1),$$

where s denotes the scalar curvature of M. Plugging in (1.26), and take inner product with ϕ, equation (1.30) becomes

$$(1.31) \qquad \frac{1}{2}d^*d|\phi|^2 + |\nabla_A\phi|^2 + \frac{1}{4}|\phi|^4 + \frac{1}{4}\langle s\phi, \phi\rangle + \frac{1}{2}\langle \hat{\mu}^0\phi, \phi\rangle = \langle D_A(\hat{\mu}^1), \phi\rangle.$$

Notice that the inequality of arithmetic and geometric means gives

$$\frac{1}{16}|\phi|^4 - \frac{1}{2}z'|\phi(x_0)| \cdot |\phi|^2 \geqslant -(z')^2|\phi(x_0)|^2$$

Therefore inequality (1.31) implies:

$$\frac{1}{2}d^*d|\phi|^2 + |\nabla_A\phi|^2 + \frac{1}{8}|\phi|^4 - z_0(1 + |\phi(x_0)|^2) \leqslant \langle D_A(\hat\mu^1), \phi\rangle.$$

Let $h \geqslant 0$ be a smooth radial function on B_{x_0} such that $h = 1$ on $B_W(x_0, \epsilon/4)$ and $\operatorname{supp} h \subset B_W(x_0, \epsilon/2)$. Let $\chi = h^4$. Let $G_{x_0} \geqslant 0$ be the Green's kernel on B_{x_0} with a pole at x_0 and equals zero on ∂B_{x_0}. Then:

$$\int_{B_{x_0}} \left(\frac{1}{2}d^*d|\phi|^2 + |\nabla_A\phi|^2 + \frac{1}{8}|\phi|^4 - z_0(1 + |\phi(x_0)|^2)\right) \cdot G_{x_0} \cdot \chi$$

$$\leqslant \int_{B_{x_0}} \langle D_A(\hat\mu^1), \phi \cdot G_{x_0}\chi\rangle,$$

which is the same as

$$(1.32) \quad \int_{B_{x_0}} \left(\left(-\frac{1}{2}\Delta(|\phi|^2\chi) + \frac{1}{2}|\phi|^2\Delta\chi + \nabla|\phi|^2 \cdot \nabla\chi\right) \cdot G_{x_0} + |\nabla_A\phi|^2 G_{x_0}\chi\right.$$

$$\left. + \frac{1}{8}|\phi|^4 G_{x_0}\chi - z_0(1 + |\phi(x_0)|^2) G_{x_0}\chi\right) \leqslant \int_{B_{x_0}} \langle\hat\mu^1, D_A(\phi G_{x_0}\chi)\rangle.$$

Therefore

$$(1.33) \quad \frac{1}{2}|\phi(x_0)|^2 \leqslant \int_{B_{x_0}} \left((-\frac{1}{2}|\phi|^2\Delta\chi - \nabla|\phi|^2 \cdot \nabla\chi) \cdot G_{x_0} - |\nabla_A\phi|^2 G_{x_0}\chi\right.$$

$$\left. - \frac{1}{8}|\phi|^4 G_{x_0}\chi + z_0(1 + |\phi(x_0)|^2) G_{x_0}\chi + \langle\hat\mu^1, D_A(\phi G_{x_0}\chi)\rangle\right).$$

By the inequality of arithmetic and geometric means,

$$-\frac{1}{2}|\phi|^2\Delta\chi - \frac{1}{16}|\phi|^4\chi$$

$$\leqslant |\phi|^2(2h^3|\Delta h| + 6h^2|\nabla h|^2) - \frac{1}{16}|\phi|^4 h^4$$

$$\leqslant z_1(|\Delta h|^2 h^2 + |\nabla h|^4),$$

and

$$|\nabla|\phi|^2| \cdot |\nabla \chi| - \frac{1}{2}|\nabla_A \phi|^2 \chi - \frac{1}{16}|\phi|^4 \chi$$

$$\leqslant 2|\phi| \cdot |\nabla_A \phi| \cdot (4h^3|\nabla h|) - \frac{1}{2}|\nabla_A \phi|^2 h^4 - \frac{1}{16}|\phi|^4 h^4$$

$$\leqslant z_2|\phi|^2 h^2 |\nabla h|^2 - \frac{1}{16}|\phi|^4 h^4$$

$$\leqslant z_3|\nabla h|^4.$$

By (1.29) and the inequality of arithmetic and geometric means,

$$\int_{B_{x_0}} \langle \hat{\mu}^1, D_A \phi \rangle G_{x_0} \chi \leqslant \int_{B_{x_0}} \frac{1}{2}|\nabla_A \phi|^2 G_{x_0}\chi + z_4 \int_{B_{x_0}} G_{x_0}\chi.$$

Thus by (1.33),

$$(1.34) \quad \frac{1}{2}|\phi(x_0)|^2 \leqslant z_0(1 + |\phi(x_0)|^2)\int_{B_{x_0}} G_{x_0}\chi$$

$$+ \int_{B_{x_0}} \left(z_1(|\Delta h|^2 h^2 + |\nabla h|^4) + z_3|\nabla h|^4 \right) G_{x_0} + z_4 \int_{B_{x_0}} G_{x_0}\chi$$

$$+ \int_{B_{x_0}} |\hat{\mu}^1||\phi||\nabla(G_{x_0}\chi)|.$$

By the assumption, $|\phi|$ attains maximum at x_0, therefore

$$\int_{B_{x_0}} |\hat{\mu}^1||\phi||\nabla(G_{x_0}\chi)| \leqslant |\phi(x_0)|\int_{B_{x_0}} |\hat{\mu}^1||\nabla(G_{x_0}\chi)|$$

$$(1.35) \qquad\qquad \leqslant \frac{1}{4}|\phi(x_0)|^2 + \left(\int_{B_{x_0}} |\hat{\mu}^1||\nabla(G_{x_0}\chi)| \right)^2.$$

Notice that the constants z_i do not depend on the choice of ϵ. Take ϵ small enough such that

$$(1.36) \qquad\qquad \int_{B_{x_0}} G_{x_0} \leqslant \frac{1}{8z_0}.$$

Since M has bounded geometry, the choice of ϵ can be made to be independent of x_0. Plug in (1.35) and (1.36) to (1.34) and rearrange, this gives

$$\frac{1}{8}|\phi(x_0)|^2 \leqslant z_0 \int_{B_{x_0}} G_{x_0}\,\chi$$
$$+ \int_{B_{x_0}} \left(z_1(|\Delta h|^2 h^2 + |\nabla h|^4) + z_3|\nabla h|^4\right) G_{x_0}$$
$$+ z_4 \int_{B_{x_0}} G_{x_0}\,\chi + \left(\int_{B_{x_0}} |\hat{\mu}^1||\nabla(G_{x_0}\,\chi)|\right)^2.$$

Therefore $|\phi(x_0)|^2 \leqslant z_5$ for some constant z_5.

Combining case 1 and case 2, this proves the C^0 bound for $|\phi|$. The bound for $|F_a^+|$ then follows from equation (1.10) and the C^0 bound of the perturbation. $\qquad\square$

1.5.3. *Exponential decay on symplectic ends.* This subsection proves the exponential decay of E on symplectic ends. Recall that given a configuration (A, ϕ) on any symplectic end G_j, the function E is defined as

$$E = |1 - |\alpha|^2 - |\beta|^2|^2 + |\beta|^2 + |\nabla_a\alpha|^2 + |\nabla'_A\beta|^2 + |F_a|^2.$$

If $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ then

$$\int_{G_j} E < +\infty.$$

The main result of this subsection is:

Proposition 1.32. *There exists a constant z such that the following holds. Let (A, ϕ) be a locally L_k^2 solution of (1.10) on a symplectic end G_j. Let d be the distance function to ∂G_j. Assume $\int_{G_j} E < +\infty$. Then there is a constant d_0 which may depend on (A, ϕ) such that*

$$E(x) < e^{-\sqrt{r}\cdot(d(x)-d_0)/z}$$

for every $x \in G_j$ with $d(x) > d_0$.

To prove proposition 1.32, one needs the following lemma:

Lemma 1.33. *Suppose (A, ϕ) satisfies the assumptions of proposition 1.32, then on the end G_j, one has $\lim\limits_{d(x) \to +\infty} E(x) = 0$.*

Proof. Assume the contrary, then there is a sequence $\{x_n\} \subset G_j$ and a constant $\delta > 0$ such that $d(x_n) \to \infty$ and $E(x_n) \geqslant \delta$ for all n. Let $B_n = B_M(x_n, \epsilon_0)$. Without loss of generality, assume that B_n are pairwise disjoint. Let g be the metric of M, consider the sequence $(M, g, x_n, \mathfrak{s}, A, \phi)$. By proposition 1.27 and the C^0 estimate of ϕ in lemma 1.30, a subsequence of it converges to some limit configuration $(\widetilde{M}, \tilde{g}, \tilde{x}, \tilde{\mathfrak{s}}, \tilde{A}, \tilde{\phi})$. Since $\nabla^k \Omega$ is bounded for all k, the form Ω passes to a limit $\widetilde{\Omega}$ in the limit space $\widetilde{M}$. The form $\widetilde{\Omega}$ is a symplectic form compatible with $\tilde{g}$, and $\tilde{\mathfrak{s}}$ is the canonical Spin^c structure given by $(\tilde{g}, \widetilde{\Omega})$. Thus $\tilde{\phi}$ decomposes into an α component and a β component as in (1.11) and the corresponding energy density function $\widetilde{E}$ will satisfy $\widetilde{E}(\tilde{x}) \geqslant \delta$. Therefore there will be a possitive constant $\delta' > 0$ such that $\int_{B_n} E > \delta'$ for sufficiently large n. This contradicts the assumption of $\int_{G_j} E < +\infty$. $\qquad \square$

The next lemma is the work horse for all the estimates in the proof of proposition 1.32. The lemma and the way it is used in the proof of proposition 1.32 are inspired by the arguments in [21].

Lemma 1.34. *Let $K, v_0 > 0$ be constants. Let $r \geqslant 1$. Let N be an n-dimensional complete Riemannian manifold with $\mathrm{Ric} \geqslant -K$, let s be a C^2 function on the ball $B_N(x_0, R)$ with radius R. If s satisfies:*

$$\frac{1}{2} d^* d\, s + rV s \leqslant h,$$

$$s|_{\partial B_M(s,R)} = t.$$

where $V \geqslant v_0 > 0$ *is a positive function, then there is a positive constant ϵ depending only on n, K, and v_0, such that the following inequality holds:*

$$s(x_0) \leqslant \left(\sup_{B_N(x_0, R)} \left| \frac{h}{rV} \right| \right) + \left(\sup_{\partial B_N(x_0, R)} |t| \right) e^{-\epsilon R \sqrt{r}}.$$

If $\partial B_M(x_0, R) = \varnothing$, then the value of $\displaystyle\sup_{\partial B_M(x_0, R)} |t|$ is understood as 0.

Proof. Let ρ be the distance function to x_0. By the Laplacian comparison theorem (cf. [32, p. 7-8]),

$$\Delta \rho \leqslant \frac{n-1}{\rho}(1 + k\,\rho).$$

in the sense of distributions, where $k = \sqrt{K/(n-1)}$. Let $f(u) \geqslant u$ be a smooth function on $\mathbb{R}$ such that $f(u) = 1$ when $u \leqslant 1/2$ and $f(u) = u$ when $u \geqslant 1$. Let $h = e^{\epsilon \sqrt{r} f(\rho)}$ be a function on M, where ϵ is a small positive constant to be determined. Notice that in the sense of distributions,

$$-\Delta h = -\left(\epsilon \sqrt{r} f''(\rho) + \epsilon^2 r (f'(\rho))^2 + \epsilon \sqrt{r} f'(\rho) \Delta \rho \right) h$$

$$\geqslant -\left(\epsilon \sqrt{r} f''(\rho) + \epsilon^2 r (f'(\rho))^2 + \epsilon \sqrt{r} f'(\rho) \Big(\frac{n-1}{\rho} + k(n-1) \Big) \right) h.$$

Therefore, there exists a constant $z > 0$ independent of r such that when $\epsilon < 1/z$, the inequality $\frac{1}{2} d^* dh + rVh \geqslant 0$ holds in the sense of distributrions. By the maximum principal for weak solutions ([15], Theorem 8.1), this implies

$$s \leqslant \sup_{B_N(x_0, R)} \left| \frac{h}{rV} \right| + \left(\sup_{\partial B_N(x_0, R)} |t| \right) \cdot \frac{g}{e^{\epsilon \sqrt{r} f(R)}},$$

on the ball $B_N(x_0, R)$, hence the lemma is proved. $\qquad\square$

Proof of proposition 1.32. The proof is divided into 7 steps:

. *Step 1.* By lemma 1.33, there is a $d_1 > 0$ such that if $d(x) > d_1$ then

$$(1.37) \qquad\qquad\qquad |\alpha(x)| > \frac{1}{2}, \quad E(x) < 1.$$

In the following steps it will always be assumed that $d(x) > d_1$.

. *Step 2: pointwise estimates on α and β.* By [37] Lemma 2.2, there exist constants z_1, z_2, $z_3 \geqslant 1$, such that when $\zeta \in (0, \frac{r}{2z_1 z_2})$, $r > z_1$, and $\delta > z_3$, set

$$u = (1 - |\alpha|^2) - \zeta|\beta|^2 + \frac{\delta}{\zeta r},$$

then the following inequality holds:

$$\frac{1}{2}d^* du + \frac{r}{4}|\alpha|^2 u \geqslant 0.$$

Therefore lemma 1.34 and (1.37) implies that

$$u \geqslant -z_5 e^{-\sqrt{r} \cdot (d - d_1)/z_6}, \quad \text{if } d > d_1.$$

Thus

(1.38) $$|\alpha|^2 \leqslant 1 + \frac{z_7}{r^2},$$

(1.39) $$|\beta|^2 \leqslant \frac{z_7}{r}\left(1 - |\alpha|^2 + \frac{z_7}{r^2}\right),$$

whenever $d > d_0 + 1$.

. *Step 3: pointwise estimates on F_a.* On G_j, the first equation of (1.10) reads as

(1.40) $$F_a^+ = -\frac{i}{8}r \cdot \left(1 - |\alpha|^2 + |\beta|^2\right)\Omega + \frac{r}{4}(\alpha^*\beta - \alpha\beta^*).$$

Thus by inequalities (1.38) and (1.39),

$$|F_a^+| \leqslant \frac{r}{4\sqrt{2}}(1 + \frac{z_{11}}{r})(1 - |\alpha|^2) + z_{11}.$$

Now estimate $|F_a^-|$. By lemma 2.5 of [37], there exists constants z_{12}, z_{13}, z_{14}, z_{15} such that if $r > z_{15}$, let

$$q_0 = \frac{r}{4\sqrt{2}}(1 + \frac{z_{12}}{r})(1 - |\alpha|^2) - z_{13} \cdot r|\beta|^2 + z_{14},$$

$$s = |F_a^-|,$$

then

$$\frac{1}{2}d^*d(s - q_0) + \frac{r}{4}|\alpha|^2(s - q_0) \leqslant |\mathcal{R}|s,$$

where $\mathcal{R}$ is a tensor defined by curvature.

Therefore when $r > 8 \sup |\mathcal{R}|$,

$$\frac{1}{2}d^*d(s - q_0) + \frac{r}{8}|\alpha|^2(s - q_0) \leqslant |\mathcal{R}|q_0.$$

Notice that if $d(x) > d_1$ then $E(x) \leqslant 1$ hence q_0 is bounded by a constant. Apply lemma 1.34, this implies:

$$|F_a^-| \leqslant \frac{r}{4\sqrt{2}}(1 + \frac{z_{16}}{r})(1 - |\alpha|^2) + z_{16}.$$

In conclusion, there is a constant z_{17} such that

$$|F_a^\pm| \leqslant \frac{r}{4\sqrt{2}}(1 + \frac{z_{17}}{r})(1 - |\alpha|^2) + z_{17}.$$

. *Step 4: pointwise estimates on $|\nabla_a \alpha|$ and $|\nabla'_A \beta|$.* Let

$$y = |\nabla_a \alpha|^2 + r|\nabla'_A \beta|^2.$$

Inequality (2.43) of [37] shows that for a constant z_{18},

$$\frac{1}{2}d^*d(y - z_{18} \cdot r \cdot u) + \frac{r}{4}|\alpha|^2(y - z_{18} \cdot r \cdot u) \leqslant 0.$$

By lemma 1.34 again and the pointwise estimates on α and β, this implies

$$|\nabla_a \alpha|^2 + r \cdot |\nabla'_A \beta|^2 \leqslant z_{19} \cdot r \cdot (1 - |\alpha|^2) + z_{19}.$$

. *Step 5: Exponential decay of $|\nabla_a \alpha|$, $|\nabla'_A \beta|$, and $|\beta|$.* Let

$$y_1 = |\nabla_a \alpha|^2 + \frac{r}{32}|\nabla'_A \beta|^2 + \frac{r^2}{16\,z_{20}}|\beta|^2.$$

Inequality (4.15) of [37] shows that[1] if z_{20} is large enough then there exists a constant z_{21} such that,

$$(1.41) \qquad \frac{1}{2}d^* dy_1 + \frac{r}{4}|\alpha|^2 y_1 \leqslant \left(z_{21} \cdot r \cdot (1 - |\alpha|^2) + \frac{r}{8}\right)y_1$$

Take $d_2 > 0$ so that when $d > d_2$,

$$|1 - |\alpha|^2| < \min\left\{\frac{1}{16\,z_{21}}, \frac{1}{8}\right\}.$$

Then (1.41) implies

$$\frac{1}{2}d^* dy_1 + \frac{r}{32}y_1 \leqslant 0.$$

By lemma 1.34 this implies

$$(1.42) \qquad y_1 < z_{22} \cdot e^{\sqrt{r}\cdot(d - d_2)/z_{23}},$$

when $d > d_2$.

. *Step 6: decay of $|1 - |\alpha|^2|$.* By equation (2.3) of [37],

$$\frac{1}{2}d^* d|\alpha|^2 + |\nabla_a \alpha|^2 + \frac{r}{4}|\alpha|^2(|\alpha|^2 - 1 + |\beta|^2) + \alpha \boxtimes \nabla'_A \beta + \alpha \boxtimes \beta = 0,$$

where $\boxtimes$ are pointwise bilinear pairings defined by the metric and the symplectic form. A straight forward calculation thus shows

$$\frac{1}{4}d^* d|1 - |\alpha|^2|^2 = \left(\frac{1}{2}d^* d(1 - |\alpha|^2)\right) \cdot (1 - |\alpha|^2) - \frac{1}{2}|\nabla_a |\alpha|^2|^2$$

$$= -\frac{r}{4}|\alpha|^2|1 - |\alpha|^2|^2 + |\nabla_a \alpha|^2 \cdot (1 - |\alpha|^2)$$

---

[1] In [37], the derivation of inequality (4.15) only used the pointwise estimates of α, β, F_a, $\nabla_a \alpha$ and $\nabla'_A \beta$ developed in section 2, and it doesn't depend on the refined pointwise estimate of F_a^- developed in section 3d. Therefore, the inequalities derived from step 2 to step 4 are sufficient for deriving inequality (1.41) here.

$$+ \frac{r}{4}|\alpha|^2|\beta|^2(1 - |\alpha|^2) + (1 - |\alpha|^2) \cdot (\alpha \boxtimes \nabla'_A \beta + \alpha \boxtimes \beta).$$

The inequality above and (1.42) shows that when $d > d_2$,

$$\frac{1}{4}d^*d|1 - |\alpha|^2|^2 + \frac{r}{4}|\alpha|^2|1 - |\alpha|^2|^2 \leqslant z_{24} \cdot e^{(d-d_3)\sqrt{r}/z_{25}}.$$

By lemma 1.34, this implies

(1.43)
$$|1 - |\alpha|^2|^2 < z_{26} \cdot e^{(d-d_3)\sqrt{r}/z_{27}},$$

when $d > d_2$.

. *Step 7: decay of $|F_a|$.* The exponential decay for $|F_a^+|$ follows from (1.40), (1.42) and (1.43). For $s = |F_a^-|$, inequality (2.19) of [37] gives

$$\frac{1}{2}d^*ds + \frac{r}{4}(|\alpha|^2 + |\beta|^2)s \leqslant |\mathcal{R}|s + \frac{r}{4\sqrt{2}}(|\nabla_a \alpha|^2 + |\nabla'_A \beta|^2)$$

$$+ z_{28} \cdot r(|\alpha||\beta| + |\alpha||\nabla'_A \beta| + |\beta||\nabla_a \alpha| + |\beta|^2).$$

Therefore (1.42) and (1.43) shows that if $|\alpha| > 7/8$ and $r > 16 \sup |\mathcal{R}|$, then

$$\frac{1}{2}d^*ds + \frac{r}{8}|\alpha|^2s \leqslant z_{29} \cdot e^{(d-d_3)\sqrt{r}/z_{30}}.$$

By lemma 1.34, this implies

(1.44)
$$s < z_{31} \cdot e^{(d-d_3)\sqrt{r}/z_{32}},$$

when $t > d_2$.

The proposition then follows from (1.42), (1.43), and (1.44). $\qquad\square$

The exponential decay in proposition 1.32 will be uniform if there is an apriori bound on the integral of E. More precisely:

Proposition 1.35. *For any constant $C > 0$, there is a constant d_0 depending on C and r such that the following holds. Let (A, ϕ) be a locally L_k^2 solution of (1.10) on a symplectic end G_j. Let d be the distance function to ∂G_j. If*

$$\text{(1.45)} \qquad \int_{G_j} E < C,$$

then

$$E(x) < e^{-\sqrt{r} \cdot (d(x) - d_0)/z}$$

for every $x \in G_j$ with $d(x) > d_0$.

To simplify notations, denote the function $\frac{1}{2}\Lambda F_a = \frac{1}{2}\langle \Omega, F_a \rangle$ by F_a^ω. The following lemma is needed for the proof of proposition 1.35. The same identity also appeared in [21].

Lemma 1.36. *Let (A, ϕ) be a L_k^2 configuration on a symplectic end G_j with the canonical $Spin^c$ structure, assume that $\operatorname{supp} \phi \subset G_j$, and*

$$\int_{G_j} E < \infty.$$

Then

$$\text{(1.46)} \quad \int_{G_j} \left(|\bar{\partial}_a \alpha + \bar{\partial}_a^* \beta|^2 + 2|iF_a^\omega - \frac{r}{8}(1 - |\alpha|^2 + |\beta|^2)|^2 + 2|F_a^{0,2} - \frac{r}{4}\bar{\alpha}\beta|^2 \right.$$

$$\left. + \frac{r}{2}iF_a^\omega - 2|iF_a^\omega|^2 - 2|F_a^{0,2}|^2 \right)$$

$$= \int_{G_j} \left(\frac{1}{2}|\nabla_a \alpha|^2 + \frac{1}{2}|\nabla_{A'+a}\beta|^2 + \frac{1}{2}\langle iF_{A'+a}^\omega \beta, \beta \rangle \right.$$

$$\left. + \frac{r^2}{32}(1 - |\alpha|^2 - |\beta|^2)^2 + \frac{r^2}{8}|\beta|^2 - 2\langle N \circ \partial_a \alpha, \beta \rangle \right).$$

Where A' is the unique unitary connection on $\Lambda^{0,2}(T^*G_j)$ such that $\nabla_{A'}^{1,0} = \partial$. N is the Nijenhuis tensor.

Proof. The identity follows from the Weitzenböck formulas (12) and (13) in [20]. Translated to the notations of this chapter, these identities are:

$$\bar\partial_a^* \bar\partial_a \alpha = \frac{1}{2}(\nabla_a^* \nabla_a \alpha - 2iF_a^\omega \alpha),$$

$$\bar\partial_a \bar\partial_a^* \beta = \frac{1}{2}(\nabla_{A'+a}^* \nabla_{A'+a} \beta + 2iF_{A'+a}^\omega \beta).$$

The finiteness of energy justifies the following integration by parts:

$$\int_{G_j} \langle \bar\partial_a \alpha, \bar\partial_a^* \beta \rangle = \int_{G_j} \langle \bar\partial_a \bar\partial_a \alpha, \beta \rangle,$$

$$\int_{G_j} \langle \bar\partial_a \alpha, \bar\partial_a \alpha \rangle = \int_{G_j} \langle \bar\partial_a^* \bar\partial_a \alpha, \alpha \rangle,$$

$$\int_{G_j} \langle \bar\partial_a^* \beta, \bar\partial_a^* \beta \rangle = \int_{G_j} \langle \bar\partial_a \bar\partial_a^* \beta, \beta \rangle,$$

$$\int_{G_j} \langle \nabla_a \alpha, \nabla_a \alpha \rangle = \int_{G_j} \langle \nabla_a^* \nabla_a \alpha, \alpha \rangle,$$

$$\int_{G_j} \langle \nabla_{A'+a} \beta, \nabla_{A'+a} \beta \rangle = \int_{G_j} \langle \nabla_{A'+a}^* \nabla_{A'+a} \beta, \beta \rangle.$$

The lemma is then proved by a direct computation. $\qquad\square$

Proof of proposition 1.35. The first step is to prove $\lim_{d(x)\to\infty} E(x) = 0$ uniformly on the symplectic end G_j. Assume the contrary, then for some δ_0 there is a sequence of locally L_k^2 solutions (A_n, ϕ_n) defined on G_j satisfying $\int_{G_j} E_n < C$, and a sequence of points $x_n \in G_j$ with $d(x_n) \to \infty$ such that $E_n(x_n) \geqslant \delta_0$ for all n. Let g be the metric on M. By proposition 1.27 and the C^0 bound of $|\phi_n|$ proved in lemma 1.30, the sequence $\{(M, g, \mathfrak{s}, x_n, A_n, \phi_n)\}_{n\geqslant 1}$ converges to a configuration $(\widetilde{M}, \tilde{g}, \tilde{\mathfrak{s}}, \tilde{x}, \tilde{A}, \tilde{\phi})$. Notice that the symplectic form $\Omega = d\Theta$ is exact and $\nabla^k \Theta$ is bounded for all k, thus by taking a subsequence, the symplectic structures converge to a limit $\widetilde{\Omega}$ on $\widetilde{M}$ such that $\widetilde{\Omega} = d\widetilde{\Theta}$ for some 1-form $\widetilde{\Theta}$ with bounded norm, and $\tilde{\mathfrak{s}}$ is the canonical Spin^c structure given by $\widetilde{\Omega}$. Therefore $\tilde{\phi}$ decomposes as in (1.11) and the corresponding energy density function $\widetilde{E}$ satisfies $\widetilde{E}(\tilde{x}) \geqslant \delta_0$.

Notice that $\widetilde{M}$ is a complete symplectic manifold with bounded geometry, and $\nabla^k \Theta$ is bounded on $\widetilde{M}$. Moreover, by (1.45),

$$\int_{\widetilde{M}} \widetilde{E} \leqslant C.$$

Let $\tilde{d}$ be the distance function to $\tilde{x}$. Then proposition 1.32 shows that there are constants $\tilde{z}, \tilde{d}_0 > 0$ such that

$$(1.47) \qquad \qquad \widetilde{E} < e^{-\sqrt{r}(\tilde{d} - \tilde{d}_0)/\tilde{z}}$$

when $\tilde{d} > \tilde{d}_0$.

Lemma 1.36 gives

$$(1.48) \quad \int_{\widetilde{M}} \left(|\bar\partial_{\tilde{a}} \tilde{\alpha} + \bar\partial_{\tilde{a}}^* \tilde{\beta}|^2 + 2|iF_{\tilde{a}}^\omega - \frac{r}{8}(1 - |\tilde{\alpha}|^2 + |\tilde{\beta}|^2)|^2 + 2|F_{\tilde{a}}^{0,2} - \frac{r}{4}\bar{\tilde{\alpha}}\tilde{\beta}|^2 \right.$$

$$\left. + \frac{r}{2}iF_{\tilde{a}}^\omega - 2|iF_{\tilde{a}}^\omega|^2 - 2|F_{\tilde{a}}^{0,2}|^2 \right)$$

$$= \int_{\widetilde{M}} \left(\frac{1}{2}|\nabla_{\tilde{a}}\tilde{\alpha}|^2 + \frac{1}{2}|\nabla_{\tilde{A}'+\tilde{a}}\tilde{\beta}|^2 + \frac{1}{2}\langle iF_{\tilde{A}'+\tilde{a}}^\omega \tilde{\beta}, \tilde{\beta}\rangle \right.$$

$$\left. + \frac{r^2}{32}(1 - |\tilde{\alpha}|^2 - |\tilde{\beta}|^2)^2 + \frac{r^2}{8}|\tilde{\beta}|^2 - 2\langle \tilde{N} \circ \partial_{\tilde{a}}\tilde{\alpha}, \tilde{\beta}\rangle \right).$$

By (1.47) and the volume comparison theorem, every term in the integrals above is integrable when r is sufficiently large. By the Seiberg-Witten equations, the first three terms on the left hand side of (1.48) are zero. Therefore

$$(1.49) \quad \int_{\widetilde{M}} \left(\frac{r}{2}iF_{\tilde{a}}^\omega - 2|iF_{\tilde{a}}^\omega|^2 - 2|F_{\tilde{a}}^{0,2}|^2 \right)$$

$$= \int_{\widetilde{M}} \left(\frac{1}{2}|\nabla_{\tilde{a}}\tilde{\alpha}|^2 + \frac{1}{2}|\nabla_{\tilde{A}'+\tilde{a}}\tilde{\beta}|^2 + \frac{1}{2}\langle iF_{\tilde{A}'+\tilde{a}}^\omega \tilde{\beta}, \tilde{\beta}\rangle \right.$$

$$\left. + \frac{r^2}{32}(1 - |\tilde{\alpha}|^2 - |\tilde{\beta}|^2)^2 + \frac{r^2}{8}|\tilde{\beta}|^2 - 2\langle \tilde{N} \circ \partial_{\tilde{a}}\tilde{\alpha}, \tilde{\beta}\rangle \right).$$

The exponential decay estimate ([1.47](#)) and volume comparison theorem justifies the following integration by parts for sufficiently large r:

$$\int_{\widetilde{M}} F_{\tilde{a}}^{\omega} = \int_{\widetilde{M}} \frac{1}{2} F_{\tilde{a}} \wedge d\widetilde{\Theta} = 0.$$

Thus equation ([1.49](#)) becomes

$$(1.50) \quad 0 = \int_{\widetilde{M}} \Big(2|iF_{\tilde{a}}^{\omega}|^2 + 2|F_{\tilde{a}}^{0,2}|^2 + \frac{1}{2}|\nabla_{\tilde{a}}\tilde{\alpha}|^2 + \frac{1}{2}|\nabla_{\tilde{A}'+\tilde{a}}\tilde{\beta}|^2 + \frac{1}{2}\langle iF_{\tilde{A}'+\tilde{a}}^{\omega}\tilde{\beta}, \tilde{\beta}\rangle$$
$$+ \frac{r^2}{32}(1 - |\tilde{\alpha}|^2 - |\tilde{\beta}|^2)^2 + \frac{r^2}{8}|\tilde{\beta}|^2 - 2\langle \tilde{N} \circ \partial_{\tilde{a}}\tilde{\alpha}, \tilde{\beta}\rangle \Big)$$

However, when r is sufficiently large the right hand side of ([1.50](#)) is bounded from below by

$$(1.51) \quad \int_{\widetilde{M}} \frac{1}{4}|\nabla_{\tilde{a}}\tilde{\alpha}|^2 + \frac{1}{4}|\nabla_{\tilde{A}'+\tilde{a}}\tilde{\beta}|^2 + \frac{r^2}{64}(1 - |\tilde{\alpha}|^2 - |\tilde{\beta}|^2)^2 + \frac{r^2}{16}|\tilde{\beta}|^2 + 2|F_{\tilde{a}}^+|^2$$

This implies the integral in ([1.51](#)) is identically 0 on $\widetilde{M}$, contradicting the assumption that $\widetilde{E}(\tilde{x}) \geqslant \delta_0$.

Now go back to the proof of theorem [1.35](#). Since the convergence

$$\lim_{d(x)\to+\infty} E(x) = 0$$

is uniform, the constants d_1 and d_2 in the proof of proposition [1.32](#) can be taken to be independent of the solutions, and therefore the exponential decay estimates proved in proposition [1.32](#) are uniform. $\qquad\square$

Remark 1.37. Although this is not used in the proof, it is worth noticing that if $M = X$ as in lemma [1.3](#), then the geometry of $\widetilde{M}$ in the proof above will have some interesting properties. In fact, the orthonormal local frame (e^1, e^2, e^3, e^4) in ([1.2](#)) will converge to an orthonormal local frame $(\tilde{e}^1, \tilde{e}^2, \tilde{e}^3, \tilde{e}^4)$ on $\widetilde{M}$. Write $x_n = (q_n, t_n)$ on X, and by taking a subsequence assume that q_n is convergent in Y. Since $|t_n| \to \infty$,

equation (1.2) gives:

$$
\begin{cases}
d\,\tilde{e}^1 &= \nu_1\,\tilde{e}^1 \wedge \tilde{e}^2 \\[2mm]
d\,\tilde{e}^2 &= \nu_4\,\tilde{e}^1 \wedge \tilde{e}^2 \\[2mm]
d\,\tilde{e}^3 &= \tilde{e}^4 \wedge \tilde{e}^3 \\[2mm]
d\,\tilde{e}^4 &= 0.
\end{cases}
$$

Therefore, if one takes $\mathcal{K}_1 = \ker\tilde{e}^1 \cap \ker\tilde{e}^2$ and $\mathcal{K}_2 = \ker\tilde{e}^3 \cap \ker\tilde{e}^4$, then $\mathcal{K}_1$ and $\mathcal{K}_2$ do not depend on the choice of (e^1, e^2) on Y, and they define two orthogonal distributions on the limit manifold $\widetilde{M}$. Frobeninus theorem shows that $\mathcal{K}_1$ and $\mathcal{K}_2$ define two foliations. The two foliations are both totally geodesic. The leaves of $\mathcal{K}_2$ are limits of the leaves of the taut foliation $\mathcal{F}$, and the leaves of $\mathcal{K}_1$ have curvature 0. The proof of lemma 1.3 can be used to show that the injectivity radius of every leaf of $\mathcal{K}_1$ is infinite, therefore the leaves of $\mathcal{K}_1$ are flat planes.

One can also write down the limit metric in local coordinate systems. Let $x_n = (q_n, t_n)$ be a sequence of points in X satisfying $t_n \to \infty$. By taking a subsequence, assume that q_n converge to a point q in Y. Since $\mathcal{F} = \ker\lambda$ is a foliation, on a small open neighborhood U of q there is a local coordinate system (x^1, x^2, x^3), such that the x^3 coordinate of q is 0, and $\lambda = f\,dx^3$ for some positive function f. Write the forms e^1, e^2, μ_1 on U as

$$
e^1 = \sigma_1\,dx^1 + \sigma_2\,dx^2 + \sigma_3\,dx^3,
$$

$$
e^2 = v_1\,dx^1 + v_2\,dx^2 + v_3\,dx^3,
$$

$$
\mu_1 = \zeta_1\,dx^1 + \zeta_2\,dx^2 + \zeta_3\,dx^3.
$$

Let $\beta > 0$ be any positive constant, consider the following functions on the space $U \times (e^{-\beta}t_n, e^{\beta}t_n)$:

$$x_n^1 = x^1,$$

$$x_n^2 = x^2,$$

$$x_n^3 = \sqrt{t_n^2 + 1}\, x^3,$$

$$x_n^4 = \ln|t| - \ln|t_n|.$$

The functions $x_n^1, x_n^2, x_n^3, x_n^4$ define a local coordinate system for $U \times (e^{-\beta}t_n, e^{\beta}t_n)$. The forms e^1, e^2, e^3, e^4 can then be written as:

$$e^1 = \sigma_1\, dx_n^1 + \sigma_2\, dx_n^2 + \frac{\sigma_3}{\sqrt{t_n^2 + 1}}\, dx_n^3$$

$$e^2 = \upsilon_1\, dx_n^1 + \upsilon_2\, dx_n^2 + \frac{\upsilon_3}{\sqrt{t_n^2 + 1}}\, dx_n^3$$

$$e^3 = \frac{\sqrt{(\exp(x_n^4)|t_n|)^2 + 1}}{\sqrt{t_n^2 + 1}}\, f\, dx_n^3$$

$$e^4 = \frac{t}{\sqrt{1 + t^2}}(\zeta_1\, dx_n^1 + \zeta_2\, dx_n^2 + \frac{\zeta_3}{1 + t_n^2}\, dx^3) + \frac{|t|}{\sqrt{1 + t^2}}\, dx_n^4.$$

Let $n \to +\infty$, the coordinate functions $(x_n^1, x_n^2, x_n^3, x_n^4)$ converge to a coordinated system $(\tilde{x}_n^1, \tilde{x}_n^2, \tilde{x}_n^3, \tilde{x}_n^4)$ on the limit manifold $\widetilde{M}$. Now it is obvious from the calculations above that the metrics on the open sets $U \times (e^{-\beta}t_n, e^{\beta}t_n)$ converge to a limit metric on $\widetilde{M}$. The limit orthonormal frame $(\tilde{e}^1, \tilde{e}^2, \tilde{e}^3, \tilde{e}^4)$ on $\widetilde{M}$ are expressed in the local coordinate system as

$$\tilde{e}^1 = \sigma_1\, dx_n^1 + \sigma_2\, dx_n^2$$

$$\tilde{e}^2 = \upsilon_1\, dx_n^1 + \upsilon_2\, dx_n^2$$

$$\tilde{e}^3 = e^{2\, x_n^4}\, f\, dx_n^3$$

$$\tilde{e}^4 = \zeta_1\, dx_n^1 + \zeta_2\, dx_n^2 + dx_n^4.$$

1.5.4. *Uniform energy bound.* In this section, the notation C_i will denote the constants that are independent of the solution (A, ϕ) but may depend on r.

This subsection proves the following uniform energy bound:

Proposition 1.38. *Let M be a manifold with each end being either symplectic or cylindrical, let $(\mathfrak{s}, \{\tau_j\})$ be an admissible Spin^c structure on M. Then there exists a constant r_0 and a constant C which depends on M and r, such that if $r > r_0$ and $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ is a solution to* (1.10)*, then*

$$(1.52) \qquad \sum_j \int_{G_j} E < C$$

An immediate corollary of proposition 1.38 and proposition 1.35 is the following result:

Theorem 1.39. *There exists a constant $r_0 > 0$ and a constant z depending on M, and a constant C depending on both M and r, such that if $r > r_0$ and $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ solves* (1.10)*, then the inequality*

$$(1.53) \qquad E < e^{-\sqrt{r} \cdot (d - C)/z}$$

holds on each symplectic end G_j whenever $d > C$. $\qquad\square$

The rest of the subsection is devoted to the proof of proposition 1.38.

First define some notation: Let Ω be the symplectic form on $\cup G_j$. Let d be a function defined on M such that $d = 0$ on $M - \cup G_j$ and d equals the distance to ∂G_j on G_j. If $\mathfrak{a} \in \mathcal{C}(Y, \mathfrak{t})$ is a critical point of $\mathcal{L}$, let $\gamma_{\mathfrak{a}} \in \mathcal{C}(Y \times [0, 1], \mathfrak{t}')$ be the configuration on $Y \times [0, 1]$ which is in temporal gauge and represents the constant path at $\mathfrak{a}$. Since there are only finitely many critical points up to gauge transformations, there is a constant z such that

$$(1.54) \qquad \|F_{\gamma_{\mathfrak{a}}}\|_{L^2}^2 < z$$

for every critical point $\mathfrak{a}$.

Let $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ be a solution to (1.10). Choose a gauge representative of (A, ϕ) such it is in temporal gauge on each cylindrical end. Let t be the function on M which is the projection of H_i to $[0, \mathbb{R}^+)$ on each cylindrical end H_i and is zero on $M - (\cup H_i)$. Recalled that A_0 is a fixed Spin^c connection on $(M, \mathfrak{s})$ such that A_0 equals the pull back of canonical connections on the symplectic ends, and A_0 is in temporal gauge and invariant under translations on each cynlindrical end. Take

$$z' = z + \int_{t \in (0,1]} |F_{A_0}|^2 + 1.$$

Let n_c be the number of cylindrical ends. By (1.54), there exists an $R > 1$ sufficiently large such that

(1.55)
$$\int_{t(x) \in [R, R+1]} |F_a(x)|^2 < z' \cdot n_c.$$

Lemma 1.40. *There is a constant $r_0 > 0$ and a function $T(\kappa, r, z) > 0$ depending on the manifold M with the following property. Let $R > 1$, $r > r_0$, let (A, ϕ) be a locally L_k^2 configuration defined on $t \leqslant R + 1$ solving the perturbed Seiberg-Witten equation (1.10) on the symplectic ends with $\int_{G_j} E < +\infty$ for each j. Suppose there are constants $\kappa > 0$, $z > 0$ such that*

(1.56)
$$\int_{t(x) \in [R, R+1]} |F_a|^2 \leqslant \kappa.$$

and suppose that whenever $t(x) \leqslant R + 1$, the norms $|\phi(x)|$ and $|F_a^+(x)|$ satisfy the following C^0 bounds:

(1.57)
$$|\phi(x)| \leqslant z \cdot \sqrt{r}, \quad |F_a^+(x)| \leqslant z \cdot r.$$

Then the following inequalities hold:

(1.58)
$$\sum_j \int_{G_j} E < T(\kappa, r, z) + 5z^2 \, r^2 \, n_c \, Vol(Y) \cdot R,$$

(1.59)
$$\int_{t(x) \in (0, R]} |F_a|^2 < T(\kappa, r, z) + 4z^2 \, r^2 \, n_c \, Vol(Y) \cdot R.$$

Proof. Use T_i to denote constants that may depend on M, κ, r, and z but are independent of the choice of (A, ϕ).

On each G_j, the equation (1.10) reads as

$$\bar{\partial}_a \alpha + \bar{\partial}_a^* \beta = 0,$$

$$F_a^\omega = -\frac{ir}{8}(1 - |\alpha|^2 + |\beta|^2),$$

$$F_a^{0,2} = \frac{r}{4}\bar{\alpha}\beta.$$

Denote the ϵ_0-neighborhood of ∂G_j by N_j. Let $\chi \geqslant 0$ be a cut-off function on M such that $\text{supp}\,\chi \subset \cup(G_j - N_j)$ and $\chi = 1$ when $d > 2\epsilon_0$. Let $\phi' = \chi\,\phi$. Apply lemma 1.36 for (A, ϕ'), notice that proposition 1.32 and the volume comparison theorem implies that each term on either side of (1.46) is integrable for sufficiently large r. Apply the C^0 bound in (1.57), there is the following inequality:

$$T_1 + \sum_j \int_{G_j} \left(\frac{r}{2}iF_a^\omega - 2|iF_a^\omega|^2 - 2|F_a^{0,2}|^2\right) \geqslant \sum_j \int_{G_j} \left(\frac{1}{2}|\nabla_a \alpha|^2\right.$$

$$+ \frac{1}{2}|\nabla_{A'+a}\beta|^2 + \frac{1}{2}\langle iF_{A'+a}^\omega \beta, \beta\rangle + \frac{r^2}{32}(1 - |\alpha|^2 - |\beta|^2)^2$$

$$\left. + \frac{r^2}{8}|\beta|^2 - 2\langle N \circ \partial_a \alpha, \beta\rangle\right).$$

For sufficiently large r, a rearrangment argument proves:

(1.60)
$$\sum_j \int_{G_j} (E - |F_a^-|^2) \leqslant T_2 + i\,r \sum_j \int_{G_j} F_a \wedge \Omega,$$

where the constant T_2 depends on r.

Now by (1.56) and lemma 5.1.2 of [22], there exists a new connection a' of the trivial $\mathbb{C}$-bundle defined on the set $\{x \in M | t(x) \in [R, R+1]\}$, such that:

(1) $\|a - a'\|_{L_1^2} < T_3$, for some constant T_3 depending on κ.

(2) $a' = a$ when $t \in [R, R + \frac{1}{3}]$.

(3) $F_{a'} = 0$, when $t \in [R + \frac{2}{3}, R + 1]$.

Notice that by definition 1.7 the symplectic form $\Omega_j = d\Theta_j$ and it can be extended to a neighborhood of G_j, therefore there exists an exact form $\Omega = d\Theta$ on M such that

(1) $\Omega = \Omega_j$ on each G_j.

(2) $\Theta = \Theta_j$ on $G_j - N_j$.

(3) $\Omega = 0$ outside a tubular neighborhood of $\cup G_j$.

On a symplectic end G_j, $|\phi(x)| \to \sqrt{r}$ as $d(x) \to \infty$. Therefore one can take a gauge representative of (A, ϕ) such that $\phi/\Phi_0 \in \mathbb{R}$ when $d(x)$ is large. By proposition 1.32, for some constants $z_0, d_0 > 0$.

$$(1.61) \qquad |a| \leqslant e^{-\sqrt{r} \cdot (d - d_0)/z_0}$$

when $d > d_0$. Now extend a' to $t \leqslant R + 1$ by taking $a' = a$ when $t < R$. For r sufficiently large, the inequality (1.61) justifies the following identities from integration by parts:

$$(1.62) \qquad \int_{t \leqslant R+1} F_{a'} \wedge \Omega = 0,$$

$$(1.63) \qquad \int_{t \leqslant R+1} F_{a'} \wedge F_{a'} = 0.$$

Let $Z_R = \{x \in M | x \notin G_j \text{ for any } j, \text{ and } t(x) \leqslant R + 1\}$. Then $\text{Vol}(Z_R) \leqslant z_1 + n_c \cdot R \cdot \text{Vol}(Y)$. By (1.60),

$$\sum_j \int_{G_j} (E - |F_a^-|^2) \leqslant T_2 + r \left| \sum_j \int_{G_j} F_a \wedge \Omega \right|$$

$$\leqslant T_3 + r \left| \sum_j \int_{G_j} F_{a'} \wedge \Omega \right|$$

Notice that (1.62) implies

$$\sum_j \int_{G_j} F_{a'} \wedge \Omega + \int_{Z_R} F_{a'} \wedge \Omega = 0.$$

Therefore,

$$
\begin{aligned}
\sum_j \int_{G_j} (E - |F_a^-|^2) &\leqslant T_3 + r \left| \sum_j \int_{G_j} F_{a'} \wedge \Omega \right| \\
&= T_3 + r \left| \int_{Z_R} F_{a'} \wedge \Omega \right| \\
&\leqslant T_3 + \frac{1}{4} \int_{Z_R} |F_{a'}|^2 + r^2 \int_{Z_R} |\Omega|^2 \\
&\leqslant T_4 + \frac{1}{4} \int_{Z_R} |F_{a'}|^2 \\
&\leqslant T_4 + \frac{1}{4} \int_{t < R+1} |F_{a'}|^2 \\
&= T_4 + \frac{1}{2} \int_{t < R+1} |F_{a'}^+|^2 \\
&\leqslant T_5 + \frac{1}{2} \int_{Z_R} |F_a^+|^2 + \frac{1}{2} \sum_j \int_{G_j} (E - |F_a^-|^2) \\
&\leqslant T_6 + \frac{1}{2} z^2 r^2 n_c \mathrm{Vol}(Y) \cdot R + \frac{1}{2} \sum_j \int_{G_j} (E - |F_a^-|^2),
\end{aligned}
$$

where the last inequality comes from assumption (1.57). Hence

$$(1.64) \qquad \sum_j \int_{G_j} (E - |F_a^-|^2) \leqslant 2\, T_6 + z^2\, r^2\, n_c \mathrm{Vol}(Y) \cdot R.$$

Therefore,

$$
\begin{aligned}
\int_{t \leqslant R+1} |F_a|^2 &\leqslant T_7 + \int_{t \leqslant R+1} |F_{a'}|^2 \\
&= T_7 + 2 \int_{t \leqslant R+1} |F_{a'}^+|^2
\end{aligned}
$$

$$\leqslant T_7 + 2 \int_{Z_R} |F_a^+|^2 + 2 \sum_j \int_{G_j} (E - |F_a^-|^2)$$

(1.65)
$$\leqslant T_8 + 4z^2 \, r^2 \, n_c \text{Vol}(Y) \cdot R.$$

The lemma follows immediately from (1.64) and (1.65). □

On the other hand, there is the following estimate:

Lemma 1.41. *For every Riemannian 3-manifold N, there are constants z_1, z_2 and a function of $R_0(T, C)$ depending on N, such that the following holds: let a be a L_4^2 unitary connection for the trivial $\mathbb{C}$ bundle on $N \times [0, R]$, where $N \times [0, R]$ is endowed with the product metric. If T, $C > 0$, $R > R_0(T, C)$ and*

(1.66)
$$\int_0^R |F_a|^2 \leqslant T + C \cdot R,$$

(1.67)
$$|F_a^+|^2 \leqslant C \text{ pointwise on } N \times [0, R],$$

then

(1.68)
$$\int_{\frac{R}{2} - \frac{1}{2}}^{\frac{R}{2} + \frac{1}{2}} |F_a|^2 < z_1 \cdot C + z_2.$$

Proof. Put a in temporal gauge, and represent a as a function $a(t)$ of t which takes value in $L_3^2(N, iT^*N)$. Then

$$|F_a^+| = \frac{\sqrt{2}}{2} \big| \dot{a}(t) + *da(t) \big|,$$

$$|F_a^-| = \frac{\sqrt{2}}{2} \big| \dot{a}(t) - *da(t) \big|.$$

Let

$$\cdots < \lambda_{-3} < \lambda_{-2} < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

be the eigenvalues of the self-adjoint operator $*d$ on N. Let

$$k = \max\left\{\frac{1}{|\lambda_{-1}|}, \frac{1}{|\lambda_1|}\right\}.$$

Decompose a as

$$a(t) = \sum_{n=-\infty}^{+\infty} a_n(t),$$

where $*da_n(t) = \lambda_n a_n(t)$. Let

$$b_n(t) = \dot{a}_n(t) + \lambda_n(t)a_n(t).$$

By (1.67),

$$\sum_{n=-\infty}^{\infty} \|b_n(t)\|_{L^2}^2 \leqslant 2C \cdot \mathrm{Vol}(N).$$

By (1.66),

$$\int_0^R \|*da(t)\|_{L^2}^2 \, dt \leqslant \frac{1}{2}\int_0^R \left(\|\dot{a}(t) + *da(t)\|_{L^2}^2 + \|\dot{a}(t) - *da(t)\|_{L^2}^2\right) dt$$

$$= \int_0^R |F_a|^2$$

$$\leqslant T + CR.$$

Thus there exits a $t_1 \in [0,1]$ such that

(1.69)
$$\sum_n \lambda_n^2 \|a_n(t_1)\|_{L^2}^2 \leqslant T + CR.$$

Now if $n > 0$,

$$\lambda_n a_n(t) = a_n(t_1) \cdot e^{(t_1-t)\lambda_n} \cdot \lambda_n + \int_{t_1}^t e^{\lambda_n(s-t)} \cdot \lambda_n \cdot b_n(s) \, ds$$

When $t > k + 1$,

$$\|\lambda_n a_n(t)\|_{L^2}^2 \leqslant 3\left(X_n^2(t) + Y_n^2(t) + Z_n^2(t)\right),$$

where

$$X_n(t) = \|a_n(t_1)\, e^{(t_1-t)\lambda_n}\lambda_n\|_{L^2}$$

$$Y_n(t) = \|\int_{t_1}^{t-k} b_n(s)\, e^{(s-t)\lambda_n}\lambda_n\, ds\|_{L^2}$$

$$Z_n(t) = \|\int_{t-k}^{t} b_n(s)\, e^{(s-t)\lambda_n}\lambda_n\, ds\|_{L^2}$$

Notice that since $|\lambda_n| \geqslant \frac{1}{k}$,

$$X_n(t) \leqslant \|\lambda_n a_n(t_1)\|_{L^2} \cdot e^{(t_1-t)/k},$$

Now assume $R > 2k + 3$. By (1.69),

$$\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \sum_{n\geqslant 1} X_n(t)^2\, dt \leqslant (T + CR) \cdot e^{-(R-3)/k}.$$

Notice that if $s - t < -k$ then the function $\lambda \mapsto e^{(s-t)\lambda}\lambda$ is decreasing when $\lambda > \lambda_1$. By Minkowski's inequality, for $t > k + 1$,

$$\left(\sum_{n\geqslant 1} Y_n(t)^2\right)^{1/2} \leqslant \left(\sum_{n\geqslant 1}\left(\int_{t_1}^{t-k}\|b_n(s)\|_{L^2}\, e^{(s-t)\lambda_n}\lambda_n\, ds\right)^2\right)^{1/2}$$

$$\leqslant \left(\sum_{n\geqslant 1}\left(\int_{t_1}^{t-k}\|b_n(s)\|_{L^2}\, e^{(s-t)\lambda_1}\lambda_1\, ds\right)^2\right)^{1/2}$$

$$\leqslant \int_{t_1}^{t-k}\left(\sum_{n\geqslant 1}\|b_n(s)\|_{L^2}^2\right)^{1/2} e^{(s-t)\lambda_1}\lambda_1\, ds$$

$$\leqslant \sqrt{2C \cdot \mathrm{Vol}(N)}.$$

Thus

$$\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \sum_{n\geqslant 1} Y_n(t)^2\, dt \leqslant 2C \cdot \mathrm{Vol}(N).$$

As for Z_n, the Minkowski inequality gives

$$\left(\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} Z_n(t)^2\, dt\right)^{1/2} \leqslant \left(\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \left(\int_0^k \|b_n(t-s)\|_{L^2} \cdot e^{-s\lambda_n}\lambda_n\, ds\right)^2 dt\right)^{1/2}$$

$$\leqslant \int_0^k \left(\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \|b_n(t-s)\|_{L^2}^2\, dt\right)^{1/2} e^{-s\lambda_n}\lambda_n\, ds$$

$$\leqslant \int_0^k \left(\int_{\frac{R}{2}-\frac{1}{2}-k}^{\frac{R}{2}+\frac{1}{2}} \|b_n(t)\|_{L^2}^2\, dt\right)^{1/2} e^{-s\lambda_n}\lambda_n\, ds$$

$$\leqslant \left(\int_{\frac{R}{2}-\frac{1}{2}-k}^{\frac{R}{2}+\frac{1}{2}} \|b_n(t)\|_{L^2}^2\, dt\right)^{1/2}.$$

Therefore

$$\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \sum_{n\geqslant 1} Z_n(t)^2\, dt \leqslant \int_{\frac{R}{2}-\frac{1}{2}-k}^{\frac{R}{2}+\frac{1}{2}} \sum_{n\geqslant 1} \|b_n(t)\|_{L^2}^2\, dt$$

$$\leqslant (k+1)2C \cdot \mathrm{Vol}(N).$$

Combining the estimates above,

$$\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \sum_{n\geqslant 1} \|\lambda_n a_n(t)\|_{L^2}^2\, dt \leqslant \int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \sum_{n\geqslant 1} 3\big(X_n^2(t) + Y_n^2(t) + Z_n^2(t)\big)$$

$$\leqslant 6(k+2)\mathrm{Vol}(N) \cdot C + 3(T+CR) \cdot e^{-(R-3)/k}.$$

On the other hand, there exists a $t_2 \in [R-1, R]$ such that

$$\sum_n \lambda_n^2 \|a_n(t_2)\|_{L^2}^2 \leqslant T + CR.$$

If $n < 0$, then

$$\lambda_n a_n(t) = a_n(t_2) \cdot e^{(t-t_2)(-\lambda_n)} \cdot \lambda_n - \int_t^{t_2} e^{(-\lambda_n)(t-s)} \cdot \lambda_n \cdot b_n(s)\, ds.$$

When $R > 2k+3$, the same argument proves

$$\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \sum_{n\leqslant -1} \|\lambda_n a_n(t)\|_{L^2}^2\, dt \leqslant 6(k+2)\mathrm{Vol}(N) \cdot C + 3(T+CR) \cdot e^{-(R-3)/k}.$$

Therefore, when $R > 2k + 3$

$$\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \| * da \|_{L^2}^2 \, dt = \int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \sum_{n=-\infty}^{+\infty} \| \lambda_n a_n(t) \|_{L^2}^2 \, dt$$

$$\leqslant 12(k+2)\mathrm{Vol}(N) \cdot C + 6(T + CR) \cdot e^{-(R-3)/k}.$$

Notice that

$$\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \left| |F_a^+| - |F_a^-| \right|^2 \leqslant \int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} 2 \| * da(t) \|_{L^2}^2 \, dt.$$

Hence

$$\int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} |F_a|^2 \leqslant 2 \int_{\frac{R}{2}-\frac{1}{2}}^{\frac{R}{2}+\frac{1}{2}} \left| |F_a^+| - |F_a^-| \right|^2 + |2F_a^+|^2$$

$$\leqslant 8C\mathrm{Vol}(N) + 48(k+2)\mathrm{Vol}(N) \cdot C$$

$$+ 24(T + CR) \cdot e^{-(R-2)/k},$$

and the lemma follows from the inequality above. $\qquad\square$

The proof of proposition 1.38 follows easily from the previous two lemmas:

Proof of proposition 1.38. Pick r_0 large enough so that lemma 1.40 is valid when $r > r_0$. Let z' be the constant in (1.55). Let the function $T(\kappa, r, z)$ be as in lemma 1.40, and constant z be as in proposition 1.29, and let the function $R_0(T, C)$ and constants z_1, z_2 be as in lemma 1.41 when N equals n_c copies of Y.

Let $C_1 = 4z^2 \, r^2 n_c \cdot \max(\mathrm{Vol}(Y), 1/4)$, take $\kappa = \max(z' \cdot n_c, \, z_1 \cdot C_1 + z_2)$.

Let

$$R_{\min} = \inf\{R > 2 | \int_{t \in [R, R+1]} |F_a|^2 < \kappa\}.$$

By (1.55), $R_{\min}$ is finite. By (1.59),

$$\int_{t \in (0, R]} |F_a|^2 < T(\kappa, r, z) + 4z^2 \, r^2 n_c \mathrm{Vol}(Y) \cdot R$$

$$\leqslant T(\kappa, r, z) + C_1 \cdot R.$$

and since the constant z is the same constant in proposition 1.29,

$$|F_a^+|^2 \leqslant z^2 \, r^2 \leqslant C_1.$$

If $R_{\min} > \max(R_0(T(\kappa, r, z), C_1), 5)$, take $R' = (R_{\min} - 1)/2$, then lemma 1.41 gives

$$\int_{t \in [R', R'+1]} |F_a|^2 < z_1 \cdot C_1 + z_2 \leqslant \kappa.$$

This contradicts the definition of $R_{\min}$. Therefore,

$$R_{\min} \leqslant \max(R_0(T(\kappa, r, z), C_1), 5),$$

hence by (1.58),

$$\sum_j \int_{G_j} E < T(\kappa, r, z) + 5z^2 \, r^2 \, n_c \mathrm{Vol}(Y) \cdot R_{\min}$$

$$\leqslant T(\kappa, r, z) + 5z^2 \, r^2 \, n_c \mathrm{Vol}(Y) \cdot \max(R_0(T(\kappa, r, z), C_1), 5).$$

And proposition 1.38 is proved. □

1.5.5. *Manifold with a stretching neck.* For later references, this subsection considers the following scenario:

Let M be a manifold with *only* symplectic ends. Let G_j be the ends of M, let d be the function on M such that $d = 0$ on $M - \cup G_j$ and d is the distance function to ∂G_j on G_j. Let $(\mathfrak{s}, \{\tau_j\})$ be an admissible Spin^c structure. Suppose the three manifold Y is embedded in M such that $Y \cap G_j = \varnothing$ for each G_j, and $M - Y$ has two components. Assume that the metric of M on a tubular neighborhood of Y is the product metric $Y \times (-\epsilon, \epsilon)$. For $R > \epsilon$, let M_R be the Riemannian manifold which is diffeomorphic to M but the metric on the tubular neighborhood of Y is changed to the product metric of $Y \times (-R, R)$. In other words, M_R is obtained from M by "stretching" the

tubular neighborhood of Y. Let $f_R : M_R \to M$ be the map that shrinks the neck. The map f_R can be chosen in such a way that at each point of M_R the norm of the tangent map of f_R is less than or equal to 1. The admissible Spin^c structure $(\mathfrak{s}, \{\tau_j\})$ and the function d then pull back to M_R via f_R.

Let $\eta(t)$ be a cutoff function on $\mathbb{R}$ supported in $(-\epsilon, \epsilon)$. Take two strongly tame perturbations $\mathfrak{p}_1$ and $\mathfrak{p}_2$ such that $\|\mathfrak{p}_1\|, \|\mathfrak{p}_2\| < 1$. Take the perturbation on the symplectic ends as in (1.8). If $\mathfrak{t}$ is the induced Spin^c structure on Y by $\mathfrak{s}$, take the perturbation on $Y \times (-R, R)$ to be $\hat{q}_{\mathfrak{t}} + \eta(t + R - \epsilon)\hat{p}_1 + \eta(t - R + \epsilon)\hat{p}_2$, and extend them to a perturbation $\hat{\mu}_R$ on M_R by using a partition of unity. Consider the equation

$$(1.70) \qquad \mathfrak{F}(A, \phi) = \hat{\mu}_R(A, \phi).$$

Recall that a Spin^c connection A_0 was chosen and fixed on $(M, \mathfrak{s})$. Let A_R be the pull back of A_0 to M_R; any Spin^c connection on M_R can be written as $A_R + a$.

Theorem 1.42. *There exists a constant $r_0 > 0$ and a constant z depending only on M and the embedding of Y, such that for every $r > r_0$ there is a constant $C > 0$ depending on r but independent of R, such that if $(A, \phi) \in \mathcal{C}_k(M_R, \mathfrak{s}, r, \{\tau_j\})$ solves the equation (1.70), then the inequality*

$$E < e^{-\sqrt{r} \cdot (d - C)/z}$$

holds on each end G_j whenever $d > C$.

The proof of theorem 1.42 is similar to the proof of theorem 1.39. Before giving the proof of theorem 1.42, several lemmas are needed.

Lemma 1.43 (C^0 estimate). *Under the assumptions of theorem 1.42, there is a constant z depending only on M and the embedding of Y, such that $|\phi| < z \cdot \sqrt{r}$, $|F_a^+| < z \cdot r$.*

Proof. The proof is the same as the proof of proposition 1.29. □

Let Z_R be the neck $Y \times (-R, R)$. The following lemma is analogous to lemma 1.40.

Lemma 1.44. *Under the assumptions of theorem 1.42, let z be the constant in lemma 1.43, then there is a constant C_0 depending on M, the embedding of Y in M, and r, such that*

$$(1.71) \qquad \sum_j \int_{G_j} E < C_0 + 10z^2\, r^2\, Vol(Y) \cdot R,$$

$$(1.72) \qquad \int_{Z_R} |F_a|^2 < C_0 + 8z^2\, r^2\, Vol(Y) \cdot R.$$

Remark 1.45. The coefficients in the inequalities (1.71) and (1.72) are twice the corresponding coefficients in lemma 1.40, because in this case the neck length is $2R$ instead of R. Of course, the exact values of the coefficients won't matter.

Proof of lemma 1.44. Let Ω be the symplectic form on $\cup G_j$. Use C_i to denote constants depending on M, the embedding of Y in M, and r, but independent of R. Integration by parts as in inequality (1.60), one obtains

$$(1.73) \qquad \sum_j \int_{G_j} (E - |F_a^-|^2) \leqslant C_1 + i\, r \sum_j \int_{G_j} F_a \wedge \Omega.$$

Notice that $c_0 = \int_{M_R} F_a \wedge F_a$ is a topological invariant of $(\mathfrak{s}, \{\tau_j\})$. Let $\Omega = d\Theta$ be the symplectic form on the ends and extend Ω to an exact form $\Omega' = d\Theta'$ on M such that $\Theta = \Theta'$ when $d > \epsilon_0$ and supp Θ' is contained in a tubular neighborhood of the symplectic ends. Pull Θ' and Ω' back to M_R and denote the pulled back forms by Θ_R and Ω_R. Since $|df_R| \leqslant 1$, the norm of Ω_R and Θ_R are uniformly bounded. Since $\Omega_R = d\Theta_R$, this implies $\int_{M_R} F_a \wedge \Omega_R = 0$. Therefore,

$$\sum_j \int_{G_j} (E - |F_a^-|^2) \leqslant C_1 + r \left| \int_{M_R - \cup G_j} F_a \wedge \Omega_R \right|$$

$$\leqslant C_1 + \frac{1}{4} \int_{M_R - \cup G_j} |F_a|^2 + r^2 \int_{M_R - \cup G_j} |\Omega_R|^2$$

$$\leqslant C_2 + \frac{1}{4} \int_{M_R} |F_a|^2$$

$$= C_2 + \frac{c_0}{4} + \frac{1}{2} \int_{M_R} |F_a^+|^2$$

$$\leqslant C_3 + z^2 \, r^2 \mathrm{Vol}(Y) \cdot R + \frac{1}{2} \sum_j \int_{G_j} (E - |F_a^-|^2),$$

Therefore

$$\sum_j \int_{G_j} (E - |F_a^-|^2) \leqslant 2C_3 + 2z^2 \, r^2 \mathrm{Vol}(Y) \cdot R.$$

On the other hand,

$$\int_{M_R} |F_a|^2 \leqslant |c_0| + 2 \int_{M_R} |F_a^+|^2$$

$$\leqslant |c_0| + 2 \int_{M_R - \cup G_j} |F_a^+|^2 + 2 \sum_j \int_{G_j} (E - |F_a^-|^2)$$

$$\leqslant C_4 + 8z^2 r^2 \mathrm{Vol}(Y) \cdot R.$$

The lemma is then proved by combining the two inequalities above. $\qquad \square$

Proof of theorem 1.42. By proposition 1.35, one needs to find a uniform upper bound for $\sum_j \int_{G_j} E$. Lemma 1.44 provides an upper bound that grows linearly with respect to R. An argument similar to the proof of proposition 1.38 can improve it to a uniform bound.

In fact, take the function $T(\kappa, r, z)$ in lemma 1.40, and take the constants z_1, z_2 and the function $R_0(T, C)$ in lemma 1.41 applied to $N = Y$. Let C_0 be the constant in lemma 1.44, let z be the constant in lemma 1.43. Take $C_1 = 8z^2 r^2 \mathrm{Vol}(N)$, take $\kappa = z_1 C_1 + z_2$ and $T = \max\big(C_0, T(\kappa, r, z_0)\big)$. If $R < R_0(T, C_1) + 3$ then the energy is bounded uniformly by lemma 1.44. Otherwise, by lemma 1.41,

$$\int_{N \times (-1/2, 1/2)} |F_a|^2 \leqslant \kappa.$$

Take

$$R_{\max} = \sup\{R' | R' \leqslant R - 2, \int_{N \times (R', R'+1)} |F_a|^2 \leqslant \kappa\},$$

then lemma 1.41 proves

$$R - R_{\max} \leqslant R_0(T, C_1) + 5.$$

Now apply lemma 1.40 to the component of $M - Y \times (0, R_{\max} - 1/2)$ which contains $Y \times \{R\}$, a uniform energy bound on the symplectic ends of this component is then obtained. The energy bound on the other component follows from the same argument by considering

$$R_{\min} = \inf\{R' | R' \geqslant -R + 2, \int_{N \times (R'-1/2, R'+1/2)} |F_a|^2 \leqslant \kappa\},$$

and obtaining an upper bound of $R_{\min} + R$. □

1.6. **Monopole Floer homology.** This section defines $c(\mathcal{F})$ for a smooth oriented and co-oriented foliation $\mathcal{F}$ wihtout transverse invariant measure, and proves the properties claimed in the introduction.

Recall that in section 1.2, a Riemannian manifold $X = (Y \times \mathbb{R}, g)$ is defined, and there is a compatible symplectic form Ω on X such that (X, Ω) is a manifold with symplectic ends.

By section 27.3 of [22], if W is a compact 4-manifold with boundary $Y_- \cup Y_+$, and $b_2^+(W) \geqslant 2$, then there is a map

$$\overrightarrow{HM}(W) : \widehat{HM}_\bullet(Y_-) \to \widetilde{HM}_\bullet(Y_+)$$

defined by attaching two cylinders $Y_- \times (-\infty, 0]$ and $Y_+ \times [0, +\infty)$ to W and counting solutions of a perturbed Seiberg-Witten equation on this extended manifold. In particular, if $Y_- = S^3$, then this map defines an element $\overrightarrow{HM}(W)(1) \in \widetilde{HM}_\bullet(Y_+)$. Another way to interpret the element $\overrightarrow{HM}(W)(1) \in \widetilde{HM}_\bullet(Y_+)$ is to attach a D^4 to the boundary $Y_- \cong S^3$ and count solutions on $D^4 \cup W \cup Y_+ \times [0, +\infty)$. By lemma

27.4.2 of [22], it is straight forward to prove that the two constructions give the same element in $\widetilde{HM}_\bullet(Y_+)$

To construct the invariant $c(\mathcal{F})$ of the foliation $\mathcal{F}$, one considers the manifold $X_0 = Y \times (-\infty, -1]$ with the restriction metric from X. After attaching a cylinder $Y \times [0, +\infty)$ and smoothing the metric, the manifold $X_0 \cup Y \times [0, +\infty)$ becomes a manifold with one cylindrical end and one symplectic end. By lemma 1.9, the canonical Spin^c structure on X_0 is extended to a Spin^c structure on $X_0 \cup Y \times [0, +\infty)$. Denote this Spin^c structure by $\mathfrak{s}$. Let $M = X_0 \cup Y \times [0, +\infty)$, and let τ be the identity map from $\mathfrak{s}|_{X_0}$ to the canonical Spin^c structure on X_0.

Theorem 1.46. *For a generic choice of the perturbation $\mathfrak{p}$ and sufficiently large r, the moduli space*

$$\left\{ (A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau\}) \,|\, (A, \phi) \text{ solves } (1.10) \right\} / \mathcal{G}_{k+1}$$

is a countable union of finite dimensional manifolds. Counting the number of solutions in the zero-dimensional components of the moduli space as in definition 27.3.1 of [22] gives an element $c(\mathcal{F})$ in $\widetilde{HM}_\bullet(Y)$. The solutions are counted with relative signs as depicted section 20 of [22], therefore the class $c(\mathcal{F})$ is only defined up to an overall sign. The element $c(\mathcal{F})$ does not depend on the choice of λ, ω, g, $\mathfrak{q}_t$, $\mathfrak{p}$, or r. Moreover, $c(\mathcal{F})$ is invariant under homotopies of $\mathcal{F}$ among foliations without transverse invariant measure.

Proof. If X_0 were a compact manifold with boundary then the regularity of moduli space would follow from proposition 24.4.7 of [22] and the definition of $c(\mathcal{F})$ would follow from proposition 27.3.2 of [22].

Because of the uniform exponential decay estimate in theorem 1.39, the machinary developed in [22] for compact manifolds can be applied to X_0 with few modifications. Whenever the compactness is used in [22] it can always be replaced by the exponential

decay estimate or the Fredholm theorey developed in section 1.4. Here is a list of the modifications:

(1) Notice that for any $(A, \phi) \in \mathcal{C}_k(M, \mathfrak{s}, r, \{\tau\})$, the spinor ϕ cannot be identically zero, therefore no reducible solution is possible and there is no need to make the assumption $b_2 \geqslant 2$.

(2) In [22] the proof of proposition 24.3.1 took integration on the compact manifold X to obtain equation (24.11). However, since there are no reducible solutions, one doesn't need to blow up the configuration space. Therefore the corresponding result for X_0, that the solution space on X_0 is a Hilbert manifold, can be proved directly by the unique continuation theorem without resorting to equation (24.11).

(3) The Fredholm theory for elliptic operators on compact manifolds was used in [22] to establish the Fredholmness on maps between Hilbert manifolds. This can be substituted by proposition 1.15.

(4) The topological energy for a solution would be infinite if the definition is copied verbatim from definition 4.5.4 of [22]. A new definition of the topological energy for solutions in $\mathcal{C}_k(M, \mathfrak{s}, r, \{\tau_j\})$ can be defined as follows: let C be the constant in theorem 1.39, let d be the function on M which equals zero on $M - X_0$ and equals the distance function to ∂X_0 on X_0. Fix a subset $M' \subset M$ such that $M' \cap X_0$ is compact, $M' \supset \{x | d(x) < C\}$, and $\partial M'$ is a smooth submanifold. Define

$$(1.74) \qquad \mathcal{E}^{top}(A, \phi) = \frac{1}{4} \int_{M'} F_{A^t} \wedge F_{A^t} - \int_{\partial M'} \langle \phi, D_B \phi \rangle + \int_{\partial M'} (H/2)|\phi|^2.$$

Here H is the mean curvature of $\partial M'$ and B is the boundary Dirac operator (cf. section 4.5 of [22]).

The topological energy defined in (1.74) is *not* invariant under homotopy of solutions. However inequality (1.53) implies that the difference of the topological energies of two solutions in the same connected component is bounded by a constant. Therefore one can still bound the topological energy by the index as in [22].

(5) The compactness of the space of brocken trajectories follows from theorem 1.39 and elliptic regularity.

Therefore the arguments of [22] can be carried over to the case considered here. The same proof as in corollary 25.3.9 of [22] proves that the homology class $c(\mathcal{F})$ defined in this theorem is invariant under homotopy of the parameters, thus $c(\mathcal{F})$ is independent of the choices of parameters and is invariant under homotopy of $\mathcal{F}$. $\qquad\square$

Let $j_* : \widetilde{HM}_\bullet(Y) \to \widehat{HM}_\bullet(Y)$ be the map in the long exact sequence of monopole Floer homologies. The next theorem proves the nonvanishing of $j_*c(\mathcal{F})$.

Theorem 1.47. *Let $\mathcal{F}$ be a smooth foliation on Y with no transverse invariant measure, then $j_*c(\mathcal{F}) \neq 0$.*

The proof uses a standard gluing argument. Consider the moduli space of solutions on X. Recall that $X_0 = Y \times (-\infty, -1] \subset X$. Let $X_1 = Y \times [1, +\infty) \subset X$, then X can be considered as a manifold with two symplectic ends X_0 and X_1. Let $\mathfrak{s}$ be the canonical Spin^c structure on X, let $\mathfrak{s}_0 = \mathfrak{s}|_{X_0}$, $\mathfrak{s}_1 = \mathfrak{s}|_{X_1}$. Let A_0 be the canonical connection for $\mathfrak{s}$, and Φ_0 be the canonical section of the spinor bundle. Let $\tau_0 : \mathfrak{s}_0 \to \mathfrak{s}_0$ be the identity map, let $\tau_1 : \mathfrak{s}_1 \to \mathfrak{s}_1$ be an isomorphism. The next lemma considers solutions of the following version of Seiberg-Witten equation on X:

(1.75)
$$\begin{cases} \rho(F_{A^t}^+) - (\phi\phi^*)_0 = \rho(F_{A_0^t}^+) - (\Phi_0\Phi_0^*)_0 \\ D_A(\phi) = 0 \end{cases}$$

On the symplectic ends X_0 and X_1, this equation is the same as (1.10).

Lemma 1.48. *Suppose r is sufficiently large. If*

$$(A, \phi) \in \mathcal{C}_k(X, \mathfrak{s}, r, \{\tau_0, \tau_1\})$$

is a solution to (1.75), then τ_1 is homotopic to identity and (A, ϕ) is gauge equivalent to $(A_0, \sqrt{r}\Phi_0)$. Moreover, the moduli space of solutions, which is a point in this case, is regular.

Proof. Theorem 1.32 proves the exponential decay of (A, ϕ) on ends. Integration by parts as in lemma 1.36 gives

$$(1.76) \quad \int_X \left(|\bar{\partial}_a \alpha + \bar{\partial}_a^* \beta|^2 + 2|iF_a^\omega - \frac{r}{8}(1 - |\alpha|^2 + |\beta|^2)|^2 + 2|F_a^{0,2} - \frac{r}{4}\bar{\alpha}\beta|^2 \right.$$
$$\left. + \frac{r}{2}iF_a^\omega - 2|iF_a^\omega|^2 - 2|F_a^{0,2}|^2 \right)$$
$$= \int_X \left(\frac{1}{2}|\nabla_a \alpha|^2 + \frac{1}{2}|\nabla_{A'+a}\beta|^2 + \frac{1}{2}\langle iF_{A'+a}^\omega \beta, \beta \rangle \right.$$
$$\left. + \frac{r^2}{32}(1 - |\alpha|^2 - |\beta|^2)^2 + \frac{r^2}{8}|\beta|^2 - 2\langle N \circ \partial_a \alpha, \beta \rangle \right).$$

Equation (1.75) implies that the first three terms of the left hand side of (1.76) are zero. Therefore when r is sufficiently large,

$$\int_X \frac{r}{2}iF_a^\omega \geqslant \frac{1}{4}\int_X (E - |F_a^-|^2).$$

However by exponential decay,

$$\int_X F_a \wedge \Omega = \int_X F_a \wedge d\Theta = 0.$$

Therefore $E - |F_a^-|^2 \equiv 0$ for any solution (A, ϕ). The regularity of moduli space follows from inequality (1.17) of section 1.4 and lemma 3.11 of [21]. $\square$

Proof of theorem 1.47. Notice that X_1 is a noncompact manifold with $-Y$ as its boundary. Here $-Y$ denotes the manifold Y with a reversed orientation. The

same construction as in theorem 1.46 applied to X_1 attached with a cylindrical end $Y \times (-\infty, 0]$ gives a cohomology class $c^*(\mathcal{F}) \in \widehat{HM}^{\bullet}(Y)$. Now stretch the normal neighborhood of $Y \times \{0\}$ of X. Theorem 1.42 justfies the standard gluing argument and gives

$$(1.77) \qquad \langle j_* c(\mathcal{F}), c^*(\mathcal{F}) \rangle = \sum_{\tau_1} SW(X, \mathfrak{s}, \tau_0, \tau_1).$$

Here the summation is taken over the homotopy classes of τ_1 such that the formal dimension of the moduli space of solutions to (1.75) in $\mathcal{C}_k(X, \mathfrak{s}, r, \{\tau_0, \tau_1\})$ is zero. The number $SW(X, \mathfrak{s}, \tau_0, \tau_1)$ is an oriented counting of points in the moduli space after a generic perturbation of the equation.

By the previous lemma, the point $(A_0, \sqrt{r}\Phi_0)$ is a regular point in the moduli space of solutions to (1.75), and it is the only point in that moduli space. Equation (1.77) then implies

$$\langle j_* c(\mathcal{F}), c^*(\mathcal{F}) \rangle = \pm 1.$$

Hence $j_* c(\mathcal{F}) \neq 0$. $\qquad \square$

The next result concerns the grading of $c(\mathcal{F})$.

Theorem 1.49. *The grading of $c(\mathcal{F})$ is represented by the homotopy class of the tangent plane field of $\mathcal{F}$.*

Proof. By the index formula in proposition 1.15, the grading of $c(\mathcal{F})$ is represented by a nowhere vanishing section $\psi \in \Gamma(Y \times \{0\}, S^+)$ such that the relative Euler number $e(X_0 \cup Y \times [0, \infty), \Phi_0|_{X_0}, \psi) = 0$. Since $\mathfrak{s}$ is the canonical Spin^c structure, ψ can be taken to be $\Phi_0|_{Y \times \{0\}}$. A straight forward calculation then shows that the plane field corresponding to (S^+, ψ) is homotopic to $\ker \alpha = \mathcal{F}$. $\qquad \square$

1.7. **Topological applications.** The following result is a corollary of theorem 1.47 and theorem 1.49.

Corollary 1.50 (Kronheimer and Mrowka [22])**.** *Let Y be an oriented three manifold. If $\mathcal{F}$ is a smooth foliation on Y without transverse invariant measure, then $HM_{[\mathcal{F}]}(Y) \neq 0$.* □

Since every foliation without transverse invariant measure is a taut foliation, the corollary above is a special case of theorem 41.4.1 of [22]. On the other hand, on rational homology spheres every foliation without transverse invariant measure is a taut foliation. Therefore, corollary 1.50 gives an alternative proof of theorem 41.4.1 of [22] for rational homology spheres without making reference to the Eliashberg-Thurston theorem.

With some more effort one can use corollary 1.50 to prove the nonvanishing theorem for taut foliations on some other three manifolds. In fact, the following lemma shows that in many cases smooth folaitions without transverse invariant measure are "generic" among smooth taut foliations. The lemma was explained to the author by Jonathan Bowden.

Lemma 1.51 (Bowden [4])**.** *Let Y be an atoroidal manifold and $\mathcal{F}$ a smooth taut foliation on Y. Then $\mathcal{F}$ can be C^0 approximated by a smooth taut folaition $\mathcal{F}'$ such that either $\mathcal{F}'$ has no transverse invariant measure or the pair $(Y, \mathcal{F}')$ is homeomorphic to a surface bundle over S^1 foliated by the fibers.*

Proof. By [3], the foliation $\mathcal{F}$ can be can be C^0 approximated by a smooth taut folaition $\mathcal{F}_1$ such that every closed leaf of $\mathcal{F}_1$ has genus 0 or 1. Since $Y \not\cong S^2 \times S^1$, by Reeb's stability theorem the foliation $\mathcal{F}_1$ has no closed leaf with genus 0. Since every closed leaf of a taut foliation is incompressible and Y is assumed to be atoroidal, the foliation $\mathcal{F}_1$ has no torus leaf. This proves that $\mathcal{F}_1$ has no closed leaf.

If $\mathcal{F}_1$ has a transverse invariant measure μ, let A be a minimal set contained in the support of μ. Since $\mathcal{F}_1$ has no closed leaf, the minimal set A is either equal to Y or is exceptional. If A is exceptional, by Sacksteder's theorem, there exists a leaf L

in A containing a curve of contracting linear holonomy. Since L is in the support of μ, on a neighborhood of L the measure μ has to be a constant multiple of the delta measure of L. This implies that L is a closed leaf, which is a contradiction. Therefore $A = Y$, and in this case $\mathcal{F}_1$ can be further perturbed in the C^0 norm to a foliation which is homeomorphic to a surface bundle over S^1 foliated by the fibers (corollary 9.5.9 of [7]). $\qquad\square$

The following nonvanishing theorem follows immediately.

Corollary 1.52 (Kronheimer and Mrowka [22]). *Let Y be an atoroidal manifold, and assume that Y is not a surface bundle over S^1. If $\mathcal{F}$ is a smooth taut foliation on Y, then $HM_{[\mathcal{F}]}(Y) \neq 0$.* $\qquad\square$

The next two corollaries follow from the fact that $HM(Y)$ is of finite rank.

Corollary 1.53 (Kronheimer and Mrowka [21]). *On a three manifold Y, there are only finitely many homotopy classes of plane fields that can be realized by smooth foliations without transverse invariant measure.*

Corollary 1.54 (Kronheimer and Mrowka [21]). *If Y is a rational homology sphere, or if Y is an atoroidal manifold and not homeomorphic to a surface bundle over S^1, then there are only finitely many homotopy classes of plane fields that can be realized by smooth taut foliations.* $\qquad\square$

Since $c(\mathcal{F})$ is non-zero and is graded by the homotopy class of $\mathcal{F}$, foliations in different homotopy classes as oriented plane fields have different values of $c(\mathcal{F})$. It turns out that the invariant $c(\mathcal{F})$ is stronger than the homotopy class itself. The following theorem is a preparation for the construction of such examples.

Theorem 1.55. *Suppose Y bounds a 4-manifold M, and assume that there is an exact symplectic form ω on M such that $\omega|_Y$ is positive on $\mathcal{F}$. Assume further that*

$2\,c_1(\omega) \neq 0$. Let $-\mathcal{F}$ be the same foliation as $\mathcal{F}$ but with orientation reversed. Then the foliation invariants $c(\mathcal{F})$ and $c(-\mathcal{F})$ are linearly independent.

Proof. Write $\mathcal{F} = \ker\lambda$ and let π_Y be the projection of $Y \times \mathbb{R}$ onto Y, the the symplectic form Ω on $Y \times \mathbb{R}$ used in the definition of $c(\mathcal{F})$ can be taken to be $\Omega = \pi_Y^*(\omega_Y) + d(t \cdot \pi_Y^*\lambda)$.

Now identify a neighborhood of $Y = \partial M \subset M$ as $(-1, 0] \times Y$, and attach a cylindrical end $[0, +\infty) \times Y$ to the boundary. Let $\widetilde{M} = M \cup [0, +\infty) \times Y$ be the resulting manifold. Fix a non-decreasing smooth function $p : [-1, +\infty) \to [-1, 0]$ such that $p(t) = 0$ for $p \geq 0$ and $p(t) = t$ on $[-1, -1/2]$. The map $p \times \mathrm{id}_Y :$ $[-1, +\infty) \times Y \to [-1, 0] \times Y$ then extends to a map $p_1 : \widetilde{M} \to M$ by identity. Let $g : [-1, +\infty) \to [0, +\infty)$ be a non-decreasing smooth function such that $g(t) = 0$ near $t = -1$, and $g(t) > 0$ when $t \geq -1/2$, and $g(t) = t$ when $t \geq 1$. The form $d(g(t)\pi_Y^*\lambda)$ is defined on $[-1, +\infty) \times Y$ and extends to $\widetilde{M}$ by zero. Denote the extended form by ν, let $\Omega_0 = \nu + p_1^*(\omega)$. Then Ω_0 is a symplectic form on $\widetilde{M}$ which coincides with Ω on $[1, +\infty) \times Y$. Let g_0 be a metric on $\widetilde{M}$ which is compatible with Ω_0 and equals the metric g defined by equation (1.1) on $[1, +\infty) \times Y$.

Remove a small ball in M, the remaining part of M forms a cobordism from Y to S^3. For any Spin^c structure $\mathfrak{s}$ on M, it induces a map $\widetilde{HM}^*(M, \mathfrak{s}) : \widetilde{HM}^*(S^3) \to \widetilde{HM}^*(Y)$. Write $\check{1} \in \widetilde{HM}^*(S^3) = \mathbb{Z}[U]$ be the generator of the cohomology.

Let $\mathfrak{s}_0$ be the canonical Spin^c structure associated to the symplectic form Ω_0 on $\widetilde{M}$, let A_0 be its canonical Spin^c connection, let Φ_0 be the canonical section of the spinor bundle of $\mathfrak{s}_0$. For any Spin^c structure $\mathfrak{s}$ on $\widetilde{M}$, consider the perturbed Seiberg-Witten equation

$$(1.78) \qquad \mathfrak{F}(A, \phi) = \Big(-\frac{ir}{4}\rho(\Omega_0) + \rho(F_{A_0^t}^+), 0\Big).$$

Use integration by parts as in lemma 1.48, one can prove that when r is sufficiently large the only solution to equation (1.78) up to gauge transformation is $\mathfrak{s} = \mathfrak{s}_0$ and

$(A, \phi) = (A_0, \sqrt{r}\Phi_0)$. Inequality (1.17) of section 1.4 and lemma 3.11 of [21] then implies that this solution represents a regular point of the moduli space of solutions. By the gluing formula:

$$\langle \widetilde{HM}^*(M, \mathfrak{s})(\check{1}), c(\mathcal{F}) \rangle = \begin{cases} \pm 1 & \text{if } \mathfrak{s} = \mathfrak{s}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Now change the symplectic form on M from ω to $-\omega$, the canonical Spin^c structure is then changed to the conjugation of $\mathfrak{s}_0$, and the gluing formula becomes:

$$\langle \widetilde{HM}^*(M, \mathfrak{s})(\check{1}), c(-\mathcal{F}) \rangle = \begin{cases} \pm 1 & \text{if } \mathfrak{s} = \overline{\mathfrak{s}_0}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $2\, c_1(\omega) \neq 0$, the Spin^c structures $\mathfrak{s}_0$ and $\overline{\mathfrak{s}_0}$ are different, therefore $c(\mathcal{F})$ and $c(-\mathcal{F})$ are linearly independent. $\qquad \square$

The next lemma will help to provide examples that satisfy the conditions of theorem 1.55. The result was explained to the author by Cheuk Yu Mak. Recall that a contact form α on Y is said to have a strong symplectic filling if Y bounds a 4-manifold (M, ω), such that there is a Liouville vector field X near Y with $(\iota_X \omega)|_Y = \alpha$.

Lemma 1.56. *Let Y be an S^1 bundle over a compact surface of genus g with Euler number $e < 0$ and $e \neq 2 - 2g$. There exists a contact form α on Y, such that α has an exact strong symplectic filling with a non-torsion first Chern class, and such that the Reeb vector field of α is the positive unit tangent vector field of the S^1-fibers.*

Proof. Let E be a holomorphic line bundle with Euler number e over a Riemann surface of genus g, denote the complex structure on E by J. Let h be an Hermittian metric on E such that its Chern connection has negative curvature. Let E_1 be the unit disk bundle of E with respect to the metric h, then E_1 is a complex manifold with a J-convex boundary. The circle bundle ∂E_1 is a principal $U(1)$-bundle and the

Chern connection on E induces a connection on ∂E_1. Let α_1 be the connection form on ∂E_1. Then $\ker \alpha_1 = T\partial E_1 \cap J(T\partial E_1)$ is a contact structure on ∂E_1, and the Reeb vector field of α_1 is exactly the positive unit tangent vector field of the S^1-fibers.

By a theorem of Bogomolov and de Oliveira [2], there exists a smooth family of integrable almost complex structures J_t, $t \in (0,1)$ on E_1, such that $J_0 = J$ and (E_1, J_t) is Stein when $t > 0$.

Let f be a J_0-convex function defined near ∂E_1, such that $\partial E_1 = f^{-1}(1)$, the value 1 is a regular value of f, and that $f < 1$ in the interiori of E_1. There exists an $\epsilon_0 > 0$ sufficiently small such that for any $0 < \delta < \epsilon_0$, the function f is J_δ-convex and the level set $f^{-1}(1 - \delta)$ is regular and diffeomorphic to $f^{-1}(1)$. Now $\alpha_{1-\delta} := df \circ J_\delta$ is a contact form on the level set $f^{-1}(1 - \delta)$.

Let $\alpha'_{1-\delta}$ be the pull back of $\alpha_{1-\delta}$ to ∂E_1. For sufficiently small δ, the contact structure $\ker \alpha'_{1-\delta}$ is C^∞ close to $\ker \alpha_1$, thus by Gray's stability theorem there exists a diffeomorphism $\iota : \partial E_1 \to \partial E_1$ which is isotopic to the identity and a positive function u on ∂E_1 such that $\iota^*(u \cdot \alpha'_{1-\delta}) = \alpha_1$. The Reeb vector field of $\iota^*(u \cdot \alpha'_{1-\delta})$ is therefore the positive unit tangent vector field of the S^1-fibers. Notice that for a sufficiently large constant C, there exists a strong symplectic cobordism from $(\partial E_1, u \cdot \alpha'_{1-\delta})$ to $(\partial E_1, \alpha'_{1-\delta}/C)$. Since $(\partial E_1, \alpha'_{1-\delta})$ is Stein fillable, this implies that the contact form $u \cdot \alpha'_{1-\delta}$ is strong exact symplecticly fillable, hence so is $\iota^*(u \cdot \alpha'_{1-\delta})$. The first Chern class of the filling is equal to the first Chern class of the complex manifold (E, J), which is not torsion when $e \neq 2 - 2g$. Since $Y \cong \partial E_1$, this proves the lemma. $\qquad \square$

Let Y be an S^1 bundle over a compact surface of genus $g > 1$ with Euler number e, such that $2 - 2g < e < 0$. By a theorem of Wood [46], there exists an oriented smooth foliation $\mathcal{F}$ on Y which is transverse to the S^1 fibers. Let $-\mathcal{F}$ be the same foliation as $\mathcal{F}$ but with the opposite orientation.

Proposition 1.57. *Let Y, e, $\mathcal{F}$, and $-\mathcal{F}$ be defined as above. Then $\mathcal{F}$, $-\mathcal{F}$ are foliations without transverse invariant measure, and $c(\mathcal{F})$ and $c(-\mathcal{F})$ are linearly*

independent. Furthermore, if $e|2g-2$, then $c(\mathcal{F})$ and $c(-\mathcal{F})$ are homotopic as oriented plane fields.

Proof. By lemma 1.56, there exists a contact form α on Y with a strong exact symplectic filling (M, ω), such that $c_1(\omega)$ is not torsion on M and the Reeb vector field of α is positively transverse to $\mathcal{F}$. Notice that the Reeb vector field being positively transverse to $\mathcal{F}$ is equivalent to the form ω being positive on $\mathcal{F}$. Since ω is exact, this implies that $\mathcal{F}$ and $-\mathcal{F}$ have no transverse invariant measure. Moreover, by theorem 1.55, $c(\mathcal{F})$ and $c(-\mathcal{F})$ are linearly independent.

It remains to prove that $\mathcal{F}$ and $-\mathcal{F}$ are homotopic as plane fields when $e|2g-2$. Let $S^1 \to Y \xrightarrow{\pi} \Sigma$ be the bundle structure of Y, let $e(Y) \in H^2(\Sigma)$ be the Euler class of the bundle. By the Gysin exact equence,

$$H^0(\Sigma) \xrightarrow{\cup e(Y)} H^2(\Sigma) \xrightarrow{\pi^*} H^2(Y)$$

is exact. Notice that $\mathcal{F}$ is isomorphic to $\pi^*(T\Sigma)$ as a plane bundle, therefore the assumption $e|2g-2$ implies that the Euler class of $\mathcal{F}$ is zero, hence $\mathcal{F}$ has a globally defined basis $\{e_1, e_2\}$. Let e_3 be the positively oriented normal vector field of $\mathcal{F}$, then for $t \in [0, 1]$ the family of plane fields $\mathcal{F}_t = \text{span}\{e_1, \cos(\pi t)\, e_2 + \sin(\pi t)\, e_3\}$ defines a homotopy from $\mathcal{F}$ to $-\mathcal{F}$. $\qquad\square$

2. Modulo 2 counting of Klein-bottle leaves in smooth taut foliations

2.1. **Introduction.** Given a smooth cooriented foliation on a three manifold, it was proved in [3] that after a generic smooth perturbation, there is no closed leaf with genus greater than 1. This article explores the other side of the story, and proves the deformation invariance of the parity for the number of Klein-bottle leaves in taut foliations. As a corollary, one can construct a taut foliation such that every smooth deformation of it through taut foliations has at least one Klein-bottle leaf.

Let $\mathcal{L}$ be a smooth cooriented 2-dimensional foliation on a smooth three manifold Y. The foliation $\mathcal{L}$ and the manifold Y are allowed to be non-orientable. By definition, the foliation $\mathcal{L}$ is called a taut foliation if for every point $p \in Y$ there exists an embedded circle in Y, which passes through p and is transverse to $\mathcal{L}$.

Let K be a leaf of $\mathcal{L}$, let $\gamma : S^1 \to K$ be a closed oriented curve on K. The holonomy of $\mathcal{L}$ along γ is defined as follows. Take a map $i : S^1 \times (-1, 1) \to Y$, such that for every $x \in S^1$, $i(x, 0) = \gamma(x)$, and the image of $\{x\} \times (-1, 1)$ is transverse to $\mathcal{L}$. The intersection of the image of i with $\mathcal{L}$ then defines a horizontal direction field on $S^1 \times (-1, 1)$, and the integration of the direction field defines a map $h_\gamma : (-\epsilon, \epsilon) \to (-1, 1)$ for ϵ sufficiently small. Up to conjugations, the germ of h_γ at 0 is well-defined and is independent of the choice of i. The holonomy of $\mathcal{L}$ along γ is defined to be the germ of h_γ at 0. The value $h'_\gamma(0)$ is called the linear holonomy of $\mathcal{L}$ along γ.

Definition 2.1. Let $K \subset Y$ be a closed leaf of $\mathcal{L}$. The leaf K is said to have non-trivial linear holonomy if there exists a closed curve γ on K, such that the linear holonomy of $\mathcal{L}$ along γ is not equal to 1.

Let K be a closed 2-dimensional submanifold of Y. If K is cooriented, one can define an element $PD[K] \in \text{Hom}(H_1(Y; \mathbb{Z}); \mathbb{Z})$ as follows. Let $[\gamma]$ be a homology class represented by a closed curve γ, then $PD[K]$ maps $[\gamma]$ to the oriented intersection

number of γ and K. Since $\mathrm{Hom}(H_1(Y;\mathbb{Z});\mathbb{Z}) \cong H^1(Y;\mathbb{Z})$, the element $PD[K]$ can be viewed as an element in $H^1(Y;\mathbb{Z})$. If both Y and K are oriented and if the orientations of Y and K are compatible with the coorientation of K, then $PD[K]$ is equal to the Poincaré dual of the fundamental class of K.

Definition 2.2. Let $A \in H^1(Y;\mathbb{Z})$. A closed leaf K of $\mathcal{L}$ is said to be in the class A if $PD[K] = A$. The foliation $\mathcal{L}$ is called A-admissible if every Klein-bottle leaf of $\mathcal{L}$ in the class A has non-trivial linear holonomy.

The following result is the main theorem of this article.

Theorem 2.3. *Let $A \in H^1(Y;\mathbb{Z})$. Let $\mathcal{L}_s$, $s \in [0,1]$ be a smooth family of coorientable taut foliations on Y. Suppose $\mathcal{L}_0$ and $\mathcal{L}_1$ are both A-admissible. For $i = 0, 1$, let n_i be the number of Klein-bottle leaves in the class A. Then n_0 and n_1 have the same parity.*

Notice that if there is no Klein-bottle leaf of $\mathcal{L}$ in the homology class A, then $\mathcal{L}$ is automatically A-admissible. Therefore, the following result follows immediately.

Corollary 2.4. *Let $A \in H^1(Y;\mathbb{Z})$, and let $\mathcal{L}$ be an A-admissible smooth coorientable taut foliation on Y. Assume that $\mathcal{L}$ has an odd number of Klein-bottle leaves in the class A. Then every smooth deformation of $\mathcal{L}$ through taut foliations has at least one Klein-bottle leaf in the class A.* □

Remark 2.5. It would be interesting to understand whether a similar result holds for torus leaves. Suppose $\mathcal{L}_0$ and $\mathcal{L}_1$ are two cooriented taut foliations on Y that can be deformed to each other through taut foliations. Suppose every closed torus leaf in a homology class A has non-trivial linear holonomy, is it always true that the numbers of torus leaves in the homology class A in $\mathcal{L}_0$ and $\mathcal{L}_1$ have the same parity? At the time of writing, the answer is not clear to the author .

This article is organized as follows. Sections 2.2 and 2.3 build up necessary tools for the proof of theorem 2.3. Sections 2.4 and 2.5 prove the theorem. Section 2.6 gives an explicit example for corollary 2.4, therefore constructs a taut foliation such that every deformation of it has at least one Klein-bottle leaf.

This work was finished when I was a graduate student at Harvard University. I would like to express my most sincere gratitude to my advisor Clifford Taubes, for his inspiration, encouragement, and patient guidance. I also thank the anonymous referee for carefully reading the manuscript and providing many insightful comments and suggestions.

2.2. **Moduli spaces of J-holomorphic tori.** This section recalls some properties for the spaces of J-holomorphic tori in a symplectic manifold. Many results in this section are essentially special cases of Taubes's theory on Gromov invariants [36].

Let X be a smooth 4-manifold. To avoid complications caused by exceptional spheres, assume throughout this section that $\pi_2(X) = 0$. This will be enough for the proof of theorem 2.3. Let J be a smooth almost complex structure on X.

Consider an immersed closed J-holomorphic curve C in X. Let N be the normal bundle of C, the fiber of N then inherits an almost complex structure from J. Let $\pi : N \to C$ be the projection from N to C. Choose a local diffeomorphism φ from a neighborhood of the zero section of N to a neighborhood of C in X, which maps the zero section of N to C. The map φ can be chosen in such a way that the tangent map is $\mathbb{C}$-linear on the zero section of N. Every closed immersed J-holomorphic curve that is C^1-close to C is the image of a section of N. Fix an arbitrary connection ∇_0 on N and let $\bar{\partial}_0$ be the $(0,1)$-part of ∇_0. If s is a section of N near the zero section, the equation for $\varphi(s)$ to be a J-holomorphic curve in X can be schematically written as

$$(2.1) \qquad \bar{\partial}_0 s + \tau(s)(\nabla_0(s)) + Q(s)(\nabla_0(s), \nabla_0(s)) + T(s) = 0.$$

Here τ is a smooth section of $\pi^*(\mathrm{Hom}_{\mathbb{R}}(T^*C \otimes_{\mathbb{R}} N, T^{0,1}C \otimes_{\mathbb{C}} N))$, and Q is a smooth section of $\pi^*(\mathrm{Hom}_{\mathbb{R}}(T^*C \otimes_{\mathbb{R}} N \otimes_{\mathbb{R}} N, T^{0,1}C \otimes_{\mathbb{C}} N))$, and T is a smooth section of $\pi^*(T^{0,1}C \otimes_{\mathbb{C}} N)$. The values of τ, Q, and T are defined pointwise by the values of J in an algebraic way, and τ, Q, T are zero when $s = 0$. The linearized equation of (2.1) at $s = 0$ is $\bar{\partial}_0(s) + \frac{\partial T}{\partial s}(s) = 0$. Define

$$(2.2) \qquad L(s) := \bar{\partial}_0(s) + \frac{\partial T}{\partial s}(s).$$

Notice that L is only an $\mathbb{R}$-linear operator. The curve C is called nondegenerate if L is surjective as a map from $L_1^2(N)$ to $L^2(N)$. By elliptic regularity, if C is nondegenerate then the operator L is also surjective as a map from $L_k^2(N)$ to $L_{k-1}^2(N)$ for every $k \geqslant 1$. The index of the operator L equals

$$(2.3) \qquad \mathrm{ind}\, L = \langle c_1(N), [C] \rangle - \langle c_1(T^{0,1}X), [C] \rangle.$$

It follows from the definition that nondegeneracy only depends on the 1-jet of J on C. Namely, if there is another almost complex structure J' such that $(J - J')|_C = 0$ and $(\nabla(J - J'))|_C = 0$, then C is nondegenerate as a J-holomorpic curve if and only if it is nondegenerate as a J'-holomorphic curve.

For a homology class $e \in H_2(X; \mathbb{Z})$, define

$$d(e) = e \cdot e - \langle c_1(T^{0,1}X), e \rangle.$$

By equation (2.3), $d(e)$ is the formal dimension of the moduli space of embedded pseudo-holomorphic curves in X in the homology class e. By the adjunction formula, the genus g of such a curve satisfies

$$e \cdot e + 2 - 2g = -\langle c_1(T^{0,1}X), e \rangle.$$

Therefore $d(e) = 2(g - \langle c_1(T^{0,1}X), e \rangle - 1)$. In general, the formal dimension of the moduli space of J-holomorphic *maps* from a genus g curve to X in the homology class e, modulo self-isomorphisms of the domain, is also $2(g - \langle c_1(T^{0,1}X), e \rangle - 1)$.

Now assume X has a symplectic structure ω. Recall that an almost complex structure J is compatible with ω if $\omega(\cdot, J\cdot)$ defines a Riemannian metric. Let $\mathcal{J}(X, \omega)$ be the set of smooth almost complex structures compatible with ω. For a closed surface Σ and a map $\rho : \Sigma \to X$, define the topological energy of ρ to be $\int_\Sigma \rho^*(\omega)$.

Definition 2.6. Let (X, ω) be a symplectic manifold. Let $E > 0$ be a constant. An almost complex structure $J \in \mathcal{J}(X, \omega)$ is called E-admissible if the following conditions hold:

(1) Every embedded J-holomorphic torus C with topological energy less than or equal to E and with $d([C]) = 0$ is nondegenerate.

(2) For every homology class $e \in H_2(X; \mathbb{Z})$, if $\langle [\omega], e \rangle \leqslant E$, and if $\langle c_1(T^{0,1}X), e \rangle > 0$ (namely, the formal dimension of the moduli space of J-holomorphic maps from a torus to X in the homology class e, modulo self-isomorphisms of the domain, is negative), then there is no somewhere injective J-holomorphic map from a torus to X in the homology class e.

The next lemma is a special case of proposition 7.1 in [37]. Recall that the C^∞ topology on $\mathcal{J}(X, \omega)$ is defined as the Fréchet topology, namely it is induced by the distance function

$$d(j_1, j_2) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{\|j_1 - j_2\|_{C^n}}{1 + \|j_1 - j_2\|_{C^n}}.$$

Lemma 2.7. *Let $E > 0$ be a constant. If (X, ω) is a compact symplectic manifold, the set of E-admissible almost complex structures form a dense subset of $\mathcal{J}(X, \omega)$ in the C^∞-topology.* □

A homology class e is called primitive if $e \neq n \cdot e'$ for every integer $n > 1$ and every $e' \in H_2(X;\mathbb{Z})$. If $e \in H_2(X;\mathbb{Z})$ is a primitive class, define $\mathcal{M}(X, J, e)$ to be the set of embedded J-holomorphic tori in X with fundamental class e.

Now consider smooth families of almost complex structures. Assume ω_s $(s \in [0,1])$ is a smooth family of symplectic forms on X. For $i = 0, 1$, let $J_i \in \mathcal{J}(X, \omega_i)$. Define

$$\mathcal{J}(X, \{\omega_s\}, J_0, J_1)$$

to be the set of smooth families $\{J_s\}$ connecting J_0 and J_1, such that $J_s \in \mathcal{J}(X, \omega_s)$ for each $s \in [0,1]$.

Lemma 2.8. *Let X be a compact 4-manifold and let ω_s $(s \in [0,1])$ be a smooth family of symplectic forms on X. Let $e \in H_2(X;\mathbb{Z})$ be a primitive class, such that $\langle c_1(T^{0,1}X), e \rangle = 0$ and $e \cdot e = 0$, and let $E > 0$ be a constant such that $E > \langle [\omega_s], e \rangle$ for every s. For $i \in \{0, 1\}$, let $J_i \in \mathcal{J}(X, \omega_i)$ be an E-admissible almost complex structure on X. Then there is an open and dense subset $\mathcal{U} \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1)$ in the C^∞-topology, such that for every element $\{J_s\} \in \mathcal{U}$, the moduli space $\mathcal{M}(X, \{J_s\}, e) = \coprod_{s \in [0,1]} \mathcal{M}(X, J_s, e)$ has the structure of a compact smooth 1-manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$.*

Proof. For general X and e the moduli space $\mathcal{M}(X, \{J_s\}, e)$ may not be compact. However, the compactness of the space $\mathcal{M}(X, \{J_s\}, e)$ follows from the assumptions that $\pi_2(X) = 0$ and e being primitive. Since $\mathcal{M}(X, \{J_s\}, e)$ only consists of tori, Gromov's compactness theorem (see for example [47]) implies that for every sequence $\{C_n\} \subset \mathcal{M}(X, \{J_s\}, e)$, there is a subsequence $\{C_{n_i}\}$ with $C_{n_i} \in \mathcal{M}(X, J_{s_i}, e)$ and $\lim_{i \to \infty} s_i = s_0$, such that the sequence C_{n_i} is convergent to the image of one of the following: (1) a possibly branched multiple cover of a somewhere injective J_{s_0}-holomorphic map, (2) a J_{s_0}-holomorphic map with at least one spherical component, (3) a somewhere injective J_{s_0}-holomorphic map from a torus. Case (1) is impossible

since e is assumed to be a primitive class. Case (2) is impossible becase there is no non-constant J_{s_0}-holomorphic maps from a sphere to X. When case (3) happens, for the limit curve the adjunction formula states that $e \cdot e + 2 - 2g = -\langle c_1(T^{0,1}X), e \rangle + \kappa$, where κ depends on the behaviour of singularities and self-intersections of the curve, and κ is always positive if the curve is not embedded (see [24]). Since $g = 1$, $e \cdot e = 0$, $\langle c_1(T^{0,1}X), e \rangle = 0$, it follows that $\kappa = 0$, hence the limit curve is an embedded curve, namely it is an element of $\mathcal{M}(X, J_{s_0}, e)$. Therefore the space $\mathcal{M}(X, \{J_s\}, e)$ is compact.

Since $\mathcal{M}(X, \{J_s\}, e)$ consists of only embedded curves, the standard transversality argument (see for example section 3.2 of [25]) shows that on a dense subset $\mathcal{V} \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1)$, the moduli space $\mathcal{M}(X, \{J_s\}, e)$ is a smooth 1-manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$.

Since $\mathcal{M}(X, \{J_s\}, e)$ is always compact, the transversality condition is an open condition, therefore there exists an open set $\mathcal{U} \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1)$, such that $\mathcal{V} \subset \mathcal{U}$, and for every $\{J_s\} \in \mathcal{U}$, the moduli space $\mathcal{M}(X, \{J_s\}, e)$ is a compact smooth 1-manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$. □

With a little more effort one can generalize lemma 2.8 to non-compact symplectic manifolds. To start, one needs the following definition.

Definition 2.9. Let (X, ω) be a symplectic manifold, not necessarily compact. Let $J \in \mathcal{J}(X, \omega)$. The pair (ω, J) defines a Riemannian metric g on X. The triple (X, ω, J) is said to have bounded geometry with bounding constant N if the following conditions hold:

(1) The metric g is complete.

(2) The norm of the curvature tensor of g is less than N.

(3) The injectivity radius of (X, g) is greater than $1/N$.

One says that a path $\{(X, \omega_s, J_s)\}$ has uniformly bounded geometry if for each s, the space (X, ω_s, J_s) has bounded geometry, and the bounding constant N is independent of s.

The following lemma is a well-known result.

Lemma 2.10. *Let* (X, ω, J) *be a triple with bounded geometry, with bounding constant* N. *Let* $e \in H_2(X; \mathbb{Z})$, *and let* $E > 0$ *be a constant such that* $E \geqslant \langle [\omega], e \rangle$. *Then there is a constant* $M(N, E)$, *depending only on* N *and* E, *such that every connected* J-*holomorphic curve* C *with fundamental class* e *must have diameter less than* $M(N, E)$ *with respect to the metric defined by* $\omega(\cdot, J\cdot)$.

Proof. By the monotonicity of area for J-holomorphic curves (see, for example, [16, section 2.3 E'_2]), the area of $B_p(1/N) \cap C$ is greater than or equal to $\frac{\pi}{N^2}$. Since C is connected, this implies that its diameter is bounded by $2\,Area(C)N/\pi$. Notice that the area of C equals $\langle [\omega], e \rangle$, which is bounded by E, hence the the diameter is bounded by $2EN/\pi$. $\qquad\square$

In the noncompact case, one needs to be more careful about the topology on the space of almost complex structures. A topology on $\mathcal{J}(X, \omega)$ can be defined as follows. Cover X by countably many compact sets $\{A_i\}_{i \in \mathbb{Z}}$. For each A_i define the C^∞-topology on $\mathcal{J}(A_i, \omega)$. Endow the product space

$$\prod_{i \in \mathbb{Z}} \mathcal{J}(A_i, \omega)$$

with the box topology, and consider the map

$$\mathcal{J}(X, \omega) \hookrightarrow \prod_{i \in \mathbb{Z}} \mathcal{J}(A_i, \omega)$$

defined by restrictions. The topology on $\mathcal{J}(X, \omega)$ is then defined as the pull back of the box topology on the product space.

The topology on $\mathcal{J}(X, \omega)$ does not depend on the choice of the covering $\{A_i\}$. When X is noncompact, the topology on $\mathcal{J}(X, \omega)$ is not first countable.

For $N > 0$, define $\mathcal{J}(X, \omega, N)$ to be the set of almost complex structures $J \in \mathcal{J}(X, \omega)$ such that (X, ω, J) has bounded geometry with bounding constant N. With the topology given above, the space $\mathcal{J}(X, \omega, N)$ is an open subset of $\mathcal{J}(X, \omega)$.

A topology on $\mathcal{J}(X, \{\omega_s\}, J_0, J_1)$ can be defined in a similar way. Cover X by countably many compact sets $\{A_i\}_{i \in \mathbb{Z}}$. For each A_i, one takes the C^∞-topology on $\mathcal{J}(A_i, \{\omega_s\}, J_0, J_1)$. The topology on the space $\mathcal{J}(X, \{\omega_s\}, J_0, J_1)$ is then defined as the pull back of the box topology on the product space. This topology does not depend on the choice of the covering $\{A_i\}$.

For $N > 0$, define the set $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ to be the set of families $\{J_s\} \in \mathcal{J}(X, \{\omega_s\}, J_0, J_1)$ such that the family $\{(X, J_s, \omega_s)\}$ has uniformly bounded geometry with bounding constant N. Then the set $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ is an open subset of the set $\mathcal{J}(X, \{\omega_s\}, J_0, J_1)$.

The following lemma is essentially a diagonal argument. It explains why the topologies defined above are the correct topologies for the perturbation arguments in this article.

Lemma 2.11. *Let $\{A_n\}_{n \geqslant 1}$ be a countable, locally finite cover of X by compact subsets. Let ω be a symplectic form on X, let ω_s be a smooth family of symplectic forms on X. Let $N > 0$ be a constant. Let $J_i \in \mathcal{J}(X, \omega_i, N)$, where $i = 0$ or 1.*

(1) *Let $\varphi : \mathcal{J}(X, \omega) \hookrightarrow \prod_n \mathcal{J}(A_n, \omega)$ be the embedding map. For every n, let $\mathcal{U}_n$ be an open and dense subset of $\mathcal{J}(A_n, \omega)$, then $\varphi^{-1}(\prod_n \mathcal{U}_n)$ is an open and dense subset of $\mathcal{J}(X, \omega)$.*

(2) *Let $\varphi : \mathcal{J}(X, \{\omega_s\}, J_0, J_1) \hookrightarrow \prod_n \mathcal{J}(A_i, \{\omega_s\}, J_0, J_1)$ be the embedding map. For every n, let*

$$\mathcal{U}_n \subset \mathcal{J}(A_n, \{\omega_s\}, J_0, J_1)$$

be an open and dense subset, then $\varphi^{-1}(\prod_n \mathcal{U}_n)$ is an open and dense subset of $\mathcal{J}(X, \{\omega_s\}, J_0, J_1)$.

Proof. For part 1, the set $\varphi^{-1}(\prod_n \mathcal{U}_n)$ is open by the definition of box topology. To prove that $\varphi^{-1}(\prod_n \mathcal{U}_n)$ is dense, let J be an element of $\mathcal{J}(X, \omega)$. Let $J_n = J|_{A_n} \in \mathcal{J}(A_n, \omega)$. For every n, let $\mathcal{V}_n \subset \mathcal{J}(A_n, \omega)$ be a given open neighborhood of J_n. One needs to find an element $J' \in \mathcal{J}(X, \omega)$ such that $J'|_{A_n} \in \mathcal{V}_n \cap \mathcal{U}_n$. For each n, let D_n be an open neighborhood of A_n such that the family $\{D_n\}$ is still a locally finite cover of X. One obtains the desired J' by perturbing J on the open sets $\{D_n\}$ one by one. To start, perturb the section J on D_1 to obtain a section J_1. Since $\mathcal{U}_1$ is dense it is possible to find a perturbation such that $J_1|_{A_1} \in \mathcal{U}_1 \cap \mathcal{V}_1$. Now assume that after perturbation on $D_1, D_2, \cdots, D_k$, one obtains a section J_k such that $J_k|_{A_i} \in \mathcal{U}_j \cap \mathcal{V}_j$ for $j = 1, 2, \cdots, k$. Then a perturbation of J_k on D_{k+1} gives a section J_{k+1} such that $J_{k+1}|_{A_{k+1}} \in \mathcal{U}_{k+1} \cap \mathcal{V}_{k+1}$. When the perturbation is small enough, it still has the property that $J_{k+1}|_{A_j} \in \mathcal{U}_j \cap \mathcal{V}_j$ for $j = 1, 2, \cdots, k$. Since $\{D_n\}$ is a locally finite cover of X, on each compact subset of X the sequence $\{J_k\}$ stabilizes for sufficiently large k. The limit $\lim_{k \to \infty} J_k$ then gives the desired J'.

The proofs for part 2 is exactly the same, one only needs to change the notation $\mathcal{J}(\cdot, \omega)$ to $\mathcal{J}(\cdot, \{\omega_s\}, J_0, J_1)$. $\qquad\square$

Remark 2.12. Lemma 2.11 is essentially a result on the box topology, and it does not use any specific properties of symplectic topology or almost complex structures. Since the lemma above is already sufficient for the purpose of this article, the most general statement is not given here.

Lemma 2.13. *Let X be a 4-manifold, let $e \in H_2(X; \mathbb{Z})$ be a primitive class with $\langle c_1(T^{0,1}X), e \rangle = 0$ and $e \cdot e = 0$. Assume ω_s $(s \in [0, 1])$ is a smooth family of symplectic forms on X. Let E be a positive constant such that $E > \langle [\omega_s], e \rangle$ for every s. For $i = 0, 1$, assume $J_i \in \mathcal{J}(X, \omega_i, N)$ is E-admissible. If the set $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ is*

not empty, then there is an open and dense subset $\mathcal{U} \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$, such that for each $\{J_s\} \in \mathcal{U}$, the moduli space

$$\mathcal{M}(X, \{J_s\}, e) = \coprod_{s \in [0,1]} \mathcal{M}(X, J_s, e)$$

has the structure of a smooth 1-manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$. Moreover, if $f : X \to \mathbb{R}$ is a smooth proper function on X, then the function defined as

$$\mathfrak{f} : \mathcal{M}(X, \{J_s\}, e) \to \mathbb{R}$$

$$C \mapsto \int_C f \, dA$$

is a smooth proper function on $\mathcal{M}(X, \{J_s\}, e)$. Here for $C \in \mathcal{M}(X, J_s, e)$, the form dA is the area form on C defined by J_s and ω_s.

Proof. First prove that $\mathfrak{f}$ is a proper function. For any constant $z > 0$, take a sequence of curves $C_n \in \mathcal{M}(X, \{J_s\}, e)$ such that $|\mathfrak{f}(C_n)| < z$. By the definition of $\mathfrak{f}$, there exists a sequence of points $p_n \in C_n$ such that $|f(p_n)| < z$. Since f is a proper function on X, the sequence p_n is bounded on X. By lemma 2.10 this implies that the curves C_n stay in a bounded subset of X. By the argument for the compact case (lemma 2.8), the sequence $\{C_n\}$ has a subsequence that converges to another point in $\mathcal{M}(X, \{J_s\}, e)$, hence the function $\mathfrak{f}$ is proper.

It remains to prove that there is an open and dense subset

$$\mathcal{U} \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1, N),$$

such that for every $\{J_s\} \in \mathcal{U}$, the space $\mathcal{M}(X, \{J_s\}, e)$ is a smooth 1-dimensional manifold. Let g_s be the metric on X compatible with J_s and ω_s. Let g be a complete metric on X such that $g_s \geqslant g$ for every s. From now on, the distance function on X is defined by the metric g. By lemma 2.10, there exists a constant $M > 0$ such

that the diameter of every J_s-holomorphic curve with topological energy no greater than E is bounded by M. Let $\{B_n\}$ be a countable locally finite cover of X by open balls of radius 1. For every n, let A_n be the closed ball with the same center as B_n and with radius $(M + 1)$. The family $\{A_n\}$ is also a locally finite cover of X. For each n, let $\mathcal{M}_n(X, \{J_s\}, e)$ be the set of curves $C \in \mathcal{M}(X, \{J_s\}, e)$ such that $C \subset A_n$, and let $\mathcal{M}'_n(X, \{J_s\}, e)$ be the set of curves $C \cap \mathcal{M}(X, \{J_s\}, e)$ such that $C \subset B_n \neq \varnothing$. By the diameter bound, $\mathcal{M}'_n(X, \{J_s\}, e) \subset \mathcal{M}_n(X, \{J_s\}, e)$. Take f to be the distance function to the center of B_n, it was proved in the previous paragraph that the corresponding function $\mathfrak{f}$ on the moduli space is proper, hence $\mathcal{M}_n(X, \{J_s\}, e)$ is a compact set, therefore the transversality condition is open on $\mathcal{M}_n(X, \{J_s\}, e)$. As a result, there is an open and dense subset $\mathcal{U}_n \subset \mathcal{J}(A_n, \{\omega_s\}, J_0, J_1, N)$, such that if $\{J_s\}|_{A_n} \in \mathcal{U}_n$, then the set $\mathcal{M}'_n(X, \{J_s\}, e) \subset \mathcal{M}_n(X, \{J_s\}, e)$ is a smooth 1 dimensional manifold. Notice that $\{\mathcal{M}'_n(X, \{J_s\}, e)\}_{n \geqslant 1}$ is an open cover of $\mathcal{M}(X, \{J_s\}, e)$. It then follows from part 2 of lemma 2.11 that there is an open and dense subset $\mathcal{U} \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ such that for every element $\{J_s\} \in \mathcal{U}$ the set $\mathcal{M}(X, \{J_s\}, e)$ is a smooth 1-manifold. $\qquad \square$

2.3. **Symplectization of taut foliations.** This section discusses a symplectization of *oriented and cooriented* taut foliations. It is the main ingredient for the proof of theorem 2.3.

Let M be a smooth 3-manifold, let $\mathcal{F}$ be a smooth oriented and cooriented taut foliation on M. Since $\mathcal{F}$ is cooriented, it can be written as $\mathcal{F} = \ker \lambda$, where λ is positive in the positive normal direction of $\mathcal{F}$. Since $\mathcal{F}$ is taut, there exists a closed 2-form ω such that $\omega \wedge \lambda > 0$ everywhere on M. Choose a metric g_0 on M such that $*_{g_0}\lambda = \omega$. By Frobenius theorem, $d\lambda = \mu \wedge \lambda$ for a unique 1-form μ satisfying $\mu \perp \lambda$. Locally, write $\omega = e^1 \wedge e^2$ where e^1 and e^2 are orthonormal with respect to the metric

g_0. Consider the 2-form $\Omega = \omega + d(t\lambda)$ on $\mathbb{R} \times M$ and the metric g defined by

$$g = \frac{1}{1+t^2}(dt + t\mu)^2 + (1+t^2)\lambda^2 + (e^1)^2 + (e^2)^2$$

The 2-form Ω is a symplectic form on $\mathbb{R} \times M$, and the metric g is independent of the choice of $\{e^1, e^2\}$ and is compatible with Ω. Let J be the almost complex structure given by (Ω, g). To simplify notations, let X be the manifold $\mathbb{R} \times M$.

Recall that by lemma 1.3,

Lemma 2.14. *The triple* (X, Ω, J) *has bounded geometry.* $\qquad \square$

Locally, let $\{e_0, e_1, e_2\}$ be the basis of TM dual to $\{\lambda, e^1, e^2\}$, and extend them to $\mathbb{R}$-translation invariant vector fields on $\mathbb{R} \times M$. Let $\hat{e}_1 = e_1 - t\mu(e_1)\frac{\partial}{\partial t}$, $\hat{e}_2 = e_2 - t\mu(e_2)\frac{\partial}{\partial t}$. The almost complex structure J is then given by

$$J\frac{\partial}{\partial t} = \frac{1}{1+t^2}e_0,$$

$$J\hat{e}_1 = \hat{e}_2.$$

Define $\widetilde{\mathcal{F}} = \text{span}\{\hat{e}_1, \hat{e}_2\}$, it is a J-invariant plane field on X.

Lemma 2.15. *The plane field* $\widetilde{\mathcal{F}}$ *is a foliation on* X. *Under the projection* $\mathbb{R} \times M \to M$, *the leaves of* $\widetilde{\mathcal{F}}$ *projects to the leaves of* $\mathcal{F}$.

Proof. Since $d\mu \wedge \lambda = d(d\lambda) = 0$, there is a μ_1 such that $d\mu = \mu_1 \wedge \lambda$. Therefore, one has $d(dt + t\mu) = (dt + t\mu) \wedge \mu + t\mu_1 \wedge \lambda$, and $d\lambda = \mu \wedge \lambda$. By the Frobenius theorem, the plane field $\widetilde{\mathcal{F}} = \ker(dt + t\mu) \cap \ker \lambda$ is a foliation. The tangent planes of $\widetilde{\mathcal{F}}$ projects isomorphically to the tangent planes of $\mathcal{F}$ pointwise, thus the leaves of $\widetilde{\mathcal{F}}$ projects to the leaves of $\mathcal{F}$. $\qquad \square$

It turns out that every closed J-holomorphic curve in X is a closed leaf of $\widetilde{\mathcal{F}}$.

Lemma 2.16. *Let $\rho : \Sigma \to X$ be a J-holomorphic map from a closed Riemann surface to X. Then either ρ is a constant map, or it is a branched cover of a closed leaf of $\widetilde{\mathcal{F}}$.*

Proof. Since ρ is J-holomorphic, $\rho^*\big((dt + t\mu) \wedge \lambda\big) \geqslant 0$ pointwise on Σ. On the other hand,

$$\int_{\Sigma} \rho^* \big((dt + t\mu) \wedge \lambda\big) = \int_{\Sigma} \rho^*(d(t\lambda)) = 0.$$

Therefore $\rho(\Sigma)$ is tangent to $\ker(dt + t\mu) \cap \ker \lambda$, hence either ρ is a constant map, or it is a branched cover of a closed leaf of $\widetilde{\mathcal{F}}$. $\qquad\square$

Lemma 2.17. *Let L be a leaf of $\mathcal{F}$ and γ a closed curve on L. Let $\pi : \mathbb{R} \times M \to M$ be the projection map. The foliation $\widetilde{\mathcal{F}}$ is then transverse to $\pi^{-1}(\gamma)$ and gives a horizontal foliation on $\pi^{-1}(\gamma) \cong \mathbb{R} \times \gamma$. The holonomy of this foliation along γ is given by the multiplication of $l(\gamma)^{-1}$, where $l(\gamma)$ is the linear holonomy of $\mathcal{F}$ along γ.*

Proof. Recall that locally (λ, e^1, e^2) is an orthonormal basis of T^*M and (e_0, e_1, e_2) is its dual basis. Let $(-\epsilon, \epsilon) \times L \subset M$ be a tubular neighborhood of L in M, let z be the first coordinate function on $(-\epsilon, \epsilon) \times L$. The parametrization of the tubular neighborhood can be chosen such that $\frac{\partial}{\partial z} = e_0$. Now λ has the form $\lambda = dz + \nu(z)$ where $\nu(z)$ is a 1-form on L depending on z and $\nu(0) = 0$. The condition that $\ker \lambda$ is a foliation is equivalent to

$$d\nu + \frac{\partial \nu}{\partial z} \wedge \nu = 0.$$

The 1-form μ satsifies $d\lambda = \mu \wedge \lambda$, therefore $\mu|_L = -\frac{\partial \nu}{\partial z}|_{z=0}$.

Suppose γ is a closed curve on L parametrized by $u \in [0, 1]$. Let $(t(u), \gamma(u))$ be a curve in $\mathbb{R} \times M$ that is a lift of γ and tangent to $\widetilde{\mathcal{F}}$. Then the function $t(u)$ satisfies $\dot{t} + t\mu(\dot{\gamma}) = 0$. Therefore

$$t(1) = \exp\left(-\int_0^1 \mu(\dot{\gamma})du\right) \cdot t(0) = \exp\left(\int_0^1 \frac{\partial \nu}{\partial z}(0)(\dot{\gamma}(u))du\right) \cdot t(0).$$

Now compute the linear holonomy of $\mathcal{F}$ along γ. If $(z(u), \gamma(u))$ is a curve in $(-\epsilon, \epsilon) \times L$ tangent to $\mathcal{F}$, then

(2.4)
$$\dot{z} + \nu(z)(\dot{\gamma}) = 0.$$

If $z_s(u)$, $s \in [0, \epsilon)$ is a smooth family of solutions to (2.4) with $z_0(u) = 0$, then the linearized part $l(u) = \frac{\partial z_s}{\partial s}|_{s=0}(u)$ satisfies

$$\dot{l} + l \cdot \frac{\partial \nu}{\partial z}\bigg|_{z=0}(\dot{\gamma}) = 0.$$

Therefore the linear holonomy of $\mathcal{F}$ along γ is

$$\exp\left(-\int_0^1 \frac{\partial \nu}{\partial z}(0)(\dot{\gamma}(u))du\right),$$

hence the linear holonomy of $\mathcal{F}$ along γ is inverse to the holonomy on $\pi^{-1}(\gamma)$ given by $\widetilde{\mathcal{F}}$. $\qquad\square$

The following result follows immediately from lemmas 2.16 and 2.17:

Corollary 2.18. *Let C be a closed embedded J-holomorphic curve on X. Then either $C \subset M \times \{0\}$ and C is a closed leaf of $\mathcal{F}$, or C does not intersect the slice $M \times \{0\}$ and it projects diffeomorphically to a closed leaf of $\mathcal{F}$ with trivial linear holonomy.* $\qquad\square$

The next lemma studies J-holomorphic tori on X.

Lemma 2.19. *Suppose T is a torus leaf of $\mathcal{F}$ with non-trivial linear holonomy. Then $T \times \{0\}$ is a nondegenerate J-holomorphic curve in X.*

Proof. Notice that $d([T]) = 0$, thus the index of the deformation operator is zero, and one only needs to prove that the operator L on T defined by equation (2.2) has a trivial kernel.

Let $T_0 = T \times \{0\}$ be the torus in X. As in lemma 2.17, let (e_0, e_1, e_2) be the dual basis of (λ, e^1, e^2). Let $(-\epsilon, \epsilon) \times T \subset M$ be a tubular neighborhood of T, let z be

the first coordinate function, and choose a parametrization such that $\frac{\partial}{\partial z} = e_0$. Then on this neighborhood, λ has the form $\lambda = dz + \nu(z)$, where $\nu(z)$ is a 1-form on T depending on z and $\nu(0) = 0$. The condition that $\ker \lambda$ is a foliation is equivalent to

$$d\nu + \frac{\partial \nu}{\partial z} \wedge \nu = 0.$$

Let $\beta = \frac{\partial \nu}{\partial z}\big|_{z=0}$. Apply $\frac{\partial}{\partial z}$ on the equation above at $z = 0$, one obtains $d\beta = 0$. Extend β to $(-\epsilon, \epsilon) \times T$ by pulling back from the second factor. Let $\lambda' = dz + z \cdot \beta$, then $\ker \lambda'$ defines another foliation near T. Let $\mu' = -\beta$.

Let e_1', e_2' be vector fields on $(-\epsilon, \epsilon) \times T$ such that they are tangent to $\ker \lambda'$, and that their projections to T form a positive orthonormal basis. Let t be the coordinate function on the $\mathbb{R}$ component, and extend e_1', e_2' to a neighborhood of T_0 in X by translations in the t-direction. Define an almost complex structure J' on $\mathbb{R} \times (-\epsilon, \epsilon) \times T$ by

$$J'\frac{\partial}{\partial t} = \frac{\partial}{\partial z},$$

$$J'(e_1' - t\mu'(e_1')\frac{\partial}{\partial t}) = e_2' - t\mu'(e_2')\frac{\partial}{\partial t}.$$

Equation (2.1) for the deformation of J'-holomorphic curves near T_0 is a linear equation. In fact, let

$$(f, g) : T \to \mathbb{R} \times (-\epsilon, \epsilon)$$

be the parametrization of a curve C near T_0. For $p \in T \cong T_0$, let $v = e_1'(p)$, $w = e_2'(p)$, then the tangent space of C at $(f(p), g(p), p)$ is spanned by $(\partial_v f, \partial_v g, v)$ and $(\partial_w f, \partial_w g, w)$. Notice that

$$\mu'(e_1')(f(p), p) = -\beta(v), \quad \mu'(e_2')(f(p), p) = -\beta(w).$$

Therefore

$$J'(\partial_v f, \partial_v g, v) = (-\partial_v g + \beta(w)f, \partial_v f - \beta(v)f, w),$$

$$J'(\partial_w f, \partial_w g, w) = (-\partial_w g - \beta(v)f, \partial_w f - \beta(w)f, -v).$$

Hence C is J'-holomorphic at $(f(p), g(p), p)$ if and only if

$$\beta(w)f = \partial_v g + \partial_w f,$$

$$\beta(v)f = \partial_v f - \partial_w g.$$

This shows that equation (2.1) is linear for curves near T_0.

On the other hand, since T has nontrivial linear holonomy, the same arguments as in lemma 2.16 and lemma 2.17 shows that T_0 is the only embedded J'-holomorphic torus in a neighborhood of T_0. Since equation (2.1) is linear for T_0, this implies that T_0 is nondegenerate as a J'-holomorphic curve. Recall that J' and J agree up to first order derivatives along the curve T_0, therefore T_0 is nondegenerate with respect to J. $\qquad\square$

2.4. **Proof of theorem 2.3.** Now let $\mathcal{L}$ be a cooriented smooth taut foliation on a smooth 3-manifold Y. Consider its orientation double cover $\widetilde{\mathcal{L}}$. It is an oriented and cooriented taut foliation on the orientation double cover of Y. Let $p : \widetilde{Y} \to Y$ be the covering map. If K is a Klein-bottle leaf of $\mathcal{L}$, then $p^{-1}(K)$ is a torus leaf of $\widetilde{L}$. Recall that in the beginning of section 2.1, a homology class $PD[K] \in H^1(Y; \mathbb{Z})$ was defined for every Klein-bottle leaf.

Lemma 2.20. *Let K be an embedded cooriented surface in Y, then $p^{-1}(K)$ is cooriented and hence inherits an orientation from $\widetilde{Y}$. Let $PD[p^{-1}(K)]$ be the Poincaré dual of the fundamental class of $p^{-1}(K)$, then $p^*(PD[K]) = PD[p^{-1}(K)]$.*

Proof. Let γ be a closed curve in $\widetilde{Y}$. Use $I(\cdot, \cdot)$ to denote the intersection number. Then

$$\langle PD[p^{-1}(K)], [\gamma] \rangle = I(p^{-1}(K), \gamma)$$

$$= I(K, p(\gamma)) = \langle PD[K], p_*[\gamma] \rangle = \langle p^*(PD[K]), [\gamma] \rangle.$$

Therefore $p^*(PD[K]) = PD[p^{-1}(K)]$. $\qquad\square$

Lemma 2.21. *The pull-back map $p^* : H^1(Y; \mathbb{Z}) \to H^1(\widetilde{Y}; \mathbb{Z})$ is injective.*

Proof. Every element in $\ker p^*$ is represented by an element $\alpha \in \mathrm{Hom}(\pi_1(Y), \mathbb{Z})$ such that α is zero on the image of $p_* : \pi_1(\widetilde{Y}) \to \pi_1(Y)$. Since $\mathrm{Im}\, p_*$ is a normal subgroup of $\pi_1(Y)$ of index 2, the map α is decomposed as

$$\alpha : \pi_1(Y) \to \pi_1(Y)/\pi_1(\widetilde{Y}) \cong \mathbb{Z}/2 \to \mathbb{Z},$$

which has to be zero. Therefore p^* is injective. $\qquad\square$

By lemma 2.20 and 2.21, a Klein-bottle leaf K has $PD[K] = A$ if and only if $PD([p^{-1}(K)]) = p^*(A)$. The next lemma shows that for every Klein-bottle leaf K of $\mathcal{L}$ the fundamental class $[p^{-1}(K)]$ is a primitive class.

Lemma 2.22. *Let $\mathcal{F}$ be an oriented and cooriented taut foliation on a smooth three manifold M, then the fundamental class of every closed leaf of $\mathcal{F}$ is a primitive class.*

Proof. Let L be a closed leaf of $\mathcal{F}$. Take a point $p \in L$. By the definition of tautness, there exists an embedded circle γ passing through p and transverse to the foliation. Let $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1) = p$ be a parametrization of γ. By transversality, $\gamma^{-1}(L)$ is a finite set. Let t_0 be the minimum value of $t > 0$ such that $\gamma(t_0) \in L$. Then for ϵ sufficiently small one can slide the part of γ on $(t_0 - \epsilon, t_0 + \epsilon)$ along the foliation, such that the resulting curve is still transverse to $\mathcal{F}$, and such that $\gamma(t_0) = p$. Now

$\gamma|_{[0,t_0]}$ defines a circle whose intersection number with L equals 1. The existence of such a curve implies that the fundamental class of L is primitive. $\qquad\square$

With the preparations above, one can now prove theorem 2.3.

Proof of theorem 2.3. Let $A \in H^1(Y; \mathbb{Z})$. Suppose $\mathcal{L}_0$ and $\mathcal{L}_1$ are two smooth A-admissible taut foliations on Y, such that they can be deformed to each other by a smooth family of taut foliations $\mathcal{L}_s$, $s \in [0,1]$. Let $\widetilde{Y}$ be the orientation double cover of Y. Then the orientation double covers $\widetilde{\mathcal{L}}_s$ of $\mathcal{L}_s$ form a smooth family of oriented and cooriented taut foliaitons on $\widetilde{Y}$.

Let $\tilde{\sigma} : \widetilde{Y} \to \widetilde{Y}$ be the deck transformation of the orientation double cover. Then the map $\tilde{\sigma}$ preserves the coorientation of $\widetilde{\mathcal{L}}_s$ and reverses its orientation for each s.

There exists a smooth family of 1-forms λ_s and closed 2-forms ω_s on $\widetilde{Y}$ such that $\widetilde{\mathcal{L}}_s = \ker \lambda_s$ and $\lambda_s \wedge \omega_s > 0$. By changing λ_s to $(\lambda_s + \tilde{\sigma}^* \lambda_s)/2$ and changing ω_s to $(\omega_s - \tilde{\sigma}^* \omega_s)/2$, one can assume that $\tilde{\sigma}^* \lambda_s = \lambda_s$, and $\tilde{\sigma}^* \omega_s = -\omega_s$. Let (Ω_s, J_s) be the corresponding symplectic structures and almost complex structures on $X = \mathbb{R} \times \widetilde{Y}$ as defined in section 2.3. Define

$$\begin{aligned} \sigma : \quad & X \quad \to X \\ & (t,x) \quad \mapsto (-t, \tilde{\sigma}(x)). \end{aligned}$$

Then $\sigma^*(\Omega_s) = -\Omega_s$, and $\sigma^*(J_s) = -J_s$. The family $\{(X, \Omega_s, J_s)\}$ has uniformly bounded geometry. This means that there is a constant $N > 0$ such that $J_s \in \mathcal{J}(X, \Omega_s, N)$ for each s.

If neither $\mathcal{L}_0$ nor $\mathcal{L}_1$ has any Klein-bottle leaf in the class A, the statement of theorem 2.3 obviously holds. From now on assume that either $\mathcal{L}_0$ or $\mathcal{L}_1$ has at least one Klein-bottle leaf in the class A. This implies either $\widetilde{\mathcal{L}}_0$ or $\widetilde{\mathcal{L}}_1$ has at least one torus leaf. By a theorem of Novikov [29] (or see for example Theorem 9.1.7 of [8]), $\pi_2(X) = \pi_2(\widetilde{Y}) = 0$. Let e be the push forward of $PD(p^*(A)) \in H_2(\widetilde{Y}; \mathbb{Z})$ to $H_2(X; \mathbb{Z})$

via the inclusion map $\widetilde{Y} \cong \{0\} \times \widetilde{Y} \hookrightarrow X$. The class e then satisfies $\sigma_*(e) = -e$. By lemma 2.22, e is a primitive class.

Take a positive constant E such that $E > \langle [\Omega_s], e \rangle$ for all s. For every $J' \in \mathcal{J}(X, \Omega_s, N)$ and every $C \in \mathcal{M}(X, J', e)$, lemma 2.10 gives a diameter bound of C by $M(N, E)$.

Let $t_0 > 0$ be a fixed positive number. For $i = 0, 1$, the union of torus leaves L in $\widetilde{\mathcal{L}}_i$ in the homology class $p^*(A)$ such that $\int_L \omega_i \leqslant E$ and L is not the lift of any Klein-bottle leaf form a compact set $\widetilde{B}_i$. The set $\widetilde{B}_i$ satisfies $\tilde{\sigma}(\widetilde{B}_i) = \widetilde{B}_i$. Let $\widetilde{U}_i$ be a neighborhood of $\widetilde{B}_i$ such that $\tilde{\sigma}(\widetilde{U}_i) = \widetilde{U}_i$ and the closure of $\widetilde{U}_i$ does not intersect the lift of any Klein-bottle leaf of $\mathcal{L}_i$. Let

$$V = \big((-\infty, -t_0) \cup (t_0, \infty) \big) \times \widetilde{Y},$$

$$U_i = \big(\mathbb{R} \times \widetilde{U}_i \big) \bigcup \big((-\infty, -t_0) \cup (t_0, \infty) \big) \times \widetilde{Y}.$$

V and U_i are open subsets of X. The following two lemmas will be proved in section 2.5.

Lemma 2.23. *For $i = 0, 1$, the almost complex structure J_i can be perturbed to $J'_i \in \mathcal{J}(X, \Omega_i, N)$, such that $J'_i = J_i$ near the lifts of Klein-bottle leaves, and J'_i is E-admissible. Moreover, one can choose J'_i such that the following are satisfied: $\sigma^*(J'_i) = -J'_i$ on U_i, and every J'_i-holomorphic torus of X in the homology class e is either contained in U_i or is the lift of a Klein-bottle leaf in $\mathcal{L}_i$ in the class A. If C is a J'_i-holomorphic curve in the homology class e contained in U_i, then $\sigma(C) \neq C$.*

Lemma 2.24. *The almost complex structures J'_0 and J'_1 given by lemma 2.23 can be connected by a smooth family of almost complex structures*

$$J'_s \in \mathcal{J}(X, \Omega_s, N),$$

such that $\sigma^*(J'_s) = -J'_s$ on V, and the moduli space

$$\mathcal{M}(X, \{J'_s\}, e) = \coprod_{s \in [0,1]} \mathcal{M}(X, J'_s, e)$$

has the structure of a smooth 1-manifold with boundary $\mathcal{M}(X, J'_0, e) \cup \mathcal{M}(X, J'_1, e)$. Moreover, let $t : X \to \mathbb{R}$ be the projection of $X = \widetilde{Y} \times \mathbb{R}$ to $\mathbb{R}$, then the function defined as

$$\mathfrak{f} : \mathcal{M}(X, \{J'_s\}, e) \to \mathbb{R}$$

$$C \mapsto \int_C t \, dA.$$

is a smooth proper function on $\mathcal{M}(X, \{J'_s\}, e)$, where for $C \in \mathcal{M}(X, J'_s, e)$, the form dA is the area form on C given by g_s.

Let $\{J'_s\}$ be the family of almost complex structures given by the lemmas above. By the bound on geometry and the diameter bound, there exists a sufficiently large $t_1 > 0$ such that for every J'_s-holomorphic torus C in the homology class e, if $|\mathfrak{f}(C)| > t_1$, then C is contained in V. Take a constant $t_2 > t_1$ such that both t_2 and $-t_2$ are regular values of $\mathfrak{f}$, and that $t_2 \notin \mathfrak{f}(\mathcal{M}(X, J'_0, e) \cup \mathcal{M}(X, J'_1, e))$. Let $S_i = \mathcal{M}(X, J'_i, e) \cap \mathfrak{f}^{-1}([-t_2, t_2])$. The set $\mathfrak{f}^{-1}(t_2) \cup \mathfrak{f}^{-1}(-t_2) \cup S_0 \cup S_1$ is the boundary of the compact 1-manifold $\mathfrak{f}^{-1}([-t_2, t_2])$, hence it has an even number of elements.

The construction of t_2 implies that every element in $\mathfrak{f}^{-1}(t_2) \cup \mathfrak{f}^{-1}(-t_2)$ is contained in V. The properties of $\{J'_s\}$ given by lemma 2.24 states that $\sigma^*(J'_s) = -J'_s$ on V, thus σ maps $\mathfrak{f}^{-1}(t_2)$ to $\mathfrak{f}^{-1}(-t_2)$, hence the set $\mathfrak{f}^{-1}(t_2) \cup \mathfrak{f}^{-1}(-t_2)$ has an even number of elements. The properties given by lemma 2.23 implies that σ acts on the set S_i, and the fixed point set consists of tori in $\{0\} \times \widetilde{Y}$ which are lifts of Klein-bottle leaves of $\mathcal{L}_i$ in the homology class A. On the other hand, let $\mathcal{K}_i$ be the set of lifts of Klein-bottle

leaves of $\mathcal{L}_i$ in the homology class A, then for every

$$C \in \{0\} \times (\mathcal{K}_0 \cup \mathcal{K}_1) \subset \mathcal{M}(X, J_0', e) \cup \mathcal{M}(X, J_1', e),$$

one has $\mathfrak{f}(C) = 0$. Hence $C \in S_0 \cup S_1$, and C is fixed by σ.

The arguments above show that the number of elements in $\mathfrak{f}^{-1}(t_2) \cup \mathfrak{f}^{-1}(-t_2) \cup S_0 \cup S_1$ has the same parity as the number of elements in $\mathcal{K}_0 \cup \mathcal{K}_1$. Therefore, the set $\mathcal{K}_0 \cup \mathcal{K}_1$ has an even number of elements, and the desired result is proved. $\square$

2.5. **Technical lemmas.** The purpose of this section is to prove lemma 2.23 and lemma 2.24. The proofs are routine and straightforward, they are given here for lack of a direct reference. Throughout this section X will be a smooth 4-manifold with $\pi_2(X) = 0$.

Definition 2.25. Let (X, ω) be a symplectic manifold. Let $B \subset X$ be a closed subset. Let $E, N > 0$ be constants. An almost complex structure $J \in \mathcal{J}(X, \omega, N)$ is called (B, E)-admissible if the following conditions hold:

(1) Every embedded torus C with topological energy less than or equal to E, $d([C]) = 0$, $[C]$ being primitive, and satisfying $C \cap B \neq \varnothing$, is nondegenerate.
(2) For every primitive homology class $e \in H_2(X; \mathbb{Z})$, if $\langle [\omega], e \rangle \leqslant E$, and if $\langle c_1(T^{0,1}X), e \rangle > 0$ (namely, the formal dimension of the moduli space of J-holomorphic maps from a torus to X in the homology class e, modulo self-isomorphisms of the domain, is negative), then there is no somewhere injective J-holomorphic map ρ from a torus to X in the homology class e such that $\mathrm{Im}(\rho) \cap B \neq \varnothing$.

The next lemma follows from Gromov's compactness theorem and the diameter bound of lemma 2.10.

Lemma 2.26. *Let* (X, ω) *be a symplectic manifold. Let* $B \subset X$ *be a closed subset, and* $E, N > 0$ *be constants. The elements of* $\mathcal{J}(X, \omega, N)$ *that are* (B, E)*-admissible form an open and dense subset of* $\mathcal{J}(X, \omega, N)$.

Proof. First consider the case when B is compact. The denseness of (B, E)-admissible almost complex structures then follows from the standard transversality argument. Now one proves the openness. Let $M(N, E)$ be the upper bound of diameter given by lemma 2.10. Let A be a compact set containing B such that the distance between ∂A and B is greater than $M(N, E) + 2$. Suppose J is a (B, E)-admissible almost complex structure. Let $\mathcal{U}$ be a sufficiently small open neighborhood of $J|_A \in \mathcal{J}(A, \omega)$, such that for every $J' \in \mathcal{J}(X, \omega, N)$, if $J'|_A \in \mathcal{U}$ then the distance between ∂A and B is greater than $M(N, E) + 1$. One claims that there is a smaller neighborhood $\mathcal{V} \subset \mathcal{U}$ containing J, such that for every $J' \in \mathcal{J}(X, \omega, N)$, if $J'|_A \in \mathcal{V}$ then J' is (B, E)-admissible. In fact, assume the claim is not true, since $\mathcal{J}(A, \omega)$ is first countable, there is a sequence $\{J_n\} \subset \mathcal{J}(X, \omega, N)$, such that $J_n|_A \to J|_A$ in the C^∞ topology, and that every J_n is not (B, E)-admissible. Therefore for every n, there exists a J_n-holomorphic curve C_n with topological energy no greater than E, such that $[C_n]$ is primitive, $C_n \cap B \neq \varnothing$. Moreover, either C_n is an embedded degenerate curve with index zero, or C_n is a curve with negative index. The diameter bound implies $C_n \subset A$ for each n. Gromov's compactness theorem then implies that there is a subsequence of C_n converging to a non-constant J-holomorphic map with possibly bubbles, nodal singularities, and branched-cover components. Since is it assumed that $\pi_2(X) = 0$ and $[C_n]$ is primitive, the limit map has to be an embedded torus. The torus given by the limit map has topological energy less than or equal to E, and it violates the assumption that J is (B, E)-admissible.

Now consider the case when B is not necessarily compact. Cover B by a locally finite family of compact subsets B_n. Let A_n be the closed $(M(N, E) + d + 2)$-neighborhood of B_n. By the argument of the previous paragraph, for each n there is

an open and dense subset $\mathcal{V}_n$ of $\mathcal{J}(A_n, \omega)$, such that for every $J' \in \mathcal{J}(X, \omega, N)$, the condition that J' is (B_n, E)-admissible is equivalent to $J'|_{A_n} \in \mathcal{V}_n$. Notice that J' is (B, E)-admissible if and only if it is (B_n, E)-admissible for every n. Therefore the result follows from part 1 of lemma 2.11. $\qquad\square$

From now on assume that $\sigma : X \to X$ is a map that acts diffeomorphically on X, such that $\sigma^2 = \mathrm{id}_X$ and the quotient map $X \to X/\sigma$ is a covering map.

Definition 2.27. Let (X, ω) be a symplectic manifold. Let $d, E, N > 0$ be constants. Let B be a closed subset of X such that $\sigma(B) = B$. An almost complex structure $J \in \mathcal{J}(X, \omega, N)$ is called (d, E)-regular with respect to B if for every J-holomorphic map ρ from a torus to X with topological energy less than or equal to E, at least one of the following conditions hold:

(1) The distance between the sets $\mathrm{Im}(\rho)$ and $\sigma(\mathrm{Im}(\rho))$ is greater than d.

(2) The distance of $\mathrm{Im}(\rho)$ and B is greater than d.

Here the distance is defined by the metric $g_J = \omega(\cdot, J\cdot)$ on X.

Remark 2.28. Notice that since the map ρ in the definition above can be a constant map, for a (d, E)-regular almost complex structure J with respect to B, one has $\mathrm{dist}(p, \sigma(p)) > d$ for every $p \in B$.

The following result is also a consequence of Gromov's compactness theorem, and the proof follows a similar strategy as lemma 2.26.

Lemma 2.29. *Let $d, E, N > 0$ be constants, and B is a closed subset of X such that $\sigma(B) = B$. The elements of $\mathcal{J}(X, \omega, N)$ that are (d, E)-regular with respect to B form an open subset of $\mathcal{J}(X, \omega, N)$.*

Proof. First consider the case when B is compact. Let $M(N, E)$ be the upper bound of diameter given by lemma 2.10. Let A be a compact set containing B such that the distance between ∂A and B is greater than $M(N, E) + d + 2$. Suppose J is a

(d, E)-regular almost complex structure with respect to B. Let $\mathcal{U}$ be a sufficiently small open neighborhood of $J|_A \in \mathcal{J}(A, \omega)$, such that for every $J' \in \mathcal{J}(X, \omega, N)$, if $J'|_A \in \mathcal{U}$ then the distance between ∂A and B is greater than $M(N, E) + d + 1$. One claims that there is a smaller neighborhood $\mathcal{V} \subset \mathcal{U}$ containing J, such that for every $J' \in \mathcal{J}(X, \omega, N)$, if $J'|_A \in \mathcal{V}$ then J' is (d, E)-regular with respect to B. In fact, assume the claim is not true, since $\mathcal{J}(A, \omega)$ is first countable, there is a sequence $\{J_n\} \subset \mathcal{J}(X, \omega, N)$, such that $J_n|_A \to J|_A$ in the C^∞ topology, and that every J_n is not (d, E)-regular with respect to B. By the definition of (d, E)-regularity, there is a sequence of J_n-holomorphic maps ρ_n from torus to X with topological energy less than or equal to E, such that the distance of $\mathrm{Im}(\rho)$ to B with respect to the metric given by J_n is less than or equal to d, and the distance between $\mathrm{Im}(\rho)$ and $\sigma(\mathrm{Im}(\rho))$ with respect to the metric given by J_n is less than or equal to d. By the diameter bound, every curve C_n is contained in the set A. Gromov's compactness theorem then implies that there is a subsequence of ρ_n converging to a non-constant J-holomorphic map with possibly bubbles, nodal singularities, and branched-cover components. Since is it assumed that $\pi_2(X) = 0$, the limit map has to be a possibly branched cover of a torus. The torus given by the limit map has topological energy less than or equal to E, and it violates the assumption that J is (d, E)-regular with respect to B.

Now consider the case when B is not necessarily compact. Let J be a (d, E)-regular almost complex structure with respect to B. Cover B by a locally finite family of compact subsets B_n such that $\sigma(B_n) = B_n$ for each n. Let A_n be the closed $(M(N, E) + d + 2)$-neighborhood of B_n. By the argument of the previous paragraph, for each n there is an open neighborhood $\mathcal{V}_n$ of $J|_{A_n}$ in $\mathcal{J}(A_n, \omega)$, such that for every $J' \in \mathcal{J}(X, \omega, N)$, if $J'|_{A_n} \in \mathcal{V}_n$ then J' is (d, E)-regular with respect to B_n. Notice that J' is (d, E)-regular with respect to B if and only if it is (d, E)-regular with respect to every B_n. By the definition of the topology on $\mathcal{J}(X, \omega, N)$, this implies that J has

an open neighborhood consisting of (d, E)-regular almost complex structures with respect to B. $\qquad\square$

The following lemma is a 1-parametrized version of lemma 2.29.

Lemma 2.30. *Let $d, E, N > 0$ be constants, and B is a closed subset of X such that $\sigma(B) = B$. Let ω_s ($s \in [0, 1]$) be a smooth family of symplectic forms on X, and let $J_i \in \mathcal{J}(X, \omega_i, N)$. Then the set of elements $\{J_s\} \in \mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ such that every J_s is (d, E)-regular with respect to B form an open subset of $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$.*

Proof. The proof is exactly the same as lemma 2.29. One only needs to change the notation J to $\{J_s\}$, and change the notation $\mathcal{J}(X, \omega, N)$ to $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$. $\qquad\square$

Lemma 2.31. *Let (X, ω) be a symplectic manifold such that $\sigma^*(\omega) = -\omega$. Let $d, E, N > 0$ be constants. Let B be a closed subset of X such that $\sigma(B) = B$. Assume $J \in \mathcal{J}(X, \omega, N)$ is (d, E)-regular with respect to B, and assume that $\sigma^*(J) = -J$ on B. Then for every open neighborhood $\mathcal{U}$ of J in $\mathcal{J}(X, \omega, N)$, there is an element J' such that J' is (d, E)-regular with respect to B and is E-admissible, and $\sigma^*(J') = -J'$ on B. Moreover, if there is a closed subset $H \subset X$ such that $\sigma(H) = H$ and J is (H, E)-admissible, then J' can be taken to be equal to J on the set H.*

Proof. By shrinking the open neighborhood $\mathcal{U}$, one can assume that every element of $\mathcal{U}$ is (d, E)-regular with respect to B, and that there is a complete metric g_0 on X such that $\sigma^*(g_0) = g_0$ and $g_0 \leqslant g_{J'}$ for every $J' \in \mathcal{U}$. For the rest of this proof, the distance function on X is defined by g_0.

Cover X by a locally finite family of closed balls with radius $d/10$. Say

$$X = \bigcup_{i=1}^{+\infty} B_i,$$

where $\{B_i\}$ are closed balls with radius $d/10$. Let D_i be the open $d/10$-neighborhood of B_i, then the diameter of D_i is less than $d/2$.

Let $A_j = \cup_{i \leqslant j} B_j$, then $A_0 = \varnothing$. The construction of J' follows from induction. Assume that J_j is already (A_j, E)-admissible with $\sigma^*(J_j) = -J_j$ on B, the following paragraph will perturb J_j to J_{j+1} such that J_{j+1} is (A_{j+1}, E)-admissible with $\sigma^*(J_{j+1}) = -J_{j+1}$ on B.

In fact, if $D_{j+1} \cap B = \varnothing$, then a generic perturbation on D_{j+1} will do the job. If $D_{j+1} \cap B \neq \varnothing$, choose a small perturbation on D_{j+1} such that the resulting almost complex structure J'_{j+1} is still in $\mathcal{U}$ and is (B_{j+1}, E)-admissible. Recall that every element in $\mathcal{U}$ is (d, E)-regular with respect to B, hence by remark 2.28 and the diameter bound on D_{j+1}, one has $\sigma(D_{j+1}) \cap D_{j+1} = \varnothing$. Now make an additional perturbation on $\sigma(D_{j+1})$ such that the resulting almost complex structure J_{j+1} satisfies $\sigma(J_{j+1}) = -J_{j+1}$ on B. One can choose the perturbation on D_{j+1} to be small enough such that J_{j+1} is also in $\mathcal{U}$.

Now J_{j+1} is (d, E)-regular with respect to B, the diameter of D_j is less than $d/2$, and $D_j \cap B \neq \varnothing$. One claims that there is no J_{j+1}-holomorphic map from a torus with topological energy less than or equal to E and passing through both D_{j+1} and $\sigma(D_{j+1})$. In fact, assume C passes through both D_{j+1} and $\sigma(D_{j+1})$, then then distance between C and $\sigma(C)$ is less than $d/2$. Since $D_j \cap B \neq \varnothing$, the distance between C and B is less than $d/2$. This is contradictory to the fact that J'_{j+1} is (d, E)-regular with respect to B.

Since no J_{j+1}-holomorphic map from a torus with topological energy less than or equal to E can pass through both D_{j+1} and $\sigma(D_{j+1})$, the almost complex structure J'_{j+1} being (D_{j+1}, E)-admissible implies that J_{j+1} is (D_{j+1}, E)-admissible. By lemma 2.26, being (A_j, E)-admissible is an open condition, thus when the perturbation is sufficiently small J_{j+1} is also (A_j, E)-admissible. Therefore the almost complex structure J_{j+1} is (A_{j+1}, E)-admissible. Since the family $\{D_n\}$ is locally finite, on each compact set the sequence $\{J_j\}$ stabilizes for sufficiently large j. The desired J' can then be obtained by taking $\lim_{j \to \infty} J_j$. Moreover, if there is a closed subset $H \subset X$

such that $\sigma(H) = H$ and J is (H, E)-admissible, then each step of the perturbation can be taken to be outside of H. $\qquad\square$

The following lemma is a 1-parametrized version of lemma 2.31, and the proof is essentially the same.

Lemma 2.32. *Let $e \in H_2(X; \mathbb{Z})$ be a primitive class. Let B be a closed subset of X such that $\sigma(B) = B$. Assume ω_s $(s \in [0, 1])$ is a smooth family of symplectic forms on X such that $\sigma^*(\omega_s) = -\omega_s$ for each s. Let $d, N > 0$ be constants. Let E be a positive constant such that $E > \langle [\omega_s], e \rangle$ for every s. For $i = 0, 1$, assume $J_i \in \mathcal{J}(X, \omega_i, N)$ is E-admissible and (d, E)-regular with respect to B. Assume $\{J_s\} \in \mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$, such that for each s, the almost complex structure J_s is (d, E)-regular with respect to B, and $\sigma^*(J_s) = -J_s$ on B. Then for every open neighborhood $\mathcal{U}$ of $\{J_s\}$ in $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$, there is an element $\{J'_s\}$ such that $\{J'_s\}$ is (d, E)-regular with respect to B and is E-admissible, and $\sigma^*(J'_s) = -J'_s$ on B for every s. Moreover, if there is a closed subset $H \subset X$ such that $\sigma(H) = H$ and $\{J_s\}$ is (H, E)-admissible, then J'_s can be taken to be equal to J_s on the set H.*

Proof. The proof follows verbatim as the proof of lemma 2.31. One only needs to change the notation J to $\{J_s\}$, and change $\mathcal{J}(X, \omega, N)$ to $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$. $\qquad\square$

Combining the results above, one obtains the following lemma.

Lemma 2.33. *Let $e \in H_2(X; \mathbb{Z})$ be a primitive class. Let B be a closed subset of X such that $\sigma(B) = B$. Assume ω_s $(s \in [0, 1])$ is a smooth family of symplectic forms on X such that $\sigma^*(\omega_s) = -\omega_s$ for each s. Let $d, N > 0$ be constants. Let E be a positive constant such that $E > \langle [\omega_s], e \rangle$ for every s. For $i = 0, 1$, assume $J_i \in \mathcal{J}(X, \omega_i, N)$ is E-admissible and (d, E)-regular with respect to B. Let $\mathcal{J}$ be the subset of elements $\{J_s\}$ of $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ such that for each s, the almost complex structure J_s is (d, E)-regular with respect to B, and $\sigma^*(J_s) = -J_s$ on B. If $\mathcal{J}$*

is not empty, let $\mathcal{U} \subset \mathcal{J}$ be the subset of $\mathcal{J}$, such that for every $\{J_s\} \in \mathcal{U}$, the moduli space $\mathcal{M}(X, \{J_s\}, e) = \coprod_{s \in [0,1]} \mathcal{M}(X, J_s, e)$ has the structure of a smooth 1-manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$. Then $\mathcal{U}$ is open and dense. Moreover, if $f : X \to \mathbb{R}$ is a smooth proper function on X, then the function defined as

$$\mathfrak{f} : \mathcal{M}(X, \{J_s\}, e) \to \mathbb{R}$$

$$C \mapsto \oint_C f \, dA.$$

is a smooth proper function on $\mathcal{M}(X, \{J_s\}, e)$, where for $C \in \mathcal{M}(X, J_s, e)$, dA is the area form of C given by g_s. $\qquad\square$

Proof. The openness of $\mathcal{U}$ follows from lemma 2.30. The fact that $\mathcal{U}$ is dense follows from lemma 2.32. The properness of the function $\mathfrak{f}$ was proved in lemma 2.13. $\qquad\square$

The following lemma controls the location of pseudo-holomorphic curves after perturbation of the almost complex structure.

Lemma 2.34. *Let (X, ω) be a symplectic manifold, let $J \in \mathcal{J}(X, \omega, N)$. Let $E > 0$ be a positive constant, and let B be a closed subset of X. Assume that there is no non-constant J-holomorphic map ρ from a torus to X, such that $Im(\rho) \cap B$ is nonempty and the topological energy of ρ is no greater than E. Then there is an open neighborhood $\mathcal{U}$ of J in $\mathcal{J}(X, \omega, N)$, such that for every $J' \in \mathcal{U}$, there is no embedded J'-holomorphic torus in X intersecting B with topological energy less than or equal to E.*

Proof. Cover the set B by a locally finite family of compact subsets B_n. Let $M(N, E)$ be the upper bound given by lemma 2.10 for geometry bound N and energy bound E. Let A_n be the closed $M(N, E) + 1$-neighborhood of B_n. One claims that there is an open neighborhood $\mathcal{U}_n$ of $J|_{A_n} \in \mathcal{J}(A_n, \omega)$ such that for every $J' \in \mathcal{J}(A_n, \omega, N)$, if $J'|_{A_n} \in \mathcal{U}_n$, then there is no embedded J'-holomorphic torus in X intersecting B_n

with topological energy less than or equal to E. Assume the result does not hold, then there is a sequence of $J_n \subset \mathcal{J}(A, \omega, N)$ such that for each n there exists a J_n-holomorphic map ρ_n from a torus to X which intersects B and has topological energy less than or equal to E, and $J_n|_{A_n} \to J|_{A_n}$. For sufficiently large n, the distance between ∂A_n and B_n is greater than $M(N, E)$ with respect to the distance given by J_n, therefore the relevent J_n-holomorphic curve is contained in A_n. By Gromov's compactness theorem, a subsequence of ρ_n will give a non-constant J-holomorphic map from a torus to A_n, such that the intersection $\mathrm{Im}(\rho) \cap B$ is nonempty, and the topological energy of ρ is less than or equal to E, which is a contradiction. Therefore, the claim holds. The result of the lemma then follows from part 1 of lemma 2.11. $\quad\square$

With the preparations above, one can now give the proofs of lemma 2.23 and lemma 2.24.

Proof of lemma 2.23. Let g_i be the metric on X given by (Ω_i, J_i). By corollary 2.18, every $C \in \mathcal{M}(X, J_i, e)$ either satisfies $\sigma(C) \cap C = \varnothing$, or C is the lift of a Klein-bottle leaf. Since the space of torus leaves in Y is compact, there exists a positive constant $d_i^{(1)} > 0$, such when $\sigma(C) \cap C = \varnothing$, the distance between C and $\sigma(C)$ with respect to g_i is greater than $d_i^{(1)}$. Let

$$d_i^{(2)} = \frac{1}{3} \inf_{p \in \overline{U}_i} d_{g_i}(p, \sigma(p)).$$

Let $d_i^{(3)}$ be the distance from $\widetilde{U}_i$ to the union of the lifts of Klein-bottle leaves. Recall that

$$U_i = \left(\mathbb{R} \times \widetilde{U}_i\right) \bigcup \left((-\infty, -t_0) \cup (t_0, \infty)\right) \times \widetilde{Y}.$$

Let

$$d = \min_{\substack{i=0,1 \\ j=1,2,3}} d_i^{(j)}.$$

For every $E > 0$, the almost complex structure J_i is (d, E)-regular with respect to $\overline{U}_i$. In fact, every J_i-holomorphic map from a torus to X is one of the following: (1)

a constant map, (2) a covering to the lift of a torus leaf, (3) a covering to the lift of a Klein-bottle leaf. Let C be its image. In case (1), either the distance from C to $\overline{U_i}$ is at least $d_i^{(2)}$, or the distance from C tp $\sigma(C)$ is at least $d_i^{(2)}/3$. In case (2), the distance from C to $\sigma(C)$ is at least $d_i^{(1)}$. In case (3), the distance from C to $\overline{U_i}$ is at least $d_i^{(3)}$. Choose E to be any positive constant such that $E > \max_i \langle \Omega_i, e \rangle$.

Apply lemma 2.31 to $B = \overline{U_i}$, there is a perturbation

$$J_i' \in \mathcal{J}(X, \Omega_i, N)$$

of J_i, such that J_i' is E-admissible and $\sigma^*(J_i') = -J_i'$ on $\overline{U_i}$. Let W_i be a small compact neighborhood of the union of lifts of Klein-bottle leaves such that $\sigma(W_i) = W_i$. The almost complex structure J_i' can be taken to be equal to J_i on W_i since J_i is already (W_i, E)-admissible. By the definition of the set U_i, every J_i-holomorphic map from a torus to X is either a lift of Klein-bottle leaf or is mapped into the set U_i. Therefore lemma 2.34 shows that when the perturbation is sufficiently small, every J_i'-holomorphic torus with homology class e is either contained in U_i or is contained in W_i. In the latter case the curve is contained in $\widetilde{Y} \times \{0\}$ and it is a lift of a Klein-bottle leaf of $\mathcal{L}_i$ in class A. Since J_i' is (d, E)-regular with respect to $\overline{U_i}$, for every J_i' holomorphic torus C in U_i one has $\sigma(C) \neq C$. $\qquad \square$

Proof of lemma 2.24. The almost complex structures J_0' and J_1' can be connected by a smooth family of almost complex structures $J_s' \in \mathcal{J}(X, \Omega_s, J_0', J_1', N)$ such that $\sigma^*(J_s') = -J_s'$ on V. Using lemma 2.33, the family J_s' can be further perturbed to satisfy the desired conditions. $\qquad \square$

2.6. **An example.** This section gives an example of a taut foliation with an odd number of Klein-bottle leaves such that every closed leaf has non-trivial linear holonomy. By corollary 2.4, every deformation of such a foliation via taut foliations has at least one Klein-bottle leaf.

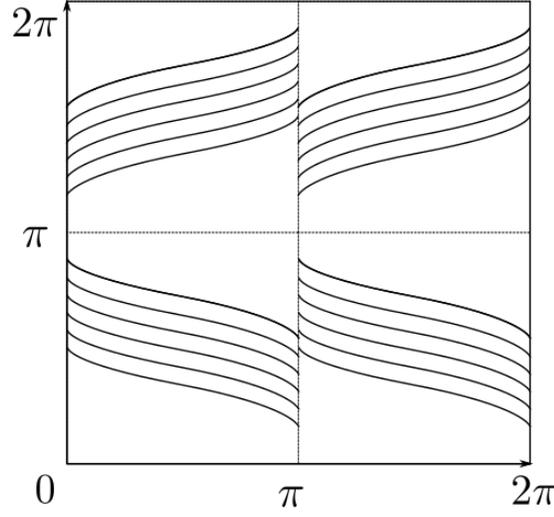FIGURE 1. The foliation $\hat{\mathcal{I}}$ on $S^1 \times S^1$

Think of the torus $T_0 = S^1 \times S^1$ as a trivial S^1-bundle over S^1. Let $z_1, z_2 \in \mathbb{R}/2\pi$ be the coordinates of the two S^1 factors, where z_1 is the coordinate for the fiber, and z_2 is the coordinate for the base. Let γ be a closed curve on the base that wraps the S^1 once in the positive direction. Take a horizontal foliation $\hat{\mathcal{I}}$ on T_0 such that the holonomy along γ has two fixed points: $z_1 = 0$ and $z_1 = \pi$, and that holonomy map has nontrivial linearization at these two points. Moreover, choose $\hat{\mathcal{I}}$ so that it is invariant under the map $(z_1, z_2) \mapsto (z_1 + \pi, \pi - z_2)$ and the map $(z_1, z_2) \mapsto (z_1, z_2 + \pi)$.

Figure 1 gives a picture for such a foliation $\hat{\mathcal{I}}$, where z_2 is the horizontal coordinate, and z_1 is the vertical coordinate.

Consider the pull back of the foliation $\hat{\mathcal{I}}$ to $T_0 \times S^1$. Let $z_3 \in \mathbb{R}/2\pi$ be the coordinate for the S^1 factor, then span$\{\hat{\mathcal{I}}, \frac{\partial}{\partial z_3}\}$ defines a foliation $\mathcal{I}$ on $T_0 \times S^1$. There are exactly two torus leaves in $\mathcal{I}$, and they are given by $z_1 = 0$ and $z_1 = \pi$.

The foliation $\mathcal{I}$ is invariant under the maps

$$\sigma_1 : (z_1, z_2, z_3) \mapsto (z_1 + \pi, \pi - z_2, z_3)$$

$$\sigma_2 : (z_1, z_2, z_3) \mapsto (z_1, z_2 + \pi, \pi - z_3)$$

$$\sigma_3 : (z_1, z_2, z_3) \mapsto (z_1 + \pi, -z_2, \pi - z_3)$$

The set $V = \{\mathrm{id}, \sigma_1, \sigma_2, \sigma_3\}$ is a group acting freely and discountinuously on $T_0 \times S^1$ and it preserves the coorientation of $\mathcal{I}$. The two torus leaves in $T_0 \times S^1$ are identified under the quotient by V, and their images give the unique Klein-bottle leaf in $\mathcal{I}/V$. Moreover, the Klein-bottle leaf has non-trivial linear holonomy. Therefore, corollary 2.4 implies the following result.

Proposition 2.35. *Every deformation of $\mathcal{I}/V$ through taut foliations must have at least one Klein-bottle leaf.* □

3. Rectifiability and Minkowski bounds for the zero loci of $\mathbb{Z}/2$ harmonic spinors in dimension 4

3.1. **Introduction.**

3.1.1. *Background.* The notion of $\mathbb{Z}/2$ harmonic spinors was first introduced by the works of Taubes [38, 40] to describe the behaviour of certain non-convergent sequences of $PSL_2(\mathbb{C})$ connections on a three manifold. It also appears in the compactifications of the moduli spaces of solutions to the Kapustin-Witten equations [39], the Vafa-Witten equations [42], and the Seiberg-Witten equations with multiple spinors [18, 41]. These equations may have important topological applications. For example, Witten [45] has conjectured that the space of solutions to the Kapustin-Witten equations can be used to compute the Jones polynomials and the Khovanov homology for knots. Haydys [17] conjectured a relation between the multiple spinor Seiberg-Witten monopoles, Fueter sections, and $G2$ instantons.

All of these applications require some understanding of the boundaries of the compactifications for the relevant moduli spaces. The "points of discontinuity" for $\mathbb{Z}/2$ harmonic spinors play a crucial role in the structure of the boundary spaces. For a $\mathbb{Z}/2$ harmonic spinor, the set of "points of discontinuity" I and the zero set Z were defined in [40]. Takahashi [34, 35] studied the moduli spaces of $\mathbb{Z}/2$ harmonic spinors with additional regularity assumptions on I, where I was assumed to be a union of embedded circles in the case of dimension 3, and an embedded surface in the case of dimension 4. In general, I may not have this regularity. The regularities of I and Z were studied extensively in [40]. For example, it was proved that I is the closure of a countable union of open submanifolds with codimension 2, and that $Z \supset I$ has Hausdorff codimension at least 2. This article improves the regularity results by proving that Z is rectifiable and has locally finite Minkowski content. The proof is inspired by [10], where a similar problem was studied for Dir-minimizing Q-valued functions.

3.1.2. *Statement of results.* Let X be a 4-dimensional Riemannian manifold. Let $\mathcal{V}$ be a Clifford bundle over X. That is, $\mathcal{V}$ is a unitary vector bundle equipped with an extra structure $\rho \in \operatorname{Hom}(TX, \operatorname{Hom}(\mathcal{V}, \mathcal{V}))$, such that $\rho(e)^2 = -\|e\|^2 \cdot \operatorname{id}$ and $\|\rho(e)(u)\| = \|e\| \cdot \|u\|$ for every $e \in T_p X$ and $u \in \mathcal{V}|_p$. Let ∇ be a connection on V that is compatible with $(X, \mathcal{V}, \rho)$. Namely, for every pair of smooth vector fields e, e', and every smooth section u of $\mathcal{V}$, one has

$$\nabla_e(\rho(e') \cdot u) = \rho(\nabla_e e') \cdot u + \rho(e') \cdot \nabla_e(u).$$

The Dirac operator on $\mathcal{V}$ is defined by

$$D(u) = \sum_{i=1}^{4} \rho(e_i) \nabla_{e_i} u,$$

where $\{e_i\}$ is a local orthonormal frame for TX.

Let Q be a positive integer. For a vector space E, define $\mathcal{A}_Q(E)$ to be the set of unordered Q-tuples of points in E. If $P_1, P_2, \cdots, P_Q$ are Q points in E, use $\sum_{i=1}^{Q} [\![P_i]\!] \in \mathcal{A}_Q(E)$ to denote the Q-tuple given by the collection of P_i's. If E is endowed with a Euclidean metric, one can define a metric on $\mathcal{A}_Q(E)$ by

$$\operatorname{dist}\left(\sum_i [\![P_i]\!], \sum_i [\![S_i]\!]\right) = \min_{\sigma \in \mathcal{P}_Q} \sqrt{\sum_i |P_i - S_{\sigma(i)}|^2},$$

where $\mathcal{P}_Q$ is the permutation group of $\{1, 2, \cdots, Q\}$. If $T \in \mathcal{A}_Q(E)$, define $|T| = \operatorname{dist}(T, Q[\![0]\!])$.

A map from X is called a Q-valued section of $\mathcal{V}$ if it maps every $x \in X$ to an element of $\mathcal{A}_Q(\mathcal{V}|_x)$. A Q-valued section is called continuous if it is continuous under local trivializations of $\mathcal{V}$.

Definition 3.1. Let U be a continuous 2-valued section of $\mathcal{V}$. Then U is called a $\mathbb{Z}/2$ harmonic spinor if the following conditions hold.

(1) U is not identically $2[\![0]\!]$.

(2) Let Z be the set of U where $U = 2[\![0]\!]$. For every $x \in X - Z$, there exists a neighborhood of x, such that on this neighborhood U can be written as $U = [\![u]\!] + [\![-u]\!]$, where u is a smooth section of $\mathcal{V}$ satisfying $D(u) = 0$.

(3) Near a point $x \in X - Z$, write U as $[\![u]\!] + [\![-u]\!]$, then the function $|\nabla u|$ is a well defined smooth function on $X - Z$. The section U satisfies

$$\int_{X-Z} |\nabla u|^2 < \infty.$$

This definition is equivalent to the definition of $\mathbb{Z}/2$ harmonic spinors given in [40].

For a point $x \in X$ and $r > 0$, use $B_x(r)$ to denote the geodesic ball in X with center x and radius r. As in (1.5) of [40], we make the following additional assumption on U.

Assumption 3.2. There exits a constant $\epsilon > 0$ such that the following holds. For every $x \in X$ with $U(x) = 2[\![0]\!]$, there exist constants $C, r_0 > 0$, depending on x, such that

$$\int_{B_x(r)} |U(y)|^2 \, dy < C \cdot r^{4+\epsilon}, \quad \text{for every } r \in (0, r_0).$$

Assume U is a $\mathbb{Z}/2$ harmonic spinor, and let Z be the set of U where $U = 2[\![0]\!]$. Taubes [40] proved the following theorem.

Theorem 3.3 (Taubes [40])**.** *If U satisfies assumption 3.2, then the Hausdorff dimension of Z is at most 2.*

This article improves theorem 3.3 to the following result.

Theorem 3.4. *If U satisfies assumption 3.2, then Z is a 2-rectifiable set. Moreover, for every compact subset $A \subset X$, there exist constants C and r_0 depending on A and Z, such that for every $r < r_0$,*

$$\text{Vol}\left(\{x : \text{dist}(x, A \cap Z) < r\}\right) < C \cdot r^2.$$

In other words, Z is a 2-rectifiable set with locally finite 2 dimensional Minkowski content. Since the Minkowski content controls the Hausdorff measure, theorem 3.4 implies that Z has locally finite 2 dimensional Hausdorff measure.

Theorem 3.4 immediately implies that the zero locus of a $\mathbb{Z}/2$ harmonic spinor on a 3-manifold is 1-rectifiable and has locally finite Minkowski content.

3.2. $\mathbb{Z}/2$ **harmonic spinors as Sobolev sections.** There is a Sobolev theory for Q-valued functions on $\mathbb{R}^m$, see for example [11]. For an open set $\Omega \subset \mathbb{R}^m$, the space $W^{1,2}(\Omega, \mathcal{A}_Q)$ is defined to be the space of Q valued functions T on Ω, such that $|T| \in L^2(\Omega)$, and that T has a distributional derivative which is also in $L^2(\Omega)$. The Sobolev theory extends to Q-valued sections of vector bundles without any difficulty. This section proves the following lemma.

Lemma 3.5. *If U is a $\mathbb{Z}/2$ harmonic spinor, then U is in $W^{1,2}(X, \mathcal{A}_2)$. Moreover, $D(U) = 0$ in the distributional sense.*

This lemma allows us to study the compactness properties of $\mathbb{Z}/2$ harmonic spinors by the Sobolev theory for Q-valued functions.

We start with the following definition.

Definition 3.6. Let T be a Q-valued section of $\mathcal{V}$. It is called a smooth Q-valued section, if for every $x \in X$, there exists a neighborhood of x on which T can be written as

$$T = \sum_{i=1}^{Q} [\![f_i]\!],$$

where f_i's are smooth sections of $\mathcal{V}$.

If T is a smooth Q-valued section and is locally written as $\sum_i [\![f_i]\!]$, then the function $\sum_i |f_i|^2 + \sum_i |\nabla f_i|^2$ is well defined on X. In this case, the $W^{1,2}$ norm of T is given by $(\int_X \sum_i |f_i|^2 + \sum_i |\nabla f_i|^2)^{1/2}$.

Proof of lemma 3.5. The proof is essentially the same as lemma 2.4 of [40].

Let χ be a smooth non-increasing function on $\mathbb{R}$, such that $\chi(t) = 1$ when $t \leqslant 1$, and $\chi(t) = 0$ when $t \geqslant 2$. For $s > 0$, let $\tau_s = \chi(\ln|U|/\ln s)$. Then $\tau_s(x) = 0$ when $|U(x)| \leqslant s^2$, and $\tau_s(x) = 1$ when $|U(x)| \geqslant s$.

The section $\tau_s U$ is a 2-valued smooth section of $\mathcal{V}$. Recall that on $X - Z$, the $\mathbb{Z}/2$ harmonic spinor U can be locally written as $U = [\![u]\!] + [\![-u]\!]$. Although u is only defined up to a sign, the functions $|u|$ and $|\tau_s \nabla u + \nabla \tau_s \cdot u|$ are well defined on $X - Z$. Thus the $W^{1,2}$ norm of $\tau_s U$ is given by

$$\|\tau_s U\|_{W^{1,2}} = \sqrt{2} \int_X (|\tau_s|^2 |u|^2 + |\tau_s \nabla u + \nabla \tau_s \cdot u|^2).$$

Notice that

$$|\nabla \tau_s| \cdot |u| \leqslant \frac{1}{|\ln s|}(\sup |\chi'|) \cdot |\nabla u|,$$

hence its L^2 norm converges to zero as $s \to 0$. Therefore,

$$(3.1) \qquad \lim_{s \to 0} \|\tau_s U\|_{W^{1,2}} = \sqrt{2} \int_{X-Z} (|u|^2 + |\nabla u|^2).$$

In particular, $\tau_s U$ is bounded in $W^{1,2}$ as $s \to 0$, thus a subsequence of it weakly converges in $W^{1,2}$ to an element $U' \in W^{1,2}$. Since $\tau_s U$ also uniformly converges to U, one must have $U' = U$. Therefore $U \in W^{1,2}$.

Since D is a smooth first-order differential operator, $D(U) \in L^2_{loc}(X)$. By the definition of $\mathbb{Z}/2$ harmonic spinors, $D(U) = 0$ on $X - Z$. By section 2.2.1 of [11], the derivatives of U are zero at the Lebesgue points of Z, hence $D(U) = 0$ on those points. That proves $D(U) = 0$ in the distributional sense. $\qquad\square$

The argument of lemma 2.1 also shows that U can be $W^{1,2}$ approximated by smooth sections. We write it as a separate lemma for later reference.

Lemma 3.7. *Let U be a $\mathbb{Z}/2$ harmonic spinor. Then there exits a sequence of smooth sections U_i, such that $U_i = -U_i$, and*

$$\lim_{i \to \infty} U_i = U \ in \ W^{1,2}.$$

Proof. Since $|U|$ and $|\nabla U|$ are zero on the Lebesgue points of Z, one has

$$\|U\|_{W^{1,2}} = \int_{X-Z} (|U|^2 + |\nabla U|^2) = \sqrt{2} \int_{X-Z} (|u|^2 + |\nabla u|^2).$$

Define τ_s as in the proof of lemma 3.5. It was proved previously that there is a sequence $s_i \to 0$, such that $\tau_{s_i} U$ converges weakly to U in $W^{1,2}$. As a consequence,

$$\liminf_{i \to \infty} \|\tau_{s_i} U\|_{W^{1,2}} \geqslant \|U\|_{W^{1,2}}$$

On the other hand, by (3.1),

$$\lim_{i \to \infty} \|\tau_{s_i} U\|_{W^{1,2}} = \sqrt{2} \int_{X-Z} (|u|^2 + |\nabla u|^2) = \|U\|_{W^{1,2}}.$$

Therefore $\tau_{s_i} U$ converges strongly to U in $W^{1,2}$. $\square$

3.3. **Frequency functions.** This section recalls some results on frequency functions from [40].

Let U be a $\mathbb{Z}/2$ harmonic spinor. On $X - Z$ the section U can be locally written as $U = [\![u]\!] + [\![-u]\!]$. As before, we will use notations like $|u|$ and $|\nabla u|$ to denote the corresponding functions on $X - Z$ if they can be globally defined. The functions $|u|$ and $|\nabla u|$ extend to X by defining them to be zero on Z.

The following C^0 estimate was established in [40].

Lemma 3.8 ([40], Lemma 2.3)**.** *Let $A \subset B$ be two open subsets of X, and assume the closure of A is compact and contained in B. Then there exists a constant K,*

depending on A, B and the norms of the curvatures of X and V, such that

$$\sup_{x \in A} |u(x)|^2 \leqslant K \int_B |u(x)|^2 \, dx.$$

Now introduce some notations. Fix a point $x_0 \in X$. Take $R > 0$ such that $B_{x_0}(500R) \subset X$ is complete, and that the injectivity radius of X is greater than $1000R$ for every point in the ball $B_{x_0}(500R)$.

Later on we will need to work on both the Euclidean space and the manifold X, so we need to differentiate the notations. We will use $B_x(r)$ to denote the geodesic ball on X with center $x \in X$ and radius $r > 0$. Use $\bar{B}_x(r)$ to denote the Euclidean ball with center x in the Euclidean space and radius $r > 0$. When the center is the origin, $\bar{B}(r)$ is also used to denote $\bar{B}_0(r)$. Use $d(x, y)$ to denote the distance function on X, and use $|x - y|$ to denote the distance function on $\mathbb{R}^4$.

For every $x \in B_{x_0}(500R)$, use the normal coordinate centered at x to identify $B_x(500R)$ with the ball $\bar{B}(500R) \subset \mathbb{R}^4$. Let g_x be the function of metric matrices on $\bar{B}(500R)$ corresponding to $B_x(500R)$. For each $z \in \bar{B}(500R)$, let $K_x(z), \kappa_x(z)$ be the largest and smallest eigenvalue of $g_x(z)$. Assume that R is sufficiently small so that for every $x \in B_{x_0}(500R)$, $z \in \bar{B}(500R)$,

$$(3.2) \qquad \left(\frac{11}{12}\right)^2 \leqslant \kappa_x(z) \leqslant K_x(z) \leqslant \left(\frac{12}{11}\right)^2$$

In order to prove theorem 3.4, one only needs to study the rectifiability and the Minkowski content of $Z \cap B_{x_0}(R/2)$.

For $x \in B_{x_0}(500R)$, $r \in (0, 500R]$, define the height function

$$H(x, r) = \int_{\partial B_x(r)} |u|^2,$$

then $H(x, r)$ is always positive [40, Lemma 3.1]. Define

$$D(x, r) = \int_{B_x(r)} |\nabla u|^2,$$

and define the frequency function

$$N(x,r) = \frac{rD(x,r)}{H(x,r)}.$$

Section 3(a) of [40] proved the following monotonicity properties for N and H:

Lemma 3.9 ([40], (3.6) and Lemma 3.2)**.** *The functions N and H are absolutely continuous with respect to r, and there exist constants $\kappa > 0$ and $r_0 > 0$, depending only on the norms of curvatures of X and $\mathcal{V}$ on $B_{x_0}(1000R)$, such that when $r \leqslant r_0$,*

$$(3.3) \qquad\qquad \frac{\partial}{\partial r}H \geqslant \frac{3}{r}H - \kappa r H,$$

$$(3.4) \qquad\qquad \frac{\partial}{\partial r}N \geqslant -\kappa r(1 + N).$$

$$(3.5) \qquad (\frac{N}{r} + \kappa r)\frac{H}{r^3} \geqslant \frac{\partial}{\partial r}(\frac{H}{r^3}) \geqslant (\frac{N}{r} - \kappa r)\frac{H}{r^3}$$

By shrinking the size of R, we assume without loss of generality that $r_0 = 500R$, hence inequalities (3.3), (3.4), and (3.5) hold for all $x \in B_{x_0}(500R)$ and $r \leqslant 500R$.

Inequality (3.3) gives the following lemma

Lemma 3.10 ([40], Lemma 3.1)**.** *There exists a constant $\kappa > 0$, such that when $s < r < 500R$,*

$$H(x,r) \geqslant \left(\frac{r}{s}\right)^3 \cdot e^{-\kappa(r^2 - s^2)} \cdot H(x,s).$$

Inequality (3.4) gives

Lemma 3.11. *There exists a constant $\kappa > 0$, such that when $s < r < 500R$,*

$$N(x,r) \geqslant e^{-\kappa(r^2 - s^2)}N(x,s) - \kappa(r^2 - s^2).$$

Since $N(x, 500R)$ is continuous with respect to x, lemma 3.11 implies that $N(x,r)$ is bounded for all $x \in B_{x_0}(500R)$, $r \leqslant 500R$. Let Λ be an upper bound for N. From now on Λ will be treated as a constant. For the rest of this article, unless otherwise

stated, C, C_1, C_2, $\cdots$ will denote positive constants that depend on Λ, R, and the norms of the curvatures of X and $\mathcal{V}$, but independent of U. The values of C, C_1, C_2, $\cdots$ may be different in different appearances.

If $|g| \leqslant C \cdot f$ for some constant C, we write $g = O(f)$.

Inequality (3.5) then implies that there exists a constant C such that

$$(3.6) \qquad \left| \frac{\partial}{\partial r} \left(\ln(\frac{H}{r^3}) \right) \right| = O(\frac{1}{r}).$$

Inequality (3.4) implies that there exists $C > 0$, such that whenever $r \geqslant s$,

$$N(x, r) \geqslant N(x, s) - C(r^2 - s^2).$$

3.4. **Smoothed frequency functions.** We need to use a modified version of frequency functions. Let ϕ be a non-increasing smooth function on $\mathbb{R}$ such that $\phi(t) = 1$ when $t \leqslant 3/4$, and $\phi(t) = 0$ when $t \geqslant 1$. From now on ϕ will be fixed, hence the values of ϕ and its derivatives are considered as universal constants. Following [10], we define the smoothed frequency functions as follows.

Definition 3.12. For $x \in X$, let ν_x be the gradient vector field of the distance function $d(x, \cdot)$. For $x \in B_{x_0}(500R)$, $r \leqslant 500R$, introduce the following functions

$$D_\phi(x, r) = \int |\nabla u(y)|^2 \phi\left(\frac{d(x, y)}{r}\right) dy,$$

$$H_\phi(x, r) = -\int |u(y)|^2 d(x, y)^{-1} \phi'\left(\frac{d(x, y)}{r}\right) dy,$$

$$N_\phi(x, r) = \frac{r D_\phi(x, r)}{H_\phi(x, r)},$$

$$E_\phi(x, r) = -\int |\nabla_{\nu_x} u(y)|^2 d(x, y) \phi'\left(\frac{d(x, y)}{r}\right) dy.$$

Inequality (3.6) has the following useful corollary.

Lemma 3.13. *There exists a constant C with the following property. Let $r \in (0, 32R]$.*
Assume $s_1 \leqslant 10r$, $s_2 \geqslant r/10$. Then for any two points x, y with $d(x,y) \leqslant r$, one has

$$H_\phi(x, s_1) \leqslant C(H_\phi(y, s_2)).$$

Proof. Since the constant K in lemma 3.1 only depends on the norms of the curvatures and the sets A, B, a rescaling argument gives

$$|u(z)|^2 \leqslant \frac{C_1}{r^4} \int_{B_z(r)} |u|^2, \quad \forall B_z(r) \subset B_{x_0}(500R).$$

Therefore for every $z \in \partial B_x(s_1)$,

$$|u(z)|^2 \leqslant \frac{C_2}{r^4} \int_{B_y(12r)} |u|^2.$$

On the other hand, inequality (3.6) and lemma 3.10 gives

$$\frac{1}{r^4} \int_{B_y(12r)} |u|^2 \leqslant \frac{C_3}{r^3} H(y, s_2).$$

Therefore

$$H(x, s_1) = O(H(y, s_2)).$$

Apply (3.6) again, one obtains

$$H(y, s_2) = O(H_\phi(y, s_2)),$$
$$H_\phi(x, s_1) = O(H(x, s_1)),$$

hence the lemma is proved. $\qquad\square$

Lemma 3.14. *For $x \in B_{x_0}(32R)$, $r \leqslant 32R$, one has*

$$\int_{B_x(r)} |u(y)|^2 dy = O(r H_\phi(x, r)),$$

$$\int_{B_x(r)} |u(y)||\nabla u(y)| dy = O(H_\phi(x, r)),$$

$$\int_{B_x(r)} |\nabla u(y)|^2 dy = O(\frac{1}{r} H_\phi(x, r)).$$

Proof. The first equation follows from inequality (3.6) and lemma 3.10. For the third,

$$\int_{B_x(r)} |\nabla u(y)|^2 dy \leqslant D_\phi(x, 2r)$$

$$= \frac{1}{2r} N_\phi(x, 2r) H_\phi(x, 2r)$$

$$= O(\frac{1}{r} H_\phi(x, r)).$$

The second equation then follows from Cauchy's inequality. $\square$

The main result of this section is the following proposition.

Proposition 3.15. *The functions D_ϕ, H_ϕ, N_ϕ, and E_ϕ are smooth in both variables. Assume $x \in B_{x_0}(32R)$, $r \leqslant 32R$, and $v \in T_x(X)$. Consider the normal coordinate centered at x with radius r, extend the vector v to a vector field on $B_x(r)$ by requiring that the coordinate functions of v are constants. Then the following equations hold*

$$(3.7) \qquad D_\phi(x, r) = -\frac{1}{r} \int \phi'\Big(\frac{d(x, y)}{r}\Big) \nabla_{\nu_x} u(y) \cdot u(y) \, dy + O(r H_\phi(x, r)),$$

$$(3.8) \qquad \partial_r D_\phi(x, r) = \frac{2}{r} D_\phi(x, r) + \frac{2}{r^2} E_\phi(x, r) + O(H_\phi(x, r)),$$

$$(3.9) \qquad \partial_v D_\phi(x, r) = -\frac{2}{r} \int \phi'\Big(\frac{d(x, y)}{r}\Big) \nabla_{\nu_x} u(y) \cdot \nabla_v u(y) \, dy + O(H_\phi(x, r)),$$

$$(3.10) \qquad \partial_r H_\phi(x, r) = \frac{3}{r} H_\phi(x, r) + 2D_\phi(x, r) + O(r H_\phi(x, r)),$$

$$(3.11) \qquad \partial_v H_\phi(x, r) = -2 \int u(y) \cdot \nabla_v u(y) \, d(x, y)^{-1} \phi'\Big(\frac{d(x, y)}{r}\Big) \, dy + O(r H_\phi(x, r)).$$

The smoothness of the functions follows from the fact that ϕ is smooth and $|u|$, $|\nabla u|$ are both in L^2.

Proof of (3.7). It was proved in [40, Section 2(c)] that

$$(3.12) \qquad \int_{\partial B_x(s)} \nabla_{\nu_x} u(y) \cdot u(y) \, dy = \int_{B_x(s)} |\nabla u(y)|^2 \, dy + \int_{B_x(s)} \langle u(y), \mathcal{R}u(y) \rangle \, dy,$$

where $\mathcal{R}$ is a bounded curvature term from the Weitzenböck formula.

Therefore, by lemma 3.14,

$$
\begin{aligned}
D_\phi(x,r) &= -\frac{1}{r}\int_0^r \phi'\left(\frac{s}{r}\right)\int_{B_x(s)}|\nabla u(y)|^2\,dy\,ds \\
&= -\frac{1}{r}\int \phi'\left(\frac{d(x,y)}{r}\right)\nabla_{\nu_x}u(y)\cdot u(y)\,dy + \frac{1}{r}\int_0^r \phi'\left(\frac{s}{r}\right)\int_{B_x(s)}\langle u,\mathcal{R}u\rangle\,dy\,ds \\
&= -\frac{1}{r}\int \phi'\left(\frac{d(x,y)}{r}\right)\nabla_{\nu_x}u(y)\cdot u(y)\,dy + O(rH_\phi(x,r)).
\end{aligned}
$$

□

Proof of (3.8).

$$
\begin{aligned}
\partial_r D_\phi(x,r) &= -\frac{1}{r^2}\int |\nabla u(y)|^2\phi'\left(\frac{d(x,y)}{r}\right)\cdot d(x,y)\,dy \\
(3.13)\qquad &= -\frac{1}{r^2}\int_0^r \phi'\left(\frac{s}{r}\right)\cdot s\int_{\partial B_x(s)}|\nabla u(y)|^2\,dy\,ds
\end{aligned}
$$

It was proved in [40, Section 2(d)] that

$$
\int_{\partial B_x(s)}|\nabla u(y)|^2\,dy = 2\int_{\partial B_x(s)}|\nabla_{\nu_x}u(y)|^2\,dy + \frac{2}{s}\int_{B_x(s)}|\nabla u(y)|^2\,dy
$$
$$
+ \frac{2}{s}\int_{B_x(s)}\langle u(y),\mathcal{R}u(y)\rangle\,dy - \int_{\partial B_x(s)}\langle \mathcal{R}_1 u(y),\nabla u(y)\rangle\,dy + \int_{\partial B_x(s)}\langle u(y),\mathcal{R}_2 u(y)\rangle\,dy,
$$

where $\mathcal{R}$, $\mathcal{R}_1$, $\mathcal{R}_2$ are smooth tensors, $\mathcal{R}$ and $\mathcal{R}_2$ are bounded, the norm of $\mathcal{R}_1$ is bounded by $C_1\cdot r$.

Notice that

$$
-\int_0^r \phi'\left(\frac{s}{r}\right)\cdot s\int_{\partial B_x(s)}|\nabla_{\nu_x}u(y)|^2\,dy\,ds = E_\phi(x,r),
$$

$$
-\frac{1}{r}\int_0^r \phi'\left(\frac{s}{r}\right)\int_{B_x(s)}|\nabla u(y)|^2\,dy\,ds = D_\phi(x,r).
$$

Plug into equation (3.13), we have

$$\partial_r D_\phi(x,r) = \frac{2}{r} D_\phi(x,r) + \frac{2}{r^2} E_\phi(x,r) - \frac{1}{r^2} \int_0^r \phi'\left(\frac{s}{r}\right) \cdot s \cdot \left[\frac{2}{s} \int_{B_x(s)} \langle u(y), \mathcal{R}u(y) \rangle \, dy \right.$$

$$\left. - \int_{\partial B_x(s)} \langle \mathcal{R}_1 u(y), \nabla u(y) \rangle \, dy + \int_{\partial B_x(s)} \langle u(y), \mathcal{R}_2 u(y) \rangle \, dy \right] ds.$$

Lemma 3.14 implies

$$-\frac{1}{r^2} \int_0^r \phi'\left(\frac{s}{r}\right) \cdot s \cdot \left[\frac{2}{s} \int_{B_x(s)} \langle u(y), \mathcal{R}u(y) \rangle \, dy + \int_{\partial B_x(s)} \langle u(y), \mathcal{R}_2 u(y) \rangle \, dy \right] ds$$

$$= O(H_\phi(x,r)).$$

On the other hand,

$$\left| -\frac{1}{r^2} \int_0^r \phi'\left(\frac{s}{r}\right) \cdot s \cdot \left[-\int_{\partial B_x(s)} \langle \mathcal{R}_1 u(y), \nabla u(y) \rangle \, dy \right] ds \right|$$

$$\leqslant C_2 \cdot \int_0^r \left| \phi'\left(\frac{s}{r}\right) \right| \int_{\partial B_x(s)} |u(y)||\nabla u(y)| \, dy \, ds$$

$$\leqslant C_3 \int_{B_x(r)} |u(y)||\nabla u(y)| dy = O(H_\phi(x,r)).$$

Hence the result is proved. $\qquad\qquad\square$

Proof of (3.9). For a function $G(x,y)$ defined on $X \times X$ and a vector field w, use $\frac{\partial x}{\partial w} G$ to denote the directional derivative of G with respect to x, use $\frac{\partial y}{\partial w} G$ to denote the directional derivative with respect to y.

The first variation formula of geodesic lengths gives

$$\frac{\partial x}{\partial v} d(x,y) + \frac{\partial y}{\partial v} d(x,y) = O(d(x,y)^2).$$

We have

$$\frac{\partial x}{\partial v} D_\phi(x,r) = \frac{1}{r} \int |\nabla u(y)|^2 \phi'\left(\frac{d(x,y)}{r}\right) \cdot \frac{\partial x}{\partial v} d(x,y) \, dy$$

$$= -\frac{1}{r} \int |\nabla u(y)|^2 \phi'\left(\frac{d(x,y)}{r}\right) \cdot \frac{\partial y}{\partial v} d(x,y) \, dy + O(r) \int_{B_x(r)} |\nabla u(y)|^2$$

$$(3.14) \qquad = -\int |\nabla u(y)|^2 \cdot \frac{\partial y}{\partial v} \phi\Big(\frac{d(x,y)}{r}\Big) \, dy + O(H_\phi(x,r)).$$

One needs to establish the following lemma.

Lemma 3.16. *Let F be the curvature of $\mathcal{V}$, and $\{e_i\}$ be an orthonormal basis of TX. Let φ be a smooth function with $\operatorname{supp}\varphi \subset B_x(r)$. Then*

$$\int |\nabla u|^2 \partial_v \varphi$$

$$= 2\int \langle d\varphi \otimes \nabla_v u, \nabla u\rangle - 2\int \sum_i \varphi\langle F(v,e_i)u, \nabla_{e_i}u\rangle - 2\int \sum_i \varphi\langle \nabla_{[v,e_i]}u, \nabla_{e_i}u\rangle$$

$$- \int |\nabla u|^2 \varphi \, div(v) + 2\int \sum_i \varphi\langle \nabla_v u, \nabla_{\nabla_{e_i}e_i}u\rangle$$

$$+ 2\int \sum_i \varphi\langle \nabla_v u, \nabla_{e_i}u\rangle \, div(e_i) + 2\int \varphi\langle \nabla_v u, \mathcal{R}_0 u\rangle,$$

where $\mathcal{R}_0$ is the curvature term in the Weitzenböck formula.

Proof of lemma 3.16. By lemma 3.7, there exists a sequence of smooth 2-valued section U_i, such that $U_i = -U_i$ and $U_i \to U$ in $W^{1,2}$. By partitions of unity, integration by parts works for U_i. For any U_i, locally write it as $[\![w]\!] + [\![-w]\!]$ where w is a smooth section of $\mathcal{V}$, then

$$\int |\nabla w|^2 \partial_v \varphi$$

$$= -\int \sum_i \varphi\nabla_v \langle \nabla_{e_i}w, \nabla_{e_i}w\rangle - \int |\nabla w|^2 \varphi \, div(v)$$

$$= -2\int \sum_i \varphi\langle \nabla_{e_i}\nabla_v w, \nabla_{e_i}w\rangle - 2\int \sum_i \varphi\langle F(v,e_i)w, \nabla_{e_i}w\rangle$$

$$- 2\int \sum_i \varphi\langle \nabla_{[v,e_i]}w, \nabla_{e_i}w\rangle - \int |\nabla w|^2 \varphi \, div(v)$$

Here F denotes the curvature of $\mathcal{V}$. For the first term in the formula above,

$$\int \sum_i \varphi\langle \nabla_{e_i}\nabla_v w, \nabla_{e_i}w\rangle$$

$$= -\int \sum_i (\nabla_{e_i}\varphi)\langle \nabla_v w, \nabla_{e_i} w\rangle - \int \sum_i \varphi\langle \nabla_v w, \nabla_{e_i}\nabla_{e_i} w\rangle$$

$$- \int \sum_i \varphi\langle \nabla_v w, \nabla_{e_i} w\rangle \operatorname{div}(e_i)$$

$$= -\int \sum_i (\nabla_{e_i}\varphi)\langle \nabla_v w, \nabla_{e_i} w\rangle + \int \sum_i \varphi\langle \nabla_v w, \nabla^\dagger\nabla w\rangle$$

$$- \int \sum_i \varphi\langle \nabla_v w, \nabla_{\nabla_{e_i} e_i} w\rangle - \int \sum_i \varphi\langle \nabla_v w, \nabla_{e_i} w\rangle \operatorname{div}(e_i)$$

For the second term in the formula above, let $\mathcal{R}_0$ be the curvature term in the Weitzenböck formula, then

$$\int \sum_i \varphi\langle \nabla_v w, \nabla^\dagger\nabla w\rangle = \int \langle \varphi\nabla_v w, D^2 w - \mathcal{R}_0 w\rangle$$

$$= -\int \varphi\langle \nabla_v w, \mathcal{R}_0 w\rangle + \int \langle \rho(\nabla\varphi)\nabla_v w, Dw\rangle - \int \langle \varphi\langle[\nabla_v, D]w, Dw\rangle + \int \varphi\langle \nabla_v(Dw), Dw\rangle$$

$$= -\int \varphi\langle \nabla_v w, \mathcal{R}_0 w\rangle + \int \langle \rho(\nabla\varphi)\nabla_v w, Dw\rangle - \int \langle \varphi\langle[\nabla_v, D]w, Dw\rangle$$

$$- \frac{1}{2}\int \partial_v\varphi|Dw|^2 - \frac{1}{2}\int \varphi|Dw|^2 \operatorname{div}(v)$$

Therefore

$$\int |\nabla w|^2 \partial_v\varphi$$

$$= -2\int \sum_i \varphi\langle F(v, e_i)w, \nabla_{e_i} w\rangle - 2\int \sum_i \varphi\langle \nabla_{[v,e_i]}w, \nabla_{e_i} w\rangle - \int |\nabla w|^2 \varphi \operatorname{div}(v)$$

$$+ 2\int \sum_i (\nabla_{e_i}\varphi)\langle \nabla_v w, \nabla_{e_i} w\rangle + 2\int \sum_i \varphi\langle \nabla_v w, \nabla_{\nabla_{e_i} e_i} w\rangle + 2\int \sum_i \varphi\langle \nabla_v w, \nabla_{e_i} w\rangle \operatorname{div}(e_i)$$

$$+ 2\int \varphi\langle \nabla_v w, \mathcal{R}_0 w\rangle - 2\int \langle \rho(\nabla\varphi)\nabla_v w, Dw\rangle + 2\int \langle \varphi\langle[\nabla_v, D]w, Dw\rangle$$

$$+ \int \partial_v\varphi|Dw|^2 - \int \varphi|Dw|^2 \operatorname{div}(v)$$

Take limit $U_i \to U$, one has

$$\int |\nabla u|^2 \partial_v\varphi$$

$$= -2 \int \sum_i \varphi \langle F(v, e_i)u, \nabla_{e_i} u \rangle - 2 \int \sum_i \varphi \langle \nabla_{[v,e_i]} u, \nabla_{e_i} u \rangle - \int |\nabla u|^2 \varphi \operatorname{div}(v)$$

$$+ 2 \int \sum_i (\nabla_{e_i} \varphi) \langle \nabla_v u, \nabla_{e_i} u \rangle + 2 \int \sum_i \varphi \langle \nabla_v u, \nabla_{\nabla_{e_i} e_i} u \rangle + 2 \int \sum_i \varphi \langle \nabla_v u, \nabla_{e_i} u \rangle \operatorname{div}(e_i)$$

$$+ 2 \int \varphi \langle \nabla_v u, \mathcal{R}_0 u \rangle - 2 \int \langle \rho(\nabla \varphi) \nabla_v u, Du \rangle + 2 \int \langle \varphi \langle [\nabla_v, D] u, Du \rangle$$

$$+ \int \partial_v \varphi |Du|^2 - \int \varphi |Du|^2 \operatorname{div}(v)$$

$$= -2 \int \sum_i \varphi \langle F(v, e_i)u, \nabla_{e_i} u \rangle - 2 \int \sum_i \varphi \langle \nabla_{[v,e_i]} u, \nabla_{e_i} u \rangle - \int |\nabla u|^2 \varphi \operatorname{div}(v)$$

$$+ 2 \int \sum_i (\nabla_{e_i} \varphi) \langle \nabla_v u, \nabla_{e_i} u \rangle + 2 \int \sum_i \varphi \langle \nabla_v u, \nabla_{\nabla_{e_i} e_i} u \rangle$$

$$+ 2 \int \sum_i \varphi \langle \nabla_v u, \nabla_{e_i} u \rangle \operatorname{div}(e_i) + 2 \int \varphi \langle \nabla_v u, \mathcal{R}_0 u \rangle$$

Notice that

$$\sum_i (\nabla_{e_i} \varphi) \langle \nabla_v u, \nabla_{e_i} u \rangle = \langle d\varphi \otimes \nabla_v u, \nabla u \rangle,$$

therefore the lemma is proved. $\qquad \square$

Back to the proof of equation (3.9). Take $\varphi(y) = \phi(d(x, y)/r)$. By Lemma 3.14,

$$-2 \int \sum_i \varphi \langle F(v, e_i)u, \nabla_{e_i} u \rangle + 2 \int \varphi \langle \nabla_v u, \mathcal{R}_0 u \rangle = O(H_\phi(x, r)).$$

On the other hand, $|\operatorname{div}(v)| = O(r)$, and one can choose $\{e_i\}$ such that $|[v, e_i]| = O(r)$, $|\operatorname{div}(e_i)| = O(r)$, and $|\nabla_{e_i} e_i| = O(r)$. Thus by lemma 3.14,

$$-2 \int \sum_i \varphi \langle \nabla_{[v,e_i]} u, \nabla_{e_i} u \rangle - \int |\nabla u|^2 \varphi \operatorname{div}(v) + 2 \int \sum_i \varphi \langle \nabla_v u, \nabla_{\nabla_{e_i} e_i} u \rangle$$

$$+ 2 \int \sum_i \varphi \langle \nabla_v u, \nabla_{e_i} u \rangle \operatorname{div}(e_i) = O(H_\phi(x, r)).$$

Equation (3.9) then follows immediately from equation (3.14) and lemma 3.16. $\quad \square$

Proof of (3.10). By [40, Equation (2.11)],

$$(3.15) \qquad \partial_s H(x,s) = \frac{3}{s} H(x,s) + 2D(x,s) + \int_{B_x(s)} \langle u, \mathcal{R}u \rangle + \int_{\partial B_x(s)} \mathfrak{t}|u|^2,$$

where $\mathcal{R}$ is a curvature term from the Weitzenböck formula, and $\mathfrak{t}$ comes from the mean curvature of $\partial B_x(s)$. The function $\mathfrak{t}$ satisfies $|\mathfrak{t}(y)| = O(d(x,y))$. Notice that

$$H_\phi(x,r) = \int_0^r -\phi'(s/r) \cdot \frac{1}{s} \cdot H(s)\, ds = \int_0^1 -\phi'(\lambda) \frac{1}{\lambda} \cdot H(\lambda r)\, d\lambda.$$

Therefore

$$\partial_r H_\phi(x,r)$$

$$= \int_0^1 -\phi'(\lambda) \cdot (\partial_r H)(\lambda r)\, d\lambda$$

$$= \int_0^1 -\phi'(\lambda) \Big[\frac{3}{\lambda r} H(x, \lambda r) + 2D(x, \lambda r) + \int_{B_x(\lambda r)} \langle u, \mathcal{R}u \rangle + \int_{\partial B_x(\lambda r)} \mathfrak{t}|u|^2 \Big]\, d\lambda$$

$$= -\frac{1}{r} \int_0^r \phi'(s/r) \Big[\frac{3}{s} H(x,s) + 2D(x,s) + \int_{B_x(s)} \langle u, \mathcal{R}u \rangle + \int_{\partial B_x(s)} \mathfrak{t}|u|^2 \Big]\, ds$$

$$= \frac{3}{r} H_\phi(x,r) + 2D_\phi(x,r) - \frac{1}{r} \int_0^r \phi'(s/r) \Big[\int_{B_x(s)} \langle u, \mathcal{R}u \rangle + \int_{\partial B_x(s)} \mathfrak{t}|u|^2 \Big]\, ds$$

$$= \frac{3}{r} H_\phi(x,r) + 2D_\phi(x,r) + O(r H_\phi(x,r)).$$

$\square$

Proof of (3.11). As in the proof of (3.9), for a function $G(x,y)$, use $\frac{\partial x}{\partial v} G$ to denote the directional derivative of G with respect to x, and use $\frac{\partial y}{\partial v} G$ to denote the directional derivative with respect to y. Recall that we have

$$\frac{\partial x}{\partial v} d(x,y) + \frac{\partial y}{\partial v} d(x,y) = O(d(x,y)^2),$$

therefore

$$\Big(\frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} \Big) \Big[d(x,y)^{-1} \phi'\Big(\frac{d(x,y)}{r} \Big) \Big] = O(1).$$

We have

$$\partial_v H(x,r)$$

$$= -\int |u(y)|^2 \frac{\partial x}{\partial v}\Big[d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\Big]dy$$

$$= \int |u(y)|^2 \frac{\partial y}{\partial v}\Big[d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\Big]dy + O\Big(\int_{B_x(r)} |u|^2\Big)$$

$$= -\int \frac{\partial}{\partial v}|u(y)|^2 d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)dy$$

$$\quad -\int |u(y)|^2 d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\,\mathrm{div}(v)dy + O(rH_\phi(x,r))$$

$$= -2\int u(y)\cdot\nabla_v u(y)\,d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\,dy + O(rH_\phi(x,r))$$

The last equality follows from $|\mathrm{div}(v)| = O(r)$ and $\int_{B_x(r)} |u|^2 = O(rH_\phi(x,r))$. □

Remark 3.17. When both X and $\mathcal{V}$ are flat, all the curvature terms in the computations above are zero. Therefore, proposition 3.15 becomes

$$D_\phi(x,r) = -\frac{1}{r}\int \phi'\Big(\frac{d(x,y)}{r}\Big)\nabla_{\nu_x}u(y)\cdot u(y)\,dy,$$

$$\partial_r D_\phi(x,r) = \frac{2}{r}D_\phi(x,r) + \frac{2}{r^2}E_\phi(x,r)$$

$$\partial_v D_\phi(x,r) = -\frac{2}{r}\int \phi'\Big(\frac{d(x,y)}{r}\Big)\nabla_{\nu_x}u(y)\cdot\nabla_v u(y)\,dy$$

$$\partial_r H_\phi(x,r) = \frac{3}{r}H_\phi(x,r) + 2D_\phi(x,r)$$

$$\partial_v H_\phi(x,r) = -2\int u(y)\cdot\nabla_v u(y)\,d(x,y)^{-1}\phi'\Big(\frac{d(x,y)}{r}\Big)\,dy$$

Corollary 3.18. *Let $\eta_x(y) = d(x,y)\cdot\nu_x(y)$. Under the assumptions of proposition 3.15, one has*

$$(3.16)\quad \partial_v N_\phi(x,r) = \frac{2}{H_\phi(x,r)}\int -\frac{1}{d(x,y)}\phi'\Big(\frac{d(x,y)}{r}\Big).$$

$$(\nabla_{\eta_x}u(y) - N_\phi(x,r)u(y))\cdot\nabla_v u(y)\,dy + O(r).$$

(3.17) $\quad \partial_r N_\phi(x, r) = \dfrac{2}{r H_\phi(x, r)} \displaystyle\int -\phi'\Big(\dfrac{d(x, y)}{r}\Big)\cdot$

$$d(x, y)^{-1} |\nabla_{\eta_x} u(y) - N_\phi(x, r) u(y)|^2 \, dy + O(r),$$

As a consequence, there exists a constant C, such that $\Big(N_\phi(x, r) + C r^2 \Big)$ is increasing in r.

Proof. The first equation follows immediately from proposition 3.15 by combining equations (3.9) and (3.11). For the first one, lemma 3.15 gives

$$\partial_r N_\phi(x, r) = \frac{2}{r H_\phi(x, r)} \Big(E_\phi(x, r) - \frac{r^2 D_\phi(x, r)^2}{H_\phi(x, r)} \Big) + O(r),$$

and we have

$$E_\phi(x, r) - \frac{r^2 D_\phi(x, r)^2}{H_\phi(x, r)}$$

$$= E_\phi(x, r) - 2r D_\phi(x, r) N_\phi(x, r) + N_\phi(x, r)^2 H_\phi(x, r)$$

$$= \int -\phi'\Big(\frac{d(x, y)}{r}\Big) d(x, y)^{-1} |\nabla_{\eta_x} u(y) - N_\phi(x, r) u(y)|^2 \, dy + O(r^2 H_\phi(x, r))$$

Hence the second equation is verified.

$\square$

3.5. **Compactness.** This section proves a compactness result for $\mathbb{Z}/2$ harmonic spinors.

Consider the ball $\Omega = \bar{B}(5) \subset \mathbb{R}^4$ centered at the origin. Let $\mathcal{V}$ be a fixed trivial vector bundle on Ω. Assume g_n is a sequence of Riemannian metrics on Ω, A_n is a sequence of connenction forms on $\mathcal{V}$, and ρ_n is a sequence of Clifford bundle structures of $\mathcal{V}$. Assume that (g_n, A_n, ρ_n) are compatible, and assume that (g_n, A_n, ρ_n) converge to (g, A, ρ) in C^∞. Assume g is the Euclidean metric on $\bar{B}(5)$. Then for sufficiently large n, the injectivity radius at each point in $B(2)$ is at least 2.5. Without loss of generality, assume that this property holds for every n.

Fix $\epsilon, \Lambda > 0$. For every n, assume U_n is a 2-valued section of $\mathcal{V}$ defined on $\bar{B}(5)$, with the following properties:

(1) The section U_n is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(5)$ with respect to (g_n, A_n, ρ_n).

(2) U_n satisfies assumption 3.2 with respect to ϵ.

(3) Let $N_\phi^{(n)}$ be the smoothed frequency function for the extended U_n. Then whenever $N_\phi(x, r)$ is defined,

$$N_\phi^{(n)}(x, r) \leqslant \Lambda.$$

(4) Let $H_\phi^{(n)}$ be the smoothed height function of U_n, then $H_\phi^{(n)}(0, 1) = 1$.

The main result of this section is the following proposition.

Proposition 3.19. *Let U_n be given as above. Then there exits a subsequence of $\{U_n\}$, such that the sequence converges strongly in $W^{1,2}(\bar{B}(2))$ to a section U. The section U is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(2)$ with respect to (g, A, ρ), and U satisfies assumption 3.2 for a possibly smaller value of ϵ. Moreover, U_n converges to U uniformly on $\bar{B}(2)$.*

Proof. Fix a trivialization of $\mathcal{V}$, and fix $s \in (0, 0.5)$. The bound on $N_\phi^{(n)}$ and the assumption that $H_\phi^{(n)}(0, 1) = 1$ implies that $\|U\|_{L^2(\bar{B}(2+s))} \leqslant C_1$ for some constant C_1. The upper bound on N_ϕ then implies $\|\nabla_{A_n} U\|_{L^2(\bar{B}(2+s/2))} \leqslant C_2$. Since $A_n \to A$ in C^∞, this implies that U_n is bounded in $W^{1,2}(\bar{B}(2 + s/2))$. Therefore, there is a subsequence of $\{U_n\}$ which converges weakly in $W^{1,2}(\bar{B}(2 + s/2))$ and converges strongly in $L^2(\bar{B}(2 + s/2))$. To avoid complicated notations, the subsequence is still denoted by $\{U_n\}$. Denote the limit of $\{U_n\}$ on $\bar{B}(2+s/2)$ by U. Let $H_\phi^{(n)}$, $D_\phi^{(n)}$, $N_\phi^{(n)}$ be the smoothed frequency functions for U_n, let H_ϕ, D_ϕ, N_ϕ be the corresponding functions for U. Since $U_n \to U$ strongly in L^2, one has $H_\phi(0, 1) = 1$, thus U is not identically $2[\![0]\!]$.

By [40, Section 3(e)], there exists constants $K > 0$ and $\alpha \in (0, 1)$, depending on ϵ, Λ, R and the C^1 norms of the curvatures of $\{g_n\}$ and A_n, such that

$$\|U_n\|_{C^\alpha(\bar{B}(2+s/2))} \leqslant K.$$

By the Arzela-Ascoli theorem, there exists a further subsequence of $\{U_n\}$ which converges uniformly to U on $\bar{B}(2 + s/2)$. Still denote this subsequence by $\{U_n\}$. Since solutions to the Dirac equation are closed under C^0 limits, U is a $\mathbb{Z}/2$ harmonic spinor. U is also Hölder continuous, so it satisfies assumption 3.2.

Locally write U_n as $[\![u_n]\!] + [\![-u_n]\!]$, and write U as $[\![u]\!] + [\![-u]\!]$. The weak convergence of U_n to U implies

$$\liminf_{n \to \infty} \int_{\bar{B}(2)} |\nabla_{A_n} u_n|^2 \geqslant \int_{\bar{B}(2)} |\nabla_A u|^2.$$

We want to prove that

$$\lim_{n \to \infty} \int_{\bar{B}(2)} |\nabla_{A_n} u_n|^2 = \int_{\bar{B}(2)} |\nabla_A u|^2.$$

Assume the contrary, then there exists a subsequence of n such that

$$\int_{\bar{B}(2)} |\nabla_{A_n} u_n|^2 \geqslant \int_{\bar{B}(2)} |\nabla_A u|^2 + \delta$$

for some $\delta > 0$. Since $\int_{\bar{B}(r)} |\nabla_A u|^2$ is continuous in r, and $\int_{\bar{B}(r)} |\nabla_{A_n} u_n|^2$ is non-decreasing in r for every n, there exists $r \in (2, 2 + s/2)$ and $\sigma \in (1, (2 + s/2)/r)$, such that for every $t \in [2, r]$,

(3.18)
$$\int_{\bar{B}(t)} |\nabla_{A_n} u_n|^2 \geqslant \int_{\bar{B}(\sigma t)} |\nabla_A u|^2 + \delta/2$$

Use $B_n(t)$ to denote the geodesic ball of center 0 and radius t with metric g_n. Since $g_n \to g$, we have $\bar{B}(t) \subset B_n(\sigma t)$ for sufficiently large n. Equation (3.18) then gives

$$(3.19) \qquad \int_{B_n(\sigma t)} |\nabla_{A_n} u_n|^2 \geqslant \int_{\bar{B}(\sigma t)} |\nabla_A u|^2 + \delta/2, \quad \text{for } t \in [2, r]$$

when n is sufficiently large.

By equation (3.15), for every t,

$$\partial_t H^{(n)}(0, t) = \frac{3}{t} H^{(n)}(0, t) + 2D^{(n)}(0, t) + \int_{B_n(t)} \langle u, \mathcal{R}^{(n)} u \rangle + \int_{\partial B_n(t)} \mathfrak{t}^{(n)} |u|^2,$$

$$\partial_t H(0, t) = \frac{3}{t} H(0, t) + 2D(0, t) + \int_{\bar{B}(t)} \langle u, \mathcal{R} u \rangle + \int_{\partial \bar{B}(t)} \mathfrak{t} |u|^2,$$

where $\mathcal{R}^{(n)}$ and $\mathfrak{t}^{(n)}$ are bounded terms that are uniformly convergent to $\mathcal{R}$ and $\mathfrak{t}$ as n goes to infinity. The uniform convergence of $|u_n|$ and g_n then imply

$$\lim_{s \to \infty} \int_{2\sigma}^{\sigma r} D^{(n)}(0, t) \, dt = \int_{2\sigma}^{\sigma r} D(0, t) \, dt,$$

which contradicts (3.19). In conclusion,

$$\lim_{n \to \infty} \int_{\bar{B}(2)} |\nabla_{A_n} u_n|^2 = \int_{\bar{B}(2)} |\nabla_A u|^2.$$

Since $(A_n, g_n) \to (A, g)$ in C^∞, this implies

$$\lim_{n \to \infty} \|U_i\|_{W^{1,2}}(\bar{B}(2)) = \|U\|_{W^{1,2}}(\bar{B}(2)),$$

therefore U_i convergence strongly to U in $W^{1,2}(\bar{B}(2))$. $\qquad \square$

Corollary 3.20. *Let $\sigma > 1$. Let g_* be a metric on $\mathbb{R}^4$ given by a constant metric matrix, such that all eigenvalues of the matrix are in the interval $[\sigma^{-2}, \sigma^2]$.*

Assume $\{(g_n, A_n, \rho_n)\}_{n \geqslant 1}$ is a sequence of geometric data on $\bar{B}(5\sigma^2)$, and assume (g_n, A_n, ρ_n) converge to (g_*, A, ρ) in C^∞. Let U_n be a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(5\sigma^2)$ with respect to (g_n, A_n, ρ_n), such that the sequence U_n satisfies conditions (2) to (4)

listed before proposition 3.19. Then a subsequence of U_n converges to a $\mathbb{Z}/2$ harmonic spinor in $W^{1,2}(\bar{B}(2))$ with respect to (g, A, ρ). The limit U satisfies assumption 3.2, and the sequence U_n converges to U uniformly.

Proof. Take a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $T^*(g_*)$ is the Euclidean metric. Then $(T^*g_n, T^*A_n, T^*\rho_n, T^*U_n)$ gives a sequence of $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(5\sigma)$. Since T^*g_n converges to the Euclidean metric, one can apply lemma 3.19 and find a convergent subsequence on $\bar{B}(2\sigma)$. Now pull back by T^{-1}, one obtains a convergent subseqence of U_n on $\bar{B}(2)$. $\qquad\square$

3.6. **Frequency pinching estimates.** For $x \in B_{x_0}(32R)$ and $0 < s < r \leqslant 32R$, define

$$W_s^r(x) = N_\phi(x, r) - N_\phi(x, s).$$

This section proves the following estimate

Proposition 3.21. *There exists a constant C with the following property. Let $r \in (0, 8R]$. Assume $x_1, x_2 \in B_{x_0}(32R)$, such that $d(x_1, x_2) \leqslant r/4$. Let x be a point on the short geodesic γ bounded by x_1 and x_2. Let v be a unit tangent vector of γ at x. Then*

$$d(x_1, x_2) \cdot |\partial_v N_\phi(x, r)| \leqslant C\Big[\sqrt{|W_{r/4}^{4r}(x_1)|} + \sqrt{|W_{r/4}^{4r}(x_2)|} + r\Big].$$

The proof is adapted from the arguments in [10, Section 4]. First, one needs to prove the following lemma.

Lemma 3.22. *There exists a constant C, such that for every $x \in B_{x_0}(32R)$ and $r \leqslant 8R$, one has*

$$\int_{B_x(3r) - B_x(r/3)} |\nabla_{\eta_x} u(y) - N_\phi(x, d(x, y)) u(y)|^2 dy \leqslant Cr H_\phi(x, r)(W_{r/4}^{4r}(x) + Cr^2).$$

Proof. By equation (3.17),

$$\int_{r/4}^{4r} \partial_s N_\phi(x, s)ds + O(r^2)$$

$$= \int_{r/4}^{4r} \frac{2}{sH_\phi(x, s)} \int -\phi'\Big(\frac{d(x, y)}{s}\Big)d(x, y)^{-1}|\nabla_{\eta_x} u(y) - N_\phi(x, s)u(y)|^2 \, dyds$$

$$\geqslant \frac{1}{C_1 r H_\phi(x, r)} \int_{r/4}^{4r} \int -\phi'\Big(\frac{d(x, y)}{s}\Big)d(x, y)^{-1}|\nabla_{\eta_x} u(y) - N_\phi(x, s)u(y)|^2 \, dyds$$

$$\geqslant \frac{1}{C_1 r H_\phi(x, r)} \int_{r/3}^{4r} \int -\phi'\Big(\frac{d(x, y)}{s}\Big)d(x, y)^{-1}|\nabla_{\eta_x} u(y) - N_\phi(x, s)u(y)|^2 \, dyds$$

$$=: (A)$$

For every pair (y, s) in the support of the integration in (A), one has $d(x, y) \in [r/4, 4r]$, hence

$$|N_\phi(x, s) - N_\phi(x, d(x, y))| \leqslant W_{r/4}^{4r}(x) + C_2 r^2.$$

Therefore,

$$(A) \geqslant \frac{1}{C_1 r H_\phi(x, r)} \underbrace{\int_{r/3}^{4r} \int -\phi'\Big(\frac{d(x, y)}{s}\Big)d(x, y)^{-1}|\nabla_{\eta_x} u(y) - N_\phi(x, d(x, y))u(y)|^2 \, dyds}_{=:I}$$

$$- \frac{C_3(W_{r/4}^{4r}(x) + C_2 r^2)}{r H_\phi(x, r)} \underbrace{\int_{r/3}^{4r} \int -\phi'\Big(\frac{d(x, y)}{s}\Big)d(x, y)^{-1}\Big[|\nabla u(y)||u(y)|d(x, y) + |u(y)|^2\Big]dyds}_{=:II}.$$

By lemma 3.14, $II = O(r H_\phi(x, 4r)) = O((r H_\phi(x, r))$. By Fubini's theorem,

$$I = \int_{B_x(4r)} |\nabla_{\eta_x} u(y) - N_\phi(x, d(x, y))u(y)|^2 \int_{r/3}^{4r} -\phi'\Big(\frac{d(x, y)}{s}\Big)d(x, y)^{-1} \, dsdy$$

Notice that

$$\inf_{\{y|d(x,y)\in[r/3,3r]\}} \int_{r/3}^{4r} -\phi'\Big(\frac{d(x, y)}{s}\Big)d(x, y)^{-1} \, ds > 0,$$

Therefore

$$I \geqslant \frac{1}{C_4} \int_{B_x(3r)-B_x(r/3)} |\nabla_{\eta_x} u(y) - N_\phi(x, d(x, y))u(y)|^2 \, dy,$$

In conclusion,

$$(A) \geqslant \frac{1}{C_5 r H_\phi(x,r)} \int_{B_x(3r) - B_x(r/3)} |\nabla_{\eta_x} u(y) - N_\phi(x, d(x,y)) u(y)|^2 \, dy$$

$$- C_6(W_{r/4}^{4r}(x) + C_2 r^2),$$

hence

$$C_7 r H_\phi(x,r)(W_{r/4}^{4r}(x) + C_8 r^2) \geqslant \int_{B_x(3r) - B_x(r/3)} |\nabla_{\eta_x} u(y) - N_\phi(x, d(x,y)) u(y)|^2 dy.$$

$\square$

One also needs the following technical lemma.

Lemma 3.23. *Assume M is a compact manifold, possibly with boundary. Let $\varphi^\zeta :$ $\Omega \subset \overline{B_{x_0}(64R)} \to \mathbb{R}^4$ be a smooth family of smooth embeddings, parametrized by $\zeta \in M$. For every $\zeta \in M$ and $x \in B_{x_0}(64R)$, one can define a vector field η_x^ζ on $B_{x_0}(64R)$ as follows. For every $y \in B_{x_0}(64R)$, let*

$$\eta_x^\zeta(y) = [(\varphi^\zeta)_*(y)]^{-1}(\varphi^\zeta(y) - \varphi^\zeta(x)).$$

Then there exists a constant $\Theta > 0$, depending on φ, such that

$$|\eta_x^\zeta(y) - \eta_x(y)| \leqslant \Theta \cdot d(x,y)^2.$$

Proof. Fix x, compute the covariant derivates of η_x^ζ and η_x at x. Since both vector fields are zero at x, their covariant derivatives at x are independent of the connections. Let $e \in T_x X$. Taking derivate in the Euclidean coordinates φ^ζ, one obtains $\nabla_e(\eta_x^\zeta)(x) = e$. Taking derivative in the normal coordinates centered at x, one obtains $\nabla_e(\eta_x)(x) = e$. Therefore, η_x^ζ and η_x have the same derivatives at x. Since we are working on compact manifolds, $|\eta_x^\zeta(y) - \eta_x(y)| \leqslant \Theta \cdot d(x,y)^2$ for some constant Θ independent of x. $\square$

Proof of proposition 3.21. Assume that v points from x_1 towards x_2. Extend v to a vector field on $B_x(r)$, such that the coordinates of v are constant under the normal coordinate centered at x. Now apply lemma 3.23. Let $M = \overline{B_{x_0}(32R)}$. For every $\zeta \in \overline{B_{x_0}(32R)}$, let φ^ζ be the exponential map centered at ζ. Then for every $z \in B_x(r)$,

$$(3.20) \qquad v(z) = \frac{\eta_{x_1}^x(z) - \eta_{x_2}^x(z)}{|\varphi^x(x_1) - \varphi^x(x_2)|}.$$

By lemma 3.23,

$$(3.21) \qquad |\eta_{x_1}^x(z) - \eta_{x_1}(z)| = O(r^2), \quad |\eta_{x_2}^x(z) - \eta_{x_2}(z)| = O(r^2)$$

Notice that since φ^x is the exponential map centered at x,

$$(3.22) \qquad |\varphi^x(x_1) - \varphi^x(x_2)| = d(x_1, x_2).$$

Combine (3.20), (3.21) and (3.22) together, one obtains

$$\left| v(z) - \frac{\eta_{x_1}(z) - \eta_{x_2}(z)}{d(x_1, x_2)} \right| = O(r^2/d(x_1, x_2)).$$

Define

$$\mathcal{E}_l(z) = \nabla_{\eta_{x_l}} u(z) - N_\phi(x_l, d(z, x_l)) u(z) \qquad \text{for } l = 1, 2.$$

Then

$$d(x_1, x_2) \nabla_v u(z) = \nabla_{\eta_{x_1}} u(z) - \nabla_{\eta_{x_2}} u(z) + O(r^2 |\nabla u|)$$

$$= \underbrace{\left(N_\phi(x_1, d(z, x_1)) - N_\phi(x_2, d(z, x_2)) \right)}_{=:\mathcal{E}_3(z)} u(z)$$

$$+ \mathcal{E}_1(z) - \mathcal{E}_2(z) + O(r^2 |\nabla u|).$$

To simplify notations, define the measure

$$d\mu_x = -d(x, y)^{-1} \phi'\left(\frac{d(x, y)}{r} \right) dy.$$

Using (3.16), one can write

$$d(x_1, x_2) \cdot \partial_v N_\phi(x, r)$$

$$= O(r^2) + \frac{2}{H_\phi(x, r)} \int \nabla_{\eta_x} u(y) \cdot (\mathcal{E}_1 - \mathcal{E}_2 + \mathcal{E}_3 u + O(r^2 |\nabla u|)) d\mu_x$$

$$- \frac{2}{H_\phi(x, r)} \int u N_\phi(x, r) \cdot (\mathcal{E}_1 - \mathcal{E}_2 + \mathcal{E}_3 u + O(r^2 |\nabla u|)) d\mu_x$$

$$= \underbrace{\frac{2}{H_\phi(x, r)} \int \nabla_{\eta_x} u(y) \cdot (\mathcal{E}_1 - \mathcal{E}_2) d\mu_x}_{=:(A)} - \underbrace{\frac{2 N_\phi(x, r)}{H_\phi(x, r)} \int u \cdot (\mathcal{E}_1 - \mathcal{E}_2) d\mu_x}_{=:(B)}$$

$$+ \underbrace{\frac{2}{H_\phi(x, r)} \int \mathcal{E}_3 u (\nabla_{\eta_x} u - N_\phi(x, r) u) \, d\mu_x}_{=:(C)} + O(r^2)$$

To bound (C), notice that

$$\mathcal{E}_3(z) = \underbrace{N_\phi(x_1, r) - N_\phi(x_2, r)}_{=:\mathcal{E}} + \underbrace{\left[N_\phi(x_1, d(z, x_1)) - N_\phi(x_1, r) \right]}_{=:\mathcal{E}_4(z)}$$

$$- \underbrace{\left[N_\phi(x_2, d(z, x_2)) - N_\phi(x_2, r) \right]}_{=:\mathcal{E}_5(z)} .$$

By (3.7),

$$\int u \cdot \nabla_{\eta_x} u \, d\mu_x = r D_\phi(x, r) + O(r^2 H_\phi(x, r))$$

$$= N_\phi(x, r) H_\phi(x, r) + O(r^2 H_\phi(x, r))$$

$$= N_\phi(x, r) \int |u|^2 \, d\mu_x + O(r^2 H_\phi(x, r)).$$

Hence

$$\int u \cdot (\nabla_{\eta_x} u - N_\phi(x, r) u) d\mu_x = O(r^2 H_\phi(x, r)),$$

therefore

$$\int \mathcal{E} u \cdot (\nabla_{\eta_x} u - N_\phi(x, r) u) d\mu_x = O(r^2 H_\phi(x, r)).$$

By lemma 3.14,

$$2\int |u|(|\nabla_{\eta_x}u| + |N_\phi(x,r)||u|)\,d\mu_x = O(H_\phi(x,r)).$$

In addition, notice that

$$\sup_{z\in\,\text{supp}\,\mu_x}|\mathcal{E}_4(z)| + |\mathcal{E}_5(z)| \leqslant W^{4r}_{r/4}(x_1) + W^{4r}_{r/4}(x_2) + C_1 r^2.$$

Therefore,

$$\int(|\mathcal{E}_4| + |\mathcal{E}_5|)\cdot\big|u(\nabla_{\eta_x}u - N_\phi(x,r)u)\big|\,d\mu_x$$

$$\leqslant C_2 H_\phi(x,r)(W^{4r}_{r/4}(x_1) + W^{4r}_{r/4}(x_2) + C_1 r^2).$$

As a result,

$$(C) \leqslant C_3(W^{4r}_{r/4}(x_1) + W^{4r}_{r/4}(x_2) + C_4 r^2).$$

To bound (A), use Cauchy's inequality to obtain

$$(A) \leqslant \frac{C_5}{H_\phi(x,r)}\Big(\int_{B_x(r)}|\nabla u|^2 dy\Big)^{1/2}\Big(\int_{B_x(r)-B_x(3r/4)}\big(\mathcal{E}_1^2 + \mathcal{E}_2^2\big)dy\Big)^{1/2}$$

$$\leqslant \frac{C_6}{r^{1/2}}\Big(\int_{B_x(r)-B_x(3r/4)}\big(\mathcal{E}_1^2 + \mathcal{E}_2^2\big)dy\Big)^{1/2}.$$

Now apply lemma 3.22,

$$\int_{B_x(r)-B_x(3r/4)}\mathcal{E}_1^2\,dy \leqslant \int_{B_{x_1}(5r/4)-B_{x_1}(r/2)}\mathcal{E}_1^2\,dy$$

$$\leqslant C_7 r H_\phi(x_1,r)(W^{4r}_{r/4}(x_1) + C_7 r^2)$$

A similar estimate works for the integral of $\mathcal{E}_2$. Therefore

$$(A) \leqslant C_8\Big[\sqrt{|W^{4r}_{r/4}(x_1)|} + \sqrt{|W^{4r}_{r/4}(x_2)|} + r\Big].$$

Similarly, applying Cauchy's inequality on (B) leads to

$$(B) \leqslant \frac{C_9}{r H_\phi(x,r)} \Big(\int_{B_x(r)} |u|^2 dy \Big)^{1/2} \Big(\int_{B_x(r) - B_x(3r/4)} \big(\mathcal{E}_1^2 + \mathcal{E}_2^2\big) dy \Big)^{1/2}$$

$$\leqslant \frac{C_{10}}{r^{1/2}} \Big(\int_{B_x(r) - B_x(3r/4)} \big(\mathcal{E}_1^2 + \mathcal{E}_2^2\big) dy \Big)^{1/2}$$

Lemma 3.22 then gives

$$(B) \leqslant C_{11}\Big[\sqrt{|W_{r/4}^{4r}(x_1)|} + \sqrt{|W_{r/4}^{4r}(x_2)|} + r \Big],$$

and the proposition is proved. $\qquad\square$

Corollary 3.24. *Assume* $x_1, x_2 \in B_{x_0}(32R)$, *assume* $r \in (0, 8R]$. *If* $d(x_1, x_2) \leqslant r/4$, *then*

$$|N_\phi(x_1, r) - N_\phi(x_2, r)| \leqslant C\Big[\sqrt{|W_{r/4}^{4r}(x_1)|} + \sqrt{|W_{r/4}^{4r}(x_2)|} + r \Big].$$

$\qquad\square$

3.7. L^2 **approximation by planes.** This section develops a distortion bound analogous to that of [10, Proposition 5.3]. Assume U satisfies assumption 3.2 with respect to $\epsilon > 0$. In this section, the constants C, C_1, C_2, $\cdots$ will denote constants that depend on Λ, R, the C^1 norms of the curvatures, as well as ϵ. The presentation of this section is adapted from section 5 of [10].

Definition 3.25. Suppose μ is a Radon measure on $\mathbb{R}^4$. For $x \in \mathbb{R}^4$, $r > 0$, define

$$D_\mu^2(x, r) = \inf_L r^{-4} \int_{B_x(r)} \mathrm{dist}(y, L)^2 \, d\mu(y),$$

where L is taken among the set of 2-dimensional affine subspaces.

For a measure μ supported in Z, we wish to bound the value of $D_\mu^2(x, r)$ in terms of the frequency functions. However, we have to be careful, since X is a Riemannian manifold, but $D_\mu^2(x, r)$ is only defined for Euclidean spaces. We identify $B_{x_0}(32R)$

with $\bar{B}(32R)$ using the exponential map centered at x_0. From now on, we will work on the Euclidean space using this identification.

The main result of this section is the following

Proposition 3.26. *There exists a positive constant $R_0 \leqslant R$ and a constant C with the following property. Let μ be a Radon measure supported in Z. For $x \in \bar{B}(R)$ and $r \leqslant R_0$, one has*

$$D_\mu^2(x, r/8) \leqslant \frac{C}{r^2} \int_{\bar{B}_x(r/8)} (W_{r/4}^{4r}(z) + Cr^2) d\mu(z).$$

First, observe that the function $D_\mu^2(x, r)$ has the following geometric interpretation. Assume $\mu(\bar{B}_x(r)) > 0$, let

$$\bar{z} = \frac{1}{\mu(\bar{B}_x(r))} \int_{\bar{B}_r(x)} z \, d\mu(z),$$

Define a non-negative bilinear form b on $\mathbb{R}^4$ as

$$b(v, w) = \int_{\bar{B}_x(r)} \big((z - \bar{z}) \cdot v\big)\big((z - \bar{z}) \cdot w\big) \, d\mu(z).$$

Let $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_4$ be the eigenvalues of b, then

$$D_\mu^2(x, r) = r^{-4}(\lambda_1 + \lambda_2).$$

Let v_i be an eigenvector with eigenvalue λ_i, a straightforward argument of linear algebra shows that

(3.23) $$\int_{B_x(r)} \big((z - \bar{z}) \cdot v_i\big) z \, d\mu(z) = \lambda_i v_i.$$

The following lemma can be understood as a version of Poincaré inequality for $\mathbb{Z}/2$ harmonic spinors.

Lemma 3.27. *There exist constants $C, R_0 > 0$ with the following property. Let v_1, v_2, v_3 be orthonormal vectors in $\mathbb{R}^4$. Let $x \in \bar{B}(R)$, $r \leqslant R_0$. Assume $Z \cap \bar{B}_x(r/8) \neq$*

$\varnothing$, *then*

$$\int_{\bar{B}_x(5r/4)-\bar{B}_x(3r/4)} \sum_{j=1}^{3} |\nabla_{v_j} u(z)|^2 \, dz \geqslant \frac{H_\phi(x,r)}{Cr}.$$

Proof. Assume such constants do not exist. Then there exists a sequence

$$\{(x_n, r_n, U_n, v_1^{(n)}, v_2^{(n)}, v_3^{(n)})\}_{n \geqslant 1},$$

such that $r_n \leqslant \frac{1}{n}$, the vectors $v_1^{(n)}, v_2^{(n)}, v_3^{(n)}$ are orthonormal in $\mathbb{R}^4$,

$$(3.24) \qquad \int_{\bar{B}_{x_n}(5r_n/4)-\bar{B}_{x_n}(3r_n/4)} \sum_{j=1}^{3} |\nabla_{v_j^{(n)}} u(z)|^2 \, dz \leqslant \frac{H_\phi(x_n, r_n)}{n r_n},$$

and $Z \cap \bar{B}_{x_n}(r_n/8) \neq \varnothing$.

Let $\sigma = (12/11)^2$. Rescale the ball $\bar{B}_{x_n}(5\sigma^2 r_n)$ to $\bar{B}(5\sigma^2)$, and normalize the restriction of U. By assumption (3.2), the pull back metrics g_n are given by matrix-valued functions on $\bar{B}(5\sigma^2)$ with eigenvalues bounded by $1/\sigma^2$ and σ^2. There is a subsequence of the pull backs of $(g_n, A_n, \rho_n, v_1^{(n)}, v_2^{(n)}, v_3^{(n)})$ that converges to some data set $(g, A, \rho, v_1, v_2, v_3)$ in C^∞, and since $r_n \to 0$, the limit data set (g, A, ρ) is invariant under translations. By corollary 3.20, after taking a subsequence, the rescaled U_n converges to a $\mathbb{Z}/2$ harmonic spinor U^* on $\bar{B}(2)$ with respect to (g, A, ρ), which satisfies assumption 3.2.

The assumption that $Z \cap \bar{B}_{x_n}(r_n/8) \neq \varnothing$ implies that U^* has at least one zero point in $\bar{B}(1/8)$. Inequality (3.24) gives

$$\int_{\bar{B}(5/4)-\bar{B}(3/4)} \sum_{j=1}^{3} |\nabla_{v_j} u^*(z)|^2 \, dz = 0$$

Theorem 3.3 implies that U^* is not identically zero on $\bar{B}(5/4) - \bar{B}(3/4)$. Since U^* solves the Dirac equation on non-zero points, the unique continuation property implies that $|U|$ is constant in 3 linearly independent directions in $\bar{B}(5/4) - \bar{B}(3/4)$, hence theorem 3.3 implies that U is everywhere non-zero in $\bar{B}(5/4)$, and that is a contradiction. $\qquad \square$

Now one can give the proof of proposition 3.26. The proof is adapted from the proof of proposition 5.3 in [10].

Proof of proposition 3.26. Let R_0 be given by lemma 3.27, and assume $r \leqslant R_0$. Without loss of generality, assume that $D_\mu^2(x, r/8) > 0$. In particular, $\mu(\bar{B}_x(r/8)) > 0$, thus $Z \cap \bar{B}_x(r/8) \neq \varnothing$. Let

$$\bar{z} = \frac{1}{\mu(\bar{B}_x(r/8))} \int_{\bar{B}_x(r/8)} z \, d\mu(z).$$

Let $0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_4$ be the corresponding eigenvalues, then $D_\mu^2(x, r/8) > 0$ implies $\lambda_2 > 0$. Let v_i be the unit eigenvector with eigenvalue λ_i. Let $\operatorname{grad} u(z)$ be the vector in $T_z \mathbb{R}^4 \otimes \mathcal{V}$, such that for every $v \in T_z \mathbb{R}^4$,

$$\langle v, \operatorname{grad} u(z) \rangle_{\mathbb{R}^4} = \nabla_v u(z).$$

By (3.2), $\|\operatorname{grad} u(z)\|_{\mathbb{R}^4} \leqslant (\frac{12}{11}) \|\nabla u\|_X$. Equation (3.23) gives

$$-\lambda_i v_i \cdot \operatorname{grad} u(y) = \int_{\bar{B}_x(r/8)} \big((z - \bar{z}) \cdot v_i\big)\big((y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big) d\mu(z)$$

for any constant α. By Cauchy's inequality

$$\lambda_i^2 |v_i \cdot \operatorname{grad} u(y)|^2$$

$$\leqslant \int_{\bar{B}_x(r/8)} \big|(z - \bar{z}) \cdot v_i\big|^2 d\mu(z) \int_{\bar{B}_x(r/8)} \big|(y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big|^2 d\mu(z)$$

$$= \lambda_i \int_{\bar{B}_x(r/8)} \big|(y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big|^2 d\mu(y)$$

Therefore, when $\lambda_i \neq 0$,

$$\lambda_i |v_i \cdot \operatorname{grad} u(y)|^2 \leqslant \int_{\bar{B}_x(r/8)} \big|(y - z) \cdot \operatorname{grad} u(y) - \alpha u(y)\big|^2 d\mu(z).$$

Integrate with respect to y on $\bar{B}_x(5r/4) - \bar{B}_x(3r/4)$, and sum up $i = 2, 3, 4,$

$$\int_{\bar{B}_x(5r/4) - \bar{B}_x(3r/4)} \sum_{i=2}^{4} \lambda_i |v_i \cdot \operatorname{grad} u(y)|^2 \, dy$$

$$\leqslant 3 \int_{y \in \bar{B}_x(5r/4) - \bar{B}_x(3r/4)} \int_{z \in \bar{B}_x(r/8)} \left| (y - z) \cdot \operatorname{grad} u(y) - \alpha u(y) \right|^2 d\mu(z) dy$$

$$(3.25) \qquad \leqslant 3 \int_{z \in \bar{B}_x(r/8)} \int_{y \in \bar{B}_z(11r/8) - \bar{B}_z(5r/8)} \left| (y - z) \cdot \operatorname{grad} u(y) - \alpha u(y) \right|^2 dy d\mu(z).$$

On the other hand,

$$r^2 D_\mu^2(x, r) \sum_{i=2}^{4} |v_j \cdot \operatorname{grad} u(y)|^2 = r^{-2}(\lambda_1 + \lambda_2) \sum_{i=2}^{4} |v_j \cdot \operatorname{grad} u(y)|^2$$

$$\leqslant \frac{2}{r^2} \sum_{i=2}^{4} \lambda_i |v_j \cdot \operatorname{grad} u(y)|^2$$

Therefore

$$r^2 D_\mu^2(x, r) \int_{\bar{B}_x(5r/4) - \bar{B}_x(3r/4)} \sum_{i=2}^{4} |v_j \cdot \operatorname{grad} u(y)|^2 \, dy$$

$$\leqslant \frac{2}{r^2} \int_{\bar{B}_x(5r/4) - \bar{B}_x(3r/4)} \sum_{i=2}^{4} \lambda_i |v_j \cdot \operatorname{grad} u(y)|^2 \, dy$$

By lemma 3.27, this implies

$$r^2 H_\phi(x, r) D_\mu^2(x, r) \leqslant \frac{C_1}{r} \int_{\bar{B}_x(5r/4) - \bar{B}_x(3r/4)} \sum_{i=2}^{4} \lambda_i |v_j \cdot \operatorname{grad} u(y)|^2 \, dy$$

Therefore inequality (3.25) gives

$$(3.26) \quad r^2 H_\phi(x, r) D_\mu^2(x, r)$$

$$\leqslant \frac{3C_1}{r} \int_{\bar{B}_x(r/8)} \underbrace{\int_{\bar{B}_z(11r/8) - \bar{B}_z(5r/8)} \left| (y - z) \cdot \operatorname{grad} u(y) - \alpha u(y) \right|^2 dy}_{=:A(z,r)} \, d\mu(z).$$

where the constant C_1 is independent of α.

Notice that

$$
A(z,r) \leqslant 3 \bigg(\underbrace{\int_{\bar{B}_z(11r/8) - \bar{B}_z(5r/8)} \big| \eta_z(y) \cdot \operatorname{grad} u(y) - N_\phi(z, d(z,y)) u(y) \big|^2 \, dy}_{=:A_1(z,r)}
$$

$$
+ \underbrace{\int_{\bar{B}_z(11r/8) - \bar{B}_z(5r/8)} |(y-z) - \eta_z(y)|^2 |\operatorname{grad} u(y)|^2 dy}_{=:A_2(z,r)}
$$

$$
+ \underbrace{\int_{\bar{B}_z(11r/8) - \bar{B}_z(5r/8)} \big(N_\phi(z, d(z,y)) - \alpha \big)^2 |u(y)|^2 dy}_{=:A_3(z,r)} \bigg)
$$

Notice that by (3.2), we have $\bar{B}_z(11r/8) - \bar{B}_z(5r/8) \subset B_z(3r/2) - B_z(r/2)$. Therefore, by lemma 3.22,

$$
A_1(z,r) \leqslant C_2 r H_\phi(z,r)(W^r_{r/4}(z) + C_2 r^2).
$$

By lemma 3.23 and lemma 3.14,

$$
A_2(z,r) = O\bigg(r^4 \int_{B_z(3r/2)} |\nabla u|^2\bigg) = O(r^3 H_\phi(x,r)).
$$

To bound $A_3(z,r)$, first break it into two parts

$$
A_3(z,r) \leqslant C_3 \underbrace{\int_{B_z(3r/2) - B_z(r/2)} \big(N_\phi(z, d(z,y)) - N_\phi(z,r) \big)^2 |u(y)|^2 dy}_{=:A_4(z,r)}
$$

$$
+ C_4 \underbrace{\int_{B_z(3r/8) - B_z(r/2)} \big(N_\phi(z,r) - \alpha \big)^2 |u(y)|^2 dy}_{=:A_5(z,r)}
$$

Here the balls $B_z(3r/2)$ and $B_z(r/2)$ are the geodesic balls on X, and the measure dy is the volume form of X. The monotonicity of N_ϕ implies that

$$
A_4(z,r) \leqslant (W^{4r}_{r/4}(z) + C_5 r^2) \int_{B_z(3r/2)} |u(y)|^2 dy
$$

$$
\leqslant C_6 r H_\phi(x,r)(W^{4r}_{r/4}(z) + C_5 r^2).
$$

Now take $p \in B_x(r/8)$, such that

$$|W_{r/4}^{4r}(p)| = \inf_{q \in B_x(r/8)} |W_{r/4}^{4r}(q)|,$$

and take $\alpha = N_\phi(p, r)$. Then by lemma 3.24, for $z \in B_x(r/8)$,

$$A_5(z, r) \leqslant \int_{B_z(3r/2) - B_z(r/2)} \left(C_7 \left(\sqrt{|W_{r/4}^{4r}(z)|} + \sqrt{|W_{r/4}^{4r}(p)|} + r \right) \right)^2 |u(y)|^2 dy$$

$$\leqslant C_8 \left(W_{r/4}^{4r}(z) + C_8 r^2 \right) \int_{B_z(3r/2) - B_z(r/2)} |u(y)|^2 dy$$

$$\leqslant C_9 r H_\phi(x, r) \left(W_{r/4}^{4r}(z) + C_8 r^2 \right)$$

In conclusion,

$$A(z, r) \leqslant C_{10} r H_\phi(x, r) \left(W_{r/4}^{4r}(z) + C_{11} r^2 \right).$$

Therefore proposition 3.26 follows from inequality (3.26). $\qquad\square$

3.8. **Approximate spines.**

Definition 3.28. Given a set of points $\{p_i\}_{i=0}^k \subset \mathbb{R}^4$ and a number $\beta > 0$, one says that $\{p_i\}_{i=0}^k$ is β-linearly independent, if for every $j \in \{0, 1, \cdots, k\}$, the distance between p_j and the affine subspace spanned by $\{p_i\}_{i=0}^k \backslash \{p_j\}$ is at least β.

Given a set $F \subset \mathbb{R}^4$, one says that F β-spans a k-dimensional affine subspace, if there exit $(k + 1)$ points in F that are β-linearly independent.

Lemma 3.29. *If F is a bounded set that does not β-span a k-dimensional affine space, then there exists a $(k - 1)$-dimensional affine space V, such that F is contained in the 2β-neighborhood of V.*

Proof. For k points $\{q_1, \cdots, q_k\}$ in $\mathbb{R}^4$, let $V(q_1, \cdots, q_k)$ be the volume of the $(k - 1)$ dimensional simplex spanned by these points. Let $\{p_1, \cdots, p_k\} \subset F$ be k points in F

such that

$$(3.27) \qquad V(p_1, \cdots, p_k) \geqslant \frac{1}{2} \sup_{q_1, \cdots, q_k \in F} V(q_1, \cdots, q_k).$$

If the volume $V(p_1, \cdots, p_k)$ is zero, then F is contained in a $(k-1)$-dimensional affine subspace, and the statement is trivial. If the volume is positive, then the set $\{p_1, \cdots, p_k\}$ spans a $k-1$ dimensional affine space V. If F is contained in the 2β neighborhood of V, then the statement is verified. Otherwise, there exists a point $p_{k+1} \in F$, such that the distance of p_{k+1} and V is greater than 2β. Let d_j be the distance between p_j and the affine subspace spanned by $\{p_i\}_{i=0}^{k+1} \setminus \{p_j\}$, then $d_{k+1} \geqslant 2\beta$. By (3.27), $2d_j \geqslant d_{k+1}$ for every j. Therefore $\{p_1, \cdots, p_{k+1}\}$ is β-linearly independent, and that contradicts the assumption on F. $\qquad\square$

As in section 3.7, use the normal coordinate centered at x_0 to identify $B_{x_0}(32R)$ with the ball $\bar{B}(32R)$ in $\mathbb{R}^4$. Recall that by assumption (3.2),

$$\left(\frac{11}{12}\right)^2 \leqslant \kappa_{x_0}(z) \leqslant K_{x_0}(z) \leqslant \left(\frac{12}{11}\right)^2,$$

where $\kappa_{x_0}(z)$ and $K_{x_0}(z)$ are the upper and lower bound of the eigenvalues of the metric matrix at $z \in \bar{B}_x(32R)$.

The compactness property of $\mathbb{Z}/2$ harmonic spinors leads to the following lemma.

Lemma 3.30. *Let $\beta, \bar{\beta}, \tilde{\beta} \in (0, 1)$ be given. Then there exits $\delta > 0$, depending on $\beta, \bar{\beta}$, the upper bound Λ of the frequency function, the value of R, the curvatures of X and $\mathcal{V}$, and the constant ϵ in assumption 3.2, such that the following holds. If $x \in \bar{B}(R)$, $r \leqslant \delta$, and $\{p_1, p_2, p_3\}$ is a set of $\bar{\beta}r$-linearly independent points in $\bar{B}_x(r)$, such that*

$$N_\phi(p_i, 2r) - N_\phi(p_i, \tilde{\beta}r) < \delta \quad i = 1, 2, 3.$$

Let V be the affine space spanned by p_1, p_2, p_3. Then the set $Z \cap \bar{B}_x(r)$ is contained in the βr neighborhood of $V \cap \bar{B}_x(r)$.

Proof. Assume such δ does not exist. Then there exist sequences $\{p_i^{(n)}\}_{i=1}^3$, x_n, and r_n, such that $r_n \to 0$, the points $\{p_i^{(n)}\}_{i=1}^3$ are contained in $\bar{B}_{x_n}(r_n)$ and are $\bar{\beta}r_n$-linearly independent, and

$$N_\phi(p_i^{(n)}, 2r_n) - N_\phi(p_i^{(n)}, \tilde{\beta}r_n) < \frac{1}{n} \quad i = 1, 2, 3,$$

and there exists $y_n \in Z$ such that the distance from y_n to the affine space spanned by $\{p_i^{(n)}\}_{i=1}^3$ is greater than βr_n.

Let $\sigma = 12/11$. Rescale the balls $\bar{B}_{x_n}(10\sigma^2 r_n)$ to radius $10\sigma^2$, and normalize the section U. Corollary 3.20 then gives a limit section U^* which satisfies the following properties:

(1) U^* is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(4)$, with respect to a translation-invariant metric, the trivial connection on $\mathcal{V}$, and a translation invariant Clifford multiplication. U^* satisfies assumption 3.2.

(2) There exist points $p_1^*, p_2^*, p_3^* \in \bar{B}(1)$, such that they are $\bar{\beta}$-linearly independent, and

(3.28) $$N_\phi(p_i^*, 2) - N_\phi(p_i^*, \tilde{\beta}) = 0 \quad i = 1, 2, 3,$$

(3) Let V^* be the affine space spanned by $\{p_i^*\}_{i=1}^3$. There exits a point $q \in \bar{B}(1)$ in the zero set of U^*, such that the distance from q to $V^* \cap \bar{B}(1)$ is at least β.

Since U^* is defined on a flat manifold with flat bundle, remark 3.17 indicates that for U^*,

$$\partial_r N_\phi(x, r) = \frac{2}{rH_\phi(x, r)} \int -\phi'\left(\frac{d(x, y)}{r}\right) d(x, y)^{-1} |\nabla_{\eta_x} u(y) - N_\phi(x, r)u(y)|^2 \, dy.$$

Therefore equation (3.28) implies that for $i \in \{1, 2, 3\}$, the section U^* is homogeneous on $\bar{B}_{p_i^*}(2) - \bar{B}_{p_i^*}(\tilde{\beta})$ with respect to the center p_i^*. The unique continuation property for solutions to the Dirac equation implies that U^* is homogeneous on $\bar{B}(2)$ with

respect to p_i^*. An elementary argument (see for example [10, Lemma 6.8]) then shows that the section U^* is zero on the affine space V^*, and that U^* is invariant in the directions parallel to V^*. Therefore, property (3) of U^* implies that U^* is zero on a 3-dimensional affine subspace, which contradicts theorem 3.3. $\qquad\square$

Similarly, one has

Lemma 3.31. *Let* $\beta, \bar{\beta}, \tilde{\beta} \in (0,1)$ *and* $\tau > 0$ *be given. Then there exits* $\delta > 0$, *depending on* $\beta, \bar{\beta}, \tilde{\beta}, \tau$, *the upper bound* Λ *of the frequency function, the value of* R, *the curvatures of* X *and* $\mathcal{V}$, *and the constant* ϵ *in assumption 3.2, such that the following holds. Assume* $x \in \bar{B}(R)$, *and* $r \leqslant \delta$, *and* $\{p_1, p_2, p_3\}$ *is a set of points in* $\bar{B}_x(r)$ *that is* $\bar{\beta}r$-*linearly independent, such that*

$$N_\phi(p_i, 2r) - N_\phi(p_i, \tilde{\beta}r) < \delta \quad i = 1, 2, 3.$$

Let V *be the affine space spanned by* $\{p_i\}$. *Then for all* $y, y' \in \bar{B}_x(r) \cap Z$, *one has*

$$|N_\phi(y, \beta r) - N_\phi(y', \beta r)| < \tau.$$

Proof. Assume such δ does not exist, then arguing as before, one obtains a 2-valued section U^* on $\bar{B}(4)$ with the following properties:

(1) U^* is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(4)$, with respect to a translation-invariant metric, the trivial connection on $\mathcal{V}$, and a translation invariant Clifford multiplication. U^* satisfies assumption 3.2.

(2) There exist points $p_1^*, p_2^*, p_3^* \in \bar{B}(1)$, such that they are $\bar{\beta}$-linearly independent, and

(3.29) $$N_\phi(p_i^*, 2) - N_\phi(p_i^*, \tilde{\beta}) = 0 \quad i = 1, 2, 3,$$

(3) Let Z^* be the zero set of U^*. There exist $y, y' \in \bar{B}(1) \cap Z^*$, such that

$$|N_\phi(y, \beta) - N_\phi(y', \beta)| \geqslant \tau.$$

However, as in the proof of the previous lemma, the first two properties imply that U^* is invariant in the directions parallel to the plane V^* spanned by p_1^*, p_2^*, p_3^*, and $Z^* \subset V^*$, which contradicts property (3). $\qquad\square$

3.9. **Rectifiability and the Minkowski bound.** This section only concerns estimates on the Euclidean space. To simplify notations, for the rest of this section, use $B_x(r)$ and $B(r)$ to denote the Euclidean balls.

Definition 3.32. Let $\mathcal{Z}$ be a Borel subset of $\bar{B}(R) \subset \mathbb{R}^4$. A function $\mathcal{I}(x, r)$ defined for $x \in \mathcal{Z}$ and $r \leqslant 128R$ is called a taming function for $\mathcal{Z}$, if the following conditions hold.

(1) $\mathcal{I}(x, r)$ is non-negative, bounded, continuous, and is non-decreasing in r.

(2) Let $\beta, \bar{\beta} \in (0, 1)$ and $\tau > 0$ be given. Then there exists $\epsilon(\beta, \bar{\beta}, \tau) > 0$, depending on $\beta, \bar{\beta}, \tau$, such that the following holds. Assume $x \in \bar{B}(R)$, $r \leqslant R$, and $\{p_1, p_2, p_3\}$ is a set of points in $\bar{B}_x(r)$ that is $\bar{\beta}r$-linearly independent, such that

$$\mathcal{I}(p_i, 2r) - \mathcal{I}(p_i, \beta r/2) < \delta \quad i = 1, 2, \cdots, m - 1.$$

Then for all $y, y' \in \bar{B}_x(r) \cap \mathcal{Z}$, one has

$$|\mathcal{I}(y, \beta r/2) - \mathcal{I}(y', \beta r/2)| < \tau.$$

(3) There exists a constant C, such that for every Radon measure μ supported in $\mathcal{Z}$, the following inequality holds for every $x \in \bar{B}(2R)$ and $r \leqslant 2R$:

$$D_\mu^2(x, r) \leqslant \frac{C}{r^2} \int_{\bar{B}_x(r)} [\mathcal{I}(z, 32r) - \mathcal{I}(z, 2r)] \, d\mu(z).$$

Theorem 3.33 ([10])**.** *Assume $\mathcal{Z}$ is a Borel subset of $B(R)$ and there exists a taming function $\mathcal{I}(x, r)$ for $\mathcal{Z}$. Then the set $\mathcal{Z} \cap B(R/2)$ is 2-rectifiable and has finite 2-dimensional Minkowski content.*

The proof of theorem 3.33 follows almost verbatim from the arguments in sections 7 and 8 of [10]. Nevertheless, a proof is given here for the reader's convenience.

The proof of theorem 3.33 makes use of two Reifenberg-type theorems. We state the special cases of the theorems for dimension 4 and codimension 2.

Theorem 3.34 ([28], Theorem 3.4)**.** *There exist universal constants $K_0 > 0$ and $\delta_0 > 0$ such that the following holds. Assume $\{B_{x_i}(r_i)\}$ is a collection of balls in $B(2R)$, such that $\{B_{x_i}(r_i/4)\}$ are disjoint. Define a measure $\mu = \sum_i r_i^2 \delta_{x_i}$. Suppose*

$$\int_{B_x(r)} \int_0^r \frac{D_\mu^2(z, s)}{s} \, ds d\mu(z) < \delta_0 r^2$$

for every $B_x(r) \subset B(2R)$, then $\mu(B(R)) \leqslant K_0 R^2$.

Theorem 3.35 ([1], Corollary 1.3)**.** *Assume $S \subset B(R)$ is a $\mathcal{H}_2$-measurable set and has finite Hausdorff measure, let λ be the restriction of $\mathcal{H}_2$ to S. Assume that for λ-a.e. z,*

$$\int_0^R \frac{D_\lambda^2(z, s)}{s} \, ds < +\infty,$$

then S is 2-rectifiable.

Proof of theorem 3.33. Assume $B_x(r) \subset B(R)$. If one rescales $B_x(r)$ to $B(R)$, then the function $\mathcal{I}'(y, s) = \mathcal{I}(x + (ry)/R, sr/R)$ is a taming function for $[(A - x) \cdot (R/r)] \cap B(R)$ with the same function $\epsilon(\beta, \bar{\beta}, \tau)$ and constant C. Therefore definition 3.32 is invariant under rescaling, thus one only needs to consider the case for $R = 2$.

Let $\beta = 1/10$. Let $\bar{\beta} \leqslant 1/100$ be a positive universal constant, let $\tau > 0$ be a constant that is defined by $\bar{\beta}$ and C, and let $\delta > 0$ be a constant that is defined

by $\bar{\beta}, \tau$, the function ϵ and the constant C. The exact values for $\bar{\beta}, \tau$ and δ will be determined later in the proof.

Let Λ be an upper bound of $\mathcal{I}$, namely $\Lambda \geqslant \sup_{x \in A, x \leqslant 128R} \mathcal{I}(x, r) = \sup_{x \in A} \mathcal{I}(x, 256)$.

Define

$$D_\delta(r) = B(R/2) \cap \{x \in \mathcal{Z} | \mathcal{I}(x, \beta r/2) \geqslant \Lambda - \delta\}.$$

Define

$$W_{r_1}^{r_2}(x) = \mathcal{I}(x, r_1) - \mathcal{I}(x, r_2).$$

If $\{B_{x_i}(r_i)\}$ is a family of balls, we call the sum $\sum_i r_i^2$ its 2-dimensional volume.

Step 1. First, require that $\delta < \epsilon(\beta, \bar{\beta}, \tau)$. For $B_x(r) \subset B(2)$, and a set $A \subset \mathcal{Z} \cap B_x(r)$, define an operator $\mathcal{F}_A$, which turns $B_x(r)$ into a finite set of balls. It has the property that either $\mathcal{F}_A(B_x(r)) = \{B_x(r)\}$, or $\mathcal{F}_A(B_x(r))$ is a family of balls with radius βr. In either case, the balls in the family $\mathcal{F}(B_x(r))$ will cover the set A. The operator $\mathcal{F}_A$ is defined as follows. If $A \cap D_\delta(r)$ does not $\bar{\beta}r$-span a 2-dimensional affine space, then it is called "bad". Otherwise, it is called "good". In the bad case, define $\mathcal{F}_A(B_x(r)) = \{B_x(r)\}$. In the good case, cover A by a family of balls $\{B_{x_i}(\beta r)\}$ with the following properties

(1) The distance between x_i and x_j is at least $\beta r/2$ for $\forall i \neq j$,

(2) Each x_i is an element of A.

Define $\mathcal{F}_A(B_x(r))$ to be the family $\{B_{x_i}(\beta r)\}$.

Obviouly the descriptions above do not uniquely specify the operator $\mathcal{F}_A$. When there are more than one possibilities, choose one arbitrarily.

If $B_x(r)$ is a good ball, let $p_1, p_2, p_3 \in D_\delta(r) \cap B_x(r)$ be three points that $\bar{\beta}r$ span a plane, let $\mathcal{F}(B_x(r)) = \{B_{x_i}(\beta r)\}$. By condition (2) of definition 3.32,

$$|\mathcal{I}(x_i, \beta r/2) - \mathcal{I}(p_i, \beta r/2)| \leqslant \tau.$$

Therefore

(3.30) $$\mathcal{I}(x_i, \beta r/2) \geqslant \Lambda - \delta - \tau$$

The operator $\mathcal{F}_A$ can be extended to act on a collection of balls. Assume $\{B_{x_i}(r)\}_{i=1}^n$ is a collection of balls with the same radius. Let $A \subset \bigcup B_{x_i}(r) \cap \mathcal{Z}$. Assume $\{B_{x_i}(r)\}_{i=1}^k$ are the good balls, and $\{B_{x_i}(r)\}_{i=k+1}^n$ are the bad balls. Then there exists a collection of balls $\{B_{y_j}(\beta r)\}$, such that

(1) $\{B_{y_j}(\beta r)\}$ covers $\bigcup_{i=1}^k (A \cap B_{x_i}(r))$.

(2) $|y_j - y_l| \geqslant \beta r/2$, for $\forall j \neq l$.

(3) $y_j \in \bigcup_{i=1}^k A \cap B_{x_i}(r)$, for $\forall j$.

Inequality (3.30) still holds when x_i is replaced by y_j. Define $\mathcal{F}_A\{B_{x_i}(r)\}$ to be the union of $\{B_{y_j}(\beta r)\}$ and $\{B_{x_i}(r)\}_{i=k+1}^n$.

Step 2. Let $N > 0$ be a positive integer. Let $A_0(x,r) = \mathcal{Z} \cap B_x(r)$. Apply the operator $\mathcal{F}_{A_0}$ to $B_x(r)$ to obain a set of balls, which we denote by $\mathcal{S}_1(x,r)$. Assume $\mathcal{S}_1(x,r)$ splits to two sets $\mathcal{S}_1(x,r) = \mathcal{S}_{1,g}(x,r) \bigcup \mathcal{S}_{1,b}(x,r)$, where $\mathcal{S}_{1,g}(x,r)$ is the collection of good balls and $\mathcal{S}_{1,b}(x,r)$ is the collection of bad balls. Let

$$A_1(x,r) = A_0(x,r) - \bigcup_{B_{x_i}(r_i) \in \mathcal{S}_{1,b}(x,r)} B_{x_i}(r_i).$$

Apply $\mathcal{F}_{A_1(x,r)}$ to $\mathcal{S}_{1,g}(x,r)$ and obtain a new set of balls

$$\mathcal{S}_2(x,r) = \mathcal{F}_{A_1(x,r)}(\mathcal{S}_{1,g}(x,r)) \bigcup \mathcal{S}_{1,b}(x,r).$$

Similarly, write $\mathcal{S}_2(x,r) = \mathcal{S}_{2,g}(x,r) \bigcup \mathcal{S}_{2,b}(x,r)$, and define

$$A_2(x,r) = A_1(x,r) - \bigcup_{B_{x_i}(r_i) \in \mathcal{S}_{2,b}(x,r)} B_{x_i}(r_i),$$

and define $\mathcal{S}_3 = \mathcal{F}_{A_2}(\mathcal{S}_{2,g}) \bigcup \mathcal{S}_{2,b}$. Repeat the procedure N times to obtain a set of balls $\mathcal{S}_N(x, r)$.

The family $\mathcal{S}_N(x, r)$ has the following property. If $B_{x_1}(r_1)$ and $B_{x_2}(r_2)$ are two distinct elements of $\mathcal{S}_N(x, r)$, then

$$(3.31) \qquad |x_1 - x_2| \geqslant (r_1 + r_2)/4.$$

Inequality (3.31) can be proved by induction. For $N = 1$, it follows from the definition of $\mathcal{F}_A$. Assume (3.31) holds for $N - 1$, and write $\mathcal{S}_N = \mathcal{F}_{A_{N-1}}(\mathcal{S}_{N-1,g}) \bigcup \mathcal{S}_{N-1,b}$. Let $B_{x_1}(r_1), B_{x_2}(r_2) \in \mathcal{S}_N$. If both $B_{x_1}(r_1), B_{x_2}(r_2) \in \mathcal{F}_{A_{N-1}}(\mathcal{S}_{N-1,g})$, then (3.31) follows from the definition of $\mathcal{F}$. If both $B_{x_1}(r_1), B_{x_2}(r_2) \in \mathcal{S}_{N-1,b}$, then (3.31) follows from the induction hypothesis. If $B_{x_1}(r_1) \in \mathcal{F}_{A_{N-1}}(\mathcal{S}_{N-1,g})$, $B_{x_2}(r_2) \in \mathcal{S}_{N-1,b}$, then $x_1 \notin B_{x_2}(r_2)$. By the construction of $\mathcal{F}$, one has $r_1 \leqslant \beta r_2$. Since $\beta = 1/10$, one has $|x_1 - x_2| \geqslant r_2 \geqslant (r_1 + r_2)/2$.

By (3.30), either $\mathcal{S}_N = \{B_x(r)\}$, or

$$(3.32) \qquad \mathcal{I}(x_i, r_i/2) \geqslant \Lambda - \delta - \tau, \quad \forall B_{x_i}(r_i) \in \mathcal{S}_N.$$

Step 3. We claim that there exists a universal constant $K_1 > 1$, such that for τ and δ sufficiently small, we have

$$(3.33) \qquad \sum_{B_{x_i}(r_i) \in \mathcal{S}_N(x, r)} r_i^2 < K_1 r^2.$$

Without loss of generality, assume $\mathcal{S}_N(x, r) \neq \{B_x(r)\}$. Let $r_j = \beta^{N-j} r$. Define Radon measures

$$\mu = \sum_{B_y(s) \in \mathcal{S}_N(x, r)} s^2 \delta_y,$$

$$\mu_j = \sum_{B_y(s) \in \mathcal{S}_N(x, r), s \leqslant r_j} s^2 \delta_y.$$

Notice that by (3.31), there exists a universal constant K_2 such that

(3.34) $$\mu_0((B_x(r_0)) \leqslant K_2 \, r_0^2, \quad \forall x.$$

Let K_0 be the constant given by theorem 3.34, let $K_3 = \max\{K_0, K_2\}$. We prove that if τ, δ is chosen sufficiently small, then for every $j = 0, 1, \cdots, N-3$, and every $B_y(r_j) \subset B_x(2r)$, one has

(3.35) $$\mu_j(B_y(r_j)) \leqslant K_3 \, r_j^2.$$

The claim is proved by induction on j. The case for $j = 0$ follows from (3.34). Assume that the claim is proved for $0, 1, \cdots, j$, and $j < N-3$. Then there exists a universal constant $M > 1$, such that for every $y \in B_x(2r)$, $k \leqslant j+1$, and $s \in [r_k/2, 2r_k]$,

(3.36) $$\mu_{k+3}((B_y(s)) \leqslant M \, (K_3 + 1) \, s^2$$

We want to use theorem 3.34 and (3.36) to prove

$$\mu_{j+1}((B_y(r_{j+1})) \leqslant K_3 \, r_{j+1}^2, \text{ for } \forall B_y(r_{j+1}) \subset B_x(2r).$$

If $\mu_{j+1}(B_y(r_{j+1})) = 0$, the inequality is trivial. From now on assume $\mu(B_y(r_{j+1})) > 0$. Since $r_{j+1} \leqslant r_{N-3} = r/8$, and $\operatorname{supp} \mu \subset B_x(r)$, we have $B_y(4r_{j+1}) \subset B_x(2r)$.

Notice that for $B_{x_i}(s_i) \in \mathcal{S}_N$, if $t < \min_{k} |x_i - x_k|$, then

$$D_\mu^2(x_i, t) = 0.$$

Define

$$\overline{W}_{2t}^{32t}(x_i) = \begin{cases} 0 & \text{if } t < s_i/4, \\[2mm] W_{2t}^{32t}(x_i) & \text{if } t \geqslant s_i/4. \end{cases}$$

Inequality (3.31) and condition (3) of definition 3.32 gives

$$(3.37) \qquad D_\mu^2(q,t) \leqslant C \int_{B_q(t)} \frac{\overline{W}_{2t}^{32t}(p)}{t^3} \, d\mu(p)$$

for every (q,t).

For $B_z(s) \subset B_y(2r_{j+1})$, assume $s \in [r_k/2, 2r_k]$ for $k \leqslant j+1$. Inequality (3.37) gives

$$\int_{B_z(s)} \int_0^s \frac{D_{\mu_{j+1}}^2(q,t)}{t} \, dt \, d\mu_{j+1}(q)$$

$$\leqslant C \int_{B_z(s)} \int_0^s \int_{B_q(t)} \frac{\overline{W}_{2t}^{32t}(p)}{t^3} \, d\mu_{j+1}(p) \, dt \, d\mu_{j+1}(q)$$

$$(3.38) \qquad \leqslant C \int_{B_z(s)} \int_0^s \int_{B_q(t)} \frac{\overline{W}_{2t}^{32t}(p)}{t^3} \, d\mu_{k+3}(p) \, dt \, d\mu_{k+3}(q)$$

$$\leqslant C \int_{B_z(2s)} \int_0^s \int_{B_p(t)} \frac{\overline{W}_{2t}^{32t}(p)}{t^3} \, d\mu_{k+3}(q) \, ds \, d\mu_{k+3}(p)$$

$$(3.39) \qquad \leqslant CM(K_3+1) \int_{B_z(2s)} \int_0^s \frac{\overline{W}_{2t}^{32t}(p)}{t} \, dt \, d\mu_{k+3}(p),$$

where inequality (3.38) follows from (3.31). For $p \in \operatorname{supp} \mu_{j+1}$, let s_p be the radius of ball in $\mathcal{S}_N$ with center p. If $s \geqslant s_p/4$, then

$$\int_0^s \frac{\overline{W}_{2t}^{32t}(p)}{t} \, dt = \int_{s_p/4}^s \frac{W_{2t}^{32t}(p)}{t} \, dt = \int_{2s}^{32s} \mathcal{I}(p,t) \, dt - \int_{s_p/a}^{16s_p} \mathcal{I}(p,t) \, dt$$

$$(3.40) \qquad \leqslant W_{s_p/2}^{32s}(p) \int_2^{32} \frac{1}{t} \, dt \leqslant \ln(16)\,(\delta + \tau).$$

The last inequality above follows from (3.32). Therefore, the right hand side of (3.39) is bounded by

$$CM(K_3+1) \int_{B_z(2s)} \int_0^s \frac{\overline{W}_{2t}^{32t}(p)}{t} \, dt \, d\mu_{k+3}(p)$$

$$\leqslant CM(K_3+1)\, \mu_{k+3}(B_z(2s)) \ln(16)\,(\tau + \delta) \leqslant 4CM^2(K_3+1)^2 \ln(16)(\tau+\delta)\, s^2$$

Let δ_0 be the constant given by theorem 3.34. Take

$$\tau < \frac{\delta_0}{8CM^2(K_3+1)^2 \ln(16)},$$

and

$$\delta < \frac{\delta_0}{8CM^2(K_3+1)^2 \ln(16)},$$

then the conditions of theorem 3.34 are satisfied, therefore $\mu_{j+1}((B_y(r_{j+1})) \leqslant K_0\, r_{j+1}^2$. By induction, (3.35) is proved. Inequality (3.33) then follows from (3.35) by the the case of $j = N - 3$.

Step 4. By lemma 3.29, the result obtained from the previous steps can be summarized as follows. For any integer $N > 0$, and any ball $B_x(r)$, there is a covering of $\mathcal{Z} \cap B_x(r)$ by a family of balls $\mathcal{S}_N(x,r) = \{B_{x_i}(r_i)\}_i$, such that the following properties hold:

(1) The radius of each ball is at least $\beta^N r$.

(2) For a all $B_{x_i}(r_i) \in \mathcal{S}_N$, either $r_i = \beta^N r$, or $r_i = \beta^j r$ for some integer $j < N$, and $B_{x_i}(r_i) \cap D_\delta(r_i)$ is contained in the $2\bar{\beta}r_i$ neighborhood of a line.

(3) $\sum_i r_i^2 \leqslant K_1\, r^2$.

As a consequence,

Lemma 3.36. *There exists a universal constant $K_1 > 1$, and a constant δ, such that the following property holds. For any $B_x(r) \subset B(2)$, and $s \in (0,r)$, there exists a covering of $\mathcal{Z} \cap B_x(r)$ by balls $\mathcal{S} = \{B_{x_i}(r_i)\}_i$, such that*

(1) The radius of each ball is at least βs.

(2) For a ball $B_{x_i}(r_i) \in \mathcal{S}$, either $r_i \leqslant s$, or $B_{x_i}(r_i) \cap D_\delta(r_i)$ is contained in the $2\bar{\beta}r_i$ neighborhood of a line.

(3) $\sum_i r_i^2 \leqslant K_1\, r^2$.

Step 5. We prove the following lemma

Lemma 3.37. *There exists a universal constant K_4, and a constant δ, such that the following property holds. For any $B_x(r) \subset B(2)$, and $s \in (0, r)$, there exists a splitting of $\mathcal{Z}$ into $\mathcal{Z} = \bigcup_i \mathcal{E}_i$, and a family of balls $\mathcal{S} = \{B_{x_i}(r_i)\}_i$, such that*

(1) $\mathcal{E}_i \subset B_{x_i}(r_i)$.

(2) *The radius of each ball is at least $4\bar{\beta}s$.*

(3) *For a ball $B_{x_i}(r_i) \in \mathcal{S}$, either $r_i \in [4\bar{\beta}s, s]$, or $B_{x_i}(r_i) \cap D_\delta(r_i) = \varnothing$*

(4) $\sum_i r_i^2 \leqslant K_4 \, r^2$.

Proof of lemma 3.37. Notice that by the assumptions on β and $\bar{\beta}$, we have $4\bar{\beta} < \beta$.

If $\{B_{x_i}(r_i)\}_i$ is a covering of $\mathcal{Z} \cap B_x(r)$ that satisfies the three properties given by lemma 3.36 with respect to s, we say that $\{B_{x_i}(r_i)\}_i$ is an s-admissible covering of $B_x(r) \cap \mathcal{Z}$. Fix $s > 0$, by lemma 3.36, s-admissible coverings of $B_x(r) \cap \mathcal{Z}$ exist.

Let $\{B_{x_i}(r_i)\}$ be an s-admissible covering of $B_x(r) \cap \mathcal{Z}$. Let $\mathcal{E}_i = \mathcal{Z} \cap B_{x_i}(r_i)$. Then the family $\{(\mathcal{E}_i, B_{x_i}(r_i))\}$ satisfies conditions (1), (2) of lemma 3.37, and $\sum_i r_i^2 \leqslant K_1 r^2$. However, it may not satisfy condition (3). In the following, we will give a procedure to adjust the family, such that at each step the covering still satisfies property (2) of s-admissibility, and after finitely many steps of adjustments, the family will satisfy property (3) of lemma 3.37. At the same time, $\sum_i r_i^2$ is being contorlled throughout the adjustments.

Assume $\{B_{x_i}(r_i)\}$ is an s-admissible covering of $B_x(r) \cap \mathcal{Z}$, and $\mathcal{E}_i \subset B_{x_i}(r_i)$, $B_x(r) \cap \mathcal{Z} = \bigcup \mathcal{E}_i$. Assume $(\mathcal{E}_0, B_{x_0}(r_0))$ does not satisfy property (3) of lemma 3.37. Then $r_0 > s$.

By property (2) of s-admissibility, $B_{x_0}(r_0) \cap D_\delta(r_0)$ is contained in the $2\bar{\beta}r_0$ neighborhood of a line. Thus one can cover $B_{x_0}(r_0) \cap D_\delta(r_0)$ by a family of no more than $[10/\bar{\beta}]$ balls with radius $4\bar{\beta}r_0$. Let $\{B_{y_j}(t_j)\}$ be this family. If $4\bar{\beta}r_0 > s$, apply lemma 3.36 again to each ball $B_{y_j}(t_j)$ and replace it with an s-admissible covering of $B_{y_j}(t_j) \cap D_\delta(r_0)$. Otherwise keep the family $\{B_{y_j}(t_j)\}$ as it is. Let $\{B_{z_j}(l_j)\}$ be

the result of this procedure. Then $\{B_{z_j}(l_j)\}$ covers $B_{x_0}(r_0) \cap D_\delta(r_0)$, and it has the following properties

(1) $4\bar{\beta}s \leqslant l_j \leqslant 4\bar{\beta}r_0$ for each j,

(2) $\sum_j l_j^2 \leqslant [10/\bar{\beta}] \cdot K_1 (4\bar{\beta}r_0)^2$.

Take $\bar{\beta} \leqslant 1/(320K_1)$, then $\sum_j l_j^2 \leqslant \frac{1}{2}r_0^2$.

The adjustment of the family $\{(\mathcal{E}_i, B_{x_i}(r_i))\}$ is defined as follows. First, remove $(\mathcal{E}_0, B_{x_0}(r_0))$ from the family, and add $(\mathcal{E}_0 \backslash D_\delta(r_0), B_{x_0}(r_0))$ into the family. Next, add the family $\{(\mathcal{E}_0 \cap B_{z_j}(l_j), B_{z_j}(l_j))\}$ constructed from the previous paragraph into this family.

This adjustment replaces an element $(\mathcal{E}_0, B_{x_0}(r_0))$ which does not satisfy property (3) of lemma 3.37 by a family of balls, such that the biggest ball in this family has the same radius r_0 and satisfies property (3). The rest of the balls have radius in the interval $[4\bar{\beta}s, 4\bar{\beta}r_0]$ and their 2-dimensional volume is bounded by $\frac{1}{2}r_0^2$. Moreover, the new family still satisfies property (2) of lemma 3.36. Therefore, after finitely many times of adjustments, we will obtain a family that satisfies conditions (1), (2), (3), with 2-dimensional volume

$$\sum_i r_i^2 \leqslant 2K_1 r^2,$$

hence the lemma is proved. $\qquad\square$

Step 6. Given $s \in (0,1)$, we use lemma 3.37 to construct a covering of $\mathcal{Z} \cap B(1)$ by a family of balls $\{B_{x_i}(r_i)\}$ with radius $r_i \in [4\bar{\beta}s, s]$, such that the 2-dimensional volume of the covering is bounded.

We call a family $\{(\mathcal{E}_i, B_{x_i}(r_i))\}$ a split-covering of a set A, if $\mathcal{E}_i \subset B_{x_i}(r_i)$, and $A = \bigcup \mathcal{E}_i$.

If a split-covering of $\mathcal{Z} \cap B_x(r)$ satisfies the properties given by lemma 3.37, we say that it is strongly s-admissible.

Let $\mathcal{S}$ be a strongly s-admissible split-covering of $\mathcal{Z} \cap B(1)$. For every $B_{x_i}(r_i) \in \mathcal{S}$, if $r_i \leqslant s$, we say it is of type I. Otherwise, we say it is of type II. Assume $B_{x_i}(r_i)$ is a ball of type II, then the function $\mathcal{I}(x, r)$ is at most $\Lambda - \delta$ for $x \in \mathcal{E}_i$, $r_i \leqslant \beta r_i/2$. There exists a universal constant L such that $\mathcal{E}_i$ can be covered by L balls $B_{y_j}(\beta r_i/512)$ with radius $(\beta r_i/512)$. Therefore, for each ball, the set $\mathcal{E}_i \cap B_{y_j}(\beta r_i/512)$ has a strongly s-admissible split-covering, with Λ replaced by $\Lambda - \delta$.

Change $(B_{x_i}(r_i), \mathcal{E}_i)$ to the union of the L strongly s-admissible split-coverings of $\mathcal{E}_i \cap B_{y_j}(\beta r_i/512)$, we obtain a split-covering of $\mathcal{E}_i$ with 2-dimensional volume at most $LK_4(\beta r_i/512)^2$. Define an operation $\mathcal{G}$ on $\mathcal{S}$, such that $\mathcal{G}(\mathcal{S})$ is constructed from $\mathcal{S}$ by replacing every type II element in $\mathcal{S}$ with the union of the L split-coverings described above.

Notice that for the balls $B_{y_j}(\beta r_i/512)$, the upper bound Λ is replaced by $\Lambda - \delta$. Therefore, this procedure can only be carried for at most $N = \lceil \frac{\Lambda}{\delta} \rceil$ times. After that, every ball in $\mathcal{G}^{(N)}(\mathcal{S})$ is of type I. Namely, every ball in $\mathcal{G}^{(N)}(\mathcal{S})$ has radius in the interval $[4\bar{\beta}s, s]$.

Let V_n be the 2 dimensional volume of $\mathcal{G}^{(n)}(\mathcal{S})$, then we have

$$V_{n+1} \leqslant (1 + LK_4(\beta/512)^2)V_n.$$

Therefore the total 2-dimensional volume of $\mathcal{G}^{(n)}(\mathcal{S})$ is bounded by

$$V_n \leqslant (1 + LK_4(\beta/512)^2)^N K_4.$$

Since s can be taken to be arbitrarily small, the Minkowski content of $\mathcal{Z} \cap B(1)$ is bounded by a contant K depending on Λ, ϵ and C.

By rescaling, we conclude that the Minkowski content of $\mathcal{Z} \cap B_x(r)$ is bounded by $K r^2$. Since the Minkowski content bounds the Hausdorff measure, there exists a

constant K' depending on Λ, ϵ and C, such that

$$(3.41) \qquad \mathcal{H}_2(\mathcal{Z} \cap B_x(r)) \leqslant K' \, r^2.$$

Step 7. Now invoke theorem 3.35. Let λ be the restriction of $\mathcal{H}_2$ to $\mathcal{Z}$. By (3.41),

$$\int_{B(1)} \int_0^1 \frac{D_\lambda^2(z,s)}{s} \, ds \, d\lambda(z) \leqslant C \int_{B(1)} \int_0^1 \int_{B_z(s)} \frac{W_{2s}^{32s}(p)}{s^3} \, d\lambda(p) \, ds \, d\lambda(z)$$

$$\leqslant C \int_{B(2)} \int_0^1 \int_{B_p(s)} \frac{W_{2s}^{32s}(p)}{s^3} \, d\lambda(z) \, ds \, d\lambda(p)$$

$$\leqslant CK' \int_{B(2)} \int_0^1 \frac{W_{2s}^{32s}(p)}{s} \, ds \, d\lambda(p)$$

The same estimate as (3.40) gives

$$\int_0^1 \frac{W_{2s}^{32s}(p)}{s} \, ds \leqslant \ln(16)\Lambda.$$

Thus

$$CK' \int_{B(2)} \int_0^1 \frac{W_{2s}^{32s}(p)}{s} \, ds \, d\lambda(p) \leqslant 4C(K')^2 \ln(16)\Lambda < \infty$$

Therefore, the conditions of theorem 3.35 are satisfied for $\mathcal{Z} \cap B(1)$, hence $\mathcal{Z} \cap B(1)$ is a rectifiable set, and the result is proved. $\qquad\square$

Proof of theorem 3.4. Let R_0 be the constant given by proposition 3.26. Cover $B_{x_0}(R)$ by finitely many Euclidean balls of radius $R_0/32$. Let $B_{x_i}(R_0/32)$ be such a ball, we claim that there exists a constant C such that

$$\mathcal{I}(x,r) = N_\phi(x,r) + Cr^2$$

is a taming function for $Z \cap B_{x_i}(R_0/16)$ on the ball $B_{x_i}(R_0/16)$.

In fact, it follows from the definition that $N_\phi(x,r)$ is non-negative and continuous. By equation (3.17), there exists $C_1 > 0$ such that $\mathcal{I}_1(x,r) = N_\phi(x,r) + C_1 r^2$ is increasing in r. By proposition 3.26, there exists C_2, such that for $\mathcal{I}_2(x,r) = \mathcal{I}_1(x,r) +$

$C_2 r^2$, one has

$$D_\mu^2(x, r) \leqslant \frac{C_1}{r^2} \int_{B_x(r)} [\mathcal{I}_2(32r) - \mathcal{I}_2(2r)] d\mu(x)$$

for every Radon measure supported in $Z \cap B_{x_i}(R_0)$ and $r \leqslant 8R_0$, thus $\mathcal{I}_2$ satisfies condition (3) of definition 3.32.

Notice that since $\mathcal{I}_1(x, r)$ is increasing in r, for $\tilde{\beta} > 0$, the inequality

$$\mathcal{I}_2(x, 2r) - \mathcal{I}_2(x, \tilde{\beta} r) < \delta$$

implies that $r < \sqrt{\delta/(4C_2)}$. Therefore, lemma 3.31 implies $\mathcal{I}_2$ satisfies condition (2) of definition 3.32.

In conclusion, $\mathcal{I}_2(x, r)$ is a taming function for Z on $B_{x_i}(R_0/16)$, therefore theorem 3.4 follows from theorem 3.33. □

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