Geodesic Currents on Hyperbolic Surfaces: Entropy, Intersection Number, and Equidistribution

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Geodesic Currents on Hyperbolic Surfaces: Entropy, Intersection Number, and Equidistribution

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Abstract

Let X be a hyperbolic surface of finite volume. In this thesis, we investigate the behavior of closed geodesics on X and their geometric intersections. Specifically, we address the following three questions:

1. How does the interaction between the intersection number and length of closed geodesics change when the surface X varies? Let us denote the length and geometric intersection number of a pair of closed geodesics on X by $\ell_X(\alpha)$ and $i(\alpha, \beta)$, respectively, and let

$$I(X) := \sup_{\alpha,\beta} \frac{i(\alpha, \beta)}{\ell_X(\alpha)\ell_X(\beta)},$$

where the supremum is taken over all pairs of closed geodesics. We refer to $I(X)$ as the *interaction strength* of X, as it controls the best upper bound on $i(\alpha, \beta)$ in terms of $\ell_X(\alpha)\ell_X(\beta)$. Let $\mathcal{M}_g$ be the moduli space of compact hyperbolic surfaces with genus g. Our main result here describes the exact asymptotic behavior of $I(X)$ on $\mathcal{M}_g$. To state it, let $\operatorname{sys}(X) := \inf_\alpha \ell_X(\alpha)$ denotes the systole of X. We show that

$$I(X) \sim \frac{1}{2\operatorname{sys}(X)\log(1/\operatorname{sys}(X))}$$

as $X \to \infty$ in $\mathcal{M}_g$.

2. How are the intersection points between all pairs of closed geodesics distributed on X? It is not difficult to see that these points are dense in X. We show that the intersection

points of closed geodesics with length $\leq T$ become equidistributed on X, as $T \to \infty$. More precisely, consider all the intersection points (including self-intersections) between closed geodesics with length $\leq T$. Let μ_T be the counting measure on these points (the points are considered with multiplicity). Our main result here shows that the probability measure $\mu_T/|\mu_T|$ converges to the normalized area measure on X, as $T \to \infty$. In the non-compact case, we carry out a careful analysis to show there is no escape of mass in the limit.

3. Can one control the measure-theoretic entropy h_μ of a geodesic current μ in terms of its self-intersection number $i(\mu, \mu)$? Geodesic currents generalize closed geodesics, and it is known that geodesic currents with zero self-intersection number have zero measure-theoretic entropy. This observation motivates our question. We show that the answer is affirmative when the geodesic current is ergodic. More precisely, we show that when μ is an ergodic geodesic current with $\ell_X(\mu) = 1$, we have

$$h_\mu = \mathcal{O}\left(\sqrt{i(\mu,\mu)}\log\left(\frac{1}{\sqrt{i(\mu,\mu)}}\right)\right).$$

The proof involves topological and combinatorial estimates for the number of closed geodesics with a small number of self-intersections relative to length.

Table of Contents

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1. Introduction

In this work, we aim to understand the behavior of closed geodesics on finite-volume hyperbolic surfaces by studying their geometric intersection number. More precisely, we investigate the interaction between the intersection number and length of closed geodesics and how this relationship changes when the surface X varies. Additionally, we find the distribution of the intersection points between closed geodesics on X. We also give a bound on the entropy of geodesic currents (a generalization of closed geodesics) in terms of their self-intersection number. These topics illustrate an interplay between geometry, dynamics, and topology of hyperbolic surfaces.

Overview. Let $\mathcal{M}_{g,n}$ be the moduli space of hyperbolic surfaces of genus g with n cusps. For $X \in \mathcal{M}_{g,n}$, define the (geometric) intersection number $i(\alpha, \beta)$ between closed geodesics α, β to be the total number of transverse intersection points between α, β. The points are counted with multiplicity; in other words, if the geodesics pass through a point (transversally) multiple times, the point is counted with a weight, see §2.1. The intersection number can be defined on the homotopy class of the closed curves as well. Namely, it is the minimum of the intersection number between the representatives in the homotopy classes. Therefore, the intersection number is a topological invariant. Note that these definitions are compatible because geodesic representatives always attain the minimum intersection number.

To delve deeper into this topic, we employ the theory of geodesic currents. A geodesic current is a measure on the unit tangent bundle $T_1(X)$, which is invariant under the geodesic flow and the involution map (the map that sends $v \in T_1(X)$ to $-v \in T_1(X)$), see §2.2. Geodesic currents represent a generalization of closed geodesics. The entropy of a geodesic current μ with length $\ell_X(\mu) = 1$ (meaning that μ is a probability measure) is its measure-theoretic entropy with respect to the geodesic flow and denoted by h_μ.

For $X \in \mathcal{M}_{g,n}$, let the systole of X, written $\mathrm{sys}(X)$, be the length of the shortest closed geodesic(s) of X. Let $\ell_X(\gamma)$ denote the length of closed geodesic γ on X.

Questions. In this thesis, we aim to study the following questions:

Question 1. *How does $I(X)$ change by the geometry of X, where*

$$I(X) := \sup_{\alpha,\beta} \frac{i(\alpha,\beta)}{\ell_X(\alpha)\ell_X(\beta)},$$

and the supremum is over all pairs of closed geodesics?

We call $I(X)$ *the interaction strength* of X because it controls the intersection number and its interaction with lengths of closed geodesics.

Question 2. *How are the intersection points between all pairs of closed geodesics distributed on X?*

These intersection points are dense on X (see Proposition 4.1). Moreover, closed geodesics are equidistributed on X [Bow][Rob][Mar]. Therefore, it is natural to expect that these intersection points become uniformly distributed too.

Question 3. *Can we control the measure-theoretic entropy h_μ of a geodesic current μ in terms of the self-intersection number $i(\mu,\mu)$?*

This question is inspired by the fact that the geodesic currents with self-intersection number zero have measure-theoretic entropy zero [Fat, Lem. 3.3][BS1][You, §5].

Main Results. We answer the questions with the following results.

Theorem 1.1. *Let $I(X)$ be the interaction strength of $X \in \mathcal{M}_g$. We have*

$$I(X) \sim \frac{1}{2\,\mathrm{sys}(X)\log(1/\mathrm{sys}(X))},$$

as $X \to \infty$ in $\mathcal{M}_g$.

Here the notation $\sim$ means that the ratio tends to 1. It is well-known that $X \to \infty$ in $\mathcal{M}_g$ is equivalent to $\mathrm{sys}(X) \to 0$.

Theorem 1.2. *Let X be a finite volume hyperbolic surface. The intersection points of closed geodesics with length $\leq T$ become equidistributed on X, as $T \to \infty$. In other words,* *the sequence of the normalized counting measures on the intersection points (the points are considered with multiplicity) converges weakly to the normalized area measure on X.*

Theorem 1.3. *Let μ be an ergodic geodesic current with $\ell_X(\mu) = 1$. Let h_μ be the measure-theoretic entropy of μ with respect to the geodesic flow on $T_1(X)$, we have:*

$$h_\mu = \mathcal{O}\left(\sqrt{i(\mu,\mu)}\log\left(\frac{1}{\sqrt{i(\mu,\mu)}}\right)\right).$$

The proofs are given in §3, 4, 5, respectively.

Discussion. Now, let us explain each topic in detail and provide a sketch of the proofs.

1. Interaction strength

The upper bound on the intersection number, in terms of the hyperbolic length and word length of the curves, has been studied in different settings [CP2][CP1][Bas][CMP]. In §3, we study the supremum of the intersection number in terms of hyperbolic length. Given $X \in \mathcal{M}_g$, we know that $i(\alpha,\beta) \leq c(X)\ell_X(\alpha)\ell_X(\beta)$ for a constant $c(X) > 0$ [Bas](also see Proposition 2.3). Recall that the *interaction strength* of X is

$$I(X) := \sup_{\alpha,\beta} \frac{i(\alpha,\beta)}{\ell_X(\alpha)\ell_X(\beta)}, \tag{1}$$

where the supremum is over all pairs of closed geodesics. We refer to it as the *interaction strength* of X because it controls the upper bound on $i(\alpha,\beta)$ in terms of $\ell_X(\alpha)\ell_X(\beta)$. We can see that $I(X) \to \infty$, as $X \to \infty$, in $\mathcal{M}_g$ (see Corollary 3.2) and that

$$I(X) \leq 4/\operatorname{sys}(X)^2,$$

by Proposition 2.3. Note that $X \to \infty$ in $\mathcal{M}_g$ is equivalent to $\operatorname{sys}(X) \to 0$ [Mum]. Our main result, Theorem 1.1, describes the exact asymptotic behavior of $I(X)$ on $\mathcal{M}_g$:

$$I(X) \sim \frac{1}{2\operatorname{sys}(X)\log(1/\operatorname{sys}(X))}$$

as $X \to \infty$ in $\mathcal{M}_g$, or equivalently, as $\operatorname{sys}(X) \to 0$. Recall that the notation $\sim$ means that the ratio tends to 1.

Let α be a closed geodesic with length $\operatorname{sys}(X)$. We can find a closed geodesic β with the length at most about $4\log(1/\operatorname{sys}(X))$ that intersects α in two points. The normalized intersection number $i(\alpha,\beta)/\ell_X(\alpha)\ell_X(\beta)$ is a lower bound for $I(X)$ and is at least about $1/(2\operatorname{sys}(X)\log(1/\operatorname{sys}(X)))$, §3.1. See Figure 1.

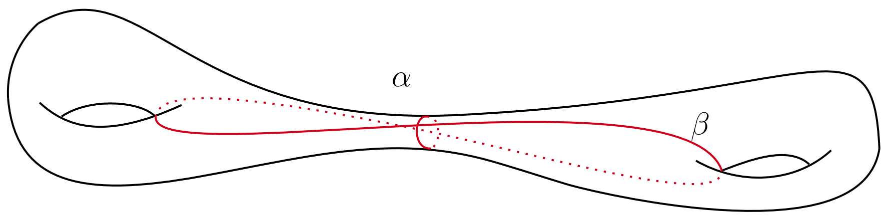

FIGURE 1

From Theorem 1.1 and continuity of $I(X)$, we can obtain an estimate of $I(X)$ up to a g-dependent multiplicative constant for all surfaces in $\mathcal{M}g$ (see Corollary 3.15).

The notation $A \asymp B$ ($A \asymp_g B$) means $A \in [B/c, cB]$ ($A \in [B/c_g, c_g B]$) for an implicit constant $c > 0$ ($c_g > 0$). We say that A and B are asymptotically equivalent when $A \sim B$ and that A and B have the same order of magnitude when $A \asymp B$.

We can see $I(X)$ attains its minimum at a point in $\mathcal{M}_g$, since it is proper (by Theorem 1.1). The order of magnitude of the minimum of I is given (see §3.3):

Proposition 1.4. *We have*

$$\min_{X\in\mathcal{M}_g} I(X) \asymp \frac{1}{(\log g)^2},$$

uniformly in g.

A similar result as Theorem 1.1 holds when X varies in $\mathcal{M}_{g,n}$, provided we restrict attention to closed geodesics in a fixed compact subset of X, see §3.4. This result also gives a global estimate of the interaction strength up to a g-dependent multiplicative constant for all $X \in \mathcal{M}_g$ (see Corollary 3.19).

One can also consider the following related quantities:

$$I_\Delta(X) := \sup_\gamma \frac{i(\gamma,\gamma)}{\ell_X(\gamma)^2},$$

Where the supremum is over all closed geodesics, and

$$I_{simple}(X) := \sup_{\alpha,\beta} \frac{i(\alpha,\beta)}{\ell_X(\alpha)\ell_X(\beta)},$$

where the supremum is over all pairs of simple closed geodesics.

We will see that they have the same order of magnitude as $I(X)$. More precisely, we have:

$$I_\Delta(X) \in [I(X)/2, I(X)],$$

as it is shown in Proposition 3.7, and

$$I_{simple} \asymp_g I(X),$$

as it is shown in Corollary 3.5.

Moreover, $I_{simple}(X)/I(X) \to 1$ as $X \to \infty$ in $\mathcal{M}_g$.

Although $I(X) \to \infty$ when $X \to \infty$ in $\mathcal{M}_g$ we can see that the interaction strength of a random hyperbolic surface in $\mathcal{M}_g$ is still finite. In other words, we have:

Corollary 1.5. *The expected value of the interaction strength on $\mathcal{M}_g$, equipped with $Weil-Petersson$ measure, is finite.*

This Corollary is an immediate consequence of Theorem 1.1 and Mirzakhani's bound on the volume of $\mathcal{M}_g^\epsilon$, the ϵ-thin part of $\mathcal{M}_g$ [Mir3, Thm. 4.1]; see §3.5. Note that the upper bound in [Bas] and Proposition 2.3 is insufficient to obtain Corollary 1.5.

Outline of the Proof of Theorem 1.1: The main steps of the proof are as follows. In order to find the lower bound for $I(X)$, we consider a systole and the shortest closed geodesic that intersects it at two points as depicted in Figure 1. In order to find the required upper bound, we consider a proper pants decomposition $\mathcal{P}$ of X. Then we split X into thin and thick parts where the thin parts are annular neighborhoods of the curves in $\mathcal{P}$ (they are a subset of their collars) and the thick parts are the complements (see §2). We have the following inequality for $I_i, \ell_i, \ell_i' \geq 0$:

$$\frac{I_1 + I_2}{(\ell_1 + \ell_2)(\ell_1' + \ell_2')} \leq \max(\frac{I_1}{\ell_1\ell_1'}, \frac{I_2}{\ell_2\ell_2'}).$$

It implies that it is enough to obtain the required upper bound in each part separately. Then we can see that the thin part containing the systole has the main contribution to $I(X)$. The heart of the proof lies in approximating the lengths and intersection numbers of geodesic arcs inside the thin parts (see §3.2).

2. Intersection points

It is not hard to see that the intersection points between pairs of closed geodesics are dense on X (see Proposition 4.1). This is also known that the closed geodesics are equidistributed on X [Bow] [Rob] [Mar]. Therefore, it is natural to expect the intersection points to be uniformly distributed on X as well. In other words, we consider all of the intersection points (including the self-intersections) between the closed geodesics with length $< T$ and show that these become equidistributed on X, with respect to the area measure, as $T \to \infty$. Note that X may have cusps, and a challenging part of the proof is to show that there is no escape of mass to the cusps in the limit.

Let us state this more mathematically. Let $\mathcal{G}$ be the set of closed geodesics on X and $\mathcal{G}_T$ the set of closed geodesics with length $\leq T$. Given $\alpha, \beta \in \mathcal{G}$, define the intersection measure

$I(\alpha,\beta)$ as follows:

$$I(\alpha,\beta) := \sum_{p\in\alpha\cap\beta} c_p\delta_p$$

where the sum is over the transverse intersection points of α and β, δ_p is the delta measure at a point p, and the weight c_p is the multiplicity of the intersection at the point p. In other words, $c_p = 1$ unless the geodesics pass through the point p multiple times (see §2.1). The intersection number of α and β, $i(\alpha,\beta)$, is the total volume of $I(\alpha,\beta)$ on X. The intersection measure can extend to geodesic currents (see §2.2).

Denote $|\mu|$ as the total volume of measure μ. Let A be the area measure on X obtained from the hyperbolic metric. Then $A(X)$ =arae$(X) = 2\pi(2g-2+n)$ where g is genus of X and n the number of cusps. Define the multi-curve γ_T as:

$$\gamma_T := \sum_{\gamma\in\mathcal{G}_T}\gamma.$$

Our main result, Theorem 1.2, states:

The sequence of probability measures $I(\gamma_T,\gamma_T)/|I(\gamma_T,\gamma_T)|$ converges weakly to $A/area(X)$ in $C_c^*(X)$, as $T\to\infty$.

Convergence in $C_c^*(X)$ means weak convergence in the dual space of the continuous compactly supported functions.

The proof also implies the growth rate of the intersection number.

Corollary 1.6. *As $T\to\infty$, we have*

$$i(\gamma_T,\gamma_T) \sim \frac{1}{\pi^2(2g-2+n)}(\sum_{\ell_X(\gamma)\leq T}\ell_X(\gamma))^2 \sim \frac{e^{2T}}{\pi^2(2g-2+n)}.$$

In Chapter 4, we prove Theorem 1.2 using the theory of geodesic currents.

Geodesic Currents. The theory of geodesic currents is a tool for studying closed geodesics. It was developed by Bonahon and has essential applications in the study of Teichmüller space and hyperbolic surfaces [Bon2][Bon3]. Given a closed geodesic γ, we can interpret it as an invariant measure of the geodesic flow on the unit tangent bundle $T_1(X)$. Namely, consider the length measure on the subset of $T_1(X)$ containing the unit tangent vectors of γ. This measure is a Borel, geodesic flow invariant and involution invariant measure on $T_1(X)$ (the involution sends a vector $v \in T_1(X)$ to $-v$). A geodesic current is a measure on $T_1(X)$ that satisfies these properties. Length of a geodesic current C is the total volume of $T_1(X)$ with respect to C. The closure of the set of weighted closed geodesics is the space of geodesic currents. The intersection number can also be extended to geodesic currents (see §2.2 for more details).

Outline of the Proof of Theorem 1.2. The proof has two main parts:

- In the first part, we prove a weaker convergence of measures (compared to the convergence in Theorem 1.2) using the theory of geodesic currents. In fact, we extend the result of Bonahon about the continuity of $i(\cdot,\cdot)$ on compact surfaces to finite volume surfaces (see Theorem 4.4). Then we obtain:

$$\frac{I(\gamma_T, \gamma_T)}{\ell_X(\gamma_T)^2} \to \frac{A}{\pi^2(2g-2+n)}, \tag{2}$$

  as $T \to \infty$, in $C_c^*(X)$. This weaker form of convergence does not guarantee that there is no escape of mass to infinity (into the cusps), in the limit.

- In the second part, we show the measures in the convergence sequence (2) are *tight* and there is no escape of mass to infinity. As a result, we obtain the required convergence of the probability measures in Theorem 1.2. Note that the intersection number has different behavior when there is a cusp. Namely, when X is compact $i(\alpha, \beta) \leq c(X)\ell_X(\alpha)\ell_X(\beta)$ for a constant $c(X)$ but when X has a cusp, the intersection number can be exponential in terms of lengths [Bas]. For example, a closed geodesic with weight c that goes around the cusp n times has length $\sim 2c \log n$ and self-intersection number $\sim c^2 n$. Therefore, having a small length in a neighborhood of the cusp does not necessarily imply a small self-intersection number in that neighborhood. In order to control the intersection number, we divide geodesic arcs inside the cusp's neighborhood into different groups based on their winding number around the cusp. Fix horoball neighborhood Y of a cusp with horocycle boundary δY of length 1. Define an n-excursion (or simply n-exc) corresponding to Y as follows. It is an arc inside Y with endpoints on δY such that its winding number around the cusp is n. Let $E_n(\gamma)$ be the number of n-excs of $\gamma \in \mathcal{G}$. Our main result in the second part is the following.

Proposition 1.7. *For $n, T > 0$ we have:*

$$\frac{E_n(\gamma_T)}{\ell_X(\gamma_T)} = \frac{\sum_{\gamma \in \mathcal{G}_T} E_n(\gamma)}{\sum_{\gamma \in \mathcal{G}_T} \ell_X(\gamma)} \leq \frac{c}{n^2},$$

for a constant $c > 0$.

In addition, in Lemma 4.8, we show that the intersection number between an n-exc and a m-exc is $\leq 2\min(n, m) + 2$. From this fact and Proposition 1.7, we conclude that the measures are tight.

3. Entropy and self-intersection number

By the work of Birman and Series, one can conclude that the measure-theoretic entropy of measured laminations is zero [BS1]. This was also proved by Young and Fathi [You] [Fat]. On the other hand, measured laminations are exactly the geodesic currents with self-intersection number zero. This observation inspires us to ask the following question:

If μ is a geodesic current with length 1, and $i(\mu, \mu)$ is small, then does this imply that h_μ, the measure-theoretic entropy of μ, is small as well?

When μ is ergodic, we prove Theorem 1.3 and conclude that the answer is yes.

Note that the opposite is not true; namely, $h_\gamma = 0$ when γ is a closed geodesic but $i(\gamma, \gamma)$ might not be small.

Outline of the Proof of Theorem 1.3. The proof has two parts.

- First, we find an upper bound on h_μ in terms of the growth rate of the number of closed geodesics with a small the self-intersection number relative to length. More precisely, let $P_\epsilon(T)$ be the number of the closed geodesics γ with $\ell_X(\gamma) \leq T$ and $i(\gamma, \gamma) \leq \epsilon T^2$. Let
$$P_\epsilon := \lim_{T\to\infty} \frac{\log p_\epsilon(T)}{T}.$$
In §5.1, we show $h_\mu \leq P_{2i(\mu,\mu)}$ (Theorem 5.2).
- In the second part, we find an upper bound for the growth rate P_ϵ using a hexagonal decomposition of the surface and symbolic coding of the closed curves based on that decomposition (see Theorem 5.3). The heart of the proof is that for each closed geodesic γ there exists a special representative that avoids a certain type of intersection (see Theorem 5.6). It helps us to obtain an upper bound on $p_\epsilon(T)$. In fact, we assign a word to each edge of a hexagon based on the outgoing arcs from that edge. Then we give a bound on the number of possible words obtained from special representatives. The details are explained in §5.2.

Open Problems.

(1) When is $I(X)$ achieved by closed geodesics?

In Lemma 3.6, we see that $I(X)$ is equal to the intersection number between two geodesic currents of length 1. One may ask whether $I(X)$ is achieved by a pair of closed geodesics, particularly simple closed geodesics. For instance, is it true that for weil-Petersson-almost every $X \in \mathcal{M}_g$ there are simple closed geodesics α, β such that $i(\alpha, \beta) = I(X)\ell_X(\alpha)\ell_X(\beta)$? If it is true then we would have $I_{simple}(X) = I(X)$ for all $X \in \mathcal{M}_g$.

This question is inspired by the work of Thurston in the article [Thu]. Let $\mathcal{T}_g$ be the Teichmüller space of hyperbolic surfaces of genus g. Given closed curve γ, define $\ell_X(\gamma)$ as the hyperbolic length of the geodesic representative of the homotopy class

$[\gamma]$ on X. Thurston proved that for generic pairs $X, Y \in T_g$, the stretch factor

$$L(X,Y) := \sup_{\gamma} \log(\frac{\ell_X(\gamma)}{\ell_Y(\gamma)})$$

is attained by a simple closed geodesic. In other words, $L(X,Y) = \log(\ell_X(\gamma)/\ell_Y(\gamma))$ for a simple closed geodesic γ [Thu, §10].

(2) Is $I_{simple}(X) = I(X)$ for all $X \in \mathcal{M}_g$?

(3) Is $I_\Delta = I(X)/2$ for all $X \in \mathcal{M}_g$?

(4) In the equidistribution result, Theorem 1.2, the points are considered with multiplicity. When we pick weight one for all the intersection points, does the result still hold?

(5) Is it true that when $i(C_n, C_n) \to 0$ then $h_{C_n} \to 0$ as well, for geodesic currents C_n? This is true by Theorem 1.3, for ergodic geodesic currents.

Notes and References.

(1) As we mentioned, intersection number is a topological invariant of the homotopy classes. For closed curves α, β, their intersection number $i(\alpha, \beta)$ is defined as the minimum number of intersections between representatives in the homotopy classes $[\alpha], [\beta]$. Interestingly, geodesics attain the minimum. The relation between self-intersection number and word length is also studied for specific topological surfaces; see [CP2][CP1][CMP].

(2) There is a similar quantity to $I(X)$ for the algebraic intersection number and is studied in [CKM][MM].

(3) The statistic of the self-intersection number is studied in [Lal2] for hyperbolic surfaces and in [CL] for topological surfaces. There are also some results in [Bas] about the infimum of the closed geodesics' length when the self-intersection number is fixed.

(4) The first part of the proof of Theorem 1.2 implies the equidistribution result for compact hyperbolic surfaces. Lalley proved this case of compact surfaces differently (not using the geodesic currents) [Lal1].

(5) A qualitative version of Theorem 1.3 for the Haussdorff dimension of the support of geodesic currents (instead of the entropy of geodesic currents) can be deduced from Sapir's work [Sap].

(6) Closed geodesics on X are crucial in the study of geometry and dynamics of X in $\mathcal{M}_g$; for instant see [Thu][Mir2][Mir1][Wol].

2. Preliminaries

2.1. **Hyperbolic Surfaces.** In this section, we briefly explain some basics of hyperbolic geometry (for a reference, see [Bas]). We also present some properties of function $\sigma(x) = \sinh^{-1}(\frac{1}{\sinh(x)})$ which is half of the length of the collar of a simple closed geodesic γ with $\ell_X(\gamma) = 2x$. From now on, for simplicity, we may write $\ell(.)$ instead of $\ell_X(.)$.

Upper half-plane. Let (X, d) be a hyperbolic surface (with metric d) and $\pi : \mathbb{H} \to X$ its universal cover map, where $\mathbb{H}$ is the upper half-plane. We denote the length of an arc α by $\ell(\alpha)$.

Collar of a closed geodesic. Assume that γ is a simple closed geodesic on X. Let $N_\gamma(r) := \{x| \, d(x, \gamma) \leq r\}$ be the $r-$neighborhood of γ.

Lemma 2.1. *Assume that $N_\gamma(r)$ is an embedded annulus in X. Then its boundary components have length $\ell(\gamma)\cosh(r)$ and γ splits $N_\gamma(r)$ into two parts of area $\ell(\gamma)\sinh(r)$.*

Proof. Apply Fermi coordinate with respect to γ, which is:

$$ds^2 = d\rho^2 + \cosh(\rho)^2 dt^2,$$

where ρ is distance from γ and t is the distance between the projection on γ and a based point $P_0 \in \gamma$ [Bus, p. 4]. □

Define $\sigma(\gamma) := \sigma(\ell(\gamma)/2)$. when $r = \sigma(\gamma)$, the neighborhood $N_\gamma(r)$ is called the *collar of* γ and denoted by $O(collar, \gamma)$. It is well-known that the collar neighborhood of a simple closed geodesic is embedded in X [Bus, Thm. 4.1.1].

Consider a pair of pants S, the boundaries of S are called *cuffs* of S. Assume that S has a hyperbolic structure. The shortest arcs between different cuffs are called *seams* of S. Any hyperbolic surface X has a decomposition into pairs of pants by cutting X along a maximal set of disjoint simple closed geodesics. Pair of pants in the decomposition may have a boundary of length 0 when X has a cusp.

Thin-thick decomposition. Consider a pants decomposition $\mathcal{P} = \{P_1, \ldots, P_{2g-2}\}$ of X by closed geodesics $\gamma_1, \ldots, \gamma_{3g-3}$. The collars of γ_k's are pairwise disjoint annuli [Bus, Thm. 4.1.1.]. Let $O(collar, k)$ be the collar of γ_k. Let $O(thin, k)$ denote *thin part* $N_{\gamma_k}(r_k) \subset O(collar, k)$ of γ_k where

$$r_k := \sinh^{-1}\left(\frac{1}{\sinh(\ell(\gamma_k/2))\sqrt{1+\cosh^2(\ell(\gamma_k)/2)}}\right).$$

The *thick parts* of X with respect to $\mathcal{P}$ are the $2g-2$ pairs of pants in the complements of the thin parts. Denote the thick part in P_j by $O(thick, j)$.

From Bers' theorem, we know that there is a pants decomposition such that $\ell(\gamma_k) \leq L_g$, where L_g is *Bers' constant* [Bus, Thm. 5.1.2.]. We call such a decomposition a *proper decomposition*.

The following lemma describes an important property of thin parts and clarifies why r_k is defined as above.

Lemma 2.2. *Assume that α is a geodesic arc in the collar of γ with endpoints on one boundary of the collar. If α does not intersect itself then it is disjoint from the interior of $O(thin, \gamma)$.*

Proof. Consider preimages $\widetilde{\gamma}$ and $\widetilde{\alpha}$ of γ and α in $\mathbb{H}$, respectively. The endpoints A, B of $\widetilde{\alpha}$ have distance $\sigma(\gamma)$ from $\widetilde{\gamma}$. Let A' and B' be the projection of A, B on $\widetilde{\gamma}$. Therefore, $\ell(AA') = \ell(BB') = \sigma(\gamma)$ and $\ell(A'B') \leq \ell(\gamma)$ since α doesn't intersect itself.

Let m_α, m_γ be the midpoints of AB and $A'B'$. Then the geodesic arc $m_\alpha m_\gamma$ is orthogonal to $\widetilde{\alpha}$ and $\widetilde{\gamma}$ and its length is the distance between $\widetilde{\alpha}$ and $\widetilde{\gamma}$ (see Figure 2).

Let ϕ be the angle between AB and AA'. Define: $m := \ell(m_\gamma m_\alpha)$, $a := \ell(AA') = \sigma(\gamma)$, and $b := \ell(A'm_\gamma)$. From the hyperbolic formula for trirectangle $Am_\alpha m_\gamma A'$ [Bus, Thm. 2.3.1.] we have:

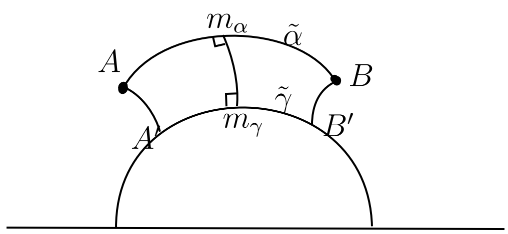

FIGURE 2

$$\sinh m = \frac{\cos\phi}{\sinh b} = \frac{1}{\sqrt{1+\tan^2\phi}\sinh b} = \frac{1}{\sqrt{1+(\coth b/\sinh a)^2}\sinh b} = \frac{\sinh a}{\sqrt{\sinh^2 a \sinh^2 b + \cosh^2 b}}.$$

Therefore, from $b = \ell(A'B')/2 \leq \ell(\gamma)/2$ we have:

$$\sinh m \geq \frac{\sinh a}{\sqrt{\sinh^2 a \sinh^2 \ell(\gamma)/2 + \cosh^2 \ell(\gamma)/2}} = \frac{1}{\sinh(\ell(\gamma)/2)\sqrt{1+\cosh^2(\ell(\gamma)/2)}}.$$

As a result, γ is disjoint from the interior of $O(thin, \gamma)$.

□

Intersection number. Given closed curves α, β, the *(geometric) intersection number* of them, $i(\alpha, \beta)$, is the minimal number of (transversal) intersection points between the representatives in the homotopy classes $[\alpha], [\beta]$ and the points are counted with multiplicity. Interestingly, geodesic representatives always attain the minimum. By multiplicity, we mean that we count a point with a weight if the curves pass through it multiple times. To find the weight at a point p we can slightly move the curves in a neighborhood of p to have only simple intersections. Then the minimum number of simple intersections is the weight. We can bound the intersection number from above in terms of the product of the lengths.

Proposition 2.3. *Let X be a compact hyperbolic surface. Then we have:*

$$i(\alpha, \beta) \leq \frac{4}{\mathrm{sys}(X)^2}\ell(\alpha)\ell(\beta),$$

for all $\alpha, \beta \in \mathcal{G}$.

Proof. Given two geodesic arcs α_1, α_2 of length $\leq \text{sys}(X)/2$, we can see that they intersect transversally at most once: Let P be an intersection point of α_1, α_2. Then α_1, α_2 are embedded inside the disk centered at P with radius $\text{sys}(X)/2$.

Now consider the set $\alpha \times \beta = \{(a, b) | a \in \alpha, b \in \beta\}$. It is a torus with the metric $d\ell_1 \times d\ell_2$ where $d\ell_i'$s are the hyperbolic length along α, β. It has volume $\ell(\alpha) \times \ell(\beta)$. Note that each intersection point can be indicated by a point in $\alpha \times \beta$. The squares with centers at the intersection points and side length $\text{sys}(X)/2$ are disjoint as explained above. Therefore

$$i(\alpha, \beta)\frac{\text{sys}(X)^2}{4} \leq area(\alpha \times \beta) = \ell(\alpha)\ell(\beta).$$

□

Define $I(X)$, the *interaction strength* of X, as the optimal constant $c(X)$ such that $i(\gamma_1, \gamma_2) \leq c(X)\ell(\gamma_1)\ell(\gamma_2)$ for all $\gamma_1, \gamma_2 \in \mathcal{G}$. In other words,

$$I(X) := \sup_{\gamma_1, \gamma_2 \in \mathcal{G}(X)} \frac{i(\gamma_1, \gamma_2)}{\ell(\gamma_1)\ell(\gamma_2)}.$$

We can see that $I(X)$ is a continuous function on $\mathcal{M}_g$.

Lemma 2.4. *The function $I : \mathcal{M} \to \mathbb{R}$ is continuous.*

Proof. The Teichmüller space $\mathcal{T}_g$ is the universal cover of $\mathcal{M}_g$. Therefore, It is equivalent to show $I(X)$ on $\mathcal{T}_g$ is continuous which is an immediate consequence of the fact that function $\ell_\gamma : \mathcal{T}_g \to \mathcal{R}^+$ is continuous where γ is a closed curve and $\ell_\gamma(X)$ is the length of the geodesic representative of the homotopy class $[\gamma]$ on $X \in \mathcal{T}_g$. □

Properties of $\sigma(t)$. Now we discuss some properties of the function $\sigma(t) = \sinh^{-1}(\frac{1}{\sinh(t)})$ which are repeatedly applied in the proofs. We may not refer to the lemma when we apply it.

Lemma 2.5. *Assume that $t > 0$. For the function $\sigma(t)$ we have:*

(1) $\sigma(t) \geq \log \frac{1}{t}$,

(2) $\sigma(t) \leq 2\log \frac{1}{t}$ *for* $t \leq 1/3$,

(3) $\sigma(t)$ *is decreasing,*

(4) $t\sigma(t)$ *is* < 1 *and increasing for* $t \leq \frac{1}{2}$,

(5) $\lim\limits_{t\to 0} \frac{\log(1/t)}{\sigma(t/2)} = 1$.

Proof. **Part 1.** If $t \geq 1$ then $\sigma(t) > 0 > \log \frac{1}{t}$. So assume $t \in (0,1]$. The function $\sinh(X)$ is increasing therefore it is enough to prove $(\frac{1}{t} - t)(e^t - e^{-t}) \leq 4$.

- When $t \in [\log 2, 1]$ we have

$$(\frac{1}{t} - t)(e^t - e^{-t}) \leq (\frac{1}{t} - t)(2te) \leq 2e(1 - (\log 2)^2) < 4.$$

- When $t \in (0, \log 2)$, then we have

$$(\frac{1}{t} - t)(e^t - e^{-t}) \leq (\frac{1}{t} - t)(2te^{\log 2}) < 4.$$

part 2. It is equivalent to show

$$4 \leq (e^t - e^{-t})(\frac{1}{t^2} - t^2)$$

which is true since we have the following:

$$(e^t - e^{-t})(\frac{1}{t^2} - t^2) \geq 2te^{\frac{-1}{3}}(\frac{1}{t^2} - t^2) \geq 2e^{\frac{-1}{3}}(3 - \frac{1}{27}) > 4.$$

Part 3. It implies from the fact that $\sinh(t)$ is increasing.

Part 4. First we show $t\sigma(t) < 1$. Note that $e^x - e^{-x} \geq 2x$ for $x \in \mathbb{R}^+$. Therefore, $\sinh(t)\sinh(1/t) \geq 1$ which implies $1/t \geq \sigma(t)$. In order to show $t\sigma(t)$ is increasing we prove its derivative is positive for $t \leq \frac{1}{2}$. In other words, we aim to prove the following:

$$\sigma(t) - \frac{t}{\sinh(t)} \geq 0.$$

We have:

$$\sigma(t) \geq \sigma(\frac{1}{2}) \geq 1 \geq \frac{t}{\sinh(t)}$$

as required.

Part 5. We have $\sinh^{-1}(N)/\log 2N \to 1$ when $N \to \infty$. As a result, we have

$$\lim_{t\to 0} \frac{\log(1/t)}{\sigma(t/2)} = \lim_{t\to 0} \frac{\log(1/t)}{\log(2/\sinh(t/2))}.$$

On the other hand, $\sinh(x) \in [x, 2x]$ when $x > 0$ is small enough. Therefore, we have:

$$\frac{\log(1/t)}{\log(2/\sinh(t/2))} \in [\frac{\log(1/t)}{\log(4/t)}, \frac{\log(1/t)}{\log(2/t)}].$$

We can see that the bounds tend to 1 as $t \to 0$. $\square$

2.2. **Geodesic Currents.** In this section, we present some basic concepts and results concerning geodesic currents and their extensions to finite-volume surfaces. For a reference see [Bon2][Bon3].

Let $X := \mathbb{H}/\Gamma$ be a hyperbolic surface of finite volume. Each geodesic of $\mathbb{H}$ is represented by its two boundary points at ∞. As a result, the space of all unoriented geodesics is homeomorphic to $G(\mathbb{H}) := (S^1 \times S^1 - \Delta)/(\mathbb{Z}/2)$. Let $\mathbb{P}(X)$ be the tangent line bundle. It is the quotient of $T_1(X)$ by the involution action where the involution sends (x, v) to $(x, -v)$. We have a foliation $\mathcal{F}$ on $\mathbb{P}(X)$ including the leaves of the geodesic flow.

A *geodesic current* can be defined and interpreted in different ways:

(1) It is a Borel, locally finite, $\Gamma-$invariant measure on $G(\mathbb{H})$.
(2) It is a Borel, locally finite, $\phi-$invariant and involution invariant measure on $T_1(X)$.
(3) It is an invariant transverse measure for the geodesic foliation $\mathcal{F}$. In other words, it assigns a measure to each codimension 1 submanifold $V \subset \mathbb{P}(X)$ transverse to the leaves of $\mathcal{F}$. In addition, it is invariant when V moves along leaves of $\mathcal{F}$.

We may not mention which interpretation we use when it is clear.

Note that the third definition shows that geodesic currents are topological invariants. In other words, we can define geodesic currents independent of the hyperbolic structure because there is a canonical correspondence between the geodesics of hyperbolic structures on a topological surface S.

The relation between these definitions is as follows. The correspondence between the first and third definitions is clear; the measure of a subset of $(\mathbb{H})$ is corresponding to the transverse measure of this set of geodesics. For the second definition, note that a measure on $T_1(X)$ which satisfies the assumptions in (ii) can be written as $C \times d\ell/2$ where $d\ell$ is the length measure along the geodesics and C can be interpreted as a transverse measure of $\mathcal{F}$ or a $\Gamma-$invarient measure on $G(\mathbb{H})$. On the other hand, the product of a transverse measure of $\mathcal{F}$ with $d\ell/2$ is a measure on $T_1(X)$ which is $\phi-$invarient. Similarly, we obtain a measure on $T_1(X)$ from a $\Gamma-$invarient measure C on $G(\mathbb{H})$. Note that we have factor $\frac{1}{2}$, because the geodesics in G and $\mathcal{F}$ are unoriented despite of the geodesics in $T_1(X)$.

Let $\mathcal{C}$ be the space of all geodesic currents. Denote $C_c^*(M)$ as the dual of compactly supported continuous functions on a topological space M. The set $\mathcal{C}$ is equipped with a weak topology. Namely, a sequence C_n of geodesic currents converge to C if

$$\lim_{n\to\infty} \int_{T_1(X)} f\, dC_n \to \int_{T_1(X)} f\, dC$$

for every $f \in C_c^*(T_1(X))$. As the introduction explains, closed geodesics are an example of geodesic currents; consider the half of the length measure on the unit tangent vectors in $T_1(X)$. Moreover, weighted closed geodesics are dense in $\mathcal{C}$. The notion of length and intersection number of closed geodesics can be extended to geodesic currents. When X is compact, we can define length and intersection number as the continuous extension of these functions on closed geodesics. But when X has cusps, the intersection number $i(.,.)$ is not continuous anymore. Therefore, we define it as follows.

The *Length* of a geodesic current C is defined as the total volume of $T_1(X)$ with respect to C and denoted by $\ell_X(C)$ (or for simplicity $\ell(C)$). In this thesis, we restrict $\mathcal{C}$ to the geodesic currents of finite length; whenever we consider a geodesic current, we assume it has a finite length.

Let $\mathbb{P}(X) \oplus \mathbb{P}(X)$ be the Whitney sum of the bundle $\mathbb{P}(X) \to X$ which is the space of triples $(x, \lambda_1, \lambda_2)$ where $x \in X$ and λ_1, λ_2 are two tangent lines at x. Assume P_1, P_2 are the projection of $\mathbb{P}(X) \oplus \mathbb{P}(X)$ to $\mathbb{P}(X)$ by forgetting the second and the third components, respectively. Let $\mathcal{F}_i$ be the foliation on $\mathbb{P}(X) \oplus \mathbb{P}(X)$ such that its leaves are the preimage of the leaves of $\mathcal{F}$ under the map P_i, for $i = 1, 2$.

The space $\mathbb{P}(X) \oplus \mathbb{P}(X)$ is a 4 dimensional manifold foliated by $\mathcal{F}_1$ and $\mathcal{F}_2$ which are transverse outside the diagonal $\triangle$. Define $\mathcal{I}_1(X) := \mathbb{P}(X) \oplus \mathbb{P}(X) \backslash \triangle$.

Every two geodesic currents C_1 and C_2 define a natural measure $I(C_1, C_2)$ on $\mathcal{I}_1(X)$ as follows. The geodesic current C_1 is an invariant transverse measure of $\mathcal{F}$ on $\mathbb{P}(X)$. Therefore, $P_1^*(C_1)$ define a measure on each plane in $\mathcal{I}_1(X)$ transverse to $\mathcal{F}_1$. We know $\mathcal{F}_1$ and $\mathcal{F}_2$ are transverse in $\mathcal{I}_1(X)$. Therefore, $P_1^*(C_1)$ induces a measure on each leaf of $\mathcal{F}_2$. Similarly, $P_2^*(C_2)$ induce a measure on each leaf of $\mathcal{F}_1$. Define $I(C_1, C_2)$ as the product measure $P_1^*(C_1) \times P_2^*(C_2)$.

The *Intersection number* $i(C_1, C_2)$ of C_1 and C_2 is the total volume of $\mathcal{I}_1(X)$ with respect to $I(C_1, C_2)$.

There is a canonical geodesic current called *Liouville measure*, denoted by L_X. We can define it as a Γ-invariant measure on $G(\mathbb{H})$ as follows. Consider the intervals $[a, b], [c, d]$ on S^1, the boundary of the upper half-plane. Then L_X-measure of the set of geodesics with endpoints in the intervals is:

$$L_X([a, b] \times [c, d]) = \left| \log \left| \frac{(a-c)(b-d)}{(a-d)(b-c)} \right| \right|.$$

Consider a geodesic h. We can see that $L_X = 1/2\sin(\theta)d\theta \wedge dx$ on the set of unoriented geodesics meeting h transversely, where θ is the angle a geodesic makes with h and x is the hyperbolic distance along h to a base point on h. See [Bon3].

For any geodesic current C, $i(C, L_X) = \ell_X(C)$; In particular, $i(L_X, L_X) = \ell_X(L_X) = 2\pi^2(g-1)$ [Bon3]. Bonahon proves it for compact surfaces [Bon1, Prop 15]. The same proof work for the finite volume surfaces.

Proposition 2.6. *Let X be a compact hyperbolic surface and L_X be the Liouville measure of X. Then $i(C, L_X) = \ell_X(C)$ for any geodesic current C and $i(L_X, L_X) = \ell_X(L_X) = 2\pi^2(g-1)$.*

Proof. Since L_X is Γ-invariant, we can see that the measure of the set of geodesics intersecting a geodesic arc α is proportional to the length of α. But we can see they are actually equal because:

$$\int_\alpha \int_0^\pi \frac{1}{2}\sin\theta d\theta dx = \int_\alpha dx.$$

Geodesic current C and L_X define a product measure on $\mathcal{I}_1(X)$. Consider a box B in $\mathbb{P}(X)$ which is the geodesic arcs between two small and close arcs on X. So $B \cong D^2 \times [0, x]$ and $p \times [0, 1]$, for each point $p \in D^2$, corresponds to a preimage of a geodesic arc in $\mathbb{P}(X)$. If we integrate the measure $I(C, L_X)$ along the third component λ_2 in $\mathcal{I}_1(X)$ then we obtain a measure μ on $\mathbb{P}(X)$ and $\mu(B)$ is obtained by integrating the length of geodesic arcs with respect to the measure induced on D^2 by C. Therefore, μ is equal to $C \times d\ell_X$ on $\mathbb{P}(X)$. So $i(C, L_X)$ is total volume of $\mathbb{P}(X)$ with respect to μ and is equal to $\ell_X(C)$.

The length of Liouvill measure can be computed similarly. If we consider coordinate (x, y, θ) on $T_1(\mathbb{H})$ (unit tangent bundle of the upper half plane) where x, y present the distance to two orthogonal geodesics (as the axis) and θ is the angle to the horizental direction then we can see that Liouville measure is equal to $1/2 d\theta dx dy$ which is the half of the canonical volume measure on $T_1(\mathbb{H})$. So $i(L_X, L_X)$ is the half of the volume of $\mathbb{P}(X)$ which is $2\pi^2(g-1)$.

$\square$

For any two geodesic currents C_1, C_2, the push forward of the measure $I(C_1, C_2)$ by the projection $P : \mathcal{I}_1(X) \to X$ which sends $(x, \pm\lambda_1, \pm\lambda_2)$ to x, gives a measure $P_*I(C_1, C_2)$ on X. We may refer to it as $I(C_1, C_2)$ as well when it is clear we talk about the push forward measure.

Lemma 2.7. *Given closed geodesics γ_1, γ_2, $P_*I(\gamma_1, \gamma_2)$ is a measure supported on the intersection points of γ_1 and γ_2 and we have:*

$$P_*I(\gamma_1, \gamma_2) = \sum_{p \in \gamma_1 \cap \gamma_2} m_P \delta_P,$$

where the sum is over the transverse intersection points of γ_1 and γ_2 and m_P is the multiplicity at the intersection P.

Proof. It is easy to see that the number of the points in $\mathcal{I}_1(X)$ with the third component equal to p is exactly m_p, the multiplicity at p. □

Lemma 2.8. *The induced measure $P_*I(L_X, L_X)$ is $\frac{\pi}{2}A$.*

Proof. Consider the following diagram:

$$\begin{tikzcd} I(X) \arrow[r, "P_2"] \arrow[dr, "P"'] & \mathbb{P}(X) \arrow[d, "P'"] \\ & X \end{tikzcd}$$

where P' sends $(x, \pm\lambda_1)$ to x.

The push forward of the measure $I(L_X, L_X)$ to $\mathbb{P}(X)$ by P_2 is the volume measure on $\mathbb{P}(X)$ and it is locally $1/2dxdy \times d\theta$ which is half of the canonical volume measure on $T_1(X)$. It is proved in Proposition 2.6. Therefore, $P_*I(L_X, L_X) = \frac{\pi}{2}A$. □

2.3. **Dynamics.** In this part, we mention a few dynamical systems results that we apply in the proofs of the main theorems.

Convergence of measures. Let M be a metric space. The set of measures $\{\mu_a\}$ are called *tight* when for each ϵ there is a compact subset $K_\epsilon \subset M$ such that $\mu_a(K_\epsilon^c) < \epsilon$ for all indices a.

Recall that $C_c^*(M)$ is the dual of the set of the compactly-supported continuous functions on M. The following theorem is a well-known result in measure theory.

Theorem 2.9. *Assume that the measures μ_n on M converge to μ in $C_c^*(M)$ and they are tight. Then The probability measures $\mu_n/|\mu_n|$ converge to $\mu/|\mu|$ in $C_c^*(M)$.*

Measure-theoretic entropy. Let M be a compact metric space with distance function d and a continuous flow $f = \{f_t\}_{t\in\mathbb{R}}$. Assume that μ is a Borel probability measure on M, which is f-invariant. Let h_μ^M be the measure theoretic of μ. For simplicity, we write h_μ. When μ is ergodic with respect to f, Katok obtained the following equivalent definition for h_μ.

Let d be the metric on $T_1(X)$ induced by the hyperbolic metric of X. Define

$$d_T(v_1, v_2) = \max_{0\le t\le T} D(\phi_t(v_1), \phi_t(v_2)).$$

Therefore, d_T is a finer metric for $T > 0$.

Given $T > 0, \epsilon > 0, 1 > \delta > 0$, define $N_\mu(T, \epsilon, \delta)$ as the minimum number of balls of radius ϵ, with respect to metric d_T, which is required to cover a set of measure $\ge 1-\delta$. When μ is ergodic, h_μ is equal to the growth rate of $N_\mu(T, \epsilon, \delta)$.

Proposition 2.10. *(Katok) Assume that μ is an ergodic measure. For every $1 > \delta > 0$ we have*

$$h_\mu = \lim_{\epsilon\to 0} \liminf_{T\to\infty} \frac{\log N_\mu(T, \epsilon, \delta)}{T} = \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{\log N_\mu(T, \epsilon, \delta)}{T}.$$

Anosov closing lemma. Let M be a negative curvature manifold, and $T_1(M)$ the unit tangent bundle of M. The geodesic flow and metric on $T_1(M)$ are denoted by ϕ_t, D, respectively. Anosov closing lemma states:

Theorem 2.11. *For any $\epsilon > 0$ there exists $\delta > 0$ such that if*

$$D(v, \phi_t(v)) < \delta$$

then there is a geodesic $\gamma = \{\phi_t(w)\}_{s=0}^{t'}$ of length t' where $|t - t'| < \epsilon$ and

$$D(\phi_s(v), \phi_s(w)) < \epsilon \quad \textit{for } 0 \leq s \leq t.$$

Note that γ is the geodesic representative of the closed curve obtained from $\{\phi_s(v)\}_{s=0}^{t}$ by connecting v and $\phi_t(v)$.

Symbolic coding of the closed curves

Consider a proper pants decomposition P of X, including $3g-3$ closed geodesics (with length of the closed geodesics $\leq L_g$). These closed geodesics and the seams of the pairs of pants split X into $4g-4$ hexagons. We call it a *proper hexagonal decomposition* of X. Assign letters $e_1, \bar{e_1}, \ldots, e_{4g_4}, e_{4\bar{g}-4}$ to the edges of the hexagons. Each oriented closed geodesic γ with a marked point is corresponding to a word $\omega(\gamma)$ with letters in $\{e_1, \bar{e_1}, \ldots, e_{4g_4}, e_{4\bar{g}-4}\}$. The word presents the edges of the hexagons that γ intersects in order. Define $\ell_{com}(\gamma)$ as the combinatorial length of γ which is the number of the letters in $\omega(\gamma)$.

Lemma 2.12. *For a closed geodesic γ we have*

$$\ell_{com}(\gamma) \leq c(X)\ell_X(\gamma),$$

where $c(X) = 2I(X)(3g-3)L_g + 2/\operatorname{sys}(X)$. Here $I(X)$ and L_g are interaction strength and Bers' constant, respectively.

Proof. We can assume γ doesn't pass through the vertices of the hexagons since; if it does, we slightly move γ to avoid intersection with vertices. The intersection points of γ and the edges of hexagons split γ into $\ell_{com}(\gamma)$ geodesic arcs. Each geodesic arc either has an endpoint on one of the cuffs or its endpoints are on seams. On the other hand, the length of an arc between two seams is at least $\operatorname{sys}(X)/2$. Therefore $\ell_{com}(\gamma)$ is ≤ 2(number of the intersections with cuffs)+ $2\ell(\gamma)/\operatorname{sys}(X)$. Let P be the union of cuffs. The first term is

$$i(\gamma, P) \leq I(X)\ell_X(\gamma)\ell_X(P) \leq I(X)\ell_X(\gamma)(3g-3)L_g,$$

where $I(X)$ is interaction strength (see §3) and L_g is Bers' constant. $\square$

3. Interaction Strength

In this section, we prove Theorem 1.1 and study the first main question of this thesis:

Question 1. *How does $I(X)$ change by the geometry of X, where*

$$I(X) := \sup_{\alpha,\beta} \frac{i(\alpha,\beta)}{\ell_X(\alpha)\ell_X(\beta)},$$

and the supremum is over all pairs of closed geodesics?

Recall that $I(X)$ is the interaction strength of X. In section 2, we showed $I(X)$ is finite and bounded above by $4/(\operatorname{sys}(X))^2$ (Proposition 2.3). In the following, first we find a lower bound for $I(X)$ which shows $I(X) \to \infty$ as $X \to \infty$. Then, we find an upper bound on $I(X)$. Combining the lower and upper bound gives the exact asymptotic of $I(X)$ on $\mathcal{M}_g$ as $X \to \infty$. Note that through this section, $c_0, c_0'(c_g, c_g')$ refer to positive constants (depending only on g). They are not necessarily staying the same, or referring to a specific value throughout the section.

3.1. **Systole.**

In this subsection, we prove a lower bound of $I(X)$ asymptotic to

$$\frac{1}{2\operatorname{sys}(X)\log(1/\operatorname{sys}(X))}.$$

From the proof, we conclude I_{simple} and I are comparable.

Let γ_1 be a systole of X. It is known that γ_1 is simple.

Lemma 3.1. *There exists a simple closed geodesic γ_2 such that either $i(\gamma_1,\gamma_2) = 2$ and $\ell(\gamma_2) < 4\log(1/\operatorname{sys}(X)) + c_g$ or $i(\gamma_1,\gamma_2) = 1$ and $\ell(\gamma_2) < 2\log(1/\operatorname{sys}(X)) + c_g$.*

Proof. Geodesic γ_1 split $N_{\gamma_1}(r)$ into two halves $N^1_{\gamma_1}(r), N^2_{\gamma_1}(r)$. They are not homotopic to an annulus when $r \geq \sinh^{-1}(\frac{4\pi(g-1)}{\ell(\gamma_1)})$. Since by Lemma 2.1, the area of $N^i_{\gamma_1}(r)$ is $\geq area(X)$.

Let r^1_0 be the smallest r such that $N^1_{\gamma_1}(r)$ is homotopic to an annulus with two identified points, call it P, on one of the boundaries. Let $e_1 \subset N^1_{\gamma_1}(r^1_0)$ be a simple arc with endpoints on γ_1, passes through P, and has length $\leq 2r_0^1 \leq 2\sinh^{-1}(\frac{4\pi(g-1)}{\mathrm{sys}(X)})$. Similarly, we define r_0^2 and e_2 in $N^2_{\gamma_1}(r_0^2)$. See Figure 3.

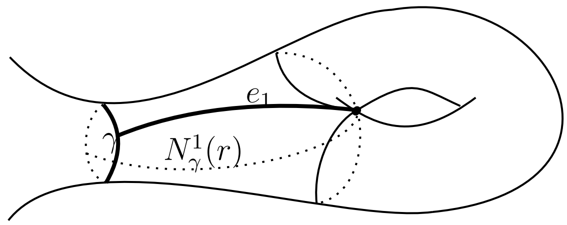

FIGURE 3

Connect the endpoints of e_1 and e_2 by disjoint subarcs of γ_1. Then we obtain a closed curve with length $< 4.\sinh^{-1}(\frac{4\pi(g-1)}{\mathrm{sys}(X)}) + \mathrm{sys}(X)$. Let γ_2 be its geodesic representative.

Note that $i(\gamma_1, \gamma_2) = 2$ and

$$\ell(\gamma_2) \leq 4\log\frac{12\pi(g-1)}{\mathrm{sys}(X)} + c_g \leq 4\log\frac{1}{\mathrm{sys}(X)} + c_g'.$$

We used the fact that $\sinh^{-1}(x) < \log 3x$ when $x \geq 1$.

Now we show that γ_2 is homotopically non-trivial. Consider the $4-$punctured sphere $S := N^1_{\gamma_1}(r_0^1) \cup N^2_{\gamma_1}(r_0^2)$. Cutting S along γ_2 splits S into two pairs of pants. Therefore, γ_2 is nontrivial (see Figure 4).

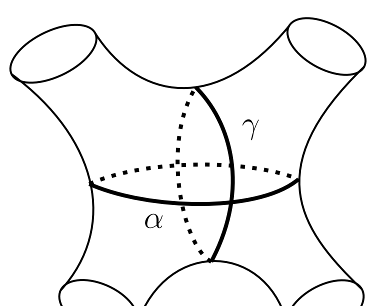

FIGURE 4

The constructed arcs e_1, e_2 are simple. Therefore, if they do not intersect then γ_2 is simple too, as required. But if e_1 and e_2 intersect at a point like $Q \in X$ then we consider the following simple closed geodesic instead. Choose subarcs $f_1 \subset e_1, f_2 \subset e_2$ from Q to γ_1 such that $\ell(f_i) < \ell(e_i)/2$, for $i = 1, 2$. We may assume f_1, f_2 are not intersecting. Connect f_1 and f_2 by a subarc of γ_1 to obtain a closed curve. The intersection number of γ_1 and the constructed curve γ_2 is 1, hence γ_2 is not trivial. Let $\widetilde{\gamma_2}$ be the geodesic representative of γ_2. We have $i(\gamma_1, \widetilde{\gamma_2}) = 1$ and $\ell(\widetilde{\gamma_2}) \leq 2\sinh^{-1}(4\pi(g-1)/\operatorname{sys}(X)) + \operatorname{sys}(X) \leq 2\log(1/\operatorname{sys}(X)) + c_g$, as required. □

Corollary 3.2. *For $g \geq 2$, $I(X) \to \infty$ as $X \to \infty$ in $\mathcal{M}_g$.*

Proof. Consider γ_1, γ_2 as explained in Lemma 3.1 □

Proposition 3.3. *For $g \geq 2$, as $X \to \infty$ in $\mathcal{M}_g$ we have*

$$2I(X)\operatorname{sys}(X)\log(\frac{1}{\operatorname{sys}(X)}) \geq 1.$$

Proof. Assume that $\operatorname{sys}(X)$ is small enough. Now consider γ_1 and γ_2 as defined in Lemma 3.1, we have:

$$2I(X)\operatorname{sys}(X)\log(1/\operatorname{sys}(X)) \geq \frac{2i(\gamma_1,\gamma_2)}{\ell(\gamma_1)\ell(\gamma_2)}\operatorname{sys}(X)\log(1/\operatorname{sys}(X)) \geq \frac{\log(1/\operatorname{sys}(X))}{\log(1/\operatorname{sys}(X)) + C_g},$$

the right hand side tends to 1 when $\operatorname{sys}(X) \to 0$, as required. □

Since the curves γ_1 and γ_2 we found in Lemma 3.1 are simple we have:

Corollary 3.4. *For $g \geq 2$, we have $2\operatorname{sys}(X)\log(1/\operatorname{sys}(X))I_{simple}(X) \to 1$ as $X \to \infty$ in $\mathcal{M}_g$ or equivalently as $\operatorname{sys}(X) \to 0$.*

Corollary 3.5. *There is a constant $c_g > 0$ such that $I_{simple}(X) \in [c_g I(X), I(X)]$ for $X \in \mathcal{M}_g$.*

We can see $I_\Delta(X)$ has the same order of magnitude as $I(X)$ too.

Lemma 3.6. *There exist $C_1, C_2 \in \mathcal{C}$ with length 1 ($\ell_X(C_1) = \ell_X(C_2) = 1$) such that $I(X) = i(C_1, C_2)$.*

Proof. The space of geodesic currents with length ≤ 1 is compact [Bon2, Prop. 4.7]. Therefore, the ratio $i(C_1, C_2)/\ell_X(C_1)\ell_X(C_2)$ attains its supremum by two geodesic currents. $\square$

Proposition 3.7. *For $X \in \mathcal{M}_g$ we have $I_\Delta(X) \in [I(X)/2, I(X)]$.*

Proof. Clearly $I_\Delta(X) \leq I(X)$. Now assume $I(X) = i(C_1, C_2)$ where C_1, C_2 are geodesic currents of length 1 (by Lemma 3.6). Then $C_1 + C_2$ is a geodesic current of length 2 and we have

$$I_\Delta(X) \geq \frac{i(C_1 + C_2, C_1 + C_2)}{4} \geq \frac{i(C_1, C_2)}{2} = \frac{I(X)}{2}.$$

We used the fact that the intersection number is a bilinear function on $\mathcal{C} \times \mathcal{C}$. $\square$

3.2. **Interaction strength in thin/thick part.** In this subsection, we prove the following result:

Proposition 3.8. *For $g \geq 2$ we have*

$$I(X) 2 \operatorname{sys}(X) \log(1/\operatorname{sys}(X)) \leq 1$$

as $X \to \infty$ in $\mathcal{M}_g$.

Consider a proper pants decomposition P containing pair of pants $P_1, \ldots, P_{2g-2}$ and its thin and thick decomposition. We want to estimate the intersection number and length of geodesic arcs in thin and thick parts in order to prove Proposition 3.8

Let $i_Y(-,-)$ and $\ell_Y(-)$ be the restriction of the intersection number and the length functions to the subset $Y \subset X$, respectively. In other words, $\ell_Y(a) = \ell(a \cap Y)$ and $i_Y(a, b)$ is the number of the intersection points in Y. Similarly, let $i_{thin}(.,.), i_{thick}(.,.), \ell_{thin}(.), \ell_{thick}(.), \ell_{collar}(.)$

be the intersection number and length in the collars, thin parts, and thick parts, correspondingly.

For $\eta, \zeta \in \mathcal{G}$ we have

$$\frac{i(\eta,\zeta)}{\ell(\eta)\ell(\zeta)} = \frac{i_{thick}(\eta,\zeta)}{\ell(\eta)\ell(\zeta)} + \frac{i_{thin}(\eta,\zeta)}{\ell(\eta)\ell(\zeta)} \leq \frac{i_{thick}(\eta,\zeta)}{\ell_{thick}(\eta)\ell_{thick}(\zeta)} + \frac{i_{thin}(\eta,\zeta)}{\ell_{collar}(\eta)\ell_{collar}(\zeta)}. \tag{3}$$

The following inequality, for $a_i, b_i, I_i \geq 0$, tells us that it is enough to find the upper bound for the interaction strength in the thin and thick parts separately:

$$\frac{I_1 + I_2}{(a_1+a_2)(b_1+b_2)} \leq \max(\frac{I_1}{a_1 b_1}, \frac{I_2}{a_2 b_2}). \tag{4}$$

In other words, we have:

$$\frac{i(\eta,\zeta)}{\ell(\eta)\ell(\zeta)} \leq \max_{1\leq j\leq 2g-2} \frac{i_{O(thick,j)}(\eta,\zeta)}{\ell_{O(thick,j)}(\eta)\ell_{O(thick,j)}(\zeta)} + \max_{1\leq k\leq 3g-3} \frac{i_{O(thin,k)}(\eta,\zeta)}{\ell_{O(collar,k)}(\eta)\ell_{O(collar,k)}(\zeta)}.$$

Now we prove an upper bound for thick and thin parts.

Thick part. The interaction strength is $< c_g$ in thick parts.

Lemma 3.9. *Consider thick part $O(thick, j)$ and the seams of the pair of pants containing it. (a) The length of an arc in $O(thick, j)$ with endpoints on the seams is $> c_g$, (b) the distances between the cuffs of $O(thick, j)$ are $> c_g$.*

Proof. For part (a), assume that AB is a geodesic arc in $O(thick, j)$ with endpoints A and B on the seams. Let A', B' be the projection of A and B to γ_k. Note that $\ell(AA'), \ell(BB') \geq r_k$. From the hyperbolic geometry formula for quadrilateral $ABB'A'$ with two right angles (see [Bus, Equ. 2.3.2.]) we know:

$$\cosh(\ell(AB)) = \cosh(\ell(AA'))\cosh(\ell(BB'))\cosh(\ell(A'B')) - \sinh(\ell(AA'))\sinh(\ell(BB')).$$

Therefore, we have

$$\cosh(\ell(AB)) \geq \cosh(\ell(AA'))\cosh(\ell(BB'))(\cosh(\ell(A'B'))-1) \geq 2\cosh(r_k)^2\sinh^2(\ell(A'B')/2).$$

We used the fact that $\cosh x = 2\sinh^2 x/2 + 1$. After simplifying the right-hand side, we obtain:

$$\cosh(\ell(AB)) \geq 2(1+\frac{1}{\sinh^2(\ell(\gamma_k/2)(1+\cosh^2(\ell(\gamma_k)/2))})\sinh^2(\ell(\gamma_k)/4) \geq \frac{\sinh^2(\ell(\gamma_k)/4)}{c_g\sinh^2(\ell(\gamma_k)/2)} > c_g',$$

as required.

For part (b), it is enough to show that the distance between the boundary of $O(thin, k)$ and $O(collar, k)$ is $> c_g$ (the boundaries which are on the same side of γ_k).

$$distance = \sigma(\gamma_k) - r_k = \sinh^{-1}(\frac{1}{\sinh(l)}) - \sinh^{-1}(\frac{1}{\sinh(l)\sqrt{1+\cosh(l)^2}}),$$

where $l = \ell(\gamma_k)/2$.

Define

$$f(x) := \sinh(x)\ ,\ y_1 := \sinh^{-1}(\frac{1}{\sinh(l)})\ ,\ y_2 := \sinh^{-1}(\frac{1}{2\sinh(l)}).$$

Then there is $y \in [y_2, y_1]$ such that:

$$y_1 - y_2 = \frac{f(y_1) - f(y_2)}{f'(y)},$$

Therefore,

$$distance \geq y_1 - y_2 \geq \frac{1}{2\sinh(l)\sqrt{1+1/\sinh(l)^2}} \geq \frac{1}{2\sqrt{1+\sinh(L_g)^2}} > c_g,$$

as required. □

Proposition 3.10. *For all $\eta, \zeta \in G$ we have:*

$$\frac{i_{O(thick,j)}(\eta,\zeta)}{\ell_{O(thick,j)}(\eta)\ell_{O(thick,j)}(\zeta)} \leq c_g.$$

Proof. Let e_1, e_2, e_3 and d_1, d_2, d_3 be the cuffs and seams of thick part $O(thick, j)$, respectively. We can split $\eta \cap O(thick, i)$ and $\zeta \cap O(thick, j)$ into subarcs of the following types:

- An arc between different cuffs that does not intersect the seams.
- An arc with endpoints on the cuffs or seams such that its interior intersects exactly one seam.

Note that the subarcs may overlap but a generic point is on at most two subarcs.

Assume $\eta \cap O(thick, j), \zeta \cap O(thick, j)$ are split into $\eta_1, \dots, \eta_n$ and $\zeta_1, \dots, \zeta_m$. We can see $i(\eta_s, \zeta_t) \leq 2$ and $\ell(\eta_s), \ell(\zeta_t) > c_g$ by Lemma 3.9. Therefore, we have:

$$\frac{i_{O(thick,ij}(\eta,\zeta)}{\ell_{O(thick,j)}(\eta)\ell_{O(thick,j)}(\zeta)} \leq \frac{4\sum\limits_{s,t} i(\eta_s,\zeta_t)}{(\sum\limits_{s}\ell(\eta_s))(\sum\limits_{t}\ell(\zeta_t))} \leq \frac{8}{c_g^2},$$

as required. $\square$

Similarly, when $\ell(\gamma_k) \geq c_0$, we can see that the term corresponding to $O(collar, k)$ in the interaction strength is bounded above by a constant. So from now on we assume $\ell(\gamma_k) \leq c_0$.

Thin part. In this part, we restrict the intersection number into thin part $O(thin, k)$. But the lengths are restricted to the collar $O(collar, k)$ instead of $O(thin, k)$ to avoid very short arcs with zero winding number around γ_k. In other words, we consider $i_{O(thin,k)}(-,-)$ and $\ell_{O(collar,k)}(-)$.

Proposition 3.11. *Assume that $\ell(\gamma_k) \leq c_0$. We have:*

$$\frac{i_{O(thin,k)}(\eta,\zeta)}{\ell_{O(collar,k)}(\eta)\ell_{O(collar,k)}(\zeta)} \leq \frac{1}{2s\sigma(s/2)} + c_g.$$

We have two types of arcs in $O(collar, k)$ with the endpoints on its boundaries:

- Type 1 subarcs have endpoints on the different boundaries of the collar.
- Type 2 subarcs have endpoints on the same boundary of the collar.

Given geodesic arc $\alpha \subset O(collar, k)$, we define *the winding number of* α, $\omega(\alpha)$, as follows. Let $\widetilde{\alpha}$ and $\widetilde{\gamma_k}$ be preimages of α and γ_k in $\mathbb{H}$, respectively. Assume that A, B are the endpoints of $\widetilde{\alpha}$ and A', B' the projection of A, B to $\widetilde{\gamma_k}$. Then $\omega(\alpha) := \lfloor \ell(A'B')/\ell(\gamma_k) \rfloor$. When α is of type 2, by Lemma 2.2, we can see that $w(\alpha) \geq 1$ and can not be zero.

In the following, we estimate the length and intersection number of arcs based on their types and winding numbers.

Lemma 3.12. *Assume that* $\alpha \subset O(collar, k)$ *is a geodesic arc of type* 1*. Then we have:*

$$\ell(\alpha) \geq 2\sigma(\gamma_k), \ \ell(\alpha) \geq 2\sigma(\gamma_k) + (\omega(\alpha)+1)\ell(\gamma_k) - 4.$$

Proof. The distance between the boundaries of $O(collar, k)$ is $2\sigma(\gamma_k)$, therefore, $\ell(\alpha) \geq 2\sigma(\gamma_k)$. Let $\widetilde{\gamma_k}$ and $\widetilde{\alpha}$ be preimage of γ_k and α, respectively, in $\mathbb{H}$. The endpoints of the geodesic arc $\widetilde{\alpha}$ are A and B, where $d(A, \widetilde{\gamma_k}) = d(B, \widetilde{\gamma_k}) = \sigma(\gamma_k)$. Let A' and B' be the projection of A and B on $\widetilde{\gamma_k}$, then the geodesic arc $A'B'$, has length $\geq w(\beta)\ell(\gamma_k)$ (see Figure 5).

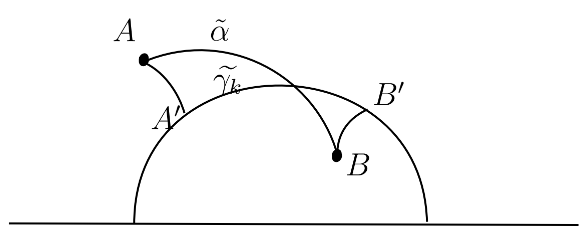

FIGURE 5

From the hyperbolic geometry formula for quadrilateral $ABB'A'$ with two right angles (see [Bus, Equ. 2.3.2.]) we have:

$$\cosh \ell(\widetilde{\alpha}) = \cosh(\sigma(\gamma_k))^2 \cosh(\ell(A'B')) + \sinh(\sigma(\gamma_k))^2.$$

As a result, we have:

$$e^{\ell(\widetilde{\alpha})} \geq \cosh \ell(\widetilde{\alpha}) \geq \frac{e^{2\sigma(\gamma_k)}}{4} \frac{e^{\omega(\alpha)\ell(\gamma_k)}}{2},$$

therefore:

$$\ell(\alpha) \geq 2\sigma(\gamma_k) + (\omega(\alpha)+1)\ell(\gamma_k) - \log 8 - 1.$$

$\square$

Lemma 3.13. *Assume that $\alpha \in O(collar, k)$ is a geodesic arc of type 2 and $\alpha \cap O(thin, k) \neq \emptyset$. Then we have:*

$$\ell(\alpha) \geq (w(\alpha)+1)\ell(\gamma_k), \;\; \ell(\alpha) \geq 2\sqrt{(w(\alpha)+1)\ell(\gamma_k)\sigma(\gamma_k)}$$

when $\ell(\gamma_k) \leq c_0$.

Proof. Similar to the proof of Lemma 3.12 define $\widetilde{\gamma_k}$, $\widetilde{\alpha}, A, A', B, B'$. Recall that $\ell(AA') = \ell(BB') = \sigma(\gamma_k)$ and $\ell(A'B') \geq w(\beta)\ell(\gamma_k)$ (see Figure 6).

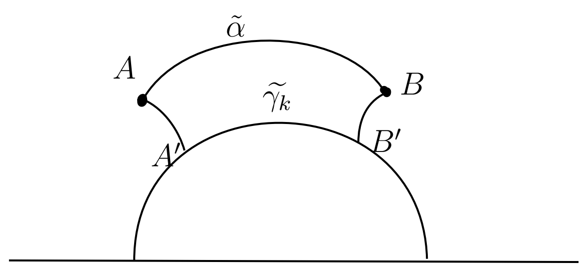

Figure 6

From the hyperbolic geometry formula for quadrilateral $ABB'A'$ with two right angles (see [Bus, Equ. 2.3.2.]) we have:

$$\cosh(\ell(\widetilde{\alpha})) = \cosh(\sigma(\gamma_k))^2 \cosh((A'B')) - \sinh(\sigma(\gamma_k))^2$$

or equivalently:

$$\sinh(\ell(\widetilde{\alpha})/2) = \cosh(\sigma(\gamma_k))\sinh(\ell(A'B')/2). \tag{5}$$

We used the fact that $\cosh^2 x - \sinh^2 x = 1$ and $\cosh x = 2\sinh^2 x/2 + 1$.

Note that $\ell(\alpha) > c_0$, therefore:

$$\ell(\alpha) \geq 2\sinh^{-1}(\cosh(\sigma(\gamma_k))\sinh(\ell(\gamma_k)/2)) = 2\sinh^{-1}(\cosh(\ell(\gamma_k)/2)) \geq \sinh^{-1}(1).$$

From equation 5 and the fact that we can assume $\ell(\gamma_k)$ is small enough, we conclude $\ell(\alpha) \geq \ell(\gamma_k)(\omega(\alpha)+1)$.

Now we aim to prove the second inequality. We can assume $w(\alpha) \geq N_0$ unless:

$$2\sqrt{(w(\alpha)+2)\ell(\gamma_k)\sigma(\gamma_k)} \to 0,$$

but $\ell(\alpha)$ is greater than a constant.

Consider the following cases:

- When $1/(w(\alpha)+1) \geq \ell(\gamma_k)/2$: From the inequalities $\cosh(\sigma(\gamma_k)) \geq \cosh(\log(2/\ell(\gamma_k))) \geq 1/\ell(\gamma_k)$, $\sinh(x) \geq x$ and Equation 5, we have:

  $$\ell(\alpha) \geq 2\sinh^{-1}(w(\alpha)/2) \geq 2\log(w(\alpha)/2).$$

  On the other hand, we know $t\sigma(t)$ is increasing when t is small enough. Therefore, we have

  $$\log(w(\alpha)/2) \geq \sqrt{4\log(w(\alpha)+1)} \geq \sqrt{2\sigma(1/(w(\alpha)+1))} \geq \sqrt{(w(\alpha)+1)\ell(\gamma_k)\sigma(\gamma_k)}$$

  So, $\ell(\alpha) \geq 2\sqrt{(w(\alpha)+1)\ell(\gamma_k)\sigma(\gamma_k)}$

- When $1/(w(\alpha)+1) < \ell(\gamma_k)/2$: In this case, we have:

$$e^{\ell(\alpha)/2} \geq \sinh \frac{\ell(\alpha)}{2} \geq \frac{e^{\sigma(\gamma_k)}}{2} \frac{e^{(\omega(\alpha)+1)\ell(\gamma_k)/2}}{c_0}.$$

So

$$\ell(\alpha) \geq 2\sigma(\gamma_k) + (\omega(\alpha)+1)\ell(\gamma_k) - c_0' \geq \sigma(\gamma_k) + (\omega(\alpha)+2)\ell(\gamma_k) \geq 2\sqrt{(\omega(\alpha)+1)\ell(\gamma_k)\sigma(\gamma_k)}.$$

as required.

□

Assume that $A_n, B_n \subset O(thin, k)$ represent geodesic arcs with winding number n of type 1 and 2, respectively. Then the intersection number between subarcs based on their type and winding number is as follows.

Lemma 3.14. *The following table (table 7) shows an upper bound on the intersection number between geodesic arcs based on the type and winding number of the arcs.*

type of subarc	A_m	B_m
A_n	$\leq n+m+2$	$\leq m+1$
B_n	$\leq n+1$	$\leq 2\min(n,m)$ $+2$

FIGURE 7. Upper bounds on the intersection number

Proof. Recall that geodesic arcs have the minimum intersection number among the curves in their homotopy class with fixed endpoints. Therefore, it is enough to prove the bounds for some representatives in their homotopy classes. The universal cover of the collar is an infinite rectangle $R = [-\infty, \infty] \times [0,1]$ with deck transformation $(x, y) \to (x+1, y)$. Hence, each rectangle between $x = k$ and $x = k + 1$ is a fundamental domain. In the following, we describe the representatives. Without loss of generality assume $n \leq m$.

- We can apply a Dehn twist around γ_k on the collar without changing the intersection number between arcs. When we have two arcs of type A_n, A_m, we can apply some Dehn twists to obtain arcs of type A_0 and A_{m+n}(or A_{m-n}). Assume that the arcs are P_1P_2 and Q_1Q_2 in R, where the coordinates of the points are $P_1 = (0,0), P_2 = (1, c_0), Q_1 = (0,0), Q_2 = (1, m + n + c_0')$, where $c_0, c_0' \in [0,1]$. It is easy to see that they intersect at $\leq n + m + 1$ points.
- Similarly, when we have arcs of type A_n, B_m, we may instead assume arcs of type A_0, B_m by applying some iteration of Dehn twist around γ_k. Then P_1P_2 is the arc of type A_0, where $P_1 = (0,0)$ and $P_2 = (1, c_o)$ for $c_0 \in [0,1)$. Now it is easy to see the intersection number between the arcs is $\leq m + 1$.
- Assume that we have arcs of type B_n, B_m. We may assume the arc of type B_n is in the lower half $((-\infty, \infty) \times [0, 1/2])$ of R, and the other arc is constructed by three subarcs as follows. The middle subarc is in the upper half of R and two ends are vertical lines from $x = 0$ to $x = 1/2$. Then similar to the previous case, we can see that each of the vertical lines intersects the arc of type B_n in $\leq n + 1$ points. Then the total intersection number is $\leq 2n + 2$ points.

$\square$

Proof of Proposition 3.11 Without lose of generality we assume $\eta_0 = \eta \cap O(collar, k)$, $\zeta_0 = \zeta \cap O(collar, k)$ *and* $\omega(\eta_0) = n$, $\omega(\zeta_0) = m$. *We can also assume* $n \leq m$. *Define:*

$$I_0 = \frac{i_{O(thin,k)}(\eta_0, \zeta_0)}{\ell_{O(collar,k)}(\eta)\ell_{O(collar,k)}(\zeta)}$$

Now we apply the above lemmas to bound intersection number and length. Consider the following cases:

- *when* η_0, ζ_0 *are of type one, we have*

$$I_0 \le \frac{m+n+2}{(2\sigma(\gamma_k)+(n+1)\ell(\gamma_k)-4)(2\sigma(\gamma_k)+(m+1)\ell(\gamma_k)-4)},$$

we aim to prove the denominator is $\ge 2\ell(\gamma_k)\sigma(\gamma_k)(m+n+2)$ *which is implied from the followings:*

(1) *If* $n+1 \ge 4(m+1)/\sigma(\gamma_k)^2$ *then*

$$4\sigma(\gamma_k)^2+(n+1)(m+1)\ell(\gamma_k)^2 \ge 16\sigma(\gamma_k)+4\ell(\gamma_k)+2\sqrt{(n+1)(m+1)\sigma(\gamma_k)\ell(\gamma_k)}$$

$$\ge 16\sigma(\gamma_k)+4(n+1)\ell(\gamma_k)+4(m+1)\ell(\gamma_k)$$

(2) *If* $4(m+1)/\sigma(\gamma_k)^2 \ge n+1 \ge \sigma(\gamma_k)/\ell(\gamma_k)$, *then*

$$\ell(\eta_0) \ge 3\sigma(\gamma_k)-4 \ge \frac{5\sigma(\gamma_k)}{2},$$

$$\ell(\zeta_0) \ge \frac{4(m+1)\ell(\gamma_k)}{5}+\frac{(m+1)\ell(\gamma_k)}{5} \ge \frac{4(m+1)\ell(\gamma_k)}{5}+\frac{(n+1)\sigma(\gamma_k)^2\ell(\gamma_k)}{20} \ge \frac{4(m+n+2)\ell(\gamma_k)}{5}$$

(3) *If* $\sigma(\gamma_k)/\ell(\gamma_k) \ge n+1$ *then*

$$\ell(\eta_0) \ge \sigma(\gamma_k)+(n+1)\ell(\gamma_k) \ge (m+n+2)\ell(\gamma_k), \quad \ell(\zeta_0) \ge 2\sigma(\gamma_k)$$

- *when* η_0 *and* ζ_0 *are of type one and two, respectively, we have*

$$I_0 \le \frac{m+1}{\ell(\eta_0)\ell(\zeta_0)},$$

and

$$\ell(\eta_0) \ge (m+1)\ell(\gamma_k), \quad \ell(\zeta_0) \ge 2\sigma(\gamma_k)$$

The case that η_0 *is type two and* ζ_0 *type one is similar.*

- *when* η_0 *and* ζ_0 *are of type two, we have*

$$I_0 \le \frac{2\min(n,m)+2}{\ell(\eta_0)\ell(\zeta_0)},$$

and we know

$$\ell(\eta_0), \ell(\zeta_0) \geq 2\sqrt{(\min(n,m)+1)\ell(\gamma_k)\sigma(\gamma_k)}$$

- *When one of the arcs is γ_k and the other arc is of type 1 then we have*

$$I_0 \leq \frac{1}{2\sigma(\gamma_k)\ell(\gamma_k)}$$

□

Proof of Proposition 3.8. It is implied from Propositions 3.10 and 3.11.

Proof of Theorem 1.1. The proof is implied from Propositions 3.3 and 3.8. □

From the continuity of $I(X)$ (Lemma 2.4) and Theorem 1.1 we obtain an estimate of $I(X)$ up to a multiplicative constant.

Corollary 3.15. *For $X \in \mathcal{M}_g$ we have*

$$I(X) \asymp_g \frac{1}{s\log(1/s)},$$

where $s = \min(\mathrm{sys}(X), 1/2)$.

3.3. **Minimum on moduli space.** In this subsection, we prove Proposition 1.4.

First, we prove the existence of a pair of pants with short cuffs.

Lemma 3.16. *Given $X \in \mathcal{M}_g$, it has a pair of pants with cuffs of length $\leq c_0 \log g$.*

Proof. Let γ_1 be a systole of X. Enlarge the collar of γ_1 from one side until it runs into itself for the first time at a point Q. It is homeomorphic to a pair of pants. Consider the geodesic representative of its cuffs to obtain a pair of pants P with geodesic boundaries $\gamma_1, \gamma_2, \gamma_3$. We know $\mathrm{sys}(X) \leq 2\log(4g-2) \leq 6\log(g)$ [Bus, Lem. 5.2.1.]. Now we show $\ell(\gamma_2), \ell(\gamma_3) \leq c_0 \log g$ too. Let BC be a shortest arc between γ_1 and the seam between γ_2

and γ_3 where $B \in \gamma_1$. Assume that AE is the seam between γ_1, γ_2 where $A \in \gamma_1$ and $D \subset \gamma_2$ the base of seam between γ_2, γ_3. Consider the right-angle pentagon $ABCDE$. See Figure 8.

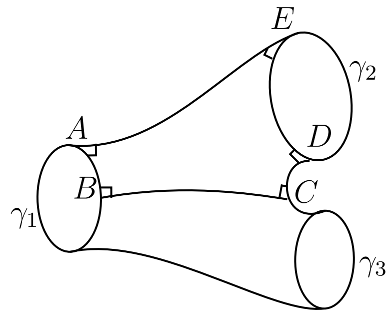

FIGURE 8

From the trigonometry formula, [Bus, Thm. 2.3.4.], we have:

$$\cosh(\ell(\gamma_2)/2) = \cosh(ED) = \sinh \ell(AB) \sinh \ell(BC).$$

Note that $\ell(AB) \leq \operatorname{sys}(X)$ and $\ell(BC) \leq d(Q, \gamma_1) \leq \sinh^{-1}(4\pi(g-1)/\operatorname{sys}(X))$ (see proof of Lemma 3.1).

Therefore, we have

$$\cosh(\ell(\gamma_2)/2) \leq \frac{\sinh(\operatorname{sys}(X))4\pi(g-1)}{\operatorname{sys}(X)}.$$

Moreover, we can see $\sinh(x)/x \leq c_0 g^2$ when $x \in (0, 2\log(4g-2)]$. As a result, $\ell(\gamma_2) \leq 2\log(c_0 g^3) \leq c_0' \log g$. Similarly, we can show $\ell(\gamma_3) \leq c_0 \log g$.

$\square$

Lemma 3.17. *For $X \in \mathcal{M}_g$, there is a figure eight closed geodesic with length $\leq c_0 \log g$.*

Proof. By Lemma 3.16, there is a pair of pants P with cuffs $\gamma_1, \gamma_2, \gamma_3$ of lengths $\leq c_0 \log g$. Let γ be a figure eight geodesic in P. Then using Trigonometry in hyperbolic geometry [Bus, Eqn. 4.2.3.] we have

$$\cosh(\frac{\ell(\gamma)}{2}) = \cosh(\frac{\ell(\gamma_3)}{2}) + 2\cosh(\frac{\ell(\gamma_1)}{2})\cosh(\frac{\ell(\gamma_2)}{2}) \leq c_0 g^{2c_0'}$$

As a result, $\ell(\gamma) \leq c \log(g)$ for a constant $c > 0$. $\square$

Proof of Proposition 1.4. It is known that there is a hyperbolic surface $X_0 \in \mathcal{M}_g$ such that $\mathrm{sys}(X_0) \geq c \log g$ for a constant $c > 0$ independent of g [KSV][BS]. Therefore by Proposition 2.3 we have $I(X_0) \leq b/(\log g)^2$ where $b = 4/c^2$. On the other hand, by Lemma 3.17, we obtain $I(X) \geq 1/(c \log g)^2$ for all $X \in \mathcal{M}_g$. $\square$

3.4. **Finite volume surfaces with cusps.** In this subsection, we prove a similar result as Theorem 1.1 when X varies in $\mathcal{M}_{g,n}$.

When X is a finite volume hyperbolic surface with cusps, we have $I(X) = \infty$. For example, a closed geodesic γ_n that turns around a cusp n times has length $\ell(\gamma_n) \sim 2 \log n$ and self-intersection number $i(\gamma_n, \gamma_n) \sim 2n$ for $n \in \mathbb{N}$. However, when the closed geodesics are restricted to a compact subset of X, we obtain a result similar to Theorem 1.1 (see §**??**). Given a subsurface $Y \subset X$, define

$$I(Y) := \sup_{\alpha, \beta \subset Y} \frac{i(\alpha, \beta)}{\ell(\alpha)\ell(\beta)},$$

where the supremum is over all pairs of closed geodesics in Y. Let X_r be the compact subset of X obtained from removing, for each cusp, the standard horoball bounded by a horocycle of length $r < 1$.

Theorem 3.18. *Fix $g > 1, n > 0$, and let X be a finite volume hyperbolic surface of genus g with n cusps. Let $s = \min(\mathrm{sys}(X_r), 1/2)$, we have*

$$I(X_r) \sim \max\left(\frac{1}{2s \log(1/s)}, \frac{1}{2r \log^2(1/r)}\right),$$

as $\min(s, r) \to 0$.

Consider a proper pants decomposition of X where it may contain a cusp as a cuff instead of closed geodesics. Similarly, we can define thin and thick decomposition. For a cusp c, the neighborhood with the horocycle boundary of length 1 is the thin part of c; since the limit of the length of the thin part's boundary is 1 when instead of a cusp we have a closed geodesic γ, as $\ell(\gamma) \to 0$. Similarly, we can see that the collar of c is a neighborhood with the horocycle boundary of length 2. We can also see that the geodesic arcs with endpoints on the horocycle of length 2 with no self-intersection are disjoint from the thin part of the cusp (similar to Lemma 2.2). In addition, the seams of the thick parts are $> c_g$ since the distance between the horocycle of length 1 and 2 is $\log 2$. In the end, we can see that the distances between the seams of a thick part are $\geq c'_g$.

Let γ_n be a geodesic arc with endpoints on the horocycle of length 2 and with winding number n around the cusp. We have:

- $\ell(\gamma_n) \in [2\log n, 2\log n + \log 24]$ using distance formula [Bus, Equ. 1.1.2].
- the intersection number $i(\gamma_n, \gamma_m)$ between γ_n and γ_m is in $[s, s+2]$ where $s = \min(n, m)$ (similar to Lemma 3.14),
- we can see $n \leq 2/r$ when γ_n doesn't intersect the horocycle of length r.

Proof of Theorem 3.18. We consider a proper thin and thick decomposition. The interaction strength inside the thick parts is $< c_g$ (similar to Proposition 3.10). If a thin part is the neighborhood of a closed geodesic then, similar to Proposition 3.11, we can see interaction strength is asymptotic to $1/2 \operatorname{sys}(X) \log(1/\operatorname{sys}(X))$ as $\operatorname{sys}(X) \to 0$. If a thin part is the neighborhood of a cusp then we can see for $n \leq m \leq 2/r$:

$$\frac{i_{O(thin,cusp)}(\gamma_n, \gamma_m)}{\ell_{O(collar,cusp)}(\gamma_n)\ell_{O(collar,cusp)}(\gamma_m)} \leq \frac{n+2}{4(\log n)^2}$$

which is asymptotic to $1/2r(\log r)^2$ and by inequality 4 it gives the required upper bound. On the other hand, the ratio for $n = m = \lfloor 2/r \rfloor$ is asymptotic to $1/2r(\log r)^2$ which gives the lower bound for the interaction strength in that thin part. $\square$

The subset of hyperbolic surfaces $X \in \mathcal{M}_{g,n}$ with $\operatorname{sys}(X) \geq \epsilon$ is compact. Therefore, we have

Corollary 3.19. *Fix $g \geq 2, n > 0$ and let X be a finite volume hyperbolic surface of genus g with n cusps. Then we have*

$$I(X_r) \asymp_{g,n} \max\left(\frac{1}{s\log(1/s)}, \frac{1}{r(\log(1/r))^2}\right)$$

where $s = \min(\mathrm{sys}(X), 1/2)$.

3.5. **Expected value on $\mathcal{M}_g$.** In this subsection, we prove Corollary 1.5.

There is a volume form $\mathrm{vol}_{\mathrm{WP}}$ on $\mathcal{M}_g$ associated with the Weil-Petersson symplectic form on T_g. It induces a probability measure $\mathbb{P}_g$ defined as the following

$$\mathbb{P}_g(A) := \frac{\mathrm{vol}_{\mathrm{WP}}(A)}{\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)}.$$

Define $\mathcal{M}_g^\epsilon := \{X \in \mathcal{M}_g |\, \mathrm{sys}(X) \leq \epsilon\}$. Mirzakhani showed there is C_0 such that

$$\frac{1}{C_0}\epsilon^2 \leq \mathbb{P}_g(\mathcal{M}_g^\epsilon) \leq C_0\epsilon^2 \tag{6}$$

for $g \in \mathbb{N}$ [MP, §5][Mir3].

Proof of Corollary 1.5. Complement of M_g^ϵ is compact. Therefore, it is enough to show the integral on $\mathcal{M}_g$ is finite. Define $\mathcal{M}_g^{[a,b]} := \{X \in \mathcal{M}_g |\, \mathrm{sys}(X) \in [a,b]\}$. From 6 we have

$$\int_{\mathcal{M}_g^{[2^{-(n+1)},2^{-n}]}} \frac{1}{\mathrm{sys}(X)}\, d\,\mathrm{vol}_{\mathrm{WP}} \leq \frac{C}{2^{n-1}}.$$

Therefore, from Theorem 1.1, if $\epsilon \leq \frac{1}{2^n}$ then

$$\int_{\mathcal{M}_g^\epsilon} I(X)\, d\,\mathrm{vol}_{\mathrm{WP}} \leq \int_{\mathcal{M}_g^\epsilon} \frac{1}{\mathrm{sys}(X)}\, d\,\mathrm{vol}_{\mathrm{WP}} \leq \sum_{m \geq n} \frac{C}{2^{m-1}} = \frac{C}{2^m},$$

which is finite as required. $\square$

Note that from (6) we have

$$\int_{\mathcal{M}_g^\epsilon} \frac{1}{\operatorname{sys}(X)^2}\, d\operatorname{vol}_{\mathrm{WP}} \geq \frac{1}{C_0}$$

for $\epsilon > 0$ small enough. Therefore $\int_{\mathcal{M}_g} \frac{1}{\operatorname{sys}(X)^2} d\operatorname{vol}_{\mathrm{WP}} = \infty$ and the upper bound $c_0/\operatorname{sys}(X)^2$ on $I(X)$ is not sufficient to conclude Corollary 3.15.

4. Intersection Points

In this section, we prove Theorem 1.2 and answer the second main question of this thesis:

Question 2. *How are the intersection points between all pairs of closed geodesics distributed on X?*

Let X be a finite volume hyperbolic surface and ϕ_t the geodesic flow. Let $\phi_{[0,t]}(v)$ be the geodesic arc in $T_1(X)$ with length t start at v.

We can see that the intersection points are dense on X.

Proposition 4.1. *The set of intersection points between closed geodesics is a dense subset of X.*

Proof. The proof is constructive. Consider a small open ball B in X. We show there is an intersection point in B. We know that the geodesic flow is recurrent. Therefore, almost every unit tangent vector in B returns very close to itself in B after a long time going along the geodesic flow. Let v_1, v_2 be two such a vector in B, which are on two transverse geodesic arcs in B. We know there are $t_1, t_2 > 0$ such that $d(v_1, \phi_{t_1}(v_1))$ and $d(v_2, \phi_{t_2}(v_2))$ are very small. If we connect v_1 and $\phi_{t_1}(v_1)$ by a small arc, then we obtain a closed curve and by Anosov closing lemma (Theorem 2.11) its geodesic representative γ_1 stays very close to $\phi_{[0,t_1]}(v_1)$ at all time $t \in [0, t_1]$. Therefore, γ_1 is very close to the geodesic passed through v_1 inside B, Similarly, we construct geodesic γ_2 from v_2 and γ_2 is very close to $\phi_{[0,t_2]}(v_2)$. Therefore, γ_1 and γ_2 intersect at a point in B. $\square$

Recall that $\mathcal{G}$ is the set of closed geodesics on X and $\mathcal{G}_T$ is the set of closed geodesics with length $\leq T$.

Given $\alpha, \beta \in \mathcal{G}$, define intersection measure $I(\alpha, \beta)$ as follows:

$$I(\alpha, \beta) := \sum_{p \in \alpha \cap \beta} c_p \delta_p$$

where the sum is over the points in X that α and β intersect transversally, δ_p is delta measure at point p and weight c_p is the multiplicity at p. In other words, $c_p = 1$ unless the geodesics pass through the point p multiple times. See §2.1 for more details. The intersection number of α and β, $i(\alpha, \beta)$, is defined as the total volume of X with respect to $I(\alpha, \beta)$. Similarly, we can define intersection measure and intersection number of multi-curves $\sum_i a_i\gamma_i$ where $a_i > 0$ and $\gamma_i \in \mathcal{G}$.

Denote $|\mu|$ as the total volume of measure μ. Let A be the area measure on X obtained from the hyperbolic metric. Then $A(X) = arae(X) = 2\pi(2g - 2 + n)$ where g is genus of X and n the number of the cusps. Define multi-curve γ_T as:

$$\gamma_T := \sum_{\gamma \in \gamma_T} \gamma.$$

In this section, we prove the following theorem.

Theorem 4.2. *The sequence of probability measures $I(\gamma_T, \gamma_T)/|I(\gamma_T, \gamma_T)|$ converges to $A/area(X)$ in $C_c^*(X)$, as $T \to \infty$.*

proof of Theorem 1.2. It is a direct consequence of theorem 4.2.

4.1. **Intersection measure.** As we explained in §2, every two geodesic currents C_1, C_2 define an intersection measure $I(C_1, C_2)$ on $\mathcal{I}_1(X)$. The intersection number $i(C_1, C_2)$ is the total volume of $\mathcal{I}_1(X)$ with respect to $I(C_1, C_2)$. In [Bon2, Prop. 4.5], Bonahon proves that the intersection number is continuous when X is compact (or when we restrict the geodesics into a compact subset of X). When X has finite volume, the intersection number is not continuous, but in this subsection, we show that a weaker similar result still holds.

Lemma 4.3. *The intersection number $i(.,.)$ is not a continuous function when X has cusps.*

Proof. Consider a neighborhood N of a cusp. Let γ_n be a closed geodesic that goes around the cusp for n times inside N and the rest of γ_n is outside N. Then the geodesic current

γ_n converges to a geodesic ray r with both ends going into the cusp as $n \to \infty$. We know that $i(\gamma_n, \gamma_n) \sim 2n$ and $i(r, r) \sim c_0$ for a constant c_0 (depending on the curve outside N). Therefore, $i(.,.)$ is not continuous.

$\square$

Theorem 4.4. *Assume that geodesic currents B_is and C_is converge to $B, C \in \mathcal{C}$ in $C_c^*(T_1(X))$, respectively. Then the intersection measures $I(B_i, C_i)$s converge to $I(B, C)$ in dual of the bounded continuous functions with support in a subset $K \cap \mathcal{I}_1(X)$ such that K is a compact subset of $\mathbb{P}(X) \oplus \mathbb{P}(X)$.*

Since $i(.,.)$ is not continuous when X has cusps, it is mandatory to restrict the convergence to the dual of the functions with support disjoint from cusp's neighborhood and being in $K \cap \mathcal{I}_1(X)$. For instance, the total measures $I(B_i, C_i)(\mathcal{I}_1(X))$ do not necessarily converge to total measure $I(B, C)(\mathcal{I}_1(X))$. The following result is the heart of the proof of Theorem 4.4.

An H-box is defined as a subset of $\mathbb{P}(X)$ containing the (tangent lines of) geodesics between two geodesic arcs on X. The geodesic arcs are small enough such that two arcs in the corresponding H-box intersect at most once. The tangent lines at the points on one edge of $H-$box b make a plane transverse to $\mathcal{F}$ in $\mathbb{P}(X)$. Therefore, any geodesic current C induces a measure on this plane. We say, it is the measure of b with respect to C.

Lemma 4.5. *Let K be a compact subset of $\mathbb{P}(X) \oplus \mathbb{P}(X)$. There are neighborhoods V_ϵ, U_ϵ of B, C, respectively, and a neighbourhood $O_\epsilon \subset \mathcal{I}_1(X)$ of diagonal $\triangle$ such that*

$$I(C', B')(O_\epsilon \cap K) < \epsilon,$$

for all $B' \in V_\epsilon$ and $C' \in U_\epsilon$.

Proof. The projection of K on X is a compact subset of X. Let K' be the set of tangent lines to this compact subset of X. Then K' is a compact subset of $\mathbb{P}(X)$. We can cover K' by compact $H-$boxes $b_1, \dots, b_m$ such that each box is of one of the following types:

(1) $C(b_i) < \delta$, or

(2) $B(b_i) < \delta$, or

(3) There is a closed geodesic γ with transverse measure $> \delta$ with respect to both C and B (atom of C and B) which passes through b_i once. Moreover, $C(b_i \backslash \gamma)$ and $B(b_i \backslash \gamma)$ are $< \delta'$.

Define $O_\epsilon = (b_1 \oplus b_1 \cup \dots \cup b_n \oplus b_n) \cap \mathcal{I}_1(X)$. Now we aim to prove that it has a small total mass with respect to B, C, and geodesic currents in a small neighborhood of B and C. We will choose δ and δ' small enough later.

Choose small neighbourhoods V_δ and U_δ of B and C, respectively, such that:

- $B'(b_i) < \delta + B(b_i), C'(b_i) < \delta + C(b_i)$
- $B'(K') < B(K') + \delta, C'(K') < C(K') + \delta$

for every $B' \in V_\epsilon$ and $C' \in U_\epsilon$.

We can see that the sum of the contribution of $(b_i \oplus b_i) \cap \mathcal{I}_1(X)$s for $H-$boxes b_i of type 1 and 2 to $I(C', B')(O_\epsilon)$ is less than

$$2\delta \sum_j C'(b_j) + 2\delta \sum_j B'(b_j) < 2\delta(C'(K') + B'(K') + 2\delta).$$

Choosing δ small enough makes the contribution less than $\epsilon/2$.

Now we show that the contribution of $b_i \oplus b_i$s are small for b_is of type 3.

Assume that b_i is of type 3 and closed geodesic $\gamma \subset \mathbb{P}(X)$ is the common atom of B and C in b_i.

Consider $\widetilde{X} = \mathbb{H}/[\gamma]$, a covering of X homeomorphic to an annulus with γ as its core curve. Each geodesic current on X induces a geodesic current on $\widetilde{X}$. Let $\widetilde{b_i}$ be the lift of b_i

to $\widetilde{X}$ (the lift that intersects γ). Define $G^i_k \subset \widetilde{b_i}$ as the set of tangent lines to the geodesics which pass through b_i for k times.

Note that γ is not in any of G^i_k's because γ passes through $\widetilde{b_i}$ infinitely many times. By Lemma 3.14, inside $\widetilde{b_i}$, each geodesic in G^i_p has at most $p+q+2$ intersection points with each geodesic in G^i_q. As a result, the contribution of the intersection points of G^i_p and G^i_q to $I(B', C')$ is at most $\frac{p+q+2}{pq} B'(G^i_p)C'(G^i_q)$.

Therefore, the contribution of $b_i \oplus b_i$ to $I(C', B')$ is at most:

$$\begin{aligned}\sum_{p\le k} B'(G^i_p)C'(b_i) + \sum_{q\le k} C'(G^i_q)B'(b_i) + \sum_{p,q>k} \frac{p+q+2}{pq} B'(G^i_p)C'(G^i_q) \\ + \sum_{p>k} \frac{B'(G^i_p)}{p} C'(\gamma) + \sum_{q>k} \frac{C'(G^i_q)}{q} B'(\gamma),\end{aligned} \tag{7}$$

where $k \in \mathbb{N}$ is a large enough constant.

Note that $\cup_{j=1}^k G^i_p \subset \mathbb{P}(X)$ has measure $< \delta'$ with respect to B and C. On the other hand, there are no atoms in the boundary of G^i_ps; atoms corresponding to a segment of a closed geodesic. Thererfore, by Lemma [Bon2, Lemma 4.3], mass of $\partial(\cup_{j=1}^k G^i_p)$ is 0 with respect to any geodesic currents. Therefore, we can assume the measure of $\cup_{j=1}^k G^i_p \subset \mathbb{P}(X)$ is $< \delta + \delta'$ with respect to B', C' too.

Now, from the upper bound (7), we can see that the contribution of $b_i \oplus b_i$ to $I(C', B')$ is less than

$$\le (\delta + \delta')(C'(b_i) + B'(b_i)) + \frac{5}{k} B'(b_i)C'(b_i) \tag{8}$$

Choose δ' small enough and k large enough. Then we can see the contribution is $< \epsilon/2$. $\square$

Proof of Theorem 4.4. Consider $U_\epsilon, V_\epsilon, O_\epsilon$ from Lemma 4.5. Assume that f is bounded above by b. Then from Lemma 4.5 we have

$$|\int_{O_\epsilon \cap K} f\, dI(B_n, C_n) - \int_{O_\epsilon \cap K} f\, dI(B, C)| \leq 2b\epsilon.$$

Moreover, the convergence of B_ns and C_ns implies that

$$|\int_{E_\epsilon} f\, dI(B_n, C_n) - \int_{E_\epsilon} f\, dI(B, C)| \leq \epsilon,$$

when n is large enough, for compact set $E_\epsilon = K/O_\epsilon$. Combining these two inequalities imply the following

$$\int_{\mathcal{I}_1(X)} f\, dI(B_n, C_n) \to \int_{\mathcal{I}_1(X)} f\, dI(B, C),$$

when $n \to \infty$, as required. □

4.2. **Excursion.** In this subsection, we find an upper bound on the number of the self-intersection points of γ_T in a neighborhood of cusps. Recall that γ_T is the sum of the closed geodesics up to length T.

Let p be a cusp of X. Define $N_p(r)$ and the horoball neighborhood of P with horocycle boundary of length of r, or in other words, $\ell(\partial N_p(r)) = r$.

Then the measure $P_* I(\gamma_T, \gamma_T)(N_p(r))$ is the number of the self-intersection points of γ_T inside $N_p(r)$. We have:

Proposition 4.6. *There is a constant c_r such that in the self-intersection number $i(\gamma_T, \gamma_T)$, the contribution of the points inside $N_p(r)$ is bounded above by $c_r e^{2T}$. Moreover, $c_r \to 0$ as $r \to 0$. In other words, we have*

$$P_* I(\gamma_T, \gamma_T)(N_P(r)) \leq c_r e^{2T},$$

for a constant c_r which tends to 0 as $r \to 0$.

In order to prove Proposition 4.6, we use the following results.

First, we give an upper bound to the number of closed geodesic passing through a point.

Lemma 4.7. *The number of the geodesics with endpoints at a point $Q \in X$, and length $\leq T$ is $\leq Ce^T$ where C is a constant.*

Proof. Let Q_0 be a preimage of $Q \in X$ in $\mathbb{H}$. The geodesics with endpoints at Q and length $\leq T$ are corresponding to the π_1 (fundamental group) orbit of Q in the disk of radius T. Consider the preimages (in $\mathbb{H}$) of a small disk of radius r around Q. They are separated disks inside a disk of radius $T + 2r$ around Q_0. Therefore, the number of these points is $< B(area\, of\, the\, disk) < Ce^T$ for constants B, C. $\square$

Define an *n-excursion* (or simply *n-exc*) as a geodesic arc with endpoints on $\partial N_p(r)$ and winding number n around the cusp.

Lemma 4.8. *The intersection number between an n-exc and m-exc is $\leq 2\min(n, m) + 2$*

Proof. It is similar to the proof of Lemma 3.14, for the case of B_n and B_m. $\square$

Lemma 4.9. *Let E_n be an n-exc in $N_p(r)$. Then we have*

$$\ell(E_n) \geq 2\log n + c$$

for a constant $c \in \mathbb{R}$ depending on r and independent of n.

Proof. Consider a preimage of E_n as a semicircle and $\partial N_p(r)$ as a horizontal line in $\mathbb{H}$. It is presented in Figure 9. Let z_1, z_2 be the endpoints of E_n on $\partial N_p(r)$ in $\mathbb{H}$. The points z_1, z_2 have the same height and the difference between their $x - coordinate$ is $\geq nr$.

From the formula of hyperbolic length, [Bus, Equ. 1.1.2] we have

$$\ell(E_n) = d(v_1, v_2) = \cosh^{-1}(1 + \frac{|v_1 - v_2|^2}{Imv_1 Imv_2}) \geq \log(1 + bn^2) \geq 2\log(n) + \log b$$

Where b is a constant depending on r. $\square$

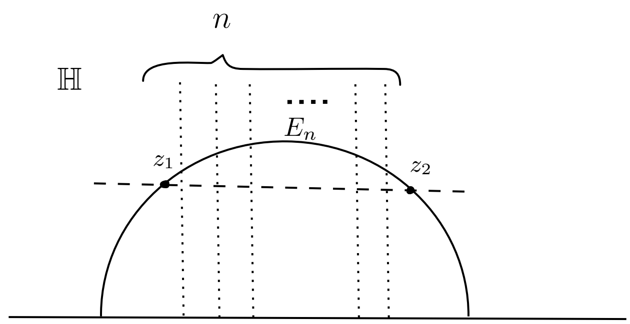

Figure 9

Denote $A_n(T)$ as the number of the n-excs in $N_p(1)$ among the closed geodesics of length $\leq T$.

Proposition 4.10. *There is a constant $C > 0$ such that $A_n(T) \leq \frac{Ce^T}{n^2}$.*

Proof. From now on, by excursion we mean excursion with respect to the horocycle of length 1. Let A_T be the set of pairs (γ, E_n) where γ is a closed geodesic with length $\leq T$ and E_n an n-exc of γ. Let $B_T(Q)$ be the set of geodesics with length $\leq T$ from a point Q to itself. We define an injective map $\psi : A_T \to B_{T-2\log n+c}$ for a constant c, as follows.

Fix a point Q on $\partial N_p(1)$. Given (γ, E_n), substitute E_n with an arc going around the cusp once without changing its endpoints on the horocycle $\partial N_p(1)$. Then move the endpoints of the arc (the endpoints of E_n on the horocycle) to Q homotopicaly. See Figure 10. We obtain a closed curve $\psi((\gamma, E_n))$ with Q as the marked point with length $\leq \ell(\gamma) - 2\log n + c$. The bound on the length obtained from Lemma 4.9 and the fact that attaching the arc and moving its endpoints are changing the length at most by a constant c.

Note that ϕ is injective because if we have a geodesic γ with endpoints at Q in the image of ϕ, then $\phi^{-1}(\gamma)$ is determined uniquely. It is homotopic to the closed curve obtained from joining an n-exc to γ at Q. It implies $A_n(T) = |A_T|$ is $\leq |B_{T-2\log n+c}|$ which is $< Ce^{T-2\log n+c} \leq C'\frac{e^T}{n^2}$ by Lemma 4.7. $\square$

Proof of Proposition 1.7. It is implied directly from Proposition 4.10. Note that $\ell(\gamma_T) \sim e^T$ by Margulis' result on counting closed geodesics [Mar].

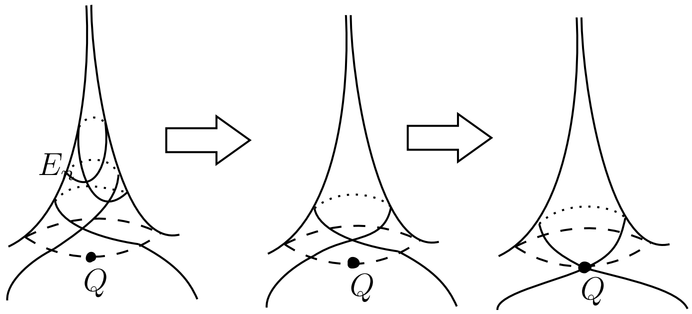

FIGURE 10

Proof of Proposition 4.6. Consider excursions in $N_p(1)$. The intersection points in $N_p(r)$ are obtained from n-excs with $n > n_r$ where $r < 1$, and $n_r \to \infty$ as $r \to 0$.

We know from Lemma 4.8 that the intersection number of an n-exc and an m-exc is $\leq 2\min(m,n) + 2 \leq 4\min(m,n)$. Therefore, we have

$$P_* I(\gamma_T, \gamma_T)(N_p(r)) \leq \sum_{i,j>n_r} 4\min(i,j) A_i(T) A_j(T) \leq \sum_{i,j>n_r} 4C^2 \min(i,j) \frac{e^{2T}}{i^2 j^2} \leq 8C^2 e^{2T} \sum_{i=n_r}^{\infty} \frac{1}{i^2} \sum_{j=n_r}^{i} \frac{1}{j}$$

$$\leq 8C^2 e^{2T} \sum_{i=n_r}^{\infty} \frac{\log i}{i^2}.$$

The sum $\sum_{i=1}^{\infty} \frac{\log i}{i^2}$ is finite and the term $\sum_{i=n_r}^{\infty} \frac{\log i}{i^2}$ tends to 0 as $r \to 0$, as required.

□

4.3. **Equidistribution.** In this subsection, we prove Theorem 4.2.

Recall that geodesic current γ_T is the sum of the closed geodesics with length $\leq T$ and L_X is the Liouville current of X.

We know closed geodesics are equidistributed on X [Mar, Thm. 9.1][Bow][Rob]. In other words, we have

Theorem 4.11 (Margulis, Bowen, Parry, Roblin). *Let X be a finite volume hyperbolic surface. We have*

$$\frac{\gamma_T}{\ell(\gamma_T)} \to \frac{L_X}{|L_X|},$$

as $T \to \infty$, *in* $C_c^*(T_1(X))$.

Proof of Theorem 4.2. From Theorem 4.11 and Theorem 4.4 we conclude, as $T \to \infty$

$$\frac{I(\gamma_T, \gamma_T)}{\ell(\gamma_T)^2} \to \frac{I(L_X, L_X)}{|L_X|^2}, \tag{9}$$

in $\mathcal{B}^*$, where $\mathcal{B}^*$ is the dual of the set of continuous bounded functions with support in a $K \subset \mathcal{I}_1(X)$ such that K is a compact subset of $\mathbb{P}(X) \oplus \mathbb{P}(X)$.

Note that the pullback of a continuous function with compact support on X to $\mathcal{I}_1(X)$ is in $\mathcal{B}_*$.

Therefore, from pushforward of the measures in conrgence (9) by map $P : \mathcal{I}_1(X) \to X$, we conclude:

$$\frac{1}{\ell(\gamma_T)^2} P_*(I(\gamma_T, \gamma_T)) \to \frac{1}{\pi^4(2g-2+n)^2} P_*(I(L_X, L_X)), \tag{10}$$

as $T \to \infty$, in $C_c^*(X)$.

Then, from Lemma 2.8 and Lemma 2.7, we have:

$$\frac{1}{\ell(\gamma_T)^2} \sum_P m_P \delta_P \to \frac{1}{2\pi^3(2g-2+n)^2} A,$$

in $C_c^*(X)$ as $T \to \infty$. The sum is over the self-intersection points of γ_T.

These measures are not necessarily probability measures; therefore, to finish the proof, we only need to show they are tight (see Theorem 2.9). Let K_ϵ be the compact subset of X obtained from removing a small horoball neighborhood $N_p(r)$ of the cusps. The measure of a horoball neighborhood $N_p(r)$ with respect to $I(\gamma_T, \gamma_T)$ is the number of the intersection points in $N_p(r)$. On the other hand, we know from Margulis' result that $\ell(\gamma_T) \sim e^T$. Therefore, from Proposition 4.6, we conclude that $N_p(r)$ has small measure with respect to all measures $I(\gamma_T, \gamma_T)/\ell(\gamma_T)^2$. As a result, the complement of K_ϵ has a small measure with respect to these measures. It means that these measures are tight, as required. $\square$

5. Entropy and Self-Intersection Number

In this section, we prove Theorem 1.3 and answer the third main question of this thesis in the case of ergodic geodesic currents:

Question 3. *Can we control the measure-theoretic entropy h_μ of a geodesic current μ in terms of the self-intersection number $i(\mu,\mu)$?*

Let X be a compact hyperbolic surface and $T_1(X)$ the unit tangent bundle of X equipped with the geodesic flow $\phi_t : T_1(X) \to T_1(X)$.

Recall that h_μ is the measure-theoretic entropy of length 1 geodesic current μ with respect to the geodesic flow of X. Note that to define the entropy of a geodesic current, we interpret the geodesic current as a measure on $T_1(X)$. See §2.2 for different interpretations of a geodesic current.

It is known that $h_\mu = 0$ when μ is a measured lamination [Fat, Lem. 3.3][BS2]. It is also implied from the work of Birman and Series [BS1]. On the other hand, measured laminations are exactly the geodesic currents with zero self-intersection number. This observation motivates the question.

We show the answer to this question is yes when μ is ergodic. More precisely, we establish an upper bound for h in terms of $i(\mu,\mu)$.

Theorem 5.1. *Let μ be an ergodic geodesic current with $\ell_X(C) = 1$. We have:*

$$h_\mu \leq 384\sqrt{2}(g-1)\sqrt{i(\mu,\mu)}\log\frac{c(X)}{\sqrt{i(\mu,\mu)}},$$

where $c(X) = \sqrt{2}eI(X)(3g-3)L_g + \sqrt{2}e/\operatorname{sys}(X)$. The notations $I(X), L_g$ refer to the interaction strength and Bers' constant, respectively.

Note that $I(X) \leq 4/(\operatorname{sys}(X))^2$ (see 2.3). We also know $L_g \leq 26(g-1)$ [Bus, Thm. 5.1.2].

Proof of Theorem 1.3. It is a direct consequence of Theorem 5.1.

Outline of the Proof of Theorem 5.1. The proof has two parts. First, we find an upper bound on h_μ in terms of the growth rate of the number of closed geodesics with a small self-intersection relative to the length. Then we find an upper bound on this growth rate with topological and combinatorial methods.

Recall that $\mathcal{G}_T$ is the set of closed geodesics with $\ell_X(\gamma) \leq T$.

Define

$$P_\epsilon(T) := |\{\gamma | \, \gamma \in \mathcal{G}_T, i(\gamma, \gamma) \leq \epsilon T^2\}|,$$

and

$$P_\epsilon := \lim_{T\to\infty} \frac{\log P_\epsilon(T)}{T}.$$

Theorem 5.2. *Let μ be an ergodic geodesic current with $\ell_X(\mu) = 1$. Then we have*

$$h_\mu \leq P_{2i(\mu,\mu)}.$$

Theorem 5.3. *For $\epsilon > 0$ we have*

$$P_\epsilon \leq 384(g-1)\sqrt{\epsilon} \log \frac{c_0(X)}{\sqrt{\epsilon}},$$

where $c_0(X) = 2eI(X)(3g-3)L_g + 2e/\operatorname{sys}(X)$.

The notations $I(X)$ and L_g refer to interaction strength and Bers' constant, respectively.

Proof of Theorem 1.3. From Theorem 5.2 and Theorem 5.3, we conclude

$$h_\mu \leq P_{2i(\mu,\mu)} \leq 384(g-1)\sqrt{2i(\mu,\mu)} \log \frac{c_0(X)}{\sqrt{2i(\mu,\mu)}}$$

□

5.1. **Entropy and Closed geodesics.** In this subsection, we prove Theorem 5.2. From now on, we assume μ is an ergodic geodesic current with $\ell_X(\mu) = 1$. It is a well-known result of Katok that $h_\mu \leq P_\infty$ [Kat, Thm. 2.1]. Recall that $N_\mu(T, \epsilon, \delta)$ is the minimum number of the balls of radius ϵ, with respect to metric d_T, which is required to cover a set with μ-measure $\geq 1-\delta$. See 2.3 for more details.

Katok showed the entropy h_μ is equal to the exponential growth rate of $N_\mu(T,\epsilon,\delta)$. He related $N_\mu(T,\epsilon,\delta)$ to the number of the closed orbits with length $\leq (1+\epsilon')T$. We show that we can also add a constraint on the self-intersection number of the orbits. More precisely, we show the self-intersection number of these obtained orbits is about $i(\mu,\mu)T^2$. Then we conclude Theorem 5.2.

Let $\phi_{[0,t]}(v)$ be the geodesic arc of length t in $T_1(X)$ start at $v \in T_1(X)$.

Proposition 5.4. *For μ-almost every $v \in T_1(X)$ we have:*

$$\lim_{n\to\infty} \frac{i(\gamma_n,\gamma_n)}{\ell(\gamma_n)^2} = i(\mu,\mu),$$

for any sequence t_n such that $d(\phi_{t_n}(v),v) \leq c_0$ and γ_n is the geodesic representative of the closed curve obtained from connecting v and $\phi_{[0,t_n]}(v)$. Here $c_0 > 0$ is a fixed constant.

Proof. From $Birkhoff'$s ergodic theorem, for $\mu-$almost every $v \in T_1(X)$ we have:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\phi_t(v))dt = \int_{T_1(X)} f d\mu \tag{11}$$

for $f \in L^1(X)$.

On the other hand, the geodesic representative γ_n is a long ray that stays close to $\phi_{[0,t_n]}(v)$ for all time. More precisely, from Anosov closing lemma 2.11, we know γ_n and $\phi_{[0,t_n](v)}$ are in ϵ_0 distance of each other and $|\ell_X(\gamma_n) - T| \leq \epsilon_0$. Assume that f is bounded above by $b > 0$. Therefore, we have

$$|\frac{1}{T}\int_0^T f(\phi_t(v))dt - \frac{1}{\ell(\gamma_n)}\int_0^{\ell(\gamma_n)} f(\gamma_n(t))dt| \leq \frac{1}{T}\int_0^T |f(\phi_t(v)) - f(\gamma_n(t))|dt + b\epsilon_0$$

$$\leq \frac{1}{T}\int_{\sqrt{T}}^{T-\sqrt{T}} |f(\phi_t(v)) - f(\gamma_n(t))|dt + \frac{1}{T}\int\limits_{I_T} |f(\phi_t(v)) - f(\gamma_n(t))|dt + b\epsilon_0,$$

where I_T is the complement of $[\sqrt{T}, T-\sqrt{T}]$ in $[0,T]$. When we increase T, the geodesic rays γ_n and $\phi_T(v)$ are getting closer and closer. Moreover, $|f(\phi_t(v)) - f(\gamma_n(t))| \leq c_T$ for a constant c_T such that $c_T \to 0$ when $T \to \infty$. Then the right-hand side of the inequality is less than:

$$\leq \frac{(T-2\sqrt{T})c_T}{T} + \frac{2\sqrt{T}b}{T} + b\epsilon_0.$$

It tends to zero when $T \to \infty$. Therefore, we have

$$\lim_{T\to\infty} \frac{1}{\gamma_n}\int_0^{\ell(\gamma_n)} f(\gamma_n(t))dt = \int_{T_1(X)} f d\mu.$$

It means that the weighted closed geodesics $\gamma_n/\ell(\gamma_n)$ converge to μ, as $n \to \infty$. The continuity of $i(.,.)$ implies $i(\gamma_n,\gamma_n)/\ell(\gamma_n)^2$ converges to $i(\mu,\mu)$ when $n \to \infty$, as required. $\square$

Define subset $A_{(T,\epsilon)} \subset T_1(X)$ as the vectors v such that $|i(\gamma_T,\gamma_T)/\ell(\gamma_T)^2 - i(\mu,\mu)| \leq \epsilon$ for any closed geodesic γ_T obtained from connecting $\phi_T(v)$ and v when $d(v,\phi_T(v)) \leq c_0$.

Let $B_{(T,\epsilon)} := \{v \in T_1(X) | \exists T \leq t \leq (1+\epsilon)T; d(v,\phi_t(v)) < \epsilon\}$.

Lemma 5.5. *For $\epsilon > 0$, as $T \to \infty$ we have:*

$$\mu(A_{(T,\epsilon)}) \to 1 \tag{12}$$

$$\mu(B_{(T,\epsilon)}) \to 1. \tag{13}$$

Proof. We can conclude Equation 12 from Lemma 5.4.

Cover $T_1(X)$ by finitely many balls of radius $\epsilon/2$. Let K be one of these balls and χ_K the characteristic function of K. By ergodicity of μ for *a.e.* $v \in K \subset T_1(X)$ we have:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \chi_K(\phi_t(v))dt = \int_{T_1(X)} \chi_K(v)d\mu = \mu(K). \tag{14}$$

Therefore for large enough T, we have:

$$\int_0^T \chi_K(\phi_t(v))dt < \int_0^{T(1+\epsilon)} \chi_K(\phi_t(v))dt.$$

We conclude for $\mu - a.e.\ v \in K$ and larg enough T there exists $t \in [T, T(1+\epsilon)]$ such that v and $\phi_t(v)$ are in K and $d(v, \phi_t(v)) < \epsilon$. Apply this to all the balls in the covering of $T_1(X)$, and it implies Equation 13. $\square$

Proof of Theorem 5.2. As we mentioned, the proof is similar to Katok's proof in [Kat, Thm 2.1].

Note that $N_{(T,\epsilon,\delta)}$ is the minimum number of $\epsilon-$balls with respect to d_T metric such that they cover a set of measure $> 1-\delta$. From Proposition 2.10, we know h_μ is equal to the exponential growth rate of $N_{T,\epsilon,1/2}$. Let $Y := A_{(T,\epsilon)} \cap B_{(T,\delta)} \subset T_1(X)$. By Lemma 5.5, for every $\epsilon, \delta > 0$ and T large enough $C(Y) \geq \frac{1}{2}$. We will choose δ later in terms of ϵ.

We say a set Λ is $(T,\epsilon)-$separated set if the d_T distance between every two points of Λ is $> \epsilon$. Let $\Lambda_{(T,\epsilon)}$ be a maximal $(T, 3\epsilon)-$separated subset of Y. Then balls of radius 3ϵ, with respect to metric d_T, with centers at the points of $\Lambda_{(T,3\epsilon)}$ cover Y. Therefore we have:

$$|\Lambda_{(T,\epsilon)}| \geq N(T, 3\epsilon, \frac{1}{2}). \tag{15}$$

Given $v \in \Lambda_{(T,\epsilon)}$. Using the fact that $v \in B_{(T,\delta)}$ and applying Anosov's closing lemma, we obtain a closed geodesic γ_v which is close to $\phi_{[0,T]}(v)$. Choose δ small enough such that γ_v is $\epsilon-$close to $\phi_{[0,T]}(v)$ and $\ell(\gamma_v) < (1+\epsilon)T$. From Anosov closing lemma, we know there exists a point q_v on γ_v such that $d(q_v, v) < \epsilon$.

When $v_1 \neq v_2$, we have $d(q_{v_1}, q_{v_2}) > \epsilon$. Moreover, if geodesics γ_{v_1} and γ_{v_2} are the same for $v_1 \neq v_2$ then we have:

$$d_t(q_{v_1}, q_{v_2}) > d_t(v_1, v_2) - d_t(v_1, q_{v_1}) - d_t(v_2, q_{v_2}) > 3\epsilon - 2\epsilon = \epsilon \tag{16}$$

Equation 16 implies that the number of q_v's on each obtained closed geodesic γ is $< \ell(\gamma)/\epsilon$. We know $\ell(\gamma) < (1+\epsilon)T$, therefore the number of obtained closed geodesics is greater than

$$\frac{|\Lambda_{(\epsilon,T)}|}{(1+\epsilon)T/\epsilon} \geq \frac{\epsilon}{T(1+\epsilon)} N(T, 3\epsilon, \frac{1}{2}). \tag{17}$$

Note that $|i(\gamma_v,\gamma_v)/\ell(\gamma_v)^2 - i(\mu,\mu)| \leq \epsilon$ and we may assume $i(\gamma_v,\gamma_v) > 0$, therefore, we have

$$i(\gamma_v,\gamma_v) \leq 2i(\mu,\mu)\ell(\gamma_v)^2$$

It implies

$$P_{2i(\mu,\mu)}((1+\epsilon)T) \geq \frac{\epsilon}{T(1+\epsilon)} N(T, 3\epsilon, \frac{1}{2})$$

Let $T' = (1+\epsilon)T$, we have

$$\frac{\log P_{2i(\mu,mu)}(T')}{T'} \geq \frac{1}{(1+\epsilon)} \frac{\log \frac{\epsilon}{T(1+\epsilon)} + \log N(T, 3\epsilon, \frac{1}{2})}{T}$$

Tending $T \to \infty$ and then $\epsilon \to 0$ implies

$$p_{2i(\mu,\mu)} \geq h_\mu$$

□

5.2. **Counting closed geodesics.** In this subsection, we prove Theorem 5.3.

Consider a proper hexagon decomposition of X. Let α, β be oriented simple arcs in a hexagon such that their endpoints are on the edges but not on the vertices. Assume that α, β intersect once. This intersection is of one of the following types based on the endpoints of α and β:

- *Type 1.* Neither their initial points nor their terminal points are on the same edge.
- *Type 2 (a).* Their initial points are on the same edge but their terminal points are not.

- *Type 2 (b).* Their terminal points are on the same edge but their initial points are not.
- *Type 3.* Their initial points are on the same edge, and their terminal points are on the same edge as well.

Assume that γ_1, γ_2 are oriented closed curves on X such that they do not pass through the vertices of the hexagons. We say γ_1, γ_2 are linearly homotopic if there is a homotopy h between γ_1, γ_2 that moves the points on the edges only along the same edge. In other words, if α is a subarc of γ_1 between edges b and b' then h moves the endpoints of α along b and b' and sends α to an arc between b, b'.

We say a closed curve is proper if it does not pass through the vertices of the hexagos and its subarcs inside hexagons are simple subarcs. The heart of the proof is the following theorem:

Theorem 5.6. *Let γ be a proper oriented closed curve. Then there is a proper oriented closed curve γ_2 linearly homotop to γ_1 such that it does not have any intersection of type* 3*. In addition, the number of the intersection points of type* 2 *for γ_2 is not larger than the corresponding number for γ_1.*

Proof. Consider the incoming arcs to an edge e. We set an order on their endpoints on e and show that these orders are compatible for all edges in the decomposition. Then we prove that a closed curve with these orderings satisfies the required constraints. Moreover, the order between the endpoints of an incoming arc and an outgoing arc does not matter.

When we fix an edge e of a hexagon, we can give an order from left to right (clockwise) to other edges of the hexagon. In addition, we have an orientation on edge e induced by the (clockwise orientation of) hexagon containing e. If we pick the other hexagon containing e then we obtain the opposite orientation on e. Now consider two incoming arcs a_1, b_1 to e with endpoint p_1, p_2 on e. Assume that a_1, b_1 stay on the same edge after k steps. In other words, the subarcs $a_1 a_2 \dots a_k$ and $b_1 b_2 \dots b_k$ are corresponding to the same words but a_{k+1}

and b_{k+1} start on the same edge e_0 and end on different edges e', e''. If e' is on the right of e'' (relative to the orientation in the hexagon with respect to edge e_0) then we set p_1 on the right of p_2 and vice versa. It is not hard to see that it gives a compatible ordering on the endpoints of the incoming edges to e on e.

Note that this ordering is compatible for all edges because in this process the ordering between two endpoints p_1, p_2 on e is determined only by the word of their forwarding path.

There is no intersection of type 3. If we have intersecting arcs $a_1 = p_1q_1, b_1 = p_2q_2$ which start and end on the same edges and p_1 is on the left of p_2 (with respect to the hexagon containing the forward paths) then we can say that the forwarding path of a_1 is on the left of b_1 until they diverge and go to different edges. Since it is how we set the orderings. Therefore, there is no intersection between them until they diverge. In particular, there is no intersection between a_1 and b_1, which is a contradiction.

Moreover, we can see that the number of intersections of type 2 does not increase. We show for each intersection point of type 2 in γ_2 there is at least one corresponding intersection point in γ_1. Consider two intersecting arcs α, β in γ_2 that either their initial points or their terminal points share the same edge. It means that there are subarcs $a_0 \dots a_k$ and $b_0 \dots b_k$ (forwarding ordered) containing α, β, respectively, such that a_i, b_i share the same starting and ending edges except a_0, b_0 and a_k, b_k. Based on how we set the orders, we can see a_0 and b_0 have to be α, β. Moreover, a_i, b_i are separate for $i > 0$. No matter how we change the order of endpoints on the edges, we can see subarcs $a_0 \dots a_k$ and $b_0 \dots b_k$ have to intersect at least once since the order of their initial points is opposite of the order of their terminal points. As a result, γ_1 has at least the same number of type 2 intersection points as γ_2.

$\square$

We say γ_2 that satisfies the properties in Theorem 5.6 is a *special representative* of γ_1.

We also apply the following combinatorial results to prove Theorem 5.3.

Given a proper oriented closed curve γ, we obtain a word on each edge e of a hexagon H corresponding to the incoming (outgoing) arcs of γ to e inside H. In other words, the word $w_1 \dots w_m$ means that the other end of the ith arc is on edge w_i. In total, a curve gives us $48g - 48$ words.

Lemma 5.7. *The word $\omega(\gamma)$ corresponding to a special representative γ is determined uniquely if we have all the $48g - 48$ assigned words to the edges for incoming and outgoing arcs.*

Proof. Based on the words, we have the relative position of the initial points and terminal points of the arcs on the edges. For instance, word $w_1 \dots w_n$ tells us that on edge e there are n outgoing arcs with terminal points on $w_1, \dots, w_n$ from left to right. Now consider the endpoints of the arcs from an edge e to another edge e' in a hexagon. Since there is no intersection of type 3 there is a unique way to connect these points. Therefor, $\omega(\gamma)$ is determined uniquely.

□

Lemma 5.8. *Assume that we have n arcs in a hexagon such that each arc has an end on edge e_1 and another end on the midpoint of one of the 5 other edges e_2, e_3, e_4, e_5, e_6. In addition, assume that their endpoints are different on e_1. Let ω be the word of length n with letters from $\{e_2, e_3, e_4, e_5, e_6\}$ corresponding to these arcs. The number of possible words w such that the corresponding arcs intersect in $\leq \epsilon n^2$ is*

$$\leq 16\epsilon^2 n^4 (\frac{e}{\sqrt{\epsilon}})^{8\sqrt{\epsilon}n}.$$

Proof. First, assume that instead of a hexagon we have a triangle and we connect arcs to midpoints of two edges e_2, e_3. Assume that e_2 is on the left of e_3 (with respect to e_1). If the number of points attached to e_2 is $\leq \sqrt{\epsilon}n$ then the number of possible words is

$$\leq \sqrt{\epsilon}n\binom{n}{\sqrt{\epsilon}n}.$$

The coefficient $\sqrt{\epsilon}n$ represents the possible number of the letters e_2 in ω. Then assume that the number of points attached to e_2 is $\geq \sqrt{\epsilon}n$. Choose the $\sqrt{\epsilon}n$ most right points attached to e_2; there are n choose $\sqrt{\epsilon}n$ possible ways. The most left chosen point split n into two blocks. The points in the right block are completely determined where to attach ($\sqrt{\epsilon}n$ of them were chosen to attach to e_2 and the rest have to be attached to e_3). Now consider the points on the left block. At most $\sqrt{\epsilon}n$ of them can be attached to e_3 unless we would have $> \epsilon n^2$ intersection points (between these arcs and the $\sqrt{\epsilon}n$ chosen points attached to e_2). Therefore, we have $\leq \sqrt{\epsilon}n$ points connected to e_3 in the left block. The number of ways to determine the word in the left block is

$$\leq \sqrt{\epsilon}n\binom{n}{\sqrt{\epsilon}n}.$$

In total, the number of possible such words is

$$\leq \sqrt{\epsilon}n\binom{n}{\sqrt{\epsilon}n} + \sqrt{\epsilon}n\binom{n}{\sqrt{\epsilon}n}^2 \leq 2\sqrt{\epsilon}n\binom{n}{\sqrt{\epsilon}n}^2 \leq 2\sqrt{\epsilon}n(\frac{e}{\sqrt{\epsilon}})^{2\sqrt{\epsilon}n}$$

The last inequality is implied by Stirling's approximation.

By iterating the process 4 times, we conclude the result for the hexagon. In other words, we find the number of such words with two letters a, b that a represents e_2 and b represents the other 4 edges. Then, determining where to attach the points with the letter b (to edges e_3, e_4, e_5) is a similar problem. Therefore, the number of possible words in the hexagon case is

$$\leq (2\sqrt{\epsilon}n(\frac{e}{\sqrt{\epsilon}})^{2\sqrt{\epsilon}n})^4.$$

$\square$

Proof of Theorem 1.3. From Lemma 2.12 we have

$$P_\epsilon(T) \leq |\{\gamma \in \mathcal{G} |\, \ell_{com}(\gamma) \leq c(X)T, i(\gamma,\gamma) \leq \epsilon T^2\}|.$$

Now instead of closed geodesics we consider their special representatives (as described in Theorem 5.6). These special representatives have length $\leq c(X)T$ and the number of their type 2 intersections are $\leq \epsilon T^2$. Therefore, from Lemma 5.7, we have

$$|\{\gamma \in \mathcal{G} |\, \ell_{com}(\gamma) \leq c(X)T, i(\gamma,\gamma) \leq \epsilon T^2\}| \leq (number\, of\, possible\, words\, on\, one\, edge)^{48g-48}.$$

Now, we apply Lemma 5.8 to each edge for n being $c(X)T$ and ϵ being $\epsilon/c(X)^2$. Therefore, the right-hand side of the last inequality is

$$\leq (2\sqrt{\epsilon}T(\frac{ec(X)}{\sqrt{\epsilon}})^{2\sqrt{\epsilon}T})^{4(48g-48)}.$$

It implies

$$P_\epsilon \leq (384g-384)\sqrt{\epsilon}\log\frac{ec(X)}{\sqrt{\epsilon}}.$$

$\square$

A qualitative version of Theorem 5.1 can be deduced from Sapir's work [Sap]. She showed the Haussdorff dimension of the set of geodesic rays with a small self-intersection number relative to length is small.

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