Holomorphically parametrized L2 Cramer's rule and its algebraic geometric applications

Holomorphically parametrized L^2 Cramer's rule and its algebraic geometric applications

A dissertation presented

by

Yih Sung

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

Harvard University
Cambridge, Massachusetts

April 2013

Dissertation Advisor: Yum-Tong Siu Yih Sung

Holomorphically parametrized L^2 Cramer's rule and its algebraic geometric applications

Abstract

Suppose $f, g_1, \cdots, g_p$ are holomorphic functions over $\Omega \subset \mathbb{C}^n$. Then there raises a natural question: when can we find holomorphic functions $h_1, \cdots, h_p$ such that $f = \sum g_j h_j$? The celebrated Skoda theorem solves this question and gives a L^2 sufficient condition. In general, we can consider the vector bundle case, i.e. to determine the sufficient condition of solving $f_i(x) = \sum g_{ij}(x) h_j(x)$ with parameter $x \in \Omega$. Since the problem is related to solving linear equations, the answer naturally connects to the Cramer's rule. In the first part we will give a proof of division theorem by projectivization technique and study the generalized fundamental inequalities. In the second part we will apply the skills and the results of the division theorems to show some applications.

Contents

Acknowledgements v

1. Introduction 1
2. Projectivization Computation 10
3. Linear Algebra Lemmas 24
4. Main Theorem 34
5. Fundamental Lemma 47
6. Algebraic Verification And Vanishing Theorems 64
7. Griffith Positivity 79
8. Effective Artin-Rees Lemma (Ideal Case) 83
9. Effective Artin-Rees Lemma (Module Case) 93

References 101

Acknowledgements

I want to thank my adviser professor Siu who has incredible patient and superb knowledge to help me finish this thesis, and professor Schmid for the useful discussion about the representation theory in integration. I also want to thank my parents. They always support my study and provide very warm back up. I want to express my gratitude to my esteemed friend and brother Chi Chen-Yu who has taught me a lot of mathematics and discussed with me with all his passion. Another special acknowledgement I want to deliver to my fiancee Chen Yu-Hsing. Without her constant encourage I hardly finish my PhD program. Finally I want to thank all my friends who communicate with me and talk to me during the aboard study. I love all of you.

1. Introduction

Solving linear equations is a very old and important subject in algebra. That is to say given a constant matrix G and a constant column vector f, we want to determine the conditions for solving

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} g_{1,1} & g_{2,1} & \cdots & g_{p,1} \\ g_{1,2} & g_{2,2} & \cdots & g_{p,2} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1,r} & g_{2,r} & \cdots & g_{p,r} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_r \end{pmatrix}.$$

When the number of unknown p is equal to the number of equations r there is a beautiful formula to describe the solutions:

$$h_j = \frac{\begin{vmatrix} g_{1,1} & \cdots & f_1 & \cdots & g_{p,1} \\ g_{1,2} & \cdots & f_2 & \cdots & g_{p,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{1,r} & \cdots & f_r & \cdots & g_{p,r} \end{vmatrix}}{\begin{vmatrix} g_{1,1} & \cdots & g_{j,1} & \cdots & g_{p,1} \\ g_{1,2} & \cdots & g_{j,1} & \cdots & g_{p,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{1,r} & \cdots & g_{j,r} & \cdots & g_{p,r} \end{vmatrix}}$$

which is known for Cramer's rule, and if $p > r$ then there always exists non-trivial solutions for a consistent system of linear equations. With this understanding, it is interesting to ask what would happen if G and f are parametrized by some variables, say $x_1, \cdots, x_n$? For each fixed value $x_1, \cdots, x_n$ this system of equations can be always solved if $p > r$, the fun part is to find a compatible solutions when $x_1, \cdots, x_n$ are varying, especially if $x_1, \cdots, x_n$ are coordinates of a Stein domain, that means we are searching solutions $h(x)$ that are holomorphic with respect to the coordinates

x. Inspired by the classic Cramer's rule, we expect to describe the solubility of this system of equations should be in terms of Cramer's-rule-type L^2 conditions.

In the $r = 1$ case, there is a satisfied answer due to Skoda[8] and later Varolin[10] generalized to some other delicate situations.

Theorem 1.1. *If f satisfies the L^2 condition*

$$\int_\Omega \frac{|f|^2 e^{-\phi}}{(\sum |g_j|^2)^{\alpha\beta+1}} \mathrm{d}V_\Omega < \infty,$$

for some constant $\alpha > 1$, where ϕ is a psh function and $\beta = \min\{n, p-1\}$, then there exists holomorphic functions g_j such that

$$f = \sum g_j h_j,$$

and h_j has L^2 estimate

$$\int_\Omega \frac{|h_j|^2 e^{-\phi}}{(\sum |g_j|^2)^{\alpha\beta}} \mathrm{d}V_\Omega \le \frac{\alpha}{\alpha - 1} \int_\Omega \frac{|f|^2 e^{-\phi}}{(\sum |g_j|^2)^{\alpha\beta+1}} \mathrm{d}V_\Omega.$$

The idea of the proof is to change the problem into a solving $\bar\partial$-equation problem:

$$\bar\partial u_j = -\bar\partial\Big(\frac{\bar g_j f}{|g|^2}\Big),$$

and the key to solve these equations is an inequality of *a priori* estimate:

$$\beta \partial_k \bar\partial_\ell \log |g|^2 v_j^k \bar v_j^\ell \ge \frac{1}{|g|^2} \Big| \sum_{j,\ell}^{1} \sum_{k=1}^{n} \bar g_\ell (g_\ell \partial_k g_j - g_j \partial_k g_\ell) v_j^k \Big|^2, \tag{1}$$

where $\beta = \min\{p-1, n\}$. This inequality can be interpreted as a comparison of the curvature of $1/|g|^2$ and the second fundamental form of the vector bundle defined by the kernel of g_j. By these two crucial observation, Skoda's result of rank 1 case is precisely the L^2-Cramer's rule.

In order to deal with the general situation, let us reformulate the problem. Let V over $\Omega \subset \mathbb{C}^n$ be a rank r trivial bundle equipped with a Hermitian metric $h_{i\bar{j}}$. We are considering the division problem:

Given holomorphic sections $f = (f_i)$, and $g_1 = (g_{i,1}), \cdots, g_p = (g_{i,p})$ of V, determine when f can be expressed as

$$f = \sum g_j h_j. \tag{2}$$

We can generalize the problem (2) one step further to the twisted setting that f is a section of $V \otimes L$ for some line bundle L over Ω and h_i would be the sections of L. For convenience, this problem can be figured by the following diagram:

$$G : \bigoplus_{1 \le j \le p} L \longrightarrow V \otimes L,$$

where G is the matrix formed by the column vectors g_j. Suppose G has generic rank r, i.e. G is full rank generically, Skoda also has a theorem as follows.

Theorem 1.2. [9] *Let X be a pseudo-convex kähler manifold. Consider the diagram*

$$G : E^p \otimes L \otimes K \longrightarrow V^r \otimes L \otimes K \ .$$

Assume G is generically surjective, $E \ge_{Nak} 0$ and

$$\sqrt{-1}\Theta(L) \ge (m + \epsilon)\sqrt{-1}\Theta(\det V)$$

for some $\epsilon > 0$, where

$$m = \min\{n, p - r\}.$$

Suppose a holomorphic section f of $V \otimes L \otimes K$ satisfying

$$L_0 := \int_X \frac{\langle \widetilde{GG^\dagger} f, f\rangle}{(\det(GG^\dagger))^{m+1+\epsilon}} \mathrm{d}V_X < \infty,$$

where $G^\dagger$ is conjugate transposed of G and $\widetilde{M}$ for a matrix M is defined by

$$\widetilde{M} = (\det M)M^{-1}.$$

Then there exist h, a section of $E \otimes L \otimes K$, such that

$$f = Gh \ \text{ and}$$

$$\int_X \frac{\|h\|^2}{(\det(GG^\dagger))^{m+\epsilon}} \mathrm{d}V_X \leq \Big(1 + \frac{m}{\epsilon}\Big) L_0.$$

Though he did not formulate the extrinsic form of the L^2-condition, by simple linear algebra we can derive

$$\frac{\langle \widetilde{GG^\dagger} f, f\rangle}{(\det(GG^\dagger))^{m+1+\epsilon}} = \frac{\sum_{1\leq j_1 < \cdots < j_{r-1} \leq p} |f \wedge g_{j_1} \wedge \cdots \wedge g_{j_{r-1}}|^2}{\big(\sum_{1 \leq k_1 < \cdots < k_r \leq p} |g_{k_1} \wedge \cdots \wedge g_{k_r}|^2\big)^{m+1+\epsilon}},$$

which is precisely the rank r L^2 Cramer's rule. This formulation has already appeared in Kelleher and Tyalor's paper [7], but in their paper more strict constrains are required. The idea of Skoda's proof is to decompose

$$E = S \oplus V,$$

as smooth vector bundles. Note that S is the kernel of G, so it associates a second fundamental form β. Similar to the $r = 1$ case the strategy is to change the problem into solving $\bar\partial$-equations. This time it turns to solve

$$\bar\partial u = -\beta^* \wedge f \tag{3}$$

on S. Again there is a corresponding key inequality in the vector bundle case:

$$\beta_1 \sum_{j,k,\ell} \big|\beta_{j,k} \lrcorner v_\ell\big|^2 \geq \sum_j \big| \sum_k \beta_{j,k} \lrcorner v_k\big|^2, \tag{4}$$

where $\beta_1 = \min\{n, p-r\}$. Once this fundamental inequality is established, the division problem can be solved by standard functional analysis.

For the case that G is generic surjective (i.e. theorem 1.2) we have a different approach. We apply the so called projectivization technique. The basic idea is to introduce extra variables $z_1, \cdots, z_r$ into (2) and write

$$\tilde{f} = \sum_i f_i z_i = \sum_j \Big(\sum_i g_{ij} z_i\Big) h_j = \sum_j \tilde{g}_j h_j. \tag{5}$$

There is a corresponding abstract way to describe the projectivization process. Let $\pi : \mathbb{P}(V^*) \longrightarrow \Omega$ be the projectivization of V^*, and $h_{i\bar{j}}$ would induce a natural metric

$$e^{-\tilde{\varphi}} = \frac{1}{\sum_{i,j} h^*_{i\bar{j}} z_i \bar{z}_j}$$

on $\mathcal{O}_{V^*}(1)$. Moreover the curvature of $\mathcal{O}_{V^*}(1)$, $\partial\bar{\partial}\tilde{\varphi}$ defines a metric on each fiber $\mathbb{P}(V^*)_x$. This metric is usually named by Fubini-Study metric, and its induced volume form is

$$\mathrm{d}V_{FS} = (r-1)! \det(h) e^{-r\tilde{\varphi}} \mathrm{d}V_z.$$

Recall the natural isomorphisms

$$H^k(\Omega, V) \cong H^k(\mathbb{P}(V^*), \mathcal{O}_{V^*}(1)),$$

$$H^q(\Omega, V \otimes F) \cong H^q(\mathbb{P}(V^*), \mathcal{O}_{V^*}(1) \otimes \pi^* F),$$

where F is a line bundle on Ω. Hence every section of V can be regarded as an holomorphic section of $\mathcal{O}_{V*}(1)$, and the diagram can be redrew by

$$\tilde{G} : \bigoplus_{1 \le j \le p} \pi^* L \longrightarrow \mathcal{O}_{V^*}(1) \otimes \pi^* L.$$

Therefore on $\mathbb{P}(V^*)$ the division problem is reduced to the line bundle situation. However if we turn to solve

$$\tilde{f}(x,z) = \sum_j \tilde{g}_j(x,z)\tilde{h}_j(x,z),$$

$\tilde{h}_j(x,z)$ might involve fiber variables. Fortunately, since $\tilde{h}_j(x,z)$ is a holomorphic sections of π^*L, $\tilde{h}_j$ is constant along fiber, i.e.

$$\tilde{h}_j(x,z) = h_j(x).$$

Therefore we can focus on the division problem (5). Nevertheless there are two issues deter us from applying Skoda's theorem directly. First of all, the curvature condition has to be satisfied. However if we use the natural weight function

$$\frac{1}{(\|\tilde{g}\|^2)^{\alpha\beta+1}}$$

then there would be a negativity term along fiber direction. In order to handle this defect, we change to use the mix weight function

$$\frac{1}{(\|\tilde{g}\|^2)^r \mu^{\alpha\beta-r+1}},$$

where

$$\mu = \sum_{1\leq i_1<\cdots<i_r\leq p} |g_{i_1}\wedge\cdots\wedge g_{i_r}|^2,$$

then the negativity would be eliminated, and by averaging technique we will encounter the Skoda-type inequality

$$\begin{aligned}(6)\qquad &\frac{1}{\alpha\mu}\sum_{1\leq i_1<\cdots<i_{r-1}\leq p}\left|u\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}}\right|^2 + \alpha\beta_1\sum_{j=1}^p\sum_{k,\ell=1}^n(\partial_k\bar{\partial}_\ell\log\mu)v_j^k\bar{v}_j^\ell \\ &\geq 2\left|\sum_{1\leq i_1<\cdots<i_{r-1}\leq p}\ \sum_{\substack{1\leq j\leq p\\ 1\leq k\leq n}}\left(u\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}},\overline{\partial_k\left(\frac{g_j\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}}}{\mu}\right)v_j^k}\right)\right.\end{aligned}$$

This could be verified by both combinatorics computation and invariant form computation. The later would connect to Skoda's fundamental inequality.

On the other hand, we need to figure out the L^2 condition w.r.t the horizontal variables (i.e. involving only x). On the total space $\mathbb{P}(V^*)$ the integral reads

$$\int_{\mathbb{P}(V^*)} \frac{|\tilde{f}|^2 e^{-\tilde{\varphi}}}{(\sum_j |\tilde{g}_j|^2 e^{-\tilde{\varphi}})^{\alpha\beta+1}} \mathrm{d}V_{FS} \wedge \mathrm{d}V_{\Omega}. \tag{7}$$

We will demonstrate how to average along fiber in the next section. After that we can combine these two techniques to conclude the L^2 Cramer's rule for the full rank case.

Our main result is to remove the assumption that

$$G : E \longrightarrow V$$

is generic surjective. Inspired by the explicit form of Skoda's fundamental inequality we can generalize it to the rank m case as following:

$$\begin{aligned} &\frac{1}{\alpha\mu} \sum_{1 \le i_1 < \cdots < i_{m-1} \le p} \left| u \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}} \right|^2 + \alpha\beta_1 \sum_{j=1}^{p} \sum_{k,\ell=1}^{n} (\partial_k \bar{\partial}_\ell \log \mu) v_j^k \bar{v}_j^\ell \\ \ge 2 &\left| \sum_{1 \le i_1 < \cdots < i_{m-1} \le p} \sum_{\substack{1 \le j \le p \\ 1 \le k \le n}} \left(u \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}, \overline{\partial_k \Big(\frac{g_j \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}}{\mu} \Big) v_j^k} \right) \right. \end{aligned} \tag{8}$$

Once the fundamental inequality is established by standard functional analysis we can get the rank m L^2-Cramer's rule. First let us list some notations.

- $\mathrm{Ric}_X = \sqrt{-1}\partial\bar{\partial}\kappa$, if the volume form is written as $e^{-\kappa}\mathrm{d}V_X$.
- $c_1(V) = \sqrt{-1}\Theta(\det V)$.
- $k = \alpha\beta - r + 1$.

Theorem 1.3. *Let X be a projective algebraic manifold of complex dimension n with a kähler metric. Let L be a holomorphic line bundle on X with a smooth metric $e^{-\chi}$. Let V be a holomorphic vector bundle on X of rank r and let $h_{\alpha\bar{\beta}}$ be a smooth* *hermitian metric of* V*. Let* $g_1, \cdots, g_p$ *be holomorphic sections of* V *over* X *such that the matrix*

$$\begin{pmatrix} f & G \end{pmatrix} = \begin{pmatrix} f_1 & g_{1,1} & g_{2,1} & \cdots & g_{p,1} \\ f_2 & g_{1,2} & g_{2,2} & \cdots & g_{p,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_r & g_{1,r} & g_{2,r} & \cdots & g_{p,r} \end{pmatrix}$$

has generic rank m *(with* $p \geq r \geq m$*), and let* f *be a holomorphic section of* $V \otimes L$ *over* X*. Assume*

$$\sqrt{-1}\partial\bar{\partial}\chi + \mathrm{Ric}_X \geq \alpha\beta c_1(V),$$

where $\beta = \min\{n, p-m\}$ *and* $\alpha > 1$ *is a constant such that* $\alpha\beta$ *is an integer. If*

$$L_0 := \int_X \frac{\sum_{1\leq j_1<\cdots<j_{r-1}\leq p} \|f \wedge g_{j_1} \wedge \cdots \wedge g_{j_{m-1}}\|^2}{\left(\sum_{1\leq k_1<\cdots<k_m\leq p} \|g_{k_1} \wedge \cdots \wedge g_{k_m}\|^2\right)^{\alpha\beta-r+2}} \, \mathrm{d}V_X < \infty, \tag{9}$$

then there exists holomorphic sections $h_1, \cdots, h_p$ *of* L *over* X *such that*

$$f = \sum_{j=1}^{p} g_j h_j$$

on X *as sections of* $V \otimes L$ *and*

$$\int_X \frac{\|h_j\|^2}{\left(\sum_{1\leq k_1<\cdots<k_r\leq p} \|g_{k_1} \wedge \cdots \wedge g_{k_m}\|^2\right)^{\alpha\beta-r+1}} \, \mathrm{d}V_\Omega < \frac{\alpha}{\alpha-1} L_0.$$

Looking closely to the proof of the division theorems, the key is to solve a $\bar{\partial}$-equation

$$\bar{\partial}u = -\beta^* g^{-1} f,$$

with appropriate *a priori* estimates of $\|f\|^2$. The key techniques are based on Hörmander's wrok [4, 5]. In the proof of division theorems we can only get the estimates for certain forms. That means some special forms have 0 class in the cohomology group. Hence we can extend this idea to show a certain type of vanishing theorem by introducing appropriate conditions involving the curvature operator $\Theta_{k\bar{\ell}}$.

Theorem 1.4. *Let X be a weakly pseudo-convex manifold of dimension n and $E = X \times \mathbb{C}^p$ be a trivial vector bundle. Consider a pair of holomorphic vector bundles and a non-trivial holomorphic sub-vector bundle (S^s, E^p), then for the cohomology class represented by a $(0,k)$-form f satisfying*

$$\int_X \frac{(\widetilde{BB^*}f, f)}{\det(BB^*)(|g|^2)^N} \mathrm{d}V_X < \infty$$

vanishes for $k \geq 1$ and $N > \min\{n, s\}$. Note that the finiteness condition requires f in the domain of Θ^{-1}. For the case of $k = 1$ and $Z \subsetneq X$ we have a Cramer's rule type condition:

$$\int_X \frac{\sum |f \wedge \varphi_{I_1} \wedge \cdots \wedge \varphi_{I_{n-1}}|^2}{\sum |\varphi_{I_1} \wedge \cdots \wedge \varphi_{I_n}|^2 (|g|^2)^N} \mathrm{d}V_X < \infty.$$

The point of the vanishing theorem is that S does not require strict positiviy. It is allowed to equip a metric which has only semi-positive curvature. The allowance of semi-positivity comes from the division theorems. For classic vanishing theorems we refer to Demailly's survey paper[2].

Another direct application of L^2-Cramer's rule is to find an effective bound in Artin-Rees lemma. In the classic commutative algebra Artin-Rees lemma (c.f.[1]) is about the induced topology on the sub-module.

Lemma 1.5. *Let R be a Noetherian ring of dimension n, M and N be finite generated modules of R, then there exists some m_0 such that*

$$I^{m_0+r} M \cap N \subseteq I^r (I^{m_0} M \cap N).$$

In the classic statement, there is no discussion about the effective number of m_0. In this field, Huneke's [6] proved that there is an uniform bound which is independent of N. Nevertheless one of the most important features of the L^2-type division theorem is to get a numerical control on the vanishing order of the divisors which enable people to investigate the effective bound. Another advantage of analytic argument is that people can obtain the right estimates without taking a resolution on the ideal. This can avoid the hard control of blowing-ups. Ein and Lazarsfeld[3] have applied this feature to get the effective version of Nullstellensatz. We use the same idea and combine the effective Nullstellensatz with our vector bundle version of L^2 division theorem. With this approach, we can obtain an effective version of the Artin-Rees lemma, which generalizes Huneke's uniform bound result.

Theorem 1.6. *Let $N = (g_1, \cdots, g_p)$ be a finite generated module of*

$$M = \mathbb{C}\left[x_1, \cdots, x_n\right]^{\oplus r}.$$

Let w be the e.m.b-rank of N in M. Assume $\deg g_{i,j} \leq d$ for every i, j and

$$m_0 = (C_w^p - w + 2)(d^w)^n.$$

Then

$$I^{m_0+k} M \cap N \subset I^k(I^{m_0} M \cap N)$$

for every integer k and every finite generated ideal I of $\mathbb{C}\left[x_1, \cdots, x_n\right]$.

2. Projectivization Computation

In this section, we will focus on the technique of computing the integral of fiber in (7). First, let us fix the convention of the Euclidean volume form of $\mathbb{C}^n$. Denote

$$\mathrm{d}V_X = n!\Big(\frac{\sqrt{-1}}{2\pi}\Big)^n \mathrm{d}x_1 \wedge \mathrm{d}\bar{x}_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}\bar{x}_n.$$

Let us recall the integral

$$\int_{\mathbb{P}^{r-1}} \frac{|\sum_{i=1}^r f_i X_i|^2 (\sum h_{i\bar{j}} X_i \bar{X}_j)^k}{(\sum_{j=1}^p |\sum_{i=1}^r g_{ji} X_i|^2)^{r+k+1}} h_0 \,\mathrm{d}V_X. \tag{10}$$

Note that in the proof of the main theorem we let $k = \alpha\beta - r$.

The basic idea is to do the integral on a space that is more symmetric than $\mathbb{P}^{r-1}$. Consider the diagram

$$\begin{array}{ccc} U(r-1) & \longrightarrow & SU(r) \\ & & \big\downarrow \pi \\ & & \mathbb{P}(V^*), \end{array}$$

So we want to pull back the integrand to $SU(r)$ and do the computation over there. Recall the Fubini theorem of Lie groups

$$\begin{aligned} \int_{SU(r)} F(g)\,\mathrm{d}g &= \int_{SU(r)/U(r-1)} \Big(\int_{U(r-1)} F(gh)\,\mathrm{d}h \Big) \mathrm{d}(gH) \\ &= \int_{SU(r)/U(r-1)} F(\bar{g})\,\mathrm{d}(gH), \end{aligned}$$

if F is a function of the cosets $\bar{g} = gH$. The integral we want to compute is on the R.H.S, so we need to figure out the function F. Notice that the denominator in (10) is a Hermitian form, i.e.

$$\sum_{j=1}^{p} |\sum_{i=1}^{r} g_{ji} X_i|^2 = ZG(ZG)^\dagger = Z(GG^\dagger)Z^\dagger.$$

(Here we treat $Z = (z_1, \cdots, z_r)$ as a row vector.) Therefore we want to choose a basis $e_1, \cdots, e_r$ of V such that the Hermitian matrix

$$GG^\dagger = I,$$

i.e. becomes identity w.r.t the new basis. On the other hand under the action of $SU(r)$, the column vector Z^t can be understood as the translation of a constant vector $v = e_1 = (1, 0, \cdots, 0)^t$ by g, i.e.

$$Z^t = gv.$$

Then we can embed $\rho : SU(r) \longrightarrow GL(V)$ and make it unitary w.r.t. $\{e_1, \cdots, e_r\}$. Now we fix the basis of V and the representation ρ or $SU(R)$. Equip $\mathbb{P}(V^*)$ with new

Fubini-metric

$$GG^\dagger \text{ and its induced volume form } \underset{\cdot}{\mathrm{V}}_G = \frac{g_0}{(ZGG^\dagger Z^\dagger)^r}\mathrm{d}V_Z,$$

where $g_0 = \det(GG^\dagger)$. Then we can rewrite the integral (10) under this setting as

$$\int_{\mathbb{P}^{r-1}} \frac{|\sum_{i=1}^r f_i X_i|^2}{\sum_{j=1}^p |\sum_{i=1}^r g_{ji} X_i|^2} \Big(\frac{\sum h_{i\bar{j}} X_i \bar{X}_j}{\sum_{j=1}^p |\sum_{i=1}^r g_{ji} X_i|^2}\Big)^k \frac{h_0}{g_0}\,\mathrm{d}V_G.$$

Note that the first term $\frac{|L(X)|^2}{XGG^\dagger X^\dagger}$ and the second term $\frac{\sum h_{i\bar{j}} X_i \bar{X}_j}{XGG^\dagger X^\dagger}$ are well defined functions on $\mathbb{P}^{r-1}$, especially for the first term, its nominator $L(X) = X.f$ is an linear functional on V^*. After change of basis, we can express it as

$$L(Z) = (Z, w),$$

where $w = Af$ is the new expression w.r.t. Z. Therefore, let

$$F(Z) = \frac{|L(Z)|^2}{ZZ^\dagger}\Big(\frac{\|Z\|_H^2}{ZZ^\dagger}\Big)^k, \text{ i.e. } F(g) = |L(gv)|^2 \|gv\|_H^2,$$

then $F(gh) = F(g)$ for every $h \in U(r-1)$, which is the desired integrand we are looking for. Hence we can conclude

Lemma 2.1.

$$\int_{\mathbb{P}^{r-1}} \frac{|\sum_{i=1}^r f_i X_i|^2 (\sum h_{i\bar{j}} X_i \bar{X}_j)^k}{(\sum_{j=1}^p |\sum_{i=1}^r g_{ji} X_i|^2)^{r+k+1}} h_0\,\mathrm{d}V_X = \frac{h_0}{g_0}\int_{SU(r)} |(gv, w)|^2 (\|gv\|_H^2)^k\,\mathrm{d}g.$$

For the future usage, let us define and list some notations.

Definition 2.2. *Let f be a vector in V^* and $H = (h_{i\bar{j}})$ be a Hermitian metric. Define*

$$\|f^\perp\|_H^2 = \sum_{i<j} \big(|f_j|^2 h_{i\bar{i}} + |f_i|^2 h_{j\bar{j}} - 2Re\, f_j \overline{f_i h_{i\bar{j}}}\big).$$

- $\mu = \sum_{1\le k_1 < \cdots < k_r \le p} |g_{k_1} \wedge \cdots \wedge g_{k_r}|^2.$

- $\sigma = \sum_{1\le j_1<\cdots<j_{r-1}\le p} |f\wedge g_{j_1}\wedge\cdots\wedge g_{j_{r-1}}|^2$
- $\tau = \sum_{m,n}\sum_{i,j,k}\sum_{(r-2)} \left| f_i|G^{(r-2)}_{(m,n)',(i,k)'}| - f_j|G^{(r-2)}_{(m,n)',(j,k)'}|\right|^2 h_k.$
- $\|\wedge^k g\|^2 = \sum_{1\le j_1<\cdots<j_k\le p} \|g_{j_1}\wedge\cdots\wedge g_{j_i}\|^2.$

Now we are ready to do the computation. First, we will present an abstract argument to prove the general statement, then two examples would follow. These two examples would show the main points of the proof. In the computation we will call some linear algebra lemmas of which the proof would be given in the next section.

Theorem 2.3. *Let rank $\{g_{ij}\} = r$.*

$$\int_{\mathbb{P}^{r-1}} \frac{|\sum_{i=1}^r f_iX_i|^2(\sum h_{i\bar{j}}X_i\bar{X}_j)^k}{(\sum_{j=1}^p|\sum_{i=1}^r g_{ji}X_i|^2)^{r+k+1}} h_0\,\mathrm{d}V_X = |f|^2A(h) - \|f^\perp\|_H^2 B(h)$$
$$= \sum_{i=1}^{k+2}\frac{\sigma h_0}{\mu^i}A_i(g,h) - \sum_{i=1}^{k+1}\frac{\tau h_0}{\mu^i}B_i(g,h),$$

where $A(h)\ge 0, B(h)\ge 0$ are semi-positive symmetric functions of entries of H and $h_0 = \det H$. Moreover, the leading term $a_{k+2}h_0\sigma/\mu^{k+2}$ is

$$\frac{(k+1)!}{r(r+1)\cdots(r+k)}\frac{\sum_{1\le i_1<\cdots<i_{r-1}\le p}|(f,i_1,\cdots,i_{r-1})|^2}{\left(\sum_{1\le i_1<\cdots<i_r\le p}|(i_1,\cdots,i_r)|^2\right)^{k+2}}\|\wedge^{r-1}g\|^{2(k+1)}(\det H)^{k+2},$$

and the top order of the leading coefficient of $\|f^\perp\|_H^2$ is $1/\mu^{k+1}$.

Remark 1. Though in the following application we only need $k=0$ case, we still want to develop the technique for the general situation for the future use.

Proof. By the above lemma and lemma 3.1, we can formulate the integral as

$$\int_{\mathbb{P}^{r-1}} \frac{|\sum_{i=1}^r f_iX_i|^2(\sum h_{i\bar{j}}X_i\bar{X}_j)^k}{(\sum_{j=1}^p|\sum_{i=1}^r g_{ji}X_i|^2)^{r+k+1}} h_0\,\mathrm{d}V_X = \frac{h_0}{\mu}\int_{SU(r)} |(gv,w)|^2(\|gv\|_H^2)^k\,\mathrm{d}g.$$

Like the explanation before lemma 2.1, let $V = \mathbb{C}^r$ be the standard $SU(r)$ representation, and denote it by

$$\square .$$

Let $\{e_1, \cdots, e_r\}$ be the basis of V such that $(GG^\dagger)^t$ is identity, or we can say that $(V, \{e_i\})$ is an irreducible unitary representation of $SU(r)$. Let $v = (1, 0, \cdots, 0) \in V$ and $w = (f_1', \cdots, f_r')$ be the expression of f w.r.t. $\{e_1, \cdots, e_r\}$.

NOTE. In the following passage, in order to simplify the notation we would abuse notation with f' and f. Hence when we write $|f|^2$ we really mean $|f'|^2$, and so for f_i (which should be f_i').

In the following we assume that

$$H = \begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_r \end{pmatrix},$$

i.e. the metric is diagonal. Though it is not the general case, the process is the same in doing the general one. Under this assumption

$$\|gv\|_H^2 = \sum_{i=1}^{r} |(gv, e_i)|^2 h_i = |\lambda_1|^2 h_1 + \cdots + |\lambda_r|^2 h_r.$$

So

$$\left(\|gv\|_H^2\right)^k = \sum_{\alpha_1 + \cdots + \alpha_r = k} \binom{k}{\alpha_1 \cdots \alpha_r} |\lambda_1|^{2\alpha_1} \cdots |\lambda_r|^{2\alpha_r} h_1^{\alpha_1} \cdots h_r^{\alpha_r}.$$

Introduce this expression into the integral:

$$\sum_{\alpha_1 + \cdots + \alpha_r = k} \binom{k}{\alpha_1 \cdots \alpha_r} \int_{SU(r)} |(gv, w)|^2 |(gv, e_1)|^{2\alpha_1} \cdots |(gv, e_r)|^{2\alpha_r} h_1^{\alpha_1} \cdots h_r^{\alpha_r} \mathrm{d}g.$$

Hence we have to decompose the tensor power representation $\square^{k+1}$ into irreducible representations. By simple rules of Young tableau, we know

$$\square^{k+1} = \underbrace{\square\square\cdots\square}_{k+1} + (\cdots).$$

Note that $\underbrace{v \otimes \cdots \otimes v}_{k+1}$ is in the symmetrized mode $\underbrace{\square\square\cdots\square}_{k+1}$. Hence $(\cdots)$ can be neglected in the computation, and the important thing is we know the new basis of the sub $SU(r)$-invariant subspace V_1 of V, which has components

$$U_{i_0,\cdots,i_k} = \sum_{\sigma\in S_r} e_{\sigma(i_0)} \otimes \cdots \otimes e_{\sigma(i_k)} \text{ (mod 2)},$$

where $i_\ell \in \{1, 2, \cdots, r\}$. Since $\{e_1, \cdots, e_r\}$ are orthogonal, $\{U_{i_0,\cdots,i_k}\}$ are also orthogonal. Note that the number i_ℓ in the subindex could be repeated α'_ℓ times and do not have to obey any order. By the definition, it is easy to see that

$$|U_{i_0,\cdots,i_k}|^2 = \binom{k+1}{\alpha'_0 \cdots \alpha'_k}$$

The remaining issue is to express $e_{i_0} \otimes \cdots \otimes e_{i_k}$ in terms of $U_{i_0,\cdots,i_k}$.

Claim 1.

$$(k+1)!\, e_{i_0} \otimes \cdots \otimes e_{i_k} = \alpha'_0! \cdots \alpha'_k!\, U_{i_0,\cdots,i_k} + (\cdots).$$

First we assume $i_0, \cdots, i_k$ are distinct. Suppose

$$\square^{k+1} = \sum_\lambda d_\lambda Y_\lambda,$$

where Y_λ are rank $k+1$ Young tableau. It is easy to see that $d_{\square\square\cdots\square} = 1$ and $a_{1,\cdots,1} = 1$ by induction. Assume

$$a\, e_{i_0} \otimes \cdots \otimes e_{i_k} = \sum a_{j_0,\cdots,j_k} U_{j_0,\cdots,j_k}.$$

What we have to do is to figure out a. In particular this identity holds for $\{j_0, \cdots, j_k\} = \{1, \cdots, 1\}$, i.e.

$$a = |U_{1,\cdots,1}| = (k+1)!.$$

For the repeated situation, we have to multiply the repeated times. Therefore the factor $\alpha'_0! \cdots \alpha'_k!$ appears on the R.H.S. Then the claim 1 is proved.

Remark 2. By above identity we got an extra combinatorial identity

$$\sum d_i^2 = (k+1)!.$$

Conventionally, d_i is called Clebsch-Gordan coefficients.

With the aid of the claim the V_1-component of $e_{i_0} \otimes \cdots \otimes e_{i_k}$ can be figured out explicitly

$$\frac{1}{\binom{k+1}{\alpha'_0 \cdots \alpha'_k}} U_{i_0,\cdots,i_k}.$$

Therefore the V_1 part projection formula of integral is

$$\sum_{\alpha_1+\cdots+\alpha_r=k} \binom{k}{\alpha_1 \cdots \alpha_r} \int_{SU(r)} |(gU_{1,\cdots,1}, \frac{\sum_{i_0} f_{i_0}}{\binom{k+1}{\alpha'_0 \cdots \alpha'_k}} U_{i_0,\cdots,i_k})|^2 h_1^{\alpha_1} \cdots h_r^{\alpha_r} \, \mathrm{d}g$$
$$= \frac{(k+1)!}{r(r+1)\cdots(r+k)} \sum_{\alpha_1+\cdots+\alpha_r=k} \sum_{i_0=1}^{r} |f_{i_0}|^2 \frac{\binom{k}{\alpha_1 \cdots \alpha_r}}{\binom{k+1}{\alpha'_0 \cdots \alpha'_k}} h_1^{\alpha_1} \cdots h_r^{\alpha_r} \frac{\det H}{\mu}.$$

Note that the factor deg H/μ comes in because we applied change of variable to make

$$\mathrm{d}V \to \mathrm{V}_G.$$

The relation between α_i and α'_i is defined as follow

$i_0 \notin \{i_1, \cdots, i_k\}$**:** Then define $\alpha'_0 = 1$ and $\alpha'_i = \alpha_i$.

$i_0 = i_\ell$**:** Then define $\alpha'_0 = 0$, $\alpha'_i = \alpha_i$ if $i \neq \ell$, and $\alpha'_\ell = \alpha_\ell + 1$.

By neglecting j such that $\alpha_j = 0$ we can reformulate the summation as

$$\frac{(k+1)!}{r(r+1)\cdots(r+k)}\sum_{\ell=1}^{k}\sum_{\substack{1\le i_1<\cdots<i_\ell\le r\\ \alpha_1+\cdots+\alpha_\ell=k}}\sum_{j=1}^{\ell}\frac{\binom{k}{\alpha_1\cdots\alpha_\ell}}{\binom{k+1}{\alpha_0'\alpha_1'\cdots\alpha_\ell'}}|f_{i_j}|^2 h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell}.$$

Fix the length ℓ we can break the summation into two parts.

$i_0 = i_j$: $\displaystyle\sum_{i_1<\cdots<i_\ell}\sum_{j=1}^{\ell}\frac{\alpha_{i_j}+1}{k+1}|f_{i_j}|^2 h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell}.$

$i_0 \neq i_j$: $\displaystyle\sum_{i_0\neq i_j}\sum_{i_1<\cdots<i_\ell}\frac{1}{k+1}|f_{i_0}|^2 h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell},$

In order to show the result we have to look at these two terms more closely.

Claim 2: Fix ℓ, $i_1,\cdots,i_\ell$ and $\alpha_1,\cdots,\alpha_\ell$, then

1) $\displaystyle\sum_{i_1\neq\cdots\neq i_\ell}\sum_{j=1}^{\ell}|f_{i_j}|^2 h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell} = |f|^2 A_s(h) + \|f^\perp\|_H^2 B_s(h)$

2) $\displaystyle\sum_{i_0\neq\cdots\neq i_\ell}|f_{i_0}|^2 h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell} = |f|^2 A_a(h) + \|f^\perp\|_H^2 B_a(h),$

for some symmetric functions $A_*(h)$ and $B_*(h)$. The statement of the theorem follows easily by the claim. So let us show these two identities now.

First of all we want to show that case (1) can be reduced to case (2). Rewrite the summation (1) as

$$\sum_{i_1\neq\cdots\neq i_\ell}\sum_{j=1}^{\ell}|f_{i_j}|^2 h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell} = \sum_{i_0}|f_{i_0}|^2\sum_{i_1\neq\cdots\neq i_\ell}h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell} - \sum_{i_0\neq\cdots\neq i_\ell}|f_{i_0}|^2 h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell},$$

because on the L.H.S. f_{i_j} is one of $\{i_1,\cdots,i_\ell\}$ and what on the R.H.S. we have to exclude $i_0 = i_j$ for some $j\in\{1,\cdots,\ell\}$ which is the last term. And the first term is a symmetric function if the subindex i of $\{\alpha_i\}$ permuted by the symmetric group $S(\ell)$:

$$\sum_{\sigma\in S(\ell)}\sum_{i_0}|f_{i_0}|^2\sum_{i_1\neq\cdots\neq i_\ell}h_{i_1}^{\alpha_{\sigma(1)}}\cdots h_{i_\ell}^{\alpha_{\sigma(\ell)}} = |f|^2 A_s(h),$$

or in short

$$(1) = |f|^2 A_s(h) - (2). \tag{11}$$

Therefore we turn to consider (2).

We show the statement by induction. If $h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell}$ is the common factor, we can pull it out and sum over the permutation acted by $S(\ell)$ which makes (2) equal to

$$\sum_{i_0}|f_{i_0}|^2\sum_{\sigma\in S(\ell)}\sum_{i_1\neq\cdots\neq i_\ell}h_{i_1}^{\alpha_{\sigma(1)}}\cdots h_{i_\ell}^{\alpha_{\sigma(\ell)}}=|f|^2A_a(h).$$

Suppose the common factor is $h_{i_1}^{a_1}\cdots h_{i_\ell}^{a_\ell}$ and $a_1+\cdots+a_\ell=k'$. Let the degree of $h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell}$ be $d=k-k'$. So we can assume $d>0$. (If not then we are done.) Suppose $d=1$. Then after dropping the common factor (2) reads

$$\sum_{i_0\neq i_1}|f_{i_0}|^2h_{i_1}=\|f^\perp\|_H^2$$

by definition. So the statement is valid when $d=1$. Now we assume the statement is effective when the degree $d<k$ and $h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell}$ does not have any common factor. Then we have can do the factorization by

$$\begin{aligned}&\sum_{i_0\neq\cdots\neq i_\ell}|f_{i_0}|^2h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell}=\sum_{i_0\neq\cdots\neq i_\ell}|f_{i_0}|^2h_{i_\ell}h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell-1}\\ &=\Big(\sum_{i_0\neq i_{\ell+1}}|f_{i_0}|^2h_{i_{\ell+1}}\Big)\Big(\sum_{i_1\neq\cdots\neq i_\ell}h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell-1}\Big)-\sum_{i_0\neq\cdots\neq i_{\ell+1}}|f_{i_0}|^2h_{i_1}^{\alpha_1}\cdots h_{i_\ell}^{\alpha_\ell-1}h_{i_{\ell+1}}.\end{aligned}\tag{12}$$

The first term equals to $\|f^\perp\|_H^2B_a(h)$ after summing over $S(\ell)$, and the second term can be factored out a common factor $h_{i_1}\cdots h_{i_\ell}h_{i_{\ell+1}}$. Hence the degree drops, then we can apply the induction hypothesis to complete the argument. This ends the proof of the claim 2.

There is a subtle observation. In dealing with (2), we take out h_{i_ℓ} to pair with $|f_{i_0}|^2$, which can produce a $\|f^\perp\|_H^2$ term. Hence the symmetric function $B_a(h)$ has one degree less than the initial. This is the reason why the top order of $\|f^\perp\|_H^2$ is $k+1$ instead of $k+2$.

The last issue is to assure the sign of terms within the factor $\|f^{\perp}\|_H^2$. Go back to (11). Since the $\|f^{\perp}\|_H^2 B_a(h)$ component of (1) comes from (2) we only need to compare the coefficients between them. By collecting α-type of (1), (11) can be rewritten as

$$\begin{aligned}\frac{\alpha+1}{k+1}(1) &= |f|^2 A_s(h) - \frac{\alpha+1}{k+1}(2)\\ &= |f|^2 A_s(h) - \frac{\alpha+1}{k+1}|f|^2 A_a(h) - \frac{\alpha+1}{k+1}\|f^{\perp}\|_H^2 B_a(h).\end{aligned}$$

And (12) can be expressed as

$$\frac{1}{k+1}(2) = \frac{1}{k+1}|f|^2 A_a(h) + \frac{1}{k+1}\|f^{\perp}\|_H^2 B_a(h).$$

Therefore, by fixing the length ℓ we can pair a α-type (1) with its asymmetric apart (2) to give

$$\frac{\alpha+1}{k+1}(1) + \frac{1}{k+1}(2) = |f|^2 A(h) - \frac{\alpha}{k+1}\|f^{\perp}\|_H^2 B_a(h),$$

which explains the negative sign of component within $\|f^{\perp}\|_H^2$.

So we know the answer of the integral is in the form of

$$\frac{h_0}{\mu}\big(|f|^2 A(h) - \|f^{\perp}\|_H^2 B(h)\big),$$

where $A(h)$ is a symmetric function of h. If we apply the above algorithm to the case $\ell = 1$, we can see the leading terms of $A(h)$ is

$$(\text{Tr } H)^{k+1} = (h_1 + \cdots + h_r)^{k+1}. \tag{13}$$

In fact by careful algebra, we can confirm the leading constant coefficient is

$$\frac{(k+1)!}{r(r+1)\cdots(r+k)}.$$

Applying the linear algebra lemmas in the next section, every symmetric function of h has intrinsic meaning. For instance

$$\mathrm{Tr}\ H = \frac{\| \wedge^{r-1} g\|^2}{\mu}.$$

In addition, $|f|^2$ has intrinsic meaning as well

$$|f|^2 = \frac{\sigma}{\mu},$$

which is the Cramer's rule matrices. Combing with (13) we can conclude the highest power of μ should be $k+2$. Therefore we finish the proof. $\square$

Let us present two examples promised before. The purpose of the first one is to demonstrate the role of decomposition of representation in computing the integral.

Proposition 2.4. *Let $r \geq 2$,*

$$\begin{aligned}\int_{\mathbb{P}^{r-1}} \frac{|\sum_{i=1}^r f_i X_i|^2 (\sum h_{i\bar{j}} X_i \bar{X}_j)}{(\sum_{j=1}^p |\sum_{i=1}^r g_{ji} X_i|^2)^{r+2}} h_0 \, \mathrm{d}V_X &= \frac{1}{r(r+1)} \frac{h_0}{\mu} (2|f|^2 . \ \mathrm{Tr}\ H - \|f^\perp\|_H^2) \\ &= \frac{1}{r(r+1)} \frac{h_0}{\mu} \big(2\frac{\sigma}{\mu^2} \| \wedge^{r-1} g\|^2 - \|f^\perp\|_H^2 \big).\end{aligned}$$

Proof. Let V be the standard representation of $SU(r)$, and denote it by the Young Tableau $\square$. Suppose $\{e_i\}_{1 \leq i \leq r}$ be the standard basis of V. It induces a tensor basis $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}$ of $\otimes^n V$. For short we will denote the tensor basis by $\{u_{i_1} v_{i_2} w_{i_3} \cdots\}$. Then let us consider the tensor representation decomposition of $\otimes^n V$. For instance, $V \otimes V$ can be decomposed as following

$$\square \times \square = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} + \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array},$$

where

1) $\boxed{i \,|\, j} = u_iv_j + u_jv_i$. If $i \neq j$ denote $U^+_{ij} = u_iv_j + u_jv_i$. Let $U_i = u_iv_i$. This is the symmetric mode. We mark this vector space by V_2 and the dimension is $r(r+1)/2$.

2) $\begin{array}{|c|}\hline i \\ \hline j \\ \hline\end{array} = u_iv_j - u_jv_i = U^-_{ij}$, which is the asymmetric mode for $i < j$. We denote the space by V_0 and the dimension is $r(r-1)/2$.

Let us go back to the integral $\displaystyle\int_{\mathbb{P}^{r-1}} \frac{|\sum_{i=1}^r f_iX_i|^2(\sum h_{i\bar{j}}X_i\bar{X}_j)}{(\sum_{j=1}^p|\sum_{i=1}^r g_{ji}X_i|^2)^{r+2}}\mathrm{d}V_X$, where $\{X_i\}$ are the homogenous coordinates of $\mathbb{P}^{r-1}$. We pull back this function to $SU(r)$ and change the basis to diagonalize and normalize $(GG^\dagger)^t$ if necessary (i.e. $(GG^\dagger)^t = I$), then the integral reads

$$\int_{SU(r)} |(gv, w)|^2 \|gv\|^2_H \,\mathrm{d}g,$$

where $v = (1, 0, \cdots, 0)$, so $gv = (X_1, \cdots, X_r), w = (f_1, \cdots, f_r) \in V$. First let us expand $\|gv\|^2_H$ to simplify the calculation. Write

$$\|gv\|^2_H = \sum_{i=1}^r |\lambda_i|^2 h_{i\bar{i}} + \sum_{i<j} 2Re\,\lambda_i\bar{\lambda}_j h_{i\bar{j}} = \Sigma_1 + \Sigma_2,$$

where $\lambda_i = (gv, e_i)$. So the integral becomes a question of decomposition of tensor product of representation. By the Young Tableaux above we know the proper decomposition of $\{u_iv_j\}_{1\leq i,j\leq r}$ should be

$$\{u_1v_1, \cdots, u_rv_r, u_1v_2 + u_2v_1, \cdots, u_iv_j + u_jv_i, \cdots\} \oplus \{u_1v_2 - u_2v_1, \cdots, u_iv_j - u_jv_i, \cdots\}$$
$$= V_2 \oplus V_0.$$

Note that in the above expression we always assume $i < j$, but since U^+_{ij} is symmetric to i, j we can remove the constrain on U^+_{ij}. According to the new basis we need to figure out the expression of $v \otimes v$ and $w \otimes e_i$ for every i. Since $v = e_1$ there is no further work for $v \otimes v = u_1v_1$. For $w \otimes e_i = \sum_j f_je_j \otimes e_i$ we have

$$w \otimes e_i = f_iU_i + \sum_{j\neq i} \frac{1}{2} f_j(U^+_{ij} - \epsilon_{ij}U^-_{ij}) = f_iU_i + \sum_{j\neq i}\frac{1}{2}f_jU^+_{ij} + (\cdots),$$

where ϵ_{ij} is the Levi-Civita tensor. Here we only keep track on the part of V_2 is because $v \otimes v = u_1 v_1 \in V_2$. So the integral is only the matter of projection of norms and inner products onto each V_i. So

Σ_1: $\sum_{i=1}^{r} \frac{2}{r(r+1)} \big(|f_i|^2 h_{i\bar{i}} + \sum_{j \neq i} \frac{1}{2} |f_j|^2 h_{i\bar{i}}\big),$

Σ_2: $\sum_{i \neq j}^{r} \frac{2}{r(r+1)} \big(\frac{1}{4} \overline{(f_j U_{ij}^+, f_i U_{ij}^+)} h_{i\bar{j}}\big) = \sum_{i \neq j} \frac{2}{r(r+1)} \frac{1}{2} \bar{f}_j f_i h_{i\bar{j}}.$

We combine these two together to get

$$\frac{2}{r(r+1)} \Big(\sum_i |f_i|^2 \sum_i h_{i\bar{i}} - \frac{1}{2} \big(\sum_{j \neq i} |f_j|^2 h_{i\bar{i}} - 2Re \sum_{i<j} \bar{f}_j f_i h_{i\bar{j}} \big) \Big)$$

$$= \frac{2}{r(r+1)} \Big(|f|^2 . \operatorname{Tr} H - \frac{1}{2} \sum_{i<j} \big(|f_j|^2 h_{i\bar{i}} + |f_i|^2 h_{j\bar{j}} - 2Re\, f_j \overline{f_i h_{i\bar{j}}} \big) \Big).$$

By our previous defined notation

$$\sum_{i<j} \big(|f_j|^2 h_{i\bar{i}} + |f_i|^2 h_{j\bar{j}} - 2Re\, f_j \overline{f_i h_{i\bar{j}}} \big) = \|f^\perp\|_H^2$$

which has an intrinsic meaning by lemma 3.8. Hence we got the formula. □

Remark 3. By this computation, we get a series of inequalities. Since the integral must be semi-positive, we have

$$|f|^2 A(h) - \|f^\perp\|_H^2 B(h) \geq 0.$$

The purpose of the second example is to demonstrate the algorithm under a concrete setting. The notations are coherent to the ones in the proof of theorem 2.3.

Example 2.5. *Let* $H = \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}$, $f = (f_0, f_1)$.

$$\int_{\mathbb{P}^1} \frac{|f_0' + f_1' z|^2 (\|(1,z)\|_{h'}^2)^4}{(\sum |b_i + a_i z|^2)^7} \mathrm{d}V_z = \frac{\det h'}{\mu} \Big(|f|^2 \big(\frac{1}{6} t^4 - \frac{2}{5} t^2 d + \frac{d^2}{10} \big) - \|f^\perp\|_H^2 \big(\frac{2}{15} t^3 - \frac{1}{5} t d \big) \Big),$$

where $t = \operatorname{Tr} H$ *and* $d = \det H$.

Note that f', h' in the integral are different from f, h on the R.H.S. What on the R.H.S are the corresponding terms after change of coordinate (which normalizes $(GG^{\dagger})^{t}$.)

We can easily see from this example that the top order of $\|f^{\perp}\|_{H}^{2}$ is 3 which is exactly one less then the order of $|f|^2$. The reason is also very clear by the following inductive computation. Generalize this process then we get the abstract argument in the proof of last theorem.

Proof. $k = 4, r = 2$ implies the leading constant coefficient is

$$\frac{5!}{2 \cdot 3 \cdots 6} = \frac{1}{6}.$$

Since $r = 2$, the length ℓ has only two possible combinatorics: $\ell = 1$ and $\ell = 2$.

$\ell = 1$: There are only one type in this case, i.e. $\alpha = 4$. The corresponding terms are h_0^4 and h_1^4. Then we put f into the pictures. if the subindex i of f_i matches the subindex j of h_j we get

$$\frac{4+1}{4+1}|f_0|^2 h_0^4 + \frac{4+1}{4+1}|f_1|^2 h_1^4 = (|f_0|^2 h_0^4 + |f_1|^2 h_1^4).$$

If $i \neq j$, we turn to get

$$\frac{1}{4+1}|f_1|^2 h_0^4 + \frac{1}{4+1}|f_0|^2 h_1^4 = \Big(\frac{1}{5}|f_1|^2 h_0^4 + \frac{1}{5}|f_0|^2 h_1^4\Big).$$

$\ell = 2$: There are three types in this case: $(3,1), (1,3)$ and $(2,2)$ We apply the same procedure to all of them. Then we get three pairs

$$\Big(\frac{4}{5}|f_0|^2 h_0^3 h_1 + \frac{4}{5}|f_1|^2 h_0 h_1^3\Big) + \Big(\frac{2}{5}|f_1|^2 h_0^3 h_1 + \frac{2}{5}|f_0|^2 h_0 h_1^3\Big)$$
$$+ \frac{3}{5}(|f_0|^2 + |f_1|^2) h_0^2 h_1^2.$$

By factoring out the common factor for each pair, there are only four kinds of summation: $|f_0|^2h_0^4+|f_1|^2h_1^4, |f_1|^2h_0^4+|f_0|^2h_1^4, |f_0|^2h_0^2+|f_1|^2h_1^2$ and $|f_1|^2h_0^2+|f_0|^2h_1^2$. Now we apply the inductive argument to decompose them into $|f|^2$ and $\|f^\perp\|_H^2$ components.

- $|f_0|^2h_0^2 + |f_1|^2h_1^2 = (|f_0|^2 + |f_1|^2)(h_0^2 + h_1^2) - (|f_1|^2h_0^2 + |f_0|^2h_1^2)$ which can be reduced to the next case.
- $|f_1|^2h_0^2 + |f_0|^2h_1^2 = (|f_0|^2h_1 + |f_1|^2h_0)(h_1 + h_0) - (|f_0|^2h_0h_1 + |f_1|^2h_0h_1) = \|f^\perp\|_H^2 t - |f|^2 d$.
- $|f_0|^2h_0^4 + |f_1|^2h_1^4 = (|f_0|^2 + |f_1|^2)(h_0^4 + h_1^4) - (|f_1|^2h_0^4 + |f_0|^2h_1^4)$ which can be reduced to the next case.
- $|f_1|^2h_0^4 + |f_0|^2h_1^4 = (|f_0|^2h_1 + |f_1|^2h_0)(h_1^3 + h_0^3) - (|f_0|^2h_0^3h_1 + |f_1|^2h_0h_1^3)$.
  $= \|f^\perp\|_H^2(h_1^3 + h_0^3) - (|f_0|^2h_0^2 + |f_1|^2h_1^2)h_0h_1$
  $= \|f^\perp\|_H^2(h_1^3 + h_0^3) - |f|^2(h_0^2 + h_1^2)h_0h_1 + \|f^\perp\|_H^2(h_1 + h_0)h_0h_1 - |f|^2h_0^2h_1^2$.

Expanding the pairs by the rules, we can compute the integral by

$$
\begin{aligned}
&\frac{1}{6}\Big(|f|^2(h_0^4 + h_1^4) + \frac{8}{5}|f|^2(h_0^2 + h_1^2)h_0h_1 + \frac{9}{5}|f|^2h_0^2h_1^2 \\
&- \frac{4}{5}\|f^\perp\|_H^2(h_1^3 + h_0^3) - \frac{6}{5}\|f^\perp\|_H^2(h_1 + h_0)h_0h_1\Big) \\
&= |f|^2A(h) - \|f^\perp\|_H^2 B(h),
\end{aligned}
$$

where $A(h) = 1/6((h_0^4 + h_1^4) + \frac{8}{5}(h_0^2 + h_1^2)h_0h_1 + \frac{9}{5}h_0^2h_1^2)$ and $B(h) = 1/6(\frac{4}{5}(h_1^3 + h_0^3) + \frac{6}{5}(h_1 + h_0)h_0h_1)$. Symmetrize $A(h)$ and $B(h)$, then we get the answer

$$
A(h) = \frac{1}{6}t^4 - \frac{2}{5}t^2 d + \frac{d^2}{10}, \text{ and } B(h) = \frac{2}{15}t^3 - \frac{1}{5}td.
$$

□

3. Linear Algebra Lemmas

In this sections, we provide the proof of some technical lemmas. First let us introduce some notations.

- $\mu = \sum_{1\le i_1<\cdots<i_r\le p} |g_{i_1}\wedge\cdots\wedge g_{i_r}|^2$, and
- $\mu_m = \sum_{1\le i_1<\cdots<i_m\le p} |g_{i_1}\wedge\cdots\wedge g_{i_m}|^2$.
- $\sigma = \sigma(f) = \sum_{1\le k_1<\cdots<k_{r-1}\le p} |f\wedge g_{k_1}\wedge\cdots\wedge g_{k_{r-1}}|^2$, and
- $\sigma_m = \sigma_m(f) = \sum_{1\le k_1<\cdots<k_{m-1}\le p} |f\wedge g_{k_1}\wedge\cdots\wedge g_{k_{m-1}}|^2$.

Lemma 3.1.

$$\det(GG^\dagger) = \mu.$$

Proof. First consider the case that G is a square matrix. Then it is obvious to see

$$\det(GG^\dagger) = \det G\cdot\overline{\det G} = |\det G|^2 = |vol(G)|^2 = \mu.$$

For the general case, change the basis by a unitary transformation U and reduce to the square matrix case. □

Lemma 3.2. *Let V be a vector space of dimension p. Let $e_1,\cdots,e_p$ be a basis of V, and equip V with the standard metric h. Let $a_1,\cdots,a_r$ be linear independent vectors in V, and has expression*

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_r \\ | & & | \end{pmatrix} = \begin{pmatrix} a_{11} & & a_{r1} \\ \vdots & \cdots & \vdots \\ a_{1p} & & a_{rp} \end{pmatrix}$$

w.r.t $e_1,\cdots,e_p$. Denote $W = \langle a_1,\cdots,a_r\rangle$ the sub-vector space spanned by a_i. Suppose

$$f(X) = f(X_1a_1+\cdots+X_ra_r) = \sum_i f_iX_i$$

is a linear functional on W, then

$$\|f\|_h^2 = \frac{\sum_{1\le i_1<\cdots<i_{r-1}\le p} |(f, A_{i_1},\cdots,A_{i_{r-1}})|^2}{\sum_{1\le i_1<\cdots<i_r\le p} |A_{i_1},\cdots,A_{i_r}|^2},$$

where A_i is the r-th row of A. Note that the expression is in the form of Cramer's law.

Remark that f' in the following paragraph is NOT consistent to the previous usage, f is the same as before and $A = G$.

Proof. First let's exam the full rank case, i.e. $p = r$. Since a_i are linear independent we can make the change of coordinate to e_i and get $f(X') = \sum_i f_i' X_i'$ w.r.t. e_i, then

$$|f|_h^2 = \sum_i |f_i'|^2,$$

which is exactly the same process in the proof of $k = 3$ case. More precisely

$$\begin{pmatrix} X_1' \\ \vdots \\ X_r' \end{pmatrix} = \begin{pmatrix} a_{11} & & a_{r1} \\ \vdots & \ddots & \vdots \\ a_{1r} & & a_{rr} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix}$$

and

$$f = \begin{pmatrix} f_1 & \cdots & f_r \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_r \end{pmatrix} = fA^{-1}AX = \begin{pmatrix} f_1' & \cdots & f_r' \end{pmatrix} \begin{pmatrix} X_1' \\ \vdots \\ X_r' \end{pmatrix}.$$

So $f = f'A$ and $|f'|^2 = |fA^{-1}|^2$, which is exactly the Cramer's law.

For the general case, i.e. $p > r.$, We can assume $a_1, \cdots, a_r = \lambda_1 e_1, \cdots, \lambda_r e_r$ by an unitary action if necessary. So the matrix A' can be expressed as

$$\begin{pmatrix} a_{11} & & a_{r1} \\ \vdots & \ddots & \vdots \\ a_{1r} & & a_{rr} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

and we can apply the full rank case to conclude that $|f|^2 = \sum_1 / \sum_2$. The denominator $\sum_2 = \det A'$ can be interpreted as the r-volume of $\langle a_1, \cdots, a_r \rangle$ in $\bigwedge^r V$, so the general expression is

$$\sum_{1 \le i_1 < \cdots < i_r \le p} |A_{i_1}, \cdots, A_{i_r}|^2.$$

For the nominator, $\sum_1$ can be interpreted as the difference of volumes as following. Extend $a_i \in V$ to $\bar{a}_i = (f_i, a_i) \in \bar{V} = e_0 \oplus V$. Then the Cramer's rule can be viewed as the difference of r-th volume of $\{\bar{a}_i\}$ and the r-th volume of $\{a_i\}$. So the general formula can be naturally expressed as

$$\sum_{1 \le i_1 < \cdots < i_{r-1} \le p} |(f, A_{i_1}, \cdots, A_{i_{r-1}})|^2.$$

Hence we are done. $\square$

Lemma 3.3. *Let V be a vector space of dimension p. Let $e_1, \cdots, e_p$ be a basis of V, and equip V with the standard metric g. Let $a_1, \cdots, a_r$ be linear independent vectors in V, and has expression*

$$A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_r \\ | & & | \end{pmatrix} = \begin{pmatrix} a_{11} & & a_{r1} \\ \vdots & \cdots & \vdots \\ a_{1p} & & a_{rp} \end{pmatrix}$$

w.r.t $e_1, \cdots, e_p$. Denote $W = \langle a_1, \cdots, a_r \rangle$ the sub-vector space spanned by a_i. Then g can be expressed as

$$\begin{pmatrix} |a_1|^2 & \bar{a}_1 a_2 & \cdots & \bar{a}_1 a_r \\ a_1 \bar{a}_2 & |a_2|^2 & \cdots & \bar{a}_1 a_r \\ \vdots & \vdots & \ddots & \vdots \\ a_1 \bar{a}_r & a_1 \bar{a}_2 & \cdots & |a_r|^2 \end{pmatrix}$$

w.r.t $\{a_i\}$. *Let* T *be a linear transformation such that* $\{T(a_i)\}$ *are orthonormal. Suppose there is another metric* $h = (h_{i\bar{j}})$. *Then*

$$\mathrm{Tr}\left((T^{-1})^{\dagger} H T^{-1}\right) = \frac{1}{\mu} \sum_j \| \bigoplus_{1 \le i_1 < \cdots < i_{r-1} \le p} \sum_{(r-1)} |(a_{i_1}^{\hat{j}}, \cdots, a_{i_{r-1}}^{\hat{j}})|^2 \|_h^2$$

where $\sum_{(r-1)} |(a_{i_1}^{\hat{j}}, \cdots, a_{i_{r-1}}^{\hat{j}})|^2$ *runs through all* $r-1$ *sub-matrices (disregarded* j*-th column) of* $(a_{i_1}, \cdots, a_{i_{r-1}})$ *which is a* $p \times (r-1)$ *matrix.*

Proof. Similar to the previous argument, first we deal with the full rank case, i.e. $r = p$. So we can assume T brings $\{a_i\}$ to $\{e_i\}$. Hence $T = A$. On the other hand compute

$$\mathrm{Tr}\,(B^{\dagger} H B) = \sum_{\ell} \sum_{i,j} \overline{b_{i\ell}} h_{i\bar{j}} b_{j\ell} = \sum_{\ell} \|b_\ell\|_h^2,$$

where b_ℓ is the ℓ-th column vector of B. In our case $B = A^{-1}$, then

$$\mathrm{Tr}\left((A^{-1})^{\dagger} H A^{-1}\right) = \frac{\sum_\ell \|\hat{A}_\ell\|_h^2}{|\det A|^2},$$

where $\hat{A}_\ell$ is the ℓ-th column vector of the adjoint matrix of A. Write $\hat{A}_{k\ell}$ explicitly

$$\hat{A}_{k\ell} = (-1)^{k+\ell} \det(a_1, \cdots, \widehat{a_\ell}, \cdots, a_r)_{\hat{k}},$$

where $(a_1, \cdots, \widehat{a_\ell}, \cdots, a_r)_{\hat{k}}$ is the sub $r-1$ matrix which skips k-th row and ℓ-th column. This is exactly the form we expect.

For the general case, i.e. $p > r$, by mimicking the argument in the proof of lemma 3.2 we interpret $\det A$ as the $r-$th volume of $a_1 \wedge \cdots \wedge a_r$ and $\sum \|\hat{A}\|_h^2$ as the summation of $(r-1)$-th volume of $a_{i_1} \wedge \cdots a_{i_{r-1}}$ for all $1 \le i_1 < i_2 < \cdots < i_{r-1} \le p$. Then the formula follows. □

Lemma 3.4. *Let*

$$G = \begin{pmatrix} g_{1,1} & g_{2,1} & \cdots & g_{p,1} \\ g_{1,2} & g_{2,2} & \cdots & g_{p,2} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1,r} & g_{2,r} & \cdots & g_{p,r} \end{pmatrix}, \text{ and } S = \begin{pmatrix} s_{1,1} & g_{2,1} & \cdots & g_{r,1} \\ g_{1,2} & g_{2,2} & \cdots & g_{r,2} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1,q} & g_{2,q} & \cdots & g_{r,q} \end{pmatrix}$$

such that

$$\begin{pmatrix} s_{1,i} & s_{2,i} & \cdots & s_{r,i} \end{pmatrix} \begin{pmatrix} g_{j,1} \\ g_{j,2} \\ \vdots \\ g_{j,r} \end{pmatrix} = 0. \tag{14}$$

Suppose the rank of G is m, then

$$\frac{((S^\dagger S + GG^\dagger)^{-1} \det(S^\dagger S + GG^\dagger) f, f)}{\det(S^\dagger S + GG^\dagger)} = \frac{\sigma_m}{\mu_m}.$$

Proof. By choosing a proper basis we can assume

$$G = \begin{pmatrix} g_{1,1} & g_{2,1} & \cdots & g_{m,1} \\ g_{1,2} & g_{2,2} & \cdots & g_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1,r} & g_{2,r} & \cdots & g_{m,r} \end{pmatrix}, \text{ and } S = \begin{pmatrix} s_{1,1} & s_{2,1} & \cdots & s_{r,1} \\ s_{1,2} & s_{2,2} & \cdots & s_{r,2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,q} & s_{2,q} & \cdots & s_{r,q} \end{pmatrix}$$

such that $q = r - m$ and $\{g_1, \cdots, g_m, s_1, \cdots, s_q\}$ would form a basis of a vector space V of dimension r. Note that the compatible conditions (14) implies that V can be decomposed into the subspace generated by $W = \langle g_1, \cdots, g_m \rangle$ and $U = \langle s_1, \cdots, s_q \rangle$, i.e.

$$V = W \oplus U.$$

Let us turn to consider the matrix $A = S^\dagger S + GG^\dagger$. In order to get a better expression of A we choose $\Lambda = \{g_1, \cdots, g_m, s_1, \cdots, s_q\}$ as a new set of basis, then we can compute

$$A \cdot g_j = GG^\dagger \cdot \begin{pmatrix} (g_1, g_j) \\ (g_2, g_j) \\ \vdots \\ (g_m, g_j) \end{pmatrix} = \sum_{i=1}^{m} g_i (g_i, g_j).$$

Similarly we have

$$A \cdot s_j = S^\dagger S \cdot \begin{pmatrix} (s_j, s_1) \\ (s_j, s_2) \\ \vdots \\ (s_j, s_m) \end{pmatrix} = \sum_{i=1}^{q} s_i (s_j, s_i).$$

Therefore we can rewrite A w.r.t Λ as

$$\begin{pmatrix} (g_1, g_1) & \cdots & (g_1, g_m) & 0 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ (g_m, g_1) & \cdots & (g_m, g_m) & & & 0 \\ 0 & & & (s_1, s_1) & \cdots & (s_q, s_1) \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 0 & (s_1, s_q) & \cdots & (s_q, s_q) \end{pmatrix}.$$

Then

$$\begin{aligned} \det A = & \begin{vmatrix} (g_1, g_1) & \cdots & (g_1, g_m) \\ \vdots & \ddots & \vdots \\ (g_m, g_1) & \cdots & (g_m, g_m) \end{vmatrix} \cdot \begin{vmatrix} (s_1, s_1) & \cdots & (s_q, s_1) \\ \vdots & \ddots & \vdots \\ (s_1, s_q) & \cdots & (s_q, s_q) \end{vmatrix} \\ & \cdot |g_1 \wedge \cdots \wedge g_r \wedge s_1 \wedge \cdots \wedge s_q| \\ = & \mu_m \cdot |g_1 \wedge \cdots \wedge g_r \wedge s_1 \wedge \cdots \wedge s_q|. \end{aligned}$$

On the other hand, we can write f w.r.t Λ as

$$\begin{pmatrix} f_1' \\ \vdots \\ f_m' \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

by the compatible conditions. Hence

$$\begin{aligned}(A^{-1}\det Af, f) &= \sigma_r(f)\cdot|g_1\wedge\cdots\wedge g_r\wedge s_1\wedge\cdots\wedge s_q| \\ &= \sigma_m(f)\cdot|g_1\wedge\cdots\wedge g_r\wedge s_1\wedge\cdots\wedge s_q|,\end{aligned}$$

which implies

$$\frac{(A^{-1}\det Af, f)}{\det A} = \frac{\sigma_m}{\mu_m}.$$

This ends the proof. ☐

Corollary 3.5. *Following all the assumptions as above, we have*

$$\begin{aligned}&\frac{((S^\dagger S+GG^\dagger)^{-1}\det(S^\dagger S+GG^\dagger)f, g)}{\det(S^\dagger S+GG^\dagger)} \\ &= \frac{1}{\mu_m}\sum_{1\le i_1<\cdots<i_{m-1}\le p}\big((f\wedge g_{i_1}\wedge\cdots g_{i_{m-1}}), \overline{(g\wedge g_{i_1}\wedge\cdots g_{i_{m-1}})}\big).\end{aligned}$$

In particular $f=g$, we get the result σ_m/μ_m.

Proof. The same argument as the proof of the lemma. ☐

Remark 4. In particular, if h is diagonal, i.e. $h=(\delta_i^j h_j)$, then

$$\mathrm{Tr}\left((T^{-1})^\dagger HT^{-1}\right) = \frac{1}{\mu}\sum_j h_j\Big|\sum_{1\le i_1<\cdots<i_{r-1}\le p}\sum_{(r-1)}|(a_{i_1}^{\hat{j}},\cdots,a_{i_{r-1}}^{\hat{j}})|^2\Big|^2.$$

Lemma 3.6. *Let A be a non-singular $r \times r$ matrix, and $B = A^{-1}$. Suppose $I = \{i_1, \cdots, i_k\}, J = \{j_1, \cdots, j_k\} \subset R = \{1, \cdots, r\}$, and $k \leq r$. Let $A_{I,J}$ be the $k \times k$ sub-matrix of A,*

$$A_{I,J} = \begin{pmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_k} \\ \vdots & & \vdots \\ a_{i_k,j_1} & \cdots & a_{i_k,j_k} \end{pmatrix}.$$

Denote I' be the complement set of I in R. Then

$$\det B_{I,J} = (-1)^{|i|+|j|} \frac{\det(A^T)_{I',J'}}{\det A}.$$

Proof. Given a matrix $A = (a_{ij})$ we can construct vectors $a_i = \sum a_{ij} e_j$, where e_i are standard basis of $\mathbb{C}^r$. Denote the inverse matrix by B, and we have $e_i = b_{ij} a_j$. Hence

$$e_{i_1} \wedge \cdots \wedge e_{i_k} = B_{I,J} e_{j_1} \wedge \cdots \wedge e_{j_k} + (\cdots).$$

In order to single out the coefficient $B_{I,J}$ we wedge $a_{j'_1} \wedge \cdots \wedge a_{j'_{r-k}}$ on both sides, where $\{j'_1, \cdots, j'_{r-k}\} = J'$. Then we get

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge a_{j'_1} \wedge \cdots \wedge a_{j'_{r-k}} = B_{IJ} a_{j'_1} \wedge \cdots \wedge a_{j'_r} = (-1)^{j_1 + \cdots + j_k - k} B_{I,J} \det A.$$

On the other hand

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge a_{j'_1} \wedge \cdots \wedge a_{j'_{r-k}} = \begin{vmatrix} 0 & \cdots & 1_{i_1} & \cdots & \cdots & 0 \\ 0 & \cdots & & 1_{i_2} & \cdots & 0 \\ 0 & & \ddots & & 1_{i_k} & 0 \\ a_{j'_1,1} & & \cdots & & a_{j'_1,r-1} & a_{j'_1,r} \\ \vdots & & \ddots & & & \vdots \\ a_{j'_{r-k},1} & & \cdots & & a_{j'_{r-k},r-1} & a_{j'_{r-k},r} \end{vmatrix},$$

where the sub-indicts indicate the positions in the matrix. So the determinant is naturally to be

$$(-1)^{i_1+\cdots+i_k-k}\det(A)_{J',I'} = (-1)^{i_1+\cdots+i_k-k}\det(A^T)_{I',J'}.$$

Hence we are done. ∎

Applying the above observations, we can get the general expression of $\operatorname{Tr}\,(\bigwedge^k H)$.

Corollary 3.7. *The notations follow lemma 3.3 and 3.6. Let* $0 < k < r$.

$$\operatorname{Tr}\Big(\bigwedge^k ((T^{-1})^\dagger H T^{-1})\Big) = \frac{1}{\mu}\sum_I \| \bigoplus_{\{j_1,\cdots,j_{r-k}\}\in I'} \sum_{(r-k)} |(a^{\hat{I}}_{j_1},\cdots,a^{\hat{I}}_{j_{r-k}})|^2\|^2_{H_k},$$

where H_k is the induced metric on $\bigwedge^k V$. Note that the order (w.r.t. g_{ij}) of $\operatorname{Tr}\,(\bigwedge^k H)$ *is $2(r-k)$ which is less than the order of μ (which is $2r$). For $k = r$,*

$$\det H = \frac{\det h}{\mu}.$$

Proof. Applying the fact that $\operatorname{Tr}\big(\bigwedge^k B\big) = \sum_{|I|=k}\det B_{I,I}$, then the result is straight forward. ∎

Lemma 3.8. *Let* $r \geq 3$.

$$\|f^\perp\|^2_H = \frac{\tau}{\mu},\ \text{and}\ \tau = \sum_{m,n}\sum_{i,j,k}\sum_{(r-2)} \Big| f_i |G^{(r-2)}_{(m,n)',(i,k)'}| - f_j |G^{(r-2)}_{(m,n)',(j,k)'}|\Big|^2 h_k.$$

Note that the order (w.r.t. g_{ij}) of nominator is $2(r-2)$ which is less than the order of μ (which is $2r$). For $r = 2$ case,

$$\|f^\perp\|^2_H = \frac{(|f_1|^2 h_0 + |f_0|^2 h_1)}{\mu}.$$

Proof. The proof is by direct computation. ∎

4. Main Theorem

In the computation of previous section, the integral reduced to the local expression

$$\int_{\mathbb{C}} \frac{|f_0 + f_1 z|^2 (\sum h^*_{k\bar{\ell}} z_k \bar{z}_\ell)^{\alpha\beta}}{(\sum_i |g_{i0} + g_{i1} z|^2)^{(\alpha\beta+1)} e^{(\alpha\beta+1)\eta}} \mathrm{d}V.$$

Let the weight function

$$e^{-\phi} = (\sum h_{k\bar{\ell}} z_k \bar{z}_\ell)^{\alpha\beta} e^{-(\alpha\beta+1)\eta}.$$

For convenience we denote the coordinate of the total space by

$$w = (w_1, \cdots, w_n, w_{n+1}, \cdots, w_{n+r-1}) = (x_1, \cdots, x_n, z_1, \cdots, z_{r-1}).$$

Let us look at the problem more closely. If we apply the line bundle version of Skoda's theorem directly on $P = \mathbb{P}(V^*)$, then we will encounter a trouble of negativity coming from the weight function

$$\left(\sum h_{k\bar{\ell}} z_k \bar{z}_\ell\right)^{\alpha\beta}.$$

In order to take care of this, we need to apply a mixed metric function. Instead of using $1/\|\tilde{g}\|^{\alpha\beta-1}$ we apply

$$\frac{1}{\|\tilde{g}\|^r \mu^{\alpha\beta-r}},$$

then we can handle the extra negativity. First let us do the reduction and can assume $X = \Omega \subseteq \mathbb{C}^N$ be a pseudo-convex domain, and the vector bundle is given by

$$V = \Omega \times \mathbb{C}^r \text{ with a hermitian metric } h_{i\bar{j}}.$$

Then V induces a projective scheme

$$P = \Omega \times \mathbb{P}^{r-1},$$

which is a direct product space and is also a pseudo-convex domain by construction. Let us list the setting bellow

- Let $(X, g_{i\bar{j}})$ be a pseudo-convex domain equipped with a given Hermitian metric.
- Let (V, h) be a vector bundle over X of rank r with a given Hermitian metric $h_{k\bar{\ell}}$, and define $\Phi = \sum h^*_{k\bar{\ell}} z_k \bar{z}_\ell$.
- Let $(P, \tilde{g}_{k\bar{\ell}})$ be the product space and $\tilde{g}_{k\bar{\ell}}$ be the product metric of $g_{i\bar{j}}$ and g_{FS} which is defined by $\sqrt{-1}\partial_k \bar{\partial}_\ell \Phi$.
- Let $(M, e^{-\chi})$ be a line bundle on X, and $(K, e^{-\kappa})$ be the canonical line bundle on X, where $e^{-\kappa} = 1/\det(g_{i\bar{j}})$ is a natural metric of K which is induced by the metric of X.
- Let $(\mathcal{O}_{V^*}(1), e^{-\tilde{\varphi}})$ be the $\mathcal{O}(1)$ bundle on P, and $e^{-\tilde{\varphi}} = 1/\Phi$.
- Let $e^{-\eta}$ be a metric of line bundle $\mathcal{O}_{V^*}(1) \otimes M^*$ over P.
- Let $G : \bigoplus M \otimes K \longrightarrow V \otimes K$ be generic surjective. (Later we will remove this assumption.)

In order to solve the division problem

$$G : \bigoplus M \otimes K \longrightarrow V \otimes K,$$

we projectivize V and consider the new division problem

$$\tilde{G} : \bigoplus M \otimes K \longrightarrow \mathcal{O}_{V^*}(1) \otimes K$$

on $P = \mathbb{P}(V^*)$. Note that in order to simplify the expressions here we abuse notations $\pi^* K$ (resp. $\pi^* M$) with K (resp. M) and its pull back metric $\pi^* e^{-\kappa}$ (resp. $\pi^* e^{-\chi}$) with $e^{-\kappa}$ (resp. $e^{-\chi}$). Since we only care about the smooth sections of $\mathcal{O}_{V^*}(1) \otimes K$ in the form of

$$\tilde{f} = \sum f_i(x) z_i$$

we don't need the full power strength of $\bar{\partial}$ estimate on P. In stead we take only partial estimate. More precisely we consider the following Hilbert spaces

- $H_0 = \bigoplus \pi^* L^2(P, M \otimes K) = \bigoplus_{p-\text{copies}}$ the completion of $\{\tilde{h} \mid \tilde{h}$ is the smooth section of $M\}$ w.r.t. the metric of $e^{-\varphi_0}$. Define

$$e^{-\varphi_0} = \frac{1}{r} \frac{1}{(\|\tilde{g}\|^2)^r} \frac{1}{\mu^{\alpha\beta - r}}.$$

- $H_1 = L^2(P, \mathcal{O}_{V^*}(1), e^{-\tilde{\varphi}})$. Note that H_2 contains $\{\tilde{f} = \sum f_i(x) z_i \big| (f_0, \cdots, f_{r-1})$ is a section of V $\}$. Define

$$e^{-\varphi_1} = \frac{1}{\Phi} \frac{1}{(\|\tilde{g}\|^2)^{r+1}} \frac{1}{\mu^{\alpha\beta - r}}.$$

- $H_2 = \bigoplus \pi^* L^2_{(0,1)}(P, M \otimes K) = \bigoplus_{p-\text{copies}}$ the completion of $\{\tilde{h} = \sum h_{\bar{m}}(x)\, \mathrm{d}\bar{x}_m \mid$ is the smooth section of $\Omega^{(0,1)}(M)\}$ w.r.t. the metric of $e^{-\varphi_0 - \kappa}$, where g is the kahler metric on P.

and the diagram

$$\begin{array}{ccc} H_0 & \xrightarrow{\tilde{G}} & H_1 \\ \Big\downarrow \bar{\partial} & & \\ H_2 & & \end{array}.$$

Then we can do the Skoda estimates in this framework. The key is to compute the conjugation $\tilde{G}$. Given a section $u = (u_1, \cdots, u_r)$ it associates a section $\tilde{u} = \sum_i u_i z_i$ of $\mathcal{O}_P(1)$. The key point is to compute the conjugate $\tilde{G}^* \tilde{u}$. Suppose $h = (h_1, \cdots, h_p)$

is a section of E. Compute

$$\int_P \overline{\sum_{j=1}^p \tilde{g}_j h_j} \tilde{u} e^{-\varphi_1} \mathrm{d}V_P = \sum_{j=1}^p \int_P \overline{\tilde{g}_j h_j} \tilde{u} e^{-\varphi_1} \mathrm{d}V_P$$

$$= \sum_{j=1}^p \int_\Omega \bar{h}_j \int_{\mathbb{P}^{r-1}} \frac{\sum_{k,\ell} \overline{g_{j,k} z_k} u_\ell z_\ell \Phi^{r+1}}{\Phi(|\tilde{g}|^2)^{r+1}} \frac{1}{\mu^{\beta-r}} \,\mathrm{d}V_{FS} \wedge \mathrm{V}_\Omega$$

$$= \sum_{j=1}^p \int_\Omega \frac{\bar{h}_j}{\mu^{\beta-r}} \int_{\mathbb{P}^{r-1}} \frac{\sum_{k,\ell} (u_k \bar{g}_{j,\ell}) z_k \bar{z}_\ell}{|\tilde{g}|^2} \frac{1}{\mu} \,\mathrm{d}V_G \wedge \mathrm{V}_\Omega.$$

The crucial part is to compute the integral

$$\int_{\mathbb{P}^{r-1}} \frac{\sum_{k,\ell} (u_k \bar{g}_{j,\ell}) z_k \bar{z}_\ell}{|\tilde{g}|^2} \,\mathrm{d}V_G = \int_{SU(r)} \|ge\|_H^2 \,\mathrm{d}g,$$

where $H = (h_{k,\ell}) = \sum_{k,\ell} (u_k \bar{g}_{j,\ell})$. By our developed average technique, this term can be computed explicitly

$$\int_{SU(r)} \|ge\|_H^2 \,\mathrm{d}g = \frac{1}{r\mu} \operatorname{Tr} \left((A^{-1})^\dagger H A^{-1} \right),$$

where $A = G^t$. By 3.5 this term can be expressed as

$$\frac{1}{\mu} \sum_{1 \le i_1 < \cdots < i_{r-1} \le p} \begin{vmatrix} u_1 & g_{i_1,1} & \cdots & g_{i_{r-1},1} \\ u_2 & g_{i_1,2} & \cdots & g_{i_{r-1},2} \\ \vdots & \vdots & \ddots & \vdots \\ u_r & g_{i_1,r} & \cdots & g_{i_{r-1},r} \end{vmatrix} \overline{\begin{vmatrix} g_{j,1} & g_{i_1,1} & \cdots & g_{i_{r-1},1} \\ g_{j,2} & g_{i_1,2} & \cdots & g_{i_{r-1},2} \\ \vdots & \vdots & \ddots & \vdots \\ g_{j,r} & g_{i_1,r} & \cdots & g_{i_{r-1},r} \end{vmatrix}},$$

so

$$\int_P (G^* \tilde{u})_j e^{-\varphi_0} \mathrm{d}V_p$$

$$= \int_\Omega \frac{1}{r\mu^{\beta-r+2}} \sum u \wedge g_{i_1} \wedge \cdots \wedge g_{i_{r-1}} \overline{g_j \wedge g_{i_1} \wedge \cdots \wedge g_{i_{r-1}}} \,\mathrm{d}V_\Omega$$

(15) $$= \int_P \frac{1}{\mu} \sum u \wedge g_{i_1} \wedge \cdots \wedge g_{i_{r-1}} \overline{g_j \wedge g_{i_1} \wedge \cdots \wedge g_{i_{r-1}}} e^{-\varphi_0} \mathrm{d}V_P.$$

Once we have the expression of conjugation $G^*\tilde{u}$, we can carry out the remaining estimates. Consider

$$\|G^*\tilde{u}+\bar{\partial}^*v\|^2 = \|\bar{\partial}^*v\|^2_{(1)} + \|G^*\tilde{u}\|^2_{(2)} + 2Re\int_P (\bar{\partial}G^*\tilde{u}, v)e^{-\varphi_0}\mathrm{d}V_{P\,(3)}. \tag{16}$$

Part (1) would be combined with Bochner-Kodaira formula later. This would associate to the curvature condition. Part (2) is about the norm of $\tilde{u}$. Compute this term explicitly, then we get

$$\begin{aligned}
\|G^*\tilde{u}\|^2 &= \sum_{j=1}^{p}\int_P |G^*\tilde{u}_j|^2 e^{-\varphi_0}\mathrm{d}V_P \\
&= \sum_{j=1}^{p}\int_P \frac{1}{\mu^2}\sum\left|u\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}}\overline{g_j\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}}}\right|^2 e^{-\varphi_0}\mathrm{d}V_P \\
&= \int_\Omega \frac{\sigma}{\mu^{\beta-r+2}}\mathrm{d}V_\Omega = \int_P \|\tilde{u}\|^2 e^{-\varphi_1}\mathrm{d}V_P.
\end{aligned}$$

Note that σ/μ is the expression of the Cramer's rule (c.f the list of symbols). For part (3), it involves the inequality of Skoda-type, and we can write this term explicitly

$$2Re\int_P \sum u\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}}\overline{\partial_k(g_j\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}})v_j^k}e^{-\varphi_0}\mathrm{d}V_P.$$

Combine (16) with the Bochner-Kodaira formula, we have

$$\begin{aligned}
\|\tilde{G}^*\tilde{u}+\bar{\partial}^*v\|^2 + \|\bar{\partial}v\|^2 = &\int_P \|\tilde{u}\|^2 e^{-\varphi_1}\mathrm{d}V + 2Re\int_P(\cdots)\mathrm{d}V \\
&+ \int_P(\partial_k\partial_{\bar{\ell}}\varphi_0)v^k\bar{v}^\ell e^{-\varphi_0}\mathrm{d}V + \int_P(\mathrm{Ric}\,_P)_{k\bar{\ell}}v^k\bar{v}^\ell e^{-\varphi_0}\mathrm{d}V \\
&+ \int_P\|\bar{\nabla}v\|^2 e^{-\varphi_0}\mathrm{d}V + \int_{\partial P}(\partial_k\partial_{\bar{\ell}}\rho)v^k\bar{v}^\ell e^{-\varphi_0}\mathrm{d}V.
\end{aligned}$$

In this identity $\int_P v_j^k\bar{v}_j^\ell(\partial_k\partial_{\bar{\ell}}\varphi_0)e^{-\varphi_0}\mathrm{d}V$ is the term containing positive components that can take over other negative terms. We can rewrite the essential components into three parts:

$$\underbrace{\sum_j r\int_P \partial_k\partial_{\bar\ell}(\log|\tilde g|^2-\log\Phi)v_j^k\bar v_j^\ell e^{-\varphi_0}\mathrm{d}V}_{(a)}+\underbrace{\int_P(\mathrm{Ric}_P-\mathrm{Ric}_X)_{k\bar\ell}u_j^k\bar u_j^\ell e^{-\varphi_0}\mathrm{d}V}_{(b)}$$
$$+\underbrace{(\beta-r)\int_P\partial_k\partial_{\bar\ell}(\log\mu)u_j^k\bar u_j^\ell e^{-\varphi_0}\mathrm{d}V}_{(c)}.$$

In part (a), there contains a negative term $-r\int_P\partial_k\partial_{\bar\ell}(\log\Phi)v_j^k\bar v_j^\ell e^{-\varphi_0}\mathrm{d}V$ which comes from the process of projectivization. We need this factor to carry out the average. Fortunately, part (b) reads

$$\mathrm{Ric}_P-\mathrm{Ric}_X=r\sqrt{-1}\partial_k\bar\partial_\ell\log\Phi-c_1(V)=r\sqrt{-1}\partial_k\bar\partial_\ell\log\Phi$$

(in our case $c_1(V)=0$), and it can take care of the negative part of (a). The remaining terms in the estimate are

$$\underbrace{\sum_{k,\ell=1}^n(\partial_k\bar\partial_\ell\log\mu)v_j^k\bar v_j^\ell}_{(c)},\ \text{ and } 2\Big|\sum u\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}}\overline{\partial_k(g_j\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}})v_j^k}\Big|_{(d)}.$$

Hence things are reduced to an inequality.

Lemma 4.1. *For any constant $\alpha\geq 1$,*

$$\frac{1}{\alpha\mu}\sum_{1\le i_1<\cdots<i_{r-1}\le p}\left|u\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}}\right|^2+\alpha\beta_1\sum_{j=1}^p\underbrace{\sum_{k,\ell=1}^n(\partial_k\bar\partial_\ell\log\mu)v_j^k\bar v_j^\ell}_{(c)}$$
$$\geq 2\left|\sum_{1\le i_1<\cdots<i_{r-1}\le p}\ \sum_{\substack{1\le j\le p\\1\le k\le n}}u\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}}\overline{\partial_k\left(\frac{g_j\wedge g_{i_1}\wedge\cdots\wedge g_{i_{r-1}}}{\mu}\right)v_j^k}\right|_{(d)},\tag{17}$$

where $\beta_1=\min\{p-m,n\}$.

This inequality can be verified directly by combinatorics, but here we would like to show that it is related to Skoda's fundamental inequality.

Lemma 4.2. *Let $q = \min\{n, p - r\}$. For all smooth testing $(n, 1)$ form v and $(1, 0)$ form β we have*

$$q(\sqrt{-1}\ \mathrm{Tr}\ \beta\beta^*\Lambda v, v) \geq |\beta \lrcorner v|^2.$$

Note that $|\beta \lrcorner v|^2$ can be calculated by

$$|\beta \lrcorner v|^2 = -(\sqrt{-1}\beta^*\beta\Lambda v, v).$$

Proof. The key point of the lemma is to show the identity

$$\partial\bar{\partial} \log |g|^2 = \ \mathrm{Tr}\ \beta \wedge \beta^*,$$

which involves the explicit algebraic computation. We will demonstrate it in section 6. □

Before we unveil the relation, Let us make a digression. There is another way to derive the inequality 5.1. We want to apply the following setting. Consider the diagrams of Hilbert spaces and linear operators

$$\begin{array}{ccc} H_0 & \xrightarrow{T} & H_1 \\ \big\downarrow{\scriptstyle D} & & \\ H_2 & & \end{array}$$

where T is continuous and D is closed and has dense domain. Let G_1 be a closed subspace of H_1, then

Lemma 4.3.

$$T(\mathrm{Ker} D) = G_1$$

iff there exists some $c > 0$ such that

$$\|T^*x_1 + D^*x_2\|^2_{H_0} \geq c\|x_1\|^2_{H_1} \tag{18}$$

for every $x_1 \in G_1$ and $x_2 \in \mathrm{Dom} D^*$.

In the case of $r > m$, we have the following setting.

- $H_0 = L^2(\Omega, E, H_0)$, $E = \mathbb{C}^p$. Here we abuse notation H_0. It represents the space and the metric at the same time.
- $H_1 = L^2(\Omega, V, H_1')$, $V = \mathbb{C}^r$, where H_1' is the quotient metric of H_0,
- $H_2 = L^2(\Omega, Q)$, $Q = \mathbb{C}^q$,
- $H_3 = L^2_{0,1}(\Omega, E)$
- T_1 is a linear map described by a $p \times r$ matrix G
- T_2 is a linear map described by a $r \times q$ matrix S
- $D = \bar{\partial}$.

and the diagram reads

$$\begin{array}{ccccc} H_0 & \xrightarrow{T_1} & H_1 & \xrightarrow{T_2} & H_2 \\ \Big\downarrow D & & & & \\ H_3 & & & & \end{array}$$

and $G_1 = \text{Ker}T_2$. Note that $S \cdot G = 0$ means S is the compatible conditions. Since we choose quotient metric H_1',

$$T_1^* x_1 = T^{-1} x_1$$

for $x_1 \in G_1$, where $T_1^{-1}u$ is the minimum solution of

$$u = Gh$$

w.r.t. the compatible conditions S. Hence $T_1^* x_1$ can be expressed as

$$T_1^{-1} x_1 = G^*(S^*S + GG^*)^{-1} x_1,$$

where G^*, S^* are adjoint of G, S w.r.t. some givien *a priori* metric H_1 on V, and (18) reads

$$\|G^*(S^*S + GG^*)^{-1} f + \bar{\partial}^* v\|^2 \geq c\|f\|^2,$$

where $f \in H_1$. Like the line bundle case, we expand this expression

$$\|G^*(S^*S + GG^*)^{-1}f + \bar{\partial}^* v\|_\varphi^2 + \|\bar{\partial} v\|_\varphi^2 = \|T_1^* f\|_\varphi^2 + \|\bar{\partial}^* v\|_\varphi^2 + \|\bar{\partial} v\|_\varphi^2 + 2Re(G^*(S^*S + GG^*)^{-1}f, \bar{\partial}^* v)_\varphi$$

and couple it with the Kodaira-Bochner formula, then we get

$$\geq (\sqrt{-1}\partial\bar{\partial}\varphi v, v)_\varphi + 2Re(\bar{\partial} G^*(S^*S + GG^*)^{-1}f, v)_\varphi + \|f\|_{H_1'}^2.$$

Note that $e^{-\varphi}$ is a weight function to be determined. At this point if we can show

$$\frac{1}{\alpha}\|f\|_{H_1'}^2 + (\sqrt{-1}\partial\bar{\partial}\varphi v, v)_\varphi \geq 2|(\bar{\partial} G^*(S^*S + GG^*)^{-1}f, v)_\varphi| \tag{19}$$

then by lemma 4.3 we are done. By corollary 3.5, we have

$$(\bar{\partial} G^*(S^*S + GG^*)^{-1}f, v) = \sum_{1 \leq i_1 < \cdots < i_{m-1} \leq p} \sum_{\substack{1 \leq j \leq p \\ 1 \leq k \leq n}} \left(f \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}, \overline{\partial_k \left(\frac{g_j \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}}{\mu_m} \right) v_j^k} \right)$$

and $\|f\|_{H_1'}^2 = \|T_1^{-1} f\|_{H_0}^2 = \dfrac{1}{\mu_m} \displaystyle\sum_{1 \leq i_1 < \cdots < i_{m-1} \leq p} \left| u \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}} \right|^2.$

Now we want to compute the quotient metric H_1' in terms of the given *a priori* metric H_1 of V. Consider

$$\begin{aligned} \|u\|_{H_1'}^2 = \|T_1^{-1}u\|_{H_0}^2 &= (G^*(GG^* + S^*S)^{-1}u, G^*(GG^* + S^*S)^{-1}u)_{H_0} \\ &= ((GG^* + S^*S)^{-1}u, GG^*(GG^* + S^*S)^{-1}u)_{H_1} \\ &= ((GG^* + S^*S)^{-1}u, u)_{H_1}, \end{aligned} \tag{20}$$

and it induces a metric on $\det V$, i.e.

$$\det(H_1') = \frac{1}{\det(GG^* + S^*S)} \det(H_1)$$

on $\det V$. Thus the inequality (19) is exactly the content of the inequality 5.1, so there is an invariant expression of the term $(\bar{\partial}G^*(S^*S + GG^*)^{-1}f, v) = (\bar{\partial}T_1^{-1}f, v)$. At this moment we assume $r = m$ and later will remove this restriction. Let β be the second fundamental form of V in E, then

$$\bar{\partial}G^*(GG^*)^{-1}f = -\beta^* f.$$

Hence we have

$$\begin{aligned} |(\bar{\partial}T_1^{-1}f, v)|^2 &= |(\beta^* T_1^{-1}f, v)|^2 = |(T_1^{-1}f, \beta \lrcorner v)|^2 \\ &\leq |T_1^{-1}f|^2 \cdot |\beta \lrcorner v|^2 \end{aligned}$$

which implies

$$2|(\bar{\partial}T_1^{-1}f, v)| = 2A = 2\sqrt{|A|^2} \leq \frac{1}{\alpha}|T_1^{-1}f|^2 + \alpha|\beta \lrcorner v|^2,$$

so inequality 5.1 is reduced to show

$$\begin{aligned} |\beta \lrcorner v|^2 &\leq \beta_1 \sum_{j=1}^{p} \sum_{k,\ell=1}^{n} (\partial_k \bar{\partial}_\ell \log \mu) v_j^k \bar{v}_j^\ell \\ &= \beta_1 \sqrt{-1}\ \mathrm{Tr}\ \beta\beta^* = \beta_1 \sqrt{-1}\Theta(\det Q), \end{aligned}$$

which is precisely the content of 4.2. We can choose

- $\varphi = \log \mu^{\alpha\beta_1}$, which implies
- $\|h\|_{H_0} = \displaystyle\int_\Omega \frac{\|h\|_E^2}{\mu^{\alpha\beta_1}} \mathrm{d}V$ and
- $\|f\|_{H_1'} = \|T_1^{-1}f\|_{H_0}^2 = \displaystyle\int_\Omega \frac{\|f\|_{H_1}^2}{\mu^{\alpha\beta_1+1}} \mathrm{d}V$,

then lemma 5.1 is a indexized version of 4.2. For $r > m$ case, let us recall the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & E^p & \xrightarrow{T_1} & V^r & \xrightarrow{T_2} & \mathbb{C}^q \\ & & & & & \searrow & \uparrow & & \\ & & & & & & W^m & & \end{array} ,$$

where W is a rank m sub-bundle of V^r. In this setting we need a generalized inequality as follows.

Lemma 4.4. *For any constant $\alpha \geq 1$,*

$$\frac{1}{\alpha\mu_m} \sum_{1\leq i_1<\cdots<i_{m-1}\leq p} \left|u \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}\right|^2 + \alpha\beta_1 \sum_{j=1}^{p} \sum_{k,\ell=1}^{n} (\partial_k \bar{\partial}_\ell \log \mu_m) v_j^k \bar{v}_j^\ell \tag{21}$$

$$\geq 2 \left| \sum_{1\leq i_1<\cdots<i_{m-1}\leq p} \sum_{\substack{1\leq j\leq p \\ 1\leq k\leq n}} \left(u \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}, \overline{\partial_k \left(\frac{g_j \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}}{\mu_m} \right) v_j^k} \right) \right|.$$

where $\beta_1 = \min\{p - m, n\}$, where

$$\mu_m = \sum_{1\leq i_1<\cdots<i_m\leq p} \left|g_{i_1} \wedge \cdots \wedge g_{i_m}\right|^2.$$

Proof. This inequality can be verified by combinatoric computation which we will present in section 5. □

Once the fundamental inequality is established, we have the division theorem.

Theorem 4.5. *Let Ω be a Stein manifold of complex dimension n with a kähler metric. Suppose $p \geq r \geq m$ are positive integers. Let $f, g_1, \cdots, g_p$ be column vectors of holomorphic functions over Ω. Let*

$$G = \begin{pmatrix} g_{1,1} & g_{2,1} & \cdots & g_{p,1} \\ g_{1,2} & g_{2,2} & \cdots & g_{p,2} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1,r} & g_{2,r} & \cdots & g_{p,r} \end{pmatrix}.$$

Assume that the generic rank of

$$\begin{pmatrix} f_1 & g_{1,1} & g_{2,1} & \cdots & g_{p,1} \\ f_2 & g_{1,2} & g_{2,2} & \cdots & g_{p,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_r & g_{1,r} & g_{2,r} & \cdots & g_{p,r} \end{pmatrix}$$

is m. If f satisfies the L^2 condition

$$L_0 := \int_\Omega \frac{\sum_{1\le j_1<\cdots<j_{m-1}\le p} |f\wedge g_{j_1}\wedge\cdots\wedge g_{j_{m-1}}|^2}{\left(\sum_{1\le k_1<\cdots<k_m\le p} |g_{k_1}\wedge\cdots\wedge g_{k_m}|^2\right)^{\alpha\beta_1+1}} \mathrm{d}V_X < \infty, \tag{22}$$

where $\beta_1 = \min\{p-m, n\}$ and $\alpha > 1$ is a constant, then the division problem can be solvable and

$$\int_\Omega \frac{|h_j|^2}{\left(\sum_{1\le k_1<\cdots<k_m\le p} |g_{k_1}\wedge\cdots\wedge g_{k_m}|^2\right)^{\alpha\beta_1}} \mathrm{d}V_X < \frac{\alpha}{\alpha-1} L_0.$$

Proof. Suppose $r = m$. The basic idea is to introduce extra variables $z_1, \cdots, z_r$ into (2) and write

$$\tilde{f} = \sum_i f_i z_i = \sum_j \Big(\sum_i g_{ij} z_i\Big) h_j = \sum_j \tilde{g}_j h_j,$$

then we can use the projectivization method and only need to compute

$$\int_P \frac{\|\tilde{f}\|^2}{(\|\tilde{g}\|^2)^r} \mathrm{d}V_P = \frac{\sigma}{\mu^2}$$

by 2.3.

Suppose $r > m$, then in the functional analysis we have

- $\varphi = \log(\mu_m)^{\alpha\beta_1}$, which implies
- $\|h\|_{H_0} = \displaystyle\int_\Omega \frac{\|h\|_E^2}{(\mu_m)^{\alpha\beta_1}} \mathrm{d}V$ and
- $\|f\|_{H_1'} = \|T_1^{-1} f\|_{H_0}^2 = \displaystyle\int_\Omega \frac{\|f\|_{H_1}^2}{(\mu_m)^{\alpha\beta_1+1}} \mathrm{d}V$,

where $\mu_m = \displaystyle\sum_{1\le i_1<\cdots<i_m\le p} \|g_{i_1}\wedge\cdots\wedge g_{i_m}\|^2_{H_0^{-1}\otimes H_1}$. Hence the statement follows. $\square$

By introducing the curvature condition, we can state the division theorem.

Theorem 4.6. *Let X be a projective algebraic manifold of complex dimension n with a kähler metric. Let L be a holomorphic line bundle on X with a smooth metric $e^{-\chi}$. Let V be a holomorphic vector bundle on X of rank r and let $h_{\alpha\bar{\beta}}$ be a smooth hermitian metric of V. Let $g_1, \cdots, g_p$ be holomorphic sections of V over X such that the matrix*

$$\begin{pmatrix} f & G \end{pmatrix} = \begin{pmatrix} f_1 & g_{1,1} & g_{2,1} & \cdots & g_{p,1} \\ f_2 & g_{1,2} & g_{2,2} & \cdots & g_{p,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_r & g_{1,r} & g_{2,r} & \cdots & g_{p,r} \end{pmatrix}$$

has generic rank m (with $p \geq r \geq m$), and let f be a holomorphic section of $V \otimes L$ over X. Assume

$$\sqrt{-1}\partial\bar{\partial}\chi + \operatorname{Ric}_X \geq \alpha\beta_1 c_1(V),$$

where $\beta_1 = \min\{p-m, n\}$ and $\alpha > 1$ is a constant. If

$$L_0 := \int_X \frac{\sum_{1\leq j_1 < \cdots < j_{r-1} \leq p} \|f \wedge g_{j_1} \wedge \cdots \wedge g_{j_{m-1}}\|^2}{\left(\sum_{1 \leq k_1 < \cdots < k_m \leq p} \|g_{k_1} \wedge \cdots \wedge g_{k_m}\|^2\right)^{\alpha\beta_1 + 1}} \, \mathrm{d}V_X < \infty,$$

then there exists holomorphic sections $h_1, \cdots, h_p$ of L over X such that

$$f = \sum_{j=1}^{p} g_j h_j$$

on X as sections of $V \otimes L$ and

$$\int_X \frac{\|h_j\|^2}{\left(\sum_{1\leq k_1 < \cdots < k_r \leq p} \|g_{k_1} \wedge \cdots \wedge g_{k_m}\|^2\right)^{\alpha\beta_1}} \, \mathrm{d}V_\Omega < \frac{\alpha}{\alpha - 1} L_0.$$

Proof. We can write the division problem by the following diagram

$$G : \bigoplus_{j=1}^{p} L \longrightarrow V \otimes L \ . \tag{23}$$

$$(\alpha\beta + 1)\Theta(L) + \operatorname{Ric}_X \geq \alpha\beta c_1(V \otimes L) = \alpha\beta(c_1(v) + \Theta(L))$$

which leads to the conclusion. □

There is another way to get the conclusion by considering the following diagram

$$0 \longrightarrow K \longrightarrow E \longrightarrow\!\!\!\!\rightarrow W \longrightarrow 0$$

and we apply the full rnak division theorem then extend the solution back to original setting. Note that in applying the full rank division theorem we need to use the explicit version, otherwise it is hard to see the relation to the initial setting. Thus the advantage of using $E \longrightarrow V$ rather than $E \longrightarrow W$ is that we get a invariant expression

$$L_0 = \int_X \frac{((G\widetilde{G^* + S}^*S)f, f)}{\det(GG^* + S^*S)^{\alpha\beta+1}} \mathrm{d}V_X < \infty$$

and the generalized fundamental inequality at the same time.

Remark 5. The technique of projectivization is introduced for dealing with a more general situation. The formulation of theorem 4.5 and Skoda's theorem have one common feature that the norm of the numerator and denominator are the same in the analytic condition. In the full rank case they are allowed to be applied by different ones by absorbing the metric of $\det V$ into the weight function. However in the general case, i.e. $m < r$, we expect that there could be two different metrics

$$\int \frac{\|f \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}\|_\theta^2}{\|g_{i_1} \wedge \cdots \wedge g_{i_m}\|^2} \mathrm{d}V,$$

where θ is a Griffith semi-positive metric. Once θ is introduced, the computation becomes more complicated. The denominator of the integral would become to have two different terms, then we cannot make them into 1 simultaneously. In this case elliptic curves and elliptic integration are expected to come into the picture.

5. Fundamental Lemma

First let us recall some notations.

- $\mu = \sum_{1 \le i_1 < \cdots < i_r \le p} |g_{i_1} \wedge \cdots \wedge g_{i_r}|^2$, and
- $\mu_m = \sum_{1 \le i_1 < \cdots < i_m \le p} |g_{i_1} \wedge \cdots \wedge g_{i_m}|^2$.
- $\sigma = \sigma(f) = \sum_{1 \le k_1 < \cdots < k_{r-1} \le p} |f \wedge g_{k_1} \wedge \cdots \wedge g_{k_{r-1}}|^2$, and
- $\sigma_m = \sigma_m(f) = \sum_{1 \le k_1 < \cdots < k_{m-1} \le p} |f \wedge g_{k_1} \wedge \cdots \wedge g_{k_{m-1}}|^2$.

Lemma 5.1. *For any constant $\alpha \ge 1$,*

$$\frac{1}{\alpha \mu_m} \sum_{1 \le i_1 < \cdots < i_{m-1} \le p} \left| u \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}} \right|^2 + \alpha \beta_1 \sum_{j=1}^{p} \sum_{k,\ell=1}^{n} (\partial_k \bar{\partial}_\ell \log \mu_m) v_j^k \bar{v}_j^\ell{}_{(c)} \tag{24}$$

$$\ge 2 \left| \sum_{1 \le i_1 < \cdots < i_{m-1} \le p} \sum_{\substack{1 \le j \le p \\ 1 \le k \le n}} \left(u \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}, \overline{\partial_k \Big(\frac{g_j \wedge g_{i_1} \wedge \cdots \wedge g_{i_{m-1}}}{\mu_m} \Big) v_j^k} \right) \right|_{(d)},$$

where $\beta_1 = \min\{p - m, n\}$.

Proof. First we verify the case of $m = r$. At this moment we want to check the sub case of $\beta_1 = p - r$. Note that

$$\sum_{j=1}^{p} \sum_{k,\ell=1}^{n} (\partial_k \bar{\partial}_\ell \log \mu_m) v_j^k \bar{v}_j^\ell = \frac{1}{\mu^2} \sum_{I<J} |(\mu_I \partial_k \mu_J - \mu_J \partial_k \mu_I) v_j^k|^2,$$

where $I = \{i_1, \cdots, i_r\}$, $J = \{j_1, \cdots, j_r\}$ are subsets of $\{1, \cdots, p\}$. Let us exam the term

$$\partial_k \Big(\frac{g_j \wedge g_1 \wedge \cdots \wedge g_{r-1}}{\mu} \Big) v_j^k$$

by explicit computation, (let $\{i_1, \cdots, i_{r-1}\} = \{1, \cdots, r-1\}$ for convenience in demonstration.) we have

$$\begin{aligned}
&\sum_j \partial_k \Big(\frac{g_j \wedge g_1 \wedge \cdots \wedge g_{r-1}}{\mu}\Big) v_j^k = \frac{1}{\mu} \sum_j \Big(\partial_k \mu_{j,1,\cdots,r-1} - \frac{\partial_k \mu}{\mu} \mu_{j,1,\cdots,r-1}\Big) v_j^k \\
&= \frac{1}{\mu^2} \sum_j \sum_{i \neq j} \overline{\mu_{i,1,\cdots,r-1}} (\mu_{i,1,\cdots,r-1} \partial_k \mu_{j,1,\cdots,r-1} - \mu_{j,1,\cdots,r-1} \partial_k \mu_{i,1,\cdots,r-1}) v_j^k \\
&\quad + \frac{1}{\mu^2} \sum_j \sum_{\substack{\text{other} \\ \text{terms}}} \overline{\mu_{i,1,\cdots,r-1}} (\mu_{i_1,\cdots,i_r} \partial_k \mu_{j,1,\cdots,r-1} - \mu_{j,1,\cdots,r-1} \partial_k \mu_{i_1,\cdots,i_r}) v_j^k.
\end{aligned}$$

In fact we can collect the terms according to the repeat index ι. For instance the terms in

$$\sum_j \sum_{i \neq j} \overline{\mu_{i,1,\cdots,r-1}} (\mu_{i,1,\cdots,r-1} \partial_k \mu_{j,1,\cdots,r-1} - \mu_{j,1,\cdots,r-1} \partial_k \mu_{i,1,\cdots,r-1}) v_j^k$$

have $\iota = r-1$. For short we denote the term by

$$(i, 1, \cdots, r-1; j, 1, \cdots, r-1).$$

If $\iota = 1$ and j is not the repeat index, then we do not have to do anything. For $\iota = 1$ and j is the repeat index we have the following situation

$$\begin{aligned}
&\begin{pmatrix} \overline{(u, j_1, \cdots, j_{r-1})(j, i_1, \cdots, i_{r-1})} (j, i_1, \cdots, i_{r-1}; j, j_1, \cdots, j_{r-1}) v_j^k \\ -\overline{(u, i_1, \cdots, i_{r-1})(j, j_1, \cdots, j_{r-1})} (j, j_1, \cdots, j_{r-1}; j, i_1, \cdots, i_{r-1}) v_j^k \end{pmatrix} \\
&= \overline{\big((u, j_1, \cdots, j_{r-1})(j, i_1, \cdots, i_{r-1}) - (u, i_1, \cdots, i_{r-1})(j, j_1, \cdots, j_{r-1})\big)} \\
&\quad (j, i_1, \cdots, i_{r-1}; j, j_1, \cdots, j_{r-1}) v_j^k.
\end{aligned}$$

Now we collect terms according to the type of $(j, I'; j, J')$. There are two such terms, and we can combine them

$$\overline{\big((u, J')(j, I') - (u, I')(j, J')\big)} (j, I'; j, J') v_j^k$$

Applying the inequality $2|AB| \le \alpha|A|^2 + 1/\alpha|B|^2$ to $A = \overline{(u, J')(j, I') - (u, I')(j, J')}$, $B = (j, I'; j, J')v_j^k$ and we will determine α later, then we can see that

$$|B|^2 = \left|(\mu_{i,I'}\partial_k\mu_{j,J'} - \mu_{j,J'}\partial_k\mu_{i,I'})v_j^k\right|^2$$

which is a term in $(\partial_k\bar\partial_\ell \log \mu)v_j^k\bar v_j^\ell$. Hence the remaining work is to figure out the summation of all possible $|A|^2$. Before we sum them up we need a crucial identity:

$$\begin{aligned}(25)\qquad &(\underline{u}, i_1, \cdots, i_{r-1})(\underline{j_1, \cdots, j_{r-1}}, j) + (\underline{u}, j, i_1, \cdots, i_{r-2})(\underline{j_1, \cdots, j_{r-1}}, i_{r-1}) + \\ &(\underline{u}, i_{r-1}, j, i_1, \cdots, i_{r-3})(\underline{j_1, \cdots, j_{r-1}}, i_{r-2}) + \cdots = \\ &(u, j_1, \cdots, j_{r-1})(i_1, \cdots, i_{r-1}, j)\end{aligned}$$

if r is odd, and

$$\begin{aligned}(26)\qquad &(\underline{u}, i_1, \cdots, i_{r-1})(\underline{j_1, \cdots, j_{r-1}}, j) - (\underline{u}, j, i_1, \cdots, i_{r-2})(\underline{j_1, \cdots, j_{r-1}}, i_{r-1}) + \\ &(\underline{u}, i_{r-1}, j, i_1, \cdots, i_{r-3})(\underline{j_1, \cdots, j_{r-1}}, i_{r-2}) + \cdots = \\ &(u, j_1, \cdots, j_{r-1})(i_1, \cdots, i_{r-1}, j)\end{aligned}$$

if r is even. Note the underline part is the indexes with fixed positions, and in the even case L.H.S is an alternating sum. By these two identities we can combine terms that have the same repeat index ι. For example

$$(u, j_1, \cdots, j_{r-1})(j, i_1, \cdots, i_{r-1}) - (u, i_1, \cdots, i_{r-1})(j, j_1, \cdots, j_{r-1}) = (\cdots).$$

In fact the sum $(\cdots)$ has an intrinsic meaning of exterior product, and we denote this determinant as

$$\begin{aligned}&(u \wedge j) \curlywedge (j_1 \wedge \cdots \wedge j_{r-1}) \curlywedge (i_1 \wedge \cdots \wedge i_{r-1}) \\ :=& \sum \epsilon(\alpha_1, \alpha_2, \beta_1, \cdots, \beta_{r-2})\epsilon(\beta_{r-1}, \gamma_1, \cdots, \gamma_{r-1}) \\ &\times (u \wedge j)^{\alpha_1,\alpha_2}(j_1 \wedge \cdots \wedge j_{r-1})^{\beta_1,\cdots,\beta_{r-1}}(i_1 \wedge \cdots \wedge i_{r-1})^{\gamma_1,\cdots,\gamma_{r-1}},\end{aligned}$$

where α, β, γ indicates the index of rows and

$$\epsilon(\alpha_1, \alpha_2, \beta_1, \cdots, \beta_{r-2})$$

means the rotation sign of $(\alpha_1, \alpha_2, \beta_1, \cdots, \beta_{r-2})$. After summing them up we have another amazing identity

$$\begin{aligned}(27) \quad & \sum_{J<I} |(u \wedge j_1) \curlywedge (j_2 \wedge \cdots \wedge j_r) \curlywedge (i_1 \wedge \cdots \wedge i_{r-1})|^2 \\ & = (r-1) \sum_{I<J} |(u, i_1, \cdots, i_{r-1}|^2 |(j_1, \cdots, j_r)|^2.\end{aligned}$$

Note the notation $J < I$ means run through all possible combinations of (I, J) such that

$$I \cup J = \{1, \cdots, p\}.$$

On the other hand terms with type $\iota = 0$

$$\overline{(u, j_1, \cdots, j_{r-1}(i_1, \cdots, i_r)}(i_1, \cdots, i_r; j, j_1, \cdots, j_{r-1}) v_j^k$$

can associate an inequality

$$\leq |(u, J')|^2 |(I)|^2 + |(I; j, J') v_j^k|^2.$$

Note that $\{j, J'\} \cap \{I\} = \emptyset$ because $\iota = 0$. Hence for every fixed J', I there are $p - (2r-1)$ choices for j. Combining with terms come from $\iota = 1$ with $j \notin \iota$ (which means j is not an repeated index) we get the inequality

$$\leq (p-r) |(u, J')|^2 |(I)|^2.$$

Once we have this we are done by choosing $\alpha = p - r$. In the general case of repeat index $\iota = s + 1$ we have a similar algorithm. We combine terms with types

1) $\iota' = \iota + 1$ and $j \in \iota'$ (which means j is a repeated index)

2) ι with $j \notin \iota$.

For terms in 2), we can compute the combinatoric easily, which is

$$p - (2r - 1 - \iota).$$

On the other hand, like the $\iota' = 1, \iota = 0$ case, we need to pin down the coefficients for the repeated indexes situation (i.e. terms in 1)). Similarly we have the following identities

$$\begin{aligned}
&(\underline{u}, i_1, \cdots, i_q, \underline{j_1, \cdots, j_s})(\underline{k_1, \cdots, k_q}, k, \underline{j_1, \cdots, j_s}) \pm \\
&(\underline{u}, k, i_1, \cdots, i_{q-1}, \underline{j_1, \cdots, j_s})(\underline{k_1, \cdots, k_q}, i_q, \underline{j_1, \cdots, j_s}) + \\
&(\underline{u}, i_q, k, i_1, \cdots, i_{q-2}, \underline{j_1, \cdots, j_s})(\underline{k_1, \cdots, k_q}, i_{q-1}, \underline{j_1, \cdots, j_s}) + \cdots \\
&= (u, k_1, \cdots, k_q, \underline{j_1, \cdots, j_s})(i_1, \cdots, i_q, k, \underline{j_1, \cdots, j_s}),
\end{aligned} \tag{28}$$

where $\pm$ depends on q. Hence we have the identity

$$\begin{aligned}
&\sum_{K<I} |(u \wedge j_1 \wedge \cdots \wedge j_s) \curlywedge (k_1 \wedge \cdots \wedge k_q) \curlywedge (i_1 \wedge \cdots \wedge i_q)|^2 \\
&= (r - \iota') \sum_{I<K} |(u, k_1, \cdots, k_q, j_1, \cdots, j_s)|^2 |(i_1, \cdots, i_q, j_1, \cdots, j_r)|^2.
\end{aligned} \tag{29}$$

Therefore the coefficient should be

$$(p - 2r + 1 + \iota) + (r - \iota') = p - r - \iota' + \iota + 1 = p - r$$

which is the expected number. Hence we are done with this case.

Let us turn to check the case $\beta_1 = n$. Study term (d) again

$$
\begin{aligned}
&2\left|\sum_{J'}\sum_{j=1}^{p}\sum_{I}\frac{1}{\mu^2}\overline{(u,J')(I)}(I;j,J')_k v_j^k\right| \\
&= \frac{2}{\mu^2}\left|\sum_{J}\sum_{I}\sum_{(j,J')}\overline{(u,J')(I)}(I;j,J')_k v_j^k\right| \\
&= \frac{2}{\mu^2}\left|\sum_{I\neq J}\sum_{(j,J')}\overline{(I)\epsilon_{j,J'}(u,J')}(I;J)_k v_j^k\right| \\
&= \frac{2}{\mu^2}\left|\sum_{I<J}\sum_{(i,I')=(j,J')}\overline{\big((I)\epsilon_{j,J'}(u,J')\bar{v}_j^k - (J)\epsilon_{i,I'}(u,I')\bar{v}_i^k\big)}(I;J)_k\right|.
\end{aligned}
$$

Note that $\sum_{(j,J')}$ means to sum over all possible partition of J and $\epsilon_{j,J'}$ is the permutation index. Hence we can choose the vectors

$$
(X_k)_J = \partial_k(j,J'), \ \text{ and } (Y_k)_I = \sum_{(i,I')}\epsilon_{i,I'}(u,I')\bar{v}_i^k,
$$

and normalize the inner product

$$
H(X,Y) = \sum_{I<J}\big((I)X_J - (J)X_I\big)\overline{\big((I)Y_J - (J)Y_I\big)}
$$

w.r.t X_k, then write term $(d) = 2A = 2\sqrt{|A|^2}$ and

$$
\begin{aligned}
|A|^2 &\leq \frac{n}{\mu^4}\sum_k\left|\sum_{I<J}\sum_{(i,I')=(j,J')}\overline{\big((I)\epsilon_{j,J'}(u,J')\bar{v}_j^k - (J)\epsilon_{i,I'}(u,I')\bar{v}_i^k\big)}(I;J)_k\right|^2 \\
&= \frac{n}{\mu^4}\sum_k\Big|H(X_k,Y_k)\Big|^2 \leq \frac{n}{\mu^4}\sum_k H(X_k,X_k)H(Y_k,Y_k) \\
&\leq \frac{n}{\mu^4}\sum_J\Big|(J)\Big|^2\sum_{k,I}\Big|\sum_{(i,I')}\epsilon_{i,I'}(u,I')\bar{v}_i^k\Big|^2.
\end{aligned}
$$

By basic inequality $2\sqrt{AB} \leq \alpha A + 1/\alpha\cdot B$ we have

$$
(d) \leq \alpha\frac{n}{\mu^3}\sum_J\Big|(J)\Big|^2 + \frac{1}{\alpha\mu}\sum_{k,I}\Big|\sum_{(i,I')}\epsilon_{i,I'}(u,I')\bar{v}_i^k\Big|^2.
$$

Recall the generalized Lagrange summation

$$\sum_{I}\Big|\sum_{(i,I')}\epsilon_{i,I'}(u,I')\bar{v}_i^k\Big|^2=\sum_{I'}|(u,I')|^2\sum_j|v_j^k|^2-\sum_{J''}\Big|\sum_{\ell=1}^{p}(u,J'',\ell)v_\ell^k\Big|^2$$

and $\sum\big|(J)\big|^2=\mu$ so we can take $\alpha=\sum_{k,i}|v_i^k|^2$ and conclude

$$(d)\le\frac{n}{\mu^2}\sum_{k,i}|v_i^k|^2+\frac{1}{\mu}\sum_{I'}\Big|(u,I')\Big|^2.$$

Let us exam term (c).

$$\frac{1}{\mu^2}\sum_{\ell=1}^{p}\sum_{I<J}|(I;J)_k v_\ell^k|^2=\frac{1}{\mu^2}\sum_{I<J}\sum_{\ell,s}(I;J)_k v_\ell^k\overline{(I;J)_s v_\ell^s}$$
$$=\frac{1}{\mu^2}\sum_{\ell,s}H(X_k,X_s)v_\ell^k\bar{v}_\ell^s=\frac{1}{\mu^2}\sum_{k,\ell}|v_\ell^k|^2.$$

Hence we finish the $m=r$ case.

For the general situation $m<r$, there are non-linear terms in the computation. Like before, we fix $(j_1,\cdots,j_{m-1})=(1,\cdots,m-1)$ for convenience in demonstration. In the computation we need to sum them up over $1\le j_1<\cdots<j_{m-1}\le p$.

$$\sum_j\partial_k\Big(\frac{1}{\mu}\mu_{j,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)}\Big)v_j^k=\frac{1}{\mu}\sum_j\Big(\partial_k\mu_{j,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)}-\frac{\partial_k\mu}{\mu}\mu_{j,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)}\Big)v_j^k$$
$$=\frac{1}{\mu^2}\sum_j\sum_{i\ne j}\overline{\mu_{i,1,\cdots,m-1}^{(\beta_1,\cdots,\beta_m)}}\big(\mu_{i,1,\cdots,m-1}^{(\beta_1,\cdots,\beta_m)}\partial_k\mu_{j,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)}-\mu_{j,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)}\partial_k\mu_{i,1,\cdots,m-1}^{(\beta_1,\cdots,\beta_m)}\big)v_j^k$$
$$+\frac{1}{\mu^2}\sum_j\sum_{\substack{\text{other}\\\text{terms}}}\overline{\mu_{i,1,\cdots,m-1}^{(\beta_1,\cdots,\beta_m)}}\big(\mu_{i_1,\cdots,i_m}^{(\beta_1,\cdots,\beta_m)}\partial_k\mu_{j,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)}-\mu_{j,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)}\partial_k\mu_{i_1,\cdots,i_m}^{(\beta_1,\cdots,\beta_m)}\big)v_j^k,$$

where $(\beta_1,\cdots,\beta_m)$ indicates the indexes of matrix. We can separate terms in two groups

1) $(\alpha_1,\cdots,\alpha_m)=(\beta_1,\cdots,\beta_m)$. In this case, everything reduces to the full rank case. By the previous computation it is verified.

2) $(\alpha_1, \cdots, \alpha_m) \neq (\beta_1, \cdots, \beta_m)$. In this case, they are cross terms. We need to apply compatible condition

$$
\begin{vmatrix}
u_{\alpha_1} & g_{i_1,\alpha_1} & g_{i_2,\alpha_1} & \cdots & g_{i_m,\alpha_1} \\
u_{\alpha_2} & g_{i_1,\alpha_2} & g_{i_2,\alpha_2} & \cdots & g_{i_m,\alpha_2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
u_{\alpha_{m+1}} & g_{i_1,\alpha_r} & g_{i_2,\alpha_r} & \cdots & g_{i_m,\alpha_{m+1}}
\end{vmatrix} = 0 \tag{30}
$$

for every $1 \leq \alpha_1 < \cdots < \alpha_{m+1} \leq r$ and $1 \leq i_1 < \cdots < i_m \leq p$.

Let us focus on case 2). For the repeat index $\iota = r-1$ and $j \notin \iota$, i.e. j is not repeated, the trouble terms are $(i, i_1, \cdots, i_{m-1}) = (j, j_1, \cdots, j_{m-1}) = (i, 1, \cdots, m-1)$. For every such term

$$
\begin{aligned}
&\overline{(u, 1, \cdots, m-1)^{(\alpha_1, \cdots, \alpha_m)} \mu_{i,1,\cdots,m-1}^{(\beta_1,\cdots,\beta_m)}} \\
&\left(\mu_{i,1,\cdots,m-1}^{(\beta_1,\cdots,\beta_m)} \partial_k \mu_{i,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)} - \mu_{i,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)} \partial_k \mu_{i,1,\cdots,m-1}^{(\beta_1,\cdots,\beta_m)}\right) v_i^k
\end{aligned}
$$

we can find its conjugate

$$
\begin{aligned}
&\overline{(u, 1, \cdots, m-1)^{(\beta_1, \cdots, \beta_m)} \mu_{i,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)}} \\
&\left(\mu_{i,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)} \partial_k \mu_{i,1,\cdots,m-1}^{(\beta_1,\cdots,\beta_m)} - \mu_{i,1,\cdots,m-1}^{(\beta_1,\cdots,\beta_m)} \partial_k \mu_{i,1,\cdots,m-1}^{(\alpha_1,\cdots,\alpha_m)}\right) v_i^k
\end{aligned}
$$

and we can combine them as

$$
\overline{((u, I')^{\beta}(i, I)^{\alpha} - (u, I')^{\alpha}(i, I')^{\beta})} (\mu_I^{\alpha} \partial_k \mu_I^{\beta} - \mu_I^{\beta} \partial_k \mu_I^{\alpha}) v_i^k.
$$

Similar to the full rank case, we apply the inequality $2|AB| \leq a|A|^2 + 1/a|B|^2$ to $A = \overline{((u, I')^{\beta}(i, I)^{\alpha} - (u, I')^{\alpha}(i, I')^{\beta})}$, $B = ((i, I')^{\alpha}; (i, I')^{\beta}) v_i^k$. Note that

$$
|B|^2 = \left|(\mu_{i,I'}^{\alpha} \partial_k \mu_{i,I'}^{\beta} - \mu_{i,I'}^{\beta} \partial_k \mu_{i,I'}^{\alpha}) v_i^k\right|^2
$$

which is a term in $(\partial_k\bar\partial_\ell \log \mu_m) v_j^k \bar v_j^\ell$. Hence the remaining work is to figure out the summation of all possible $|A|^2$. Let us collect all terms in the form of

$$\overline{((u,I')^\beta (i,I')^\alpha - (u,I')^\alpha (i,I')^\beta)}$$

and sum them up over α,β. The result would be divided by

$$\det(u,I',i)$$

which is the relation in (30), hence is zero. Therefore i could only takes values such that

$$i \neq j \neq j_1 \neq \cdots \neq j_{m-1},$$

which means that there are $p-m$ choices. That is equal to β_1, so we are done with this case.

For the case of $j \in \iota$, by coupling the compatible conditions (30) we have a generalized identities of (28)

$$\begin{aligned}
&(\underline{u}, i_1, \cdots, i_q, \underline{j_1, \cdots, j_s}))^\alpha (\underline{k_1, \cdots, k_q}, k, \underline{j_1, \cdots, j_s})^\beta \pm \\
&(\underline{u}, k, i_1, \cdots, i_{q-1}, \underline{j_1, \cdots, j_s})^\alpha (\underline{k_1, \cdots, k_q}, i_q, \underline{j_1, \cdots, j_s})^\beta + \\
&(\underline{u}, i_q, k, i_1, \cdots, i_{q-2}, \underline{j_1, \cdots, j_s})^\alpha (\underline{k_1, \cdots, k_q}, i_{q-1}, \underline{j_1, \cdots, j_s})^\beta + \cdots \\
&= (u, k_1, \cdots, k_q, \underline{j_1, \cdots, j_s})^\beta (i_1, \cdots, i_q, k, \underline{j_1, \cdots, j_s})^\alpha,
\end{aligned} \tag{28'}$$

where $\pm$ depends on q. Denote

$$\big((u \wedge J) \curlywedge (K) \curlywedge (I)\big)^{\alpha,\beta} = (u,K,J)^\beta (I,J)^\alpha - (u,I,J)^\alpha (K,J)^\beta = (\cdots)$$

and we have

$$\begin{aligned}
&\sum_{K<I} |\big((u \wedge j_1 \wedge \cdots \wedge j_s) \curlywedge (k_1 \wedge \cdots \wedge k_q) \curlywedge (i_1 \wedge \cdots \wedge i_q)\big)^{\alpha,\beta}|^2 \\
&= (r-\iota') \sum_{I<K} |(u, k_1, \cdots, k_q, j_1, \cdots, j_s)^\alpha|^2 |(i_1, \cdots, i_q, j_1, \cdots, j_r)^\beta|^2.
\end{aligned} \tag{29'}$$

The coefficient is as expected, so we are done with this case.

For $\beta_1 = n$ sub case, like the full-rank case we take

$$X_J = \partial_k (j, J')^{(\alpha_1, \cdots, \alpha_m)}, \text{ and } Y_I = (u, I')^{(\alpha_1, \cdots, \alpha_m)} \bar{v}_i.$$

By the similar arguments in the full rank case, we can verify the inequality. Thus we finish the justification of (24). □

Let us use one example to explain the algorithm.

Example 5.2. *Let*

$$\begin{pmatrix} u & G \end{pmatrix} = \begin{pmatrix} u_1 & g_{1,1} & g_{2,1} & \cdots & g_{p,1} \\ u_2 & g_{1,2} & g_{2,2} & \cdots & g_{p,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_r & g_{1,r} & g_{2,r} & \cdots & g_{p,r} \end{pmatrix}$$

has generic rank 1, i.e. the image is generic a line bundle of $\mathbb{C}^r$*, and we want to verify the fundamental inequality*

$$\frac{1}{\mu_1} \sum_{\alpha=1}^{r} |u_\alpha|^2 + \beta_1 \sum_{j=1}^{p} \sum_{k,\ell=1}^{n} (\partial_k \bar{\partial}_\ell \log \mu_1) v_j^k \bar{v}_j^\ell \geq 2 \left| \sum_{\alpha=1}^{r} \sum_{\substack{1 \leq j \leq p \\ 1 \leq k \leq n}} u_\alpha \overline{\partial_k \left(\frac{g_{j,\alpha}}{\mu_1} \right) v_j^k} \right|, \tag{31}$$

where

$$\mu_1 = \sum_{\alpha=1}^{r} \sum_{j=1}^{p} |g_{j,\alpha}|^2$$

and $\beta_1 = \min\{p-1, n\}$.

Proof. First we focus on $\beta_1 = (p-1)$ case. On one hand we can compute

$$\begin{aligned} \sum_{j=1}^{p} \sum_{k,\ell=1}^{n} (\partial_k \bar{\partial}_\ell \log \mu_1) v_j^k \bar{v}_j^\ell = & \frac{1}{\mu_1^2} \sum_{\alpha=1}^{r} \sum_{\ell=1}^{p} \sum_{1 \leq i < j \leq p} \left| (g_{i,\alpha} \partial_k g_{j,\alpha} - g_{j,\alpha} \partial_k g_{i,\alpha}) v_\ell^k \right|^2 \\ & + \frac{1}{\mu_1^2} \sum_{1 \leq \alpha < \beta \leq r} \sum_{\ell=1}^{p} \sum_{i,j=1}^{p} \left| (g_{i,\alpha} \partial_k g_{j,\beta} - g_{j,\beta} \partial_k g_{i,\alpha}) v_\ell^k \right|^2 . \end{aligned}$$

On the other hand we can compute R.H.S of (31)

$$2\left|\sum_{\alpha=1}^{r}\sum_{\substack{1\le j\le p\\1\le k\le n}} u_\alpha\overline{\partial_k\Big(\frac{g_{j,\alpha}}{\mu_1}\Big)v_j^k}\right| \le \frac{2}{\mu_1^2}\sum_{\alpha=1}^{r}\left|\sum_{i,j=1}^{p}\overline{u_\alpha g_{i,\alpha}}(g_{i,\alpha}\partial_k g_{j,\alpha}-g_{j,\alpha}\partial_k g_{i,\alpha})v_j^k\right|$$

$$+\frac{2}{\mu_1^2}\left|\sum_{1\le\alpha\ne\beta\le r}\sum_{i,j=1}^{p}\overline{u_\alpha g_{i,\beta}}(g_{i,\beta}\partial_k g_{j,\alpha}-g_{j,\alpha}\partial_k g_{i,\beta})v_j^k\right|.$$

For the regular terms, we have the original Skoda's inequalities

$$\sum_{i=1}^{p}|u_\alpha|^2|g_{i,\alpha}|^2+\beta_1\sum_{\ell=1}^{p}\sum_{1\le i<j\le p}\Big|(g_{i,\alpha}\partial_k g_{j,\alpha}-g_{j,\alpha}\partial_k g_{i,\alpha})v_\ell^k\Big|^2$$

$$\ge 2\left|\sum_{i,j=1}^{p}\overline{u_\alpha g_{i,\alpha}}(g_{i,\alpha}\partial_k g_{j,\alpha}-g_{j,\alpha}\partial_k g_{i,\alpha})v_j^k\right|.$$

Hence what remaining is to compare the cross terms.

Claim 1:

$$\sum_{\alpha=1}^{r}\sum_{i=1}^{p}|u_\alpha|^2\sum_{1\le\beta\ne\alpha\le r}|g_{i,\beta}|^2+(p-1)\sum_{1\le\alpha<\beta\le r}\sum_{\ell=1}^{p}\sum_{i,j=1}^{p}\Big|(g_{i,\alpha}\partial_k g_{j,\beta}-g_{j,\beta}\partial_k g_{i,\alpha})v_\ell^k\Big|^2$$

$$\ge 2\left|\sum_{\alpha=1}^{r}\sum_{1\le\beta\ne\alpha\le r}\sum_{i,j=1}^{p}\overline{u_\alpha g_{i,\beta}}(g_{i,\alpha}\partial_k g_{j,\beta}-g_{j,\beta}\partial_k g_{i,\alpha})v_j^k\right|.$$

Once this inequality is established then we are done. The key feature is to observe that when $i=j$ we can combine two terms

$$\overline{u_\alpha g_{i,\beta}}(g_{i,\alpha}\partial_k g_{i,\beta}-g_{i,\beta}\partial_k g_{i,\alpha})v_i^k+\overline{u_\beta g_{i,\alpha}}(g_{i,\beta}\partial_k g_{i,\alpha}-g_{i,\alpha}\partial_k g_{i,\beta})v_i^k$$
$$=\overline{(u_\alpha g_{i,\beta}-u_\beta g_{i,\alpha})}(g_{i,\alpha}\partial_k g_{i,\beta}-g_{i,\beta}\partial_k g_{i,\alpha})v_i^k=0,$$

because

$$(u_\alpha g_{i,\beta}-u_\beta g_{i,\alpha})=\begin{vmatrix}u_\alpha & g_{i,\alpha}\\ u_\beta & g_{i,\beta}\end{vmatrix}=0 \tag{32}$$

by the compatible condition. Hence we can rewrite the R.H.S of the claim

$$
\begin{aligned}
&2\left|\sum_{\alpha=1}^{r}\sum_{1\le\beta\ne\alpha\le r}\sum_{i,j=1}^{p}\overline{u_\alpha g_{i,\beta}}(g_{i,\alpha}\partial_k g_{j,\beta}-g_{j,\beta}\partial_k g_{i,\alpha})v_j^k\right|\\
\le&2\sum_{\alpha=1}^{r}\sum_{1\le\beta\ne\alpha\le r}\sum_{i=1}^{p}\left|\sum_{1\le j\ne i\le p}\overline{u_\alpha g_{i,\beta}}(g_{i,\alpha}\partial_k g_{j,\beta}-g_{j,\beta}\partial_k g_{i,\alpha})v_j^k\right|\\
\le&\sum_{\alpha=1}^{r}\sum_{1\le\beta\ne\alpha\le r}\sum_{i=1}^{p}\left(\left|u_\alpha g_{i,\beta}\right|^2+\left|\sum_{1\le j\ne i\le p}(g_{i,\alpha}\partial_k g_{j,\beta}-g_{j,\beta}\partial_k g_{i,\alpha})v_j^k\right|^2\right)\\
\le&\sum_{\alpha=1}^{r}\sum_{1\le\beta\ne\alpha\le r}\sum_{i=1}^{p}|u_\alpha g_{i,\beta}|^2+(p-1)\sum_{1\le\alpha<\beta\le r}\sum_{i=1}^{p}\sum_{1\le j\ne i\le p}\left(\left|\Delta_{i,j}^{\alpha,\beta}v_i^k\right|^2+\left|\Delta_{i,j}^{\alpha,\beta}v_j^k\right|^2\right),
\end{aligned}
$$

where

$$\Delta_{i,j}^{\alpha,\beta}:=(g_{i,\alpha}\partial_k g_{j,\beta}-g_{j,\beta}\partial_k g_{i,\alpha}).$$

Note that what on the L.H.S of the claim

$$\sum_{1\le\alpha<\beta\le r}\sum_{\ell=1}^{p}\sum_{i,j=1}^{p}\left|(g_{i,\alpha}\partial_k g_{j,\beta}-g_{j,\beta}\partial_k g_{i,\alpha})v_\ell^k\right|^2$$

contains all possible $\left|\Delta_{i,j}^{\alpha,\beta}v_\ell^k\right|^2$ for $1\le\ell\le p$ which is obvious more than $\left|\Delta_{i,j}^{\alpha,\beta}v_i^k\right|^2+\left|\Delta_{i,j}^{\alpha,\beta}v_h^k\right|^2$. Therefore we verify the case $\beta_1=(p-1)$.

For the case of $\beta_1=n$, we study the R.H.S of (31) again. Collect terms and by (32) we can rewrite

(33)

$$
\begin{aligned}
&2\left|\sum_{\alpha,\beta=1}^{r}\sum_{i,j=1}^{p}\overline{u_\alpha g_{i,\beta}}(g_{i,\beta}\partial_k g_{j,\alpha}-g_{j,\alpha}\partial_k g_{i,\beta})v_j^k\right|\\
=2&\left|\left(\sum_{\alpha=\beta=1}^{r}\sum_{1\le i<j\le p}+\sum_{1\le\alpha<\beta\le r}\sum_{1\le i\ne j\le p}\right)\overline{(u_\alpha\bar{v}_j^k g_{i,\beta}-u_\beta\bar{v}_i^k g_{j,\alpha})}(g_{i,\beta}\partial_k g_{j,\alpha}-g_{j,\alpha}\partial_k g_{i,\beta})\right|
\end{aligned}
$$

Observe what inside the absolute value is an inner product defined as

$$H(X,Y)=\left(\sum_{\alpha=\beta=1}^{r}\sum_{1\le i<j\le p}+\sum_{1\le\alpha<\beta\le r}\sum_{1\le i\ne j\le p}\right)\overline{(g_{i,\beta}x_j^\alpha-g_{j,\alpha}x_i^\alpha)}(g_{i,\beta}y_j^\alpha-g_{j,\alpha}y_i^\alpha)$$

and we can choose the vectors

$$(X_k)^\alpha_j = \partial_k g_{j,\alpha}, \text{ and } (Y_k)^\alpha_j = u_\alpha \bar{v}^k_j.$$

Hence (33) can be rewritten as

$$\begin{aligned} 2\sqrt{\Big|\sum_{k=1}^n H(X_k, Y_k)\Big|^2} &\leq 2\sqrt{\sum_{k=1}^n H(X_k, X_k)\cdot\sum_{k=1}^n H(Y_k, Y_k)} \\ &\leq a\sum_{k=1}^n H(X_k, X_k) + \frac{1}{a}\sum_{k=1}^n H(Y_k, Y_k) \end{aligned} \tag{34}$$

for any constant a. We can normalize the inner product $H(X, Y)$ w.r.t to $\{X_k\}$, so

$$H(X_k, X_k) = 1$$

for every k. Let us compute

$$H(Y_k, Y_k) = \Big(\sum_{\alpha=\beta=1}^r \sum_{1\leq i<j\leq p} + \sum_{1\leq\alpha<\beta\leq r}\sum_{1\leq i\neq j\leq p}\Big)|(g_{i,\beta}(y_k)^\alpha_j - g_{j,\alpha}(y_k)^\beta_i)|^2.$$

In order to simplify the notation we will drop the sub-index of y_k in the following computation. For the regular part, by the original Skoda's inequality technique we have

$$\sum_{\alpha=\beta=1}^r \sum_{1\leq i<j\leq p} |(g_{i,\beta}y^\alpha_j - g_{j,\alpha}y^\beta_i)|^2 \leq \sum_{\alpha=\beta=1}^r \sum_{i,j=1}^p |g_{i,\alpha}|^2 |y^\alpha_j|^2.$$

For the cross term part we apply the similar technique to the index i, j.

$$\begin{aligned} &\sum_{1\leq\alpha<\beta\leq r}\sum_{1\leq i\neq j\leq p} |(g_{i,\beta}y^\alpha_j - g_{j,\alpha}y^\beta_i)|^2 \\ &= \sum_{1\leq\alpha<\beta\leq r}\sum_{1\leq i\neq j\leq p} |g_{i,\beta}y^\alpha_j|^2 + |g_{j,\alpha}y^\beta_i|^2 - 2Re\, g_{i,\beta}y^\alpha_j\overline{g_{j,\alpha}y^\beta_i} \\ &= \sum_{1\leq\alpha\neq\beta\leq r}\sum_{i,j=1}^p |g_{i,\beta}y^\alpha_j|^2 - \sum_{1\leq\alpha<\beta\leq r}\Big(\sum_{i=1}^p |g_{i,\beta}y^\alpha_i|^2 + |g_{i,\alpha}y^\beta_i|^2 + \sum_{1\leq i\neq j\leq p} 2Re(\cdots)\Big). \end{aligned}$$

CLAIM 2:

$$\sum_{1\le\alpha<\beta\le r}\Big(\sum_{i=1}^{p}|g_{i,\beta}y_i^{\alpha}|^2+|g_{i,\alpha}y_i^{\beta}|^2+\sum_{1\le i\neq j\le p}2Re\, g_{i,\beta}y_j^{\alpha}\overline{g_{j,\alpha}y_i^{\beta}}\Big)\ge 0.$$

If the claim holds, we have

$$\begin{aligned}
H(Y_k,Y_k)&=\Big(\sum_{\alpha=\beta=1}^{r}\sum_{1\le i<j\le p}+\sum_{1\le\alpha<\beta\le r}\sum_{1\le i\neq j\le p}\Big)|(g_{i,\beta}(y_k)_j^{\alpha}-g_{j,\alpha}(y_k)_i^{\beta})|^2\\
&\le\sum_{\alpha=1}^{r}\sum_{j=1}^{p}|g_{j,\alpha}|^2\cdot\sum_{\alpha=1}^{r}\sum_{j=1}^{p}|u_\alpha|^2|v_j^k|^2\\
&=\mu_1\cdot\sum_{\alpha=1}^{r}|u_\alpha|^2\sum_{j=1}^{p}|v_j^k|^2.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\sum_{j=1}^{p}\sum_{k,\ell=1}^{n}(\partial_k\bar\partial_\ell\log\mu_1)v_j^k\bar v_j^\ell=&\frac{1}{\mu_1^2}\sum_{\alpha=1}^{r}\sum_{\ell=1}^{p}\sum_{1\le i<j\le p}\Big|(g_{i,\alpha}\partial_k g_{j,\alpha}-g_{j,\alpha}\partial_k g_{i,\alpha})v_\ell^k\Big|^2\\
&+\frac{1}{\mu_1^2}\sum_{1\le\alpha<\beta\le r}\sum_{\ell=1}^{p}\sum_{i,j=1}^{p}\Big|(g_{i,\alpha}\partial_k g_{j,\beta}-g_{j,\beta}\partial_k g_{i,\alpha})v_\ell^k\Big|^2\\
&\ge\frac{1}{\mu_1^2}\sum_{\ell=1}^{p}\sum_{k,s=1}^{n}H(X_k,X_s)v_\ell^k\bar v_\ell^s=\frac{1}{\mu_1^2}\sum_{\ell=1}^{p}\sum_{k=1}^{n}|v_\ell^k|^2.
\end{aligned}$$

The inequality is because in the inner product $H(X,Y)$ it only takes the summation

$$\sum_{1\le i\neq j\le p}\Big|(g_{i,\alpha}\partial_k g_{j,\beta}-g_{j,\beta}\partial_k g_{i,\alpha})v_\ell^k\Big|^2.$$

Thus if we choose $a=\sum_{j,k}|v_j^k|^2$ in (34) then we are done.

What remaining is to verify claim 2. Expand the term carefully

$$\begin{aligned}
&\sum_{i=1}^{p}|g_{i,\beta}y_i^\alpha|^2+|g_{i,\alpha}y_i^\beta|^2+\sum_{1\le i\neq j\le p}2Re\, g_{i,\beta}y_j^\alpha\overline{g_{j,\alpha}y_i^\beta}\\
&=\sum_{i=1}^{p}|g_{i,\beta}y_i^\alpha|^2+\sum_{1\le i<j\le p}2Re\, g_{i,\beta}y_j^\alpha\overline{g_{j,\alpha}y_i^\beta}\\
&\quad+\sum_{i=1}^{p}|g_{i,\alpha}y_i^\beta|^2+\sum_{1\le i<j\le p}2Re\, g_{j,\beta}y_i^\alpha\overline{g_{i,\alpha}y_j^\beta}.
\end{aligned}$$

Observe

$$\left|\sum_{i=1}^{p}g_{i,\alpha}\bar{y}_i^\beta\right|^2=\sum_{i=1}^{p}|g_{i,\alpha}y_i^\beta|^2+\sum_{1\le i<j\le p}2Re\, g_{i,\beta}y_j^\alpha\overline{g_{j,\beta}y_i^\alpha}$$

and

$$g_{j,\beta}y_i^\alpha=g_{j,\alpha}y_i^\beta$$

by the compatible condition

$$g_{j,\beta}y_i^\alpha-g_{j,\alpha}y_i^\beta=\begin{vmatrix}u_\alpha\bar{v}_i & g_{j,\alpha}\\ u_\beta\bar{v}_i & g_{j,\beta}\end{vmatrix}=0. \tag{35}$$

Therefore

$$\sum_{i=1}^{p}|g_{i,\beta}y_i^\alpha|^2+\sum_{1\le i<j\le p}2Re\, g_{i,\beta}y_j^\alpha\overline{g_{j,\alpha}y_i^\beta}=\left|\sum_{i=1}^{p}g_{i,\alpha}\bar{y}_i^\beta\right|^2.$$

Similarly

$$\sum_{i=1}^{p}|g_{i,\alpha}y_i^\beta|^2+\sum_{1\le i<j\le p}2Re\, g_{i,\alpha}y_j^\beta\overline{g_{j,\beta}y_i^\alpha}=\left|\sum_{i=1}^{p}g_{i,\beta}\bar{y}_i^\alpha\right|^2.$$

Thus claim 2 is verified. □

Remark 6. A quick glance of the computation. The fundamental inequality is a mixture of a Cauchy-Schwartz inequality with a completion of square. What on the

R.H.S. of (31) is

$$2\left|\sum_{\alpha=1}^{r}\sum_{\substack{1\le j\le p\\1\le k\le n}} u_\alpha \overline{\partial_k\Big(\frac{g_{j,\alpha}}{\mu_1}\Big)v_j^k}\right| = \frac{1}{\mu_1^2}\cdot 2\sqrt{\Big|\sum_{k=1}^{n} H(X_k,Y_k)\Big|^2}.$$

Here comes the Cauchy-Schwartz

$$2\sqrt{\Big|\sum_{k=1}^{n} H(X_k,Y_k)\Big|^2} \le a\sum_{k=1}^{n} H(X_k,X_k) + \frac{1}{a}\sum_{k=1}^{n} H(Y_k,Y_k), \tag{36}$$

where a is any constant, especially we choose

$$a = \sum_{k=1}^{n}\sum_{j=1}^{p} |v_j^k|^2.$$

By choosing a orthonormal basis we can compute the R.H.S of (36). It is an identity

$$\frac{1}{\mu^2}\cdot a\sum_{k=1}^{n} H(X_k,X_k) \le n\sum_{j=1}^{p}\sum_{k,\ell=1}^{n}(\partial_k\bar{\partial}_\ell \log\mu_1)v_j^k\bar{v}_j^\ell$$

and a completion of square

$$\begin{aligned} H(Y_k,Y_k) =& \sum_{\alpha,\beta=1}^{r}\sum_{i,j=1}^{p}|g_{i,\beta}|^2|y_j^\alpha|^2 - \sum_{\alpha=1}^{r}\left|\sum_{i=1}^{p} g_{i,\alpha}\bar{y}_i^\alpha\right|^2 \\ &- \sum_{1\le\alpha<\beta\le r}\left(\left|\sum_{i=1}^{p} g_{i,\alpha}\bar{y}_i^\beta\right|^2 + \left|\sum_{i=1}^{p} g_{i,\beta}\bar{y}_i^\alpha\right|^2\right) \\ \le&\mu_1\sum_{\alpha=1}^{r}\sum_{j=1}^{p}|u_\alpha|^2|v_j^k|^2 \end{aligned}$$

for every k. Note that we drop the sub-index of y_k and $(y_k)_j^\alpha := u_\alpha v_j^k$. In the completion of square

$$\sum_{\alpha=1}^{r}\left|\sum_{i=1}^{p} g_{i,\alpha}\bar{y}_i^\alpha\right|^2$$

corresponds to the regular part, i.e. the original Skoda's work. The new part is

$$\sum_{1\leq\alpha<\beta\leq r}\left(\left|\sum_{i=1}^{p} g_{i,\alpha}\bar{y}_i^{\beta}\right|^2+\left|\sum_{i=1}^{p} g_{i,\beta}\bar{y}_i^{\alpha}\right|^2\right)$$

which corresponds to the cross terms. The indexes $\alpha\neq\beta$ means they are in the different rows. Therefore the R.H.S of (36)

$$\leq n\sum_{j=1}^{p}\sum_{k,\ell=1}^{n}(\partial_k\bar{\partial}_\ell\log\mu_1)v_j^k\bar{v}_j^\ell+\frac{1}{\mu_1}\sum_{\alpha=1}^{r}|u_\alpha|^2$$

which is precisely the L.H.S of (31).

6. Algebraic Verification And Vanishing Theorems

In this section we want to use indexes to verify the identities

$$\partial\bar{\partial}\log\Big(\sum_{1\leq i_1<\cdots<i_r\leq p}|g_{i_1}\wedge\cdots\wedge g_{i_r}|^2\Big)= \text{ Tr }\beta\wedge\beta^*,$$

then use this identity to get a vanishing theorem. First let use compute the second fundamental form explicitly. In this section we assume

$$E=\Omega\times\mathbb{C}^p,$$

i.e. E is trivial, and there is an exact sequence

$$0\longrightarrow S\longrightarrow E\longrightarrow V\longrightarrow 0\ .$$

Let us recall two notations

- $\mu=\displaystyle\sum_{1\leq i_1<\cdots<i_r\leq p}|g_{i_1}\wedge\cdots\wedge g_r|^2,$
- $\mu_m=\displaystyle\sum_{1\leq i_1<\cdots<i_r\leq p}|g_{i_1}\wedge\cdots\wedge g_m|^2,$

and define the projection operator

$$\Pi(v)_k = \frac{1}{\mu} \sum_{1 \le i_1 < \cdots < i_{r-1} \le p} (G(v) \wedge g_{i_1} \wedge \cdots \wedge g_{r-1}) \overline{(g_k \wedge g_{i_1} \wedge \cdots \wedge g_{r-1})},$$

where any section v of $E = \Omega \times \mathbb{C}^p$. The second fundamental form of S in E is

$$\beta(v)_\ell = \Pi(\partial(v - \Pi(v)))_\ell,$$

so

$$\begin{aligned}
\beta_{\ell j} &= \beta(e_j)_\ell = -\Pi(\partial(\Pi(e_j)))_\ell \\
&= -\Pi\Big(\frac{1}{\mu} \sum_{1 \le i_1 < \cdots < i_{r-1} \le p} (g_j \wedge g_{i_1} \wedge \cdots \wedge g_{r-1}) \overline{(g_k \wedge g_{i_1} \wedge \cdots \wedge g_{r-1})}\Big) \\
&= \frac{1}{\mu} \sum_{\substack{1 \le i_1 < \cdots < i_{r-1} \le p \\ 1 \le j_1 < \cdots < j_{r-1} \le p}} \sum_k \mu_{k, j_1, \cdots, j_{r-1}} \overline{\mu_{\ell, j_1, \cdots, j_{r-1}} \mu_{k, i_1, \cdots, i_{r-1}}} \partial\Big(\frac{\mu_{j, i_1, \cdots, i_{r-1}}}{\mu}\Big) \\
&= \sum_{1 \le i_1 < \cdots < i_{r-1} \le p} \overline{\mu_{\ell, i_1, \cdots, i_{r-1}}} \partial\Big(\frac{\mu_{j, i_1, \cdots, i_{r-1}}}{\mu}\Big).
\end{aligned}$$

The third equality is by the identity of multi-linear algebra

(37)
$$\sum_k \sum_{1 \le j_1 < \cdots < j_{r-1} \le p} \mu_{k, j_1, \cdots, j_{r-1}} \overline{\mu_{\ell, j_1, \cdots, j_{r-1}} \mu_{k, i_1, \cdots, i_{r-1}}} = \mu \cdot \overline{\mu_{\ell, i_1, \cdots, i_{r-1}}}.$$

Hence

- $\displaystyle\sum_\ell |\beta_{\ell j} \lrcorner\, v_j|^2 = \sum_\ell \Big| \sum_{1 \le i_1 < \cdots < i_{r-1} \le p} \sum_j \overline{\mu_{\ell, i_1, \cdots, i_{r-1}}} \partial_\nu \Big(\frac{\mu_{j, i_1, \cdots, i_{r-1}}}{\mu}\Big) v_j^\nu \Big|^2.$
- $\displaystyle(T^{-1}u)_k = \frac{1}{\mu} \sum_{1 \le i_1 < \cdots < i_{r-1} \le p} (u \wedge g_{i_1} \wedge \cdots \wedge g_{r-1}) \overline{(g_k \wedge g_{i_1} \wedge \cdots \wedge g_{r-1})}.$

By these two identities we can verify

Lemma 6.1.

$$(\bar\partial T^{-1} u, v) = (T^{-1} u, \beta \lrcorner\, v).$$

Proof. By explicit computation, we have

$$
\begin{aligned}
(\bar\partial T^{-1}u, v) =&\sum_j \sum_{1\le i_1<\cdots<i_{r-1}\le p} (u, i_1, \cdots, i_{r-1})\overline{\partial_\nu\Big(\frac{\mu_{j,i_1,\cdots,i_{r-1}}}{\mu}\Big)v_j^\nu} \\
=&\frac{1}{\mu}\sum_j \sum_{1\le i_1<\cdots<i_{r-1}\le p} (u, i_1, \cdots, i_{r-1})\overline{\mu_{j,i_1,\cdots,i_{r-1}}} \\
&\cdot\sum_k \sum_{1\le j_1<\cdots<j_{r-1}\le p} \mu_{j,i_1,\cdots,i_{r-1}}\overline{\partial_\nu\Big(\frac{\mu_{j,i_1,\cdots,i_{r-1}}}{\mu}\Big)v_j^\nu} \\
=&(T^{-1}u, \beta\lrcorner v)
\end{aligned}
\qquad (38)
$$

The second equality we applied the variation of (37)

$$
\sum_j \sum_{1\le j_1<\cdots<j_{r-1}\le p} (u, j_1, \cdots, j_{r-1})\mu_{j,i_1,\cdots,i_{r-1}}\overline{\mu_{j,j_1,\cdots,j_{r-1}}} = \mu\cdot(u, i_1, \cdots, i_{r-1}). \qquad (37')
$$

□

We want to use these explicit expressions to verify the other identity

Lemma 6.2.

$$
\mathrm{Tr}\ \beta\wedge\beta^* = \partial\bar\partial\log\mu.
$$

Proof. On one hand we can compute

$$
\partial_\nu\bar\partial_\lambda\log\mu = \frac{\mu\partial_\nu\bar\partial_\lambda\mu - \partial_\nu\mu\bar\partial_\lambda\mu}{\mu^2}.
$$

On the other hand, by the expression of $\beta_{\ell j}$ we can compute

$$
\begin{aligned}
\beta\wedge\beta^* =& \sum_{j,\ell}\bar\beta_{\ell j}\beta_{\ell j} \\
=&\sum_{j,\ell}\sum_{\substack{1\le i_1<\cdots<i_{r-1}\le p\\ 1\le k_1<\cdots<k_{r-1}\le p}} \overline{\mu_{\ell,i_1,\cdots,i_{r-1}}}\partial_\nu\Big(\frac{\mu_{j,i_1,\cdots,i_{r-1}}}{\mu}\Big)\mu_{\ell,k_1,\cdots,k_{r-1}}\overline{\partial_\lambda\Big(\frac{\mu_{j,k_1,\cdots,k_{r-1}}}{\mu}\Big)}.
\end{aligned}
$$

By the help of geometry we can crack this verification into several steps. By the explicit datum we have

- The second fundamental form

$$\text{Tr } \beta \wedge \beta^* = \sum_{j,\ell} \sum_{I',K'} \overline{\mu_{\ell,I'}} \partial_\nu \Big(\frac{\mu_{j,K'}}{\mu} \Big) \mu_{\ell,I'} \overline{\partial_\lambda \Big(\frac{\mu_{j,K'}}{\mu} \Big)}.$$

- Quotient metric

$$h_{i\bar{j}} = \langle T^{-1} e_i, T^{-1} e_j \rangle_{H_0} = \frac{1}{\mu} \sum_{K'} (e_i, K') \overline{(e_j, K')}.$$

- Connection

$$\nabla u = T \d{T}^{-1} u = \sum_k \sum_{I'} g_{k,j} (\partial_\nu + \bar{\partial}_\lambda) \Big(\frac{(u, I')}{\mu} \overline{\mu_{k,I'}} \Big).$$

- Curvature

$$(\Theta_V)_{ij\nu\bar{\lambda}} = - \sum_{k,I'} g_{k,j} \bar{\partial}_\lambda \Big(\partial_\nu \Big(\frac{(e_i, I')}{\mu} \Big) \overline{\mu_{k,I'}} \Big) + \sum_{k,I'} \partial_\nu \Big(g_{k,j} (e_i, I') \overline{\partial_\lambda \Big(\frac{\mu_{k,I'}}{\mu} \Big)} \Big)$$
$$- \theta_{i\ell}^{(0,1)} \theta_{\ell j}^{(1,0)} + \theta_{i\ell}^{(1,0)} \theta_{\ell j}^{(0,1)}.$$

Geometrically we expect

$$\text{Tr } \beta \wedge \beta^* = \text{ Tr } \Theta_V = \partial \bar{\partial} \log \mu.$$

The first equality comes from $\Theta_V = \beta \wedge \beta^*$ by choosing a normal coordinate and direct computation, and the second identity comes from the curvature of the determinant of the quotient bundle by choosing a holomorphic frame. Thus it is not obvious these two terms are equal. Let us verify the first identity.

Expand $\partial_\nu(\mu_{j,I'}/\mu)$ and we get

$$\begin{aligned}
&\sum_{j,\ell}\sum_{I',K'}\overline{\mu_{\ell,I'}}\partial_\nu\Big(\frac{\mu_{j,K'}}{\mu}\Big)\mu_{\ell,I'}\overline{\partial_\lambda\Big(\frac{\mu_{j,K'}}{\mu}\Big)}\\
&=\sum_{j,\ell}\sum_{I',K'}\overline{\mu_{\ell,I'}}\Big(\frac{\partial_\nu\mu_{j,K'}}{\mu}-\frac{\mu_{j,K'}\partial_\nu\mu}{\mu^2}\Big)\mu_{\ell,I'}\overline{\partial_\lambda\Big(\frac{\mu_{j,K'}}{\mu}\Big)}\\
&=\sum_{j,\ell}\sum_{I',K'}\frac{\overline{\mu_{\ell,I'}}\partial_\nu\mu_{j,K'}\mu_{\ell,I'}}{\mu}\overline{\partial_\lambda\Big(\frac{\mu_{j,K'}}{\mu}\Big)}.
\end{aligned}$$

The second identity comes from moving the term $\mu_{j,K'}$ into $\overline{\partial_\lambda(\mu_{j,K'}/\mu)}$. We expand the numerator and get

$$\sum_{\ell,I'}\overline{\mu_{\ell,I'}}\mu_{\ell,K'}\partial_\nu\mu_{j,I'}\equiv\mu\cdot(\partial_\nu g_j,K')\mod \mu_{j,K'}.$$

Hence

$$\sum_{j,\ell}\sum_{I',K'}\overline{\mu_{\ell,I'}}\partial_\nu\Big(\frac{\mu_{j,K'}}{\mu}\Big)\mu_{\ell,I'}\overline{\partial_\lambda\Big(\frac{\mu_{j,K'}}{\mu}\Big)}=\sum_{j,K'}(\partial_\nu g_j,K')\overline{\partial_\lambda\Big(\frac{\mu_{j,K'}}{\mu}\Big)}.$$

Let us compute $\operatorname{Tr}\Theta_V$ from the induced (quotient) metric $h_{i\bar{j}}$. By the explicit expressions above the connection metric can be written as

$$\theta_{ij}=\sum_k\sum_{I'}g_{k,j}(\partial_\nu+\bar{\partial}_\lambda)\Big(\frac{(e_i,I')}{\mu}\overline{\mu_{k,I'}}\Big),$$

which implies

$$\begin{aligned}
\operatorname{Tr}\Theta_V&=-\sum_k\sum_{j=1}^{r}\sum_{I'}g_{k,j}\bar{\partial}_\lambda\Big(\partial_\nu\Big(\frac{(e_j,I')}{\mu}\Big)\overline{\mu_{k,I'}}\Big)\\
&=-\bar{\partial}_\lambda\Big(\sum_k\sum_{I'}\partial_\nu\Big(\frac{\mu_{k,I'}}{\mu}\Big)\overline{\mu_{k,I'}}-(\partial_\nu g_k,I')\frac{\overline{\mu_{k,I'}}}{\mu}\Big)\\
&=\sum_k\sum_{I'}(\partial_\nu g_k,I')\overline{\partial_\lambda\Big(\frac{\mu_{k,I'}}{\mu}\Big)}.
\end{aligned}$$

Note the first identity is based on the explicitly computation and one can found $\partial_\nu \bar{\partial}_\lambda (\cdots) = 0$ and $-\theta^{(0,1)}_{i\ell}\theta^{(1,0)}_{\ell j} + \theta^{(1,0)}_{i\ell}\theta^{(0,1)}_{\ell j} = 0$. By further expansion we have

$$\begin{aligned}
&\sum_k \sum_{I'} (\partial_\nu g_k, I') \overline{\partial_\lambda \Big(\frac{\mu_{k,I'}}{\mu} \Big)} \\
&= \sum_{k<I'} \Big((\partial_\nu g_k, I') + (k, \partial_\nu i_1, I'') + \cdots \Big) \overline{\partial_\lambda \Big(\frac{\mu_{k,I'}}{\mu} \Big)} \\
&= \sum_{k<I'} \partial_\nu \mu_{k,I'} \frac{\mu \overline{\partial_\lambda \mu_{k,I'}} - \overline{\mu_{k,I'}} \bar{\partial}_\lambda \mu}{\mu^2} = \frac{\mu \partial_\nu \bar{\partial}_\lambda \mu - \partial_\nu \mu \bar{\partial}_\lambda \mu}{\mu^2}.
\end{aligned}$$

Hence we are done. □

We have another related observation.

Lemma 6.3.

$$\det(h_{i\bar{j}}) = \det \Big(\frac{1}{\mu} \sum_{K'} (e_i, K') \overline{(e_j, K')} \Big) = \frac{1}{\mu}.$$

Proof. By linear algebra.

$$\begin{aligned}
\langle e_i, e_j \rangle_{H_1'} &= \langle G^*(GG^*)^{-1} e_i, G^*(GG^*)^{-1} e_j \rangle_{H_0} \\
&= \langle (GG^*)^{-1} e_i, GG^*(GG^*)^{-1} e_j \rangle_{H_1} \\
&= \langle (GG^*)^{-1} e_i, e_j \rangle_{H_1},
\end{aligned}$$

especially when $(V, H_1) = (\mathbb{C}^r, I_r)$ then $H_1' = (GG^*)^{-1}$. Therefore

$$\det(GG^*)^{-1} = \frac{1}{\mu}.$$

□

The algebraic verification has a nice corollary. The rest of the section will be devoted into the related vanishing theorem. For $E = X \times \mathbb{C}^p$, i.e. E is trivial we have verified

$$\partial \bar{\partial} \log \mu = \operatorname{Tr} \beta \wedge \beta^*.$$

By explicit computation we have an observation.

Lemma 6.4. *Let X be a compact complex algebraic manifold, $E = X \times \mathbb{C}^p$ be a trivial vector bundle and S is a holomorphic sub vector bundle of E which associates an exact sequence*

$$0 \longrightarrow S \longrightarrow E \xrightarrow{g} Q \longrightarrow 0 \ .$$

Set

$$Z = \{x \in X \big| \det(\partial_k \bar{\partial}_\ell \log |g|^2) = 0\},$$

then Z is an analytic sub manifold of X. (Z can be equal to X.)

Proof. This is a merit of explicit computation. Instead of using μ, it is equivalent to compute the case

$$|g|^2 = \sum_{i=1}^{p} |g_i|^2.$$

Writing $h_{k\bar{\ell}} = \partial_k \bar{\partial}_\ell \log |g|^2$ explicitly, there are two cases:

- *If $k = \ell$.* $h_{k\bar{k}} = \sum_{i<j} \frac{1}{(|g|^2)^2} |g_i \partial_k g_j - g_j \partial_k g_i|^2.$
- *If $k \neq \ell$.* $h_{k\bar{\ell}} = \sum_{i<j} \frac{1}{(|g|^2)^2} (g_i \partial_k g_j - g_j \partial_k g_i)\overline{(g_i \partial_\ell g_j - g_j \partial_\ell g_i)}.$

For short, write

$$\varphi_{I,k} = \frac{1}{|g|^2}(g_i \partial_k g_j - g_j \partial_k g_i)$$

where $I = (i, j)$, then $h_{k\bar{k}} = \sum_I |\varphi_{I,k}|^2$ and $h_{k\bar{\ell}} = \sum_I \varphi_{I,k}\overline{\varphi_{I,\ell}}$. Hence

$$(h_{k\ell}) = BB^*,$$

where

$$B = \begin{vmatrix} \varphi_{I_1,1} & \varphi_{I_2,1} & \cdots & \varphi_{I_N,1} \\ \varphi_{I_1,2} & \varphi_{I_2,2} & \cdots & \varphi_{I_N,2} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{I_1,n} & \varphi_{I_2,n} & \cdots & \varphi_{I_N,n} \end{vmatrix} \tag{39}$$

and $N = C_2^p$. We can also compute the determinant

$$\det(h_{k\bar{\ell}}) = \sum_{I_1,\cdots,I_n} \begin{vmatrix} |\varphi_{I_1,1}|^2 & \varphi_{I_2,1}\overline{\varphi_{I_2,2}} & \cdots & \varphi_{I_n,1}\overline{\varphi_{I_n,n}} \\ \varphi_{I_1,2}\overline{\varphi_{I_1,1}} & |\varphi_{I_2,2}|^2 & \cdots & \varphi_{I_n,2}\overline{\varphi_{I_n,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{I_1,n}\overline{\varphi_{I_1,1}} & \varphi_{I_n,n}\overline{\varphi_{I_n,2}} & \cdots & |\varphi_{I_n,n}|^2 \end{vmatrix}$$

$$= \sum_{I_1,\cdots,I_n} \overline{\varphi_{I_1,1}\varphi_{I_2,2}\cdots\varphi_{I_n,n}} \begin{vmatrix} \varphi_{I_1,1} & \varphi_{I_2,1} & \cdots & \varphi_{I_n,1} \\ \varphi_{I_1,2} & \varphi_{I_2,2} & \cdots & \varphi_{I_n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{I_1,n} & \varphi_{I_2,n} & \cdots & \varphi_{I_n,n} \end{vmatrix}$$

$$= \sum_{I_1<\cdots<I_n} \left| \begin{vmatrix} \varphi_{I_1,1} & \varphi_{I_2,1} & \cdots & \varphi_{I_n,1} \\ \varphi_{I_1,2} & \varphi_{I_2,2} & \cdots & \varphi_{I_n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{I_1,n} & \varphi_{I_2,n} & \cdots & \varphi_{I_n,n} \end{vmatrix} \right|^2.$$

For short, denote

$$\Phi_{I_1,\cdots,I_n} = \begin{vmatrix} \varphi_{I_1,1} & \varphi_{I_2,1} & \cdots & \varphi_{I_n,1} \\ \varphi_{I_1,2} & \varphi_{I_2,2} & \cdots & \varphi_{I_n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{I_1,n} & \varphi_{I_2,n} & \cdots & \varphi_{I_n,n} \end{vmatrix},$$

then $\det(h_{k\bar{\ell}}) = \sum_{I_1<\cdots<I_n} |\Phi_{I_1,\cdots,I_n}|^2$. Therefore

$$Z = \{x \in X \big| \Phi_{I_1,\cdots,I_n}(x) = 0\},$$

which is analytic. □

Though we only show the case that $g: \ E \longrightarrow L$, where L is a line bundle, the general case can be easily deduced from the same formula. By the explicit computation we only need to replace $|g|^2$ by μ and g_j by μ_J, then the same conclusion follows.

In the following we want to show the special class $-\beta^*\tilde{f}$ in Skoda's theorem satisfies the finiteness condition

$$(\beta^*\tilde{f}, \Theta^{-1}\beta^*\tilde{f}) < \infty.$$

Lemma 6.5.

$$(\beta^*\tilde{f},\ \text{Tr}\ (\beta\beta^*)^{-1}\beta^*\tilde{f}) \leq n|\tilde{f}|^2,$$

where n is the dimension of Ω.

Proof. Compute $(\beta^*\tilde{f},\ \text{Tr}\ (\beta\beta^*)^{-1}\beta^*\tilde{f})$ by

$$\begin{aligned} &\sum_i \Big(\sum_{k,\ell} \sum_j \overline{\beta_{ij,k}} \beta_{ij,\ell} \Theta^{-1}_{k\bar{\ell}} \Big) |\tilde{f}_i|^2 \\ \leq &\sum_i \Big(\sum_{k,\ell} \sum_j \overline{\beta_{ij,k}} \beta_{ij,\ell} \Theta^{-1}_{k\bar{\ell}} \Big) \sum_i |\tilde{f}_i|^2 \end{aligned}$$

because $\sum_{k,\ell} \overline{\beta_{ij,k}} \beta_{ij,\ell} \Theta^{-1}_{k\bar{\ell}} \geq 0$ for every j. Note the location of conjugation is by the functorial of column vector. Let us compute

$$\begin{aligned} &\sum_i \Big(\sum_{k,\ell} \sum_j \overline{\beta_{ij,k}} \beta_{ij,\ell} \Theta^{-1}_{k\bar{\ell}} \Big) = \sum_{k,\ell} \sum_{i,j} \overline{\beta_{ij,k}} \beta_{ij,\ell} \Theta^{-1}_{k\bar{\ell}} \\ = &\ \text{Tr}\ (\Theta\Theta^{-1}) \leq n. \end{aligned}$$

Hence we are done. □

Therefore if $\tilde{f} = g^{-1}f$, then

$$\|g^{-1}f\|^2_\varphi = \int \frac{\sum |f \wedge g_{i_1} \wedge \cdots \wedge g_{i_{r-1}}|^2}{(\sum |g_{i_1} \wedge \cdots \wedge g_{r_r}|^2)^{N+1}} < \infty$$

would imply the $\bar{\partial}$-equation $\bar{\partial}h = \beta^*(g^{-1}f)$ is solvable, and this is exactly the L^2-condition in Skoda's theorem.

Example 6.6. *In the $n = 1$ case we have the following identity*

$$\sum_{k,\ell} \Theta^{-1}_{k\ell} \Big(\partial_k \frac{g_j}{|g|^2}\Big) \overline{\Big(\partial_\ell \frac{g_j}{|g|^2}\Big)} = \frac{1}{|g|^2},$$

by explicit expression. This implies $(\beta^* \tilde{f}, \Theta^{-1}\beta^* \tilde{f}) = |f|^2/|g|^2 = |\tilde{f}|^2$. For $n = 1$ case, it is easy to see the identity.

Proof. It means we have to compute the ratio of

$$\begin{aligned}
&\frac{\sum_j \big|\sum_i (g_i\partial_z g_j - g_j\partial_z g_i)\bar{g}_i\big|^2}{(|g|^2)^4} \\
&= \frac{\sum_j \sum_i |(g_i\partial_z g_j - g_j\partial_z g_i)|^2 \sum_m |g_m|^2 - \sum_{i<m} |(\cdots)|^2}{(|g|^2)^4} \\
&= \frac{\sum_{i<m} |(g_i\partial_z g_m - g_m\partial_z g_i)|^2}{(|g|^2)^3}
\end{aligned}$$

and

$$\frac{\sum_{i<j} |(g_i\partial_z g_j - g_j\partial_z g_i)|^2}{(|g|^2)^2},$$

which is exactly $1/|g|^2$. □

Let us recall a notation

$$\frac{(\widetilde{BB^*}h, h)}{\det(BB^*)} = ((BB^*)^{-1}h, h) = (\Theta^{-1}h, h).$$

We use this notation in order to agree the one applied in Skoda's paper.

Theorem 6.7. *Let X be a weakly pseudo-convex manifold of dimension n and $E = X \times \mathbb{C}^p$ be a trivial vector bundle. Consider a pair of holomorphic vector bundles and a non-trivial holomorphic sub-vector bundle (S^s, E^p), then for the cohomology class represented by a $(0, k)$-form f satisfying*

$$\int_X \frac{(\widetilde{BB^*}f, f)}{\det(BB^*)(|g|^2)^N} \mathrm{d}V_X < \infty$$

is vanishing for $k \geq 1$ and $N > \min\{n, s\}$. Note that the finiteness condition requires f in the domain of Θ^{-1}. For the case of $k = 1$ and $Z \subsetneq X$ we have a Cramer's rule type condition:

$$\int_X \frac{\sum |f_i \wedge \varphi_{I_1} \wedge \cdots \wedge \varphi_{I_{n-1}}|^2}{\sum |\varphi_{I_1} \wedge \cdots \wedge \varphi_{I_n}|^2 (|g|^2)^N} \mathrm{d}V_X < \infty.$$

Proof. The proof is essentially repeat Skoda's computation. For technique reason first we will assume $Z \subsetneq X$, and take a exhaustion Ω_ν of $X - Z$. Thus we can solve the $\bar\partial$-equation on each Ω_ν with estimates, then take the limit to get the solution on X. Let us now focus on the pseudo-convex domain Ω_ν. For short we will denote this domain by Ω. Let us go back to the estimates. Provided an extra line bundle M such that $M \geq N \,\mathrm{Tr}\, \beta \wedge \beta^*$ we have

$$(\sqrt{-1}(S \otimes M)\Lambda v, v) \geq (N - \beta_1)(\sqrt{-1}\,\mathrm{Tr}\, \beta \wedge \beta^* v, v).$$

Later M can be chosen as $\det(E/S)^{\otimes N}$. By the identity

$$\mathrm{Tr}\, \beta \wedge \beta^* = \partial\bar\partial \log \mu = BB^*$$

we can take the fundamental inequality and Cauchy-Schwartz inequality to get the estimates

$$\begin{aligned}
|(\tilde f, v)_\varphi|^2 &\leq (BB^* v, v)_\varphi (\tilde f, (BB^*)^{-1} \tilde f)_\varphi \\
&= (N - \beta_1)(\sqrt{-1}\,\mathrm{Tr}\, \beta \wedge \beta^* v, v)_\varphi \cdot \frac{1}{N - \beta_1} (\tilde f, (BB^*)^{-1} \tilde f)_\varphi \\
&\leq C^2 \cdot (\|\bar\partial v\|_\varphi^2 + \|\bar\partial^* v\|_\varphi^2),
\end{aligned}$$

where $C^2 = 1/(N - \beta_1) \cdot (\tilde f, (BB^*)^{-1} \tilde f)_\varphi$. More precisely the above inequality can be written with indexes

$$\Big|\sum_{i,\alpha} \bar f_{i,\alpha} v_i^\alpha\Big|^2 \leq \Big(\sum_i \bar f_{i,\beta} \Theta_{\alpha\bar\beta} f_{i,\alpha}\Big)\Big(\sum_i \bar v_i^\alpha \Theta^{\beta\bar\alpha} v_i^\beta\Big)$$

which is the practice of the inequality $|\text{ Tr } A^*B|^2 \leq \text{ Tr } (A^*\Theta A)\cdot \text{ Tr } (B^*\Theta^{-1}B)$. Thus the $\bar\partial$-equation

$$\bar\partial h = \tilde f$$

can be solved with the estimate $\|h\|^2_\varphi \leq C^2$. Note that $(BB^*)^{-1}$ exists on Ω because we exclude its zero locus before taking the exhaustion.

If $Z = X$, then we have to take a different exhaustion. Let W be the common zero loci of g_j, let Ω_σ be the exhaustion of $X - W$. Like before we would drop the index of Ω for short. On Ω we collect all holomorphic functions with finite L^2_1 norm

$$H(\Omega) = \{g = (g_1, \cdots, g_p)\big|\|g\|^2_1 < \infty\}.$$

Let us consider all holomorphic functions g such that its associated curvature matrix $\Theta = \partial\bar\partial \log |g|^2 = BB^*$ has 0 eigen value, i.e. $\det(BB^*) = 0$. Or it is equivalent to $Z = X$. By the explicit computation, g_j satisfy

$$\Phi_{I_1,\cdots,I_n} = 0$$

for every $I_1, \cdots, I_n$, so it is a closed subset of $H(\Omega)$. Let us denote the set by $Z(\Omega)$. Suppose $g \in Z(\Omega)$, then we can choose a sequence $\langle g_t\rangle$ in $H(\Omega)\backslash Z(\Omega)$ such that $g_t \to g$ in the sense of L^2_1. Thus for each g_t we can apply the previous estimates, then pass to g by taking limit. Let use verify $\Theta_t = B_tB_t^*$ would converge to $\Theta = BB^*$. It is enough to show that $B_t \to B$, and this can be examed by entry-wise verification. Denote $g_{j,t}$ by g'_j for short. Compute

$$\begin{aligned}
&\Big|\frac{g_i\partial_k g_j - g_j\partial_k g_i}{|g|^2} - \frac{g'_i\partial_k g'_j - g'_j\partial_k g'_i}{|g'|^2}\Big| \\
&\leq \Big|\frac{g_i\partial_k g_j}{|g|^2} - \frac{g'_i\partial_k g_j}{|g|^2}\Big| + \Big|\frac{g'_i\partial_k g'_j}{|g'|^2} - \frac{g'_i\partial_k g_j}{|g'|^2}\Big| + \Big|\frac{g'_i\partial_k g_j}{|g|^2} - \frac{g'_i\partial_k g_j}{|g'|^2}\Big| \\
&= \frac{|\partial_k g_j||g_i - g'_i|^2}{|g|^2} + \frac{|g'_i||\partial_k g_j - \partial_k g'_j|}{|g|^2} + \frac{|g'_i\partial_k g_j|(|g'|^2 - |g|^2)}{|g|^2|g'|^2}
\end{aligned}$$

which would go to zero as g' converges to g in $H(\Omega)$. Note that the control of the third term depends on the choice of the exhaustion Ω_σ away from the common zero of g, so there is a lower bound of $|g|^2$ which makes the estimate work. In other words, given $\epsilon > 0$

$$\left|\frac{g_{i,t}\partial_k g_{j,t}}{|g_t|^2} - \frac{g_i \partial_k g_j}{|g|^2}\right| < \epsilon$$

for every $i \neq j, k$ and $t \gg 0$. This would imply the analytic finiteness condition

$$\int_X \frac{(f, (B_t B_t^*)^{-1} f)}{(|g|^2)^N} \mathrm{d}V_X < \infty$$

would hold by taking an appropriate sub-sequence of $\langle t \rangle$ because $(f, (B_t B_t^*)^{-1} f)$ would converge to $(f, (BB^*)^{-1} f)$. For simplicity of notation we still denote the sub-sequence by t. Thus by previous argument we can solve

$$\bar{\partial} h_t = f$$

on Ω with estimates $\|h_t\|^2 \leq C_t^2$, where

$$C_t^2 = (f, (B_t B_t^*)^{-1} f).$$

By construction and assumption, $(f, (B_t B_t^*)^{-1} f)$ would converge to a constant $C(f) = (f, \Theta^{-1} f)$ as $t \to \infty$ on Ω, and h_t would converge to h. Since $C(f)$ depends only on f, we can take limit again on the domain Ω_ν, then we get a global solution on X. Hence we are done. $\square$

Now we want to introduce a multiplier ideal sheaves version of vanishing theorem. Let h is a germ of E, define

- $\mathcal{I}_1(S, E, N) = \left\{ h \middle| \frac{\|h\|^2}{(\det(BB^*) \sum |g_{j_1} \wedge \cdots \wedge g_{i_r}|^2)^N} \text{ is locally integrable.} \right\}$, where is defined in (39).
- $\mathcal{I}_2(S, E, N) = \left\{ h \middle| \frac{\|h\|^2}{(\sum |g_{j_1} \wedge \cdots \wedge g_{i_r}|^2)^N} \text{ is locally integrable.} \right\}$

Note that both $\mathcal{I}_1$ and $\mathcal{I}_2$ are coherent sheaves. Since E is equipped with the trivial metric,

$$\|h\|_\varphi^2 = \sum_i \int |h_i|^2 e^{-\varphi}.$$

Thus $\|h\|_\varphi^2 < \infty$ would imply h_i in the multiplier ideal sheaf defined by $e^{-\varphi}$, and this multiplier ideal sheaf is coherent by the classic Nadel's theorem. Observing that $\mathcal{I}_1 \subseteq \mathcal{I}_2$, it is enough to show that the forms f such that $|f|^2/\det BB^*$ would form a sub-module of $\mathcal{I}_2$, and this is obvious.

Corollary 6.8. *Let X be a weakly pseudo-convex manifold of dimension n and $E = X \times \mathbb{C}^p$ be a trivial vector bundle. Consider a pair of holomorphic vector bundles and a non-trivial holomorphic sub-vector bundle (S^s, E^p), and a analytic set*

$$Z = \{x \in X \big| \det(\partial_k \bar{\partial}_\ell \log \mu) = 0\}.$$

Suppose $Z \subsetneqq X$ then the image of

$$H^k(X, \mathcal{I}_1(S, E, N) \otimes \det(E/S)^{\otimes N} \otimes K_X) \longrightarrow H^k(X, \mathcal{I}_2(S, E, N) \otimes \det(E/S)^{\otimes N} \otimes K_X)$$

vanishes for $k \geq 1$ and $N > \min\{n, s\}$.

Proof. This is a corollary of the proof of the theorem.

Noticing that in solving the $\bar{\partial}$-equation

$$\bar{\partial} h = \tilde{f},$$

h and $\tilde{f}$ are applied with different metrics, so we need to impose two multiplier ideal sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$. In computation of the cohomology of $\mathcal{I}_i$ we need to take a fine complex resolution. Here we take a common resolution of $\mathcal{I}_1$ and $\mathcal{I}_2$, i.e.

$$\pi : \hat{\Omega} \longrightarrow \Omega$$

such that $\pi^*\mathcal{I}_i = \mathcal{O}_{\hat{\Omega}}(\sum D_{i,j})$, where $D_i = \sum_j D_{i,j}$ are simple normal crossing divisors. This corresponds to get a fine complex resolution of $\pi^*\mathcal{I}_i$ on $\hat{\Omega}$. Hence the norm condition would induce a inclusion map

$$\pi^* S \otimes \mathcal{A}^{p,q} \otimes \pi^*\mathcal{I}_1 \longrightarrow \pi^* S \otimes \mathcal{A}^{p,q} \otimes \pi^*\mathcal{I}_2$$

on $\hat{\Omega}$, then it induces a map between the cohomology groups. Consider the following diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \pi^* S \otimes \mathcal{A}^{n,1} \otimes \pi^*\mathcal{I}_2 & \xrightarrow{\bar{\partial}} & \pi^* S \otimes \mathcal{A}^{n,2} \otimes \pi^*\mathcal{I}_2 & \xrightarrow{\bar{\partial}} & \cdots \\
& \uparrow & & \uparrow & & \\
0 \longrightarrow & \pi^* S \otimes \mathcal{A}^{n,1} \otimes \pi^*\mathcal{I}_1 & \xrightarrow{\bar{\partial}} & \pi^* S \otimes \mathcal{A}^{n,2} \otimes \pi^*\mathcal{I}_1 & \xrightarrow{\bar{\partial}} & \cdots .
\end{array}$$

Suppose f is a close (n,k) form which represents a cohomology class satisfying the finiteness assumption, i.e. f is a section of $\pi^* S \otimes \mathcal{A}^{n,k} \otimes \pi^*\mathcal{I}_1$ which would imply f satisfying

$$L_0 = \int_X \frac{(f, \Theta^{-1} f)}{(|g|^2)^N} < \infty.$$

Hence we can solve the $\bar{\partial}$-equation

$$\bar{\partial} h = f$$

with estimates

$$\int_X \frac{\|h\}^2}{(|g|^2)^N} < \frac{\alpha}{\alpha - 1} L_0$$

which implies $h \in \Gamma(\pi^* S \otimes \mathcal{A}^{n,k-1} \otimes \pi^*\mathcal{I}_2)$. Hence we can conclude the image of

$$H^k(\hat{X}, \pi^*(\mathcal{I}_1 \otimes \det(Q)^{\otimes N} \otimes K_X)) \longrightarrow H^k(\hat{X}, \pi^*(\mathcal{I}_2 \otimes \det(Q)^{\otimes N} \otimes K_X))$$

vanishes. Note that $Q = E/S$. Since $\pi : \hat{\Omega} \longrightarrow \Omega$ is a resolution, we have

$$H^k(\hat{X}, \pi^*(\mathcal{I}_i \otimes Q^{\otimes N} \otimes K)) = H^k(X, \pi^*(\mathcal{I}_i \otimes Q^{\otimes N} \otimes K)).$$

Therefore we are done. □

Remark 7. The vanishing theorem 6.8 in fact does not use the full power of the estimates of solving the $\bar{\partial}$-equation $\bar{\partial}h = f$. We made a stronger assumption of finiteness condition which is point-wise defined. It does not detect the variance of the direction on the tangent space TX. However the finiteness condition in theorem 6.7 indeed captured the variance. In summary there are two points that can be improved.

(1) *Remove the assumption of $Z \subsetneq X$.* In order to define the multiplier ideal sheaf $\mathcal{I}_1$, $\det\Theta = \det BB^*$ cannot be 0 entirely. Thus we have to make this extra assumption. In the general case of $Z = X$, the foliation define by $\ker\Theta$ should be introduced and is needed to study.

(2) *Find a better coherent sheaf that contains Skoda's class $\beta^* f$.* By point-wise inspection the condition $\|h\|^2/\det\Theta) < \infty$ is weaker than $(h, \Theta^{-1}h)$, so the Skoda's class is not in the section of $\mathcal{I}_1$. The reason why we take a weaker version is because this is the simplest way to define a coherent sheaf. Looking closely to the expression $(f, \Theta^{-1}f)$, it involves the tangent directions and is not possible to be recorded by a multiplier ideal sheaf with point-wise finiteness condition. It is expected to introduce the jet space to solve this problem.

7. Griffith Positivity

In this section we want to combine the projectivization technique and the fundamental inequality we derived to reduce the requirement of the positivity of the vector bundle E. Originally, the issue of positivity occurs in the applying of Bochner-Kodaira formula. If we work on E, V over Ω then the term involving curvature would take form of Nakano positivity. But if we work over $\mathbb{P}(E^*)$ then the requirement of positivity would become the positivity of the line bundle $\mathcal{O}_{E^*}(1)$ which corresponds to the Griffith positivity of E.

Theorem 7.1. *Let X be a pseudo-convex kähler manifold. Consider the diagram*

$$G : E^p \otimes M \otimes K \longrightarrow V^r \otimes M \otimes K \ .$$

Assume $G = (g_j)$ is generically surjective, $E \geq 0$ in the sense of Griffith and

$$\sqrt{-1}\Theta(M) \geq \alpha\beta_1\sqrt{-1}\Theta(V)$$

where

$$\beta_1 = \min\{n, p - r\}.$$

Suppose a holomorphic section f of $V \otimes M \otimes K$ satisfying

$$L_0 := \int_X \frac{\|f\|^2}{(\|g\|^2)^{\alpha\beta_1+1}} \mathrm{d}V_X < \infty,$$

then there exist h, a section of $E \otimes M \otimes K$, such that

$$f = Gh \ \ and$$

$$\int_X \frac{\|h\|^2}{(\|g\|^2)^{\alpha\beta_1}} \mathrm{d}V_X \leq \frac{\alpha}{\alpha - 1} L_0.$$

Let us recall the notations.

- (V, H_1), where H_1 is a given metric of V.
- $(E, h_{k\bar{\ell}})$, where $h_{k\bar{\ell}}$ is a given inner product. It is also denoted by H_0.
- $(M, e^{-\psi})$, where $e^{-\psi}$ is a given metric of M.
- $Lin(P, \mathcal{O}_{E^*}(1)) = \{\tilde{h} = \sum h_j z_j \big| (h_j(x))$ is a smooth section of $E\}$.
- $\|g\|^2 = \|g\|^2_{E^* \otimes V}$.
- $P = \mathbb{P}(E^*)$, and $\pi : P \longrightarrow \Omega$.
- $\Phi = \sum_{k,\ell} h_{k\bar{\ell}} z_k \bar{z}_\ell, e^{-\varphi_0} = \dfrac{re^{-\psi}}{\Phi(\|g\|^2)^{\alpha\beta_1}}$. Note the function of the factor r is to balance the coefficient generated from averaging.
- let $k = \alpha\beta_1$ when k is in the exponent of $\|g\|^2$.

- $H_0 = Lin^2(P, \mathcal{O}_{E^*}(1) \otimes M)$ is a Hilbert space with inner product

$$\begin{aligned}\langle \tilde{h}_1, \tilde{h}_2 \rangle_{\varphi_0} &= \int_P (\tilde{h}_1, \tilde{h}_2)_{\varphi_0} \mathrm{d}V_P = \int_P \frac{\tilde{h}_1 \overline{\tilde{h}_2} e^{-\psi}}{\Phi(\|g\|^2)^k} \mathrm{d}V_P \\ &= \frac{1}{r} \int_\Omega \frac{r(h_1, h_2)_E e^{-\psi}}{(\|g\|^2_{E^* \otimes V})^k} \mathrm{d}V_\Omega \\ &= \langle h_1, h_2 \rangle_{H_0}.\end{aligned}$$

  Note the identity comes from the average technique we developed before. Here we abuse the notation H_0. It represents the space and the metric at the same time.

- $H_1 = L^2(P, \pi^* V)$ is a Hilbert space with the inner product induced from E. In fact it is the quotient metric and denoted by H_1'. Hence

$$\begin{aligned}\langle f_1, f_2 \rangle_{H_1'} &= \int_P (f_1, f_2)_{H_1'} \mathrm{d}V_P = \int_P (T^{-1} f_1, T^{-1} f_2)_{\varphi_0} \mathrm{d}V_P \\ &= \frac{1}{r} \int_\Omega \frac{r(T^{-1} f_1, T^{-1} f_2)_E e^{-\psi}}{(\|g\|^2)^k} \mathrm{d}V_\Omega = \int_\Omega \frac{(f_1, f_2)_\varphi e^{-\psi}}{\|g\|^2} \frac{1}{(\|g\|^2)^k} \mathrm{d}V_\Omega \\ &= \langle T^{-1} f_1, T^{-1} f_2 \rangle_{H_0}.\end{aligned}$$

  Note we abuse notation H_1 again.

- $H_2 = Lin^2_{0,1}(P, \mathcal{O}_{E^*}(1))$ is a Hilbert space.

Initially we have the diagram

$$0 \longrightarrow K \otimes M \longrightarrow E \otimes M \xrightarrow{G} V \otimes M \longrightarrow 0$$

over Ω. After taking projectivization it becomes

$$Lin(P, \mathcal{O}_{E^*}(1) \otimes \pi^* M) \xrightarrow{T} C^\infty(P, \pi^* V \otimes M)$$

over $\mathbb{P}(E^*)$. Though there is no map between $\mathcal{O}_{E^*}(1)$ and $\pi^* V$, the map between sections is enough. The extra M is needed in general for the reason of curvature estimate. Like the functional analysis we used in last section let use consider the diagram

$$\begin{array}{ccc} H_0 & \xrightarrow{T} & H_1 \\ \Big\downarrow D & & \\ H_2 & & \end{array} ,$$

and compute

$$\begin{aligned} \|T^*f + \bar\partial^* v\|^2_{\varphi_0} + \|\bar\partial v\|^2_{\varphi_0} = &\|T^*f\|^2_{\varphi_0} + \|\bar\partial^* v\|^2_{\varphi_0} + \|\bar\partial v\|^2_{\varphi_0} \\ &+ 2Re\langle T^*f, \bar\partial^* v\rangle_{\varphi_0} \end{aligned}$$

and couple it with the Kodaira-Bochner formula, then we get

$$\geq \|f\|^2_{H_1'} + \langle \sqrt{-1}\partial\bar\partial\varphi_0 v, v\rangle_{\varphi_0} + 2Re\langle \bar\partial T^* f, v\rangle_{\varphi_0}.$$

Look closely to the curvature term. It can be decomposed into three parts

$$\begin{aligned} \partial\bar\partial\varphi_0 = &\partial\bar\partial \log \Phi + (\partial\bar\partial\psi - k\partial\bar\partial\varphi) \\ &+ \alpha\beta_1 \partial\bar\partial \log \|g\|^2_{E^*}. \end{aligned}$$

$\partial\bar\partial \log \Phi \geq 0$ is equivalent to $E \geq 0$ in the sense of Griffith. The second term implies the curvature condition of M, so we require

$$\sqrt{-1}\Theta(M) \geq \alpha\beta_1\sqrt{-1}\Theta(V).$$

The last term is for the fundamental inequality. We have to verify the fundamental inequality with vector bundle metric

$$\frac{1}{\alpha}\|f\|^2_{H_1'} + \alpha\beta_1\langle \sqrt{-1}\partial\bar\partial \log \|g\|^2_{E^*} v, v\rangle_{\varphi_0} \geq 2|\langle \bar\partial T^* f, v\rangle_{\varphi_0}|, \tag{40}$$

for some appropriate constant β_1. For short, we abuse notation $\|g\|^2 = \|g\|^2_{E^*}$. Since we use the quotient metric H_1' for $\pi^* V$, the minimum solution for $Gx = u$

$$T^{-1}u = T^* u,$$

where T^* is the adjoint w.r.t H_1'. Like the argument in the previous section, we have

$$\begin{aligned}
2|\langle \bar{\partial} T^* f, v\rangle_{\varphi_0}| &= 2\sqrt{|\langle \bar{\partial} T^* f, v\rangle_{\varphi_0}|^2} \\
&= 2\sqrt{|\langle \beta^* T^{-1} f, v\rangle_{H_0}|^2} = 2\sqrt{|\langle T^{-1} f, \beta \lrcorner v\rangle_{H_0}|^2} \\
&\leq 2\sqrt{\|T^{-1} f\|^2_{H_0} \cdot \|\beta \lrcorner v\|^2_{H_0}} \\
&\leq \frac{1}{\alpha}\|f\|^2_{H_1'} + \alpha\|\beta \lrcorner v\|^2_{H_0},
\end{aligned}$$

so it reduces to the Skoda's inequality.

$$\beta_1(\sqrt{-1}\ \mathrm{Tr}\ \beta \wedge \beta^* \Lambda v, v)_{\varphi_0} \geq \|\beta \lrcorner v\|^2,$$

and $\beta_1 = \min\{n, p-1\}$. Therefore we verify (40) and complete the proof of theorem 7.1.

8. Effective Artin-Rees Lemma (Ideal Case)

In this and next section, we are going to present another application of L^2-Cramer's rule. The form of Cramer's rule and primary decomposition in the commutative algebra can be combined together to get the effective uniform bound of Artin-Rees Lemma. Before we are going to the general module case, we want to revisit the ideal case in this section.

First let us define a terminology.

Definition 8.1. *[Analytic Nullstellensatz] Let $J = (b_1, \cdots, b_p)$ be a finite generated ideal of a Notherian ring $R = \mathbb{C}[x_1, \cdots, x_n]$. Let $f \in R$ such that $f(x) = 0$ for every $x \in V(J)$. Projectivize $\mathbb{C}^n \subset \mathbb{P}^n$, and equip $\mathbb{C}^n$ with the standard Fubini-Study* *metric. Suppose* $\Omega = \Omega_\nu \subset \mathbb{C}^n$ *be an exhaustion. Assume* m_0 *be a number satisfying*

$$\int_\Omega \frac{|f^{m_0}|^2}{(\sum |b_i|^2)^\xi} \mathrm{d}V_{FS} < \infty$$

for every f*.* **NOTE.** *In this section, we wouldn't write the power of the denominator of the integrand explicitly. Since it is not important in the arguments, we simply denote it by* ξ*.*

By Skoda's theorem $f \in J$*. In particular,*

$$\sqrt{J}^{m_0} \subset J.$$

Then we call the Hilbert Nullstellensatz with a hidden integrability condition by analytic Hilbert Nullstellensatz.

Recall the effective analytic Nullstellensatz.

Theorem 8.2 (Effective Hilbert Nullstellensatz[3])**.** *Let* $J = (b_1, \cdots, b_p) \lhd \mathbb{C}[x_1, \cdots, x_n]$ *be a finite generated ideal. Assume* $\deg b_i \leq d$ *for every* i*. Then*

$$\sqrt{J}^{(\alpha\beta+1)d^n} \subseteq J,$$

where $\alpha > 1$ *is a fixed constant and* $\beta = \min\{n, p-1\}$ *is the Skoda constant.*

Proof. This is the affine version of effective Hilbert Nullstellensatz of projective space $\mathbb{P}^n$. The polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$ can be treated as one affine piece $\mathbb{C}^n$ of $\mathbb{P}^n$. Homogenize $b_1, \cdots, b_p$ to $B_1, \cdots, B_p$ that are the sections of $\mathcal{O}(d)$, i.e. degree d homogeneous polynomials of $R_0' = \mathbb{C}[X_0, X_1, \cdots, X_n]$, and generate a homogeneous ideal

$$\mathcal{J} = (B_1, \cdots, B_p) \lhd R_0'.$$

Then we can apply the projective version of Hilbert Nullstellensatz.

For every element $f \in R$, consider its homogenization F. By the assumption that $f^m = \sum b_i g_i \in J$ for some m. This means $f(x) = 0$ for every $x \in V(J)$. Therefore we want to find the proper number of m to make the following integral finite

$$\int_\Omega \frac{|f^m|^2}{(\sum |b_i|^2)^\xi} \mathrm{d}V_\Omega.$$

m depends on the vanishing order of the loci of $V(J)$ (this order can be computed by blowing-up and comparing the coefficients with the exceptional divisors), and the degree has the upper bound d^n. Hence we know $F^{(\alpha\beta+1)d^n} \in \mathcal{J}$, i.e.

$$F^{(\alpha\beta+1)d^n} = \sum B_i H_i,$$

where $F^{(\alpha\beta+1)d^n}$ is a section of $\mathcal{O}(e)$ (e is the degree of $F^{(\alpha\beta+1)d^n}$) and H_i are sections of $\mathcal{O}(e-d)$. In order to apply Skoda's theorem, it is required to check the curvature condition

$$e + (n+1) \geq (\alpha\beta + 1)d.$$

If we choose

$$e \geq (\alpha\beta+1)d^n \geq (\alpha\beta+1)d, \tag{41}$$

then the inequality is obviously satisfied. Hence

$$F^{(\alpha\beta+1)d^n}(1, x_1, \cdots, x_n) = f^{(\alpha\beta+1)d^n} = \sum b_i h_i \in J,$$

where h_i is the dehomogenized section of $\mathcal{O}(e-d)$, hence a polynomial in $\mathbb{C}[x_1, \cdots, x_n]$. This ends the proof. □

First we want to illuminate the link between Nullstellensatz and ideal case of Artin-Rees lemma. Note that we assume $N \subseteq M = \mathcal{O}^{\oplus r}$ as the same as in the previous section. $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n}$ is the sheaf of holomorphic function over $\mathbb{C}^n$.

Lemma 8.3 (Effective Artin-Rees Lemma of Ideal Case)**.** *Let $R = \mathbb{C}[x_1, \cdots, x_n] \subset \mathcal{O}$ be a Noetherian ring of dimension n, $I = (g)$ and $J = (b_1, \cdots, b_p)$ be finite generated ideals of R. There exists $m_0 = m_0(n, J)$ such that*

$$I^{m_0+r} \cap J \subseteq I^r(I^{m_0} \cap J)$$

for every $r \geq 0$. In particular m_0 can be chosen by the effective power in the effective Nullstellensatz theorem, i.e. m_0 satisfies

$$f^{m_0} \in J$$

for every $f \in \sqrt{J}$.

Proof. Since J is finite generated, J has a primary decomposition

$$J = \bigcap Q_j,$$

where Q_j are q_j-primary. For every element $f \in I^{m+r} \cap J$

$$f = g^{m+r}h$$

for some $h \in R$. Since $f \in J$, $f \in Q_j$ for every j. In the following we make argument on every Q_j. So we simply writing $Q = Q_j$. As Q is primary, either $g^{(m+r)\alpha} \in Q$ or $h \in Q$. If $h \in Q$,

$$g^{m+r}h = g^r(g^m h) \in I^r(I^m \cap Q). \tag{42}$$

Otherwise $g^{(m+r)\alpha} \in Q = (b'_1, \cdots, b'_{p'})$. Thus $g^{m+r}(x) = \sum b'_i(x)h_i(x) = 0$ for every $x \in V(Q)$,and

$$g(x) = 0$$

for every $x \in V(Q)$. By Nullstellensatz, there exists an effective $m = m_j$ such that $g^m \in Q$. In this case we can conclude (42) as well. Take

$$m_0 = \max_j \{m_j\},$$

then we get

$$g^{m_0+r}h = g^r(g^{m_0}h) \in I^r(I^{m_0} \cap Q_j) \text{ for every } j.$$

Hence we are done.

Now we want to study m_j more closely.

CLAIM: m_j can be chosen to be m_0, where m_0 is the effective power of Nullstellensatz of J.

Note that $g(x) = 0$ so m_j is the necessary power to make g^{m_j} can be divided by the generators of Q_j which we denote it by $\{b'_1, \cdots, b'_p\}$. This is the meaning of Nullstellensatz. In other words the divisible condition can be described by the L^2 integrability

$$\int_\Omega \frac{|g^{m_j}|^2}{(\sum_i |b'_i|^2)^\xi} \mathrm{d}V_\Omega < \infty, \tag{43}$$

where $\xi = \alpha \cdot \min\{n, p-1\} + 1$. Since L^2 condition is local, Ω can be taken by the neighborhood of the common zeroes of $\{b'_i\}$. Since latter the effective Hilbert Nullstellensatz we are going to apply is derived from the projective space case, we will equip $\mathbb{C}^n$ with the Fubini-Study metric on $\mathbb{P}^n$. Hence $\mathrm{d}V_\Omega$ in the integral is exactly the restriction of the volume form of $\mathrm{d}V_{FS}$ on $\mathbb{P}^n$. In fact we homogenize g^{m_j} to G_{m_j} and b'_i to B'_i as holomorphic sections of $\mathcal{O}(d)$ bundle on $\mathbb{P}^n$. If (43) is satisfied,

$$G^{m_j} = \sum B'_i H_i \text{ on } \mathbb{P}^n,$$

especially it can be dehomogenized to $g^{m_j} = \sum b'_i h_i$. By the last line of the poof of theorem 8.2, we know $\{h_i\}$ are polynomials.

After that we can translate the finiteness into vanishing order estimates. Take the resolution $\pi : Y \longrightarrow \Omega$ to resolve the ideal Q_j, i.e.

$$\pi^{-1}Q_j \cdot \mathcal{O}_Y = \mathcal{O}_Y(F_j),$$

where F_j is a divisor on Y. Take one affine piece of Y and the corresponding scheme is $Spec\, R'$. Then (43) is equivalent to

$$K_{Y/X} + m_j(\pi^* g) \geq \xi F_j.$$

Hence we want to trace the coefficients

$$-K_{Y/X} + F_j = \sum \alpha_{k,j} E_k.$$

Then (43) can be interpreted as

$$m_j \cdot \mathrm{ord}_{E_k}(\pi^* g) \geq \xi \alpha_{k,j} \text{ for every } k.$$

Similarly we can do the same thing to the ideal J, and we can chose $\pi : Y \longrightarrow \Omega$ as the common resolution of J and every Q_j. Suppose $f \in \sqrt{J}$ (or equivalently $f(x) = 0$ for every $x \in V(J)$) and $\pi^{-1}J \cdot \mathcal{O}_Y = \mathcal{O}_Y(F)$. Write

$$-K_{Y/X} + F = \mathcal{O}_Y(\sum \alpha_k E_k).$$

Then

$$\int_\Omega \frac{|f^{m_0}|^2}{(\sum_i |b_i|^2)^\xi} \mathrm{d}V_\Omega < \infty$$

is equivalent to

(44) $$m_0 \cdot \mathrm{ord}_{E_k} f \geq \xi \alpha_k \text{ for every } k.$$

In the proof of effective Nullstellensatz, we do not have the information of vanishing order of f in general. So (44) actually is stronger,

$$m_0 \geq \xi \alpha_k \text{ for every } k. \tag{44'}$$

Next we want to explore the relation between α_k and $\alpha_{k,j}$. It is obvious to see

$$JR' = \left(\bigcap Q_j\right) \cdot R' \subseteq \bigcap \left(Q_j \cdot R'\right) \subseteq Q_j R'. \tag{45}$$

Thus $F_j \subseteq F$ for every j, then $-K_{Y/X} + F_j \subseteq -K_{Y/X} + F$. It implies

$$\alpha_{k,j} \leq \alpha_k \text{ for every } k \text{ and every } j. \tag{46}$$

Combing (44') and (46), we can summarize

$$m_0 \cdot \mathrm{ord}_{E_k}(\pi^* g) \geq m_0 \geq \alpha_k \geq \alpha_{k,j}.$$

This implies $\displaystyle\int_\Omega \frac{|g^{m_0}|^2}{(\sum_i |b_i'|^2)^\xi} \mathrm{d}V_\Omega < \infty$ for every j. Therefore the effective power m_0 of Nullstellensatz can be used to get the divisibility of g w.r.t. Q_j for every j. Thus we finish the proof. □

Next, we want to show the reduction to the principle ideal case.

Lemma 8.4. *It is sufficient to prove the effective Artin-Rees lemma of the principle ideal case.*

Proof. Let $m_0 = m_0(J')$ be the number such that

$$(w)^{m_0+r} \cap J' \subset (w)^r((w)^{m_0} \cap J').$$

Suppose $I = (a_1, \cdots, a_s)$, $J = (b_1, \cdots, b_p)$ are finite generated ideal of $R = \mathbb{C}[x_1, \cdots, x_n]$. Let

$$\begin{aligned} R' = R[It, 1/t] &= \mathbb{C}[x_1, \cdots, x_n][a_1 t, \cdots, a_s t, 1/t] \\ &= \mathbb{C}[x_1, \cdots, x_n, z_1, \cdots, z_s, w]/(z_1 w - a_1(x), \cdots, z_s w - a_s(x)) \\ &= R'_0/\sim. \end{aligned}$$

be the blow-up algebra of R w.r.t. I. Pull back I to R' and denote

$$I' = IR' = (1/t) \lhd R'.$$

R' is Noetherian by construction. In this setting, we have

$$(I')^n \cap R = I^n.$$

Instead working on $Spec(R')$, we turn to work on the ambient space

$$R'_0 = \mathbb{C}[x_1, \cdots, x_n, z_1, \cdots, z_s, w],$$

which has dimension $n + s + 1$. In this setting

$$I'_0 = (w) \lhd R'_0$$

is a principle ideal. Pullback J to R'_0 and denote it by J'_0. Note that $J' = (J'_0/\sim) \lhd R'$. We can summarize the setting by the following diagram

$$\begin{array}{ccccc} (w) & \lhd & R'_0 & \rhd & J'_0 \\ \Big| & & \Big\downarrow & & \Big| \\ (1/t) & \lhd & R' & \rhd & J' \\ \Big| & & \Big\uparrow & & \Big| \\ I & \lhd & R & \rhd & J \end{array}$$

Hence we can apply the principle ideal version of effective Artin-Rees lemma to conclude

$$(47)\qquad (I_0')^{m_0+r}\cap J_0' \subset (I_0')^r\big((I_0')^{m_0}\cap J_0'\big) \Longrightarrow (I')^{m_0+r}\cap J' \subset (I')^r\big((I')^{m_0}\cap J'\big).$$

The idea is to restrict (47) with R, then we have

$$I^{m_0+r}\cap J \subset (1/t)^{m_0+r}\cap J'\cap R \subset (1/t)^r\big((1/t)^{m_0}\cap J'\big)\cap R.$$

Claim.

$$(1/t)^r\big((1/t)^{m_0}\cap J'\big)\cap R \subset I^{m_0+r}J.$$

Let us assume this for this moment, then it implies

$$I^{m_0+r}\cap J \subset I^r(I^{m_0}J) \subset I^r(I^{m_0}\cap J),$$

which is what we intend to prove. So the whole thing is reduced to show the claim.

Let F be an element of $(1/t)^r\big((1/t)^{m_0}\cap J'\big)\cap R$, then

$$F = \big((1/t)^r g\cdot (1/t)^{m_0}h\big)\in R,$$

where $g,h\in R'$ and $(1/t)^{m_0}h\in J'$. Since $F\in R$,

$$gh = t^{r+m_0}\cdot\zeta$$

for some $\zeta\in R$. More precisely

$$\begin{aligned} gh &= \Big(\frac{a_{-n}}{t^n}+\cdots+a_0+\cdots+a_kt^k\Big)\Big(\frac{b_{-p}}{t^p}+\cdots+b_0+\cdots+b_\ell t^\ell\Big)\\ &= a_{-n}b_{-p}\frac{1}{t^{n+p}}+\cdots+a_kb_\ell t^{k+\ell} = t^{m_0+r}\cdot\zeta\in R, \end{aligned}$$

so we can assume $g=a_kt^k$ and $h=b_\ell t^\ell$ satisfying $a_kb_\ell t^{k+\ell}=t^{m_0+r}\cdot\zeta\in R$, which implies

$$k+\ell = m_0+r.$$

What remaining is to discuss all possible situations of k and ℓ.

- $k > 0; \ell > 0$. Thus $a_k \in I^k; b_\ell \in I^\ell J \Longrightarrow F = a_k b_\ell \in I^{k+\ell} J = I^{m_0+r} J$.
- $k \leq 0; \ell > 0$. This implies $\ell \geq m_0 + r$, so $b_\ell \in I^{m_0+r} J \Longrightarrow F \in I^{m_0+r} J$.
- $k \geq 0; \ell < 0$. This implies $k \geq m_0 + r$, so $a_k \in I^{m_0+r}$, plus $b_\ell \in J \Longrightarrow F \in I^{m_0+r} J$.

Hence it ends the proof of the claim, so does the lemma. $\square$

Now we are ready to show the effective uniform Artin-Rees lemma.

Theorem 8.5. *Let $J = (b_1, \cdots, b_p)$ be a finite generated ideal of*

$$R = \mathbb{C}[x_1, \cdots, x_n].$$

Assume $\deg b_i \leq d$ *for every i and*

$$m_0 = (p+1)d^n,$$

then

$$I^{m_0+r} \cap J \subset I^r (I^{m_0} \cap J)$$

for every integer r and every finite generated ideal I of R.

Proof. The only thing we need to do is to figure out the numerical number m_0. Let $I = (a_1, \cdots, a_s)$. In the reduction lemma 8.4, we work on the ambient space

$$Spec\, R_0' = \mathbb{C}^{n+s+1},$$

so the Skoda's constant becomes $\beta = \min\{n+s+1, p-1\}$. Since we want the uniform result, we take $\beta = p-1$. On R_0', $I' = (w)$, so we can apply lemma 8.3, and by the effective Hilbert Nullstellensatz (theorem 8.2),

$$m_0 = (\alpha\beta + 1)d^{s+n+1},$$

where $\alpha > 1$ is a constant such that $\alpha\beta$ is an integer. Actually the power of d can be chosen as n. This is because this number depends on the bound of the degree of singularity generated by $b_1, \cdots, b_p$. Since $b_i(x)$ are functions of x, it won't increase the degree of discriminates of $J = (b_1, \cdots, b_p)$ by adding more formal variables $z_1, \cdots, z_s, w$. Therefore

$$m_0 = (p+1)d^n.$$

□

9. Effective Artin-Rees Lemma (Module Case)

Based on the technique we developed in the previous section, we are going to introduce L^2-Cramer's rule to show the module case. Let $\Omega \subseteq \mathbb{C}^n$ be a domain of dimension n, and $\mathcal{O} = \mathcal{O}_{\Omega,0}$ be a holomorphic germ of 0. Assume

$$M = \mathcal{O}^{\oplus r} = \mathcal{O}e_1 \oplus \cdots \oplus \mathcal{O}e_r$$

be a free $\mathcal{O}$-module. Let $N = (g_1, \cdots g_p) \subset M$ be a finite generated sub $\mathcal{O}$-module of M. The basis has expression

$$G_N = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,p} \\ \vdots & \vdots & \ddots & \vdots \\ g_{r,1} & g_{r,2} & \cdots & g_{r,p} \end{pmatrix}$$

w.r.t. $\{e_1, \cdots, e_r\}$.

Definition 9.1. *[Embedded Rank.] Assume the matrix has generic rank m over Ω. Then we say that the embedded rank (e.m.b-rank) of N in M is m.*

Note that we can take out several analytic hypersurfacees H such that there exists a holomorphic frame $e'_1, \cdots, e'_m$ such that N is generated by

$$\begin{pmatrix} g'_{1,1} & g'_{1,2} & \cdots & g'_{1,p} \\ \vdots & \vdots & \ddots & \vdots \\ g'_{m,1} & g'_{m,2} & \cdots & g'_{m,p} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence we can work on the domain $\Omega' = \Omega - H$ where the embedded rank of N equals to the rank of M which is m. For short we will abuse notation and denote $\Omega' = \Omega$, and assume $p \geq m$. Before we move on to the module case, let us introduce a notation.

Definition 9.2.

$$\bigwedge^r N := \langle g_{k_1} \wedge \cdots \wedge g_{k_r} \rangle_{1 \leq k_1 < \cdots < k_r \leq p},$$

i.e. the determinant ideal of N is generated by all possible determinant of $r \times r$ sub-matrix of the generator matrix of N.

In general we have the exterior algebra on module. Given a module N, we can define $\bigwedge^k N$. If $N = \langle g_1, \cdots, g_p \rangle$, then

$$\bigwedge^k N = \langle g_{i_1} \wedge \cdots \wedge g_{i_k} \rangle_{1 \leq i_1 < \cdots < i_k \leq p}.$$

In particular, in our setting that $N \subseteq M = \mathcal{O}^{\oplus r}$,

$$\bigwedge^r M = \bigwedge^r N = \mathcal{O} e_1 \wedge \cdots \wedge e_r,$$

and $\det N = \bigwedge^r N$ is a finite generated module of rank 1, hence an ideal. The definition is independent of choice of basis. If $e'_1, \cdots, e'_r$ is another set of basis then

$$e_1 \wedge \cdots \wedge e_r = f e'_1 \wedge \cdots \wedge e'_r,$$

where f is a nowhere zero holomorphic function on Ω, hence an unit. Therefore

$$\langle g_{k_1} \wedge \cdots \wedge g_{k_r} \rangle = \langle f g'_{k_1} \wedge \cdots \wedge g'_{k_r} \rangle,$$

which means $\det N$ is a well-defined notion.

Now we are ready to state the effective Artin-Rees lemma of module case. Similar to the ideal case, we want to reduce the module situation to ideal by the aid of L^2 Cramer's rule.

Lemma 9.3 (Effective Artin-Rees Lemma of Module Case)**.** *Let Ω be a domain of dimension n. Let $M = \mathbb{C}[x_1, \cdots, x_n]^{\oplus r} \subset \mathcal{O}^{\oplus r}$ be a free $\mathcal{O}$-module, $N = (g_1, \cdots, g_p) \subseteq M$ be a finite generated sub $\mathcal{O}$-module. Let N has embedded rank r and $p \geq r$. Suppose $m_0 = m_0(n, N)$ is the effective power of analytic Nullstellensatz w.r.t. the ideal* $\det N$*. Then we have*

$$I^{m_0+k} M \cap N \subseteq I^k (I^{m_0} M \cap N).$$

for every integer k.

Since we have a module version of lemma 8.4 and the reduction in the last section, we can assume $I = (g)$, i.e. I is a principle ideal. Note that the assumption $p \geq r$ is not special. The reason is as the remark before definition 9.2.

Proof. Mimic the proof of the ideal case. Let $N = \bigcap Q_j$ be the primary decomposition of N. For $f \in M$, suppose we have

$$g^{m+k} f = \begin{pmatrix} g^{m+k} f_1 \\ \vdots \\ g^{m+k} f_r \end{pmatrix} \in I^m M \cap N.$$

So $g^{m+k} f \in Q_j$ for every j. For short we write

$$Q = Q_j = (y_1, \cdots, y_\ell).$$

By the property of primary module either

1) $g^{m+k} : M \longrightarrow M$ is injection or
2) g^{m+k} is nilpotent, i.e. $(g^{m+k})^\alpha : M \longrightarrow Q$ for some α.

For case (1) $g^{m+k} f \in Q$ implies $f \in Q$. Hence

$$g^{m+k} f = g^k (g^m f) \in g^k Q.$$

For case (2) we will use the advantage that $g^{(m+k)\alpha} w \in Q$ for every $w \in M$. For convenience let

$$t = (m+k)\alpha.$$

Recall the L^2 Cramer's Rule. $g^m f \in Q$ can be homogenized to $G^m \tilde{F} \in \bigoplus \mathcal{O}(e)$ over $\mathbb{P}^n$, because we can take $e = \max\{e_i\}$, where e_i is the degree of each component of $g^m f \in M$. In order to apply the analytic Hilbert Nullstellensatz, we need to check the curvature condition. Let us write the diagram explicitly

$$\bigoplus^p \mathcal{O}(e-d) \xrightarrow{\mathcal{O}(d)} \bigoplus \mathcal{O}(e) \ ,$$

where e is the degree of the homogenized function of g^{m_0}. For the L^2 Cramer's rule, the constrain reads

$$e \geq (\alpha\beta + 1)d.$$

Note that what on the R.H.S is fixed, so if the power m_0 is big enough, the corresponding degree e would be greater than d. The argument is as the same as the one in the proof of ideal case. Since $m_0 \geq (\alpha\beta+1)(d^r)^n$,

$$e \geq m_0 \geq (\alpha\beta + 1)d.$$

Therefore we can apply the L^2-Cramer's rule. If $g^m f$ satisfies

$$\int_\Omega \frac{\sum_{1\leq i_1<\cdots<i_{r-1}\leq \ell} |g^m f \wedge y_{i_1} \wedge \cdots \wedge y_{i_{r-1}}|^2}{\left(\sum_{1\leq i_1<\cdots<i_r\leq \ell} |y_{i_1} \wedge \cdots \wedge y_{i_r})|^2\right)^\xi} \mathrm{d}V_\Omega < \infty, \tag{48}$$

then $g^m f$ can be divided by $\{y_1, \cdots, y_\ell\}$. Observe (48), g can be pulled out the summation. So if $g(x) = 0$ for every $x \in V(\det Q)$, raising to an appropriate power the integral would become finite. Then by L^2 Cramer's rule $g^m f$ can be divided by $\{y_1, \cdots, y_\ell\}$. Then we can conclude again that

$$g^{m+k} f = g^k(g^m f) \in g^k Q.$$

Hence $g^{m+k} f \in g^k Q_j$ for every j, which implies

$$g^{m+k} f \in g^k N.$$

Then we are done. So the key is to show

Claim: $g(x) = 0$ for every $x \in V(\det Q)$ in case (2).

The point is to use the assumption $g^t\zeta \in Q$ for every $\zeta \in M$. Take the constant vectors

$$e_i = (0, \cdots, 1_{i\text{-th position}}, \cdots, 0)$$

in M, and consider

$$\left|g^t e_1 \wedge \cdots \wedge g^t e_r\right|^2 = g^{rt}\left|e_1 \wedge \cdots \wedge e_r\right|^2.$$

On one hand, since $g^t e_i \in Q$

$$\left|g^t(x)e_1 \wedge \cdots \wedge g^t(x)e_r\right|^2 = \sum_{1 \le i_1 < \cdots < i_s \le p} \left|a_{i_1,\cdots,i_r} y_{i_1}(x) \wedge \cdots \wedge y_{i_r}(x)\right|^2 = 0$$

for every $x \in V(\det Q)$. On the other hand

$$g^{rt}(x)\left|e_1 \wedge \cdots \wedge e_r\right|^2 = \left|g^t(x)e_1 \wedge \cdots \wedge g^t(x)e_r\right|^2 = 0.$$

Therefore $g(x) = 0$ for every $x \in V(\det Q)$. Then the claim follows and the lemma is proved.

What remaining is to determine the effective number of m_0. By the same argument of ideal case, we can see that the number dependents on the effective Nullstellensatz. By L^2-Cramer's rule, it depends on the vanishing order of the ideal $\det Q$. Denote

$$\det Q = (b_1, \cdots, b_q), \text{ and } q = \binom{p}{r}.$$

Look at the above argument closely. The whole point is to show that g^{m_0} has enough vanishing order at x, which is equivalent to the integrability

$$\int_\Omega \frac{|g^{m_0}|^2}{(\sum |b_i|^2)^\xi} \mathrm{d}V_\Omega < \infty.$$

Remark that the V_Ω is chosen as the same as the one in the proof of theorem 8.3. By the same reduction in the proof of ideal case, we want to conclude that the effective power of Nullstellensatz w.r.t. $\det N$ is enough to apply to $\det Q$. Look at (45) closely, the key point is

$$JR' \subseteq Q_j R' \text{ for every } j.$$

We have similar situation here. Since $N \subseteq Q_j$, the e.m.b-rank $N \leq$ e.m.b-rank $Q_j \leq r$. Hence e.m.b-rank $Q_j = r$. Therefore

$$\det N \subseteq \det Q_j \text{ for every } j,$$

which implies $(\det N)R' \subseteq (\det Q_j)R'$ for every j. Then by the same argument in the proof of ideal case, the effective power m_0 of Nullstellensatz w.r.t $\det N$ can dominant the effective power of Q_j for every j. Hence we are done. □

Now we have all ingredients to show the module version of effective uniform Artin-Rees lamma.

Theorem 9.4. *Let $N = (g_1, \cdots, g_p)$ be a finite generated module of*

$$M = \mathbb{C}[x_1, \cdots, x_n]^{\oplus r}.$$

Suppose the embedded rank is w. Let $q = C^p_w$. Assume $\deg g_{i,j} \leq d$ for every i, j and

$$m_0 = (q - w + 2)(d^w)^n,$$

then

$$I^{m_0+k} M \cap N \subset I^k (I^{m_0} M \cap N)$$

for every integer k and every finite generated ideal I of $\mathbb{C}[x_1, \cdots, x_n]$.

Proof. Apply the same idea as the one in the proof of theorem 8.5. What we need is to pin down every numerical factors. Since $\deg x_{i,j} \leq d$ for every i, j and

$$\bigwedge^w N = (b_1, \cdots, b_q) \lhd \mathbb{C}[x_1, \cdots, x_n],\ q = \binom{p}{w},$$

which implies

$$\deg b_i \leq d^w, \text{ for every } i.$$

Let $I = (a_1, \cdots, a_s)$. For the Skoda's constant, we have $\beta = \min\{n+s+1, q-1\}$. Since we want an effective uniform result, we take

$$\beta = q - 1.$$

By effective Hilbert Nullstellensatz 8.2, we know m_0 can be chosen as

$$m_0 = (\alpha\beta - w + 2)(d^w)^{n+s+1}.$$

Applying the same argument as the one in the proof of 8.5, we can replace the power of d^r by n. Hence finally we can choose

$$m_0 = (q - w + 2)(d^w)^n.$$

□

References

[1] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[2] J.-P. Demailly. L^2 vanishing theorems for positive line bundles and adjunction theory. In *Transcendental methods in algebraic geometry (Cetraro, 1994)*, volume 1646 of *Lecture Notes in Math.*, pages 1–97. Springer, Berlin, 1996.

[3] L. Ein and R. Lazarsfeld. A geometric effective Nullstellensatz. *Invent. Math.*, 137(2):427–448, 1999.

[4] L. Hörmander. L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.*, 113:89–152, 1965.

[5] L. Hörmander. *An introduction to complex analysis in several variables.* North-Holland Publishing Co., Amsterdam, revised edition, 1973. North-Holland Mathematical Library, Vol. 7.

[6] C. Huneke. Desingularizations and the uniform Artin-Rees theorem. *J. London Math. Soc. (2)*, 62(3):740–756, 2000.

[7] J. J. Kelleher and B. A. Taylor. Finitely generated ideals in rings of analytic functions. *Math. Ann.*, 193:225–237, 1971.

[8] H. Skoda. Application des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids. *Ann. Sci. École Norm. Sup. (4)*, 5:545–579, 1972.

[9] H. Skoda. Morphismes surjectifs de fibrés vectoriels semi-positifs. *Ann. Sci. École Norm. Sup. (4)*, 11(4):577–611, 1978.

[10] D. Varolin. Division theorems and twisted complexes. *Math. Z.*, 259(1):1–20, 2008.