Application of Reading and Writing Program Methodology to Teaching Fractions: A Curriculum Design

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A Thesis in the Field of Mathematics for Teaching

for the Degree of Master of Liberal Arts in Extension Studies

Harvard University

February 2025

Abstract

In 2023, only approximately 26% of New Jersey students passed the state standardized test for Algebra 1, New Jersey Student Learning Assessments (NJSLA) (State of New Jersey, 2023). While many speculate on possible explanations for this predicament, this thesis posits a potential contributor could be the language of mathematics.

The idea of teaching math as a language, although not a new one, is not necessarily straightforward for educators as they often encounter difficulty with connecting students' lived experiences (including natural language) with symbolic mathematical expressions. It can be arugued that teachers could not have taught math as language because the nature of any natural language did not take definitive form until 1981 when Chomsky established Universal Grammar in *Lectures On Government And Binding: The Pisa Lectures*.

While it is hard pressed to find math textbooks that have taken the challenge of teaching math linguistically, language arts curricula have. Through language arts that support the development of fluency through spelling, grammar, and vocabulary, students may have opportunities to enhance their thinking and comprehension of what is read, spoken, and written. This thesis project proposes a language arts approach to teaching one topic in mathematics, fraction. The project is a design of a curriculum for a two-week workshop, FracSi, oriented to support ninth grade students' learning of the language of fraction and their conceptual understanding of fraction concepts.

Dedication

This thesis is dedicated to my students, past, present, and future.

Acknowledgments

I want to thank my advisor, Dr. Carolyn Gardner-Thomas. This thesis was completed due to her care and attention to details. I also need to thank the faculty at Harvard Extension School, in particular Dr. Oliver Knill, Dr. Eric Towne, Dr. David Arias, and Dr. Thomas Judson, whose insight into mathematics inspired me. I wish to acknowledge the leadership of the Extension School, Dean Nancy Coleman, Dean Suzanne Spreadbury, and Assistant Dean Andrew Engelward. Their work enables me to empower my students. Special thanks to Dr. Engelward who taught me how to teach math by sharing joy.

My gratitude goes to Mr. Phillip Millstein, my high school English teacher, who wrote in the margins of one of my essays that education needs motivated teachers. My heart holds a place for Sister Mary Margaret Buffin who educated me in not only reading and math, but she also taught the importance of sharing what is most precious to us.

My love goes to my family. Anna and Lisa, thank you for begrudgingly accepting the fact that your mother had to write a thesis. Francis and Theresa Sung, thank you for bringing me into this world and working hard throughout your lives to ensure my wellbeing. Last but not least, Ed Belbruno, you have been the wind in my sails throughout this journey.

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Chapter I

Introduction

Learning mathematics is a cumulative process, and if you fail to understand one stage, then anything that is built upon that stage is going to be rather fragile.

— Brian Butterworth

Fractions help form a foundation upon which advanced mathematics is built. Barbieri et al. (2021) found fraction arithmetic to be an important predictor of performance in Algebra 1. The researchers stated that there is a relationship between understanding fractional magnitudes and more advanced mathematical knowledge. Also, they noted that fluency with fractions may build the underpinnings of algebra, because working with both algebra and fractions requires understanding symbols and procedures. The National Mathematics Advisory Panel (2008) also emphasized that difficulties with fractions are a significant barrier to progress in algebra and related occupational fields. Furthermore, college students who struggle with fractions tend to need math remediation (Stigler et al., 2010).

If students are not proficient in fractions, they may experience significant challenges in future academics and careers involving mathematics. Ngo (2019) found college students who are placed in lower math classes because of missing fractions problems on a placement diagnostic may struggle to persist in college. Ngo also suggested that supporting students' understanding of basic math skills with fractions prior to college placement testing will likely have a positive impact on their college math experiences.

Data from New Jersey Student Learning Assessment (NJSLA) for 2021-2022 show that over 50% of students in grades three through eight in the state of New Jersey did not meet Common Core (Council of Chief State School Officers, 2008) aligned standards in mathematics (New Jersey Department of Education, 2022). Eighty-five percent of the eighth graders in New Jersey had scores falling below Level 4 (meeting grade level expectations). For Newark Public Schools (NPS) eighth graders, the percentage of students falling below the level 4 was higher than the state at 89.1%. The assessment results show that NPS students' average score during the 2021-2022 school year was lower than the overall average for students in the state of New Jersey. According to the New Jersey Assessment Resource Center, more than 25% (8 out of 30) of a sample NJSLA Grade 8 Math Test, require fractional knowledge. This thesis hypothesizes that the ability to correlate spoken English expressions for fractions to the written mathematical expression may support students' sense-making of fractions.

Chapter II

Literature Review

The Common Core State Standards document (Council of Chief State School Officers, 2008) suggests that students should be introduced to fractional knowledge using unit fractions (a fraction with numerator of one), because students can more fluidly apply and extend previous understandings of operations on whole numbers. The NCTM holds the position that curricular expectations focus on whole numbers but do not always acknowledge that a similar conceptual base is necessary for fractions.

The National Council of Teachers of Mathematics (2000) advocates for visual representations to support students' conceptual understanding of fractions. As such, visual representations now permeate math education materials. For example, Zhang et al. (2015) found that area-model representations, a form of visual depictions, dominate teaching and learning of fractions at all levels of schooling, particularly at the elementary level. Whether or not visual representations are effective for learning fractions needs to be better established. According to Zhang et al., "the emphasis on area-model approaches does not appear to have been conspicuously successful in helping elementary school students to construct strong and flexible fraction concepts" (p. 236).

However, visual representations may not be the only concern related to students' challenges with fractions. Stigler et al. (2010) suggests that students' struggles with fractions may be associated with procedural learning. The Council of Chief State School Officers (2008) also expressed concern that "students who lack understanding of a topic may rely on procedures too heavily" (p. 8). As noted by Hiebert (1999), historically, there has been a dominance of procedural learning in math classrooms. Kilpatrick et al. (2001)

also found procedural learning dominated over comprehension in school math programs. This form of teaching is particularly prevalent in schools that serve students from low socioeconomic populations (Bachman et al, 2015). Teaching procedurally often employs didactic instruction or direct instruction, a form of explicit instruction. Direct Instruction involves teacher demonstrations or talking followed by students practicing problems that follow the steps of the demonstration (Ray, 1961).

Discovery-based or constructivist instruction, on the other hand, involves students exploring and developing their own understanding of math concepts. Discovery-based instruction includes two main approaches: 1) enhanced discovery (also called guided learning) that requires the provision of necessary knowledge before setting up a learner to complete learning tasks; and 2) unassisted teaching where the learner is not provided with any knowledge before the given tasks.

Alfieri et al. (2010) reviewed 164 studies to compare the efficacy of explicit instruction and discovery-based instruction. The results of the review indicated that explicit instruction was more effective than unassisted discovery, while enhanced discovery-based instruction demonstrated as the most effective method to support students' attainment of conceptual understanding. Schwartz et al.'s (2011) study corroborated Alfieri et al.'s findings. Schwartz et al. investigated student outcomes on learning ratios and found that teaching students through enhanced discovery supported students' deep structure recall and transfer performance.

Another possible cause of students' challenges with fractions may be attributed to language. Chomsky (1986) noted that math has all the properties of a language. For instance, both the letters of the alphabet and the numbers in a counting sequence, according to Chomsky, can create infinite thoughts. Also, much like mathematics, natural language has a recursive property, where "words" can be nested in infinitely many ways (Hauser et al., 2002). Wynn (1995) describes the process of mapping linguistic ordinal representations to magnitudinal representations as non-trivial. Wynn's thoughts reflect the idea of a mental lexicon for math by connecting words to quantities. As noted by Neuman et al. (2007), literacy extends to math, and like Wynn, the researchers advocate for developing a robust and effective mental lexicon that enables self-learning.

Learning Fractions

Educators and school administrators look to intervention and remediation programs to improve standardized test scores. Remediation during the school year might use the same curriculum or separate teaching approaches and materials to students who have not mastered a specific topic, doing such students are not taught any new material until the deficient topic is mastered. (Michigan Department of Education, 2023). The report stated remediation is detrimental to student learning and performance because it leaves students further behind. More often than not, students in remediation cannot move forward with their cohort. As for summer-based programs, McCombs et al. (2019) found math remediation programs completed in one summer did not have strong positive outcomes for general math remediation. The researchers surmised that one possible reason for the low results might be that remediation programs cannot accomplish much in a short period of time.

Intervention programs can be generally placed into three categories: social-emotional, environmental, and academic. Social-emotional and environmental remediation delve into aspects of Maslow's (1943) hierarchy of needs that provides a hierarchical model for human needs: physiological, safety, love and belonging needs, esteem, and self-actualization. Social-emotional interventions address esteem and self-actualization. The results of Cohen et al.'s (2009) study that reaffirmed important values such as morality or creativity with students from a sample of low and middle income Black and White 7th-grade students' class grades increased by the end of one semester. In another study, Aronson et al. (2002) had groups of Black and White college students composing letters to middle school students to explain and endorse the belief that intelligence is malleable. At the end of the academic year, the GPAs of the college students rose significantly as compared to control groups. In addition, Black students reported an increased sense of belonging at the school.

Environmental intervention impacts physiological and safety needs. The National School Lunch Program that provides meals for K-12 students for low socioeconomic populations can be classified as an environmental intervention, because it attends to a basic physiological need (United States Department of Agriculture, 2017). Another example of environmental remediation is the large-scale initiative implemented in Newark, New Jersey, to create a unified school system composed of over 50 different schools from which parents can choose to best accommodate their children's psychological and safety needs (Chin et al., 2019). Environmental remediation programs can also include professional development programs like the Enhancing Missouri's Instructional Networked Teaching Strategies of Missouri (eMINTS) that supports teachers' implementation of new classroom methods. Meyers et al. (2020) found the eMINTS to be effective at strengthening teachers' pedagogical content knowledge. The researchers concluded that teachers who participated in the eMINTS showed significant shifts from procedural to conceptual teaching approaches.

Academic intervention as defined by the State of Oklahoma's Department of Education (2014) is "additional instruction and support that supplements the general curriculum (regular classroom instruction) and are necessary to improve academic performance for some students" (p.# 3). One project-based summer remediation program, Boston Red Sox Summer Math Program (BRSSM) showed positive outcomes for students in the program. Children in BRSSM were matched with a comparison counterpart who had the same pretest score level on NWEA MAP. After BRSSM ended, the NWEA MAP scores of participants and their matched counterparts were compared, and the results showed the NWEA MAP scores of the participants were significantly higher. While research (Patall et al. 2010) suggests that extending school time can be an effective means to support student learning, particularly for students at-risk of failing, many summer remediation programs do not show efficacy for mathematics (McCombs et al. 2019).

Researchers (Hutchinson, 2013; Kelly, 2018) also investigated online and technology-based approaches as intervention programs to bolster students' understanding of mathematical concepts, many of which also show no efficacy. For example, Kelly investigated the efficacy of utilizing Khan Academy for ninth graders at two rural high schools for math intervention and found no significant difference between the posttest scores of students who utilized the online videos along with regular classroom instructions and the posttest scores of students who received only regular instructions. In another study by Hutchinson (2013) that investigated the efficacy of a commercially available online product, SuccessMaker, which is an adaptive learning software-based math intervention. It was used in a Louisiana middle school, and the results were inconclusive for success on the Louisiana Educational Assessment Program (LEAP).

Brown et al. (1975) created an algorithm called BUGGY which allowed teachers to analyze the steps students take to solve a fractions problem. The researchers underscored the challenge of designing a representation that facilitates the discovery of misconceptions or bugs existing in students' encoding of knowledge. Brown and his team noted that one aspect of the challenge is that while some bugs can be categorized as syntactic bugs (miscalculating $\frac{1}{2} + \frac{1}{3}$, for example), other bugs such as inappropriate analogy or incorrect generalization may be considered semantic (not understanding a word problem). As Brown et al. reasoned, due to the numerous steps in logic that students need to consider when answering math questions, solving fraction-related problems requires investigating steps that students take to come to solutions.

Despite the unsatisfactory outcomes documented by the research studies on fraction-intervention programs listed above, visuals on paper and moving visuals on videodiscs appear to show some promise. Miller et al. (1989) found Mastering Fractions, a set of videodiscs from Systems Impact, to be more effective than teaching without visuals. Additionally, Bottge et al. (1993) found contextualized fractions problems on videos yielded significantly better learning results than standard word problems on paper.

A research review (Misquitta, 2011) of ten studies conducted between 1990 and 2007 and proven to be successful fraction interventions found that the effective approaches involve one of three methods of instruction: graduated sequence, strategy instruction, and direct instruction. Graduated sequence instructional methods guide students through a series of thoughtfully designed sequential stages that progressively build up conceptual understanding of fractions. Concrete Representational Abstract (CRA) is an example of a graduated sequence instruction method. CRA begins with students completing concrete mathematical tasks with the assistance of manipulatives, then using representational drawings, and finally abstracting to numbers. Flores et al. (2023) found fractions intervention using CRA instructional sequence of instruction to be effective for developing conceptual understanding of fractional numbers.

Strategy instruction involves teaching a variety of methods to solve computational problems. For example, Joseph et al. (2001) used cue cards to help students of low math achievement solve fractions computations problems. Students were allowed to look at cue cards with solved problems and procedures when they were asked to solve new problems. Other researchers, Test et al. (2005) utilized strategy instruction with a mnemonic device called LAP (Look Ask Pick).

During guided practice, students and teacher read aloud the above steps together to help students solve fraction computation problems. LAP has eight steps to identify an appropriate strategy to use for fraction computational problems.

Step 1. Look at the denominator and sign

Step 2. Ask "Will the smallest denominator evenly divide into the largest denominator?"

Step 3. Pick your fraction type.

Step 4. Type 1 - Bottom numbers are the same. Its sign is addition or subtraction.

Step 5. Type 2 - Bottom numbers are different and the smallest will evenly divide into the largest bottom number. Its sign is addition or subtraction.

Step 6. Type 3 - Bottom numbers are different and the smallest will not evenly divide into the largest bottom number.

Step 7. Identity denominator - Student points to denominator

Step 8. Divide denominator - The student is shown two denominators. They divide the smallest denominator into the largest denominator.

Miller et al. (1989) found intervention programs using direct instruction focused on fraction computations may prove effective. As noted, direct instruction involves teachers' explicit demonstrations of procedures that students mimic to solve math problems. Flores et al. (2007) successfully implemented the direct instruction curriculum called Corrective Mathematics, Basic Functions to improve student performance on computation. The curriculum has step-by-step lessons that a teacher models and students emulate. Mastering Fractions, a set of videodiscs from Systems Impact, also employed direct instruction and yielded successful results for computational problems. There was no indication that Mastering Fractions led to conceptual understanding (Bottge et al., 1993), because students improved at adding and subtracting fractions, but they did not improve at solving word problems.

A Different Approach to Teaching Fractions

This thesis introduces the idea that fractions have four components: a mathematical symbol, written English symbols, English acoustic events, and a mental picture. There is a dynamic relationship between these four components.

The first aspect of this relationship is semiotics, which is loosely defined as a framework for analyzing how signs and symbols create meaning and facilitate communication in different contexts and disciplines. Peirce (1894) clearly stated the relationship between thought and symbol. Ogden (1925) developed the triangle of meaning, which is a model of communication that shows the relationship between thought, symbol, and referent. The symbol, which is a written word in a language, represents a thought, and the referent is the object or idea to which the symbol refers. Ogden highlights the indirect relationship between the symbol and referent. As an example, two children want ice cream. The written symbol in English is “ice cream”. The thought both children have is eating ice cream. However, one child is thinking about eating chocolate ice cream, and the other is thinking about strawberry ice cream. They both have thoughts about ice cream, but their referents are different.

Furthermore, Vygotsky (1965) stated that language has two functions, 1) inner speech which is used for mental reasoning and 2) external speech which is used to communicate with people. It is highly plausible that a math student’s inner speech for fractions does not match the external speech due to the polysemy of fractions where one fraction symbol can have multiple related by different meanings.

The second aspect is between the English symbol and the English acoustic event. Broca's area, the language-generating structure located in the left inferior frontal cortex, above and behind the left eye, responded to reading silently much in the same manner auditory neurons respond to reading aloud. In an experiment led by neuroscientist Maximilian Riesenhuber of Georgetown University Medical Center, the MRI activity of

12 subjects showed that Broca's area acted like it was generating the sound of words, so the readers heard them internally (Sutherland, 2015).

The third aspect of the relationship, math and the language area of the brain are connected, as proven when neuroscientists observed activity in Broca’s area when people solved math problems (Trafton, 2015). As common sense would dictate, when math problems are read aloud or silently, it would be read in the native language of the reader.

From the research on the Broca area, it is possible that Vgotsky’s (1965) internal speech is in actuality the Broca area at work. Figure A.1 represents a possible framework for the process of decoding and synthesizing math problems involving fractions. A person received three different kinds of inputs, acoustic, English symbols, and math symbols. These three inputs are processed as internal speech that is mapped to a mental image.

It is possible that the deep structure Schwartz et al. (2011) discussed is the relationship between Vgotsky’s (1965) internal speech and a mental image. Schwartz et al. found children who invented their own indices for explaining the number of passengers and the number of vans transferred their understanding of density to speed as a comparison of distance and time. In short, the children could apply ratios to varied situations, instances of knowledge transference.

Children in the study might have established strong connections between internal speech and mental image when they invented their own vocabulary for understanding ratios. This might explain why the Schwartz et al. study showed transference when the deep structure was attained.

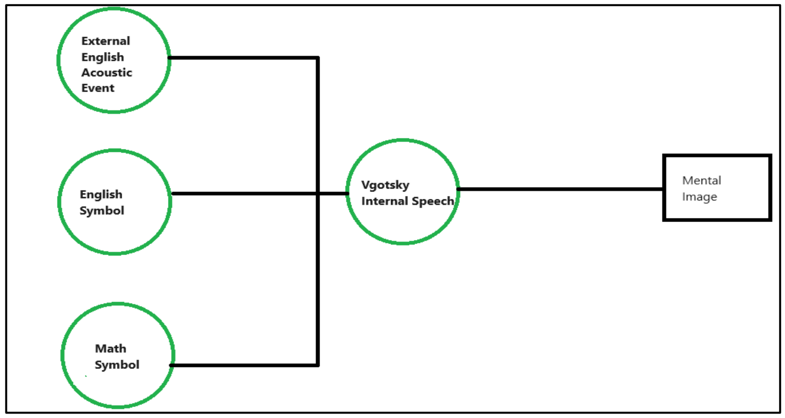


Figure A.1 A proposal for language learning framework

It is important to note that Schwartz et al.'s children invented their own quantitative index, a ratio. In Figure A.1, the Math Symbol would have been invented by the child. Their invented indices would most likely have been mapped to acoustic events. However, their invented indices may or may not have mapped to standardized vocabulary used for math tests. This might explain why the experiment did not have significant impact on children solving word problems. This possibility instantiated the idea that a language-based method to teaching fractions may improve students' understanding of fractions, because it would address all three inputs in Figure A.1.

Fractions as Language

Recently, there have been more inquiries into how language and math are related. Researchers, including Bingham et al. (2019) advocate for promoting literacy strategies to teach fractions. LeFevre et al. (2010) found that linguistic composite (a measure of language ability) of preschool-aged children predicted mathematical outcomes (e.g., whole-number computations, numeration, geometry, measurement, and number line estimation). Bailey et al. (2014) found language skills predict subsequent whole-number computations and number line estimation, which in turn predicted later fraction concepts. The researchers signaled the need to ensure mastery of mathematical language in the early years to provide a solid foundation for learning both whole numbers and fractions.

Other researchers (including Moleko, 2021; Molina, 2012) also noted that the English language poses problems to learning math, and that the problem with teaching fractions lies with how the written symbolic representation correlates specifically to English. Molina noted that fractions, "the 'F' word in mathematics" (p. 105), are "the area in fundamental mathematics most plagued with obstacles in the language and symbolism" (p. 105). The researcher suggests that the many different meanings of the symbols depend on the context. This can lead to misunderstanding and misconceptions of fractions. This is referred to as polysemy.

Over a century ago, Montessori (1914) noted the same thoughts. Her observations on how children navigate the world when they encounter words spoken to them may offer some insights for teaching fractions. According to Montessori,

> Language now comes to fix by means of exact words the ideas which the mind has acquired. These words are few in number and have reference, not to separate objects, but rather to the order of the ideas which have been formed in the mind. (p. 84)

Mathematics borrows the acoustic events (the transmission and reception of sounds such as spoken words) or visual events (like signed language) of a person's natural spoken language (Anderson et al., 2022). For example, the math statement 1 + 1 = 2 has different acoustic events depending on the native language of the speaker (including "One plus one equals two" in English, and "Uno mas uno es igual a dos" in Spanish). The written symbolic representation, the syntax and semantics are the same regardless of the spoken or signed language that represents the statement in the speaker's native language. As opposed to treating math as a separate language, numbers (1, 2, 3, …) in the Indo-Arabic system and operators (+, -, ÷, ×, for example) may be considered as another set of written symbols much in the same sense that Japanese has three different sets of written symbols (Hiragana, Katakana, and Kanji). Hiragana is the native Japanese alphabet system for native Japanese words, Katakana consists of different symbols for words not native to Japanese, and Kanji are Chinese symbols that have been adopted to represent Japanese spoken words.

Montessori (1914) observed that "the minds and hands of our children are already prepared for writing, and ideas of quantity, of identity, of differences, and of gradation, which form the bases of all calculation, have been maturing for a long time" (p. 85). Additionally, as noted by Wynn (1995), non-symbolic representations of numerical magnitudes across species can be measured in early infancy of humans. In short, it appears that human beings are born with an innate understanding of math. Many have noted that human beings possess an innate sense of ratios (Piaget,1966; Davis,1990; Kieren,1992; Wynn, 1995; Pitkethly, 1996; Cwikla, 2014). Hence, the approximate number system (ANS), which allows individuals to represent and process numerical magnitude information might be innate however the language to express it needs to be developed.

The modality for English is vocal, meaning it is spoken as opposed to signed. A word in English or any other language involves recognizing its form, an acoustic event of the spoken word, written symbols, and meaning(s) upon which a mental lexicon is built (Anderson et al., 2022). The mental lexicon functions much like a dictionary that connects speech sounds, grammar, and meanings. It contains the representation of language in the mind/brain. For example, the image of the written word “apple” and the speech sounds of saying the the word map to an individual’s mind/brain mental reprensentation whether it be an image, a taste, crunching sound, memory, or anything. The first step to establishing fluency in fractions is establishing a healthy and robust mental lexicon. In other words, identifiable ways of writing and saying fractions need to be established.

Researchers like Mamolo (2010), Molina (2012), and Tai (1999) proposed that syntax, semantics, and pragmatics pose difficulties for fraction learners. Syntax, understanding how words can or cannot be combined to make phrases and sentences, becomes automatic with fluency. Fluent speakers of a language automatically know which words to use in a sentence to convey a thought in that language. Formal semantics (mental operations which speakers perform when they compute a sentence's meaning) are also automatic when a speaker is fluent. Syntax and semantics help with comprehension of meanings in different contexts. As an example, the verb phrase “to be” in English can take on many meanings depending how it is used. In the imperative sentence, “be happy”, the speaker may be advising a person to accommodate a negative situation. However, the imperative sentence, “be quiet”, commands that a person enter a state of not necessarily low decibel volume but rather silence. When reading William Shakespeare’s *Hamlet,* the phrase “to be or not to be” uses the word “be” as a substitute for the word “live”.

In English, there are many ways to express a single concept (Tai 1999). For example, while the concept of fractions and ratios are similar, there are multiple English expressions to represent these concepts. As shown in Figure A.2 below, the written symbol can be read in English as “three fourths” or “three quarters” where three serves as a modifier to specify the number of fourths or the number of quarters. The fraction also reads as “three out of four,” with a prepositional phrase (out of four) describing the relationship of the noun (three) to the other words in the sentence. It also can be read as “three shared by four” and “three divided by four”.

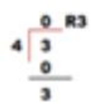
Three Fourths (Fourths - plural noun Three - modifier)	
Three Out of Four (Three - noun Out of Four - prepositional phrase)	
Three Quarters (Quarters - plural noun Three - modifier)	
Three Shared by Four (Three - subject Shared - verb by Four - prepositional phrase)	
Three Divided by Four (Three - subject Divided - verb by Four - prepositional phrase)	

Figure A.2 English Expressions for Fractions

The morphology of the expressions that the written symbols represent causes difficulty with instruction. As a result, the mental lexicon for various applications of fractions from a native English speaker could be one reason for confusion on fractions. Furthermore, students who have inefficient, inaccurate, or incomplete mental lexicons for basic verbal expressions of fractions may struggle with the concept and will likely perform poorly on math assessments.

In a comparison by Ng et al., (2010), the same fraction in English appears to suggest different meanings in Chinese. When saying the fraction $\frac{1}{4}$ in English, children learn it as a unit fraction, which is a quantity, therefore a noun. The same fraction in Mandarin Chinese can be seen as a complete set of procedures,"四分之一", which could be translated as "divide into four equal parts, then take one part." In a sense, fractions in Chinese are imperative sentences that tell how a quantity needs to be partitioned and how many portions to take.

In Chinese, there is one way to say $\frac{1}{4}$ as a unit fraction; however, in English it can be read as one quarter, one fourth, one out of four, one divided four, among others. This polysemy of fraction symbols in English, according to Mamolo (2010), makes fractions confusing. It is important to note that the Programme for International Student Assessment (PISA) test results show the top five countries in math results have Chinese as the only or one of main national languages (Schleicher, 2018).

Linguistic Approach

Glazer's (1994) approach to literacy may support students' comprehension of fractions given the challenges of the English language comprehension of math concepts. The Glazer's approach has four basic components: Comprehension, Composition, Reading and Writing, and Expanding Language. Comprehension is the primary focus of the program. Composition as well as Reading and Writing have meaning only if the student understands what is read in order for the student to write effectively. Glazer strongly emphasized that expanding language, reading, and writing are intricately intertwined in a feedback loop. “Reading and writing, listening and speaking are inseparable. (Glazer, 1994, p. 5)”

Glazer (1994) also stated that pictures support children’s learning process. Oral language serves as a primer for reading and writing. A method called Picture Talk creates a productive platform for children to articulate their thoughts on what they see or experience with words by promoting discussion about pictures in books, photos, paintings, and artistic drawings. It is important to note that math textbooks that teach fractions tend to have pictorial diagrams that prescribe mental images of fractions. However, as noted by Schwartz et al. (2011), “Giving students the end-product of expertise too soon short-cuts the need to find the deep structure that the expertise describes. (p. 770)” It is possible that textbooks giving students the inherent mental image that depicts a fraction cuts off the process for students to discover the inherent underlying structure needed for a strong and flexible understanding (Zhang et al.) of fractions.

Furthermore, research studies on neuroscience indicate that self-generated pictures are better remembered and understood than ones presented (Gregory et al., 2016). In the case of an artist, Lonni Sue Johnson, referenced by Rabin (2011), who suffered from viral encephalitis that damaged her hippocampus, she remembered her own paintings but not those of others. The research scientist, Dr. Michael McCloskey of Johns Hopkins University, found that while she could not remember the artwork of famous artists like Vincent Van Gogh, she immediately recognized her own work.

Utilizing visuals to support learning, Glazer's approach can clarify differing referents when viewing fractions. Along the veins of cognitive narratology, visuals could potentially create memorable images of stories that clarify fractions. Fractions can be used to bring a happy ending to a story about how things are shared and the proportional relationship between things. The narratological approach posed by Todorov (1969) can employ short stories with characters who encounter problems with fractions in daily life. The flow of events in a story as proposed by Todorov are the following:

Equilibrium - character lives with routine daily activities

Disruption - character experiences problem or disturbance in the character's life

Recognition - character realizes a problem or disturbance that affects the character's life

Repair - the character attempts to rectify the situation

Equilibrium - character successfully rectifies the situation, and the character is having a normal life as in the beginning or adjusting to the new situation

This structure of a story as proposed by Todorov (1969) resembles the developmental model of Piaget (1977). Piaget proposed children engage in adaptation, which involves a child changing due to situational demands. Adaptation involves assimilation, which is the process of merging previously held ideas or concepts to new ones. It also entails accommodation, which is altering previously held ideas or concepts when encountering new information. Piaget stated that once the child finds balance between self and the world, equilibrium is attained.

Chapter III

Methodology

It is likely that students' mental lexicon for mathematical concepts contributes to their performance on math assessments. As noted earlier, according to Neumann et al. (2011), students who have low math test results on fractions concepts may have inefficient, inaccurate, or incomplete mental lexicons for basic verbal expressions of fractions. The primary goal of the thesis is to develop a two-week workshop that will support the development of a more efficient mental lexicon for fractions.

Students engage with signed fractions expressed with words, phrases, and sentences to support the development of what Neuman et al. (2011) refers to as a robust and effective mental lexicon. The workshop aims to leverage the attainment of language to strengthen students' conceptual understanding of fractions and to develop their computational problem-solving skills.

Curriculum Approach

The core of the curriculum creates a language arts approach to teaching fractions. As such, the curriculum employs practices commonly found in language arts classrooms. In this curriculum, mathematical problems appear as a variety of pictorial stories, word problems, and physical tasks, to provide a variety of inputs that are needed to teach polysemous languages such as fractions (Li, 2022).

Teaching languages in general requires pictorial support (Glazer, 1996), and students respond to the pictorial problems that are accompanied by English. Student express solutions in fraction form, as English sentences and phrases, and as representational drawings of the fraction solution. Students are expected to draw their own representational diagrams of fractions and write their own English descriptions of each scene rather than rely upon static representations of fractions like a circle split into wedges. The curriculum emphasizes self-generated English and diagrams because people recall their own depictions better than those created by others (Rabin, 2011) and recall and understand their own phraseology better than those presented by others (Chomsky, 2000; Schwartz et al., 2011).

As another aspect of a language arts approach, the idea of near and far transfer, commonly called critical thinking, plays a pivotal role in this curriculum. Successful reading instruction or interventions develop analogical reasoning ability (Cartwright, et al. 2020). Hence, an important question that arises, what constitutes analogical reasoning?

In language arts, transfer often means the ability to understand concepts of one book based upon what is learned in another book. Near transfer occurs two readings have similar situations such as *Diary of a Young Girl* by Ann Frank and *I Am Malala* by Malala Yousafzai which are both autobiographies of young women who were persecuted. In mathematics, it can be said that near transfer occurs when someone solves problems that have similar or familiar superficial characteristics. Students demonstrate far transfer when they find commonalities between two vastly books, such as the history textbook *The History of the Decline and Fall of the Roman Empire* by Richard Gibbon and William Shakespear's play *Hamlet*.

In this curriculum, near transfer in math can be seen when a student calculates the per pound cost of salmon and then applies the same ability to calculate the per pound cost of trout. Far transfer occurs when a student applies a concrete activity to a problem that has abstract, dissimilar, or unfamiliar concepts. Using the same example, the same student applies the same ability to find the per child cost of movie tickets based upon what a parent paid for all tickets of all children.

This curriculum considers the varying degrees of ability to transfer as environmental biproducts. While Ceci (1991) recognized schools and parents' education level as part of the environment, Barnett (2002) found everyday life activities of particular environments impacting transfer ability. The "Kind World Hypothesis" (Gentner et al., 2018) states that for the sake of survival, people who live in predictable environments can safely transfer prior knowledge. If they know of a non-threatening animal, they will also perceive new animals with similar features to be non-threatening. If people know of a threatening animal, they will also perceive another animal with similar features to be threatening. They choose to transfer previous knowledge to new situations (Gentner et al., 2018).

On the flip side of the same coin, "Deep Learning" (Ohlsson, 2011) is based on an ever-changing environment. In this hypothesis, people who live in an environment that is highly unpredictable survive because they do not take for granted surface level similarities. If a person in unstable surroundings know of an animal that is non-threatening and sees another animal with superficial similarities, that person will not assume the new animal is safe. They reject analogical transfer (Ohlsson, 2011).

Despite the varying degrees of critical thinking skills, the ability to transfer can be learned (Richland et al., 2010). This curriculum supports analogic diversity by following the path set by Gentner (1998), which is to develop concrete understanding of fractions into more abstract concepts. As such, the curriculum uses the Concrete Representational

Abstract (CRA) model for lessons. Each lesson is divided into three phases. The initial phase involves a concrete activity using manipulatives. For example, students make a right triangle and write in fraction form the comparison of height to width ($\frac{height}{width}$). In the second phase, students have opportunities for near transfer by solving problems in situations that have some superficial similarities to the activities in the first phase. For example, a roof can be drawn as two right triangles, and the height of one right triangle can be compared to the width as a fraction. Far transfer occurs when a student finds the slope of a line on a cartesian grid by visualizing a right triangle with a segment of the line being a hypotenuse.

To further develop near and far transfer ability, the curriculum creates bridges that connect previous knowledge to new situations by providing time for activities called Prediction and Focused Free Writes. Prediction requires students to make educated guesses based upon previous lessons as to what will need to be accomplished in the next activity or problem. After completing the task, students will self-monitor by discovering whether their predictions were reasonable.

Focused Free Write gives opportunities to connect new knowledge to previous knowledge by comparing various activities within the lesson or between multiple lessons. An offshoot of Elbow's (1973) Freewriting, Focused Free Writes give students unfettered free flow of thought unhindered by disregarding grammar and spelling. Students do not stop for editing purposes. If a student finds drawings to better reflect their thought, then they draw. Focused Free Writes demonstrate efficacy in developing analogical thinking skills when there are no interruptions (Li, 2007). Teachers and students do not talk during Focused Free Write, because the objective is to give each individual student freedom to explore their own thoughts without interruption by others and themselves, the rational for ignoring grammar and spelling. At the end of the Focused Free Writes, students who wish to share their thoughts can do so.

This curriculum promotes a self-monitoring classroom, where students can take note of their own development and improvement. This curriculum emphasizes students starting with whole class activities, where all students work on the same activities. However, at the end of the lesson, students freely choose amongst a collection of activities at varying levels of difficulty or requirements for abstraction. When students select problems based upon their own comfort level with the topic, they observe where they are in the continuum of understanding. With each positive feedback for completed work, students decide whether or not to pursue more difficult questions. They examine their own success based upon the number of remaining questions and perhaps comparing their own progress to their peers.

Overview

The curriculum has four units with eight lessons and two projects: Language of Fractions, Operations with Fractions, Algebra and Fractions, and Projects.

Unit 1: The Language of Fractions unit consists of two lessons. Lessons 1, Fractions as Verbs, and Lesson 2, Fractions as Prepositional Phrases. The unit aims at supporting the development of students' mental lexicon related to fractions, starting with unit fractions. The first lesson teaches how to read the vinculum as verb phrases such as 'divided by', 'divided into' and 'shared-by'. The second lesson introduces when to read the vinculum as prepositional phrases including "out of", "to", "for", and "in".

Unit 2, The Operations with Fractions, has the first two lessons, Lessons 3 and 4 designed to foster computational fluency with multiplication and division of fractions. Lessons 5 and 6 teach addition and subtraction of fractions with like and unlike denominators.

Unit 3, Algebra, extends ratios to slopes of linear equations. Lessons 7 introduces ratios as slopes. Lesson 8 gives students an opportunity to apply linear equations in y-intercept form in a digital art project.

Unit 4, Projects, provides four individual or group projects that reinforce fractional concepts. The projects use real life applications relevant to students' lives. Students choose from the four projects based upon individual level of learning.

The eight lessons in Units 1- 3 follow a similar format with the following components:

- Agenda
- Lesson Opener
- Main Activities
- Free Choice Activities
- Teacher's Notes

In Unit 4, designed for a differentiated final summative assessment, students choose amongst three projects designed at varying levels of difficulty. These projects are not lessons. Each lesson simultaneously fosters expansion of language, reading, writing, comprehension, and composition with fractions concepts in 45 minutes. The presentation of slides (Appendix A) with activities for the lessons is the primary mode of delivery of content throughout the lessons.

Agenda

The agenda gives a possible timeline for the class. Activities that are not completed during class can be assigned for homework.

Lesson Opener

FracSi starts lessons that teach the language of fractions with Montessori (1914) styled activities. This provides the base scenarios for analogical transfer. In the Language of Fractions unit, there is a second activity that leverages auditory stimulus. Reading and writing is embedded in the Lesson Opener which requires students to articulate scenarios in natural language, whether it be English or their native language, but be able to write it in math. It also supports vocabulary expansion because students must find different ways to express the tasks presented.

Main Activities:

This presents target scenarios to give an opportunity for analogical transfer. Students engage in activities that are whole class, but the students lead the activities with teacher guidance. How a teacher decides to guide is determined by the teacher preparatory training received. Teacher training programs differ in their approach, and this paper does not address teacher training. Appendix C contains solutions to some of the problems.

Free Choice Activities:

Free Choice Activities give students an opportunity in differentiated learning by working independently. The activities are geared to further promote analogic reasoning.

Students can select just one or more activities, and they can choose to work in a group or by themselves. Classroom discussion is encouraged. Appendix C contains solutions to the problems, and Appendix B contains additional problems and their solutions.

While this phase of the lesson can be used for assessment purposes, its main function is to offer students occasions for self-monitoring. Also, it gives teachers opportunities to observe and support student progress by enabling teachers to circulate the room to answer questions or assist with work. This phase helps students achieve solutions to situations that require Far Transfer by completing a gradient of activities with varying degrees of similarity to the Lesson Opener.

Teacher's Notes

For each activity, teachers are given information on signs of mastery of a lesson and expectations on points of confusion.

Modification for Diverse Learners

The curriculum directly considers English Language Learners (ELL). This curriculum does not directly address the need of Special Education students requiring Individualized Education Plans (IEP) or 504 plans, however, there are recommendations that can assist in learning.

English Language Learners

English Language Learners, are defined as students who are unable to communicate fluently or learn effectively in English. These students could be students from outside the United States, students whose family primarily speaks a language other than English, or a student with delayed language. The curriculum supports these students by providing pictorial scenarios that can be described in the language of their choice or by creating representational drawings.

Abiding by the philosophies of University of Wisconsin-Madison School of Education (2020) for multilingual learners, all instructions and responses can be in English, native languages or a mix. In addition, technology like Google Translate, Amazon's Alexa, or Apple's Siri allows students to speak in their native language can be of assistance. More recently, artificial intelligence like ChatGPT can become a conversation partner because it can see pictures and listen to speech in different languages.

Extra Support

Because all lessons begin at a concrete lesson, this curriculum offers itself for a myriad of learners. The Lesson Opener offers Montessori (1914) inspired motor sensory based activities which can be beneficial to all students. While this curriculum does not address disabilities with sight and hearing, there are general recommendations. For the sight impaired, the story becomes the focus of the problem. Small groups with more teacher guidance, assistance, and interaction would be recommended. Products like tactile drawing boards, lightbox drawing pads, or tablets with appropriate apps easily integrate into each lesson. For the hearing impaired, the photos and pictures become the focus of the problem. Small groups with more teacher guidance, assistance, and interaction can maximize student support. Also, technology like Google Translate, Amazon's Alexa, or Apple's Siri can be of assistance. More recently, artificial intelligence like ChatGPT can become a conversation partner or tutor, because it can see pictures and listen to speech.

Chapter IV

Lesson Units

Chapter IV provides that actual lessons that are used in classrooms. Appendix A gives the slides for each lesson. Appendix B contains additional problems. Appendix C provides answers to the problems and practice sets.

Language of Fractions

The Language of Fractions Unit consists of two lessons: Lesson 1, Fractions as Verbs "To Divide/ To Share" and Lesson 2, Fractions as Prepositional Phrases "To/Out Of/In/For".

Lesson 1: Fraction as Verbs

Students demonstrate mastery of divide/share identify the numerator as the number of objects being partitioned or shared, and the denominator is the number of portions or groups. The portions or groups form by the number of people sharing or other entities requiring equitable distributions. Students understand that once a problem is set up as a fraction that fraction becomes the result. For example, if there were three pizzas that are to be shared by four people, the problem would be set up as $\frac{3}{4}$. As a result, each person would receive $\frac{3}{4}$ of a pizza.

Students read the vinculum as "divided by", "divide into", or "shared by", and the directionality is from the top down. The unit fraction $\frac{1}{3}$ can be read as "one divided by three", "one divided into three", or "one shared by three." Students also read the unit fraction $\frac{1}{3}$ as an imperative sentence, expressing a command or request “divide into three parts and take one part.”

The lesson addresses fractions using a discrete approach by dividing and sharing multiple countable objects. This lesson also supports students’ understanding of fractions language through the continuous approach by taking one object and cutting or partitioning it to be equitably shared by different numbers of people or things.

This lesson introduces the word and concept of “per”. While mastery of using this word is not expected, students need to be able to know that it is used to express the result of division or sharing.

Agenda

Section	Activity Name	Mode	Instruction Time (minutes)
Lesson Opener	Jellybean Sort	Whole class	15
Main Activities	Tik Tik Tik	Whole class	10
	Giant Brownie	Whole class	5
	Focused Free Write	Individual	5
Free Choice Activities	Pizza Pizza Skateboard Heartbeats	Individual or Small Group	10

Resources and Materials:

6 jellybeans of any color

3 paper plates

Lesson Opener: Jellybean Sort

As the objective, students learn to describe the situation in English, with a representational diagram, and as a fraction. The lesson starts with a discussion prompted by the question, “Can someone give me a sentence or phrase that uses the word “per”?

Given three plates and six jellybeans, students evenly distribute the jellybeans onto the three plates with any color combinations. Each plate represents a group. Students make drawings of their groups using colored pencils to best represent the groups and compare their drawing with their peers. Students can share their drawings for feedback and discussion.

Students express in the situation in their own words in English. They learn to read the vinculum as “divided by”, “divided into”, and “shared by”. Then they learn to express the result in English by reading the vinculum as “per”.

Students connect the English language statements such as “six jellybeans per three plates” or “six jellybeans divided by three” or “six jellybeans divide into three” or “six jellybeans shared by three” to the mathematical symbol $\frac{6}{3}$.

Students discuss how to set up problem when they understand the results are read as “per” followed by an item. (see Figure 1.1) They notice the denominator consists of the items that come after the word “per”. In the example, the result reads “per plate”, therefore, to set up the problem, the number of plates must be in the denominator.

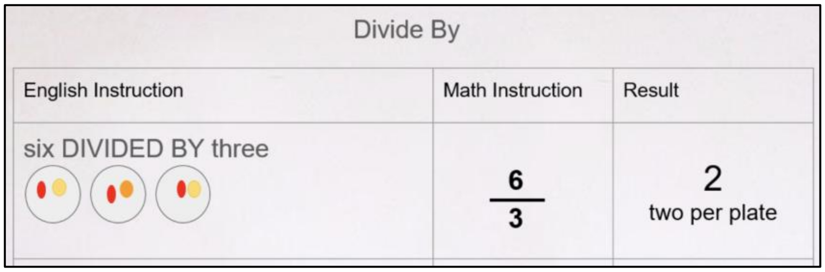


Figure 1.1 Divided Jellybeans

Main Activities

Students start with Prediction, then engage in two activities called Tik Tik Tik and Giant Brownie, and end this portion with a Focused Free Write.

Prediction. Presented with an image of a stopwatch, the question, “How many ticks per second?”, and the sound of ticking clock, student predict what they will do in the next activity, Tik Tik Tik.

Tik Tik Tik. In this activity, a ticking sound will be played six times within three seconds. Students volunteer to either silently count the seconds or count the number of ticking sounds. They are asked, “What would be in the denominator?” They are tasked to count the number of ticks that will be heard in three seconds. They are to draw a diagram that represents the task, write a sentence in English that describes the task, and write it as a fraction.

The students are asked “If there were six ticks in three seconds, how many ticks were there in one second?” There would most likely be discussion as to how to answer that question. They are then asked how they would express that in English and as a fraction.

Giant Brownie. There are three siblings, Alsha, Amir and Abdul. Their father gives them a giant brownie and tells them to share it. What do you think needs to happen?

Focused Free Write. How were all the activities, Jellybean Sort, Tik Tik Tik and Giant Brownie, similar? How are they different?

Free Choice Activities

Students choose from two activities geared for conceptual understanding, Pizza Pizza and Skateboard.

Pizza Pizza. Three children share two pizzas. How much pizza does each child receive?

Skateboard. Two brothers, Zachary and Joseph share one skateboard for one hour. Draw a diagram that represents the problem. State the problem in English, write the fraction that represents the problem, write how to read the fraction as a result.

Teachers Notes

In the Lesson Opener, Jellybean Sort, teachers circulate the room to observe and support students to make representational drawings that show comprehension and the ability to abstract. A student that draws actual plates and jellybeans indicates the intention to recreate the situation in a drawing in a more concrete fashion. Utilizing whiteboards, Smartboards, Jamboards, etc., teachers can present their own representative drawings or choose the drawings to recreate to start discussion on how best to create a representative drawing.

Another important element in the Lesson Opener, the term “per” in the English language causes confusion. It is important to explain that it is a word that can be defined or replaced by “for every”. Understanding the term per leads to clearer understanding of common daily situations such as gas mileage of cars can be expressed as 25 miles per gallon or 25 miles for every gallon of gas. Heart rate can be measured as 70 beats per minute or 70 beats for every minute. The cost of produce like pears can be said as two dollars per pound or two dollars for every pound.

Ask students to think about these sentences and think of a different way to say them:

1. The teacher gives five notebooks to EVERY student.
2. EVERY student receives five notebooks.

The Tik Tik Tik activity employs the hearing sense to help abstraction. Students no longer have something tangible to represent. Teachers need to circulate the room to observe if the students are making representational drawings that show abstraction by using shapes to denote groups and lines or symbols to denote each ticking sound.

In Focused Free Write, when students share, observe if students comprehension that anything tangible (e.g.,Jellybeans), auditory (e.g.,ricking sounds), or abstract (e.g. , time) can be placed into equal portions. To prepare students for the Skateboard activity, teachers can ask “Can time be shared?”

During the last phase, Free Choice Activities, teachers need to circulate the room to support students with their questions or points of confusion.

Lesson 2: Fractions as Prepositional Phrases

In this lesson, students demonstrate mastery by knowing when to read the vinculum as “to”, “for”, “out of” and “in.” The lesson addresses how to use the word “to” or “for” when comparing objects that are countable in one set to countable objects in another set. For example, in a set of ten pencils, to compare six pencils having erasers with four pencils without erasers, a fraction can be written as $\frac{6}{4}$ and read as six pencils with erasers “for” four pencils without eraser, or six “to” four.

When a comparison is between a part of a set and the whole set, the vinculum is read as “out of.” Considering the above example with the ten pencils, one can compare the number of pencils having erasers with the total number of pencils as six “out of” ten pencils have erasers and written in fractional form as $\frac{6}{10}$. Similarly, a comparison between the number of pencils without erasers and the total number of pencils can be written as a fraction, $\frac{4}{10}$, and read as four “out of” ten pencils.

Students understand to read the vinculum as “in” when comparing measurements against time. For example, expressing 126 miles a car traveled in two hours can be written as $\frac{126\,miles}{2\,hours}$ hours which is read as 126 miles “in” two hours. Applying what was learned in Lesson 1, students can further extrapolate that this can mean division, in which case, a car traveled at a speed of 63 miles per hour.

The lesson also addresses how a single object can be cut or partitioned, and the number of parts in one set can be compared to another number of parts in another set such as in a proportion. In addition, mastery is indicated when a student recognizes that when the numerator and denominator are the same, this indicates a whole unit (i.e., $\frac{2}{2}$, two halves of a cookie is one cookie) or a whole set (i.e., $\frac{8}{8}$, eight jellybeans are presented, and all eight were eaten, the whole set was eaten).

Agenda

Section	Activity Name	Mode	Instruction Time (minutes)
Lesson Opener	Jellybean Sort	Whole class	10
Main Activities	Ping (Intensity of sounds)	Whole class	5
	I'm Late! (Travel to Interview)	Whole class	10
	Focused Free Write	Individual	5
Free Choice Activities	Cupcake for Two	Individual or Small Group	15
	Snallygaster		
	Heartbeats		

Resources and Materials:

12 jellybeans (six of one color and two of another color, four in a third color)

6 paper plates

Lesson Opener: Jellybean Sort

The first activity is Jellybean Sort. Each student receives six plates, six jellybeans of one color (e.g, red) , two of another color (e.g, yellow), and four jellybeans of a third color (e.g, orange) At first, students use only three of the plates to sort the jellybeans by color and draw a representational diagram of red jellybeans to orange ones. Students compare their drawings to their peers' drawings.

Comparing One Subset to another Subset. Students describe the relationship between red and orange jellybeans in English words and symbolic fractional notation. "There are ________ red jellybeans to _________ orange jellybeans," and $\frac{?\,red}{?\,orange}$and they learn to write the ratio as number of red: number of orange jellybeans, $\frac{6\,red}{2\,orange}$.

Students then compare the number of red jellybeans to yellow jellybeans and orange jellybeans to yellow jellybeans. How would the fractions change if the colors are reversed? For example, what about the comparison or orange to red, yellow to red, and yellow to orange? Students then explore simpler altneratives of expressing the same fraction. Students receive prompts to recall that fractions can also be read as divided by.

Compare Subset to Whole Set. After comparing subsets to subsets, students compare subsets to whole set. Students transfer all jellybeans onto a single plate and write in English words and fractional notation how they would describe the relationship between red jellybeans and all jellybeans. They learn to express it as the number of one color "out of" the total number of jellybeans, i.e, how many "***out of***" the twelve jellybeans are red? Students also write and read the fraction of jellybeans that are red, for example, "six twelfths are red." Students draw a representational diagram of the situation and express it as a fraction.

Main Activities

There are three activities, Ping, I'm Late, and Focused Free Write. The first two activities, Ping and I'm Late, foster conceptual understanding of similarity in using fractional notations in different situations that are explained with different words. The Focused Free Write helps students see the underlying mathematical structure and connections between the Jellybean Sort, Ping, and I'm Late, to develop abstraction.

Ping (Intensity of Sounds). Students listen to two sound files, PingA.wav and PingB.wav. PingB is has a higher decibel intensity (louder) than PingA. Students draw a line that represents PingA and another line that represents PingB. Using English words, student write statements that states the ratio of volume of PingA to the volume of Ping

They also write the ratio as a fraction. Students discuss their choices of words to describe the volume of the sounds.

I’m late! (Travel to Interview). Students read and make sense of the following scenario:

Hasaan has a job interview and needs to drive from Newark, NJ, to Philadelphia, PA. The two cities are 86 miles apart. He is running late, because he slept through his alarm clock. His interview is at 12:45pm. He leaves his house at 10:45 pm. How fast does need to drive in the time available? Figure 2.1 below is a picture story to help students understand the scenario. Students describe in English words, draw pictures, and write fraction to express how fast Hasaan needs to drive.

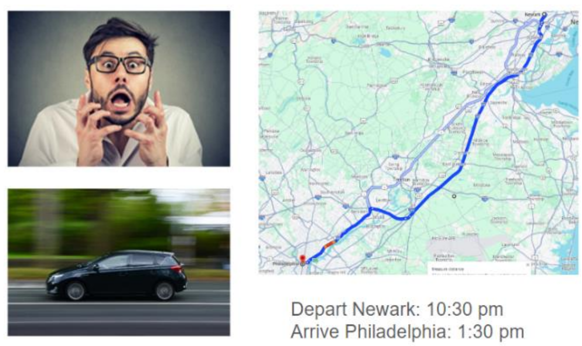


Figure 2.1 I’m Late! (Travel to Interview)

Focused Free Write. For the Focused Free Write, students attend to the following prompt individually with no support. How are Jellybean Sort, Ping (Intensity of Sounds), and I’m Late (Travel to Interview) activities similar? How are they different?

Free Choice Activities

Students choose from three activities, Cupcake for One, Snallygaster and Heartbeats. These activities give opportunities to practice using the prepositions, “to”, “out of”, and “in” as fractions.

Cupcake for One. Amor and Elijah are given just one cupcake. They were supposed to share equally, and the cupcake was cut into three equal parts. However, Amor quickly grabbed two parts. Using English words, a representational diagram, and a fraction, explain how much Amor took.

Snallygaster. In the early 1900s, people in Maryland reported sightings of a monster called Snallygaster. It is half reptile and half bird. Snallygaster has one eye, razor sharp teeth, and tentacles. In the 1980s in New Jersey, students returning from a field trip to the Pine Barrens got on a van to go back home, and they saw baby snallygasters seated in their van. See Figure 2.2 below. Look at the total number of seats. Look at how many seats are taken. Using English words, a representational diagram, and a fraction answer the following questions:

1. How would you describe the number of occupied seats to empty seats?
2. How full is the van?
3. How empty is the van?

Figure 2.2 Snallygasters

Heartbeats. Students view Figure 2.3 below and then read the following text.

Figure 2.3 Heartbeats

When Michelle sits still, her heart beats 140 times in two minutes.

1. Draw a representational drawing of Michelle's heart when she is sitting still
2. Express the situation in English and write it as a fraction.
3. Using the word "per", what is Michelle's heart rate in one minute?

When Michelle runs, her heart beats 320 times in two minutes.

1. Draw a representational drawing of Michelle's heart when she is running
2. Express the situation in English and write it as a fraction.
3. Using the word "per", what is Michelle's heart rate in one minute?

Teachers Notes

During the last phase, Free Choice Activities, teachers need to circulate the room to support students with their questions or points of confusion.

Students may struggle with understanding “out of” used for continuous conditions in Cupcake for two. If the scenario of taking two portions, some students will confuse the denominator as one, because it is correct to say in English, “I took two portions from that one cupcake.” To clarify, teachers explain that this scenario requires comparison of like objects. The two portions of cupcakes need to be compared to other portions of cupcakes, whether the other portions are those that remain or what was originally there.

Unit 2: Operations of Fractions

This unit consists of four lessons. Lesson 3 engages students with fraction multiplication; Lesson 4 provides the opportunity for students to learn fraction division; in Lesson 5, students add and subtraction fractions with like denominators; in Lesson 6, students extend their learning of addition and subtraction of fractions to expressions with different denominators. This unit also provides opportunities for students to manipulate fractions in simple equations representing simple scenarios.

Lesson 3: Fraction Multiplication

Student demonstrates mastery of this lesson when they realize that multiplying by fractions is the same as To Divide/To Share, but the numerator now gives instructions as to how many parts are to be taken. For example, there is one pizza and six friends. The pizza is divided into equal portions for six friends. Thus, each friend should get one out of six slices of the pizza or $\frac{1}{6}$ of the pizza. Because one friend, Jazmini, is not hungry, another friend, named Ben, is allowed to take two portions instead of taking one portion. Ben takes $2 \times \frac{1}{6}$ of a pizza, or $\frac{2}{1} \times \frac{1}{6}$. Ben takes two out of six slices or $\frac{2}{6}$ of a pizza.

For another example, if there were 18 jellybeans to be shared equally among six friends, each friend should $\frac{1}{6}$ of the jellybeans, but because Ben can take two portions, he gets $\frac{2}{1} \times \frac{1}{6}$ of the jellybeans or $\frac{2}{6}$ of the 18 jellybeans. Taking two out of six portions from 18 jellybeans may be represented as $\frac{2}{6} \times \frac{18}{1}$. How many jellybeans will Ben get? Students using their own intuition of how 18 jellybeans can be divided among six friends, and one person (Ben) getting two portions because another friend (Jazmini) does not want any should lead to the conclusion that each portion is 18 divided by six. Each portion is three jellybeans, and Ben taking two portions means he would take six jellybeans. Therefore, the expression $\frac{2}{6} \times \frac{18}{1}$ must be equal to six.

Agenda

Section	Activity Name (Level)	Mode	Instruction Time (minutes)
Lesson Opener	Folding	Whole class	15
Whole Class Activities	Jellybean Sort	Whole class	15
	Focused Free Write	Individual	5
Free Choice Activitites	Practice	Individual or Small Groups	10
	Road Trip		
	Lemonade		

Resources and Materials:

1 sheet of 10-inch x 10-inch origami paper

Lesson Opener: Paper Folding

The word "of" plays an important role in this lesson. Given one sheet of origami paper, students draw a square shape in their notebook to represent the origami paper and write $\frac{1}{1}$ on the square. Then, they fold the paper in half. See Figure 3.1. The first image is of the entire sheet of origami paper. The second image shows how the origami sheet is folded into two equal parts. The third image presents the sheet after the folding completes.

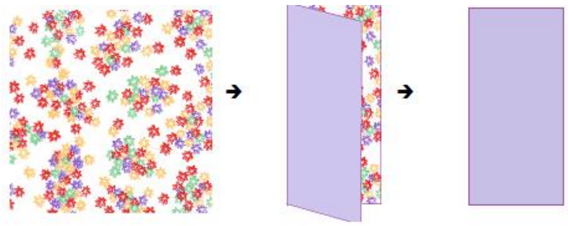


Figure 3.1 Origami Paper - Half

Students draw a representative diagram of the situation and describe what they did in English and as a fraction to represent the part of the paper shown. How can this part of the paper be written as a multiplication of fractions with the number 1 to represent the whole paper?

$$\frac{\mathbf{1}}{\mathbf{1}} \times \frac{?}{?} = \frac{\mathbf{1}}{\mathbf{2}}$$

Students make sense of dividing one paper into two equal parts (ie., dividing by two) is the same as multiplying by $\frac{1}{2}$. This activity strengthens students' understanding of the inverse relationship between multiplication and division.

Without undoing the first fold, students fold the paper in two equal parts again by making a parallel fold as shown in Figure 3.2 below. Again, students draw a representational diagram of the situation and describe what they did in English and as a fraction to represent the part of the paper shown. They state what fractions should be written on each section of the paper. Students may unfold the paper to check their answers. How can this part of the paper be written as a multiplication of fractions starting with $\frac{\mathbf{1}}{\mathbf{2}}$ of the paper?

$$\frac{\mathbf{1}}{\mathbf{2}} \times \frac{?}{?} = \frac{\mathbf{1}}{\mathbf{4}}$$

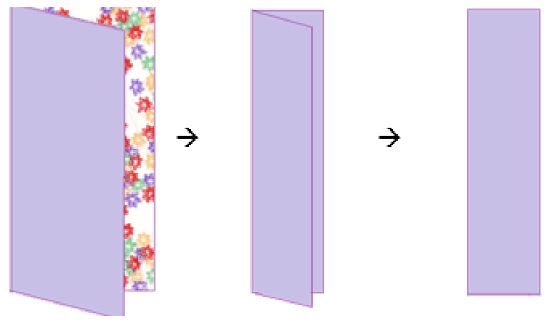


Figure 3.2 Origami Paper - Quarters

Once again, students make sense of dividing a half of the paper into two equal parts (i.e., dividing by two twice) is the same as multiplying by half twice (i.e., $\frac{1}{2} \times \frac{1}{2}$).

Thus far, students have folded one paper in half and take the half and folded it in half again.

$$\frac{\mathbf{1}}{\mathbf{1}} \times \frac{\mathbf{1}}{\mathbf{2}} \times \frac{\mathbf{1}}{\mathbf{2}} = \frac{\mathbf{1}}{\mathbf{4}}$$

Next, without undoing the folds, students fold the paper into two equal parts again, this time in the perpendicular direction. See Figure 3.3 below.

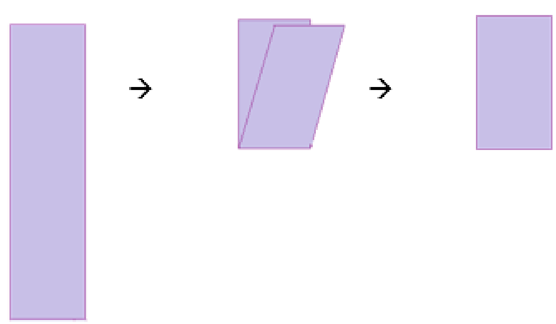


Figure 3.3 Origami Paper - Eighths

Students draw a representational diagram of the situation and describe what they just did in English and as a fraction to represent the part of the paper shown Students should recognize they have one sheet of paper folded in half, folded in half a second time, and folded in half a third time. They state what fractions should be written on each section of the origami paper. Students may unfold the paper to check their answers and recognize that $\frac{1}{1} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Finally, they fold the paper into two equal parts a fourth time. This results in a small square. See Figure 3.4.

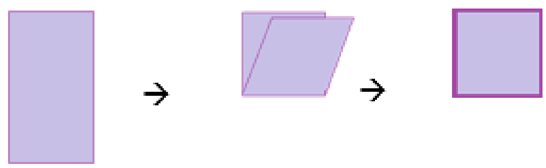


Figure 3.4 Origami Paper - Sixteenths

Students draw a representational diagram of the situation and describe what they just did in English and as a fraction to represent the part of the paper shown. They state what fractions should be written on each section of the origami paper. Students may unfold the paper to check their guess. Students discuss and reason about the pattern they notice: $\frac{1}{1} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$. At this point, students are supported through the process of multiplying fractions if they do not recognize the pattern and can come to the process on their own.

Finally, students unfold the paper, cut out each segment and reserve the pieces for the next lesson. See Figure 3.5.

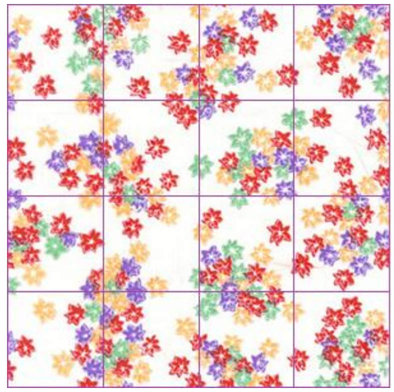


Figure 3.5 Origami Paper Pieces

What mathematical expression could students write to represent the pieces together to form a single sheet of paper? During a whole class discussion, students are introduced to the correct mathematical terminologies for each piece of paper as a “sixteenth” or “one sixteenth.”. Students are introduced to different ways these are used.

Main Activities

For the lesson’s main activities, students start with making predictions, then complete a Jellybean Sort activity, and end with a Focused Free Write.

Prediction – Presented with Figure 3.6, students predict the task at hand by connecting their understanding of the paper folding activity during the lesson opener to the jellybeans and children shown.

Figure 3.6 Prediction Activity

Jellybean Sort - Given a bag with six jellybeans and three plates, students equitably distribute the jellybeans to the three plates and draw representative diagrams of the situation. Directed to "take two thirds of the jellybeans," students interpret "of" as multiplication, students draw representative diagrams of the situation and write English and correlating math sentence with fractions that explain what they did. Students compare their drawings and statements with their peers.

Finally, students will discuss and express the mathematical process of equally dividing their six jellybeans on three plates, showing each plate having $\frac{6}{3}$ of the total jellybeans. Then, taking two of the $\frac{6}{3}$ portions lead to $\frac{6}{3} \times 2$ or $\frac{6}{3} \times \frac{2}{1}$. The direction to "take two thirds of the jellybeans compared with the process of the jellybean sort leads to two ways of thinking about the process.

Focused Free Write – The following prompt directs students' focus for the individual Free Write: How are the activities involved in Paper Folding, Jellybean Sort, and Pizza Pizza from Lesson 1 (Fraction as Verbs) similar? How are they different?

Free Choice Activities

Students choose from one practice set, Road Trip activity and Lemonade activity, to work individually or in small groups. Students who have already developed conceptual understanding of fraction multiplication may choose the practice to develop procedural fluency. The Road Trip and Lemonade activities continue to support the development of conceptual understanding.

Practice

This activity develops procedural fluency and exposes students to multiplicative inverse.

1. $\frac{3}{10} \times -20 =$	14. $\frac{1}{-5} \times -\frac{5}{2} =$
2. $\frac{1}{5} \times \frac{5}{2} =$	15. $-\frac{4}{6} \times -\frac{3}{8} =$
3. $\frac{4}{6} \times \frac{3}{8} =$	16. $\frac{20}{-6} \times \frac{-3}{-10} =$
4. $\frac{20}{6} \times \frac{3}{10} =$	17. $\frac{1}{5} \times \frac{5}{1} =$
5. $\frac{-3}{-10} \times 20 =$	18. $\frac{2}{5} \times \frac{5}{2} =$
6. $\frac{1}{-5} \times \frac{5}{2} =$	19. $\frac{4}{5} \times \frac{5}{4} =$
7. $-\frac{4}{6} \times \frac{3}{8} =$	20. $\frac{1}{6} \times \frac{6}{1} =$
8. $\frac{20}{-6} \times \frac{3}{10} =$	21. $\frac{5}{6} \times \frac{-6}{5} =$
9. $-\frac{3}{10} \times -20 =$	22. $\frac{7}{1} \times -\frac{1}{7} =$
10. $\frac{-1}{-5} \times \frac{5}{-2} =$	23. $\frac{5}{7} \times \frac{-7}{5} =$
11. $-\frac{4}{6} \times \frac{3}{-8} =$	24. $\frac{6}{7} \times \frac{-7}{6} =$
12. $\frac{20}{-6} \times \frac{-3}{10} =$	25. $\frac{-5}{8} \times \frac{-8}{5} =$
13. $\frac{-3}{10} \times 20 =$	26. $\frac{11}{8} \times \frac{8}{11} =$

Road Trip. Three friends are making a road trip from New Jersey to North Carolina. The entire trip will take 15 hours. How many hours will each friend drive if everyone drives the same amount of time? Write expressions in English words and multiplication equation with a fraction and the result as a fraction.

Figure 3.7 Pictorial Story for Road Trip Activity

Lemonade. A pitcher of lemonade contains eight cups. Two out of eight cups are lemon juice, and six cups are ice water. Use fraction multiplication to solve and answer the questions.

1. If you want to make 16 cups of lemonade, how many cups of lemon juice do you need and how many cups of ice water?
2. If you want to make 12 cups of lemonade, how many cups of lemon juice do you need and how many cups of ice water?
3. If you want to make 10 cups of lemonade, how many cups of lemon juice do you need and how many cups of ice water?
4. If you want to make 6 cups of lemonade, how many cups of lemon juice do you need and how many cups of ice water?
5. If you want to make 4 cups of lemonade, how many cups of lemon juice do you need and how many cups of ice water?

6. If you want to make 2 cups of lemonade, how many cups of lemon juice do you need and how many cups of ice water?

Teachers Notes

Students will need support with vocabulary and mathematical terminologies. For example, using "sixteenths" and "one sixteenth" to describe of each piece of the origami paper during the Paper Folding lesson opener may be new to some students. It is important that students understand the different uses in contextual situations. If the word sixteenth is used, the word needs to be plural, because you have 16 of them not just one.

The phrase would be "sixteen sixteenths" and the math expression would be $\frac{16}{16} = 1$. If using one sixteenth, the sentence would be "sixteen of one sixteenth make one whole", where the word "***of***" in this context refers to multiplication. Therefore, the math sentence for the English sentence is $16 \times \frac{1}{16}$. Using fraction multiplication, students can be supported through the mathematical algorithm for sixteen "***of***" one sixteenth.

$$16 \times \frac{1}{16} = \frac{16}{1} \times \frac{1}{16} = \frac{16 \times 1}{1 \times 16} = \frac{16}{16} = 1$$

For the Jellybean Sort during Main Activities, students are encouraged to think about two ways to complete fraction multiplication. Per student conceptual understanding, it is important to support their thinking about the connection between the following two interpretations:

1. "two thirds of ALL jellybeans"

$\frac{2}{3} \times$ ALL the jellybeans

$\frac{2}{3} \times 6$

$$\frac{2}{3} \times \frac{6}{1} = \frac{2\times6}{3\times1} = \frac{12}{3} = 4$$

six divided by three is two

2. "divide **ALL jellybeans** **into three equal parts** and take two parts"

$$\frac{6}{3} \times \frac{2}{1} = \frac{6\times2}{3\times1} = \frac{12}{3}, = 4$$

In the Focused Free Write (How are Folding, Jellybean Sort, and Pizza Pizza from Lesson 1, Fraction as Verbs similar), students need to explore the idea that multiplying a number by a fraction is similar to giving instructions. A number is split into equal portions as instructed by the denominator. Then the numerator directs how many portions to take.

In the Practice activity, numbers 17 through 26, students may struggle with understanding the multiplicative inverse property ($\frac{A}{1} \times \frac{1}{A} = 1$). To support students who struggle with these exercises, students may be encouraged to think about scenarios pf multiplication of parts to a whole yield a whole. The following are two examples:

- Let's pretend I have one third of a pizza. You have one third of a pizza. The principal has one third of a pizza. How many thirds are there altogether? How many pizzas does three thirds make?
- Let's pretend I have a sheet of paper. Tell me how many equal parts you want to cut it into? (eg. 8). As a fraction we say, we took one sheet of paper and cut it up into eight parts. What fraction is that? (eg. $\frac{1}{8}$) Do you agree we have eight little pieces of one eighths? We can write this as $\frac{1}{8} \times \frac{8}{1} = \frac{1\times8}{8\times1} = \frac{8}{8} = 1$.

During Free Choice Activities, students will be supported as needed by teachers circulating the room to answer questions or clarify points of confusion.

Lesson 4: Division

Students demonstrate mastery of division with fractions through their understanding of the inverse relationship between the arithmetic operations of multiplication and division. Students will take the reciprocal of the divisor and multiply (eg. $6 \div \frac{2}{3} = \frac{6}{1} \times \frac{3}{2}$). This lesson teaches how to use fractions as divisors. For example, 6 divided by a half.

Agenda

Section	Activity Name (Language Level)	Mode	Instruction Time (minutes)
Lesson Opener	Jellybean Giveaway	Whole class	10
Main Activities	Folding	Whole class	10
	Focused Free Write	Individual	5
Free Choice Activities	Practice Tip Lemonade Car Ride	Individual or Small Group	20

Resources and Materials:

6 jellybeans or any color

3 paper plates

Lesson Opener: Jellybean Giveaway

With six jellybeans and three paper plates, students equally divide the six jellybeans into three plates as shown in Figure 4.1 below and write an English statement, draw a representation of the plate as rectangle split into three equal parts, and write a fraction that shows the action. Students are familiar with this routine as they have done this several times in previous lessons. They record the number of jellybeans on one plate by using the word “per”.

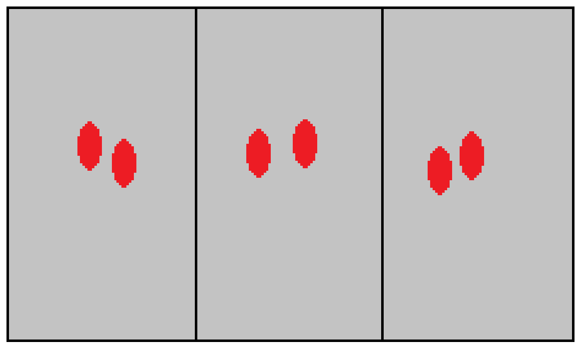


Figure 4.1 Jellybeans on Three Plates

Students repeat the process by equally dividing the six jellybeans into two plates, provide an English sentence, draw a representation of the plate as rectangle split into two equal parts, indicate the process with a fraction statement, and use the word “per” to describe the number of jellybeans on each plate.

Students remove another paper plate to repeat the process for a third time, and state how many jellybeans on one plate by using the word “per”. They draw a representation of the plate as a rectangle.

Next, students rip a paper plate into halves and represent six jellybeans per half a plate. If there are six jellybeans per half of a plate, how many jellybeans would be on the whole plate? Students draw a representation of the two halves of the plate, each with six jellybeans and state how many jellybeans are on one whole plate by using the word “per”.

How can this scenario be represented as a fraction division? After students reason and discuss the six jellybeans per half plate scenario, they compare their representations with Figure 4.2 below.

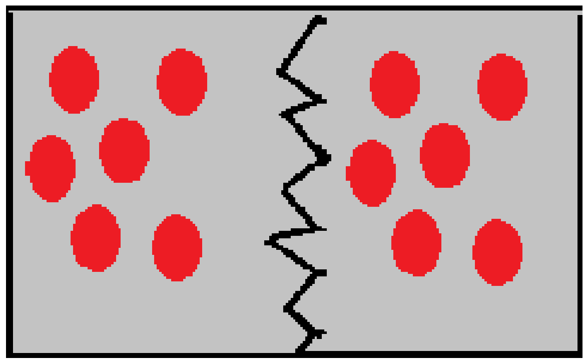


Figure 4.2 Jellybeans on Half a Plate

Students rip a second paper plate into thirds, draw a representation of a third of a plate, and write a mathematical expression with fractions to represent division. They state how many jellybeans are on one whole plate by using the word "per".

Finally, students rip the last paper plate into quarters and draw a representation of a quarter of a plate with six jellybeans. They use mathematical expressions with fractions to represent division and state how many jellybeans are on one whole plate by using the word "per". Students may eat the jellybeans and throw away all plates.

If there are "six jellybeans on two thirds of a plate", how many jellybeans are on one whole plate? Students represent this problem situation by providing drawings and fraction division and state how many jellybeans are on one whole plate by using the word "per".

Main Activities

There are two main activities for this lesson, Fractions Divided by Fractions and Focused Free Write.

Fractions Divided by Fractions - Using the 16 origami cutout pieces from the Lesson 3 Paper Folding activity, students solve the fraction divided by a fraction problem: $\frac{\frac{1}{16}}{\frac{1}{2}}$. They draw a diagram and use fraction division to solve the problem, state how the result should be read when using the word "per", and determine how many sixteenths are in the result. Students compare their results with Figure 4.3 below.

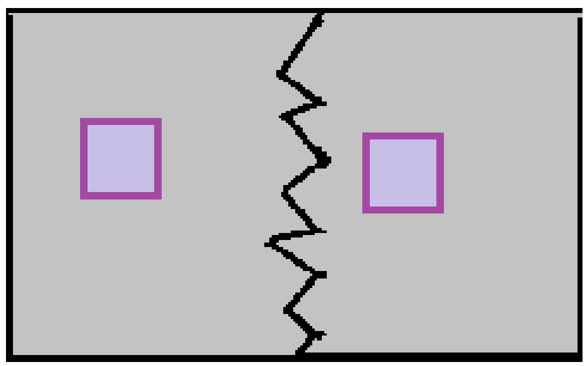


Figure 4.3 Fraction Divided by a Fraction

Students solve the following problems by showing and explaining how they arrived at the answers.

1. $\frac{\frac{1}{16}}{\frac{1}{2}} = ?$
2. $\frac{\frac{1}{16}}{\frac{1}{3}} = ?$
3. $\frac{\frac{1}{16}}{\frac{1}{4}} = ?$

Students then use the relationship between the operations of multiplication and division to rewrite the fraction divided by a fraction expression as the numerator multiplied by the reciprocal of the denominator and solve using their understanding of fraction multiplication in Lesson 3. They compare their answers using the conceptual understanding of fraction division to fraction multiplication.

Focused Free Write. How are Fractions Divided by Fractions, the Jellybean activity and Lesson 3 - Multiplication similar? How are they different?

Free Choice Activities

Students choose from one practice set, Tip, Lemonade, and Car Ride. The practice set develops procedural fluency based on students' conceptual understanding of fraction division. The Tip, Lemonade and Car Ride activities further develop students' conceptual understanding.

Practice. Students may use representation drawings or the algorithm for the following problem set.

1. $\frac{\frac{3}{10}}{-\frac{1}{20}} =$	*5.* $\frac{\frac{-3}{-10}}{\frac{1}{20}} =$	*9.* $\frac{-\frac{3}{10}}{-\frac{1}{20}} =$	*13.* $\frac{\frac{-3}{10}}{\frac{1}{20}} =$
2. $\frac{\frac{1}{5}}{\frac{2}{5}} =$	*6.* $\frac{\frac{1}{-5}}{\frac{2}{5}} =$	*10.* $\frac{\frac{-1}{-5}}{\frac{-2}{5}} =$	*14.* $\frac{\frac{1}{-5}}{-\frac{5}{2}} =$
3. $\frac{\frac{4}{6}}{\frac{8}{3}} =$	*7.* $\frac{-\frac{4}{6}}{\frac{8}{3}} =$	*11.* $\frac{-\frac{4}{6}}{\frac{-8}{3}} =$	*15.* $\frac{-\frac{4}{6}}{-\frac{3}{8}} =$
4. $\frac{\frac{20}{6}}{\frac{10}{3}} =$	*8.* $\frac{\frac{20}{-6}}{\frac{10}{3}} =$	*12.* $\frac{\frac{20}{-6}}{\frac{10}{-3}} =$	*16.* $\frac{\frac{20}{6}}{\frac{3}{10}} =$

Tip. In a quarter of an hour, Brigitte makes $18 in tips. How much money does Brigitte make in one hour?

Lemonade. Five liquid ounces of lemonade has $\frac{5}{2}$ tablespoons of sugar. How would you express this as a ratio of liquid to sugar as a fraction? How much liquid is there per tablespoon of sugar?

Car Ride. A driver drives an average of 60 miles per hour. He drove for three quarters of an hour and used three halves or one and a half gallons of gas. Fuel costs $3.15 for every <u>nine tenths</u> of a gallon, not a full gallon. See Figure 4.4.

Figure 4.4 Pictorial Story for Car Activity

1. How far did he drive
2. What is the fuel efficiency of this car in miles per gallon?
3. How much money does one gallon of gas cost?
4. How much gasoline money did he spend on this trip?

Teachers Notes

Developing conceptual understanding requires students to understand the word "per" has a non-negotiable characteristic, which is important to emphasize when explaining fractions like $\frac{3}{\frac{1}{4}}$. Using cookies and plates, the fraction is read as "three cookies per one quarter plate", and it means "there must be three cookies on every quarter plate." Some students will think that there are three cookies in the first quarter and no cookies on the others. Therefore, for four quarters, there must be three cookies in the first quarter, three cookies in the second quarter, three cookies on the third quarter, and three cookies on the fourth quarter. There are no exceptions.

Supporting students with this non-negotiable nature of the use of "per" during the Free Choice Activities, teachers need to circulate the room to address students' questions.

Lesson 5: Adding and Subtracting (Same Denominator)

Student demonstrates mastery of this lesson by successfully adding and subtracting fractions with the same denominator using their understanding of adding integers. They add and subtract the numerators of the fractions when the fractions are represented in the same unit. Students also solidify their understanding of the inverse relationship between addition and subtraction. What are two things that are inversely related (i.e., two things that undo each other)?

Agenda

Section	Activity Name	Mode	Instruction Time (minutes)
Lesson Opener	Class of Jellybeans	Whole class	15
Main Activities	Navigating the Number Line	Whole class	15
	Eating Jellybeans		
Free Choice Activities	Problem Set Cookies Meeting a Friend Space Joy Ride	Individual lor small groups	10

Resources and Materials:

20 jellybeans

Lesson Opener: Class of Jellybeans

The lesson opener activity is Jellybean Sort. Twenty jellybeans are unevenly distributed among students.

Students write "There are twenty jellybeans in total." Then each student writes a fraction that compares their number of jellybeans to the total number of jellybeans (e.g., $\frac{5}{20}$) . After discussion, students write an English sentence and explain how to determine the number of jellybeans in the room. Students write fractional expressions to show there are 20 jellybeans in the room. Having completed the previous four lessons, students should be ready to represent each student's number of jellybeans as a fractional representation of the 20 total jellybeans. What is $\frac{1}{20} + \frac{2}{20} + \frac{3}{20} + \frac{5}{20} + \frac{2}{20} + \frac{1}{20} + \frac{2}{20} + \frac{4}{20} = ?$

In English, the equation can be read as "There is one twentieth combined with two twentieth combined with three twentieth…etc". Adding all the numerators together gives fraction form $\frac{20}{20}$.

Students then draw a portion of a number line from $\frac{0}{20}$ to $\frac{20}{20}$, spacing each number half an inch apart. Using a penny, students move the coin along the number line according to the fraction they are adding. The penny lands on $\frac{20}{20}$ to show that there are indeed 20 jellybeans in the room.

Main Activities

There are two main activities in lesson 5, Navigating the Number Line and Eating Jellybeans. The learning objective is adding and subtracting negative fractions. Students extend their understanding of addition and subtraction using the number line they drew during the lesson opener to perform these arithmetic operations with negative fractions.

Navigating the Number Line. Using the penny, students navigate the number line according to the given addition statements. See Figure 5.1 below. Students understand adding a positive to a positive means placing the penny on the first number and moving the penny towards the right by the increments stated in the second number. What is $\frac{1}{8} + \frac{1}{8}$? Then students are presented with the problem $\frac{1}{8} + \frac{-2}{8}$. Students place the penny on $\frac{1}{8}$ and move the penny to the left two units.

Students solve the problem $\frac{-1}{8} + \frac{-2}{8}$. Students understand the appearance of an addition symbol means to move the penny to the right, but the negative sign in the second fraction is read as "opposite". Students move the penny to the "opposite of right" which is left $\frac{2}{8}$. The penny is on $\frac{-1}{8}$, therefore the answer is $\frac{-3}{8}$.

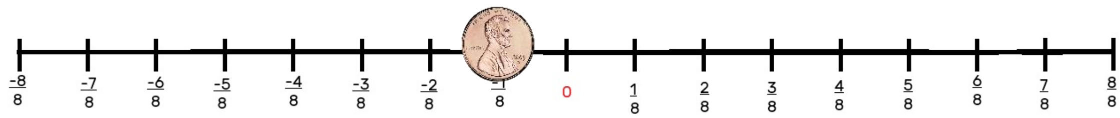


Figure 5.1 Number Line - Eighths

Presented with $\frac{-1}{8} - \frac{2}{8}$, students describe where the penny should be placed and how to move the penny and solve mathematically. Students are then presented with $\frac{-1}{8} - \frac{-2}{8}$. They are to explain the equation in English as "place the penny on negative one eighth and move the penny in the direction opposite of subtraction", which means move it to the right.

Eating Jellybeans. After completing the penny activity, students eat no more than two jellybeans on their plates. Students state as a fraction how much of the jellybeans are left. They write English sentence and a fraction that compares the number of jellybeans they ate as compared to the entire original count. They express eaten jellybeans as negative fractions (i.e., eating three jellybeans out of twenty can be written as $\frac{-3}{20}$). Students ask their classmates what fraction they all ate. They then calculate what fraction of jellybeans are left.

Free Choice Activities

Students choose from one Practice set and three activities, Money Problems, Cookies, and Meeting a Friend. The Practice set develops procedural fluency, and the three activities promote conceptual understanding.

Practice

1. $\frac{1}{2}+\frac{1}{2}=$	11. $-\frac{1}{2}-\frac{-2}{2}=$	21. $\frac{-2}{2}+\frac{1}{2}=$	31. $-\frac{6}{2}-\frac{17}{2}=$
2. $\frac{1}{2}+\frac{-1}{2}=$	12. $\frac{1}{2}-\frac{2}{2}=$	22. $\frac{-2}{2}+\frac{-1}{2}=$	32. $-\frac{6}{2}-\frac{-17}{2}=$
3. $\frac{1}{2}-\frac{1}{2}=$	13. $\frac{1}{2}-\frac{-2}{2}=$	23. $\frac{-2}{2}-\frac{1}{2}=$	33. $\frac{17}{2}+\frac{6}{2}=$
4. $\frac{1}{2}-\frac{-1}{2}=$	14. $-\frac{1}{2}+\frac{2}{2}=$	24. $\frac{-2}{2}-\frac{-1}{2}=$	34. $\frac{17}{2}+\frac{-6}{2}=$
5. $\frac{-1}{2}+\frac{1}{2}=$	15. $-\frac{1}{2}+\frac{-2}{2}=$	25. $\frac{6}{2}+\frac{17}{2}=$	35. $\frac{17}{2}-\frac{6}{2}=$
6. $\frac{-1}{2}+\frac{-1}{2}=$	16. $-\frac{1}{2}-\frac{2}{2}=$	26. $\frac{6}{2}+\frac{-17}{2}=$	36. $\frac{17}{2}-\frac{-6}{2}=$
7. $\frac{-1}{2}-\frac{1}{2}=$	17. $\frac{2}{2}+\frac{1}{2}=$	27. $\frac{6}{2}-\frac{17}{2}=$	37. $-\frac{17}{2}+\frac{6}{2}=$
8. $\frac{-1}{2}-\frac{-1}{2}=$	18. $\frac{2}{2}+\frac{-1}{2}=$	28. $\frac{6}{2}-\frac{-17}{2}=$	38. $-\frac{17}{2}+\frac{-6}{2}=$
9. $\frac{1}{2}+\frac{2}{2}=$	19. $\frac{2}{2}-\frac{1}{2}=$	29. $-\frac{6}{2}+\frac{17}{2}=$	39. $-\frac{17}{2}-\frac{6}{2}=$
10. $\frac{1}{2}+\frac{-2}{2}=$	20. $\frac{2}{2}-\frac{-1}{2}=$	30. $-\frac{6}{2}+\frac{-17}{2}=$	40. $-\frac{17}{2}-\frac{-6}{2}=$

Money Problems. Lillianna owes her brother one fifth of what she made as a barista for one week.

1. Write the amount she owes as a signed fraction.

2. Write an addition or subtraction expression that shows the relationship between how much she has, how much she owes, and what fraction remains if she pays him back.
3. If she made $300 in one week, how much money can she keep?

Cookies. There are 36 cookies in a box. Jaime has seven people in his family: Mom, Dad, Abuelito, Abuelita, his older sister Mia, his younger brother Juan and himself. His parents ate two sixths of the cookies. His grandparents ate one sixth. What fraction was left for him to share with his two siblings? If they equally shared what was remaining, how many cookies did Jaime eat?

Meeting a Friend. Sarai lives three and three fourths of a mile from her friend, Jennifer. They agreed to meet to study together. Sarai is to bring chips and Jennifer will have pretzels. Sarai takes the bus that is right outside her apartment and goes straight to Jennifer's house. After getting on the bus, Sarai realizes she forgot the chips. She gets off the bus one quarter of a mile from the stop. She walks back to her place to get the chips. When she goes outside, Jennifer texts her and tells her that they will meet at Deysi's house, which is three fourths of a mile away from Sarai's apartment in the opposite direction. Sarai decides to walk to Deysi's house. What is the distance between Jennifer's house and Deysi's house? How many miles did Sarai travel that day?

Space Joy Ride. Velocity is speed with a sign. Moving along a number line, velocity is positive when moving to the right. Moving to the left, the velocity is negative. Earth is 12,756 kilometers in diameter. The spacecraft left Earth and traveled $\frac{33{,}333}{3}$ km, and then it traveled $\frac{-84{,}333}{3}$ km. Where is it?

Teachers Notes

Because elementary schools teach reading the subtraction operand as “take away”, subtraction of negative numbers can cause confusion. Students’ understanding of subtracting integers is likely to transfer to their sense-making of subtracting fractions. They may process the math sentence 2 – 5 as “take away five from two.” Similarly, the sentence 3 - -2 could be read as “take away take away two from three” which leads misunderstandings of the meaning of subtracting negative numbers, including negative fractions. Students reading the math statement -2 - -3 as “take away take away three from negative two” seems like an impossible scenario.

The linguistic aspect tempts teachers to use double negative grammar structure to explain subtracting a negative. However, this can lead to confusion because some dialects of English, common non-English languages like French, and Shakespearean English use double negatives for the purpose of emphasis. For example, Shakespear’s Hamlet V.i.131

[Hamlet] What man dost thou dig it for?

[First Clown] For no man, sir.

[Hamlet] What woman then?

[First Clown] For none neither.

The phrase “For none neither” can be interpreted to mean “Also, not a woman.”

Due to ambiguity in the use of double negatives in natural languages, the word “opposite” is emphasized rather than “not”. Students learn the negative sign associated with number is read as “opposite of” Therefore, subtracting a negative like 1 - -2 is read as “start at positive one then move in the direction *opposite of subtraction* two spaces” on a number line.

Because English is read from left to right, some students want to read number sentences in order from left to right. In this situation, it may be more efficient to teach the subtraction operand as opposite. For example, 2 + 5 would be read as “start at two and move five spaces to the right on a number line.” The plus sign is read as “move to the right”.

However, direction changes with the appearance of negative symbols. Every instance of the negative sign, whether it is a subtraction operand or a negative sign in front of a number, the symbol is to be read as opposite. For example, 2 - 5 would be read as “start at two, move *in the opposite direction* five units to the right”, in this case, move five units to the left. The number sentence, 2 - -5, would be read as “start at positive two, move in the opposite of the opposite of right five units”, in this case, move five units to the right.

As with previous lessons, during Free Choice Activities, teachers need to circulate the room to support students with their questions or to clarify points of confusion.

Lesson 6: Addition and Subtraction with Different Denominators

Students demonstrate mastery of this lesson when they can convert fractions into equivalent fractions with a common denominator $(i.e., \frac{1}{6}+\frac{1}{3}=\frac{1}{6}+\frac{1\times2}{3\times2})$. A multiplication table can be provided for students to facilitate finding equivalent fractions.

Agenda

Section	Activity Name	Mode	Instruction Time (minutes)
Lesson Opener	Don't Let Me Down	Whole class	10
Main Class Activities	Prediction	Individual	5
	Practice	Individual	5
	Focused Free Write	Individual	5
	Eating Jellybeans	Whole class	10
Free Choice Activities	Practice Sets	Individual or Small Group	10
	Michael's Money Problems		
	Madison's Money Problems		
	Meeting a Friend		

Resources and Materials:

4 strips of paper with identical dimensions

Number lines with positive integers expressed as improper fractions

Lesson Opener: Don't Let Me Down

The opener is designed to support students' conceptual understanding of equivalent fractions.

Don't Let Me Down. Students gain an understanding that finding equivalent fractions that are less than one is similar to finding fractions that represent integers. Presented with number lines of fractions that are equivalent to integers. For example, students place a penny on the number three on the top number line. Students move the penny down to the next number line in the same position. That number has a value of three also, because 12 divided by four is three. If the students keep moving it down number lines, the value of the number will remain three. These numbers are all equivalent. See Figure 6.1 below.

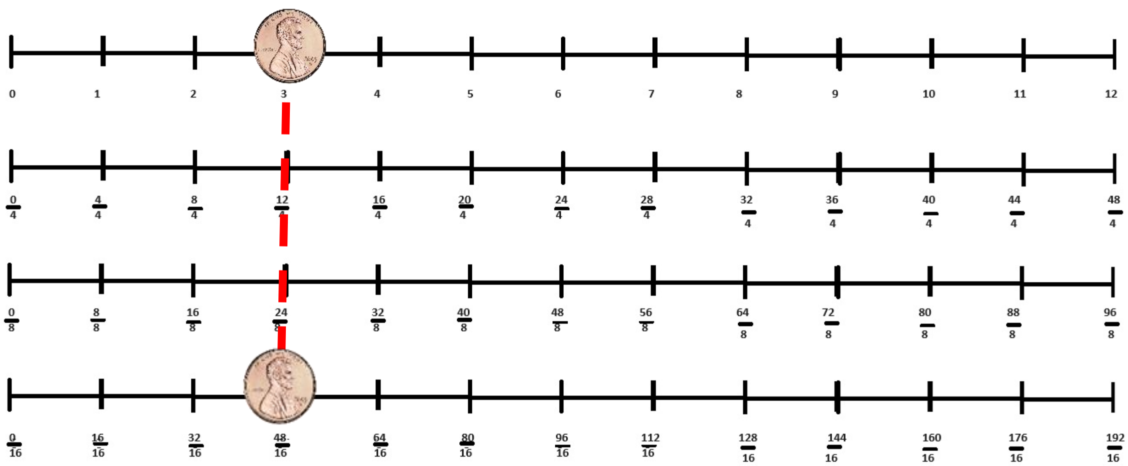


Figure 6.1 Number Lines - Equivalent

Students state fractions that are equivalent to 10, when the denominator is four, eight and sixteen.

Receiving four strips of paper, students fold one strip into halves. They fold the second strip into quarters, the third into eighths, and the last into sixteenths. They label the sections of each strip accordingly. See Figure 6.2 below.

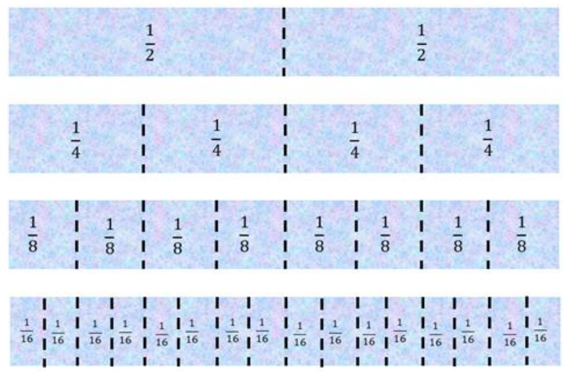


Figure 6.2 folded strips of paper

Using an 11 x 17 sheet of paper, students orient the paper landscape and draw each strip from figure 6.2 (above) as a number line. The folded line marks point zero. They replicate the folded strips in both positive and negative fractions. See Figure 6.3.

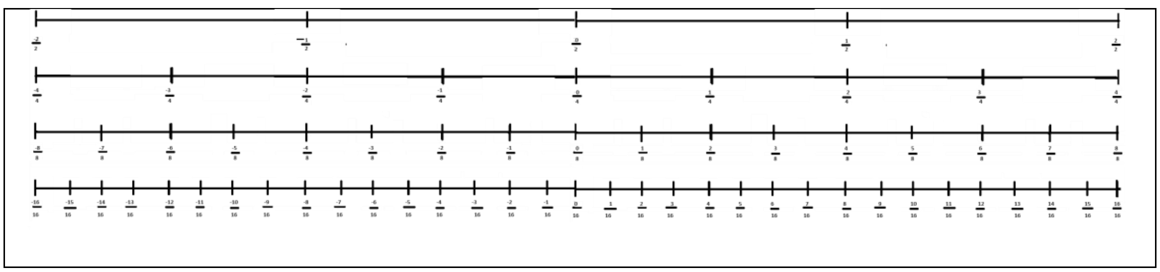


Figure 6.3 Sample Hand Drawn Number Line

Student state equivalent fractions for $\frac{1}{2}$ when then denominator is four, eight, and sixteen.

Main Activities

There are four activities, Prediction, Practice, Focused Free Write, and Eating Jellybeans. While Practice provides opportunities to develop procedural fluency, Eating Jellybeans promotes conceptual knowledge.

Prediction. Students are asked to predict on the methodology to solve $\frac{3}{8}+\frac{2}{16}$

Practice. Students solve $\frac{3}{8} + \frac{2}{16}$ by placing the penny on the fraction $\frac{3}{8}$. Because the other fraction has a larger denominator, they move the penny directly down to number below where all denominators are 16. Having found the common denominator, students use their knowledge from the previous lesson (Lesson 5) to add fractions with like denominators.

Students solve the following problems using the number lines they created.

1. $\frac{3}{8} + \frac{-2}{16}$ 2. $\frac{3}{8} - \frac{2}{16}$ 3. $\frac{3}{8} - \frac{-2}{16}$	4. $-\frac{3}{8} + \frac{2}{16}$ 5. $-\frac{3}{8} + \frac{-2}{16}$	6. $-\frac{3}{8} - \frac{2}{16}$ 7. $-\frac{3}{8} - \frac{-2}{16}$

Focused Free Write. How is adding and subtracting fractions with the same denominator similar to adding and subtracting fractions with like denominators? How are they different?

Eating Jellybeans. The Eating Jellybeans activity utilizes students' understanding of equivalent fractions to add and subtract fractions with different denominators. Students draw three number lines. All three number lines start at zero and end at the value of 12. The first number line has a denominator of one, the second has a denominator of two, and the third has a denominator of four.

Students eat one fourth of the jellybeans they are given. They write down a negative fraction that shows how much jellybeans were eaten (e.g., $\frac{-12}{4}$). Then they eat half of what is left and write down a negative fraction that expresses how many jellybeans were eaten. (eg. $\frac{-8}{2}$) Students write a fraction addition math expression to show what is left. They may use the penny provided and the number lines they drew.

Students learn the following algorithm to find a common denominator in Figure 6.4 below. The first step is to find the largest denominator. Then, students decide if the other denominators are factors of the largest one. If yes, the largest denominator is the "common denominator". Students then choose one fraction to start altering. The first fraction's denominator is a multiplicand, and they establish the multiplier that will increase the multiplicand to the common denominator. The same multiplier is then used for the numerator. This process is repeated for all the fractions. When all fractions have been altered to have like denominators, students use their understanding from Lesson 5 to find the sum.

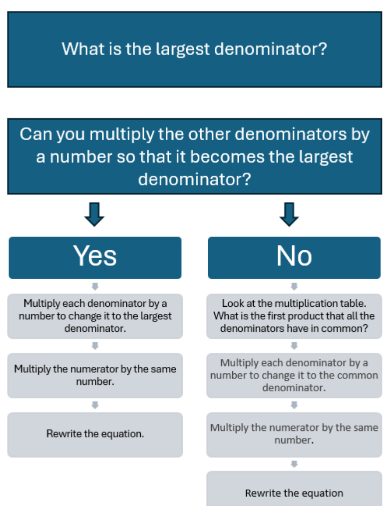


Figure 6.4 Algorithm for finding common denominator

If the other denominators are not factors of the largest denominator, then they must establish a common denominator. They use a multiplication table to find a common product of the denominators. The common product becomes the common denominator.

Free Choice Activities

Students choose between four different practice sets, Michael Owes Money, Madison Owes Money and Meeting a Friend. Students gain procedural fluency with the practice sets. In sets one and three, students solve problems with two fractions, and in sets two and four, students work with three fractions. The other activities develop conceptual understanding with the other activities.

Practice Set 1

$\frac{2}{4} + \frac{1}{8} =$	$\frac{2}{3} - \frac{1}{9} =$	$\frac{2}{81} - \frac{2}{9} =$
$\frac{-3}{21} + \frac{2}{7} =$	$\frac{-3}{9} - \frac{2}{18} =$	$\frac{-2}{21} - \frac{2}{3} =$
$\frac{2}{3} + \frac{-4}{24} =$	$\frac{2}{40} - \frac{-4}{20} =$	$\frac{2}{36} - \frac{-2}{6} =$
$\frac{-5}{8} + \frac{-2}{72} =$	$\frac{-5}{42} - \frac{-2}{7} =$	$\frac{-2}{54} - \frac{-2}{9} =$

Practice Set 2

$\frac{1}{2} + \frac{2}{4} + \frac{2}{8} =$	$\frac{1}{2} + \frac{2}{4} - \frac{2}{8} =$	$\frac{-1}{2} + \frac{2}{8} - \frac{2}{4} =$
$\frac{1}{2} + \frac{-2}{4} + \frac{2}{8} =$	$\frac{1}{2} + \frac{-2}{4} - \frac{2}{8} =$	$\frac{-1}{2} + \frac{-2}{8} - \frac{2}{4} =$
$\frac{1}{2} + \frac{2}{4} + \frac{-2}{8} =$	$\frac{1}{2} + \frac{2}{4} - \frac{-2}{8} =$	$\frac{-1}{2} + \frac{2}{8} - \frac{-2}{4} =$
$\frac{1}{2} + \frac{-2}{4} + \frac{-2}{8} =$	$\frac{1}{2} + \frac{-2}{4} - \frac{-2}{8} =$	$\frac{-1}{2} + \frac{-2}{8} - \frac{-2}{4} =$

Practice Set 3

$\frac{2}{3} + \frac{2}{8} =$	$\frac{2}{3} - \frac{2}{4} =$	$\frac{2}{2} - \frac{2}{11} =$
$\frac{-2}{4} + \frac{2}{5} =$	$\frac{-2}{2} - \frac{2}{5} =$	$\frac{-2}{4} - \frac{2}{7} =$
$\frac{2}{7} + \frac{-2}{8} =$	$\frac{2}{7} - \frac{-2}{9} =$	$\frac{2}{2} - \frac{-2}{9} =$
$\frac{-2}{4} + \frac{-2}{9} =$	$\frac{-2}{5} - \frac{-2}{9} =$	$\frac{-2}{6} - \frac{-2}{4} =$

Practice Set 4

$\frac{1}{3} + \frac{2}{4} + \frac{3}{8} =$	$\frac{1}{3} - \frac{2}{4} - \frac{3}{8} =$	$\frac{1}{3} - \frac{2}{6} - \frac{3}{5} =$
$\frac{1}{3} + \frac{2}{4} + \frac{-3}{8} =$	$\frac{1}{3} - \frac{2}{4} - \frac{-3}{8} =$	$\frac{1}{3} - \frac{2}{6} - \frac{-3}{5} =$
$\frac{1}{3} + \frac{-2}{4} + \frac{3}{8} =$	$\frac{1}{3} - \frac{-2}{4} - \frac{3}{8} =$	$\frac{1}{3} - \frac{-2}{6} - \frac{3}{5} =$
$\frac{-1}{3} + \frac{2}{4} + \frac{3}{8} =$	$\frac{-1}{3} - \frac{2}{4} - \frac{3}{8} =$	$\frac{-1}{3} - \frac{2}{6} - \frac{3}{5} =$
$\frac{-1}{3} + \frac{-2}{4} + \frac{3}{8} =$	$\frac{-1}{3} - \frac{-2}{4} - \frac{3}{8} =$	$\frac{-1}{3} - \frac{-2}{6} - \frac{3}{5} =$
$\frac{-1}{3} + \frac{-2}{4} + \frac{-3}{8} =$	$\frac{-1}{3} - \frac{-2}{4} - \frac{-3}{8} =$	$\frac{-1}{3} - \frac{-2}{6} - \frac{-3}{5} =$

Michael Owes Money. Michael has been borrowing money from his friends, Malachai, John, Ancell, and Madison. Michael just got his paycheck. However, he owes Malachai one fourth of that amount. He owes John on third, and he owes Ancell and Madison each one eighth. Does he have enough money to pay back his friends? Write a math equation with fractions to state whether Michael has enough money to pay back his friends.

Madison Owes Money. If we take a look at Madison’s situation, she owes three-fourths of her $240 paycheck to all her friends combined. However, Michael forgave his amount, which is one third of the total of what she owes. How would you write that situation as a math expression? Use a negative number to note money that is owed. How much of her paycheck is left, if she paid back all that she owed after Michael forgave his loan to her?

Meeting a Friend. Sarai lives one mile from her friend, Jennifer. They agreed to meet to study together. Sarai is to bring chips and Jennifer will have pretzels. Sarai takes the bus that is right outside her apartment and goes straight to Jennifer's house. After getting on the bus, Sarai realizes she forgot the chips. She gets off the bus one third of a mile from the stop. She walks back to her place to get the chips. When she goes outside, Jennifer texts her and tells her that they will meet at Deysi's house, which is four fifths of a mile away from Sarai's apartment in the opposite direction. Sarai decides to walk to Deysi's house. What is the distance between Jennifer's house and Deysi's house? How many miles did Sarai travel that day?

Teachers Notes

While there are different algorithms for finding common denominators, the conceptual understanding of how they are found plays an important role in this lesson. A second algorithm for finding a common denominator is to simply multiply all the denominators. For example, $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$, the common denominator would $2 \times 3 \times 4 = 24$. Then, for each fraction, find a factor that will result in the common denominator. Multiply the numerator by that factor. Using this example, we derive new fractions.

$$24 \div 2 = 12 \qquad 24 \div 3 = 8 \qquad 24 \div 4 = 6$$

$$\frac{1 \times 12}{24} + \frac{2 \times 8}{24} + \frac{3 \times 6}{24} = \frac{12}{24} + \frac{16}{24} + \frac{18}{24}$$

Then the new numerators are added together. $\frac{12+16+18}{24} = \frac{46}{24}$.

During the last phase, Free Choice Activities, teachers need to circulate the room to support students with their questions or points of confusion.

Unit 3: Algebra

Students demonstrate mastery of this lesson by picking two points on a line and determine the signed slope of the line. The Algebra unit has two lessons. Lesson 7 teaches slope as ratios. Lesson 8 is a project that develops understanding of linear equations in y-intercept form.

Lesson 7: Slope

Students demonstrate mastery of this lesson with ability to determine the signed slope of the line. This lesson primes students for linear equations in y-intercept and point slope form by introducing slope as a ratio of vertical to horizontal lengths of a right triangle in fraction form.

Agenda

Class Section	Activity	Format	Instruction Time (minutes)
Lesson Opener	Inclines as Fractions	Whole class	10
Main Activities	Prediction	Individual	25
	Sloping Roof Walks	Whole class	
	Focused Free Write	Individual	
	Slides in a Park	Individual or Small Group	
Free Choice Activities	Lines on Cartesian Planes	Individual or Small Group	10
	Walking to Grandma's	Individual or Small Group	

Resources and Materials:

5 markers (three of one color and two of another color)

24-inch string

Lesson Opener: Inclines as Fractions

Students gain insight into how inclines are slopes. Given three dry-erase markers of one color (e.g., red) two of another color (e.g., orange) and a 24-inches long string, students will create inclined planes using their understanding of right triangles. For example, students may connect the three red markers as the longer leg of the triangle and connect the two orange markers as the shorter leg. They use the string to create the hypothenuse by tying the string to the caps of end markers, as shown in Figure 7.1 below. Students may stand the red markers upright and lay the orange one on the table so that the red and orange markers form a right angle of the triangle. Students will describe the incline or steepness or slope of the string by using English word and mathematical symbols.

Figure 7.1 Incline

In whole-class discussion, students will connect their understanding of incline or steepness of the string as described using ratios in fractional form. Students will represent the length of the vertical leg of the triangle as the numerator of the fraction and the length of the horizontal leg as the denominator. The three markers for the vertical legs and two markers for the horizontal leg to lead to the fractional representation of $\frac{3\,markers}{2\,markers}$ or $\frac{3}{2}$.

Students learn to assign positive signs to slope by scanning from left to right and observing if the string rises. Students can create their own configurations and write the fractional form of the slopes they create. They can be challenged to create configurations that would create negative slopes like $-\frac{3}{2}$.

Main Activities

Students engage in four activities during this main segment of the lesson. The first activity, Prediction, gives students an opportunity to apply the learning from the Lesson Opener. “Sloping Roof Walks” provides opportunities to connect a real-life scenario to the Lesson Opener – “Inclines as Fractions”. “Slides at a Park” gives opportunity to apply slope to objects that superficially do not look like triangles. The last activity, Focused Free Write, allows students to digest and contemplate similarities and differences between the activities.

Prediction - Students are presented with a picture of three houses, Figure 7.2. They are to predict how the roofs are related to the Lesson Opener “Inclines as Fractions.”

Figure 7.2 Inclines and Roofs

Sloping Roof Walks - Presented with a picture of three houses, Figure 7.2, students create graphs that represent each roof of each house and determine the appropriate fraction for slope and whether the slope of the roof is positive or negative. The first house (left) has a width of 20 feet. The roof is 10 feet tall. The second house (middle) is 22 feet wide, and the roof is eight feet tall. The third house (right) is 16 feet wide, and the roof is eight feet tall. Students will then imagine taking a walk on the roofs from left to right and describe the incline or steepness as a positive, negative, or zero slope.

Figure 7.3 Finding the Incline

Slides in a Park. A story about Rashaud, who is five years old, introduces other real-world objects with slopes. Rashaud's father takes him, his older brother, Micah, who is seven, and his little sister, Zariah, who is three years old, to the playground. All three wanted to go down different slides with different steepness.

Rashaud: I want to go down a slide!

Zariah: I want to go down on a slide, too!

Micah: I'll go down one, but only if it's a fun one!

There are four slides as shown in Figure 7.4

Figure 7.4 Pictorial Story for Slides in the Park Activity

- Slide 1 is six feet tall and 12 feet wide.
- Slide 2 is 14 feet high and 12 feet wide.
- Slide 3 is four feet tall and five feet wide.
- Slide 4 is 10 feet tall and 12 feet wide.

Students will draw representative pictures of each slide on a Cartesian grid and use fractions to explain the steepness of each one. Which slide will Rashaud choose? Which slide will Zariah choose? How about Micah? Students will explain reasons based upon the fractional slopes.

Focused Free Write – Students respond to “How are the Sloping Roof Walk and Slides in a Park activities for today similar?”

Free Choice Activities

There are two opportunities, “Lines on Cartesian Grids” and “Going to Grandma’s”, for teachers to observe and support learning. The “Lines on Cartesian Grid” activity aims to support students’ understanding of the slopes of line in linear equations. “Going to Grandma’s” offers application of slope as speed.

Lines on a Cartesian Grid - Students will use their understanding of slopes from the previous activities to practice finding the slopes of lines on cartesian grids. What is the slope of each line?

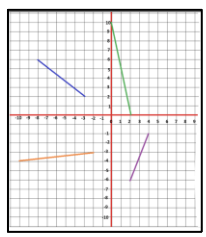


Figure 7.5 Lines in Cartesian Plane

Going to Grandma’s. Jaelynn needs to go to her grandmother’s home. She will take the bus. The purple line represents Jaelynn walking from her home to the bus stop. What is the slope of the purple line? Based upon the slope, what is Jaelynn’s speed?

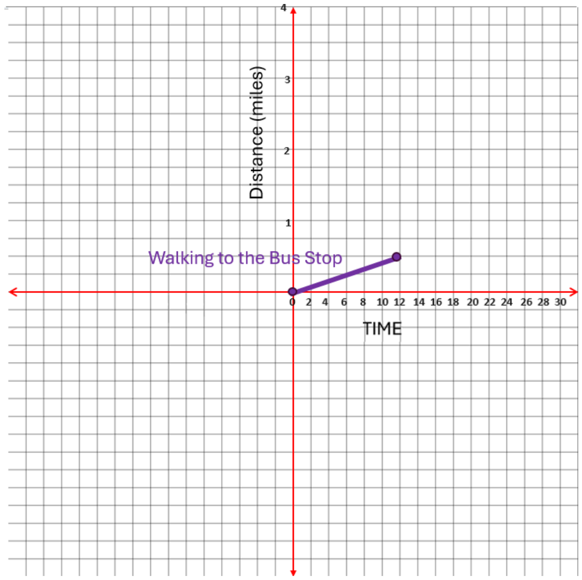


Figure 7.6 Going to Grandma's

Teacher's Notes

The orientation of cartesian grids have numbers increase left to right. Vertically, numbers increase going upwards. When scanning a cartesian grid from left to right, if a line ascends, the line has positive slope. Hence, students scan cartesian grids, photos of slopes, and real-life slopes from left to right to check for the sign of a slope.

Students may struggle with positive and negative slopes on a roof. When students imagine walking on the roof, have them pretend they start on the far left. If they move right, and they ascend, that means the slope is positive. If they walk towards the left and they ascend, the slope is negative.

Recommended questions to ask students: Pretend you are on a ladder on the right side of the house, and you walk up to the peak. What direction are you walking, left or right? Since you are walking left, not right, and you are ascending, is the slope positive or negative?

If students still struggle, they can engage in a kinesthetic activity. Students pair up and face each other. The first student tells the second student to pretend to be a positive slope. The second student raises the left hand and lowers the right hand. Then the students switch roles. The second student tells the first student to pretend to be a positive slope. The students switch roles again. The first student tells the second student to pretend to be a negative slope. The second student complies by raising the right hand and lowering the left. Lastly, they switch roles, and the second student tells the first student to pretend to be a negative slope.

Students may also struggle with the concept of a slope being zero. When a student pretends that when he stands, he feels completely upright, like standing on the classroom floor, that is when there is no slope. When a student stands in a slanted fashion, there is a slope. Recommended questions to ask students: Pretend you are at the peak of the roof. Would you feel like you are upright or slanted?

Lesson 8: Y-intercept form

Students demonstrate mastery of this lesson by writing linear equations in y-intercept form that create a digital graph of lines intercepting the origin (0,0). See Figure 8.1

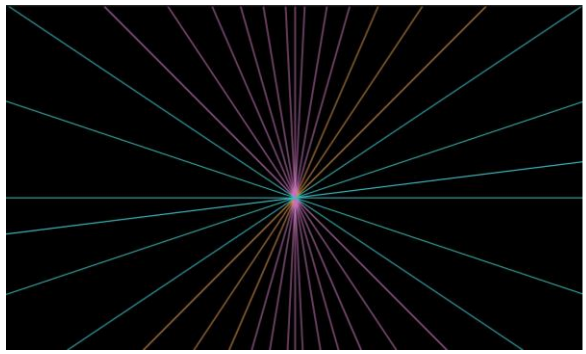


Figure 8.1 Linear Equations in y-intercept Form

Agenda

Class Section	Activity	Format	Instruction Time (minutes)
Lesson Opener	Introduction to Desmos	Whole class	10
Main Activities	y-intercept form	Individual or Small Group	35

Lesson Opener: How to use Desmos

The Lesson Opener teaches students how to use Desmos. Students learn how to use Desmos by following the steps in Appendix A Lesson 8.

Main Activities have only one activity, Y-Intercept Form. Students are to create the image in Figure 8.1 in Desmos.

Y-intercept form - Start students first type into Desmos y = 1x + 0. They learn the number one is the slope of the line. The “+ 0” means the line goes through the origin. After successfully typing in the equation y = 1x + 0, they then learn that the formal way writing the equation is y= x + 0. Students receive printouts of the black line drawing of the image they are to create. See Figure 8.2.

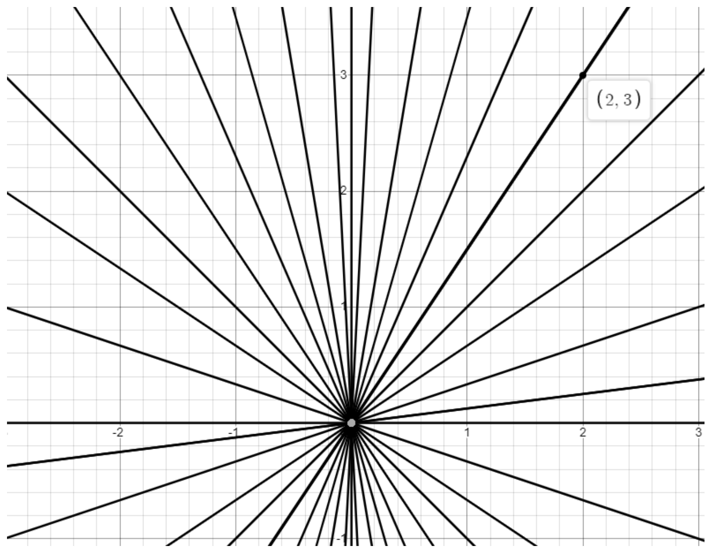


Figure 8.2 Black line drawing

They make a right triangle from the points (0,0), (2,3), and (2,0). They establish the height of the triangle they drew, and that becomes the numerator of the slope. They establish the width of the triangle they drew and that becomes the denominator of the slope. They type in $y = \frac{3}{2}x + 0$ and verify that they created the line with the point (2,3).

Students find one point on another line that they will use to create triangles. They establish slope for line and type into Desmos the linear equations in y-intercept form to recreate the lines. They do this for each line.

Teacher's Notes

As students try to input y-intercept linear equations, teachers need to circulate the room. Students may struggle with lines of negative slope. Students can be reminded of the Sloping Roof Walks in Lesson 7. Also, remind students that slope is positive ONLY when a line ascends from left to right.

Students may struggle with the horizontal line y = 0. Asking students to try making a triangle with two points that are right next to each other. There is no third point. Does the triangle have height? Does it have width?

Students may struggle with the vertical line x = 0. Recommended questions to ask students: Asking students to try making a triangle with two points that are on top of each other. There is no third point. Does the triangle have height? Does it have width?

Unit 4: Projects

Flexible in nature, this unit can serve many purposes. This unit can be a summative assessment where students demonstrate mastery of Lesson 1 – 8. It can also be used as another learning opportunity where more observations on student performance give clarity as to whether certain concepts were successfully learned.

Lesson 9

Lesson 9, the first day of the Projects unit, gives students choices for projects they wish to complete. Because students freely choose the project, the Lesson Opener introduces the projects and explains the expectations for each project. In an ideal situation, the student chooses one project and finishes it in one day. However, some students may need two days.

Section	Activity Name	Mode	Instruction Time (minutes)
Lesson Opener	Introduction of projects		5
Main Activity	Go Fly a Kite	multiple small groups or individual effort	40
	Lemonade	one group between 3 and 5 people	
	Snack Mix	one small group between 3 and 5 people	

Lesson Opener: Introduction of Projects

Each project is briefly introduced. The calculations table for each project is shown, and students silently and independently predict how to complete the calculations table. Students may need clarification on various aspects of each project before choosing.

The project, Go Fly a Kite, results in an actual kite that can be flown. The ratios of all the lengths of the frame are provided. Using the bamboo sticks for the frame, you create a cross from the frame. Using the scotch tape to secure the frame to the newspaper. Predict how to complete the calculations table.

Lemonade, the second project, results in actual lemonade that can be served to all students in the classroom. There is a recipe for lemonade, but it makes 60 ounces. In the classroom, there are x number of students, and each student will get 10 ounces of lemonade. How would the recipe be adjusted so that every student is served 10 ounces? Predict how to complete the calculations table.

The last project, Snack Mix, results in a snack mix that can be served to all students in the classroom. There is a recipe for four people. In the classroom, there are x number of students, and each student will get two cups of mix. How would the recipe be adjusted so that every student is served 2 dry cups? Predict how to complete the calculations table.

Main Activities

Once students choose their project, they receive materials to complete the project of their choice.

Go Fly a Kite

Resources and Materials: Newspaper, scissors, scotch tape, one spool of polyester sewing thread, a small hack saw, and thin bamboo sticks of varying lengths

Using the materials they received, students make a kite of the same proportions as the outline sketch. They decide what length bamboo stick would be best for the vertical segment (b + c) of the frame. Then, they determine the lengths of the other segments of the frame. After completing the calculation table and receiving feedback, the kite can be made. Calculators are allowed.

Side	**Original length**	**Factor**	**Your Length**
a	1		
b	2		
c	$\frac{1}{2}$		
b + c			

Lemonade

Resources and Materials: Materials: manual juicer, 24 ounce measuring cup, empty one gallon container, ladle for stirring

Ingredients: one bag of whole lemons (approximately three pounds of 12-15 lemons), one gallon of water, three water pitchers, one container of stevia

Students determine how to scale the recipe for the number of people in the classroom. Everyone will be served 10 ounces of lemonade. After completing the calculation table and receiving feedback, the lemonade can be made. Calculators are allowed. Upon completion, all students can enjoy lemonade together at the end.

Total ounces 60 ounces Water 35 ounces Lemon juice 25 ounces Stevia 6 tablespoons	Total ounces: Factor = $\frac{\_\_\_\_\_\_\_\_}{60}$ Water Calculation: ___________ounces Lemon juice Calculation: ___________ounces Stevia Calculation: ___________ tablespoons

Snack Mix

Resources and Materials:

Materials - large mixing bowl, one dry measuring cup, one half dry measuring cup, one quarter dry measuring cup

Ingredients - two boxes of Chex, two boxes of Pop cereal, two boxes of Cheez-Its, two containers of peanuts, two bags of m&m's, two bags of bite sized pretzels

Students determine how to scale the recipe for the number of people in the classroom. Each person will be served two dry cups of snack mix. After completing the calculation table and receiving feedback, the snack mix can be made. Calculators are allowed. Upon completion, all students can enjoy snack mix at the end.

Servings:	4	Servings needed:
Total cups	8 cups	Factor: $\frac{}{}$
		Total cups needed:
Chex	$\frac{5}{2}$ cups	Chex:
Pop	$\frac{3}{2}$ cups	Pop:
Cheez-Its	$\frac{7}{4}$ cups	Cheez-Its:
Peanuts	1 cup	Peanuts:
m&m's	$\frac{3}{4}$ cup	m&m's:
Pretzels	$\frac{1}{2}$ cup	Pretzels:

Teachers Notes

While this unit can act as a summative assessment, it is important for teachers to circulate the room to support students.

Challenges in Go Fly a Kite will be to first determine the length of the vertical segment of the frame. Students determine the vertical length by first adding b and c. Then, the student determines what length of bamboo stick would correlate to b + c. For example, b + c = $\frac{5}{2}$. If there is a stick that measures two and a half feet, then that would be appropriate for the vertical section. The horizontal segment would be easily made two feet. However, if the longest piece of bamboo is one and a quarter foot, then the factor is $\frac{1}{2}$, which means the horizontal piece needs to be one feet.

For Lemonade and Snack Mix, students will need to understand how to find the factor that will be used as the multiplier for all the ingredients. For Lemonade, the numerator of the factor is the total ounces needed, and the denominator is 60 ounces. After the factor is determined, the amounts of all ingredients are multiplied by the factor. For Snack Mix, the numerator of the factor is the total servings needed, and the denominator is 4 servings. Then the amounts of all ingredients are multiplied by the factor.

Lesson 10

Lesson 10, the second day of the Projects unit, gives students choices for projects they wish to complete. Students can also complete a project from the previous day. If they start a new project, students will predict how to complete the calculation tables. If they continue with a project from Lesson 9, the Lesson Opener reviews what still needs to be completed.

Section	Activity Name	Mode	Instruction Time (minutes)
Lesson Opener	Establish what will be accomplished		5
Main Activity	Go Fly a Kite	multiple small groups or individual effort	40
	Lemonade	one group between 3 and 5 people	
	Snack Mix	one small group between 3 and 5 people	

Lesson Opener

Students state what project they will work on and confirm they know what they will work on first.

Main Activities

Students engage in the project they stated they would concentrate on. If working on Lemonade or Snack Mix, upon completion, all students in the classroom can enjoy lemonade and snack mix together at the end.

Teachers Notes

While this unit can act as a summative assessment, it is important for teachers to circulate the room to support students.

Challenges in Go Fly a Kite will be to first determine the length of the vertical segment of the frame. Student determines the vertical length by first adding b and c. Then, the student determines what length of bamboo stick would correlate to b + c. For example, b + c = $\frac{5}{2}$. If there is a stick that measures two and a half feet, then that would be appropriate for the vertical section. The horizontal segment would be easily made two feet. However, if the longest piece of bamboo is one and a quarter feet, then the factor is $\frac{1}{2}$, which means the horizontal piece needs to be one feet.

For Lemonade and Snack Mix, students will need to understand how to find the factor that will be used as the multiplier for all the ingredients.

Conclusion

While innate math capacities present themselves in infants (Wynn, 1995) and language appears as a “common human possession” (Chomsky, 2000, p. 77), these two nature endowed gifts have not been addressed as covalent abilities. This curriculum hopes to create a rich language environment for students to develop a healthy and robust mental lexicon for fractions. Much like an organ, language is grown and “the environment triggers and to a limited extent shapes an internally-directly process of growth (Chomsky, 2000, p. 78).” As such, this curriculum strives to shape and drive the growth of math capabilities to enable the potential aptitude of each student to display itself.

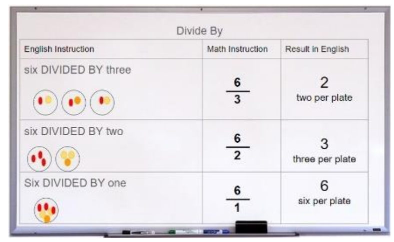
Appendix A

Appendix A contains slides for all lessons.

Lesson 1

Slide 1

Slide 2



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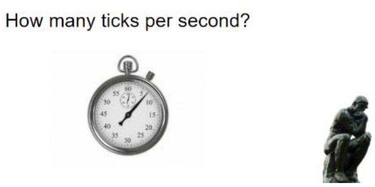

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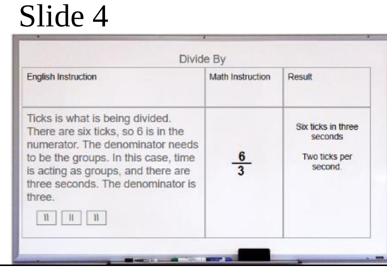

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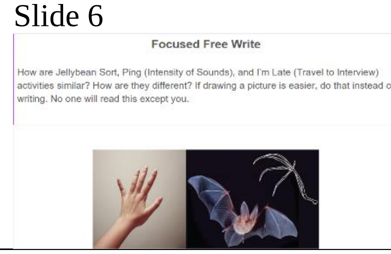

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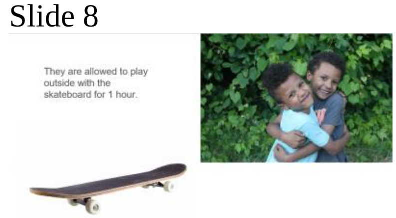
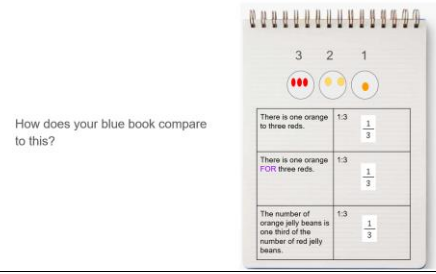


Lesson 2

Slide 1

Slide 2



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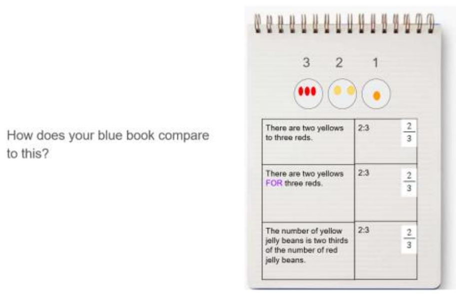

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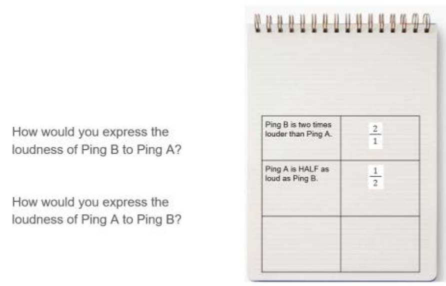

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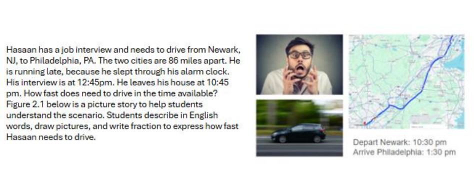

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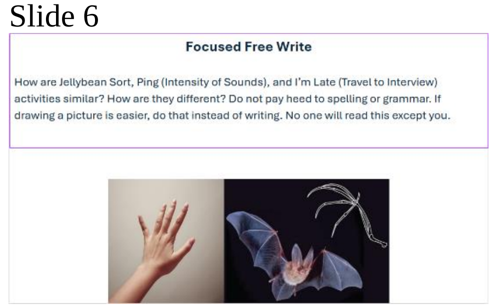

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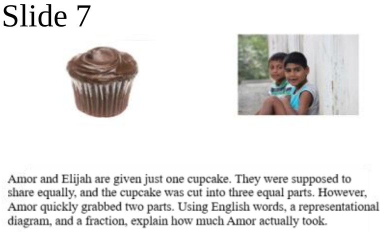

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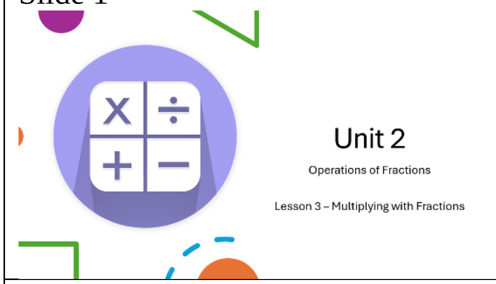
Slide 9

Slide 1



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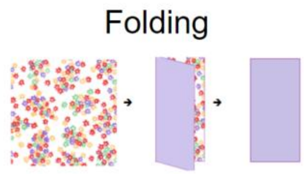

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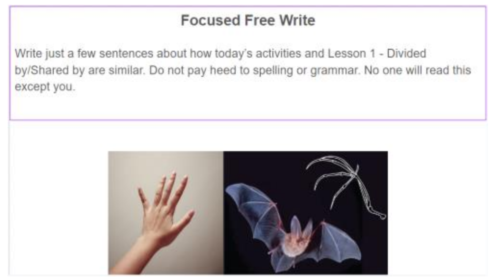


Slide 5

1. $\frac{3}{10} \times -20 =$
2. $\frac{1}{5} \times \frac{5}{2} =$
3. $\frac{4}{6} \times \frac{3}{8} =$
4. $\frac{20}{6} \times \frac{3}{10} =$
5. $\frac{-3}{-10} \times 20 =$
6. $\frac{1}{-5} \times \frac{5}{2} =$
7. $-\frac{4}{6} \times \frac{3}{8} =$
8. $\frac{20}{-6} \times \frac{3}{10} =$
9. $-\frac{3}{10} \times -20 =$
10. $\frac{-1}{-5} \times \frac{5}{-2} =$
11. $-\frac{4}{6} \times \frac{3}{-8} =$
12. $\frac{20}{-6} \times \frac{-3}{10} =$
13. $\frac{-3}{10} \times 20 =$
14. $\frac{1}{-5} \times -\frac{5}{2} =$
15. $-\frac{4}{6} \times -\frac{3}{8} =$
16. $\frac{20}{-6} \times \frac{-3}{-10} =$
17. $\frac{1}{5} \times \frac{2}{2} =$
18. $\frac{1}{5} \times \frac{3}{3} =$
19. $\frac{1}{5} \times \frac{4}{4} =$
20. $\frac{1}{5} \times \frac{5}{5} =$
21. $\frac{1}{5} \times \frac{-2}{2} =$
22. $\frac{1}{5} \times -\frac{3}{3} =$
23. $\frac{1}{5} \times \frac{-4}{4} =$
24. $\frac{1}{5} \times \frac{-5}{5} =$
25. $\frac{-1}{5} \times \frac{-4}{4} =$
26. $\frac{1}{5} \times \frac{-5}{-5} =$

Slide 6

Slide 7

Slide 8

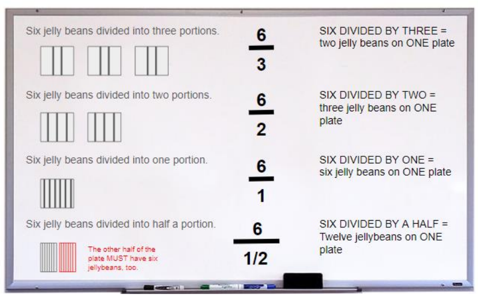
Cups of lemonade	Cups of lemon juice	Cups of water
8	2	6
16 (factor is $\frac{16}{8} = 2$)	2×2 = 4	6×2 = 12
12 (factor is $\frac{12}{8} = \frac{3}{2}$)	$2 \times \frac{3}{2} = 3$	$6 \times \frac{3}{2}$
10 (factor is $\frac{10}{8} = \frac{5}{4}$)	$2 \times \frac{5}{4} = \frac{5}{2}$	$6 \times \frac{5}{4} = \frac{30}{4} = \frac{15}{2}$
6 (factor is $\frac{6}{8} = \frac{3}{4}$)	$2 \times \frac{3}{4} = \frac{6}{4} = \frac{3}{2}$	$6 \times \frac{3}{4} = \frac{18}{4} = \frac{9}{2}$
4 (factor is $\frac{4}{8} = \frac{1}{2}$)	$2 \times \frac{1}{2} = 1$	$6 \times \frac{1}{2} = 3$
2 (factor is $\frac{2}{8} = \frac{1}{4}$)	$2 \times \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$	$6 \times \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$

Lesson 4

Slide 1

Slide 2

Slide 3



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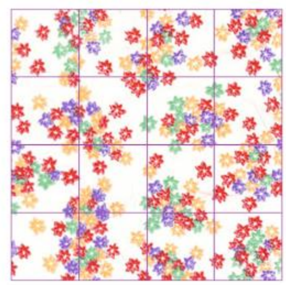
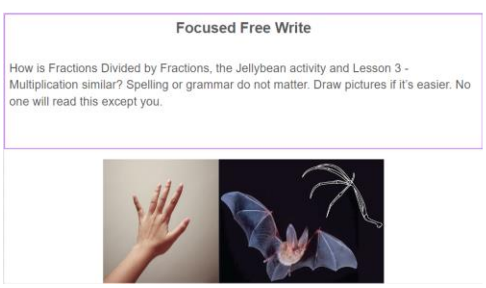


1. $\dfrac{\frac{1}{16}}{\frac{1}{2}} = ?$

2. $\dfrac{\frac{1}{16}}{\frac{1}{3}} = ?$

3. $\dfrac{\frac{1}{16}}{\frac{1}{4}} = ?$

Slide 5



Slide 6

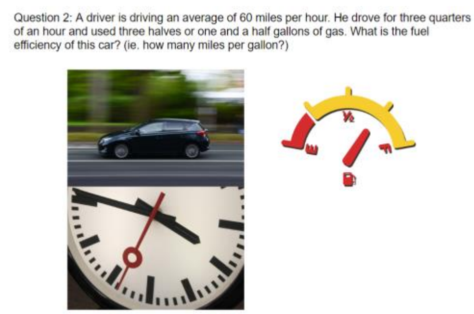
1. $\dfrac{\frac{3}{10}}{-\frac{1}{20}} =$	5. $\dfrac{\frac{-3}{-10}}{\frac{1}{20}} =$	9. $\dfrac{-\frac{3}{10}}{-\frac{1}{20}} =$	13. $\dfrac{\frac{-3}{10}}{\frac{1}{20}} =$
2. $\dfrac{\frac{1}{5}}{\frac{2}{5}} =$	6. $\dfrac{\frac{1}{-5}}{\frac{2}{5}} =$	10. $\dfrac{\frac{-1}{-5}}{\frac{-2}{5}} =$	14. $\dfrac{\frac{1}{-5}}{-\frac{5}{2}} =$
3. $\dfrac{\frac{4}{6}}{\frac{8}{3}} =$	7. $\dfrac{-\frac{4}{6}}{\frac{8}{3}} =$	11. $\dfrac{-\frac{4}{6}}{\frac{-8}{3}} =$	15. $\dfrac{-\frac{4}{6}}{-\frac{3}{8}} =$
4. $\dfrac{\frac{20}{6}}{\frac{10}{3}} =$	8. $\dfrac{\frac{20}{-6}}{\frac{10}{3}} =$	12. $\dfrac{\frac{20}{-6}}{\frac{10}{-3}} =$	16. $\dfrac{\frac{20}{6}}{\frac{3}{10}} =$

Slide 7

Slide 8

Slide 9

Slide 10



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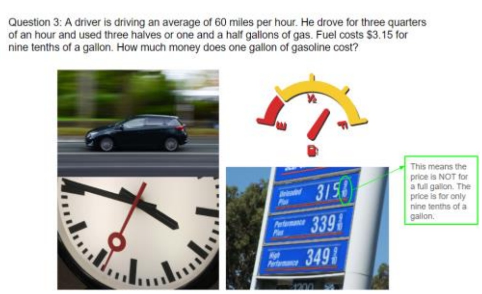

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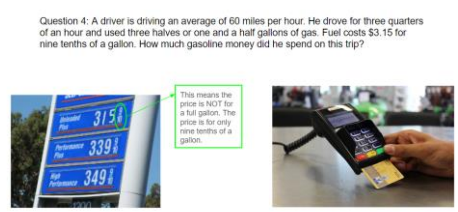

Lesson 5

Slide 1

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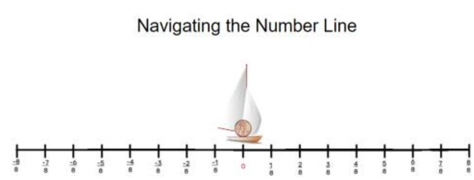

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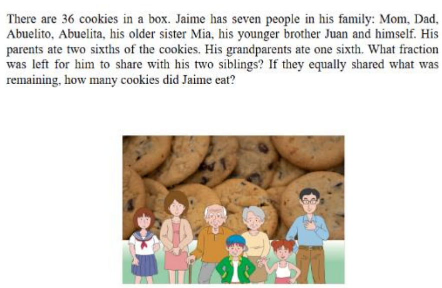

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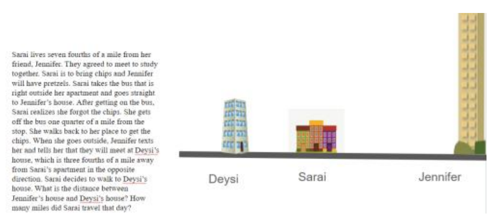

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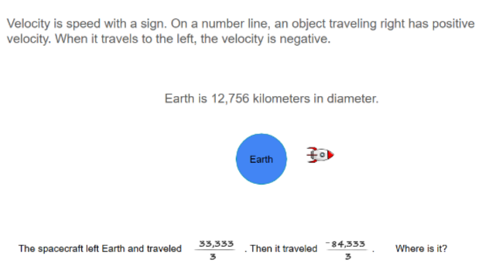

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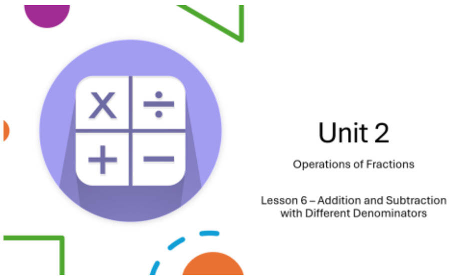

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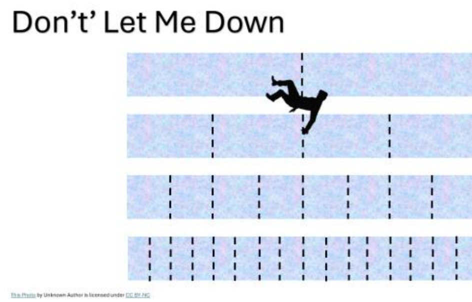

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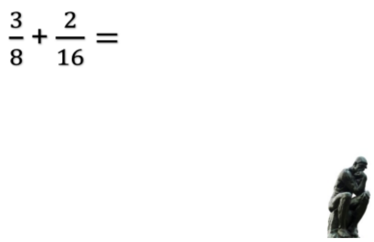

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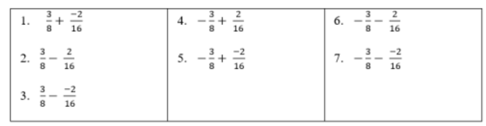

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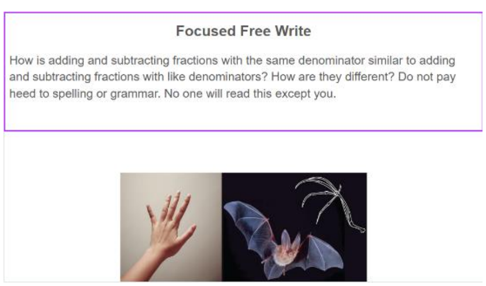

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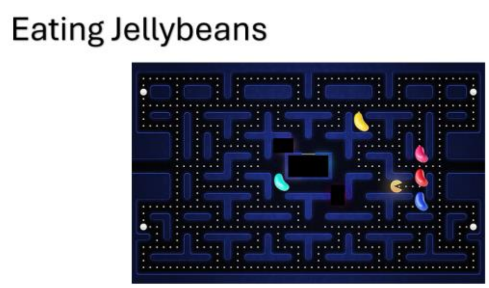

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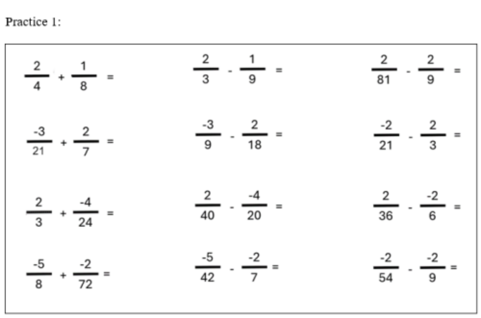

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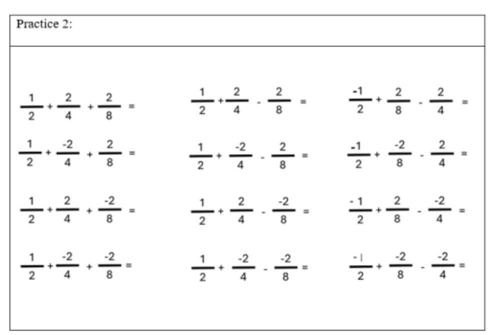


Slide 9

Practice 3:

$\frac{2}{3} + \frac{2}{8} =$	$\frac{2}{3} - \frac{2}{4} =$	$\frac{2}{2} - \frac{2}{11} =$
$\frac{-2}{4} + \frac{2}{5} =$	$\frac{-2}{2} - \frac{2}{5} =$	$\frac{-2}{4} - \frac{2}{7} =$
$\frac{2}{7} + \frac{-2}{8} =$	$\frac{2}{7} - \frac{-2}{9} =$	$\frac{2}{2} - \frac{-2}{9} =$
$\frac{-2}{4} + \frac{-2}{9} =$	$\frac{-2}{5} - \frac{-2}{9} =$	$\frac{-2}{6} - \frac{-2}{4} =$

Slide 10

Practice 4:

$\frac{1}{3} + \frac{2}{4} + \frac{3}{8} =$	$\frac{1}{3} - \frac{2}{4} - \frac{3}{8} =$	$\frac{1}{3} - \frac{2}{6} - \frac{3}{5} =$
$\frac{1}{3} + \frac{2}{4} + \frac{-3}{8} =$	$\frac{1}{3} - \frac{2}{4} - \frac{-3}{8} =$	$\frac{1}{3} - \frac{2}{6} - \frac{-3}{5} =$
$\frac{1}{3} + \frac{-2}{4} + \frac{3}{8} =$	$\frac{1}{3} - \frac{-2}{4} - \frac{3}{8} =$	$\frac{1}{3} - \frac{-2}{6} - \frac{3}{5} =$
$\frac{-1}{3} + \frac{2}{4} + \frac{3}{8} =$	$\frac{-1}{3} - \frac{2}{4} - \frac{3}{8} =$	$\frac{-1}{3} - \frac{2}{6} - \frac{3}{5} =$
$\frac{-1}{3} + \frac{-2}{4} + \frac{3}{8} =$	$\frac{-1}{3} - \frac{-2}{4} - \frac{3}{8} =$	$\frac{-1}{3} - \frac{-2}{6} - \frac{3}{5} =$
$\frac{-1}{3} + \frac{-2}{4} + \frac{-3}{8} =$	$\frac{-1}{3} - \frac{-2}{4} - \frac{-3}{8} =$	$\frac{-1}{3} - \frac{-2}{6} - \frac{-3}{5} =$

Slide 11

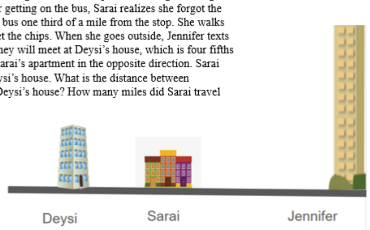
Michael has been borrowing money from his friends, Malachai, John, Ancell, and Madison. Michael just got his paycheck. However, he owes Malachai one fourth of that amount. He owes John on third, and he owes Ancell and Madison each one eighth. Does he have enough money to pay back his friends? Write a math equation with fractions to state whether Michael has enough money to pay back his friends.

Slide 12

If we take a look at Madison's situation, she owes three-fourths of her $240 paycheck to all her friends combined. However, Michael forgave his amount, which is one third of the total of what she owes. How would you write that situation as a math expression? Use a negative number to note money that is owed. How much of her paycheck is left, if she paid back all that she owed after Michael forgave his loan to her?

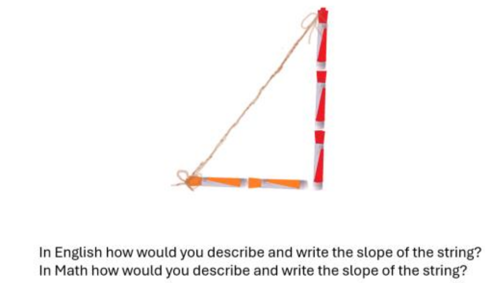
Slide 13

Sarai lives one a mile from her friend, Jennifer. They agreed to meet to study together. Sarai is to bring chips and Jennifer will have pretzels. Sarai takes the bus that is right outside her apartment and goes straight to Jennifer's house. After getting on the bus, Sarai realizes she forgot the chips. She gets off the bus one third of a mile from the stop. She walks back to her place to get the chips. When she goes outside, Jennifer texts her and tells her that they will meet at Deysi's house, which is four fifths of a mile away from Sarai's apartment in the opposite direction. Sarai decides to walk to Deysi's house. What is the distance between Jennifer's house and Deysi's house? How many miles did Sarai travel that day?



Slide 1

Slide 2



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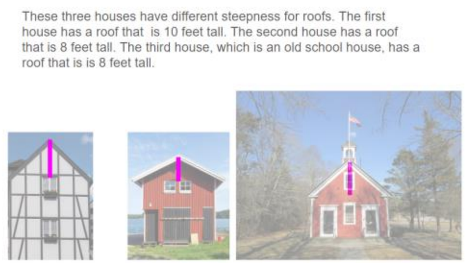

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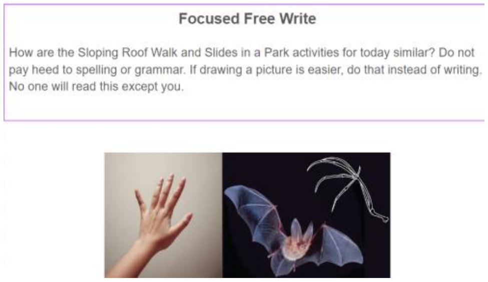

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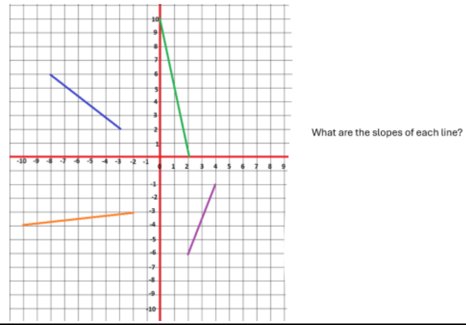

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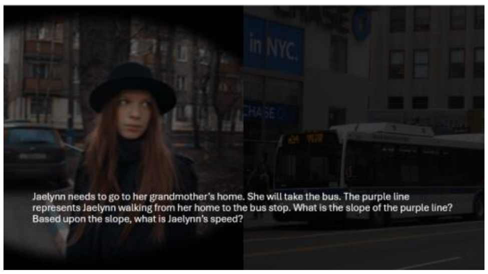

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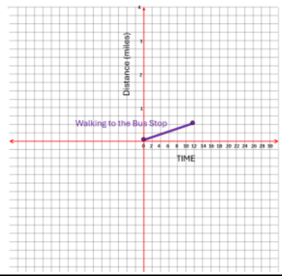

Lesson 8

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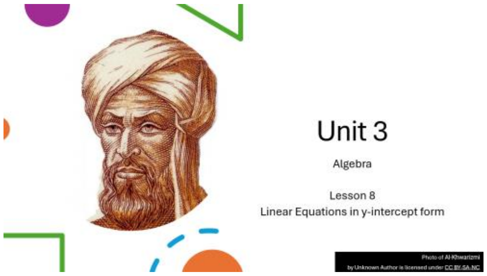
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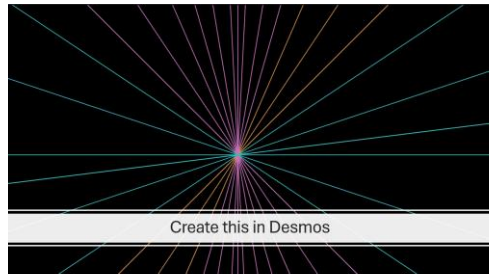
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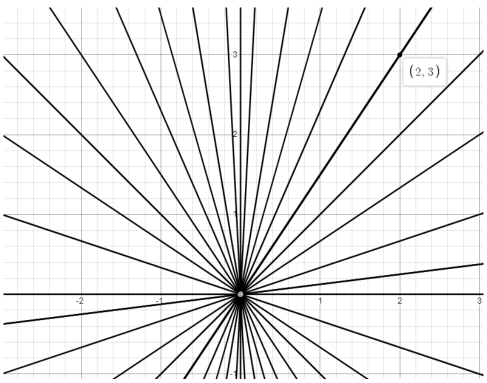
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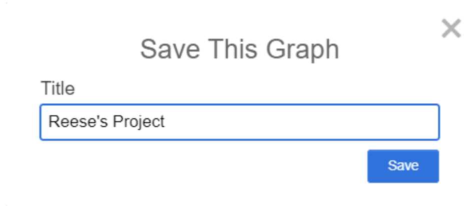
How to use Desmos:

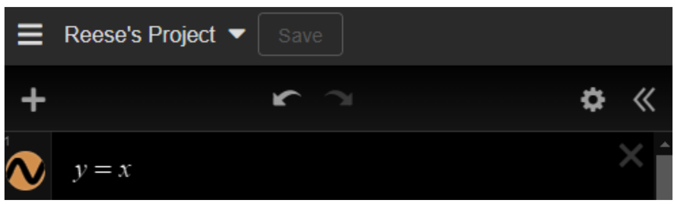
Steps:

1. Go to Desmos.com. Create an account.
2. Click on Graphing Calculator.
3. Click on Save.

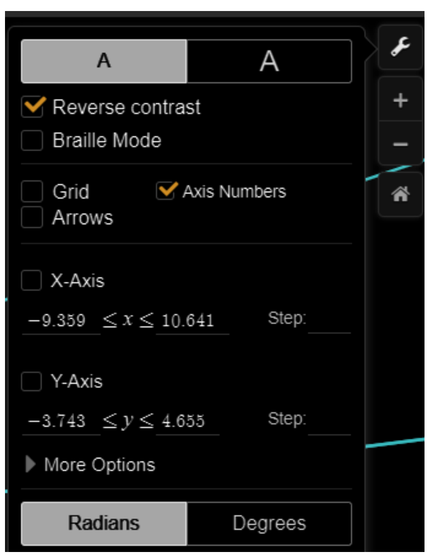
4. Name your project and click save.



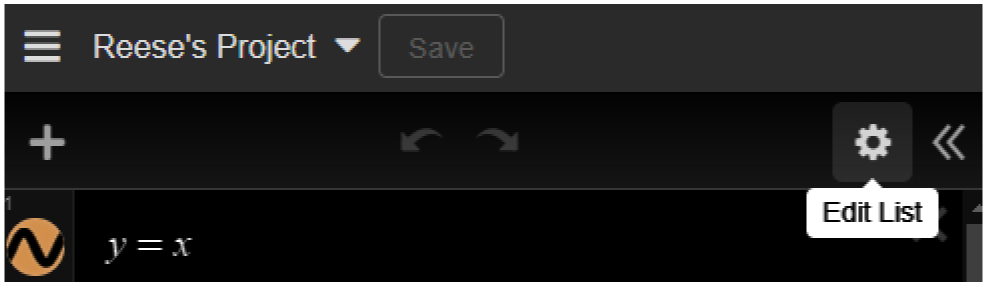
5. On the left you type in an equation to get a line: y = x. You will see a graph appear on the right.



6. When you complete all the equations, click on the wrench icon on the upper right hand. Select Reverse Contrast, and uncheck Grid, x-axis, and y-axis.

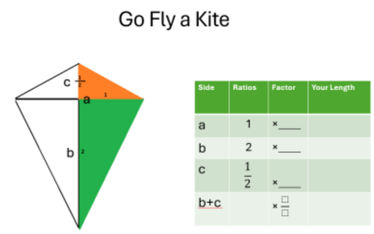


7. Click on the Edit List icon.



Click on the colored circle on the right and select the color you want the line to be and click Done.


Unit 4
Projects


Go Fly a Kite
c
a
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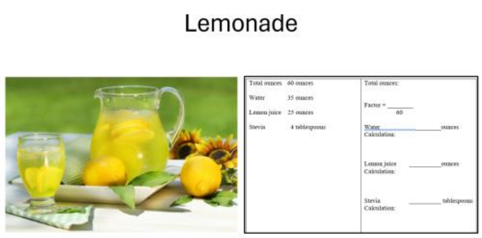
Side
Ratios
Factor
Your Length
a 1 ×
b 2 ×
c 1/2 ×
b+c


Lemonade


Snack Mix

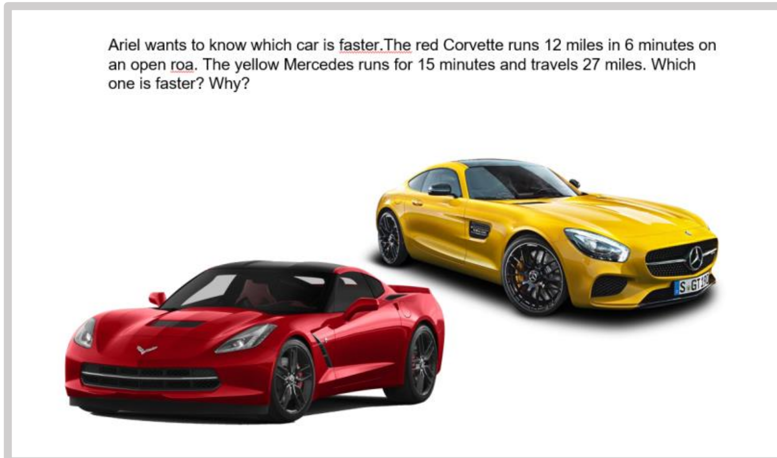
Appendix B

Lesson 1

Four friends rent a power washer for 3 hours. They decide to share the time equally between the 4 homes. How much time does each friend get to use the powerwasher for his home?

$\frac{3\ hours}{4\ friends}$ Each friend uses the power washer for three quarters of an hour.

Lesson 2



Red Corvette: $\frac{12\ miles}{6\ minutes} \rightarrow \frac{12 \times 10\ miles}{6 \times 10\ minutes} \rightarrow \frac{120\ miles}{60\ minutes} \rightarrow \frac{120\ miles}{1\ hour}$ 120 miles per hours

Yellow Mercedes: $\frac{27\ miles}{15\ minutes} \rightarrow \frac{27 \times 4\ miles}{15 \times 4\ minutes} \rightarrow \frac{108\ miles}{60\ minutes} \rightarrow \frac{108\ miles}{1\ hour}$ 108 miles per hours

The Covette is faster, because it runs 12 miles per hour faster than the Mercedes.

Fabric needed : = $100 \times \frac{3}{2} = \frac{300}{2}$ = 150 yards

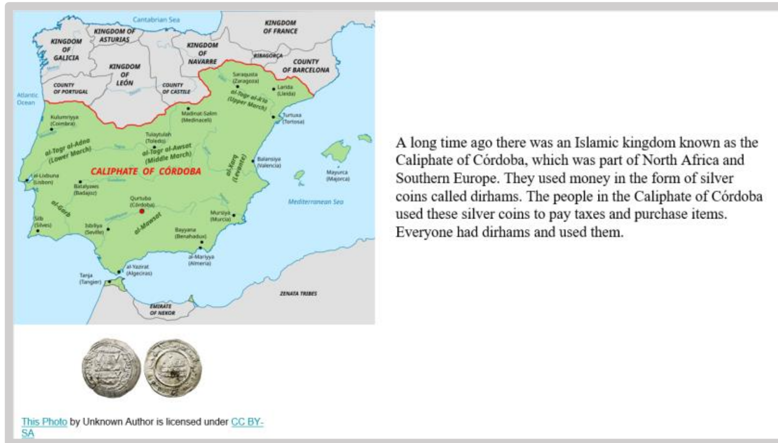
He cannot make all 100 vests, because he has only 120 yards. He needs another 30 yards.

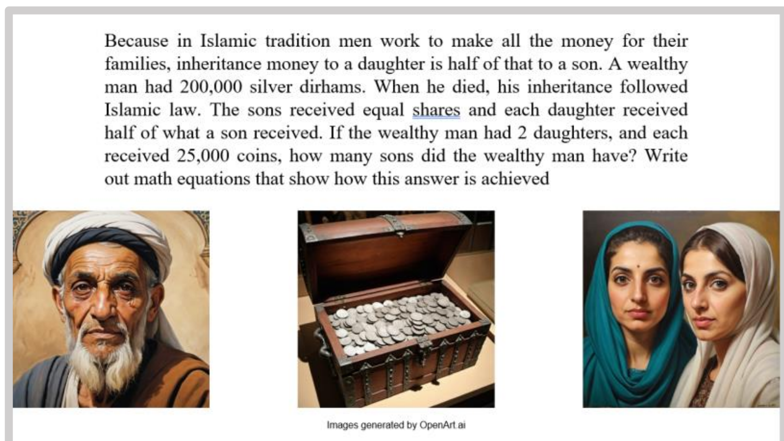
Lesson 3

$$\$21 \times \frac{3}{7} = \frac{63}{7} = \$9$$

$$\frac{3\ sticks}{1} \times \frac{8\ tbsp}{1\ stick} \times \frac{3}{4} = \frac{72}{4}\ tbsp = 18\ tablespoons$$

Lesson 4





First establish how much money is left after the two daughters take their portion. 200,000 – 25,000 – 25,000 = 150,000. Since daughters take half of what sons take, each son must have taken twice as much as one daughter. Therefore, each son received 50,000 dirhams. To find the number of sona, $\frac{150{,}000\ dirhams}{50{,}000\ dirhams\ per\ son}$, which means the man had three sons. Another way to do this is to recognize that one part is 25,000. Then establish how many parts can be made from 200,000 with 25,000. $\frac{200{,}000\ dirhams}{25{,}000\ dirhams\ per\ part} = 8\ parts$. If there are eight parts, and the two daughters took two parts, that means there are six parts left. Since each son receives two parts each, you can divide 6 by 2 parts $\frac{6\ parts}{2\ parts\ per\ son}$, and that means there are three sons.

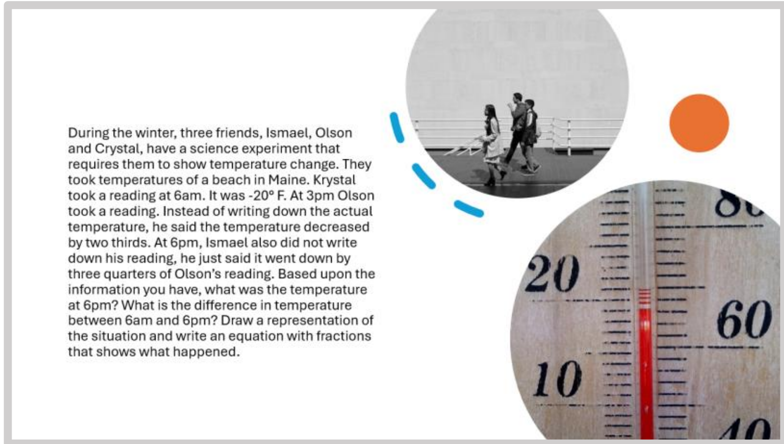
Lesson 5

$$0 + 500 - 100\frac{2}{5} + 200\frac{1}{5} - 300\frac{4}{5} = ?$$

The piggy bank represents a family savings fund. What does this math expression mean?

The savings started at zero, someone made $500 contribution. Then someone else took out $100.40. The someone added $200.20. Then someone spent $300.80. Now all that is left is $299.

Lesson 6



Time	Calculations	Temperature
6 am		-20° F
3 pm	$\frac{3}{3} - \frac{2}{3} = \frac{1}{3}$ $-\frac{20°F}{1} \times \frac{1}{3} = -\frac{20°F}{3}$ $-\frac{20°F}{3} = -6\,\frac{2}{3}°F$	$-6\,\frac{2}{3}°F$
6 pm	$\frac{4}{4} - \frac{3}{4} = \frac{1}{4}$ $-\frac{20°F}{3} \times \frac{1}{4} = -\frac{20°F}{12}$ $-\frac{20°F}{12} = -1\,\frac{8}{12}°F = -1\,\frac{2}{3}°F$	$-1\,\frac{2}{3}°F$

If using Start of Day – End of Day

$-20°F - \left(-1\,\frac{2}{3}\right) =$

$-\frac{60°F}{3} - \left(-\frac{4}{3}\right) = -\frac{56}{3}°F$

$-\frac{56}{3}°F = -18\,\frac{2}{3}°F$

If using End of Day – Start of Day

$-1\,\frac{2}{3}°F - (-20) =$

$-\frac{4}{3}°F - \left(-\frac{60}{3}\right) = +\frac{56}{3}°F$

$-\frac{56}{3}°F$

$-\frac{56}{3}°F = -18\,\frac{2}{3}°F$

Appendix C

Lesson 1

Tik Tik Tik

- "What would be in the denominator?" Answer: three
- They are to draw a diagram that represents the task, write a sentence in English that describes the task, and write it as a fraction. Answer: "six ticks divided into three groups", $\frac{12}{3}$)
- "If there were six ticks in three seconds, how many ticks were there in one second?" Answer: two
- They are then asked how they would express that in English and as a fraction. Answer: "two ticks per second" $\frac{2}{1}$

Pizza Pizza - Answer: $\frac{2}{3}$

Skateboard - Answer: Students may decide to not evenly share time. However, the objective is to ensure that students know how to equally distribute time between two people. One hour of skateboard time is shared equally between two brothers; $\frac{1}{2}$; Each brother uses the skateboard for half an hour.

Lesson 2

Comparing Subset to Subset - Answer: three reds to one yellow, three red per one yellow, or there are three times more reds than yellows, $\frac{3\ red}{1\ yellow}$

Comparing Subset to Whole Set - Answer: $\frac{6\ red}{12\ total}$

Ping - Possible Answer: $\frac{1}{2}$, Ping A is half as loud as Ping B. $\frac{2}{1}$, Ping B is twice as loud as Ping A.

I'm late!

- He leaves his house at 10:45 pm. Describe in English, as a picture, and as a fraction, how far he needs to drive in the time available. Answer: "86 miles in two hours", $\frac{86\ miles}{2\ hours}$
- If he runs into absolutely no traffic and does not have any stop lights, how fast does he need to drive to arrive by 12:45pm? Answer: "43 miles per hour"

Cupcake for One - Answer: $\frac{2}{3}$, Amor took two out of three parts

Snallygasters

- How would you describe the number of occupied seats to empty seats? Answer: two to ten, $\frac{2}{10}$
- How full is the van? Answer: two out of twelve, $\frac{2}{12}$
- How empty is the van? Answer: ten out of twelve, $\frac{10}{12}$

Heartbeats

- Express the situation in English and write it as a fraction. Answer: Her heartbeats 140 times in two minutes, $\frac{140}{2\ minutes}$.
- What is the heart rate of Michelle sitting? 70 beats per minute
- $\frac{140\ beats}{2\ minutes} = \frac{70\ beats}{1 minute} = 70\frac{beats}{minute}$
- Express the situation in English and write it as a fraction. Answer: Her heartbeats 320 times in two minutes, $\frac{320\ beats}{2\ minutes}$.
- What is the heart rate of Michelle running? Answer: 160 beats per minute

$\frac{320\ beats}{2\ minutes} = \frac{160\ beats}{1 minute} = 160\frac{beats}{minute}$

Lesson 3

Practice

7. $\frac{3}{10} \times -20 =$ -6	20. $\frac{1}{-5} \times -\frac{5}{2} = \frac{1}{2}$
8. $\frac{1}{5} \times \frac{5}{2} = \frac{1}{2}$	21. $-\frac{4}{6} \times -\frac{3}{8} = \frac{1}{6}$
9. $\frac{4}{6} \times \frac{3}{8} = \frac{1}{6}$	22. $\frac{20}{-6} \times \frac{-3}{-10} = -1$
10. $\frac{20}{6} \times \frac{3}{10} = 1$	23. $\frac{1}{5} \times \frac{5}{1} = 1$
11. $\frac{-3}{-10} \times 20 = 6$	24. $\frac{2}{5} \times \frac{5}{2} = 1$
12. $\frac{1}{-5} \times \frac{5}{2} = \frac{-1}{2}$	25. $\frac{-4}{5} \times \frac{-5}{-4} = 1$
13. $-\frac{4}{6} \times \frac{3}{8} = -\frac{1}{6}$	26. $\frac{1}{6} \times \frac{6}{1} = 1$
14. $\frac{20}{-6} \times \frac{3}{10} = -1$	27. $\frac{5}{6} \times \frac{-6}{5} = -1$
15. $-\frac{3}{10} \times -20 = 6$	28. $\frac{7}{1} \times -\frac{1}{7} = -1$
16. $\frac{-1}{-5} \times \frac{5}{-2} = -\frac{1}{2}$	29. $\frac{5}{7} \times \frac{-7}{5} = -1$
17. $-\frac{4}{6} \times \frac{3}{-8} = \frac{1}{6}$	30. $\frac{6}{7} \times \frac{-7}{6} = -1$
18. $\frac{20}{-6} \times \frac{-3}{10} = 1$	31. $\frac{-5}{8} \times \frac{-8}{5} = 1$
19. $\frac{-3}{10} \times 20 = -6$	32. $\frac{11}{8} \times \frac{8}{11} = 1$

Road Trip - Answer: one third of fifteen hours, $\frac{1}{3} \times 15\ hours = \frac{5}{1}\ hours$

Lemonade

- 16 cups of lemonade - Answer: First find the ratio of the new amount compared to the original amount, $\frac{16}{8}$, which means the ratio is $\frac{2}{1}$. Multiply the original amount of lemon juice to the ratio, 2 cups of lemon juice × $\frac{2}{1}$ = 4 cups of lemon juice. Then, multiply the ratio to the amount of water 6 cups of water ×$\frac{2}{1}$ = 12 cups of water.
- 12 cups of lemonade - Answer: First find the ratio of the new amount compared to the original amount, $\frac{12}{8}$, which means the ratio is $\frac{3}{2}$. Multiply the original amount of lemon juice to the ratio, 2 cups of lemon juice × $\frac{3}{2}$ = 3 cups of lemon juice. Then, multiply the ratio to the amount of water 6 cups of water ×$\frac{3}{2}$ = 9 cups of water.
- 10 cups of lemonade - Answer: First find the ratio of the new amount compared to the original amount, $\frac{10}{8}$, which means the ratio is $\frac{5}{4}$. Multiply the original amount of lemon juice to the ratio, 2 cups of lemon juice × $\frac{5}{4}$ = $\frac{10}{4}$, which is two and half cups of lemon juice. Then, multiply the ratio to the amount of water 6 cups of water × $\frac{5}{4}$ = $\frac{30}{4}$, which seven and a half cups of water.
- 6 cups of lemonade - Answer: First find the ratio of the new amount compared to the original amount, $\frac{6}{8}$, which means the ratio is $\frac{3}{4}$. Multiply the original amount of lemon juice to the ratio, 2 cups of lemon juice × $\frac{3}{4}$ = $\frac{3}{2}$, which is one and half cups

of lemon juice. Then, multiply the ratio to the amount of water 6 cups of water × $\frac{3}{4}$ = $\frac{18}{4}$, which is four and a half cups of water.

- 4 cups of lemonade - Answer: First find the ratio of the new amount compared to the original amount, $\frac{4}{8}$, which means the ratio is $\frac{1}{2}$. Multiply the original amount of lemon juice to the ratio, 2 cups of lemon juice × $\frac{1}{2}$ = 1 , which is one cup of lemon juice. Then, multiply the ratio to the amount of water 6 cups of water × $\frac{1}{2}$ = 3, which is three cups of water.
- 2 cups of lemonade - Answer: First find the ratio of the new amount compared to the original amount, $\frac{2}{8}$, which means the ratio is $\frac{1}{4}$. Multiply the original amount of lemon juice to the ratio, 2 cups of lemon juice × $\frac{1}{4}$ = $\frac{1}{2}$, which is half a cup of lemon juice. Then, multiply the ratio to the amount of water 6 cups of water × $\frac{1}{4}$ = $\frac{3}{2}$, which is one and a half cups of water.

Lesson 4

Fractions Divided by Fractions

- $\frac{\frac{1}{16}}{\frac{1}{2}}$ = ? Answer: $\frac{2}{16}$
- $\frac{\frac{1}{16}}{\frac{1}{3}}$ = ? Answer: $\frac{3}{16}$
- $\frac{\frac{1}{16}}{\frac{1}{4}}$ = ? Answer: $\frac{4}{16}$

Practice

1. $\frac{\frac{3}{10}}{-\frac{1}{20}} = -6$	*5.* $\frac{\frac{-3}{-10}}{\frac{1}{20}} = 6$	*9.* $\frac{-\frac{3}{10}}{-\frac{1}{20}} = 6$	*13.* $\frac{\frac{-3}{10}}{\frac{1}{20}} = -6$
2. $\frac{\frac{1}{5}}{\frac{2}{5}} = \frac{1}{2}$	*6.* $\frac{\frac{1}{-5}}{\frac{2}{5}} = -\frac{1}{2}$	*10.* $\frac{\frac{-1}{-5}}{\frac{-2}{5}} = -\frac{1}{2}$	*14.* $\frac{\frac{1}{-5}}{-\frac{5}{2}} = \frac{2}{25}$
3. $\frac{\frac{4}{6}}{\frac{8}{3}} = \frac{1}{4}$	*7.* $\frac{-\frac{4}{6}}{\frac{8}{3}} = -\frac{1}{4}$	*11.* $\frac{-\frac{4}{6}}{\frac{-8}{3}} = \frac{1}{4}$	*15.* $\frac{-\frac{4}{6}}{-\frac{3}{8}} = \frac{32}{18}$
4. $\frac{\frac{20}{6}}{\frac{10}{3}} = 1$	*8.* $\frac{\frac{20}{-6}}{\frac{10}{3}} = -1$	*12.* $\frac{\frac{20}{-6}}{\frac{10}{-3}} = 1$	*16.* $\frac{\frac{20}{6}}{\frac{3}{10}} = \frac{200}{18}$

Tip - Answer: $72

Lemonade –Answer:$\frac{5}{\frac{5}{2}}$, 2 ounces per tablespoon of sugar

Car Ride

- How far did he drive? Answer: 45 miles
- What is the fuel efficiency of this car in miles per gallon? Answer: 90 miles per gallon

- How much money does one gallon of gas cost? $2.835 per gallon
- How much gasoline money did he spend on this trip? $2.126

Lesson 5

Navigating the Number Line - Answer: place the penny on negative five eighths and you move to left by two increments , $\frac{-7}{8}$.

Practice

1. $\frac{1}{2}+\frac{1}{2}=\frac{2}{2}=1$	11. $\frac{1}{2}-\frac{2}{2}=\frac{-1}{2}$	21. $\frac{-2}{2}+\frac{1}{2}=\frac{-1}{2}$	31. $-\frac{6}{2}-\frac{17}{2}=\frac{-23}{2}$
2. $\frac{1}{2}+\frac{-1}{2}=0$	12. $\frac{1}{2}-\frac{-2}{2}=\frac{3}{2}$	22. $\frac{-2}{2}+\frac{-1}{2}=\frac{-3}{2}$	32. $-\frac{6}{2}-\frac{-17}{2}=\frac{11}{2}$
3. $\frac{1}{2}-\frac{1}{2}=0$	13. $-\frac{1}{2}+\frac{2}{2}=\frac{1}{2}$	23. $\frac{-2}{2}-\frac{1}{2}=\frac{-3}{2}$	33. $\frac{17}{2}+\frac{6}{2}=\frac{23}{2}$
4. $\frac{1}{2}-\frac{-1}{2}=\frac{2}{2}=1$	14. $-\frac{1}{2}+\frac{-2}{2}=\frac{-3}{2}$	24. $\frac{-2}{2}-\frac{-1}{2}=\frac{-1}{2}$	34. $\frac{17}{2}+\frac{-6}{2}=\frac{11}{2}$
5. $\frac{-1}{2}+\frac{1}{2}=0$	15. $-\frac{1}{2}-\frac{2}{2}=\frac{-3}{2}$	25. $\frac{6}{2}+\frac{17}{2}=\frac{23}{2}$	35. $\frac{17}{2}-\frac{6}{2}=\frac{11}{2}$
6. $\frac{-1}{2}+\frac{-1}{2}=\frac{-2}{2}=$ -1	16. $-\frac{1}{2}-\frac{-2}{2}=\frac{1}{2}$	26. $\frac{6}{2}+\frac{-17}{2}=\frac{-11}{2}$	36. $\frac{17}{2}-\frac{-6}{2}=\frac{23}{2}$
	17. $\frac{2}{2}+\frac{1}{2}=\frac{3}{2}$	27. $\frac{6}{2}-\frac{17}{2}=\frac{-11}{2}$	37. $-\frac{17}{2}+\frac{6}{2}=\frac{-11}{2}$
7. $\frac{-1}{2}-\frac{1}{2}=\frac{-2}{2}=-1$	18. $\frac{2}{2}+\frac{-1}{2}=\frac{1}{2}$	28. $\frac{6}{2}-\frac{-17}{2}=\frac{23}{2}$	38. $-\frac{17}{2}+\frac{-6}{2}=\frac{-23}{2}$
8. $\frac{-1}{2}-\frac{-1}{2}=0$	19. $\frac{2}{2}-\frac{1}{2}=\frac{1}{2}$	29. $-\frac{6}{2}+\frac{17}{2}=\frac{11}{2}$	39. $-\frac{17}{2}-\frac{6}{2}=\frac{-23}{2}$
9. $\frac{1}{2}+\frac{2}{2}=\frac{3}{2}$	20. $\frac{2}{2}-\frac{-1}{2}=\frac{3}{2}$	30. $-\frac{6}{2}+\frac{-17}{2}=\frac{-23}{2}$	40. $-\frac{17}{2}-\frac{-6}{2}=\frac{-11}{2}$
10. $\frac{1}{2}+\frac{-2}{2}=\frac{-1}{2}$			

Money Problems

- Write the amount she owes as a signed fraction. Answer: $-\frac{1}{5}$

- Write an addition/subtraction expression that shows the relationship between how much she has, how much she owes, and what fraction remains if she pays him back. Answer: $\frac{5}{5} - \frac{1}{5} = \frac{4}{5}$
- If she made \$300 in one week, how much money can she keep? Answer: $\$300 \times \frac{4}{5}$ =\$240

Cookies

- What fraction was left for him to share with his two siblings? Answer: $\frac{6}{6} - \frac{2}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$
- If they equally shared what was remaining, how many cookies did Jaime eat? Answer: $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}; \frac{1}{4} \times 36 = \frac{36}{4} = 9$

Meeting a Friend

- What is the distance between Jennifer's house and Deysi's house? Answer: The distance between the two homes is one and a half mile. $\frac{3}{4} - \left(-\frac{3}{4}\right) = \frac{6}{4}$
- How many miles did Sarai travel that day? Answer: Sarai traveled one and a quarter mile. $\frac{1}{4} + \frac{1}{4} + \frac{3}{4} = \frac{5}{4}$

Space Joy Ride - Answer: 17,000 km to the left of where it started, which is the far right side of Earth. Because the Earth is 12,756 km in diameters, the spacecraft is 4,244 km to the left of the picture of Earth.

Lesson 6

Don’t Let Me Down

- Students state fractions that are equivalent to 10, when the denominator is four, eight and 16. Answer: $\frac{40}{4}, \frac{80}{8}, \frac{160}{16}$
- Student state equivalent fractions for $\frac{1}{2}$ when then denominator is four, eight, and 16. Answers: $\frac{2}{4}, \frac{4}{8}, \frac{8}{16}$

Main Activity Practice

1. $\frac{3}{8} + \frac{-2}{16} = \frac{4}{16} = \frac{1}{4}$	4. $-\frac{3}{8} + \frac{2}{16} = \frac{-4}{16} = -\frac{1}{4}$	6. $-\frac{3}{8} - \frac{2}{16} = \frac{-8}{16} = -\frac{1}{2}$
2. $\frac{3}{8} - \frac{2}{16} = \frac{4}{16} = \frac{1}{4}$	5. $-\frac{3}{8} + \frac{-2}{16} = \frac{-8}{16} = -\frac{1}{2}$	7. $-\frac{3}{8} - \frac{-2}{16} = \frac{-4}{16} = -\frac{1}{4}$
3. $\frac{3}{8} - \frac{-2}{16} = \frac{8}{16} = \frac{1}{2}$		

Practice Set 1

$\frac{2}{4} + \frac{1}{8} = \frac{5}{8}$	$\frac{2}{3} - \frac{1}{9} = \frac{5}{9}$	$\frac{2}{81} - \frac{2}{9} = \frac{-16}{81}$
$\frac{-3}{21} + \frac{2}{7} = \frac{3}{21} = \frac{1}{7}$	$\frac{-3}{9} - \frac{2}{18} = \frac{-8}{18} = \frac{-4}{9}$	$\frac{-2}{21} - \frac{2}{3} = \frac{-16}{21}$
$\frac{2}{3} + \frac{-4}{24} = \frac{12}{24} = \frac{1}{2}$	$\frac{2}{40} - \frac{-4}{20} = \frac{10}{40} = \frac{1}{4}$	$\frac{2}{36} - \frac{-2}{6} = \frac{14}{36} = \frac{7}{18}$
$\frac{-5}{8} + \frac{-2}{72} = \frac{-47}{72}$	$\frac{-5}{42} - \frac{-2}{7} = \frac{7}{42} = \frac{1}{6}$	$\frac{-2}{54} - \frac{-2}{9} = \frac{10}{54} = \frac{5}{27}$

Practice Set 2

$\frac{1}{2} + \frac{2}{4} + \frac{2}{8} = \frac{10}{8}$ $\quad \frac{1}{2} + \frac{2}{4} - \frac{2}{8} = \frac{6}{8}$ $\quad \frac{-1}{2} + \frac{2}{8} - \frac{2}{4} = \frac{-6}{8}$

$\frac{1}{2} + \frac{-2}{4} + \frac{2}{8} = \frac{2}{8}$ $\quad \frac{1}{2} + \frac{-2}{4} - \frac{2}{8} = \frac{-2}{8}$ $\quad \frac{-1}{2} + \frac{-2}{8} - \frac{2}{4} = \frac{-10}{8}$

$\frac{1}{2} + \frac{2}{4} + \frac{-2}{8} = \frac{6}{8}$ $\quad \frac{1}{2} + \frac{2}{4} - \frac{-2}{8} = \frac{10}{8}$ $\quad \frac{-1}{2} + \frac{2}{8} - \frac{-2}{4} = \frac{2}{8}$

$\frac{1}{2} + \frac{-2}{4} + \frac{-2}{8} = \frac{-2}{8}$ $\quad \frac{1}{2} + \frac{-2}{4} - \frac{-2}{8} = \frac{2}{8}$ $\quad \frac{-1}{2} + \frac{-2}{8} - \frac{-2}{4} = \frac{-2}{8}$

Practice Set 3

$\frac{2}{3} + \frac{2}{8} = \frac{22}{24} = \frac{11}{12}$ $\quad \frac{2}{3} - \frac{2}{4} = \frac{14}{12} = \frac{7}{6}$ $\quad \frac{2}{2} - \frac{2}{11} = \frac{18}{22} = \frac{9}{11}$

$\frac{16}{24} + \frac{6}{24}$ $\quad \frac{8}{12} + \frac{6}{12}$ $\quad \frac{22}{22} - \frac{4}{22}$

$\frac{-2}{4} + \frac{2}{5} = \frac{-2}{20} = \frac{-1}{10}$ $\quad \frac{-2}{2} - \frac{2}{5} = \frac{-14}{10} = \frac{-7}{5}$ $\quad \frac{-2}{4} - \frac{2}{7} = \frac{-22}{28} = -\frac{11}{14}$

$\frac{-10}{20} + \frac{8}{20}$ $\quad \frac{-10}{10} - \frac{4}{10}$ $\quad \frac{-14}{28} - \frac{8}{28}$

$\frac{2}{7} + \frac{-2}{8} = \frac{2}{56} = \frac{1}{28}$ $\quad \frac{2}{7} - \frac{-2}{9} = \frac{32}{63}$ $\quad \frac{2}{2} - \frac{-2}{9} = \frac{22}{18} = \frac{11}{9}$

$\frac{16}{56} + \frac{-14}{56}$ $\quad \frac{18}{63} - \frac{-14}{63}$ $\quad \frac{18}{18} - \frac{-4}{18}$

$\frac{-2}{4} + \frac{-2}{9} = \frac{-26}{36} = \frac{-13}{18}$ $\quad \frac{-2}{5} - \frac{-2}{9} = \frac{-8}{45}$ $\quad \frac{-2}{6} - \frac{-2}{4} = \frac{2}{12} = \frac{1}{4}$

$\frac{-18}{36} + \frac{-8}{36}$ $\quad \frac{-18}{45} - \frac{-10}{45}$ $\quad \frac{-4}{12} - \frac{-6}{12}$

Practice Set 4

$\frac{1^{8}}{3} + \frac{2^{12}}{4} + \frac{3^{9}}{8} = \frac{29}{24}$	$\frac{1}{3} - \frac{2}{4} - \frac{3}{8} = \frac{-7}{24}$	$\frac{1\,{10}}{3} - \frac{2\,{10}}{6} - \frac{3\,{18}}{5} = \frac{-18}{30}$
$\frac{1}{3} + \frac{2}{4} + \frac{-3}{8} = \frac{17}{24}$	$\frac{1}{3} - \frac{2}{4} - \frac{-3}{8} = \frac{-1}{24}$	$\frac{1}{3} - \frac{2}{6} - \frac{-3}{5} = \frac{18}{30}$
$\frac{1}{3} + \frac{-2}{4} + \frac{3}{8} = \frac{5}{24}$	$\frac{1}{3} - \frac{-2}{4} - \frac{3}{8} = \frac{17}{24}$	$\frac{1}{3} - \frac{-2}{6} - \frac{3}{5} = \frac{2}{30}$
$\frac{-1}{3} + \frac{2}{4} + \frac{3}{8} = \frac{7}{24}$	$\frac{-1}{3} - \frac{2}{4} - \frac{3}{8} = \frac{-23}{24}$	$\frac{-1}{3} - \frac{2}{6} - \frac{3}{5} = \frac{-3}{30}$
$\frac{-1}{3} + \frac{-2}{4} + \frac{3}{8} = \frac{-17}{24}$	$\frac{-1}{3} - \frac{-2}{4} - \frac{3}{8} = \frac{1}{24}$	$\frac{-1}{3} - \frac{-2}{6} - \frac{3}{5} = \frac{-3}{30}$
$\frac{-1}{3} + \frac{-2}{4} + \frac{-3}{8} = \frac{-23}{24}$	$\frac{-1}{3} - \frac{-2}{4} - \frac{-3}{8} = \frac{7}{24}$	$\frac{-1}{3} - \frac{-2}{6} - \frac{-3}{5} = \frac{3}{30}$

Michael Owes Money – Answer: Michael has enough money to pay back his friends. $1 - \frac{1}{4} - \frac{1}{3} - \frac{1}{8} - \frac{1}{8} = \frac{24}{24} - \frac{6}{24} - \frac{8}{24} - \frac{3}{24} - \frac{3}{24} = \frac{4}{24}$

Madison Owes Money

- How would you write that situation as a math expression? Use a negative number to note money that is owed. Answer: $\frac{4}{4} + \frac{-3}{4} - \frac{-1}{3}\left(\frac{3}{4}\right) = \frac{4}{4} + \frac{-3}{4} + \frac{3}{12}$
- How much of her paycheck is left, if she paid back all that she owed after Michael forgave his loan to her? $\frac{4}{4} + \frac{-3}{4} + \frac{3}{12} = \frac{12}{12} + \frac{-9}{12} + \frac{3}{12} = \frac{6}{12} = \frac{1}{2}$ Since one half the paycheck is kept, Madison keeps half of \$240, which \$120.

Meeting a Friend

- What is the distance between Jennifer's house and Deysi's house? Answer: The distance between the two homes is one and a half mile. Answer: they live one and one fifth mile apart. $\frac{5}{5} - \left(-\frac{4}{5}\right) = \frac{9}{5}$
- How many miles did Sarai travel that day? Answer: Sarai traveled one and five sevenths of mile. $\frac{1}{3} + \frac{1}{3} + \frac{3}{4} = \frac{4}{12} + \frac{4}{12} + \frac{9}{12} = \frac{17}{12}$

Lesson 7

Sloping Roof Walks

- The first house (left) has a width of 20 feet. The roof is 10 feet tall. Answer: slopes are ±1
- The second house (middle) is 22 feet wide, and the roof is eight feet tall. slopes are $\pm\frac{8}{11}$
- The third house (right) is 16 feet wide, and the roof is eight feet tall. Answer: slopes are ±1
- Students will then imagine taking a walk on the roofs from left to right and describe the incline or steepness as a positive, negative, or zero slope. Answer: Starting on the left, positive slope until the peak. At the peak, slope is zero. Walking towards the right side, slope is negative.

Slides at a Park – Answer: Slide 1 has a slope of $\frac{6}{12}$. Slide 2 has a slope of $\frac{14}{12}$. Slide 3 has a slope of $\frac{4}{5}$. Slide 4 has a slope of $\frac{4}{10}$.

Lines on a Cartesian Plane – Answer: Blue line is $-\frac{4}{5}$. Orange line is $\frac{1}{8}$. Green line is $-\frac{10}{2}$ or $-\frac{5}{1}$. Purple line is $\frac{5}{2}$.)

Going To Grandma's - Answer: The speed can be expressed as $\frac{\frac{1}{2}\,mile}{12\,minutes} = \frac{1\ \ mile}{24\,minutes}$ which is one mile in 24 minutes. The slope is $\frac{1}{24}$. You can also express speed as $\frac{\frac{1}{2}\,mile}{12\,minutes} = \frac{\frac{5}{2}\,mile}{60\,minutes} = \frac{\frac{5}{2}\,mile}{1\,hour}$ which is two and a half miles per hour. However, the slope is still $\frac{1}{24}$.

Lesson 8

Painting with Desmos

y = $\frac{1}{8}$x	y = $\frac{2}{3}$x	y = $\frac{3}{2}$x	y = $\frac{35}{10}$x	y = 20x
y = -$\frac{1}{8}$x	y = -$\frac{2}{3}$x	y = -$\frac{3}{2}$x	y = -$\frac{35}{10}$x	y = -20x
y = $\frac{1}{3}$x	y = x	y = $\frac{23}{10}$x	y = 6x	y = 0
y = -$\frac{1}{3}$x	y = -x	y = -$\frac{23}{10}$x	y = -6x	x=0

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