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On an approach to automorphic Euler systems

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On an approach to automorphic Euler systems

Abstract

We describe a novel approach towards establishing Hecke algebra valued horizontal norm relations between pushforwards of integral cohomology classes that appear in motives associated with Shimura varieties. These relations then give rise to Euler systems for appropriate automorphic Galois representations that arise in the cohomology of such varieties. The key innovation in our approach is a precise axiomatic formulation of a refined relation that yields the desired horizontal norm relations in a universal sense. A crucial ingredient in the execution of this approach is a systematic and explicit decomposition recipe for Hecke operators of unramified reductive groups over local fields obtained using Bruhat-Tits theory. We use our approach to establish such relations in a variety of examples, some previously studied and some new.

Contents

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غالبؔ

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Synopsis

Euler systems are objects of an arithmetic-algebraic-geometric nature that are meant to provide a handle on the Selmer groups of p-adic Galois representations. An indispensable part of any Euler system construction (in the style of, say [Rub00], [Kat99], [PR98]) is establishing the so-called *horizontal norm relations* (HNR) between certain Galois cohomology classes. Roughly speaking, the requirement is to exhibit a family of integral cohomology classes c_α, one for each layer E_α in a prescribed lattice of layers of an infinite abelian extension of a number field, such that the trace of the class c_α down to a layer E_β is equal to the $P_{\alpha,\beta} \cdot c_\beta$. Here, $P_{\alpha,\beta}$ are certain elements in the group algebra of $\mathrm{Gal}(E_\beta/E)$ referred to as 'Euler factors' since the exact description of $P_{\alpha,\beta}$ is in terms of certain characteristic polynomials of Frobenii. Since these Euler factors also appear in zeta functions, the classes c_α are sometimes referred to as *zeta elements* ([Kat04]). Though Euler systems are objects typically associated with Galois representations, the tools involved in their construction are often of an automorphic nature; the Galois representation at hand arises via a motive of a Shimura variety whose zeta function is known to be automorphic and the classes used for the purposes of crafting such a system are pushforwarded from the motives of a sub-Shimura variety. It is therefore fruitful to formulate a corresponding notion which is tailored towards such situations and which specializes to its Galois counterpart after projection to suitable isotypical components in the cohomology of such varieties.

In this thesis, we propose such a notion (Definition 3.1.4) in an elementary, self-contained framework of functors on locally profinite groups which is modelled on these situations and which we hope to further specialize to the settings mentioned above in future. We moreover put together various tools from representation theory & structure theory of reductive algebraic groups over local fields (Chapters 4 & 5) that can be used to study it in practice. Examples constructed in works as early as [Kat04], [Kol90] and as recent as [LSZ17], [GS21] are shown to be instances of this more fundamental notion, appropriately reformulated. In particular, we recover the horizontal norm relations for these settings. Key results of this work are Theorems 9.5.1, 10.3.1, 11.4.1, 12.5.1 and 13.5.1, the last involving pushforwards of GSp_4-Eisenstein classes into GSp_6-Shimura varieties being the most novel in terms of technical innovation. Their arithmetic applications will be studied elsewhere. All of these results are obtained as per the format described in Chapter 3.

Chapter 0

A motivating example

This preliminary chapter is written with the intention of making accessible some of the essential ideas introduced in this thesis in the more familiar setting of modular curves. For the reader not familiar with the field, it may also serve as a first introduction to the automorphic aspects of the subject of Euler systems. We note that Theorem 0.2.22 – which is the main result of this chapter – only involves combinatorics of matrices over the field $\mathbb{Q}_\ell$ (where ℓ is a rational prime), so we do hope that the main argument will be accessible to a wide audience.

In [Kol90], Kolyvagin studied a collection of distinguished points on modular curves known as 'Heegner points' in order to create an Euler system for modular elliptic curves and used them to establish stunning results towards the conjectures of Birch & Swinnerton Dyer. We revisit Kolyvagin's construction in a language that is well-adapted to generalization in higher dimensional settings - the example will also serve as a convenient model to draw analogies from later on.

The exposition is divided into two sections. In §0.1, we recall classical modular curves in the language developed by Deligne [Del71] that puts the underlying algebraic group $\mathrm{GL}_{2,\mathbb{Q}}$ at the center of attention rather than the individual curves themselves. Our exposition in this part is mostly along the lines of [Kü17, §2.2-2.3] which we encourage the reader to consult; section 2.1 of *op. cit.* also provides a clean and coincise review of CFT and the references therein can be used for further background reading, if necessary. In §0.2, we establish the horizontal norm relations by recasting the statement into one of combinatorics of lattices in 2-dimensional vector spaces over a local field $\mathbb{Q}_\ell$. It is this combinatorics that turns out to be the major obstacle when one moves to higher dimensional settings and the bulk of this thesis is dedicated towards addressing this question for groups more complicated than GL_2.

0.1 Review of modular curves

This section reviews the theory of modular curves with goal of highlighting the underlying group-theoretic setup to define various objects of interest. Since our main theorem is a result in combinatorics, the general reader may take the material of this section as motivation and fixing notation for the aforementioned setup. For the number theorist more familiar with the language used e.g. in [DS05], we have added clarifying remarks at various junctures in order to make the connection with the language used here clearer.

0.1.1 The Shimura data

Fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. The modular curves (as algebraic curves over $\mathbb{Q}$) arise by what is known as a *Shimura datum* for $\mathbf{G} := \mathrm{GL}_{2,\mathbb{Q}}$. This is the data of a family of algebraic homorphisms from the *Deligne torus* $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ (the Weil restriction of scalars of $\mathbb{G}_m$ from $\mathbb{C}$ to $\mathbb{R}$) to the real group $\mathbf{G}_\mathbb{R} = \mathrm{GL}_{2,\mathbb{R}}$ satisfying a few properties. Rather than giving a definition in general (for which we refer the reader to Chapter 6), let us proceed with the example at hand and say that the relevant datum here is the pair $(\mathbf{G}, \mathcal{X}_{\mathrm{std}})$, where $\mathcal{X}_{\mathrm{std}}$ is the $\mathbf{G}(\mathbb{R})$-conjugacy class of the homomorphism

$$h_{\mathrm{std}} : \mathbb{S} \to \mathbf{G}_\mathbb{R} \qquad a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{0.1.1}$$

i.e. $\mathcal{X}_{\mathrm{std}} = \left\{ g \cdot h_{\mathrm{std}} \cdot g^{-1} \,|\, g \in \mathbf{G}(\mathbb{R}) \right\}$ as a set. The stabilizer $\left\{ g \in \mathbf{G}(\mathbb{R}) \,|\, g \cdot h_{\mathrm{std}} \cdot g^{-1} = h \right\}$ of h_{std} in $\mathbf{G}(\mathbb{R})$ is the image of h_{std}[1], and we can therefore identify $\mathcal{X}_{\mathrm{std}}$ with the upper and lower half plane $\mathcal{H}^\pm := \mathbb{C} - \mathbb{R}$ by sending h_{std} to $\left\{ \sqrt{-1} \right\} \in \mathcal{H}^+$. The $\mathbf{G}(\mathbb{R})$ action on $\mathcal{X}_{\mathrm{std}}$ then identifies with the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathcal{H}^\pm$ by fractional linear transformations.

Let E be an imaginary quadratic field, and let $\mathbf{H} = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$. Fix an isomorphism $\varphi : E \to \mathbb{Q}^2$ of $\mathbb{Q}$-vector spaces, or equivalently, the choice of an ordered basis $\{e_1, e_2\} \subset E$ over $\mathbb{Q}$. Given $e \in E$, multiplication by e induces a homomorphism $E \to E$ of $\mathbb{Q}$-algebras. In other words, the choice of φ induces an inclusion $\iota : E \hookrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{Q})$ of $\mathbb{Q}$-algebras and hence an inclusion of algebraic groups

$$\iota : \mathbf{H} \hookrightarrow \mathbf{G} \tag{0.1.2}$$

over $\mathbb{Q}$. We observe that the $\mathbf{G}(\mathbb{Q})$ conjugacy class of this embedding is independent of φ (by Skolem-Noether theorem, say). If we fix an embedding $\Phi : E \hookrightarrow \mathbb{C}$ (also known as a CM-type), then we have an isomorphism $E \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{C}$ of $\mathbb{R}$-algebras and hence an induced isomorphism $h_0 : \mathbb{S} \xrightarrow{\sim} \mathbf{H}_\mathbb{R}$. The pair $(\mathbf{H}, \{h_0\})$ is then itself a Shimura datum and the mapping

$$\iota : (\mathbf{H}, \{h_0\}) \hookrightarrow (\mathbf{G}, \mathcal{X}_{\mathrm{std}}) \tag{0.1.3}$$

[1] $h_{\mathrm{std}}(\mathbb{S})$ is a maximal torus (Cartan subgroup) in $\mathrm{GL}_2(\mathbb{R})$

constitutes an (injective) *morphism of Shimura data*.

Remark 0.1.4. Note however that since we are choosing an arbitrary $\mathbb{Q}$-basis e_1, e_2 of E, the point h_0 does not necessarily map to h_{std}.

0.1.2 The reflex fields

Each Shimura datum has an associated number field given as a subfield of $\mathbb{C}$ called the *reflex field* of the datum. It is defined as follows. The Deligne torus $\mathbb{S}$ splits over $\mathbb{C}$ i.e. $\mathbb{S}_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ and the isomorphism is uniquely determined by requiring that the inclusion $\mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C}) \simeq \mathbb{C}^\times \times \mathbb{C}^\times$ identifies with

$$\mathbb{C}^\times \hookrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \quad z \mapsto (z, \bar{z})$$

The reflex field, say in the case of $(\mathbf{G}, \mathcal{X}_{\mathrm{std}})$, is defined to be the field of definition of the $\mathbf{G}(\mathbb{C})$-conjucacy class of the cocharacter $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ associated to a choice of $h \in \mathcal{X}_{\mathrm{std}}$ obtained by restricting $h_{\mathbb{C}} : \mathbb{G}_m \times \mathbb{G}_m \to \mathbf{G}_{\mathbb{C}}$ to the first component. In practice, this means that in the matrices $h(z)$ for $z \in \mathbb{C}^\times$, one formally replaces $\bar{z}$ with 1. The reflex field is independent of the choice of h as the $\mathbf{G}(\mathbb{C})$-conjugacy class of μ_h, denoted $\mu_{\mathcal{X}_{\mathrm{std}}}$, is independent of $h \in \mathcal{X}_{\mathrm{std}}$. We have a similar (but easier) description for the reflex field of μ_{h_0}: the reflex field is the field of definition of the map μ_{h_0}.

The cocharacter $\mu_{h_0} : \mathbb{G}_m \to \mathbf{H}_{\mathbb{C}} \simeq \mathbb{G}_m \times \mathbb{G}_m, z \mapsto (z, 1)$ associated with h_0 is defined over any field over which $\mathbf{H}_{\mathbb{Q}}$ splits and thus the reflex of $(\mathbf{H}, \{h_0\})$ is equal to $\Phi(E)$. For $(\mathbf{G}, \mathcal{X}_{\mathrm{std}})$, the reflex field is $\mathbb{Q}$. Indeed, the cocharacter $\mu_{h_{\mathrm{std}}} : \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}(\mathbb{C})$ is given by

$$z \mapsto \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2i} \\ \frac{1-z}{2i} & \frac{z+1}{2} \end{pmatrix}.$$

where $i = \sqrt{-1}$. When conjugated by $\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}$, we get the cocharacter

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \tag{0.1.5}$$

which is itself defined over $\mathbb{Q}$, and therefore so is the $\mathbf{G}(\mathbb{C})$-conjugacy class $\mu_{\mathcal{X}_{\mathrm{std}}}$ of $\mu_{h_{\mathrm{std}}}$.

0.1.3 Canonical models

Let $\mathbb{A}_f$ (resp. $\mathbb{A}_{E,f}$) denote the group of finite adeles of $\mathbb{Q}$ (resp. E) and let $K \subset \mathbf{G}(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{A}_f)$ be a compact open subgroup which we fix in all of what follows. Such a subgroup K may be specified as the stabilizer in $\mathbf{G}(\mathbb{A}_f)$ of a $\widehat{\mathbb{Z}}$-lattice contained in the free $\mathbb{A}_f$-module $\mathbb{A}_{E,f} = \mathbb{A}_f e_1 \oplus \mathbb{A}_f e_2$. Then the double quotient

$$\mathcal{S}_K(\mathbb{C}) := \mathbf{G}(\mathbb{Q}) \backslash \mathcal{X}_{\mathrm{std}} \times \mathbf{G}(\mathbb{A}_f)/K \tag{0.1.6}$$

is the set of $\mathbb{C}$-points of an algebraic curve defined over $\mathbb{Q}$ (which is the reflex field for $\mathbf{G}$) known as its *canonical model*. We will refer to it as the *modular curve of level* K, and denote it by $\mathcal{S}_K$. The points of $\mathcal{S}_K(\mathbb{C})$ will be denoted as $[x, g]_K$ for $x \in \mathcal{X}_{\mathrm{std}}$, $g \in \mathbf{G}(\mathbb{A}_f)$. Similarly, for any compact open subgroup $U \subset \mathbf{H}(\mathbb{A}_f)$,

$$\mathcal{T}_U(\mathbb{C}) := \mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}_f)/U$$

is the set of $\mathbb{C}$-points of an étale scheme over Spec E and it has an explicit description in terms of the Shimura datum for $\mathbf{H}$ as we now describe. Let $\mu_{h_0} : \mathbb{G}_{m,E} \to \mathbf{H}_E$ be the cocharacter defined over E (as in §0.1.2). The *reciprocity law* for $(\mathbf{H}, \{h_0\})$ is the morphism

$$r(\mathbf{H}, h_0) : \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \xrightarrow{\mathrm{Res}_{E/\mathbb{Q}}(\mu_{h_0})} \mathrm{Res}_{E/\mathbb{Q}}(\mathbf{H}_E) \xrightarrow{\mathrm{Norm}_{E/\mathbb{Q}}} \mathbf{H} \tag{0.1.7}$$

where the map $\mathrm{Norm}_{E/\mathbb{Q}}$ is induced by the natural trace map $E \to \mathbb{Q}$. Unwinding definitions[2], this map is easily computed to be the identity map. The Galois action of $\sigma \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$ on a point $\gamma \in \mathcal{T}_U(\mathbb{C})$ is then declared to be translation by $a_f \in \mathbb{A}_f^\times$ for any $a = (a_\infty, a_f)$ such that $a \mapsto \sigma$ under the *Artin homomorphism* (that sends uniformizers to *geometric Frobenii*)

$$\mathrm{Art}_E : E^\times\backslash\mathbb{A}_E^\times \to \mathrm{Gal}(E^{\mathrm{ab}}/E) \tag{0.1.8}$$

from class field theory. More precisely, if $\mathrm{Art}([a]) = \sigma$ for some $a = (a_f, a_\infty)$, the action of σ on $\gamma = [h_f] \in \mathcal{T}_U(\mathbb{C})$ is by $\gamma \mapsto [a_f h_f] \in \mathcal{T}_U(\mathbb{C})$. This description of Galois action on $\mathcal{T}_U(\mathbb{C})$ constitutes the canonical model $\mathcal{T}_U$ over E for $\mathcal{T}_U(\mathbb{C})$.

Remark 0.1.9. Since E is imaginary, the ideles $\mathbb{C}^\times \hookrightarrow \mathbb{A}_E^\times$ are in the kernel of the Artin map. We can therefore consider Art_E as a map from $E^\times\backslash\mathbb{A}_{E,f}^\times = \mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}_f)$ and in fact, $\mathrm{Art}_E : \mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}_f) \simeq \mathrm{Gal}(E^{\mathrm{ab}}/E)$. See [Kü17, §2.1] and references therein for details.

Remark 0.1.10. That $\mathcal{S}_K(\mathbb{C})$ are the $\mathbb{C}$-points of algebraic curves over $\mathbb{C}$ follows from the theorem of Baily-Borel. That they have models over $\mathbb{Q}$ with prescribed properties does not at all follow from the formalism introduced so far and the meat of that argument still lies in the techniques for studying moduli of elliptic curves. However, one of the several advantages of Deligne's reformulation is that it explicitly ties in class field theory with modular curves. In our case, it allows for the aforementioned recasting of the problem of norm relations in terms of $\mathbb{Z}_\ell$-lattices.

0.1.4 Galois action on components

The curve $\mathcal{S}_K$ is not geometrically connected in general, and one can describe its geometrically connected components as follows. Let $\det : \mathbf{G} \to \mathbb{G}_m$ be the determinant map. Consider the map $\mathcal{X}_{\mathrm{std}} \to \{\pm\}$ given

[2] we need to translate what the norm map looks like when we identify $\mathrm{Res}_{E/\mathbb{Q}}\mathbf{H}_E$ with $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m \times \mathbb{G}_m)$, since the natural target for μ_{h_0} is the latter torus

by sending $h \mapsto \mathrm{sign}(\det(h(\sqrt{-1})))$. Then, the induced map

$$\mathcal{S}_K(\mathbb{C}) \to \mathbb{Q}^\times \backslash \{\pm\} \times \mathbb{A}_f^\times / \det(K)$$

provides a bijection from the geometrically connected components of $\mathcal{S}_K(\mathbb{C})$ with the set on the right. In other words, the geometric curve $\mathcal{S}_K \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}}$ decomposes as a disjoint union $\mathcal{S}_{K,\alpha}$ indexed by $\alpha \in \mathbb{Q}^\times \backslash \{\pm\} \times \mathbb{A}_f^\times / \det(K)$. The curves $\mathcal{S}_{K,\alpha}$ are not defined over $\mathbb{Q}$ but are defined over abelian extensions of $\mathbb{Q}$. These extensions are determined by a Galois action on these components defined via a similar reciprocity law as follows. Let $(\alpha_\infty, \alpha_f) \in \{\pm\} \times \mathbb{A}_f^\times$ a representative of α, $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ and $a = (a_\infty, a_f) \in \mathbb{A}_\mathbb{Q}^\times$ be any element such that $\sigma = \mathrm{Art}(a)$. Then $\sigma(\mathcal{S}_{K,\alpha}) = \mathcal{S}_{K,\alpha'}$ with α' being represented by $(\mathrm{sign}(a_\infty)\alpha_\infty, a_f \alpha_f) \in \mathbb{Q}^\times \backslash \{\pm\} \times \mathbb{A}_f^\times / \det(K)$.

0.1.5 Classical modular curves

One can obtain the classical modular curves from this description as follows: given $(a_\infty, a_f) \in \mathbb{Q}^\times \backslash \{\pm\} \times \mathbb{A}_f^\times / \det(K)$, let $\beta_f \in \mathbf{G}(\mathbb{A}_f)$ be any element such that $\det(\beta_f) = a_f$. Then, the map

$$\mathcal{H}^+ \to \mathcal{S}_K(\mathbb{C}), \qquad \tau \mapsto [\alpha_\infty \tau, \beta_f]_K$$

induces a holomorphic covering of $\mathcal{S}_{K,\alpha}(\mathbb{C})$ and induces an isomorphism $\Gamma \backslash \mathcal{H}^+ \xrightarrow{\simeq} \mathcal{S}_{K,\alpha}(\mathbb{C})$, with $\Gamma = \mathrm{SL}_2(\mathbb{Q}) \cap \beta_f K \beta_f^{-1}$. The groups Γ that arise in this fashion are known as *congruence subgroups* of $\mathrm{SL}_2(\mathbb{Q})$.

Example 0.1.1. For $N \geq 1$ an integer, let $K = \widehat{\Gamma}_0(N)$ be the stabilizer of the $\widehat{\mathbb{Z}}$-lattice $L_{f,N} := \widehat{\mathbb{Z}} e_1 \oplus \widehat{\mathbb{Z}} N e_2 \subset \mathbb{A}_{E,f}$. Then

$$K = \widehat{\Gamma}_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \,|\, c \equiv 0 \pmod{N} \right\}.$$

We have $\det(K) = \widehat{\mathbb{Z}}$ and so $\mathcal{S}_K(\mathbb{C}) = \Gamma_0(N) \backslash \mathcal{H}^+$ where $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ is the congruence subgroup of level N. For $N \geq 3$, $\mathcal{S}_K$ is the smooth, geometrically connected (quasi-projective) modular curve $Y_0(N)$ over $\mathbb{Q}$

0.1.6 CM points

Recall that $\mathbf{G}(\mathbb{Q})$ acts on $\mathcal{X}_{\mathrm{std}}$ via conjugation. There is a unique point $x_\iota \in \mathcal{X}_{\mathrm{std}}$ (depending on the choice of ι) with $\det(x_\iota(\sqrt{-1})) > 0$ whose stabilizer in $\mathbf{G}(\mathbb{Q})$ is $\iota(E^\times)$. Indeed, the point x_ι is equal to $\iota(h_0)$ or its complex conjugate, depending on whether the basis $e_1 \otimes 1, e_2 \otimes 1$ is positively oriented or not. For example, if $E = \mathbb{Q}(\sqrt{-d})$, $d > 1$ square free and we choose the basis to be $e_1 = 1$, $e_2 = \sqrt{-d}$, then $x_\iota = \sqrt{-d} \in \mathcal{H}^+ = \mathcal{X}_{\mathrm{std}}$. The set of *CM-points* on the modular curve of level K are then defined to be the elements of the set

$$\mathcal{T}_K(\mathbb{C}) := \{[x_\iota, g]_K \,|\, g \in \mathbf{G}(\mathbb{A}_f)\} \subset \mathcal{S}_K(\mathbb{C}) \tag{0.1.11}$$

We observe that the set of CM points depends only on the $\mathbf{G}(\mathbb{Q})$-conjugacy class of φ and is therefore independent of the choice of the basis e_1, e_2. Indeed, if we change φ by $q\varphi q^{-1}$ for $q \in \mathbf{G}(\mathbb{Q})$, then we change x_ι by $qx_\iota q^{-1}$ and the point $[x_\iota, g]_K$ is now written as $[qx_\iota q^{-1}, qg]_K$. We may reinterpret the set of CM-points as the images of all possible *twisted embeddings*

$$\iota_g : \mathcal{T}_{U_g}(\mathbb{C}) \hookrightarrow \mathcal{S}_{gKg^{-1}}(\mathbb{C}) \xrightarrow{[g]} \mathcal{S}_K(\mathbb{C}), \qquad [h] \mapsto [x_\iota, \iota(h)g] \tag{0.1.12}$$

where $U_g := gKg^{-1} \cap \mathbf{H}(\mathbb{A}_f)$. More generally, for any two levels $U \subset \mathbf{H}(\mathbb{A}_f)$, $K \subset \mathbf{G}(\mathbb{A}_f)$ such that $U \subset K$, there is an induced finite unramified morphism

$$\iota_{U,K} : \mathcal{T}_U \to \mathcal{S}_{K,E} \tag{0.1.13}$$

of E-schemes induced by the map of sets $\mathcal{T}_U(\mathbb{C}) \to \mathcal{S}_K(\mathbb{C})$ given by $[h] \mapsto [x_\iota, h]_K$. In Deligne's formalism, the requirement that the $\mathbb{Q}$-scheme $\mathcal{S}_K$ constitutes a canonical model of $\mathcal{S}_K(\mathbb{C})$ entails in particular that ι_g in (0.1.12) are morphisms of E-schemes for any $g \in \mathbf{G}(\mathbb{A}_f)$, and that the action of Galois group $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ on the set $\mathcal{T}_K(\mathbb{C}) \subset \mathcal{S}_K(\mathbb{C})$ is prescribed by the aforementioned reciprocity law (0.1.7): if $\sigma \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$ and if $a = (a_\infty, a_f)$ is such that $\mathrm{Art}(a) = \sigma$, then

$$\sigma[x_\iota, g]_K = [x_\iota, \iota(a_f)g]_K.$$

In particular, the field of definition of the point $[x_\iota, g]$ is E_{U_g} associated to the group $U_g \subset \mathbf{H}(\mathbb{A}_f)$ via (0.1.8) i.e. E_{U_g} is the fixed field of the subgroup $\mathrm{Art}_E(E^\times \backslash E^\times U_g) \subset \mathrm{Gal}(E^{\mathrm{ab}}/E)$ (see Remark 0.1.9).

Remark 0.1.14. The action of Galois group on the CM points in the formalism above is essentially theory of complex multiplication in disguise. A point to take note of is that the formalism only allows us to work with compact open subgroups of $\mathbf{H}(\mathbb{A}_f)$ rather than those of the quotient $\mathbf{H}(\mathbb{Q})\backslash\mathbf{H}(\mathbb{A}_f)$. It encodes the classical fact that the smallest field of definition for the CM points is the *Hilbert class field* of E.

Remark 0.1.15. In the absence of the so-called *Heegner hypothesis*, the two elliptic curves appearing in the cyclic N-isogeny associated with a CM point will not have complex multiplication by the same order in E. See [Dar04, Proposition 3.8.].

0.2 The Euler system of CM divisors

Let S be any subset of rational primes ℓ of $\mathbb{Q}$ such that

- ℓ *unramified* in E,
- the $\mathbb{Z}_\ell$ lattice generated by $e_1 \otimes 1, e_2 \otimes 1$ inside $E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is $\mathcal{O}_\ell := \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$,

- K is *hyperspecial* at ℓ i.e. $K = K^\ell K_\ell$, $K^\ell \subset \mathbf{G}(\mathbb{A}_f^\ell)$, $K_\ell = \mathbf{G}(\mathbb{Z}_\ell)$,
- $\mathbf{H}$ is unramified at ℓ i.e. $\ell \nmid \mathrm{disc}(E)$ (so that $\mathbf{H}$ has a smooth model over $\mathbb{Z}_\ell$ and $\mathbf{H}(\mathbb{Z}_\ell)$ is the maximal compact open subgroup of $\mathbf{H}(\mathbb{Q}_\ell)$),
- if ℓ is inert, K^ℓ contains the elements $\mathrm{diag}(\ell,\ell) \in \mathbf{G}(\mathbb{Q}) \hookrightarrow \mathbf{G}(\mathbb{A}_f^\ell)$.[3]

If ℓ is *inert* in E, we let λ denote the unique prime in E above ℓ. If ℓ is *split* in E, we let λ be any one of the two primes above ℓ in which case we denote the conjugate of λ by $\bar{\lambda}$. Let Λ be the set of all primes λ of E above S obtained by this procedure and let $\mathcal{N}$ the set of all square-free products of primes in S. We consider $1 \in \mathcal{N}$ as the empty product of primes in S. For $n \in \mathcal{N}$, we denote by $\mathbb{A}_f^{[n]}$ the group of finite adeles away from primes dividing n and by $K^{[n]}$ the subgroup of $\mathbf{G}(\mathbb{A}^{[n]})$ such that $K = K^{[n]} \cdot \prod_{\ell|n} K_\ell$. We let $\mathbb{A}_{f,[n]} := \prod_{\ell|n} \mathbb{Q}_\ell$ and $K_{[n]} := \prod_{\ell|n} K_\ell$.

0.2.1 CM divisors

On an algebraic curve, a divisor is a finite linear combination of points on the curve. Since $\mathcal{T}_K(\mathbb{C})$ is a set of points on $\mathcal{S}_K(\mathbb{C})$, we can use them to define a special class of divisors.

Definition 0.2.1. We call the free $\mathbb{Z}$-module $\mathcal{Z} = \mathcal{Z}_K := \mathbb{Z}[\mathcal{T}_K(\mathbb{C})]$ on elements of $\mathcal{T}_K(\mathbb{C})$ the group of *CM divisors* on $\mathcal{S}_K(\mathbb{C})$.

The group of CM divisors inherits a left action of the Galois group, or what amounts to the same thing by (0.1.8) as the action of $\mathbf{H}(\mathbb{A}_f)$. It is given via left multiplication of elements of $\mathbf{H}(\mathbb{A}_f)$ on the second component of elements of $\mathcal{S}_K(\mathbb{C})$ i.e.

$$h_f[x_\iota, g_f]_K = [x_\iota, h_f g_f]_K$$

where $h_f \in \mathbf{H}(\mathbb{A}_f)$, $g_f \in \mathbf{G}(\mathbb{A}_f)$. If $V \subset \mathbf{H}(\mathbb{A}_f)$ is a compact open subgroup, we let $\mathcal{Z}(V) = \mathcal{Z}^V$ be the $\mathbb{Z}$-submodule of all V-invariant linear combinations. This is then the subgroup of divisors defined over the field E_V associated to V via (0.1.8). We say that a divisor $\xi = \sum_\gamma a_\gamma [x_\iota, \gamma] \in \mathcal{Z}$ where $\gamma \in \mathbf{G}(\mathbb{A}_f)$ is *unramified* at a prime $\ell \in S$ if it's stabilizer V in $\mathbf{H}(\mathbb{A}_f)$ contains the subgroup $\mathcal{O}_\ell^\times$ of units of $\mathcal{O}_\ell$. Here $\mathcal{O}_\ell^\times$ is considered as a subgroup of $\mathbf{H}(\mathbb{A}_f)$ via $\mathcal{O}_\ell^\times \subset \mathbf{H}(\mathbb{Q}_\ell) \hookrightarrow \mathbf{H}(\mathbb{A}_f)$. We say that $\xi \in \mathcal{Z}$ is *unramified at* $n \in \mathcal{N}$ if it is unramified at all $\ell \mid n$.

The group of CM divisors also inherits a right action of *Hecke operators*. A Hecke operator of level K is the characteristic function of a double coset KgK where $g \in \mathbf{G}(\mathbb{A}_f)$ and is denoted $\mathrm{ch}(KgK)$. The free $\mathbb{Z}$-module of elements $\mathrm{ch}(KgK)$ for $g \in \mathbf{G}(\mathbb{A}_f)$ varying over representatives of $K\backslash\mathbf{G}(\mathbb{A}_f)/K$ is then an

[3] This condition is imposed to reflect the behaviour of the Frobenius above an inert prime ℓ in the so-called *anticyclotomic extension*. It holds whenever K is given as the stabilizer of an adelic lattice.

algebra under convolution. This is called the *Hecke algebra* of $\mathbf{G}(\mathbb{A}_f)$ of level K and we will denote it by $\mathbb{Z}[K\backslash\mathbf{G}(\mathbb{A}_f)/K]$. One may also define local variants of this algebra.

Now, given a Hecke operator $\mathrm{ch}(KgK) \in \mathbb{Z}[K\backslash\mathbf{G}(\mathbb{A}_f)/K]$, its action on $\mathcal{Z}$ is obtained by summing over a set of representatives $\gamma \in KgK/K$ with γ acting by multiplication on the right of the second component i.e.

$$\mathrm{ch}(KgK)\cdot[x_\iota, g_1]_K = \sum_{\gamma\in KgK/K}[x_\iota, g_1\gamma]_K$$

This is a finite sum since KgK has only finitely many left cosets contained in KgK owing to the compactness of K. It is also easily seen that this action is well-defined i.e. it does not depend on the choice of representatives γ and $[x_\iota, g_1]$.

The two actions we have defined – that of $\mathbf{H}(\mathbb{A}_f)$ and of $\mathbb{Z}[K\backslash\mathbf{G}(\mathbb{A}_f)/K]$ – clearly commute with each other. We collectively denote this action by

$$(h, \mathrm{ch}(KgK))\cdot[x_\iota, g_1]_K = \sum_{\gamma\in KgK/K}[x_\iota, hg_1\gamma]_K \tag{0.2.2}$$

where $h\in\mathbf{H}(\mathbb{A}_f), g, g_1 \in \mathrm{GL}_2(\mathbb{A}_f)$. We may equivalently describe the actions above in a function theoretic way[4]. Let $C_c(\mathbf{G}(\mathbb{A}_f)/K)$ denote the $\mathbb{Z}$-module of functions $\xi : \mathbf{G}(\mathbb{A}_f)\to\mathbb{Z}$ that are constant on left cosets of K in G and have compact support. It is then a free $\mathbb{Z}$-module on the basis $\mathrm{ch}(gK)$ of characteristic functions of cosets gK, $g\in\mathbf{G}(\mathbb{A}_f)$. We similarly define $C_c(\mathbf{G}(\mathbb{Q}_\ell)/K_\ell)$ for $\ell\in S$, $C_c(\mathbf{G}(\mathbb{A}_f^{[n]})/K^{[n]})$ for $n\in\mathcal{N}$ etc. Now let $\mathcal{F}$ be the $\mathbb{Z}$-module quotient of $C_c(\mathrm{GL}_2(\mathbb{A}_f)/K)$ by the relations $\xi - \xi(h^{-1}(-))$ for all $h\in\mathbf{H}(\mathbb{Q}) = E^\times$, $\xi\in C_c(\mathbf{G}(\mathbb{A}_f)/K)$. We denote the class of ξ in $\mathcal{F}$ by $[\xi]$. The module $\mathcal{F}$ inherits the actions of $\mathbf{H}(\mathbb{A}_f)$ and of Hecke operators $\mathrm{ch}(KgK)$, $g\in\mathbf{G}(\mathbb{A}_f)$ in an obvious manner such that induced isomorphism of abelian groups

$$\psi : \mathcal{F}\to\mathcal{Z} \qquad [\mathrm{ch}(g_1K)]\mapsto[x_\iota, g_1]_K \tag{0.2.3}$$

respects these two actions.

0.2.2 The Hecke polynomial

We will be interested in a particular linear combination of Hekce operators. For $\ell\in S$, let

$$\sigma_\ell := \begin{pmatrix}\ell & 0\\ 0 & 1\end{pmatrix} \qquad \tau_\ell := \begin{pmatrix}\ell & 0\\ 0 & \ell\end{pmatrix} \tag{0.2.4}$$

be elements of $\mathbf{G}(\mathbb{Q}_\ell)$ and consider them as elements of $\mathbf{G}(\mathbb{A}_f)$ via the natural embedding $\mathbf{G}(\mathbb{Q}_\ell)\hookrightarrow\mathbf{G}(\mathbb{A}_f)$.

Definition 0.2.5. The *Hecke polynomial at a prime* $\ell\in S$ is the polynomial

$$\mathfrak{H}_\ell(X) = \ell\cdot\mathrm{ch}(K) - \mathrm{ch}(K\sigma_\ell K)X + \mathrm{ch}(K\tau_\ell K)X^2 \tag{0.2.6}$$

[4]which is to emphasize the combinatorial nature of the relations we are going to establish

in the ring $\mathbb{Z}[K\backslash\mathbf{G}(\mathbb{A}_f)/K][X]$.

Remark 0.2.7. The Hecke polynomial we have defined above arises from the cocharacter (0.1.5) by certain representation theoretic considerations. See §4.3 and [BR94, §5.1] for more details.

By our discussion in the previous subsection, the expression $\mathfrak{H}_\ell(\gamma)$ for $\gamma \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$ acts on module $\mathcal{Z} = \mathcal{Z}_K$ via commuting action of Hecke operators and Galois group. If $\gamma = \mathrm{Frob}_\lambda \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$ is a choice of (arithmetic) Frobenius element above λ, then for any abelian extension F/E in which λ is unramified, the Frobenius Frob_λ restricts to the *Frobenius substitution* $\mathrm{Fr}_\lambda \in \mathrm{Gal}(F/E)$. The action of Frob_λ on $\mathcal{Z}_n$ for $\lambda \nmid n$ is therefore independent of the choice of $\mathrm{Frob}_\lambda \in \mathrm{Gal}(E^{\mathrm{ab}}/E)$.

We would like to explicitly describe elements in $\mathbf{G}(\mathbb{A}_f)$ that correspond to the Frobenii elements in $\mathbf{H}(\mathbb{A}_f)$. Notice that if ℓ is split in E, $\mathbf{H}(\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^\times \times \mathbb{Q}_\ell^\times$, but the local embedding $\iota_\ell : \mathbf{H}(\mathbb{Q}_\ell) \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_\ell)$ is not diagonal. Let $\beta_1, \beta_2 \in E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ be the local idempotents, with β_1 corresponding to our choice of λ above ℓ. Since $e_1, e_2, \beta_1, \beta_2$ are both basis of the lattice $\mathcal{O}_\ell = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, we can write $e_1 = a\beta_1 + c\beta_2$, $e_2 = b\beta_1 + d\beta_2$ for some $a, b, c, d \in \mathbb{Z}_\ell$. Thus, if $k_\ell := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_\ell)$, the embedding $k_\ell^{-1}\iota_\ell k_\ell : \mathbf{H}(\mathbb{Q}_\ell) \hookrightarrow \mathrm{GL}_2(\mathbb{Q}_\ell)$ is diagonal with the top left corner entry corresponding to β_1. Consequently, the action of arithmetic Frobenius Frob_λ corresponds, via (0.1.8), to the action of h_ℓ where

$$\mathbf{H}(\mathbb{Q}_\ell) \ni h_\ell = \begin{cases} \mathrm{diag}(\ell^{-1}, \ell^{-1}) & \text{if } \ell \text{ is inert in } E \\ k_\ell \cdot \mathrm{diag}(\ell^{-1}, 1) \cdot k_\ell^{-1} & \text{if } \ell \text{ is split in } E. \end{cases} \tag{0.2.8}$$

0.2.3 The layers $E[n]$

For each $\ell \in S$, set

$$\mathrm{GL}_2(\mathbb{Q}_\ell) \ni g_\ell := \begin{cases} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \ell \text{ is inert} \\ k_\ell \begin{pmatrix} 1 & \ell^{-1} \\ 0 & 1 \end{pmatrix} k_\ell^{-1} & \text{if } \ell \text{ is split} \end{cases} \tag{0.2.9}$$

and let $H_{g_\ell} := \mathbf{H}(\mathbb{Q}_\ell) \cap g_\ell K_\ell g_\ell^{-1}$. We note that $H_{g_\ell} \subset \mathcal{O}_\ell^\times$ necessarily, since the latter is a maximal compact open subgroup of $\mathbf{H}(\mathbb{Q}_\ell)$. We set $\Delta_\ell := \mathcal{O}_\ell^\times / H_{g_\ell}$.

For $n \in \mathcal{N}\backslash\{1\}$, let $g_n := \prod_{\ell|n} g_\ell \in \mathbb{A}_{f,[n]}$ and set $g_1 = 1$. We consider g_n as elements of $\mathbb{A}_f$ via the natural inclusion $\mathbb{A}_{f,[n]} \hookrightarrow \mathbb{A}_f$ where the components away from primes dividing n are 1. Set $H_n := \mathbf{H}(\mathbb{A}_f) \cap g_n K g_n^{-1}$. Then H_n are compact open subgroups of $\mathbf{H}(\mathbb{A}_f)$ and

$$H_n = \left(\mathbf{H}(\mathbb{A}_f^{[n]}) \cap K^{[n]}\right) \cdot \prod_{\ell|n} H_{g_\ell} \tag{0.2.10}$$

The groups H_n form a 'lattice' (in the sense of order theory) where $m|n \implies H_n \subset H_m$. Moreover, $H_m/H_n \simeq \Delta_{n/m}$ where

$$\Delta_k := \prod_{\ell|k} \Delta_\ell$$

for $k \in \mathcal{N}$. For $n \in \mathcal{N}$, let $E[n]$ the abelian field extension of E corresponding to H_n via (0.1.8) i.e. $E[n]$ is the field such that $\mathrm{Gal}(E/E[n])$ is identified with $E^\times \backslash E^\times H_n \subset \mathbf{H}(\mathbb{Q})\backslash \mathbf{H}(\mathbb{A}_f)$ via the Artin map (see Remark 0.1.9). Then $E[m] \subset E[n]$ for $m|n$. In order to describe $\mathrm{Gal}(E[m]/E[n])$, we need to take elements of $\mathcal{O}_E^\times$ into account. Let

$$\nu_n := H_n \cap E^\times \subset \mathcal{O}_E^\times \tag{0.2.11}$$

and set $v_n := |\nu_n|$. Note that $v_n|12$ since the group of units of imaginary quadratic fields are of order 2, 4 or 6. Again, the groups ν_n form a lattice with $m|n \implies \nu_n \subset \nu_m$. Set $\nu_n^m := \nu_m/\nu_n$. Then $\nu_n^m = \nu_m H_n/H_n$ is a subgroup of $H_m/H_n \simeq \Delta_{n/m}$.

Lemma 0.2.12. *For all $m, n \in \mathcal{N}$ with $m|n$, $\mathrm{Gal}(E[n]/E[m]) \simeq (\Delta_{n/m})/\nu_n^m$. In particular, the degree of extension $E[n]/E[m]$ is $|\Delta_{n/m}| - (v_m - v_n)$.*

Proof. We have $\mathrm{Gal}(E^{\mathrm{ab}}/E[n]) \simeq H_n E^\times/E^\times$, $\mathrm{Gal}(E^{\mathrm{ab}}/E[m]) \simeq H_m E^\times/E^\times$. Then,

$$\begin{aligned}
\mathrm{Gal}(E[n]/E[m]) &\simeq H_m E^\times/H_n E^\times \\
&\simeq (H_m \cdot H_n E^\times)/H_n E^\times \\
&\simeq H_m/(H_m \cap H_n E^\times). \\
&\simeq H_m/(\nu_m H_n) \\
&\simeq (H_m/H_n)/(\nu_m H_n/H_n) \\
&\simeq \Delta_{m/n}/\nu_n^m.
\end{aligned}$$

The claim on cardinality is then immediate. □

Remark 0.2.13. Although it is not necessary for what follows, we note that $E[n]$ are analogs of what are known as *ring class extensions* of E. More precisely, say K is the stabilizer of an adelic lattice $L_K \subset \mathbb{A}_{E,f} \cong \mathbb{A}_f e_1 \oplus \mathbb{A}_f e_2$. Let $\varphi_{\mathbb{A}_f} : \mathbb{A}_{E,f} \hookrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{A}_f)$ denote the natural map induced by φ. Then, $H_n = \mathbf{H}(\mathbb{A}_f) \cap g_n K g_n^{-1}$ is equal to group of units of the subring

$$R_{n,K} := \left\{ r \in \mathbb{A}_{E,f} \,|\, \varphi_{\mathbb{A}_f}(r) \cdot g_n L_K \subset g_n L_K \right\} \subset \mathbb{A}_{E,f}.$$

As $R_{n,K}$ is compact and open, $R_{n,K} \subset \widehat{\mathcal{O}}_E$ as $\widehat{\mathcal{O}}_E$ is the maximal compact subring of $\mathbb{A}_{E,f}$. Since the image $\widehat{\mathbb{Z}} = \mathbb{A}_f \hookrightarrow \mathbb{A}_{E,f}$ under $\varphi_{\mathbb{A}_f}$ is contained in diagonal matrices and the lattices $g_n L_K$ are stabilized by $\widehat{\mathbb{Z}}$, $R_{n,K}$

contains $\widehat{\mathbb{Z}}$. The upshot is that $R_{n,K}$ is the profinite completion of an *order* of $\mathcal{O}_E$ i.e. a subring of $\mathcal{O}_E$ which is free of rank 2 as a module over $\mathbb{Z}$. Consequently, the image of $\widehat{\mathbb{Z}}^\times \hookrightarrow \mathbb{A}_{E,f}$ is contained in H_n and therefore the fields $E[n]$ are stabilized by the image of the *Verlagerung map* $\mathrm{Ver} : \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$. One says that $E[n]$ is a *transfer extension* i.e. it is fixed under the image of Galois group of $\mathbb{Q}$ in the Galois group of E. The Galois group of any such abelian extension is *generalized dihedral* i.e. the action of $\mathrm{Gal}(E/\mathbb{Q})$ on $\mathrm{Gal}(E[n]/\mathbb{Q})$ is via inversion. We refer the reader to [Kü17, §3.2] for more detailed results describing various interrelated extensions of E.

0.2.4 Lattice Counting

We shall prove our norm relations by interpreting a formal sum of lattices in two ways. In this subsection, we recall some basic facts and establish a combinatorial lemma on trace maps with respect to $\Delta_\ell = \mathcal{O}_\ell^\times/H_{g_\ell}$.

Definition 0.2.14. Let ℓ be any rational prime and $V = V_{\mathrm{std}} = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$ be the standard vector space of dimension 2. A *lattice* in V is a $\mathbb{Z}_\ell$-submodule spanned by a $\mathbb{Q}_\ell$-basis for V. We let $\mathcal{L}$ denote the set of lattices in V. The *standard lattice* $L_{\mathrm{std}} \in \mathcal{L}$ is the lattice generated by the standard basis.

Each $g \in \mathbf{G}(\mathbb{Q}_\ell)$ acts on V by linear transformations, and sends a lattice to a lattice, whence we have an action $\mathbf{G}(\mathbb{Q}_\ell) \times \mathcal{L} \to \mathcal{L}$. The stabilizer of the standard lattice is precisely $K_\ell = \mathbf{G}(\mathbb{Z}_\ell)$, and therefore one obtains a bijection

$$\begin{aligned} \mathbf{G}(\mathbb{Q}_\ell)/K_\ell &\xrightarrow{\sim} \mathcal{L} \\ gK_\ell &\mapsto g \cdot L_{\mathrm{std}} \end{aligned} \tag{0.2.15}$$

Under this bijection, we can identify $C_c(\mathbf{G}(\mathbb{Q}_\ell)/K_\ell)$ with $\mathbb{Z}[\mathcal{L}]$, the set of all formal linear combination of lattices in the standard vector space V.

Let $\ell \in S$ and consider $E \otimes \mathbb{Q}_\ell$ as the standard vector space with basis $e_1 \otimes 1, e_2 \otimes 1$. The standard lattice then coincides with $\mathcal{O}_\ell := \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. We note that $\mathcal{O}_\ell = \mathcal{O}_\lambda$ (the ring of integers of E_λ) if ℓ is inert, and

$$\mathcal{O}_\ell = \mathbb{Z}_\ell e_1 \oplus \mathbb{Z}_\ell e_2 = \mathbb{Z}_\ell \beta_1 \oplus \mathbb{Z}_\ell \beta_2 = \mathcal{O}_\lambda \oplus \mathcal{O}_{\bar{\lambda}}$$

if ℓ is split. Recall that $\Delta_\ell = \mathcal{O}_\ell^\times/H_{g_\ell}$.

Lemma 0.2.16. *For $\ell \in S$, let*

$$\xi_0 := \sum_{\gamma \in \Delta_\ell} \mathrm{ch}(\gamma g_\ell K_\ell) \in C_c(\mathbf{G}(\mathbb{Q}_\ell)/K_\ell). \tag{0.2.17}$$

Then $\xi_0 \in \mathbb{Z}_\ell[\mathcal{L}]$ represents the formal sum with coefficients 1 of all lattices ηK_ℓ where *a)* $\eta \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \ell & [\kappa] \\ 0 & 1 \end{pmatrix} \mid \kappa \in \mathbb{F}_\ell \right\}$ *if ℓ is inert.*

b) $\eta \in \left\{ k_\ell \begin{pmatrix} 1 & \ell^{-1}[\kappa] \\ 0 & 1 \end{pmatrix} k_\ell^{-1} \mid \kappa \in \mathbb{F}_\ell^\times \right\}$ *if ℓ is split.*

Here $[\kappa] \in \mathbb{Z}_\ell$ denotes a lift of $\kappa \in \mathbb{F}_\ell$. In particular, $|\Delta_\ell| = \ell + 1$ (resp. $\ell - 1$) if ℓ is inert (resp. split) in E.

Proof. First observe that $\mathcal{O}_\ell^\times = \mathbf{H}(\mathbb{Q}_\ell) \cap K_\ell$ is the stabilizer in $\mathbf{H}(\mathbb{Q}_\ell)$ of the standard lattice $L_{\text{std}} = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, where $\mathbf{H}(\mathbb{Q}_\ell)$ acts on $\mathcal{L}$ via ι. Similarly, $H_{g_\ell} = \mathbf{H}(\mathbb{Q}_\ell) \cap g_\ell K_\ell g_\ell^{-1}$ is the stabilizer in $\mathbf{H}(\mathbb{Q}_\ell)$ of the lattice

$$L_{g_\ell} := g_\ell(L_{\text{std}}) = \langle g_\ell e_1, g_\ell e_2 \rangle \in \mathcal{L}$$

and ξ_0 represents the formal sum of lattices in the $\mathcal{O}_\ell^\times$ orbit of L_{g_ℓ}.

a) If ℓ is inert, $L_{g_\ell} = \langle \ell e_1, e_2 \rangle$. Since $\ell\mathcal{O}_\ell \subsetneq L_{g_\ell} \subsetneq \mathcal{O}_\ell$, the lattices L in the orbit of L_{g_ℓ} under the action of $\mathcal{O}_\ell^\times$ must also be such that $\ell\mathcal{O}_\ell \subsetneq L \subsetneq \mathcal{O}_\ell$. As $\mathcal{O}_\ell = \mathcal{O}_\lambda = L_{\text{std}}$ by our convention, the lattices in this orbit correspond to a subset of the set of lines[5] in $\mathbb{F}_{\ell^2} = \mathbb{F}_\lambda := \mathcal{O}_\lambda / \ell\mathcal{O}_\lambda$ where $\mathbb{F}_\lambda$ is considered as a 2-dimensional vector space over $\mathbb{F}_\ell$. Since $\mathcal{O}_\ell^\times = \mathcal{O}_\lambda^\times$ acts transitively on $\mathbb{F}_\lambda^\times$, the orbit is *all* the lattices L such that $\ell\mathcal{O}_\ell \subset L \subset \mathcal{O}_\ell$. The number of lines in $\mathbb{F}_\lambda$ is exactly $|\mathbb{F}_\lambda^\times / \mathbb{F}_\ell^\times| = \ell + 1$ (each element $x \in \mathbb{F}_\lambda^\times$ spans a line $\mathbb{F}_\ell x$ and any $y \in \mathbb{F}_\ell^\times x$ determines the same line). A set of representatives is then easily seen to be the one in the claim.

b) On the other hand, if ℓ is split, H_ℓ is the stabilizer in $\mathcal{O}_\ell^\times$ of the lattice $L_{g_\ell} = \langle \beta, \ell^{-1}\beta_1 + \beta_2 \rangle$. Now, $\mathcal{O}_\ell^\times = \mathcal{O}_\lambda^\times \times \mathcal{O}_{\bar{\lambda}}^\times \cong \mathbb{Z}_\ell^\times \times \mathbb{Z}_\ell^\times$ acts on $\mathcal{O}_\ell = \mathbb{Z}_\ell\beta_1 \oplus \mathbb{Z}_\ell\beta_2$ componentwise, so if $\gamma = (\gamma_1, \gamma_2) \in \mathcal{O}_\lambda^\times \times \mathcal{O}_{\bar{\lambda}}^\times$, then

$$(\gamma_1, \gamma_2) \cdot L_{g_\ell} = \langle \gamma_1\beta_1, \ell^{-1}\gamma_1\beta_1 + \gamma_2\beta_2 \rangle = \langle \beta_1, \ell^{-1}\gamma_1\gamma_2^{-1}\beta_1 + \beta_2 \rangle,$$

which is equal to L_{g_ℓ} if and only if $\gamma_1\gamma_2^{-1} \in 1 + \ell\mathbb{Z}_\ell$. Thus, there are exactly $\mathbb{Z}_\ell^\times / (1 + \ell\mathbb{Z}_\ell)$ distinct lattices in the orbit of $\mathcal{O}_\ell^\times$ on L_{g_ℓ} and we easily find representatives by taking the translates under $([\kappa], 1) \in \mathcal{O}_\lambda^\times \times \mathcal{O}_\lambda^\times$ for κ running over lifts of $\mathbb{F}_\ell^\times$ in $\mathbb{Z}_\ell$. □

Lemma 0.2.18. *For all $\ell \in S$, $\text{ch}(K_\ell \sigma K_\ell) = \sum_{\gamma \in K_\ell \sigma_\ell K_\ell / K_\ell}$ in $\mathbb{Z}[\mathcal{L}]$ represents the same formal sum of lattices as in Lemma 0.2.16 (a).*

Proof. This amounts to describing the orbit of K_ℓ acting on the lattice $\langle \ell e_1, e_2 \rangle$ which amounts to the same counting argument as in part (a). □

0.2.5 Norm Relations

We begin by defining what are sometimes called 'test vectors' for an Euler system in literature. We would however like to refer to them as *abstract zeta elements*. These will be the objects that we wish to axiomatize

[5] i.e. 1-dimensional vector subspaces

and study in this thesis systematically. For any prime $\ell \in S$, define

$$\zeta_\ell := \mathrm{ch}(K_\ell) - \mathrm{ch}(g_\ell K_\ell) \in C_c(\mathbf{G}(\mathbb{Q}_\ell)/K_\ell). \tag{0.2.19}$$

and for $n \in \mathcal{N}$, set

$$\zeta_n := \bigotimes_{\ell | n} \zeta_\ell \in C_c(\mathbf{G}(\mathbb{Q}_{[n]}/K_{[n]}) \tag{0.2.20}$$

Abusing notation, we also denote by ζ_n the 'restricted tensor product' element $\mathrm{ch}(K^{[n]}) \otimes \zeta_n \in C_c(\mathbf{G}(\mathbb{A}_f)/K)$ which consists of 2^n terms of the form $\mathrm{ch}(gK)$ with coefficients in $\{\pm 1\}$

Definition 0.2.21. For $n \in \mathcal{N}$, the n-th *Euler system divisor class* to be

$$y_n = \psi\big(|\mathcal{O}_E^\times| \cdot v_n^{-1} \cdot [\zeta_n]\big) \in \mathcal{Z} = \mathbb{Z}[\mathcal{T}_K(\mathbb{C})]$$

where ψ is the map in (0.2.3) and $v_n = |\nu_n|$ (see 0.2.11). The *bottom class* of the system is defined to be $y_1 = \psi(|\mathcal{O}_E^\times| \cdot v_1^{-1} \cdot [\mathrm{ch}(K)])$.

The divisors y_n are defined over $E[n]$ or equivalently, $y_n \in \mathcal{Z}(H_n)$. Indeed, H_ℓ stabilizes ζ_ℓ for all $\ell \mid n$ and $\mathbf{H}(\mathbb{A}_f^{[n]} \cap K^{[n]})$ acts trivially on $\mathrm{ch}(K^{[n]})$. Moreover, for any $\lambda \in \Lambda$ above a $\ell \in S$ such that $\ell \nmid n$, y_n is unramified over λ as $H_n = H_\ell H^\ell$ with $H_\ell = \mathcal{O}_\ell^\times$. Thus, the action of the arithmetic Frobenius Frob_λ at a prime λ on the divisor y_n is well-defined for any such λ.

Theorem 0.2.22. *For all $n \in \mathcal{N}$ and $\ell \in S$ with $\ell \nmid n$, we have*

$$\mathfrak{H}_\ell(\mathrm{Frob}_\lambda) y_n = \mathrm{Tr}_{E[n\ell]/E[n]}(y_{n\ell})$$

as elements of $\mathcal{Z}_n \subset \mathcal{Z}$. Here, Tr *denotes the induced trace map.*

Proof. By the equivariance properties of ψ and Lemma 0.2.12, it suffices to establish that $\mathfrak{H}_\ell(h_\ell) \cdot [\zeta_n] = \sum_{\gamma \in H_n/H_{n\ell}} \gamma \cdot [\zeta_{n\ell}]$ as elements of $\mathcal{F}$ where h_ℓ are the elements in (0.2.8). Explicitly, we need to show that

$$\Big(\ell \cdot \mathrm{ch}(K) - (h_\ell, \mathrm{ch}(K\sigma_\ell K)) + (h_\ell^2, \mathrm{ch}(K\tau_\ell K))\Big) \cdot [\mathrm{ch}(K)] = \sum_{\gamma \in \Delta_\ell} [\mathrm{ch}(\gamma K) - \mathrm{ch}(\gamma g_\ell K)] \tag{†}$$

for all $\ell \in S$. Set $\gamma_0 = \begin{pmatrix} 1 & \\ & \ell \end{pmatrix}$, $\gamma_i = \begin{pmatrix} \ell & i \\ & 1 \end{pmatrix}$ for $i = 1, \ldots, \ell$. Then $\mathrm{ch}(K_\ell \sigma_\ell K_\ell) = \sum_{i=0}^{\ell} \mathrm{ch}(\gamma_i K_\ell)$ by Lemma 0.2.18.

Case 1: ℓ is inert. Since $h_\ell \cdot [\mathrm{ch}(K)] = h_\ell^{-1} \cdot [\mathrm{ch}(K)] = [\mathrm{ch}(K)]$ in $\mathcal{F}$ (by definition), (†) would follow from the equality

$$\ell \cdot \mathrm{ch}(K) - \left(\sum_{i=0}^{\ell} \mathrm{ch}(\gamma_i K)\right) + \mathrm{ch}(K) = (\ell + 1) \cdot \mathrm{ch}(K) - \sum_{\gamma \in \Delta_\ell} \mathrm{ch}(\gamma \kappa_\ell K).$$

as elements of $C_c(\mathbf{G}(\mathbb{A}_f)/K)$. It suffices to establish this with K replaced by K_ℓ. Cancelling the obvious terms, we are reduced to showing that $\sum_{i=0}^{\ell} \mathrm{ch}(\gamma_i K_\ell) = \sum_{\gamma\in\Delta_\ell} \mathrm{ch}(\gamma g_\ell K_\ell)$. This was established in Lemma 0.2.16 (a).

Case 2: ℓ is split. Arguing similarly as in the inert case, (†) would follow from the local equality

$$\ell \cdot \mathrm{ch}(K_\ell) - \mathrm{ch}(h_\ell K_\ell \sigma_\ell K_\ell)) + \mathrm{ch}(h_\ell^2 K_\ell \tau_\ell K) = (\ell-1)\cdot \mathrm{ch}(K_\ell) - \sum_{\gamma\in\Delta_\ell} \mathrm{ch}(\gamma g_\ell K_\ell)$$

as elements of $C_c(\mathbf{G}(\mathbb{Q}_\ell)/K_\ell)$. Since $k_\ell \in K_\ell$, we see that $\mathrm{ch}(K_\ell\sigma_\ell K_\ell) = \sum_{i=0}^{\ell} \mathrm{ch}(k_\ell\gamma_i K_\ell)$ as well. Now note that $h_\ell k_\ell \gamma_0 K_\ell = h_\ell^2 k_\ell \tau_\ell K_\ell = h_\ell^2 \tau_\ell K_\ell$ and $h_\ell k_\ell \gamma_1 K_\ell = K_\ell$ as elements in $\mathbf{G}(\mathbb{Q}_\ell)/K_\ell$. Therefore, the left hand side of the claimed equality above is

$$\ell \cdot \mathrm{ch}(K_\ell) - \left(\sum_{i=0}^{\ell} \mathrm{ch}(h_\ell k_\ell \gamma_i K)\right) + \mathrm{ch}(h_\ell^2 \tau_\ell K_\ell) = (\ell-1)\cdot \mathrm{ch}(K_\ell) - \sum_{i=2}^{\ell} \mathrm{ch}\big(h_\ell(k_\ell\gamma_i k_{\ell^{-1}})K_\ell\big).$$

Thus (†) would follow from $\sum_{i=2}^{\ell} \mathrm{ch}(h_\ell(k_\ell\gamma_i k_\ell^{-1})K_\ell) = \sum_{\gamma\in\Delta_\ell} \mathrm{ch}(\gamma g_\ell K_\ell)$. Thus was established in Lemma 0.2.16 (b). □

Chapter 1

Introduction

With the example of CM divisors on modular curves at our disposal, we begin our formal introduction to this thesis.

The cohomology of Shimura varieties is a rich source of p-adic Galois representations whose zeta functions are conjectured – and in many cases known – to be automorphic and of a specific shape [Kot92]. The unramified local L-factors of these functions can be 'computed' via certain Hecke algebra valued polynomials, often referred to as *Hecke polynomials* [BR94] (see also Remark 9.2.3). These are prescribed by representation theoretic data that arises out of a given Shimura-Deligne data for such varieties. See Chapter 4 for a definition suitable for our purposes together with some techniques for computing them and Proposition 10.1.2, 11.2.3, 12.2.1, 13.2.4 for explicit examples. In the case of modular curves, this polynomial is the one appearing in Definition 0.2.5.

While local L-factors are objects of an analytic nature, Hecke polynomials are better understood as 'universal L-factors' that can specialize to any such analytic L-factor and are inherently combinatorial in nature. These polynomials evaluated at Frobenii act naturally via correspondences on the cohomology of Shimura varieties. The study of Selmer groups of the underlying Galois representations is therefore enhanced by explicit knowledge of the underlying L-factors and access to operations the cohomology of any Shimura variety enjoys.

As first shown in [Kol90], one way of studying Selmer groups is by exhibiting an infinity of carefully curated classes that live in a Galois stable lattice and satisfy a web of intricate trace relations involving Euler factors. The framework in which this intricate web is created is often of the following format: one has a pair H, G of reductive algebraic groups enriched with Shimura data and an embedding $\iota : H \to G$ preserving the said data. The setup induces a collection of pushforwards maps from the cohomology of

Shimura varieties of H to those of G. The setup may be enhanced by introducing p-adic sheaves on these varieties and using them as coefficients. One then picks a collection of classes in the source cohomology of appropriate degree for which one has an explicit description (e.g. fundamental classes of special cycles or Siegel units) and hopes that their images in the target can be made to sit in such a web with prescribed Euler factors.

Our aim is to describe a general process by means of which one can attempt to create this web in the setting just described with Hecke polynomials playing the role of Euler factors, very much in the spirit of Theorem 0.2.22. To wit, say $M_G(K_\alpha)$ denotes the cohomology (in some degree) of the Shimura variety of level K_α a compact open subgroup of the adelic group G_f of G. The levels K_α are allowed to vary over a prescribed lattice of compact open subgroups of G_f (e.g. neat subgroups) to ensure the behaviour of these varieties is not pathological. For each inclusion $K_\beta \subset K_\alpha$, one has trace maps $M_G(K_\beta) \to M_G(K_\alpha)$. Say v, w are two finite places, K_v, K_w are two local compact open subgroups and $\mathfrak{H}_v$, $\mathfrak{H}_w$ are Hecke polynomials at places v, w which for the moment the reader may think of as just linear sums of local Hecke correspondences or even just as endomorphisms that commute with each other. Let $L_v \subset K_v$, $L_w \subset K_w$ be deeper (smaller) levels and K^{vw} be a subgroup of the adelic group of G at places away from v, w. Set $K_\alpha = K^{vw} L_v L_w$, $K_\beta = K^{vw} L_v K_w$, $K_\gamma = K^{vw} K_v L_w$, $K_\delta = K^{vw} K_v K_w$, so that we have a diagram of trace maps on the left (with solid arrows).

$$
\begin{array}{ccccc}
M_H(V_\alpha) & \dashrightarrow & & M_G(K_\alpha), & \\
& & \swarrow^{\mathrm{tr}_{\alpha,\beta}} & & \searrow^{\mathrm{tr}_{\alpha,\gamma}} \\
M_H(V_\beta) & \dashrightarrow & M_G(K_\beta) & & M_G(K_\gamma) \\
& & \searrow_{\mathrm{tr}_{\beta,\delta}} & & \swarrow_{\mathrm{tr}_{\gamma,\delta}} \\
M_H(U) & \overset{\iota_*}{\dashrightarrow} & & M_G(K_\delta) &
\end{array}
\qquad
\begin{array}{ccc}
& y_\alpha & \\
\swarrow & & \searrow \\
\mathfrak{H}_w(y_\beta) & & \mathfrak{H}_v(y_\gamma) \\
\searrow & & \swarrow \\
& \mathfrak{H}_v\mathfrak{H}_w(y_\delta) &
\end{array}
$$

Let $M_H(V)$ denote the cohomology (in some degree) of Shimura varieties of H of level V from which there are maps to the cohomology M_G for suitably chosen pairs of levels. Let $y_\delta \in M_G(K_\delta)$ be a class at the 'bottom' level that is obtained as the image of a class x_U from $M_H(U)$ where $U \subset K_\delta$ is some level of the adelic group H_f of H. We seek classes $y_\square \in M_G(K_\square)$ for $\square \in \{\alpha, \beta, \gamma\}$ such that

- $\mathrm{tr}_{\alpha,\beta}(y_\alpha) = \mathfrak{H}_w(y_\beta)$,
- $\mathrm{tr}_{\alpha,\gamma}(y_\alpha) = \mathfrak{H}_v(y_\gamma)$,
- $\mathrm{tr}_{\beta,\delta}(y_\beta) = \mathfrak{H}_v(y_\delta)$,
- $\mathrm{tr}_{\gamma,\delta}(y_\gamma) = \mathfrak{H}_w(y_\delta)$.

This toy setup may be extended to an infinite web by deepening the levels at an infinite subset of all finite places which is indeed the actual situation of interest. In practice, the Galois theoretic aspect required in these norm relations may be encoded by working with a slightly enlarged group '$\mathcal{G}$' that is the product of G with a torus that takes into account the Frobenius action. The deepening of the levels is done place by place on the levels of this torus only, which corresponds to field extensions (e.g. the commonly seen ring or ray class fields, see Remark 0.2.13) by appropriate Shimura reciprocity laws. In the setup of 0.1.1, the role of this torus was taken up by $\mathbf{H}$ itself. Once these classes are pushed, say via Abel-Jacobi maps (an analogue of Kummer map) and projected to appropriate isotypical components (which in essence replaces Hecke operators by their eigenvalues on automorphic forms), one recovers the desired relations at the level of Galois cohomology.

The insight that makes the search for these classes tractable is to require $y_\square$ to also be the (sum of) images under ι_* of classes from various levels of M_H that enjoy properties of restricted tensor products (see Definition 2.6.8). To be more precise, we require the classes to be the sum of certain *mixed Hecke correspondences* (Definition 2.5.3) at local places. These can be thought of as correspondences between the cohomologies of H and G and enjoy many properties of ordinary Hecke correspondences[1]. Thus for each level say K_α, our paradigm will be to supply (usually several) levels V_α and classes x_α in the cohomology $M_H(V_\alpha)$ that we will transfer along prescribed mixed correspondences (visualized as dashed arrows in the diagram above) from level V_α to level K_α. The class y_α that we seek will then be a suitable linear sum of these classes x_α procured from levels V_α. Working with classes that enjoy properties of restricted tensor products has the added benefit of making the problem of finding these classes one of a local nature. In §0.2.5 for example, we achieved this by only varying one 'component' of the parameter space for CM points at a prime ℓ at a time.

1.1 The process

Let us now describe the process of finding these classes, or rather their parameters, at a local place, say v. The very first step to be taken is to decompose the various Hecke operators $\mathrm{ch}(K_v\sigma K_v)$ of Hecke polynomial $\mathfrak{H}_v$ (or rather it's transpose 2.4.1) into a sum mixed Hecke operators $\sum_{\sigma_j}\mathrm{ch}(U_v\sigma_j K_v)$ where U_v is the local group of U at v. In the proof of Theorem 0.2.22, this was done in disguise by studying the action of $\mathcal{O}_\ell^\times$ in Lemma 0.2.16 on certain lattices. Listing the summands that each Hecke operator breaks into amounts to studying the orbit structure of the action of U_v on a decomposition $K_v\sigma_j K_v = \bigsqcup_\gamma \gamma K_v$. We point out for the reader that achieving this decomposition directly is a rather forbidding process in general. Fortunately, there exists a rather efficient recipe for describing this decomposition in great generality and is the subject Chapter 5 of this thesis, although the recipe does require a refined knowledge of the structure of the reductive

[1]Let us point out early for the reader that our conventions for mixed ones differ slightly from those for the ordinary ones (Remark 2.5.4)

group G at hand.

Each mixed Hecke operator $\mathrm{ch}(U_v\sigma K_v)$ has an associated *degree* which is an analogue of the degree of usual Hecke operators. The degree of an ordinary Hecke operator $\mathrm{ch}(K_v\sigma K_v)$ is defined to be the cardinality of $K_v\sigma K_v/K_v$ and measures the 'size' of its action on a class. The bigger the size, the more summands that appear in its action on a class. For instance, the degree of the operator $\mathrm{ch}(K_\ell\sigma_\ell K_\ell)$ appearing in Definition 0.2.5 is $\ell+1$ while the degree of $\mathrm{ch}(K_\ell\tau_\ell K_\ell)$ is 1. Their corresponding actions reflected this quantity in the proof of Theorem 0.2.22 rather evidently. The degree of a mixed Hecke operator serves a similar purpose and is a means of quantifying the 'size' of the image of classes pushed along these operators.

Once the decomposition of $\mathfrak{H}_v$ into mixed cosets is achieved and one has a hold of the degrees of these operators, the next step is ready to be taken. The properties of pushforwards maps (developed in Chapter 2) allow us to 'twist' these mixed Hecke operators and the class x_U while keeping their image equal to $\mathfrak{H}_v(y_\delta) \in M_G(K_\delta)$: for any $h_v \in H_v$, we have

$$[(h_vU_vh_v^{-1})(h_v\sigma_j)K_v]_*(h_vx_U) = [U_v\sigma_jK_v]_*(x_U)$$

Here, $[V\sigma K_v]_*$ denotes the mixed Hecke correspondence form a local level V to K_v with twist σ, and h_vx_U denotes the pullback of x_U to level $h_vUh_v^{-1}$. The goal of the process now is to use these local twists to make as many elements σ_j the same as possible by replacing the 'old' twist σ_j with the 'new' twist $h_v\sigma_j$. One then collects the resulting terms together, one for each of the resulting new twists. This twisting and rearrangement of terms in turn modifies the source class x_U, creating various copies/translates of it enumerated by degrees of mixed Hecke operators $\mathrm{ch}(U_v\sigma_jK_v)$. It quite often happens – as is documented in the multitude of examples recorded in Part II of this thesis – that these modified classes turn out to be traces of elements from slightly 'deeper' levels $V_i \subset H$ sitting in conjugates of the level L_v where L_v are as in the diagram above. Once these deeper classes are shown to exist, the desired norm relations are obtained by pushing them along the 'new' twist $h_v\sigma_j$. This process maybe performed for each place in a suitable (infinite) subset of all finite places. One then 'glues' the resulting norm relations using the restricted tensor product properties of x_U (3.5.2) by using deeper levels obtained by deepening source levels at two places etc.

The reason why this process is able to modify classes into traces remain somewhat elusive to us, though there are many features in the examples we have studied that suggest why it does happen. To illustrate the phenomenon, say the action of all of H_v on x_U is trivial. A Hecke operator by its definition acts by creating several copies/translates on the class it acts upon (2.3.1). If the action of the group on the class at hand is trivial, the Hecke operator multiplies that class by its degree. Applying this observation to our x_U, the modification process creates certain multiples of x_U. Let's say when these multiplies are collected together, the total coefficient turns out to be divisible by a power of p, say p^k. In practice, the exact power that appears depends on the cardinality of the residue field $\Bbbk_v$ at the place v e.g. it is determined by the p-adic valuation of $|\Bbbk_v| - 1$ or $|\Bbbk_v| + 1$ depending on the torus chosen in the setup and the divisibility in practice is checked in terms of $|\Bbbk|_v \pm 1$. Now here's the main point: the class $p^k x_U$ is a trace of an integral class from *any* level V such that $[U : V]$ divides p^k. Indeed, one can pull the class x_U to level V whose trace down to level U is just the class $[U : V]x_U$, and one is free to normalize this by elements of $\mathbb{Z}_p^\times$. This property is labelled '(Co)' in Definition 2.1.1, and it encapsulates the phenomenon seen in numerous places that the effect of composing a pullback with a pushforward is just multiplication by the degree of the maps between the objects at hand.

Example 1.1.1. Consider the case where ℓ is a rational prime and $\mathfrak{H}_\ell = \ell \cdot \mathrm{ch}(K_\ell) - T_\ell + S_\ell$ where

- $K_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$,
- T_ℓ is the characteristic function of $K_\ell \begin{pmatrix} \ell & \\ & 1 \end{pmatrix} K_\ell$
- S_ℓ is the characteristic function of $K_\ell \begin{pmatrix} \ell & \\ & \ell \end{pmatrix} K_\ell$.

as in 0.2.2. As already noted, the degrees of T_ℓ, S_ℓ are $\ell + 1$, 1 respectively. Therefore, the action of $\mathfrak{H}_\ell$ on any class y_K on which $G_\ell := \mathrm{GL}_2(\mathbb{Q}_\ell)$ acts trivially ends up annihilating it, since

$$\ell - (\ell + 1) + 1 = 0.$$

Thus, $\mathfrak{H}_\ell(y_K)$ is trivially a trace from as deep a level as one desires. For a less trivial scenario, consider the case where $U_\ell = \mathbb{Z}_\ell^\times \times \mathbb{Z}_\ell^\times$, $H_\ell = \mathbb{Q}_\ell^\times \times \mathbb{Q}_\ell^\times$ embedded diagonally in G_ℓ and the class x_U is assumed to have trivial H_ℓ action (but not necessarily a trivial G_ℓ action). Then using Lemma 0.2.18, we easily see that $T_\ell = \mathrm{ch}(U_\ell \sigma_1 K_\ell) + \mathrm{ch}(U_\ell \sigma_2 K_\ell) + \mathrm{ch}(U_\ell \sigma_3 K_\ell)$ and it's transpose is $T_\ell^t = \sigma_0^{-1} T_\ell$, where

$$\sigma_0 = \begin{pmatrix} \ell & \\ & \ell \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} \ell & \\ & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \ell & 1 \\ & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & \ell \end{pmatrix}$$

Let V_ℓ denote the deeper sublevel of U_ℓ of elements $(x, y) \in U_\ell$ such that $xy^{-1} \in 1 + \ell\mathbb{Z}_\ell$. Then

$$\begin{aligned} \mathfrak{H}_\ell \circ \iota_*(x_U) &= \ell[U_\ell K_\ell]_*(x_U) + [U_\ell \sigma_0^{-1} K_\ell]_*(x_U) - \sum_{i=1}^{3} [U_\ell \sigma_0^{-1} \sigma_i K_\ell]_*(x_U) \\ &= [U_\ell K_\ell]_*\big((\ell - 1)x_U\big) - [V_\ell \sigma_1^{-1} \sigma_2 K_\ell]_*\big([V_\ell \sigma_3 U_\ell](x_U)\big) \end{aligned}$$

by twisting, rearranging terms and noting that $[U_\ell \sigma_0^{-1} \sigma_2 K_\ell]_* = [V_\ell \sigma_1^{-1} \sigma_2 K_\ell]_* \circ [V_\ell \sigma_3 U_\ell]$. The input of the $[U_\ell K_\ell]_*$ is now a multiple of $\ell - 1$ which makes it a trace from the level V_ℓ, while $[U_\ell \sigma_2 K_\ell]_*(x_U)$ factors via $[V_\ell \sigma_1^{-1} \sigma_2 K_\ell]_*$ and therefore ends up pushing the deeper class $[V_\ell \sigma_3 U_\ell](x_U) \in M_{H_\ell}(V_\ell)$. The 'new' twists that we push these classes along are 1_{G_ℓ} and $\sigma_1^{-1}\sigma_2$. Notice that

$$\sigma_1^{-1} \sigma_2 = \begin{pmatrix} 1 & \ell^{-1} \\ 0 & 1 \end{pmatrix}.$$

Compare this matrix and the coefficients of $\mathfrak{H}_\ell \circ \iota_*(x_U)$ with the element 0.2.19 at a split prime.

In essence, the amalgamation of the action of several mixed Hecke operators that arise as summands in various closely related (ordinary) Hecke operators present in the aforementioned Hecke polynomials has the overall effect of 'balancing things out' i.e. the rearrangement required to amalgamate them creates and annihilates just the right amount of copies of x_U that allows one to write these copies as traces of elements from deeper levels in M_H. In other words, it is not so much the individual Hecke operators that we are, for the lack of better knowledge, forced to study at the moment but rather the manner in which their actions complement each other that ultimately yields norm relations. There is therefore some room for formulating general criteria that can help bypass the need for analyzing individual Hecke operators as described in the process above, especially in the case where one pushes cycle classes where proving norm relation can be reduced to verifying a certain set of congruences (3.2.5).

Let us now attempt to shed some light on this phenomenon in slighly more technical language. In the examples studied in this thesis where the target group is split, the congruences that we obtain are reminicent of the features of the 'Satake polynomial' (Definition 4.3.2). The Satake polynomial is a polynomial defined in terms of certain representation theoretic data. In the setting Chapter 0.1, the polynomial has the form

$$\begin{aligned}\mathfrak{S}_\ell(X) &= (1-\alpha X)(1-\beta X) \\ &= 1-(\alpha+\beta)X+\alpha\beta X^2\end{aligned}$$

where α, β are the so-called 'Satake parameters' of some unramified representation of $\mathrm{GL}_2(F)$. The formula for the degrees of spherical Hecke operators (see Proposition 5.2.8) together with a theorem of S. Kato & G. Lusztig (Theorem 4.4.2) that gives a q-analogue of multiplicities of weight spaces do indeed explain some of these congruences. Speaking somewhat informally, the congruences arise since the act of plugging $\ell = \alpha = \beta = X = 1$ 'matches up' the linear sum of Satake and Hecke polynomials. In our GL_2 example, this matching corresponds to the similarity of the expressions

$$\ell - (\ell+1) + 1 \qquad\qquad 1-2+1$$

where on the left hand side, we see the sum of degrees of Hecke operators seen in Example 1.1.1 and on the right hand side, we see the expansion of $(1-1)^2$. See for instance the proof of Theorem 10.3.1 where these congruences are more evident. We wish to however also emphasize that these observations alone cannot provide a complete explanation as there are several variables at play here besides the target group G that dictate this phenomenon. In fact, the choice of the smaller group H and various other relevant data, such as the torus chosen for parametrizing the tower of number fields, play a decisive role in whether such an endeavour actually bears any fruit (see Remark 9.5.4).

1.2 Axiomatics

This twisting and rearrangement of operators we described earlier is axiomatized in Definition 3.1.4 by means of a certain linear combination we refer to as a *zeta element* (for that place). The summands of the zeta element record the data of the aforementioned 'deeper' classes in the source cohomology together with the set of twisted embeddings of H in G which we refer to as *twists* of zeta. These are the twists mentioned in the previous section that one needs to push the deeper classes along in order to obtain norm relations in M_G. For Theorem 0.2.22, the zeta element was constructed in (0.2.19).

As formulated in Definition 3.1.4, zeta elements in general only provide norm relation up to torsion (which is what one ends up needing in practice anyway, see Remark 3.3.3), but which nevertheless can be upgraded to an equality under certain additional assumptions (Corollary 3.3.2).

One salient feature of the axioms of zeta elements is that they do not require any knowledge of the target cohomology M_G. As such, zeta elements as formulated here are certain 'universal norm relations', independent of the pushforward chosen from the source of these elements. They however do require the knowledge of what is being pushed. The supply of source classes that turn out to be useful for these purposes is at the moment restricted to special cycles and Eisenstein classes associated with Shimura varieties. Both types of classes behave as spaces of functions on suitable topological spaces often referred to as *Schwartz spaces*. This is to say that these classes satisfy 'distribution relations' analogous to the ones seen on function spaces. See Chapter 7 for a description of higher dimensional Eisenstein classes on Siegel modular varieties in terms of such spaces. The distribution properties of Schwartz spaces are studied quite generally in §3.4. Their explicit nature is one particular reason for using them for purposes of constructing Euler system or zeta elemetns.

Let us conclude this discussion noting that the zeta element machinery built here is essentially a means of transferring the norm relation problems posed for G to several ones for H, the motivation being that the latter are easier to handle than the former owing to explicit knowledge of the classes H acts upon. This perspective is subsumed in Proposition 3.2.1, its corollaries 3.2.2, 3.2.3 and perhaps most evidently in Proposition 3.3.1. We will invoke these results on several occasions in Part II of this thesis. See Note 3.1.2 for further motivation.

Remark 1.2.1. Let us highlight that one peculiar feature of Euler systems (which makes them useful after all) is the condition that the underlying classes dwell in a Galois stable lattice. In other words, the coefficients to be used in the construction of the web of relations in the discussion above need to be p-adic integers rather than p-adic fields. The requirement is imposed to make the resulting Galois cohomology classes amenable to reduction modulo p arguments and in particular to allow for Kolyvagin's 'derivative' machinery to be executed. As an independent testament for the need take into account the coefficients, we observe that the problem of constructing such a web becomes trivial when posed over a field.

1.3 Structure and layout

This thesis is divided into two parts. The first part may be considered as 'theory' where techniques relevant to horizontal norm relations (HNR) problems are recorded in generality. Part two records examples of norm relations problems that arise in the context of symplectic and unitary Shimura varities.

We briefly outline the content of each individual chapter. In Chapter 2, we recall and expand upon the formalism of functors on locally profinite groups developed in [GS21, §2]. The formalism provides a convenient framework to study HNR problems with focus on the underlying coefficients. Chapter 3 devoted to defining and studying the notion of zeta elements which is the central idea developed in this thesis. In Chapter 4, we review the definition of Hecke polynomials associated to certain representations of Langlands dual group. These are the central object of interest of this thesis in terms of which zeta elements are constructed. In Chapter 5, we derive a recipe for decomposing parahoric double cosets into their constituent left cosets. The main ideas of this chapter are due to [Lan01]. This recipe is the primary tool needed to execute the machinery developed in Chapter 3. Chapter 6 recalls basic properties of the cohomology of Shimura varieties with the primary purpose of establishing the applicability of the formalism of Chapter 2 to them. The final Chapter 7 of Part I reviews the theory of arithmetic Eisenstien classes as described in [Kin16] and [HK18]. The main result here is a characterization of the distribution relations of Eisenstein classes for Siegel modular varieties in terms of certain function spaces. This result is needed for studying zeta element problems for pushforwards in the cohomology of these varieties later on.

Part II of this thesis consists of five examples that have individual chapters devoted to them. In Chapter 8, we review the relevant background material that motivates the particular formulation of the zeta element problems studied in the following chapters. Chapter 9 revisits the setup of [GS21] and proves the corresponding horizontal norm relations at split primes by the methods developed in the first part of this thesis. Chapter 10 studies the exterior square L-factor of GL_4. This example is motivated by a closely related one of actual arithmetic interest but is slightly simpler to study as the degree of Hecke polynomial becomes 6 as opposed to 12. Our primary motivation for including it is to have another instance of an L-factor of type A that does not enjoy the structural properties of the standard L-factor of GL_n. Chapter 11 studies the inert case for $n = 2$ of the setup of [GS21]. This has not been previously studied. In Chapter 12, we revisit the setup of [LSZ22]. The main result can be seen as a strengthening of the HNR of *op. cit.* which are shown to hold at the level of zeta elements and in particular, before passage to Galois cohomology.

The final Chapter 13 of this thesis studies the spin L-factor of GSp_6. The HNR problems in this context are completely unexplored in literature. This is also the most technically demanding chapter of this thesis and almost everything developed in this thesis is made use of. Some key features of this computation is the use of Kazhdan-Lusztig theory in forgoing the computation of actual coefficients of Hecke polynomials and the use GSp_4-Eisenstein classes as source of zeta elements. Along the way, several steps are needed in carrying out individual computations and attention needs to be paid non-hyperspecial compact open subgroups of GSp_4. The final result is however strikingly simple and suggests alternative proofs might be possible.

Part I

Generalities

Chapter 2

RIC functors

In this chapter, we recall and expand on the abstract formalism of functors on compact open subgroups of locally profinite groups as introduced in [GS21, §2][1] which we will use to study norm relations problems encountered in the settings of Shimura varieties. Note however that a few conditions of *loc. cit.* have been relaxed for generality while others have been dropped as they do not pertain to the questions addressed in this thesis. Note also that the terminology in some places has been modified to match what seems to be the standard in pre-existing literature on such functors e.g. [Thi11], [BB04], [TW95], [GM92], [Dre73]. The material in this section however builds the theory in a different direction and is developed completely independently[2] of the aforementioned sources.

2.1 Basic properties

For G a locally profinite group, let $\Upsilon = \Upsilon_G$ be a non-empty collection of compact open subgroups of G satisfying the following conditions

(T1) For all $g \in G$, $K \in \Upsilon$, $gKg^{-1} \in \Upsilon$.

(T2) For all $K, L \in \Upsilon$, there exists a $K' \in \Upsilon$ such that $K' \triangleleft K$, $K' \subset L$.

(T3) For all $K, L \in \Upsilon$, we have $K \cap L \in \Upsilon$.

We refer to elements of Υ as *levels* of G. To such a collection, we associate a *category of compact opens* $\mathcal{P}(G) = \mathcal{P}(G, \Upsilon)$ whose objects are elements of Υ and whose morphisms are given by $\mathrm{Hom}_{\mathcal{P}(G)}(L, K) =$

[1] which was in turn inspired by [Loe19].

[2] and in greater generality: most of the sources cited are only concerned with finite groups

$\{g \in G \mid g^{-1}Lg \subset K\}$ for $L, K \in \Upsilon$. Composition is given by

$$(L \xrightarrow{g} K) \circ (L' \xrightarrow{h} L) = (L' \xrightarrow{h} L \xrightarrow{g} K) = (L' \xrightarrow{hg} K).$$

A morphism $(L \xrightarrow{g} K)$ will be denoted by $[g]_{L,K}$, and if e denotes the identity of G, the inclusion $(L \xrightarrow{e} K)$ will also denoted by $\mathrm{pr}_{L,K}$. Throughout, R denotes a commutative ring with identity.

Definition 2.1.1. A *RIC functor* M on (G, Υ) valued in R-Mod is a pair of covariant functors

$$M^* : \mathcal{P}(G)^{\mathrm{op}} \to R\text{-Mod} \qquad M_* : \mathcal{P}(G) \to R\text{-Mod}$$

satisfying the following three conditions:

(C1) $M^*(K) = M_*(K)$ for all $K \in \Upsilon$. We will denote the common R-module by $M(K)$.

(C2) For all $L, K \in \Upsilon$ such that $g^{-1}Lg = K$,

$$(L \xrightarrow{g} K)^* = (K \xrightarrow{g^{-1}} L)_* \in \mathrm{Hom}(M(K), M(L)).$$

Here, for $\phi \in \mathcal{P}(G)$ a morphism, we denote $\phi_* := M_*(\phi)$, $\phi^* := M^*(\phi)$.

(C3) $[\gamma]_{K,K,*} = \mathrm{id}$ for all $K \in \Upsilon$, $\gamma \in K$.

We refer to the maps ϕ^* (resp. ϕ_*) in (C2) above as the *pullbacks* (resp. *pushforwards*) induced by ϕ. If moreover $\phi = \mathrm{id}$, we also refer to $\phi^* = \mathrm{pr}^*$ (resp. $\phi_* = \mathrm{pr}_*$) as *restrictions* (resp. *inductions*). We call M $\mathbb{Z}$*-torsion free* if $M(K)$ is $\mathbb{Z}$-torsion free for all $K \in \Upsilon$. We say that M is

(G) *Galois* if for all $L, K \in \Upsilon$, $L \triangleleft K$,

$$\mathrm{pr}^*_{L,K} : M(K) \xrightarrow{\sim} M(L)^{K/L}.$$

Here the (left) action $K/L \times M(L) \to M(L)$ is given by $(\gamma, x) \mapsto [\gamma]^*_{L,L}(x)$.

(Co) *cohomological* if for all $L, K \in \Upsilon$ with $L \subset K$,

$$(L \xrightarrow{e} K)_* \circ (L \xrightarrow{e} K)^* = [K : L] \cdot (K \xrightarrow{e} K)^*$$

i.e. the composition is multiplication by index $[K : L]$ on $M(K)$.

(M) *Mackey* if for all $K, L, L' \in \Upsilon$ with $L, L' \subset K$, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_\gamma M(L_\gamma) & \xrightarrow{\sum \mathrm{pr}_*} & M(L) \\ {\scriptstyle \oplus[\gamma]^*}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}^*} \\ M(L') & \xrightarrow[\mathrm{pr}_*]{} & M(K) \end{array} \tag{2.1.2}$$

where the direct sum in the top left corner is over a fixed choice of coset representatives $\gamma \in K$ of the double quotient $L\backslash K/L'$ and $L_\gamma = L \cap \gamma L' \gamma^{-1} \in \Upsilon$. The condition is then satisfied by any such choice of representatives of $L\backslash K/L'$.

If M satisfies both (M) and (Co), we will say that M is *CoMack*. If S is a R-algebra, the mapping $K \mapsto M(K) \otimes_R S$ is a RIC functor, which is cohomological or Mackey if M is such.

In what follows, we will often say that $M : G \to R$-Mod is a functor when we mean to say that M is a RIC functor on (G, Υ) and suppress Υ if it is clear from context.

Remark 2.1.3. The letters RIC stand for restriction, induction & conjugation and the terminology is borrowed from [Thi11]. Cf. the terminology of [NSW08, Definition 1.5.10].

Definition 2.1.4. A *morphism* $\varphi : N \to M$ of RIC functors is a pair of natural transformations $\varphi^* : N^* \to M^*, \varphi_* : N_* \to M_*$ such that $\varphi_*(K) = \varphi^*(K)$ for all $K \in \Upsilon$. We denote this common morphism by $\varphi(K)$. The category of R-Mod valued RIC functors on (G, Υ) is denoted $\mathrm{RIC}_R(G, \Upsilon)$ and the category of CoMack functors by $\mathrm{CoMack}_R(G, \Upsilon)$.

We record some straightforward implications. Let $M : \mathcal{P}(G, \Upsilon) \to R$-Mod be a RIC functor.

Lemma 2.1.5. *The functor M is Mackey if and if only if for all $K, L, L' \in \Upsilon$ with $L, L' \subset K$, we have a commutative diagram*

$$\begin{array}{ccc} \bigoplus_\delta M(L'_\delta) & \xrightarrow{\sum [\delta]_*} & M(L) \\ {\scriptstyle \bigoplus \mathrm{pr}^*} \uparrow & & \uparrow {\scriptstyle \mathrm{pr}^*} \\ M(L') & \xrightarrow{\mathrm{pr}_*} & M(K) \end{array} \tag{2.1.6}$$

where the direct sum in the top left corner is over a fixed choice of coset representatives $\delta \in K$ of $L'\backslash K/L$ and $L'_\delta = L' \cap \delta L \delta^{-1} \in \Upsilon$.

Proof. If $\gamma \in K$ runs over representatives of $L\backslash K/L'$, then $\delta = \gamma^{-1}$ runs over a set of representatives for $L'\backslash K/L$. For each $\gamma \in K$, we have a commutative diagram

$$\begin{array}{ccccc} & & M(L'_\delta) & \xrightarrow{[\delta]_*} & M(L) \\ & \overset{\mathrm{pr}_*}{\nearrow} & & \nearrow{\scriptstyle \mathrm{pr}_*} & \\ M(L') & \xrightarrow[{[\gamma]^*}]{} & M(L_\gamma) & & \end{array}$$

where $\delta = \gamma^{-1}$. Indeed, the two triangles obtained by sticking the arrow $M(L_\gamma) \xrightarrow{[\gamma]_*} M(L'_\delta)$ in the diagram above are commutative. From this, it is straightforward to see that diagram (2.1.2) commutes if and only if diagram (2.1.6) does. ∎

Remark 2.1.7. We will refer to the commutativity of the diagram (2.1.6) as axiom (M′).

Definition 2.1.8. We say that M has *injective restrictions* if $\mathrm{pr}^*_{L,K} : M(K) \to M(L)$ are injective for all $L, K \in \Upsilon$, $L \subset K$.

Lemma 2.1.9. *Suppose M is either Galois or cohomological and $\mathbb{Z}$-torsion free. Then M has injective restrictions.*

Proof. Let $L, K \in \Upsilon$ with $L \subset K$. If M is Galois, pick K' such that $K' \triangleleft K$, $K' \subset L$ (using axiom (T2)). Then $\mathrm{pr}^*_{K',L} \circ \mathrm{pr}^\circ_{L,K} : M(K) \to M(K')$ is injective by definition which implies the same for $\mathrm{pr}_{L,K}$. If M is cohomological, then $\mathrm{pr}_{L,K,*} \circ \mathrm{pr}^*_{L,K} = [K : L]$ which is injective if $M(K)$ is $\mathbb{Z}$-torsion free. □

Lemma 2.1.10. *Suppose M is cohomological. If $[K : L] \in R^\times$, then $\mathrm{pr}^*_{L,K}$ is injective and $\mathrm{pr}_{L,K,*}$ is surjective.*

Lemma 2.1.11. *Suppose M is Mackey. Let $L, K \in \Upsilon$ with $L \subset K$ and let $K' \in \Upsilon$ be such that $K' \triangleleft K$, $K' \subset L$[3]. Then $\mathrm{pr}^*_{K',K} \circ \mathrm{pr}_{L,K,*} = \sum_{\gamma \in K/L} [\gamma]^*_{K',L}$.*

Proof. Since $K' \triangleleft K$, the right multiplication action of K' on $L\backslash K$ is trivial i.e. $L\backslash K/K' = L\backslash K$. By axiom (M′) obtained in Lemma 2.1.5, we see that

$$\begin{array}{ccc} \bigoplus_\delta M(K') & \xrightarrow{\sum[\delta]_*} & M(K') \\ {\scriptstyle \bigoplus \mathrm{pr}^*}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}^*} \\ M(L) & \xrightarrow{\mathrm{pr}_*} & M(K) \end{array}$$

where δ runs over $L\backslash K$. Since $[\delta]_{K',K',*} = [\delta^{-1}]^*_{K',K'}$, we may replace δ with $\delta = \gamma^{-1}$. Then $\gamma \in K$ runs over K/L as δ runs over $L\backslash K$ and the claim follows. □

Corollary 2.1.12. *Suppose R is a $\mathbb{Q}$-algebra. Then M is Galois if it is CoMack.*

Proof. Let $L, K \in \Upsilon$ with $L \triangleleft K$. That $\mathrm{pr}^*_{L,K}$ is injective follows by Lemma 2.1.10. Say $y \in M(L)$ is invariant under the $[\gamma]^*_{L,L}$ for all $\gamma \in K/L$. By Lemma 2.1.11, $\mathrm{pr}^*_{L,K} \circ \mathrm{pr}_{L,K,*}(y) = \sum_{K/L} [\gamma]^*_{L,L}(y) = [K : L] \cdot y$. Therefore, $x := \frac{1}{[K:L]} \cdot \mathrm{pr}_{L,K,*}(y) \in M(K)$ is such that $\mathrm{pr}^*_{L,K}(x) = y$, and so $\mathrm{pr}^*_{L,K} : M(K) \to M(L)^{K/L}$ is surjective as well. □

2.2 Inductive completions

The category $\mathrm{CoMack}_R(G, \Upsilon)$ is closely related to the category of smooth representations. We show that when R is a field, there is an equivalence between the two. However, RIC functors are better suited for studying arithmetic questions such as horizontal norm relations since the axiom (G) does not necessarily hold for functors of interest when the coefficient ring R is not a field.

[3] such a K' always exists by axiom (T2)

Definition 2.2.1. Let π be a left module over $R[G]$. We say that π is a *smooth representation* of G if for any $x \in \pi$, there is a compact open $K \subset G$ such that x is fixed under the (left) action of K. A *morphism* of smooth representations is a R-linear map respecting the G-actions. The category of smooth representations of G is denoted $\mathrm{SmthRep}_R(G)$.

Suppose $\pi \in \mathrm{SmthRep}_R(G)$. Let Υ be the collection of all compact open subgroups of G. For $K \in \Upsilon$, let $M_\pi : \Upsilon \to R\text{-Mod}$ be the mapping given by $K \mapsto \pi^K$. For $g \in G$ and $(L \xrightarrow{g} K) \in \mathcal{P}(G, \Upsilon)$, let

$$[g]^*_{L,K} : M(K) \to M(L) \qquad\qquad [g]_{L,K,*} : M(L) \to M(K)$$
$$x \mapsto g \cdot x \qquad\qquad x \mapsto \sum_{\gamma \in K/g^{-1}Lg} \gamma g^{-1} \cdot x$$

Here, $g \cdot x \in \pi$ on the left is considered as an element of $M(L)$ as it is invariant under $L \subset gKg^{-1}$ and similar remarks apply to the expression on the right above. In particular, the map $[1]^*_{L,K} : M(K) \to M(L)$ is the inclusion $\pi^K \hookrightarrow \pi^L$.

Lemma 2.2.2. *The mapping M_π is a RIC functor that is CoMack and Galois.*

Proof. This is straightforward to establish as all modules are contained in π. □

Definition 2.2.3. We refer to M_π as the *RIC functor associated to π*. If $\pi = R$ is the trivial representation, we denote the associated functor by M_{triv} and refer to it as the *trivial functor*.

Definition 2.2.4. Let $M : G \to R\text{-Mod}$ be a functor. The *inductive completion* $\widehat{M}$ is defined to be the limit $\varinjlim_{K \in \Upsilon} M(K)$ with respect to the pullback maps induced by inclusion. We let $j_K : M(K) \to \widehat{M}$ denote the natural map.

There is an induced smooth action $G \times \widehat{M} \to \widehat{M}, (g, x) \mapsto g \cdot x$ where $g \cdot x$ is defined as follows. Let $K \in \Upsilon_G$, $x_K \in M(K)$ be such $j_K(x_K) = x$. Then $g \cdot x$ is defined to be the image of x_K under the composition $M(K) \xrightarrow{[g]^*} M(gKg^{-1}) \to \widehat{M}$. It is a routine check that is well-defined and that the action so-defined is smooth, as the image of $j_K : M(K) \to \widehat{M}(K)$ is contained in the K-invariants $\widehat{M}^K$. If M is also Galois, j_K identifies $M(K)$ with $\widehat{M}^K$. Moreover, if $\varphi : M \to N$ is a morphism of functors, the induced map $\hat{\varphi} : \hat{M} \to \hat{N}$ respects the G-actions.

Lemma 2.2.5. *Suppose M is cohomological. Then $\ker(j_K)$ is contained in $M(K)_{\mathbb{Z}\text{-tors}}$. In particular, if R is a field of characteristic zero, j_K is injective.*

Proof. Let $x \in \ker(j_K)$. By definition, there exists $L \in \Upsilon$, $L \subset K$ such that $\mathrm{pr}^*_{L,K}(x) = 0$. Since $\mathrm{pr}_{L,K,*} \circ \mathrm{pr}^*_{L,K} = [K : L]$, we must have $[K : L] \cdot x = 0$. □

Proposition 2.2.6. *Let R be a $\mathbb{Q}$-algebra. Then the functor* $\mathrm{SmthRep}_R(G) \to \mathrm{CoMack}_R(G)$ *given by* $\pi \mapsto M_\pi$ *induces an equivalence of categories with (quasi) inverse given by* $M \mapsto \widehat{M}$.

Proof. By Lemma 2.1.12, any CoMack functor valued in a $\mathbb{Q}$-algebra is Galois and therefore one can recover a functor M from the representation $\widehat{M}$. Similarly, $\varinjlim_{K\subset G} \pi^K = \pi$ by smoothness of π. □

2.3 Hecke operators

For background material on Haar measures and further reading, the reader may consult [Vig89, Ch. 1, §3] or [BH06, Ch. 1]. Let (G, Υ) be as in §2.1.

Definition 2.3.1. Let μ be a left invariant Haar measure on G valued in R[4] and let $K \in \Upsilon$. The *Hecke algebra* $\mathcal{H}_R(K\backslash G/K)$ *of level* K is defined to be the convolution algebra locally constant K-bi-invariant functions valued in R. The convolution product is denoted by $*$. The *Hecke algebra of* G over Υ is defined to be $\mathcal{H}_R(G) = \mathcal{H}_R(G,\Upsilon) = \bigcup_{K\in\Upsilon} \mathcal{H}_R(K\backslash G/K)$. The *transposition* on $\mathcal{H}_R(G,\Upsilon)$ is the mapping $\xi \mapsto \xi^t = (g \mapsto \xi(g^{-1}))$, $\xi \in \mathcal{H}_R(G)$.

The convolution $\xi_1 * \xi_2$ where $\xi_1, \xi_2 \in \mathcal{H}_R(G,\Upsilon)$ is given by $(\xi_1 * \xi_2)(g) = \int_{x\in G} \xi_1(x)\xi_2(x^{-1}g)d\mu(g)$. If G is unimodular, then one also has $(\xi_1 * \xi_2)(g) = \int_G \xi_1(gy^{-1})\xi_2(y)d\mu(y)$ obtained by replacing x with gy^{-1}. The transposition map is an anti-involution of $\mathcal{H}_R(G)$ i.e. $(\xi_1 * \xi_2)^t = \xi_2^t * \xi_1^t$ for all $\xi_1, \xi_2 \in \mathcal{H}_R(G)$ and stabilizes $\mathcal{H}_R(K\backslash G/K)$ for any $K \in \Upsilon$.

The Hecke algebra $\mathcal{H}_R(K\backslash G/K)$ has a R-basis given by the characteristic functions of double cosets $K\sigma K$ for $\sigma \in K\backslash G/K$ denoted $\mathrm{ch}(K\sigma K)$ and referred to as *Hecke operators*. The *degree* of $\mathrm{ch}(K\sigma K)$ is defined to be $|K\sigma K/K|$ or equivalently, the index $[K : K\cap \sigma K\sigma^{-1}]$. The product $\mathrm{ch}(K\sigma K) * \mathrm{ch}(K\tau K)$ may be described explicitly as follows: if $K\sigma K = \bigsqcup_i \alpha_i K$, $K\tau K = \bigsqcup_j \beta_j K$, then

$$\mathrm{ch}(K\sigma K) * \mathrm{ch}(K\tau K) = \mu(K) \cdot \sum_i \mathrm{ch}(\alpha_i K\tau K) = \mu(K) \cdot \sum_{i,j} \mathrm{ch}(\alpha_i\beta_j K). \tag{2.3.2}$$

The convolution may also be written $\sum_{\upsilon\in\sigma K\tau} c^{\upsilon}_{\sigma,\tau}\mathrm{ch}(K\upsilon K)$ with $c^{\upsilon}_{\sigma,\tau} = |(K\sigma K \cap \upsilon K\tau^{-1}K)/K|$. If $\mu(K) = 1$, then $\mathcal{H}_R(K\backslash G/K)$ is unital and the mapping $\mathcal{H}_R(K\backslash G/K) \to R$ given by $\mathrm{ch}(K\sigma K) \mapsto |K\sigma K/K|$ is a homormorphism of rings.

Any smooth left representation $\pi \in \mathrm{SmthRep}_R(G)$ inherits a left action of the Hecke algebra $\mathcal{H}_R(G,\Upsilon)$. The action of $\mathrm{ch}(K'\sigma K) \in \mathcal{H}_R(G,\Upsilon)$ on an element $x \in \pi$ invariant under K is given by $\mathrm{ch}(K'\sigma K) \cdot x = \mu(K) \sum_{\alpha\in K'\sigma K/K} \alpha \cdot x$. Similary, if $K \in \Upsilon$, the set π^K is stable under the action of $\mathcal{H}_R(K\backslash G/K)$ and is

[4] e.g. if μ is $\mathbb{Q}$-valued and R is a $\mathbb{Q}$-algebra

therefore a module over it. In particular, if M is a RIC functor, then $\widehat{M}$ is a module over $\mathcal{H}_R(G, \Upsilon)$ and if M is Galois, $M(K) = \widehat{M}^K$ is naturally a module over $\mathcal{H}_R(K\backslash G/K)$.

We note that $\mathcal{H}_R(G, \Upsilon)$ is a smooth left representation of G under both right and left translation actions. It is therefore a module over under itself in two ways. Let

$$\begin{aligned} \lambda : G \times \mathcal{H}_R(G, \Upsilon) &\to \mathcal{H}_R(G, \Upsilon) \qquad & \rho : G \times \mathcal{H}_R(G, \Upsilon) &\to \mathcal{H}_R(G, \Upsilon) \\ (g, \xi) &\mapsto \xi(g^{-1}(-)) & (g, \xi) &\mapsto \xi((-)g) \end{aligned}$$

When $\mathcal{H}_R(G, \Upsilon)$ is considered as a G-representation under λ, the induced action of $\mathcal{H}_R(G, \Upsilon)$ on itself is that of the convolution product $*$. When $\mathcal{H}_R(G, \Upsilon)$ is considered as a G-representation under ρ, the induced action of $\mathcal{H}_R(G, \Upsilon)$ will be denoted by $*_\rho$. There is a relation between $*$ and $*_\rho$ that is useful to record.

Lemma 2.3.3. *For $\xi_1, \xi_2 \in \mathcal{H}_R(G, \Upsilon)$, $\xi_1 *_\rho \xi_2 = \xi_2 * \xi_1^t$.*

Proof. By definition, we have for all $g \in G$

$$\begin{aligned} (\xi_1 *_\rho \xi_2)\,(g) &= \int_G \xi_1(x)\,\xi_2(gx)\,d\mu(x) \\ &= \int_G \xi_2(y)\,\xi_1(g^{-1}y)\,d\mu(y) = \int_G \xi_2(y)\,\xi_1^t(y^{-1}g)\,d\mu(y) = \left(\xi_2 * \xi_1^t\right)(g) \end{aligned}$$

where in the second equality, we used the change of variables $x = g^{-1}y$. □

2.4 Hecke correspondences

On RIC functors, one may abstractly define correspondences in the same manner as one does for the cohomology of Shimura varieties.

Definition 2.4.1. Let $M\colon G \to R$-Mod be a functor. For every $K, K' \in \Upsilon$ and $\sigma \in G$, the *Hecke correspondence* $[K'\sigma K]$ is defined to be the composition

$$[K'\sigma K]\colon M(K) \xrightarrow{\mathrm{pr}^*} M(K \cap \sigma^{-1}K'\sigma) \xrightarrow{[\sigma]^*} M(\sigma K \sigma^{-1} \cap K') \xrightarrow{\mathrm{pr}_*} M(K').$$

If $\mathcal{C}_R(K'\backslash G/K)$ denotes the free R-module on functions $\mathrm{ch}(K'\sigma K)$, $\sigma \in K'\backslash G/K$, there is a R-linear mapping $\mathcal{C}_R(K'\backslash G/K) \to \mathrm{Hom}_R(M(K), M(K'))$ given by $\mathrm{ch}(K'\sigma K) \mapsto [K'\sigma K]$. The *transpose* of $[K'\sigma K]$ is defined to be the correspondence $[K\sigma^{-1}K'] : M(K') \to M(K)$. The *degree* of $[K'\sigma K]$ is defined to be the cardinality of $K'\sigma K/K$ or equivalently, the index $[K' : K' \cap \sigma K \sigma^{-1}]$.

Lemma 2.4.2. *Let $M : G \to R$-Mod be a Mackey functor and μ be a Haar measure on G, $K, K' \in \Upsilon$ such that $\mu(K) = 1$. Then for any $\tau \in G$, the actions of $[K'\tau K]$ and $\mathrm{ch}(K'\tau K)$ on $\widehat{M}$ agree i.e. if $x \in M_G(K)$, then*

$$j_K \circ [K'\tau K](x) = \mathrm{ch}(K'\tau K) \cdot x.$$

In particular, if M is Galois, the R-linear mapping $\mathcal{H}_R(K\backslash G/K) \to \mathrm{End}_R M(K)$, $\mathrm{ch}(K\tau K) \mapsto [K\tau K]$ is a R-algebra homomorphism.

Proof. Let $L' := \tau K\tau^{-1} \cap K' \in \Upsilon$. By axiom (T2), there exists $L \in \Upsilon$ satisfying $L \triangleleft K'$ and $L \subset L'$. Then for any $\gamma \in K'$, set $L_\gamma = \gamma L' \gamma^{-1} \cap L = L$ and $L\backslash K'/L' = K'/L'$. Since M is Mackey, we have a commutative diagram

$$\begin{array}{ccccc} & & \bigoplus_\gamma M(L) & \xrightarrow{\sum \mathrm{pr}_*} & M(L) \\ & & {\scriptstyle \sum[\gamma]^*}\uparrow & & \uparrow{\scriptstyle \mathrm{pr}^*} \\ M(K) & \xrightarrow{[\tau]^*} & M(L') & \xrightarrow{\mathrm{pr}_*} & M(K') \\ & & \underbrace{\qquad\qquad\qquad\qquad}_{M(K)\xrightarrow{[K'\tau K]} M(K')} & & \end{array}$$

where $\gamma \in K'$ in the top left corner runs over a set of representatives of K'/L'. From the diagram, we see that $\mathrm{pr}^*_{L,K'} \circ [K'\tau K] = \sum_\gamma [\gamma\tau]^*$. As $K'/L' \to K'\tau K/K$ given by $\gamma L' \mapsto \gamma\tau K$ is a bijection, we obtain the first claim.

For the second claim, let $[K''\sigma K'] : M(K') \to M(K'')$ be another Hecke correspondence. Let $J \in \Upsilon$ be such that $J \triangleleft K''$ and $J \subset \tau L \tau^{-1}$. Say $K'\tau K/K = \bigsqcup \alpha K$ and $K''\sigma K'/K' = \bigsqcup \beta K'$. Then the composition $\mathrm{pr}_{K'',J} \circ [K''\sigma K'] \circ [K'\tau K]$ is given by

$$M(K) \xrightarrow{\sum[\alpha]^*} M(L) \xrightarrow{\sum[\beta]^*} M(J)$$

The second claim then follows by (2.3.2). $\square$

Lemma 2.4.3. *Suppose that $G = G_1 \times G_2$, $\sigma_i \in G_i$, $K_i \subset G_i$ with $K = K_1 \times K_2 \in \Upsilon_G$, and $M : G \to R$-Mod be a Mackey functor. Denote $\tau_1 = (\sigma_1, 1)$, $\tau_2 = (1, \sigma_2)$. Then*

$$[K\tau_1 K] \circ [K\tau_2 K] = [K\tau_2 K] \circ [K\tau_1 K]$$

as endomorphisms of $M(K)$. We denote the common endomorphism as $[K_1\sigma_1 K_1] \otimes [K_2\sigma_2 K_2]$.

Proof. Let $L_i = K_i \cap \sigma_i K_i \tau_i^{-1}$ for $i = 1, 2$. Then $L_1 K_1 = K \cap \tau_1 K \tau_1^{-1}, K_1 L_2 = K \cap \tau_2 K \tau_2^{-1}$, $L := L_1 L_2 = K \cap \tau_1\tau_2 K(\tau_1\tau_2)^{-1}$ all belong to Υ. Since $L_2 K_1 \backslash K / L_1 K_2 = \{1_K\}$ and M is Mackey, we get a commutative diagram

$$\begin{array}{ccccccccc} & & & & M(L) & & & & \\ & & & {\scriptstyle [\tau_2]^*}\nearrow & & \searrow{\scriptstyle \mathrm{pr}_*} & & & \\ & & M(L_1K_2) & & & & M(L_2K_1) & & \\ & {\scriptstyle [\tau_1]^*}\nearrow & & \searrow{\scriptstyle \mathrm{pr}_*} & & {\scriptstyle [\tau_2]^*}\nearrow & & \searrow{\scriptstyle \mathrm{pr}_*} & \\ M(K) & & \xrightarrow{[K\tau_1K]} & & M(K) & & \xrightarrow{[K\tau_2K]} & & M(K) \end{array}$$

which implies that $[K\tau_2K] \circ [K\tau_1K] = [\tau_2\tau_1]^*_{L,K} \circ \mathrm{pr}_{L,K,*}$ where $L = L_1L_2$. Since right hand side of this equality is symmetric in τ_1 and τ_2 (as $\tau_1\tau_2 = (\sigma_1, \sigma_2) = \tau_2\tau_1$), the claim follows by internchangin the roles of τ_1, τ_2. □

2.5 Pushforwards and mixed Hecke correspondences

In the situations that we are going to consider, the classes used for establishing HNR are pushforwarded from a functor which has an action of a smaller group. In this section, we study this scenario abstractly. Let $\iota : H \hookrightarrow G$ be a closed subgroup, and Υ_H, Υ_G be a collection of compact opens satisfying (T1)-(T3) and such that $K \cap H \in \Upsilon_H$ whenever $K \in \Upsilon_G$.

Definition 2.5.1. We say that $(U, K) \in \Upsilon_H \times \Upsilon_G$ forms a *compatible pair* if $U \subset K$. A *morphism of compatible pairs* $h : (V, L) \to (U, K)$ is a pair of morphisms $(V \xrightarrow{h} U)$, $(L \xrightarrow{h} K)$ for some $h \in H$. Let M_H, M_G be R-valued functors on H, G respectively. A *pushforward* $M_H \to M_G$ is a family of morphisms $\iota_{U,K,*} : M_H(U) \to M_G(K)$ for all compatible pairs $(U, K) \in \Upsilon_H \times \Upsilon_G$ such that $\iota_{U,K,*}$, $\iota_{V,L,*}$ commute with the pushforward $[h]_*$ induced by any morphism $h : (V, L) \to (U, K)$ of compatible pairs. We say that ι_* is *Mackey* if for all $U \in \Upsilon_H$, $L, K \in \Upsilon_G$, $U, L \subset K$, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_\gamma M_H(U_\gamma) & \xrightarrow{\sum[\gamma]_*} & M_G(L) \\ \oplus\,\mathrm{pr}^* \big\uparrow & & \big\uparrow \mathrm{pr}^* \\ M_H(U) & \xrightarrow{\iota_*} & M_G(K) \end{array}$$

where $\gamma \in U\backslash K/L$ is a fixed set of representatives, $U_\gamma = U \cap \gamma L\gamma^{-1}$ and $[\gamma]_* : M_H(U_\gamma) \to M_G(L)$ denotes the composition $M_H(U_\gamma) \xrightarrow{\iota_*} M_G(\gamma L\gamma^{-1}) \xrightarrow{[\gamma]_*} M_G(L)$.

If $\varphi_G : N_G \to M_G$ is a morphism of functors, then it may be viewed as a pushforward in the sense of Definition 2.5.1. We will say that φ is *Mackey* if it is so as a pushforward.

Lemma 2.5.2. *If M_G is Mackey, then so is any morphism $\varphi_G : N_G \to M_G$.*

Proof. If M is Mackey, then M satisfies the axiom (M′) given in Lemma 2.1.5. Using its notation, the commutativity of 2.1.6 implies that

$$\begin{array}{ccc} \bigoplus_\delta N_G(L'_\delta) & \xrightarrow{\varphi_G} & \bigoplus_\delta M_G(L'_\delta) \\ \oplus\,\mathrm{pr}^* \big\uparrow & & \big\uparrow \oplus\,\mathrm{pr}^* \\ N_G(L') & \xrightarrow[\varphi_G]{} & M_G(L') \end{array}$$

is commutative as well. □

Definition 2.5.3. Let $\iota_* : M_H \to M_G$ be a pushforward. For $U \in \Upsilon_H$, $K \in \Upsilon_G$ and $\sigma \in G$ the *mixed Hecke correspondence* $[U\sigma K]_*$ is defined as

$$[U\sigma K]_* : M_H(U) \xrightarrow{\mathrm{pr}^*} M_H(U \cap \sigma K \sigma^{-1}) \xrightarrow{\iota_*} M_H(\sigma K \sigma^{-1}) \xrightarrow{[\sigma]_*} M_G(K).$$

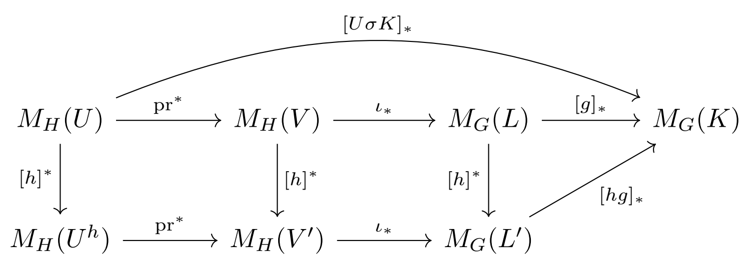
One can verify that $[U\sigma K]_*$ depends only on the double coset $U\sigma K$. The *degree* of $[U\sigma K]_*$ is defined to be the index $[H \cap \sigma K \sigma^{-1} : U \cap \sigma K \sigma^{-1}]$.

Remark 2.5.4. Suppose that $H = G$, $\iota = \mathrm{id}$ and ι_* is the identity pushforward on M_G. Then one can verify that ι_* is Mackey iff M_G is. Moreover, if $U, K \in \Upsilon_G$ and $\sigma \in G$, $[U\sigma K]_* = [U\sigma K]^t = [K\sigma^{-1}U]$ and the degrees of $[U\sigma K]_*$, $[K\sigma^{-1}U]$ agree. The '$*$' in the notation of mixed Hecke correspondence is meant to emphasize its 'pushforward nature' and its dependence on ι_*. We note that $\deg [U\sigma K]_*$ is however independent of ι_*.

Lemma 2.5.5. *Let $\iota_* : M_H \to M_G$ be a pushforward, $U \in \Upsilon_H$, $K \in \Upsilon_G$ and $\sigma \in G$. For $h \in H$, $g \in G$, denote $U^h = hUh^{-1}$, $K^g = gKg^{-1}$. Then*

$$[U\sigma K]_* = [U^h(h\sigma)K] \circ [h]^*_{U,U^h} = [g]_{K^g,K,*} \circ [U(\sigma g^{-1})K^g]_*.$$

Proof. Let $V = U \cap \sigma K \sigma^{-1}$, $V' = U^h \cap h\sigma K(h\sigma)^{-1}$, $L = \sigma K \sigma^{-1}$, $L' = h\sigma K \sigma^{-1} h^{-1}$. By definition, $hVh^{-1} = V'$, $hLh^{-1} = L'$, $V \subset L$ and $V' \subset L'$. One then easily verifies that the diagram



is commutative which implies the first equality. The proof for the other equality is similar. □

Lemma 2.5.6. *Let $\iota_* : M_H \to M_G$ be a Mackey pushforward, $\sigma \in G$, $U \in \Upsilon_H$, $K, K' \in \Upsilon_G$ with $U \subset K$. Suppose that $K\sigma K' = \bigsqcup_i U\sigma_i K'$. Then $[K\sigma K']_* \circ \iota_{U,K,*} = \sum_i [U\sigma_i K']_*$.*

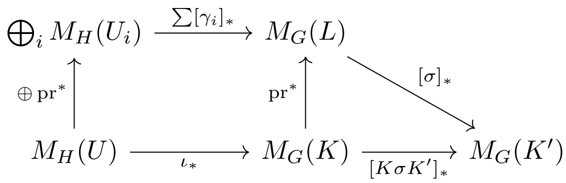
Proof. Let $L := K \cap \sigma K' \sigma^{-1}$. As $K/L \to K\sigma K'/K'$, $\gamma L \mapsto \gamma\sigma K'$ is a bijection, we may assume that $\sigma_i = \gamma_i \sigma$ where $\gamma_i \in K$ forms a set of representatives of $U\backslash K/L$. Let

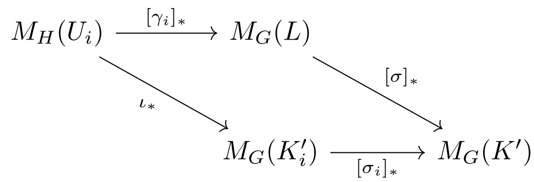
$$K'_i := \sigma_i K' \sigma_i^{-1}, \quad L_i := \gamma_i L \gamma_i^{-1}, \quad U_i := U \cap K'_i.$$

Then $L_i = K \cap \sigma_i K' \sigma_i^{-1} = K \cap K'_i$ and therefore $U_i = U \cap L_i$. As ι_* is Mackey, we see that

$$\mathrm{pr}^*_{L,K} \circ \iota_{U,K,*} = \sum_i [\gamma_i]_{U_i,L,*} \circ \mathrm{pr}^*_{U_i,U}$$

where $[\gamma_i]_{U_i,L,*} := [\gamma_i]_{L_i,L,*} \circ \iota_{U_i,L_i,*}$ (diagram on the left below).





For each fixed i, we have

$$\begin{aligned}\iota_{U_i,K'_i,*} = \mathrm{pr}_{L_i,K'_i,*} \circ \iota_{U_i,L_i,*} &= \left([\gamma_i^{-1}]_{L,K'_i,*} \circ [\gamma_i]_{L_i,L,*}\right) \circ \iota_{U_i,L_i,*} \\ &= [\gamma_i^{-1}]_{L,K'_i,*} \circ [\gamma]_{U_i,L,*}\end{aligned}$$

by definition of $[\gamma_i]_{U_i,L,*}$. Therefore,

$$[\sigma]_{L,K',*} \circ [\gamma_i]_{U_i,L,*} = \left([\sigma_i]_{K'_i,K',*} \circ [\gamma_i^{-1}]_{L,K'_i,*}\right) \circ [\gamma_i]_{U_i,L,*} = [\sigma_i]_{K'_i,K'} \circ \iota_{U_i,K'_i,*}$$

(diagram on the right above). Since $[K\sigma K']_* = [\sigma]_{L,K',*} \circ \mathrm{pr}^*_{L,K}$ by definition, we have

$$\begin{aligned}[K\sigma K']_* \circ \iota_{U,K,*} &= [\sigma]_{L,K',*} \circ \sum_i \left([\gamma_i]_{U_i,L,*} \circ \mathrm{pr}^*_{U_i,U}\right) \\ &= \sum_i \left([\sigma]_{L,K',*} \circ [\gamma_i]_{U_i,L,*}\right) \circ \mathrm{pr}^*_{U_i,U} \\ &= \sum_i \left([\sigma_i]_{K'_i,K',*} \circ \iota_{U_i,K'_i}\right) \circ \mathrm{pr}^*_{U_i,U} = \sum_i [U\sigma_i K']_*\end{aligned}$$

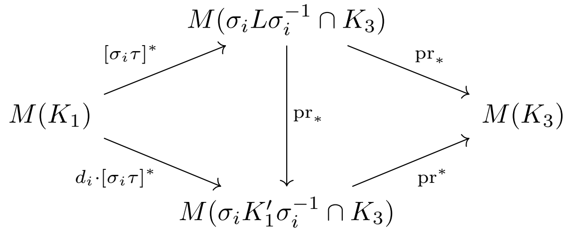
which proves the claim. □

Corollary 2.5.7. *Let $M = M_G$ be a CoMack functor on G, $K_1, K_2, K_3 \in \Upsilon$ and $\sigma, \tau \in G$. Then $[K_3\sigma K_2] \circ [K_2\tau K_1]$ is a finite $\mathbb{Z}$-linear combination of Hecke correspondences from $M(K_1)$ to $M(K_3)$*

Proof. Let $L = \tau K_1 \tau^{-1} \cap K_2 \in \Upsilon_G$ and suppose that $K_2\sigma^{-1}K_3 = \bigsqcup_i L\sigma_i^{-1}K_3$ for some $\sigma_i \in G$. Since M is Mackey, we see by Lemma 2.5.6 that

$$\begin{aligned}[K_3\sigma K_2] \circ [K_2\tau K_1] &= [K_3\sigma K_2] \circ \mathrm{pr}_{L,K_2,*} \circ [\tau]^*_{L,K_1} \\ &= \left([K_2\sigma^{-1}K_3]_* \circ \mathrm{pr}_{L,K_2,*}\right) \circ [\tau]^*_{L,K_1} \\ &= \sum_i [L\sigma_i^{-1}K_3]_* \circ [\tau]^*_{L,K_1} = \sum_i [K_3\sigma_i L] \circ [\tau]^*_{L,K_1}.\end{aligned}$$

Let $K'_1 = \tau K_1 \tau^{-1}$. For each i, let $d_i := [\sigma_i K'_1 \sigma_i^{-1} \cap K_3 : \sigma_i L \sigma_i^{-1} \cap K_3]$. Since M is cohomological, we see that the diagram



is commutative. So $[K_3\sigma_i L] \circ [\tau]^*_{L,K_1} = d_i \cdot [K_3\sigma_i\tau K_1]$ and therefore

$$[K_3\sigma K_2] \circ [K_2\tau K_1] = \sum_i d_i [K_3\sigma_i\tau K_1].$$

The claim follows. □

2.6 Morphisms on a base

In this section, we study morphisms of RIC functors that are determined by maps on certain subcollections of Υ. We also formalize the notion of parametrizating classes of one functor by another functor. The results of this section will be particularly useful in Ch 7.

Definition 2.6.1. Let $N : G \to R$-Mod be a RIC functor, $S \subset \Upsilon$ any subcollection and $\mathscr{G} = \{B_K \subset N(K)\}_S$ a family of R-submodules indexed by S. We say that $\mathscr{G}$ is *compatible under conjugation* if for all $K \in S$, $g \in G$ such that $gKg^{-1} \in S$, $[g]^*_{K,gKg^{-1}} : N(K) \to N(gKg^{-1})$ sends B_K to $B_{gKg^{-1}}$. We say that $\mathscr{G}$ is *compatible under restrictions* (resp. *induction*) if for all $L, K \in S$ with $L \subset K$, $\mathrm{pr}^*_{L,K} : N(K) \to N(L)$ (resp. $\mathrm{pr}_{L,K,*} : N(L) \to N(K)$) sends B_K to B_L (resp. B_L to B_K). We say that $\mathscr{G}$ is *compatible under pullbacks* if it is compatible under restriction and conjugation.

By definition, $\mathscr{G}$ constitutes a sub-functor of N if $S = \Upsilon$ and all three compatibilities are satisfied.

Definition 2.6.2. Let M, $N : G \to R$-Mod be RIC functors, $S \subset \Upsilon$ be any sub-collection and $\mathscr{G} = \{B_K \subset N(K)\}_S$ a family compatible under pullbacks. Let $\mathscr{F} = \mathscr{F}_S = \{\varphi_K : B_K \to M(K)\}_{K\in S}$ be a collection of R-module homorphisms. We say that $\mathscr{F}$ is *compatible under conjugation* if for all $g \in G$ and $K \in S$ such that $gKg^{-1} \in S$, $[g]^*_{K,gKg^{-1}} \circ \varphi_K = \varphi_{gKg^{-1}} \circ [g]^*_{K,gKg^{-1}}$. We say that $\mathscr{F}$ is *compatible under restriction* if for all $L, K \in \Upsilon'$ with $L \subset K$, $\mathrm{pr}^*_{L,K} \circ \varphi_K = \varphi_L \circ \mathrm{pr}^*_{L,K}$. We will say that $\mathscr{F}$ is *compatible under pullbacks* if it is compatible under restriction and conjugation. Suppose moreover that $\mathscr{G}$ is also compatible under induction. Then we say that $\mathscr{F}$ is *compatible under induction* if $\mathrm{pr}_{L,K,*} \circ \varphi_L = \varphi_K \circ \mathrm{pr}_{L,K,*}$.

By definition, $\mathscr{F}$ constitutes a morphism $\varphi : N \to M$ of RIC functors if $S = \Upsilon$, $B_K = N(K)$ for all $K \in S$ and all three compatibilities are satisfied.

Lemma 2.6.3. *Let M, $N : G \to R$-Mod be RIC functors such that M is Mackey and has injective restrictions. Let $\mathscr{F} = \{\varphi_K : N(K) \to M(K)\}_{K\in\Upsilon}$ be a family compatible under pullbacks. Then $\mathscr{F}$ is compatible under induction.*

Proof. Let $L, K \in \Upsilon$ with $L \subset K$. By (T2), there exists a $K' \in \Upsilon$ such that $K' \lhd K$, $K' \subset L$. Since $\mathrm{pr}^*_{K',K} : M(K) \to M(K')$ is injective, $\mathrm{pr}_{L,K,*} \circ \varphi_L = \varphi_K \circ \mathrm{pr}_{L,K,*}$ if and only if

$$\mathrm{pr}^*_{K',K} \circ \mathrm{pr}_{L,K,*} \circ \varphi_L = \mathrm{pr}^*_{K',K} \circ \varphi_K \circ \mathrm{pr}_{L,K,*}$$

as maps $M(L) \to M(K')$. Since $\mathscr{F}$ is compatible under pullbacks, $\mathrm{pr}^*_{K',K} \circ \varphi_K = \varphi_{K'} \circ \mathrm{pr}^*_{K',K}$ and since M is Mackey, the equality above is equivalent to

$$\sum_{\gamma \in K/L} [\gamma]^*_{K',L} \circ \varphi_L = \sum_{\gamma \in K/L} \varphi_{K'} \circ [\gamma]^*_{K',L}$$

by Lemma 2.1.11. But this holds since $\mathscr{F}$ is compatible under conjugation and restrictions. □

Lemma 2.6.4. *Let M, $N : G \to R$-Mod be RIC functors such that M has injective restrictions. Let $\mathscr{F} = \{\varphi_K : N(K) \to M(K)\}_{K \in \Upsilon}$ be a family such that $\mathscr{F}$ satisfies the restriction compatibility condition for pairs $L, K \in \Upsilon$ such that $L \triangleleft K$. Then $\mathscr{F}$ is compatible under restrictions.*

Proof. Let $L, K \in \Upsilon$. By (T2), there exists a $K' \in \Upsilon$ such that $K' \triangleleft K$, $K' \subset L$. Since $\mathrm{pr}^*_{K',L}$ is injective, $\mathrm{pr}^*_{L,K} \circ \varphi_K = \varphi_L \circ \mathrm{pr}_{L,K,*}$ if and only if

$$\mathrm{pr}^*_{K',L} \circ \mathrm{pr}^*_{L,K} \circ \varphi_K = \mathrm{pr}^*_{K',L} \circ \varphi_L \circ \mathrm{pr}_{L,K,*}.$$

As K' is normal in L, $\mathrm{pr}^*_{K',L} \circ \varphi_L = \varphi_{K'} \circ \mathrm{pr}^*_{K',L}$. Thus, the equality above is equivalent to $\mathrm{pr}^*_{K',K} \circ \varphi_K = \varphi_{K'} \circ \mathrm{pr}^*_{K',K}$. But this is true since $K' \triangleleft K$. □

Definition 2.6.5. Let $S \subset \Upsilon$ be a sub-collection. We say S forms a *base* for Υ if for all $K \in \Upsilon$, there exists $K' \in S$ such that $K' \subset K$.

Proposition 2.6.6. *Let $M, N : G \to R$-Mod be two Mackey functors such that M has injective restrictions. Let $S \subset \Upsilon$ be a base, $\mathscr{G} = \{B_K \subset N(K)\}$ be a family of submodules and $\mathscr{F} = \{\varphi_K : B(K) \to M(K)\}_{K \in S}$ be a family of R-module homomorphisms such that both $\mathscr{F}$ and $\mathscr{G}$ are compatible under pullbacks. Suppose that for all $K \in \Upsilon$ and $\phi \in N(K)$, there exist a finite index set I and $K_i \in S$, $\sigma_i \in G$, $c_i \in R$, $\phi_i \in N(K_i)$ for $i \in I$ such that $\phi = \sum_i c_i [K\sigma_i K_i](\phi_i)$. Then there exists a unique morphism $\varphi : N \to M$ of functors such that $\varphi(K) = \varphi_K$ for all $K \in S$.*

Proof. We first show that such a φ is necessarily unique. Suppose $\varphi_1, \varphi_2 : N \to M$ are two morphisms such that $\varphi_1(L)_{|B_L} = \varphi_2(L)_{|B_L} = \varphi_L$ for all $L \in S$. Let $K \in \Upsilon$ and $\phi \in N(K)$ be any element. Then $\phi = \sum_i c_i [K\sigma_i K_i](\phi_i)$ for some $c_i, K_i, \sigma_i, \phi_i \in B_{K_i}$. Since Hecke correspondences commute with morphisms of functors and φ_1, φ_2 agree on ϕ_i, we see that

$$\begin{aligned}
\varphi_1(K)(\phi) &= \sum_i c_i \cdot \varphi_1(K) \circ [K\sigma_i K_i](\phi_i) \\
&= \sum_i c_i [K\sigma_i K_i] \circ \varphi_1(K_i)(\phi_i) \\
&= \sum_i c_i [K\sigma_i K_i] \circ \varphi_2(K_i)(\phi_i) \\
&= \sum_i c_i \varphi_2(K) \circ [K\sigma_i K_i](\phi_i) = \varphi_2(K)(\phi)
\end{aligned}$$

As K and ϕ were arbitrary, this implies that $\varphi_1 = \varphi_2$.

We now show existence. Let $K \in \Upsilon$ and $\phi \in N(K_i)$. Let $K_1, \ldots, K_m \supset K$, $K_i \in S$, $\sigma_i \in G$ and $\phi_i \in N(K_i)$ be such that $\phi = \sum_i [K\sigma_i K_i](\phi_i)$. We define

$$\varphi_K(\phi) := \sum_{i=1}^{n} [K\sigma_i K_i] \circ \varphi_{K_i}(\phi_i) \in M(K).$$

To show this is well-defined, suppose that $L_1, \ldots, L_n \in S$, $\rho_j \in G$, $b_j \in R$ and $\psi_j \in M(L_j)$ also satisfy $\phi = \sum_{j=1}^{n} b_j [K\rho_j L_j](\psi_j) = \phi$. Let $K^\natural \in \Upsilon$ be such that $K^\natural \triangleleft K$ and $K^\natural \subset \sigma_i K_i \sigma_i^{-1}$ and let $K' \in S$ such that $K' \subset K^\natural$. Denote $K_{\sigma_i} := K \cap \sigma_i K_i \sigma_i^{-1}$. Since M is Mackey, our choice of $K^\natural$ implies that

$$\mathrm{pr}^*_{K^\natural, K}\left(\sum_{i=1}^{m} c_i [K\sigma_i K_i] \circ \varphi_{K_i}(\phi_i) \right) = \sum_{i=1}^{m} \sum_{\gamma \in K/K_{\sigma_i}} c_i \cdot [\gamma\sigma_i]^*_{K^\natural, K_i} \circ \varphi_{K_i}(\phi_i)$$

Since $\mathrm{pr}^*_{K',K} = \mathrm{pr}^*_{K',K^\natural} \circ \mathrm{pr}^*_{K^\natural,K}$, $\varphi_{K_i}, \varphi_{K'} \in \mathscr{F}$ commutes with $[\gamma\sigma_i]^*_{K',K_i}$ for all $\gamma \in K/K_{\sigma_i}$ and N is Mackey, we see that

$$\begin{aligned}
\mathrm{pr}^*_{K', K}\left(\sum_{i=1}^{m} c_i [K\sigma_i K_i] \circ \varphi_{K_i}(\phi_i) \right) &= \sum_{i=1}^{m} \sum_{\gamma \in K/K_{\sigma_i}} c_i \cdot [\gamma\sigma_i]^*_{K', K_i} \circ \varphi_{K_i}(\phi_i) \\
&= \varphi_{K'}\left(\sum_{i=1}^{m} \sum_{\gamma \in K/K_{\sigma_i}} c_i \cdot [\gamma\sigma_i]^*_{K', K_i}(\phi) \right) \\
&= \varphi_{K'} \circ \mathrm{pr}^*_{K', K}\left(\sum_{i=1}^{m} c_i [K\sigma_i K_i](\phi_i) \right) \\
&= \varphi_{K'} \circ \mathrm{pr}^*_{K', K}(\phi)
\end{aligned}$$

Similarly,

$$\mathrm{pr}^*_{K', K}\left(\sum_{j=1}^{n} b_j [K\rho_j L_j] \circ \varphi_{L_j}(\psi_j) \right) = \varphi_{K'} \circ \mathrm{pr}^*_{K', K}(\phi).$$

Since $\mathrm{pr}^*_{K',K}$ is injective on M, we see that $\varphi_K(\phi)$ does not depend on the choice of K_i, σ_i, c_i, ϕ_i and therefore $\varphi_K(\phi)$ is well-defined. By construction, φ_K are in $\mathscr{F}$ if $K \in S$, since the procedure above can be done by taking $K_1 = K$ and $\phi_1 = \phi$ for any $\phi \in M(K)$. Since $\mathscr{F}$ is compatible under conjugation and restriction, the collection $\{\varphi_K\}_{K \in \Upsilon}$ obtained above inherits these properties as φ_K are uniquely determined. Lemma 2.6.3 then implies that $\{\varphi_K\}_{K \in \Upsilon}$ are also compatible under induction and therefore constitute a morphism $\varphi : N \to M$. $\square$

Definition 2.6.7. Let $M : G \to R$-Mod be a functor, $K_i \in \Upsilon_G$ be a family of compact opens and $x_i \in M(K_i)$ be elements. We say the family $\{x_i\}_i$ has a *parametrization* by a functor $N : G \to R$-Mod if there exists a morphism $\varphi : N \to M$ and classes $\phi_i \in N(K_i)$ such that $\varphi_{K_i}(\phi_i) = x_i$. We say that the parametrization is Galois (resp. cohomological, resp. Mackey) if N is so.

Definition 2.6.8. Suppose that $G = \prod_\ell G_\ell$ is a finite product, $\Upsilon_\ell = \Upsilon_{G_\ell}$ is the pullback of Υ_G to G_ℓ and for each ℓ, $\{K_{i_\ell,\ell}\}$ is a collection of compact open subgroups of G_ℓ indexed by $i_\ell \in I_\ell$ such that $K_j := \prod_{i_\ell} K_{i_\ell,\ell} \in \Upsilon_G$ for any choice of $j := (i_\ell)_\ell \in I_\ell$. Let $I = \prod_\ell I_\ell$, $J \subset I$ and $x_j \in M(K_j)$ be classes. Let $J_\ell \subset I_\ell$ be the image of the projection from J to I_ℓ. We say that the family $\{x_j\}_{j\in S}$ is *compatibly parametrized* if there exists functors $N_\ell : G_\ell \to R$-Mod, a morphism $\bigotimes_\ell N_\ell \to M$ of functors of $\mathcal{P}(G, \prod_\ell \Upsilon_\ell)$ and classes $\phi_{i_\ell,\ell} \in N_\ell(K_{i,\ell})$ for $i_\ell \in J_\ell$ such that $\varphi(\otimes_\ell \phi_{i_\ell,\ell}) = x_j$ for all $j = (i_\ell)_\ell$.

2.7 Completed pushforwards

Let $\iota : H \to G$ be as in previous section and assume moreover that H, G are unimodular for this section. Let μ_H, μ_G Haar measures on H, G respectively with $\mu_H(\Upsilon_H), \mu_G(\Upsilon_G) \in R^\times$. Let $\mathcal{H}_R(G, \Upsilon_G)$ denote the Hecke algebra of G over Υ_G.

Definition 2.7.1. Given smooth representations τ of H, σ of G, we consider $\tau \otimes \mathcal{H}_R(G, \Upsilon_G)$ and σ smooth representations of $H \times G$ under the following *extended action*.

- $(h, g) \in H \times G$ acts on $x \otimes \xi \in \tau \otimes \mathcal{H}_R(G, \Upsilon_G)$ via $x \otimes \xi \mapsto hx \otimes \xi(\iota(h)^{-1}(-)g)$.
- $(h, g) \in H \times G$ acts on $y \in \sigma$ via $y \mapsto g \cdot y$.

An *intertwining map* $\Psi : \tau \otimes \mathcal{H}_R(G, \Upsilon_G) \to \sigma$ is defined to be a morphism of $H \times G$ representations.

Lemma 2.7.2. *Let $\Psi : \tau \otimes \mathcal{H}_R(G, \Upsilon_G) \to \sigma$ be an intertwining map. For any $\xi_1, \xi_2 \in \mathcal{H}_R(G, \Upsilon_G)$ and $x \in \tau$,*

$$\xi_1 \cdot \Psi(x \otimes \xi_2) = \Psi(x \otimes \xi_2 * \xi_1^t),$$

where ξ_1^t is the transpose of ξ_1.

Proof. Since Ψ is an intertwining map, it is also a morphism of $\mathcal{H}_R(G, \Upsilon_G)$ modules under the induced actions. Thus, $\xi_1 \cdot \Psi(x \otimes \xi_2) = \Psi(x \otimes \xi_1 *_\rho \xi_2)$. But Lemma 2.3.3 implies that $\xi_1 *_\rho \xi_2 = \xi_2 * \xi_1^t$. □

Lemma 2.7.3 (Frobenius Reciprocity)**.** *Let $\sigma^\vee$ denote the smooth dual of σ and $\langle \cdot, \cdot \rangle : \sigma^\vee \times \sigma \to R$ denote the induced pairing. Consider $\tau \times \sigma^\vee$ as a smooth H-representation via $h(x \otimes f) = hx \otimes \iota(h)f$. For any Ψ as above, there is a unique morphism $\psi : \tau \times \sigma^\vee \to R$ of smooth H-representations such that*

$$\langle f, \Psi(x \otimes \xi) \rangle = \psi(x \otimes (\xi \cdot f))$$

for all $x \in \tau$, $f \in \sigma^\vee$ and $\xi \in \mathcal{H}_R(G, \Upsilon_G)$. The mapping $\Psi \mapsto \psi$ thus defined induces a bijection between $\mathrm{Hom}_{H\times G}(\tau \otimes \mathcal{H}_R(G, \Upsilon_G), \sigma)$ *and* $\mathrm{Hom}_H(\tau \otimes \sigma^\vee, R)$.

Proof. For $K \in \Upsilon_G$, let $e_K \in \mathcal{H}_R(G, \Upsilon_G)$ denote the idempotent element $\frac{1}{\mu_G(K)} \cdot \mathrm{ch}(K)$. Given Ψ, $x \in \tau$, $f \in \sigma^\vee$, choose a compact open subgroup $K \in \Upsilon_G$ such that $f \in (\sigma^\vee)^K$ and set $\psi(x \otimes f) := \langle f, \Psi(x \otimes e_K)\rangle$. This is well-defined since if $L \subset K$,

$$\begin{aligned}\langle f, \Psi(x \otimes e_K)\rangle &= \sum_{\gamma \in K/L} \mu_G(K)^{-1} \langle f, \Psi(x \otimes \mathrm{ch}(L\gamma^{-1}))\rangle \\ &= \sum_{\gamma \in K/L} \mu(K)^{-1} \langle \gamma \cdot f, \Psi(x \otimes \mathrm{ch}(L))\rangle = \langle f, \Psi(x \otimes e_L)\rangle\end{aligned}$$

Since $\mu_G(K) = \mu_G(gKg^{-1})$ for $g \in G$ by unimodularity of G, we see that for all $h \in H$,

$$\begin{aligned}\psi(hx \otimes \iota(h)f) &= \left\langle \iota(h)f, \Psi(hx \otimes e_{hKh^{-1}})\right\rangle \\ &= \mu(hKh^{-1}) \cdot \left\langle \iota(h)f, \Psi(x \otimes \mathrm{ch}(Kh^{-1})\right\rangle \\ &= \mu(hKh^{-1}) \cdot \left\langle f, \iota(h)^{-1} \cdot \Psi(x \otimes \mathrm{ch}(Kh^{-1}))\right\rangle \\ &= \mu(K)^{-1} \cdot \langle f, \Psi(x \otimes \mathrm{ch}(K)) \\ &= \psi(x \otimes f)\end{aligned}$$

by equivariance properties of Ψ. Extending linearly, we obtain a H-equivariant map. Using the same relation, one defines Ψ given $\psi : \tau \times \sigma^\vee \to R$. It is easily seen that any such ψ determines a Ψ by the compatibility property and the association are inverses of each other. ☐

Proposition 2.7.4 (Completed Pushforward)**.** *Suppose M_H, M_G are RIC functors with M_H cohomological, M_G Mackey. Consider $\widehat{M}_H \otimes \mathcal{H}_R(G, \Upsilon)$ and $\widehat{M}_G$ as smooth $H \times G$ representations via the extended action. Then for any pushforward $\iota_* : M_H \to M_G$, there is a unique intertwining map of $H \times G$ representations*

$$\hat{\iota}_* : \widehat{M}_H \otimes \mathcal{H}_R(G, \Upsilon_G) \to \widehat{M}_G$$

satisfying the following compatibility condition: for all compatible pairs $(U, K) \in \Upsilon_H \times \Upsilon_G$, $x \in M_H(U)$, we have $\hat{\iota}_*(j_U(x) \otimes \mathrm{ch}(K)) = \mu_H(U) \cdot j_K \circ \iota_{U,K,*}(x)$.

Proof. For $K \in \Upsilon_G$, let $\mathcal{C}_R(G/K) \subset \mathcal{H}_R(G, \Upsilon_G)$ denote the free R-module of functions with basis $\mathrm{ch}(gK)$ for $g \in G/K$. Then $\mathcal{H}_R(G, \Upsilon_G) = \bigsqcup_{K \in \Upsilon} \mathcal{C}_R(G/K)$.

Step 1. We first define a map $\bar{\iota}_{*,K} : \widehat{M}_H \otimes \mathcal{C}_R(G/K) \to \left(\widehat{M}_G\right)^K$ of R-modules. Let $x \in \widehat{M}_H$ and $g \in G$. Choose a compact open subgroup $U \in \Upsilon_H$ such that U fixes x, $U \subset gKg^{-1} \cap H$, and let $x_U \in M_H(U)$ be an element that maps to x under j_U. We define $\bar{\iota}_{*,K}\,(x \otimes \mathrm{ch}(gK))$ to be the image of $\mu_H(U) \cdot x_U$ under the composition

$$M_H(U) \xrightarrow{[UgK]_*} M_G(K) \xrightarrow{j_K} \left(\widehat{M}_G\right)^K.$$

We claim that this is independent of our choice of x_U (and U). Say that U is replaced by another open compact subgroup V and x_U by $x_V \in M_H(V)$. By definition of $\widehat{M}_H$, there exists $W \in \Upsilon_H$ with $W \subset U \cap V$ such that $\mathrm{pr}^*_{U,V}(x_U) = \mathrm{pr}^*_{W,V}(x_V)$. We denote this common element by x_W. Since M_H is cohomological, $\mathrm{pr}_{W,U,*}(x_{V'}) = \mathrm{pr}_{W,U,*} \circ \mathrm{pr}^*_{W,U}(x_U) = [U : W]x_U$. Since $\mu_H(U), \mu_H(W) \in R^\times$, we may rewrite the last equality as $\mu_H(U)x_U = \mathrm{pr}_{W,U,*}\big(\mu_H(W)x_{V_1}\big)$. We then see from the commutative diagram,

$$\begin{array}{ccc} M_H(W) & & \\ {\scriptstyle \mathrm{pr}_*}\downarrow & \searrow^{[VgK]_*} & \\ M_H(U) & \xrightarrow[{[UgK]_*}]{} & M(K) \end{array}$$

that $\mu_H(U)x_U \in M_H(U)$ and $\mu_H(W)x_W \in M_H(W)$ are mapped to the same element under $\bar{\iota}_{*,K}$. The same argument applies to V and therefore the map is independent of the choice of U . We then extend $\bar{\iota}_{*,K}$ linearly to all $\widehat{M}_H \otimes \mathcal{C}_R(G/K)$.

Step 2. Note that both $K \mapsto \widehat{M}_H \otimes \mathcal{C}_R(G/K)$ and $K \mapsto \big(\widehat{M}_G\big)^K$ are CoMack functors. We claim that the maps $\bar{\iota}_{*,K}$ constitute a morphism of RIC functors. That the maps $\bar{\iota}_{*,K}$ for $K \in \Upsilon_G$ are compatible under conjugation follows by the second equality of Lemma 2.5.5. As both functors have injective restrictions, it suffices by Lemma 2.6.4 to show that $\bar{\iota}_*$ respects pullbacks on pairs of normal subgroups i.e. for all $K, L \in \Upsilon_G$ with $L \triangleleft K$,

$$\begin{array}{ccc} \widehat{M}_H \otimes \mathcal{C}_R(G/L) & \xrightarrow{\hat{\iota}_{L,*}} & (\hat{M}_G)^L \\ \uparrow & & \uparrow \\ \widehat{M}_H \otimes \mathcal{C}_R(G/K) & \xrightarrow{\hat{\iota}_{*,K}} & \big(\hat{M}_H\big)^K \end{array}$$

commutes. So, we need to show that for all $x \in \widehat{M}_H$, $g \in G$,

$$\bar{\iota}_{*,K}(x \otimes \mathrm{ch}(gK)) = \sum_{\gamma \in K/L} \bar{\iota}_{*,L}(x \otimes \mathrm{ch}(g\gamma L)) \tag{2.7.5}$$

as elements of $(\widehat{M}_G)^L$. Conjugating by g and replacing K, L with gKg^{-1}, gLg^{-1} respectively, we may assume without loss of generality that $g = 1_G$. Choose $U \in \Upsilon_H$ such that U fixes x and $U \subset L \cap H$. Note that as $L \triangleleft K$, $\gamma L \gamma^{-1} \cap H = L \cap H$ for any $\gamma \in K$. As before, let $x_U \in M_H(U)$ be an element mapping to x. As $\big(\widehat{M}_G\big)^K \hookrightarrow \big(\widehat{M}_G\big)^L$, we consider j_K, j_L as maps with target $\big(\widehat{M}_G\big)^L$. By definition

$$\bar{\iota}_{*,K}(x \otimes \mathrm{ch}(K)) = j_K \circ \iota_{U,K,*}\big(\mu_H(U)x_U\big),$$

$$\bar{\iota}_{*,L}\big(x \otimes \mathrm{ch}(\gamma L)\big) = j_L \circ [\gamma]_{L,L,*} \circ \iota_{U,L,*}\big(\mu_H(U)x_U\big)$$

As $j_K = j_L \circ \mathrm{pr}^*_{L,K}$, (2.7.5) would follow if $\mathrm{pr}^*_{L,K} \circ \iota_{U,K,*} = \sum_\gamma [\gamma]_{L,L,*} \circ \iota_{U,L,*}$ are equal as maps $M_H(U) \to M_G(L)$. As $\iota_{U,K,*} = \mathrm{pr}_{L,K,*} \circ \iota_{U,L,*}$, this in turn would follow from $\mathrm{pr}^*_{L,K} \circ \mathrm{pr}_{L,K,*} = \sum_\gamma [\gamma]_{L,L}$, which does hold by Lemma 2.1.11. Therefore, $\bar{\iota}_*$ is a morphism of RIC functors.

Step 3. We define $\hat{\iota}_* : \widehat{M}_H \otimes \mathcal{H}_R(G) \to \widehat{M}_G$ as the map induced by $\bar{\iota}_*$ on the inductive completions of the two G-functors. Thus $\hat{\iota}_*$ is G-equivariant and it therefore suffices to establish H-equivariance. Let $h \in H$ and $v = x \otimes \mathrm{ch}(gK) \in \widehat{M}_H \otimes \mathcal{H}_R(G)$. Choose $U \in \Upsilon_H$ such that U fixes x, $U \subset gKg^{-1} \cap H$ and $x_U \in M_H(U)$ that maps to x. Then $x_{hUh^{-1}} := [h]^*_{U,hUh^{-1}}(x_U) \in M_H(hUh^{-1})$ maps to $h \cdot x \in \widehat{M}_H$ under $j_{hUh^{-1}}$. By definition,

$$(h,1) \cdot \hat{\iota}_*(v) = \mu_H(U) \circ j_K \circ [UgK]_*(x_U)$$
$$\hat{\iota}_*((h,1) \cdot v) = \mu_H(hUh^{-1}) \cdot j_K \circ [(hUh^{-1})hgK]_*(x_{hUh^{-1}})$$

As H is unimodular, $\mu_H(U) = \mu_H(hUh^{-1})$ and the claim follows by Lemma 2.5.5. This establishes the equivariance of $\hat{\iota}_*$. That $\hat{\iota}_*$ is uniquely determined with the prescribed properties is straightforward. □

Corollary 2.7.6. *Let $\hat{\iota}$ be as above. For any $U \in \Upsilon_H$, $K \in \Upsilon_G$ and $g \in G$,*

$$\hat{\iota}_*(j_U(x_U) \otimes \mathrm{ch}(gK)) = \mu_H(U \cap gKg^{-1}) \cdot j_K \circ [UgK]_*(x_U).$$

Remark 2.7.7. In the definition of $\hat{\iota}_*$, we may replace $\mathcal{H}_R(G)$ with $\mathcal{C}_R(G)$ which is $\mathcal{H}_R(G)$ considered as a R-module with G-action given by right translation, since the definition of $\hat{\iota}_*$ does not require the convolution operation. In particular, $\hat{\iota}_*$ is independent of μ_G.

Chapter 3

Abstract zeta elements

In this chapter, we begin by giving ourselves a certain setup that one encounters in, but which it is not necessarily limited to, questions involving pushforwards of elements in the cohomology of Shimura varieties and we formulate a general problem in the style of Euler system norm relations within that setup. We then propose an abstract resolution for it by defining a notion we refer to as 'zeta elements' and study its various properties. In the context of §0.2, the element of interest is (0.2.19) and the reader is encouraged to refer to this example while reading this section. Another example based on Kato's Siegel units is provided in §3.6.

3.1 Setup and definition

Suppose for all of this section that we are given

- $\iota : H \hookrightarrow G$ a closed immersion of unimodular locally profinite groups,
- Υ_H, Υ_G collections of compact opens satisfying (T1)-(T3),
- $\mathcal{O}$ an integral domain with field of fractions a $\mathbb{Q}$-algebra
- $M_{H,\mathcal{O}} : \mathcal{P}(H) \to \mathcal{O}$-Mod, $M_{G,\mathcal{O}} : \mathcal{P}(G) \to \mathcal{O}$-Mod CoMack functors
- $\iota_* : M_{H,\mathcal{O}} \to M_{G,\mathcal{O}}$ a pushforward,
- $U \in \Upsilon_H$, $K \in \Upsilon_G$ compact opens such that $U = K \cap H$, which we call the *bottom levels*,
- $x_U \in M_{H,\mathcal{O}}(U)$ which we call the *source bottom class*,
- $\mathfrak{H} \in \mathcal{C}_{\mathcal{O}}(K \backslash G / K)$ an element which we call the *Hecke polynomial*
- $L \in \Upsilon_G$, $L \triangleleft K$ a normal compact open which we call a *compactum of field extension* of degree $d = [K : L]$.

As in 2.4.1, $\mathfrak{H}$ induces an $\mathcal{O}$-linear map $M_{G,\mathcal{O}}(K) \to M_{G,\mathcal{O}}(K)$ by Hecke correspondences which we continue to denote by $\mathfrak{H}$. Let $y_K := \iota_{U,K,*}(y_K) \in M_{G,\mathcal{O}}(L)$ which we call the *target bottom class.*

Problem 3.1.1. Does there exist a class $y_L \in M_{G,\mathcal{O}}(L)$ such that

$$\mathfrak{H}(y_K) = \mathrm{pr}_{L,K,*}(y_L)$$

as elements of $M_{G,\mathcal{O}}(K)$?

Note 3.1.2. Let us first make a few general remarks and provide some motivation for the notion we are going to describe before giving its formal definition. Notice first of all that if $d \in \mathcal{O}^\times$, the class $d^{-1} \cdot \mathrm{pr}^*_{L,K}(y_K) \in M_{G,\mathcal{O}}(L)$ solves the problem above. Thus, the non-trivial case occurs only when d is not invertible, and in particular when $\mathcal{O}$ is not a field. Second, 3.1.1 is meant to be posed as a family of analogous problems where one varies L over a prescribed lattice of fields (i.e. compact open subgroups) together with the other parameters above and ask for classes that satisfy such relations compatibly in a tower. One achieves this in practice by breaking the norm relation problem into components and varying the parameters componentwise. For instance, this happens when both H, G are the groups of adelic points of reductive algebraic groups over a number field and the class x_U has the features of a restricted tensor product (see Definition 2.6.8) and one is able to pose the problem above for each local place in a (infinite) subset of all finite places. The problem is therefore to be seen as one of a local nature that is extracted from a global setting. See §3.5 for an abstract formulation of this scenario.

The underlying premise of Problem 3.1.1 is that $y_K \in M_{G,\mathcal{O}}(K)$ is the image of a class x_U that one can vary over the levels of the functor $M_{H,\mathcal{O}}$ and for which one has a better description as compared to their counterparts in $M_{G,\mathcal{O}}$. If ι_* is also Mackey, then Lemma 2.5.6 tells us that $\mathfrak{H}(y_K)$ is the image of certain mixed Hecke correspondences applied to x_U. The class y_L we are seeking is therefore required to be of a similar form. As experience suggests, we assume that $y_L = \sum_{i=1}^r a_i [V_i g_i L]_*(x_{V_i})$ where

- $a_i \in \mathcal{O}$,
- $g_i \in G$,
- $V_i \subset g_i L g_i^{-1}$, $V_i \in \Upsilon_H$
- $x_{V_i} \in M_{H,\mathcal{O}}(V_i)$

are unknown quantities that we need to pick to obtain the said equality. If we only require equality up to $\mathcal{O}$-torsion (which suffices for applications, see 3.3.3), then one can use Lemma 2.4.2 and Proposition 2.7.4 to guide these choices. More precisely, let μ_H be a $\mathbb{Q}$-valued Haar measure on H, Φ a field containing $\mathcal{O}$, and $M_{H,\Phi}$, $M_{G,\Phi}$ denote the functors obtained by tensoring with Φ. Let $\widehat{M}_{H,\Phi}$, $\widehat{M}_{G,\Phi}$ be the completions of $M_{H,\Phi}$, $M_{G,\Phi}$ respectively and $j_U : M_{H,\Phi}(U) \to \widehat{M}_{H,\Phi}$, $j_K : M_{G,\Phi}(K) \to \widehat{M}_{G,\Phi}$ denote (abusing notation) the natural maps. We denote the images of x_U, x_{V_i} in $M_{H,\Phi}(U)$, $M_{H,\Phi}(V_i)$ etc by the same symbols. As $M_{G,\Phi}$ is cohomological, j_K is injective and therefore $\mathfrak{H}(y_K) - \mathrm{pr}_{L,K,*}(y_L)$ is $\mathcal{O}$-torsion if and only if

$$\mu_H(U)^{-1}\hat{\iota}_*(j_U(x_U) \otimes \mathfrak{H}^t) = \sum_{i=1}^{r} a_i \mu_H(V_i)^{-1}\hat{\iota}_*(j_{V_i}(x_{V_i}) \otimes \mathrm{ch}(g_i K)) \tag{3.1.3}$$

as elements of the Φ-vector space $\widehat{M}_{G,\Phi}$ (see the proof of 3.1.6 below). The H-equivariance $\hat{\iota}_*$ then provides a way of achieving this equality by expanding $\mathfrak{H}^t$ into left cosets, twisting by H and collecting/replacing terms as necessary. Under certain additional assumptions, we can upgrade the resulting norm relation to an equality. See §3.3.

For a ring R, let $R\langle H/U\rangle$ (resp. $R\langle G/K\rangle$) be the free R-modules with basis on H/U (resp. G/K). Let Φ be a field containing $\mathcal{O}$, and let $\mathcal{M}$ be the quotient of $\Phi\langle H/U\rangle \otimes \Phi\langle G/K\rangle$ by the subspace generated by relations $h'hU \otimes h'gK - hU \otimes gK$ where $h, h' \in H$, $g \in G$. We denote the class of $hU \otimes gK$ in $\mathcal{M}$ by the same symbol. For $\mathrm{ch}(K\sigma K) \in \mathcal{C}_{\mathcal{O}}(K\backslash G/K)$, we denote by $U \otimes \mathrm{ch}(K\sigma K) \in \mathcal{M}$ the element $\sum_{\gamma \in K\sigma K/K} U \otimes \gamma K \in \mathcal{M}$. For X a group, θ an element of the group algebra $R[X]$, we let $\deg\theta$ denote the image of θ under the augmentation map $R[X] \to R$. As in the discussion above, we let $j_V : M_{H,\Phi}(V) \to \widehat{M}_{H,\Phi}$ denote the natural map for any $V \in \Upsilon_H$ which we abuse to also denote the map $M_{H,\mathcal{O}}(V) \to M_{H,\Phi}(V) \to \widehat{M}_{H,\Phi}$.

Definition 3.1.4. A *zeta element* for $(x_U, \mathfrak{H}, L)$ with coefficients in Φ is an element in $\widehat{M}_{H,\Phi} \otimes \Phi\langle G/K\rangle$ of the form

$$\zeta = \sum_{i=1}^{r} b_i (x_i \otimes g_i K),$$

where $b_i \in \Phi$, $x_i \in \widehat{M}_{H,\Phi}$, $g_i \in G$, that satisfies the following conditions: for all $i = 1, \ldots, r$,

(Z1) there exists $\eta_i \in \Phi[H]$ such that $U \otimes \mathfrak{H}^t = \sum_{i=1}^{r} \eta_i U \otimes g_i K$ in $\mathcal{M}$.

(Z2) there exists $V_i \in \Upsilon_H$ with $V_i \subset H \cap g_i L g_i^{-1}$ such that $b_i \cdot \mu_H(V_i)/\mu_H(U) \in \mathcal{O}$.

(Z3) there exists $x_{V_i} \in M_{H,\mathcal{O}}(V_i)$ such that $j_{V_i}(x_{V_i}) = x_i$.

(Z4) there exists $\theta_i \in \Phi[H \cap g_i K g_i^{-1}]$ such that $\deg\theta_i \neq 0$ and $\eta_i \cdot j_U(x_U) = b_i(\theta_i/\deg\theta_i)x_i$.

We say that the cosets $g_i K$ *span* the element ζ and refer to $g_i K$ (or just the elements g_i) for which $b_i x_i \neq 0$ as the *twists* of ζ. We will refer to collection of 4-tuples $(\eta_i, V_i, x_{V_i}, \theta_i)_i$ indexed by $i = 1, \ldots, r$ as the *data* of ζ. We will say that the data $(\eta_i, V_i, x_{V_i}, \theta_i)_i$ is *trace-like* if for all i, $\theta_i = \sum_\gamma \gamma$ with γ running over a set of representatives of $(H \cap g_i K g_i^{-1})/V_i$. We will say that $(\eta_i, V_i, x_{V_i}, \theta_i)$ is *optimal* if for all i, $V_i = H \cap g_i L g_i^{-1}$. We will say that ζ is *trace-like* (resp. *optimal*) if there exists data that for ζ that is trace-like (resp. optimal).

Definition 3.1.5. Let ζ be zeta element for $(x_U, \mathfrak{H}, L)$ and $(\eta_i, V_i, x_{V_i}, \theta_i)_i$ be a data. The *associated class for* $(\eta_i, V_i, x_{V_i}, \theta_i)$ *and the pushforward* ι_* is defined to be

$$y_L := \sum_{i=1}^{r} a_i [V_i g_i L]_*(x_{V_i}) \in M_{G,\mathcal{O}}(L)$$

where $a_i := b_i \cdot \mu_H(V_i)/\mu_H(U)$. When a data for ζ is implied, we will abuse terminology and say that y_L is the class associated with (ζ, ι_*).

Proposition 3.1.6. *Suppose* $\zeta = \sum_{i=1}^{r} b_i(x_i \otimes g_i K)$ *is a zeta element for* $(x_U, \mathfrak{H}, L)$ *and* $(\eta_i,\ V_i,\ x_{V_i}, \theta_i)_i$ *a data. Then the associated class* $y_L \in M_{G,\mathcal{O}}(L)$ *is such that* $\mathfrak{H}(y_K) - \mathrm{pr}_{L,K,*}(y_L)$ *is* $\mathcal{O}$*-torsion.*

Proof. Let $\hat{\iota}_*$ denote the completed pushforward of Proposition 2.7.4 and $j_K : M_{G,\mathcal{O}}(K) \to \widehat{M}_{G,\Phi}$ denote the natural map. Let μ_G be a Haar measure on G such that $\mu_G(K) = 1$. Since $M_{G,\mathcal{O}}$ is Mackey, we see by Lemma 2.4.2 and the properties of $\hat{\iota}_*$ (Corollary 2.7.1) that

$$\mu_H(U) \cdot j_K(\mathfrak{H}(y_K)) = \mathfrak{H} \cdot \hat{\iota}_*(j_U(x_U) \otimes \mathrm{ch}(K)) = \hat{\iota}_*(j_U(x_U) \otimes \mathfrak{H}^t)$$

For $\phi = \sum_j c_j(h_j U \otimes g_j K) \in \Phi\langle H/U\rangle \otimes \Phi\langle G/K\rangle$ a finite sum, let $\phi(x_U) := \sum_j c_j \hat{\iota}_*(h_j j_U(x_U) \otimes \mathrm{ch}(g_j K)) \in \widehat{M}_{G,\Phi}$. Then $\phi \mapsto \phi(x_U)$ is Φ-linear. If ϕ belongs to the subspace $\Phi\langle h'hU \otimes h'gK - hU \otimes gK \mid h, h' \in H, g \in G\rangle$, then $\phi(x_U) = 0$ by the H-equivariance of $\hat{\iota}_*$. Therefore, $\phi(x_U)$ is independent of the class of ϕ in $\mathcal{M}$. In particular, if $\phi = U \otimes \mathfrak{H}^t$, $\phi' = \sum_{i=1}^{r} \eta_i U \otimes g_i K$, $\phi(x_U) = \phi'(x_U)$ by (Z1). By (Z4), we see that

$$\begin{aligned}
\hat{\iota}_*(j_U(x_U) \otimes \mathfrak{H}^t) &= \sum_{i=1}^{r} \hat{\iota}_*(\eta_i j_U(x_U) \otimes \mathrm{ch}(g_i K)) \\
&= \sum_{i=1}^{r} \frac{b_i}{\deg \theta_i} \hat{\iota}_*(\theta_i x_i \otimes \mathrm{ch}(g_i K))
\end{aligned}$$

Now note that for any $\gamma \in H \cap g_i K g_i^{-1}$, $\hat{\iota}_*(\gamma x_i \otimes \mathrm{ch}(g_i K)) = \hat{\iota}_*(x_i \otimes \mathrm{ch}(\gamma^{-1} g_i K) = \hat{\iota}_*(x_i \otimes \mathrm{ch}(g_i K))$. Applying this to each $\theta_i = \sum_\gamma c_\gamma \gamma$, the last sum above is equal to $\sum_{i=1}^{r} b_i \cdot \iota_*(x_i \otimes \mathrm{ch}(g_i K))$. By definition of x_i and a_i, this sum in turn equals $\sum_{i=1}^{r} a_i\,(\mu_H(U)/\mu_H(V_i)) \cdot \hat{\iota}_*\,(j_{V_i}(x_{V_i}) \otimes \mathrm{ch}(g_i K))$. As $V_i \subset g_i L g_i^{-1}$, $[V_i g_i K]_* = \mathrm{pr}_{L,K,*} \circ [V_i g_i L]_*$. Corollary 2.7.6 then implies that

$$\begin{aligned}
\mu_H(U) \cdot j_K(\mathfrak{H}(y_K)) &= \mu_H(U) \cdot j_K \circ \mathrm{pr}_{L,K,*}\left(\sum_{i=1}^{r} a_i [V_i g_i L]_*(x_{V_i})\right) \\
&= \mu_H(U) \cdot j_K\left(\mathrm{pr}_{L,K,*}(y_L)\right)
\end{aligned}$$

Thus, $j_K(\mathfrak{H}(y_K) - \mathrm{pr}_{L,K,*}(y_L)) = 0$ which implies the claim since the kernel of $M_{G,\mathcal{O}}(K) \to M_{G,\Phi}(K) \to \widehat{M}_{G,\Phi}$ is $M_{G,\mathcal{O}}(K)_{\mathcal{O}\text{-tors}}$ (2.2.4). $\square$

Lemma 3.1.7. *Suppose* $\zeta = \sum_{i=1}^{r} b_i(x_i \otimes g_i K)$ *is a zeta element for* $(x_U, \mathfrak{H}, L)$.

(a) *For* $h_1, \ldots, h_r \in H$, $\zeta' = \sum_{i=1}^r b_i(h_i x_i \otimes h_i g_i K)$ *is a zeta element.*

(b) *Suppose* ζ *is trace-like. Then there exists an optimal zeta element.*

(c) *Suppose* ζ *is optimal and trace-like. Then there exists a zeta element whose twists form distinct classes in* $H\backslash G/K$.

Proof. Let $\eta_i \in \Phi\langle H\rangle$, $V_i \subset g_i L g_i^{-1} \cap H$, $x_{V_i} \in M_{H,\mathcal{O}}(V_i)$, $\theta_i \in \Phi[H \cap g_i K g_i^{-1}]$ be such that (Z1)-(Z4) are satisfied for ζ.

(a) Let $\eta_i' = h_i \eta_i$, $V_i' := h_i V_i h_i^{-1}$, $x_{V_i}' := [h_i]^*_{V_i', V_i}(x_{V_i}) \in M_{H,\mathcal{O}}(V_i')$ and $\theta_i' := h_i \theta_i h_i^{-1} \in \Phi[H \cap hg_i K (hg_i)^{-1}]$ for all $i = 1, \ldots, r$. With this data, (Z1-Z4) are easily verified for ζ'.

(b) Let $V_i' := H \cap g_i L g_i^{-1}, H_{g_i} := H \cap g_i K g_i^{-1}$. Since ζ is trace-like, we may that for all $i = 1, \ldots, r$, $\theta_i = \left(\sum_{\gamma \in H_{g_i}/V_i'} \gamma\right)\left(\sum_{\gamma \in V_i'/V_i} \gamma\right)$. Set

- $x_{V_i'} = \mathrm{pr}_{V_i, V_i', *}(x_{V_i}) \in M_{H,\mathcal{O}}(V_i')$
- $b_i' = [V_i' : V_i]^{-1} \cdot b_i$,
- $x_i' = \delta_i \cdot x_i$ where $\delta_i = \sum_{\gamma \in V_i'/V_i} \gamma \in \mathbb{Z}[V_i']$
- $\eta_i' = \eta_i$
- $\theta_i' = \sum_{\gamma \in H_{g_i}/U_i} \gamma$

Then we claim that $\zeta' = \sum_{i=1}^r b_i'(x_i' \otimes g_i K)$ is a zeta element for $(x_U, \mathfrak{H}, L)$ with $\eta_i', V_i', x_{V_i'}, \theta_i'$ as its data. (Z1) holds since we did not change η_i. (Z2) holds since $b_i' \cdot \mu_H(V_i')/\mu_H(U) = b_i \cdot \mu_H(V_i)/\mu_H(U) \in \mathcal{O}$. (Z3) holds since

$$j_{V_i'}(x_{V_i'}) = j_{V_i} \circ \mathrm{pr}^*_{V_i, V_i'}\left(\mathrm{pr}_{V_i, V_i', *}(x_{V_i})\right) = j_{V_i}(\delta_i \cdot x_{V_i}) = \delta_i \cdot j_{V_i}(x_{V_i}) = x_i'$$

where the second equality follows since $M_{H,\mathcal{O}}$ is Mackey. Finally, (Z4) holds since

$$\begin{aligned}
\eta_i' \cdot j_U(x_U) &= \eta_i \cdot j_U(x_U) \\
&= b_i(\theta_i/\deg\theta_i) \cdot x_i \\
&= \frac{b_i}{[V_i' : V_i]}(\theta_i'/\deg(\theta_i'))x_i' = b_i'(\theta_i'/\deg(\theta_i'))x_i'
\end{aligned}$$

where we used that $\theta_i = \theta_i' \cdot \delta_i$ and $\deg(\theta) = \deg(\theta_i') \cdot \deg(\delta_i)$

(c) Say $Hg_1K = Hg_2K$. Since ζ is trace like and optimal, there is a $h \in H$ such that $g_2K = hg_1K$, $\theta_2 = h\theta_1 h^{-1}$ and $V_2 = hV_1h^{-1}$. Let $a_i = b_i \cdot \mu_H(U)/\mu_H(V_i)$ and let

- $\eta' = h\eta_1 + \eta_2$,

- $V' = V_2$,
- $x_{V'} = a_1[h]^*_{V_2,V_1}(x_{V_1}) + a_2 x_{V_2}$
- $g' = g_2$,
- $\theta' = \theta_2$,
- $b' = \mu_H(U)/\mu_H(V')$,
- $x' = j_{V'}(x'_V) = a_1 h x_1 + a_2 x_2$,

Then we claim that $\zeta' = b'(x' \otimes g'K) + \sum_{i=3}^{r} b_i(x_i \otimes g_i K)$ is a zeta element. (Z1) is immediate and (Z2), (Z3) hold by definition. We only need to verify (Z4) for g' since the other twists are kept the same. We have

$$\begin{aligned}\eta' \cdot j_U(x_U) &= (h\eta_1 + \eta_2) j_U(x_U) \\ &= b_1 h(\theta_1/\deg\theta_1)\cdot x_i + b_2(\theta_2/\deg\theta_2)x_2 \\ &= b'(\theta_2/\deg\theta_2)(a_1[h]^*_{V_2,V_1}x_1 + a_2x_2) = b'(\theta'/\deg\theta')x\end{aligned}$$

and therefore (Z4) holds as well. Repeating this argument if necessary, we can ensure that the twists of ζ are as claimed. □

Remark 3.1.8. Some comments are in order. As the reader would have noticed, the notion of a zeta element is built only out of the formal equivariance properties of pushforwards. In particular, if $\iota'_* : M_{H,\mathcal{O}} \to M'_{G,\mathcal{O}}$ is any other pushforward, the class associated with (ζ, ι'_*) solves the corresponding problem modulo torsion. We will always take Φ to be $\mathrm{Frac}(\mathcal{O})$ in this article, though one may wish to work in more convenient fields. In many situations that interest us, not all the quantities that appear in the axioms (Z1)-(Z4) need to be computed to exhibit a zeta element. See the criteria provided in the next section.

Axiom (Z1) restricts the choice of the twists g_i to elements in G which are the in the H-orbit of some left coset G/K contained in a double coset that appears with a non-zero coefficient in $\mathfrak{H}^t$. Thus, if $\mathfrak{H}^t = \sum_\sigma c_\sigma \mathrm{ch}(K\sigma K) = \sum_j c_j \mathrm{ch}(\sigma_j K)$ (where J is some finite set), the twists of a zeta element are in the set $\{h\sigma_j K \mid h \in H, j \in J\}$. Axiom (Z2) may be removed entirely, as one may always assume $a_i = 1$ (and therefore $b_i = \mu_H(U)/\mu_H(V_i)$) by replacing x_{V_i} by $a_i x_{V_i}$, b_i by $b_i a_i$. The reason to keep the coefficient b_i arbitrary is mostly a matter of taste; in case of trivial functor, the 'natural' choice for elements x_{V_i} is the identity element $1_{\mathcal{O}}$ and the required coefficients needed to satisfy the axioms are recorded by a_i.

The formulation of axiom (Z4) can be motivated by considering the case of $H = G$, $\iota_* = \mathrm{id}$, and assuming an affirmative answer to Problem 3.1.1. One can interpret, under certain conditions, a trace-like zeta element as a collection of norm relations on the source functor $M_{H,\mathcal{O}}$, one for each of its twists (see §3.3) and the formulation of (Z4) is a generalization of this observation after passage to the limit. The reason for allowing non trace like choices is that if $\vartheta_i \in \Phi[H \cap g_i K g_i^{-1}]$ with $\deg \vartheta_i \neq 0$ are any elements and $(\eta_i, V_i, x_{V_i}, \theta_i)_i$ is a data for ζ, then $\left(\deg \vartheta_i^{-1} \vartheta_i \eta_i, V_i, x_{V_i}, \vartheta_i \theta_i\right)_i$ is another. One may however assume that θ_i is a linear combination of distinct elements in H_{g_i}/V_i since x_i is invariant under V_i. For the purposes of this article, trace-like zeta elements (in fact, *uniform*, see 3.2.4) will suffice. The more general notion is a theoretical curiosity.

By Lemma 3.1.7(a), one may consider a zeta element as an element of the quotient $H\backslash(M_{H,\Phi} \otimes \Phi\langle G/K\rangle)$ by the diagonal action of H. Part (b) of *ibid* shows that if one is interested in only trace-like zeta elements, it suffices to restrict attention to the 'shallowest possible' levels $V_i = H \cap g_i L g_i^{-1}$ for possible candidates to push from, and part (c) of *ibid* shows that for such elements one may collect as many summands in $U \otimes \mathfrak{H}^t$ as possible in (Z1) before looking for classes x_{V_i}. In certain cases however, it might be more convenient to have more summands in ζ than the minimum possible or to work with a non-optimal choice of V_i (see Example 3.6.2 for an instance where the trace is initially taken from a non-optimal level W). The terminology 'zeta element' is inspired by [Kat04] and the fact that the Hecke polynomials we are interested in 'compute' the zeta functions of Shimura varieties e.g. see [BR94], [Lan79].

3.2 Existence criteria

We would like to execute the axioms of zeta element in terms of explicit summands of the Hecke polynomial and derive some criteria that will be useful later on. The goal in the manipulations that follow is to ferret out information that distinguishes the non-trivial cases from the trivial ones (e.g. it is not obvious from the axioms that ζ exists when $d \in \mathcal{O}^\times$) and to reduce the quantities one needs to compute in practice to exhibit zeta elements.

Retain the setup and notations of §3.1. We moreover also adopt the following notations/conventions for this section:

- for $\sigma \in G$, $X \subset H$ a subgroup, $X_\sigma = X_{\sigma,K} := X \cap \sigma K \sigma^{-1}$,
- for $\sigma \in G$, we denote $\deg\,[U\sigma K]_* := [H_\sigma : U_\sigma] = \mu_H(H_\sigma)/\mu_H(U_\sigma)$ (see also Definition 2.5.3),
- we let $\mathfrak{H}^t = \sum_{j \in J} c_j \operatorname{ch}(U\sigma_j K)$ where J is a finite set (by pulling back $\mathfrak{H}^t$ along $U\backslash G/K \to K\backslash G/K$),
- we let $d_j := \mu_H(H_{\sigma_j})/\mu_H(H \cap \sigma_j L \sigma_j^{-1})$ for $j \in J$. Then d_j is an integer that divides $d = [L : K]$.

We observe that both $\mu_H(H_{\sigma_j})$, $\mu_H(H \cap \sigma_j L \sigma_j^{-1})$ and hence d_j are independent of the class of σ_j in $H\backslash G/K$ (using $L \triangleleft K$ for the latter). Let $J = J_1 \sqcup J_2 \sqcup \ldots \sqcup J_r$ be a partition of J such that $j, j' \in J_i \implies H\sigma_j K = H\sigma'_j K$. We denote by d_i the integer d_j for any $j \in J_i$. We may assume, by replacing σ_j by $\sigma_j \gamma$ for $\gamma \in K$ if necessary, that $H\sigma_j = H\sigma'_j$ for all $j, j' \in J_i$. We then fix any elements $g_i \in H\sigma_j$, $j \in J_i$ and we pick/set

- $V_i = H \cap g_i L g_i^{-1}$,
- $h_j = g_i \sigma_j^{-1}$ for $j \in J_i$,
- $\varsigma_j \in \Phi[H_{g_i}]$ for $j \in J_i$ any[1] elements such that $\deg(\varsigma_j) = \deg[U\sigma K]_*$,
- $\eta_i = \frac{\mu_H(U)}{\mu_H(H_{g_i})} \cdot \sum_{j \in J_i} c_j \varsigma_j h_j$
- $b_i = \mu_H(U)/\mu_H(V_i)$
- $\theta_i = \sum_{H_{g_i}/V_i} \gamma$, so that $\deg(\theta_i) = d_i$

With these choices made, we get the following

Proposition 3.2.1. *With notation as above, there exists a zeta element for $(x_U, \mathfrak{H}, L)$ if for all $i = 1, \ldots, r$, there exist $x_{V_i} \in M_{H,\mathcal{O}}(V_i)$ such that $\sum_{j \in J_i} (c_j \varsigma_j h_j) \cdot j_U(x_U) = j_{H_{g_i}} \circ \mathrm{pr}_{V_i, H_{g_i}, *}(x_{V_i})$ as elements of $\widehat{M}_{H,\Phi}$.*

Proof. We claim $\zeta = \sum_{i=1}^r b_i (x_i \otimes g_i K)$ is the desired zeta element with η_i, V_i, θ_i as above and $x_i := j_{V_i}(x_{V_t})$. We have in $\mathcal{M}$

$$\begin{aligned} U \otimes \mathfrak{H}^t &= \sum_{j \in J} c_j [U : U_{\sigma_j}] \ (U \otimes \sigma_j K) \\ &= \sum_{j \in J} c_j \cdot \mu_H(U)/\mu_H(H_{\sigma_j}) \cdot \deg[U\sigma_j K]_* (U \otimes \sigma_j K) \\ &= \sum_{i=1}^r \mu_H(U)/\mu_H(H_{g_i}) \cdot \sum_{j \in J_i} c_j \deg[U\sigma_k K]_* (h_j U \otimes g_i K) \end{aligned}$$

where the second equality follows by applying the twist h_j to $U \otimes \sigma_j K$ and that $\mu_H(H_{\sigma_j}) = \mu_H(H_{g_i})$. Now since the degree $\varsigma_j \in \Phi[H_{g_i}]$ is equal to $\deg[U\sigma K]_*$ and twisting $h_j U \otimes g_i K$ by $\gamma \in H_{g_i}$ does not affect the component $g_i K$, we see that the last sum above is equal to $\sum_{i=1}^r \eta_i U \otimes g_i K$, and thus (Z1) holds. (Z2) and (Z3) hold by our definitions. By our hypothesis on x_{V_i} and that $M_{G,\mathcal{O}}$ is Mackey,

$$\begin{aligned} \eta_i \cdot j_U(x_U) &= \frac{\mu_H(U)}{\mu_H(H_{g_i})} \sum_{j \in J_t} (c_j \varsigma_j h_j) \cdot j_U(x_U) \\ &= \frac{b_i}{d_i} \cdot j_{H_{g_i}} \circ \mathrm{pr}_{V_i, H_{g_1}, *}(x_{V_t}) = b_i (\theta_i / \deg \theta_i) \cdot x_i \end{aligned}$$

and therefore (Z4) holds as well. □

Corollary 3.2.2. *Fix $i \in \{1, \ldots, r\}$ and suppose that for all $j \in J$, there exists $\alpha_j \in \Phi$ such that*

- *$h_j \cdot j_U(x_U) = \alpha_j \cdot j_U(x_U)$ for all $j \in J_i$,*
- *$j_U(x_U)$ lies in the image of $M_{H,\mathcal{O}}(H_{g_i}) \to \widehat{M}_{H,\Phi}$[2],*

[1] See Corollary 3.2.3 for a 'natural' choice that deals with trivial cases.

[2] This condition is automatic if $H_{g_i} \subset U$.

- $\sum_{j\in J_i} c_j\alpha_j \deg[U\sigma_j K]_* \in d_i\mathcal{O}$.

Then there exists $x_{V_i} \in M_{H,\mathcal{O}}(V_i)$ satisfying the criteria of Proposition 3.2.1. In particular, if the conditions above hold for all $i = 1,\dots,r$, there exists a zeta element for $(x_U, \mathfrak{H}, L)$ whose twists are g_i for those i for which the sum in the third bullet is not zero. If moreover $M_{H,\mathcal{O}}$ is the trivial functor on H, it suffices to check that $\sum_{j\in J_i} c_j \deg[U\sigma_j K]_* \in d_i\mathcal{O}$ for $i = 1,\dots,r$ in order to exhibit a zeta element.

Proof. Let $u \in \mathcal{O}$ be such that $\sum_{j\in J_i} c_j \deg[U\sigma_j K]_* \cdot \alpha_j = d_i u$. and $x_{H_{g_i}} \in M_{H,\mathcal{O}}(H_{g_i})$ be such that $j_{H_{g_i}}(x_{H_{g_i}}) = j_U(x_U)$. Set $x_{V_i} := \mathrm{pr}^*_{V_i,H_{g_i}}(u\cdot x_{H_{g_i}}) \in M_{H,\mathcal{O}}(V_i)$ and pick $\varsigma_j = 1_H + \cdots + 1_H = \deg[U\sigma_j K]_* \cdot 1_H$ for all $j \in J_i$. Then $j_{H_{g_i}} \circ \mathrm{pr}_{V_i,H_{g_i},*}(x_{V_i}) = d_i u \cdot j_{H_{g_i}}(x_{H_{g_i}}) = d_i u \cdot j_U(x_U)$. The criteria of Proposition 3.2.1 for the chosen i therefore reads

$$\sum_{j\in J_i} \left(c_j \deg[U\sigma_j K]_*\, \alpha_j\right) \cdot j_U(x_U) = d_i u \cdot j_U(x_U),$$

which holds by definition of u. The zeta element prescribed by Proposition 3.2.1 is then $\sum_i b_i(x_i \otimes g_i K)$ where $x_i = j_{V_i}(x_{V_i})$ and

$$b_i = \mu_H(U)/\mu_H(H_{g_i}) \sum_{j\in J_i} c_j\alpha_j \deg[U\sigma_j K]_*.$$

The claim on the twists of ζ is then clear. If $M_{H,\mathcal{O}} = M_{H,\mathcal{O},\mathrm{triv}}$ and $x_U = a\cdot 1_{\mathcal{O}} \in M_{H,\mathcal{O}}(U)$ for some $a \in \mathcal{O}$, we can pick $x_{H_{g_i}} = a \cdot 1_{\mathcal{O}} \in M_{H,\mathcal{O}}(H_{g_i})$ and $h_j \cdot j_U(x_U) = j_U(x_U)$, whence the last claim. ☐

Corollary 3.2.3. *Fix $i \in \{1,\dots,r\}$ and suppose that*

- $\sum_{j\in J_i} c_j\varsigma_j h_j \cdot j_U(x_U)$ *lies in the image of* $d_i \cdot M_{H,\mathcal{O}}(H_{g_i})$ *under* $M_{H,\mathcal{O}}(H_{g_i}) \to \widehat{M}_{H,\Phi}$

Then there exists $x_{V_i} \in M_{H,\mathcal{O}}(V_i)$ satisfying the criteria of Proposition 3.2.1. The condition of the bullet holds automatically if $d_i \in \mathcal{O}^\times$ and $\varsigma_j = \sum_{\gamma\in H_{g_i}/(h_j U_{\sigma_j} h_j^{-1})} \gamma$ for all $j \in J_i$. In particular, if the condition above holds for all $i = 1,\dots,r$, there exists a zeta element for $(x_U, \mathfrak{H}, L)$.

Proof. If $x_{H_{g_i}} \in M_{H,\mathcal{O}}(H_{g_i})$ is such that $j_{H_{g_i}}(d_i x_{H_{g_i}}) = \sum_{j\in J_i}(c_j\varsigma_j h_j)\cdot j_U(x_U)$, then the criteria of Proposition 3.2.1 is satisfied by taking $x_{V_i} := \mathrm{pr}^*_{V_i,H_{g_i}}(x_{H_{g_i}}) \in M_{H,\mathcal{O}}(V_i)$ as $j_{H_{g_i}}(x_{V_i}) = d_i j_{H_{g_i}}(x_{H_{g_i}})$. Since $M_{H,\mathcal{O}}$ is Mackey, we have $j_{H_{g_i}}([H_{g_i} h_j U](x_U)) = \varsigma_j h_j \cdot j_U(x_U)$ where ς_j is chosen as in the statement. Now $[H_{g_i} h_j U](x_U) \in M_{H,\mathcal{O}}(H_{g_i})$ and $\deg \varsigma_j = [H_{g_i} : h_j U_{\sigma_j} h_j^{-1}] = \deg[U\sigma_j K]_*$ whence the condition in the bullet above is automatic for $d_i \in \mathcal{O}^\times$. ☐

Definition 3.2.4. Let $\zeta = \sum_{i=1}^{r'} b'_i(x'_i \otimes g'_i K)$ be any zeta element for $(x_U, \mathfrak{H}, L)$ and assume that $\mathfrak{H} = \sum_{j\in J} \mathrm{ch}(U\sigma_j K)$ as above. Let J'_i be the partition of the index set J such that $H\sigma_j K = Hg'_i K$ for all $j \in J'_i$. We say that a data $(\eta'_i, V'_i, x'_{V_i}, \theta'_i)$ for ζ is *uniform* if

- the data is trace-like and optimal,

- for $i = 1, \ldots, r'$ and $j \in J'_i$, there exists $h'_j \in H$ such that $h'_j \sigma_j K = g'_i K$ and

$$\eta'_i = \mu_H(U)/\mu_H(H_{g'_i}) \sum_{j \in J'_i} c_j \varsigma'_j h'_j$$

where ς'_j is a sum of coset representatives of $H_{g'_i}/h'_j U_{\sigma_j} h'_j$ for each $j \in J'_i$.

We will say that ζ is *uniform* if there exists data for it that is uniform.

Remark 3.2.5. As also already noted, the motivation behind these criteria – and of zeta element in general – is that in practice the source $M_{H,\mathcal{O}}$ is much better understood (e.g. cycles or Eisenstein classes) than the target $M_{G,\mathcal{O}}$ as a Mackey functor. The results above are meant to provide an efficient means for parlaying this knowledge (and that of the Hecke polynomial) for Euler system style relations. In fact, in all the case we will consider $M_{H,\mathcal{O}}$ would be a space of functions on a suitable topological space that would parametrize classes in the cohomology of Shimura varieties. In §3.4 we study such $M_{H,\mathcal{O}}$ in detail. See also Ch. 7.

Remark 3.2.6. For the case of cycles coming from a sub-Shimura datum, the collection of fundamental classes of the sub-Shimura variety constitutes the trivial functor on H (see [GS21] for a concrete instance). In this case, Corollary 3.2.2 applies and proving norm relations amounts to verifying certain congruence conditions. One may of course use the finer structure of the connected components of a Shimura variety as prescribed by the reciprocity laws of [Del71]. We however point out that for the case considered in [GS21], working with the trivial functor turns out to be necessary as the failure of axiom (SD3) precludes the possibility of describing the geometric connected components of the source Shimura variety.

We will study zeta elements for groups G that are product of two groups, one of which is abelian and it would be useful to record some auxiliary results that would be helpul in applying the criteria to such group. Suppose for the rest of this section only that $G = G_1 \times T$ where T is abelian with a unique maximal compact subgroup C. Suppose also that $K = K_1 \times C$, $L = K_1 \times D$ where $K_1 \subset G_1$, $D \subset C$[3] and that

$$\mathfrak{H}^t = \sum_{\kappa \in I} e_\kappa \operatorname{ch}(K \gamma_\kappa \phi_\kappa C) \in \mathcal{C}_{\mathcal{O}}(K \backslash G/K)$$

where $e_\kappa \in \mathcal{O}$, $\gamma_\kappa \in G_1$ and $\phi_\kappa \in T$. Let $\iota_1 : H \to G_1$, $\nu : H \to T$ denote the compositions $H \xrightarrow{\iota} G \to G_1$, $H \to G \to T$ respectively. We suppose that ι_1 is injective, so we may consider H, U as a subgroup of G_1 as well as G. When we consider H, U as subgroups of G_1, we denote them by H_1, U_1 respectively.

Lemma 3.2.7. *Suppose that $K_1 \gamma_\kappa K_1 = \bigsqcup_{j \in J_\kappa} U_1 \sigma_j K_1$ where J_κ is an indexing set and $\sigma_j \in G_1$. Denote $\sigma_{j,\kappa} = \sigma_j \phi_\kappa$ and $H_{1,\sigma_j} = H_1 \cap \sigma_j K_1 \sigma_j^{-1}$ Then*

(a) $\mathfrak{H}^t = \sum_{\kappa \in I} \sum_{j \in J_\kappa} e_\kappa \operatorname{ch}(U \sigma_{j,\kappa} K)$.

[3] so $d = [C : D]$

(b) $\deg [U\sigma_{j,\kappa}K]_* = \deg [U_1\sigma_j K_1]_*$,

(c) $[H \cap \sigma_{j,\kappa}K\sigma_{j,\kappa} : H \cap \sigma_{j,\kappa}L\sigma_{j,\kappa}^{-1}] = [H_{1,\sigma_j} : H_{1,\sigma_j} \cap \nu^{-1}(D)]$.

Proof. Since ν is continuous and C is the unique maxixmal compact subgroup of T, the image under ν of any compact subgroup of H is contained in C. For (a), it suffices to note that

$$K_1\gamma_k K_1 = \bigsqcup_{j\in J_\kappa} U_1\sigma_\kappa K_1 \implies K\gamma_\kappa\phi_\kappa K = \bigsqcup_{j\in J_\kappa} U\sigma_j\phi_\kappa K$$

since $K\gamma_\kappa K = K_1\gamma_\kappa K_1 \times \phi_\kappa C$ and $\nu(U) \subset C$. For (b), note that $H \cap \sigma_{j,\kappa}K\sigma_{j,\kappa}^{-1} = H \cap \sigma_j K\sigma_j^{-1}$ as T is abelian. Since $H_1 \cap \sigma_j K_1\sigma_j^{-1}$ is compact, $\nu(H_1 \cap \sigma_j K_1 \sigma_j^{-1}) \subset C$ and therefore

$$H \cap \sigma_{j,\kappa}K\sigma_{j,\kappa}^{-1} = \iota_1^{-1}\big(\sigma_j K_1 \sigma_j^{-1}\big) \cap \nu^{-1}(C) = H_1 \cap \sigma_j K_1 \sigma_j^{-1}.$$

Similarly $U \cap \sigma_j\phi_\kappa K(\sigma_j\phi_\kappa)^{-1} = U_1 \cap \sigma_j K_1\sigma_j^{-1}$ and (b) follows. The argument for (c) is similar. □

3.3 Handling torsion

We now address the equality of norm relation asked for in Problem 3.1.1 without forgoing torsion. The set up is the same as in start of §3.1 and notations that appear before Proposition 3.2.1 are maintained i.e. $\mathfrak{H}^t = \sum_{j\in J} c_j \operatorname{ch}(U_j\sigma_j K), H_\sigma, U_\sigma$ denote $H \cap \sigma K\sigma^{-1}, U \cap \sigma K \sigma^{-1}$ respectively for $\sigma \in G$, $J = J_1 \sqcup \ldots \sqcup J_r$ is a partition such that for $j, j' \in J_i \implies H\sigma_j = H\sigma_{j'}, g_i$ are arbitrary but fixed elements in the coset $H\sigma_j$ for any $j \in J_i$, $h_j = g_i\sigma_j^{-1} \in H$, $V_i = H \cap g_i L g_i^{-1}$, and $d_i = [H_{g_i} : V_i]$

Proposition 3.3.1. *Suppose that*

- ι_* *is Mackey,*
- *there exists* $x_{V_i} \in M_{H,\mathcal{O}}(V_i)$ *such that* $\sum_{j\in J_i} c_j [H_{g_i}h_jU](x_U) = \operatorname{pr}_{V_i,H_{g_i},*}(x_{V_i})$ *as elements of* $M_{H,\mathcal{O}}(H_{g_i})$ *for all* $i = 1, \ldots, r$

Then $y_L = \sum_{i=1}^r [V_i g_i L]_*(x_{V_i})$ *is such that* $\mathfrak{H}(y_K) = \operatorname{pr}_{L,K,*}(y_L)$.

Proof. Since ι_* is Mackey, $\mathfrak{H}(y_K) = \sum_{j\in J}[U\sigma_j K]_*(x_U) = \sum_{i=1}^r \sum_{j\in J_i} c_j [U\sigma_j K]_*(x_U)$ by Lemma 2.5.6 Let $W_j := h_j U_{\sigma_j} h_j^{-1} = h_j U h_j^{-1} \cap g_i K g_i^{-1}$. Then

$$\begin{aligned}
\mathfrak{H}(y_K) &= \sum_{i=1}^r \sum_{j\in J_i} c_j \left[U_{\sigma_j}\sigma_j K\right]_* \circ \operatorname{pr}^*_{U_{\sigma_j},U}(x_U) \\
&= \sum_{i=1}^r \sum_{j\in J_i} c_j [W_j g_i K]_* \circ [h_j]^*_{W_j,U}(x_U) \\
&= \sum_{i=1}^r \sum_{j\in J_i} c_j [H_{g_i} g_i K]_* \circ \operatorname{pr}_{W_j,H_{g_i},*} \circ [h_j]^*_{W_j,U}(x_U)
\end{aligned}$$

Now note that $U \cap h_j^{-1} H_{g_i} h_j = U_{\sigma_j}$, $h_j U h_j^{-1} \cap H_{g_i} = W_j$ which gives us that $\mathrm{pr}_{W_j, H_{g_i}, *} \circ [h_j]^*_{W_j, U}(x_U) = [H_{g_i} h_j U](x_U)$.

$$\begin{array}{ccccc} & M_{H,\mathcal{O}}(U_{\sigma_j}) & \xrightarrow{[h_j]^*} & M_{H,\mathcal{O}}(W_j) & M_{H,\mathcal{O}}(V_i) \\ \nearrow^{\mathrm{pr}^*} & & & \searrow^{\mathrm{pr}_*} & \downarrow \mathrm{pr}_* \\ M_{H,\mathcal{O}}(U) & & \xrightarrow[{[H_{g_i} h_j U]}]{} & & M_{H,\mathcal{O}}(H_{g_i}) \end{array}$$

Therefore,

$$\begin{aligned} \mathfrak{H}(y_K) &= \sum_{i=1}^{r} [H_{g_i} g_i K]_* \left(\sum_{j \in J_i} c_j [H_{g_i} h_j U](x_U) \right) \\ &= \sum_{i=1}^{r} [H_{g_i} g_i K]_* \circ \mathrm{pr}_{V_i, H_{g_i}, *}(x_{V_i}) \\ &= \sum_{i=1}^{r} \mathrm{pr}_{L,K,*} \circ [V_i g_i L]_*(x_{V_i}) = \mathrm{pr}_{L,K,*}(y_L) \end{aligned}$$

the second equality following by our assumption on x_{V_i}. □

Corollary 3.3.2. *Suppose that*

- $M_{H,\mathcal{O}}$ *is torsion free*
- ι_* *is Mackey.*

If there exists a uniform zeta element ζ *for* $(x_U, \mathfrak{H}, L)$*, the class* y_L *associated with a uniform data for* ζ *and* ι_* *satisfies* $\mathfrak{H}(y_K) = \mathrm{pr}_{L,K,*}(y_L)$*. If* $\mathfrak{H}' \in \mathcal{C}_{\mathcal{O}}(K \backslash G / K)$ *is any element such that* $\mathfrak{H} - \mathfrak{H}' \in d \cdot \mathcal{C}_{\mathcal{O}}(K \backslash G / K)$*, then a uniform zeta element exists for* $(x_U, \mathfrak{H}, L)$ *if and only if such an element exists for* $(x_U, \mathfrak{H}', L)$*.*

Proof. By Lemma 3.1.7(i) and (iii), we may assume that the $\zeta = \sum_{i=1}^{r} b_i (x_i \otimes g_i K)$ for some $b_i \in \Phi$, $x_i \in \hat{M}_{H,\Phi}$. We may assume that $b_i = \mu_H(U)/\mu_H(V_i)$ by replacing x_i with a multiple if necessary. Let $\theta_i = \sum_{\gamma \in H_{g_i}/V_i} \gamma$ and $x_{V_i} \in M_{H,\mathcal{O}}(V_i)$ be the classes mapping to x_i. Since $M_{H,\mathcal{O}}$ is Mackey, $j_{H_{g_i}}([H_{g_i} h_j U](x_U) = \varsigma_j h_j \cdot j_U(x_U)$. As $\eta_i x_i = b_i(\theta_i/d_i) x_i$ by (Z4), we have

$$j_{H_{g_i}} \Big(\sum_{j \in J_i} c_j [H_{g_i} h_j U](x_U) \Big) = j_{H_{g_i}} \circ \mathrm{pr}_{H_{g_i}, V_i, *}(x_{V_i})$$

by the choice of η_i. Since $M_{H,\mathcal{O}}$ is cohomological and torsion free, $j_{H_{g_i}} : M_{H,\mathcal{O}}(H_{g_i}) \to \widehat{M}_{H,\Phi}$ is injective, and therefore $\sum_{j \in J_i} c_j [H_{g_i} h_j U](x_U) = \mathrm{pr}_{V_i, H_{g_i}, *}(x_{V_i})$, whence by Proposition 3.3.1, $\mathfrak{H}(y_K) = \mathrm{pr}_{L,K,*}(y_L)$. If $\mathfrak{H}' = \sum_j c'_j \mathrm{ch}(U \sigma_j K)$ and $c'_j - c_j \in d_i \mathcal{O}$ for all $j \in J_i$, $i = 1, \ldots, r$, then $\sum_{j \in J_i} c_j [H_{g_i} h_j U](x_U)$ is in the image of $\mathrm{pr}_{V_i, H_{g_i}, *}$ if and only if $\sum_{j \in J_i} c'_j [H_{g_i} h_j U](x_U)$ is, since $M_{H,\mathcal{O}}$ is cohomological. We can therefore take $\eta'_i = \mu_H(H_{g_i})/\mu_H(U) \sum_{j \in J_i} c'_j \varsigma_j h_j$ and show that $\sum_{i=1}^{r} b_i (x'_i \otimes g_i K)$ is a trace-like, optimal zeta element for $(x_U, \mathfrak{H}', L)$ for some x'_i, thus obtaining the claimed equality. □

Remark 3.3.3. In applications to Shimura varieties, one eventually projects the norm relations to a π_f-isotypical component of the cohomology of the target Shimura variety, where π_f is (the finite part of) an irreducible cohomological automorphic representation of the target reductive group, in order to land in the first Galois cohomology H^1 of a Galois representation ρ_π in the multiplicity space of π. The projection step, to our knowledge, requires the coefficients to be in a field. Thus the information about torsion is lost anyway i.e. one apriori obtains norm relations in the *image* of H^1 of a Galois stable lattice $T_\pi \subset \rho_\pi$ inside H^1 of the Galois representation ρ_π. One way to retrieve the torsion in the norm relation after projecting to Galois representation is to to use Iwasawa theoretic arguments e.g. see [GS21], [LSZ22]. We will however not address this question here.

3.4 Schwartz spaces

In the problems studied in Part II, the source functor M_H will be modelled on certain function spaces. In this section, we derive a criteria for determining when elements in such functors are in the image of trace maps.

Let H be locally profinite group and X a locally compact Hausdorff totally disconnected space endowed with a continuous left H-action $H \times X \to X$. By definition, X carries a basis of compact open subsets. For a ring R, we denote by $\mathcal{S}_R(X)$ the R-module of locally constant compactly supported functions valued in R. Under the left translation action on functions, $\mathcal{S}_R(X)$ becomes a smooth left representation of H. In what follows, we will frequently use the following fact: the set of all compact open subsets of X is closed under finite unions, finite intersection and relative complements. Moreover, if $U \subset H$ is a compact open subgroup, then the set of compact open subsets that are invariant under U is such a collection as well.

Definition 3.4.1. Let $W, V \subset H$ be compact open subgroups with $V \subset W$. We say that $x \in X$ is (W, V)-*smooth* if there exist a V-invariant compact open neighbourhood Z of x such that the γZ are distinct open subsets for $\gamma \in W/V$ and are pairwise disjoint. A W-invariant compact open subset $Y \subset X$ is said to be (W, V)-*smooth* if $Y = \bigsqcup_{\gamma \in W/V} Y_\gamma$ such that Y_{id} is V-invariant compact open subsets of X and $Y_\gamma = \gamma Y_{\mathrm{id}}$ for all $\gamma \in W/V$.

If $x \in X$ is (W, V)-smooth, it is contained in a (W, V)-smooth neighbourhood all points of which are then (W, V)-smooth. The converse is also true in compact open neighbourhoods.

Lemma 3.4.2. *Suppose that $S \subset X$ is a W-invariant compact open subset such that all points in S are (W, V)-smooth. Then S is (W, V)-smooth.*

Proof. It suffices to show that S is a finite disjoint union (W,V)-smooth compact open subsets. For all $x \in S$, let Z_x denote the V-invariant compact open neighbourhood of x satisfying the smoothness condition. Since S is a W-invariant compact open, $S \cap Z_x$ is a V-invariant compact open. We can therefore assume that $Z_x \subset S$ for all $x \in S$ and that $Y_x := \bigsqcup_{\gamma \in W/V} \gamma Z_x \subset S$. Since S is compact and $S = \bigcup_{x \in S} Y_x$, we have $S = \bigcup_{i=1}^n Y_i$ where Y_i form a finite subcollection of Y_x. Let $Z_i \subset Y_i$ denote the corresponding V-invariant compact open. Thus S is a finite union of n of (W,V)-smooth compact open subsets of X. We proceed by induction on n. If $n = 1$, we are done. For $n > 1$, let $S' := S - Y_1$, $Y_i' := Y_i - (Y_i \cap Y_1)$ and $Z_i' = Z_i - (Z_i \cap Y_1)$. Since Y_1, S & Y_i are W-invariant compact open subsets of X, S', Y_i' are as well. Similarly, $Z_i' \subset Y_i'$ are V-invariant compact open subsets of X contained in Y_i'. Moreover Y_i' are (W,V)-smooth:

$$Y_i = \bigsqcup_{\gamma \in W/V} \gamma Z_i \implies Y_i' = \bigsqcup_{\gamma \in W/V} \gamma Z_i - (\gamma Z_i \cap Y_1) = \bigsqcup_{\gamma \in W/V} \gamma Z_i'$$

Thus S' is a union of $n-1$ smooth (w.r.t. (W,V)) compact open subsets of X. By induction hypothesis, S' can be written as a finite disjoint union of such subsets and therefore so can $S = S' \sqcup Y_1$. □

We have the following criteria for checking (W,V)-smoothness of point.

Lemma 3.4.3. *A point x is (W,V)-smooth if and only if* $\mathrm{Stab}_W(x) \subset V$.

Proof. The only if direction is clear, so assume that $\mathrm{Stab}_W(x) \subset V$. Let $U \subset V$ be a compact open subgroup that is normal in W. For $\sigma \in W$, let $C_\sigma := U\sigma x$ denote the U-orbit of σx. By continuity of $H \times X \to X$, C_σ are compact and therefore closed in X. Since $U \triangleleft W$, $C_\sigma = \sigma U x$ and W/U acts transitively on the orbit space $\{C_\sigma \mid \sigma \in W\}$ via $(\tau U, C_\sigma) \mapsto C_{\tau\sigma}$. Let U° denote the inverse image in W under $W \twoheadrightarrow W/U$ of the stabilizer of C_{id} under this action. Note that the condition on $\mathrm{Stab}_W(x)$ implies that $U^\circ \subset V$. It therefore suffices to show that x is (W,U°)-smooth. Let $\gamma_1, \ldots, \gamma_n \in W$ be a set of representatives for W/U°, $\delta_1, \delta_2, \ldots, \delta_m \in U^\circ$ be a set of representatives for U°/U and let $C_i = C_{\gamma_i}$. Then C_i for $i = 1, \ldots, n$ are pairwise disjoint and $\gamma_i \delta_j \gamma_i^{-1} C_i = C_i$. For any compact open neighbourhood T of x, $X' := WT$ is a compact open subset of X that contains C_i for all i. Since X' is compact Hausdorff, it is normal and we may therefore choose compact open subsets S_i of X'[4] such that S_i contains C_i and $S_1, \ldots, S_k$ are pairwise disjoint. For each $k = 1, \ldots n$, $\ell = 1, \ldots, m$, let $\delta_{\ell,k} := \gamma_k \delta_\ell \gamma_k^{-1}$ and

$$Z_{k,\ell} := \delta_{\ell,k} U S_k - \bigcup_{i \neq k} \bigcup_{j=1}^{m} \delta_{j,i} U S_i$$

Since $\delta_{j,i} U S_j = U \delta_{j,i} S_j$ and $\{U \delta_{j,i} S_i \mid j = 1, \ldots, m, i = 1, \ldots, n\}$ is a collection of U-invariant compact open subsets, $Z_{k,\ell}$ are U-invariant compact opens neighbourhoods as well. By construction, $Z_{k,\ell}$ intersects $Z_{k',\ell'}$ if and only if $k = k'$. Moreover, we claim that $\gamma_k \delta_\ell x \in Z_{k,\ell}$. Suppose for the sake of a contradiction that

[4] and therefore of X

$\gamma_k\delta_\ell x \in \delta_{j,i}US_i$ for $i \neq k$. Then $U\delta_{i,j}^{-1}\gamma_k\delta_\ell x = \delta_{i,j}^{-1}\gamma_k Ux = C_{\delta_{i,j}^{-1}\gamma_k}$ intersects S_i. As C_{γ_i} is the only element in $\{C_\sigma \,|\, \sigma \in W\}$ contained in S_i, this can only happen if $\delta_{i,j}^{-1}\gamma_k Ux = \gamma_i Ux$. But this means that $\gamma_k Ux = \gamma_i Ux$, whence $i = k$, a contradiction. We therefore see that

$$Z := \bigcap_{k=1}^{n}\bigcap_{\ell=1}^{n} \delta_\ell^{-1}\gamma_k^{-1} Z_{k,\ell}$$

is a U-invariant compact open neighbourhood of x such that $\gamma_k\delta_\ell Z$, $\gamma_{k'}\delta_{\ell'} Z$ are disjoint for any $1 \leq \ell_1, \ell_2 \leq m$, $1 \leq k, k' \leq n$ with $k \neq k'$, since $\gamma_k\delta_\ell Z \subset Z_{k,\ell}$, $\gamma_{k'}\delta_{\ell'} \subset Z_{k',\ell'}$. If we now let $Z^\circ := \bigcup_{\ell=1}^{n} \delta_\ell Z$, then Z° is U°-invariant such that $\gamma_1 U^\circ, \ldots, \gamma_n U^\circ$ are pairwise disjoint, whence x is (W, U°)-smooth. □

For each $x \in X$, we let V_x denote the subgroup of W generated by V and $\mathrm{Stab}_W(x)$. By Lemma 3.4.3 V_x is the unique smallest subgroup of W such that x is (W, V_x)-smooth. Let $\mathcal{U}$ be the lattice of subgroups of W that contain V. For $\mathcal{T} \subset \mathcal{U}$ a sub-collection, we denote by $\max(\mathcal{T})$ the set of maximal elements of $\mathcal{T}$ i.e. $U \in \max \mathcal{T}$ if no $U' \in \mathcal{T}$ properly contains U. We have a filtration

$$\mathcal{U} = \mathcal{U}_0 \supsetneq \mathcal{U}_1 \supsetneq \ldots \supsetneq \mathcal{U}_N = \{V\}$$

defined inductively as $\mathcal{U}_{k+1} := \mathcal{U}_k - \max \mathcal{U}_k$ for $k = 0, \ldots, N-1$. We let $\mathrm{dep} : \mathcal{U} \to \{0, \ldots, N\}$ be the function $U \mapsto k$ if k is the largest integer such that $U \in \mathcal{U}_k$ i.e. $U \in \max \mathcal{U}_k$. We let

$$\begin{aligned} \mathrm{dep} = \mathrm{dep}_{W,V} : X &\to \{1, 2, \ldots, N\} \\ x &\mapsto \mathrm{dep}(V_x) \end{aligned}$$

and refer to $\mathrm{dep}(x)$ as the *depth of* x. We say that $S \subset X$ *has depth* k if $\inf\{\mathrm{dep}(x) \,|\, x \in S\} = k$.

Lemma 3.4.4. *If $S \subset X$ has depth k, then the set of depth k points in S is closed in S.*

Proof. Let $T \subset S$ be the set of depth k points. For $x \in S - T$, choose Y_x a (W, V_x)-smooth neighbourhood of x in X. Then for all $y \in Y_x$, $V_y \subseteq V_x$, whence $\mathrm{dep}(y) \geq \mathrm{dep}(x) > k$. Therefore, $Y_x \cap S \subset S - T$, whence $Y_x \cap S$ is an open (relative to S) neighbourhood of x contained in $S - T$. As x was arbitrary, $S - T$ is open in S, whence T is closed. □

Proposition 3.4.5. *Suppose R is an integral domain and let $\phi \in \mathcal{S}_R(X)^W$. Then there exists $\psi \in \mathcal{S}_R(X)^V$ such that $\phi = \sum_{\gamma \in W/V} \gamma \cdot \psi$ if and only if for all $x \in \mathrm{Supp}(\phi)$, $\phi(x) \in [V_x : V]R$.*

Proof. ($\implies$) Let $\psi \in \mathcal{S}_R(X)^V$ be an element satisfying the trace condition. For $x \in X$, let V_x be as above, $\delta_1, \ldots, \delta_m \in V_x$ be a set of representatives of V/V_x and $\gamma_1, \ldots, \gamma_n \in W$ be a set of representatives for W/V_x. By definition, $Vx = V_x x$. Thus

$$\phi(x) = \sum_{i,j} \psi(\gamma_i^{-1}\delta_j^{-1}x) = \sum_i [V_x : V]\psi(\gamma_i^{-1}x) \in [V_x : V]R$$

($\Longleftarrow$) Set $S := \operatorname{Supp}\phi$. By definition S is a W-invariant compact open subset X. We inductively define a partition $S = S_0 \sqcup \cdots \sqcup S_N$ by W-invariant compact open subsets such that all depth k points in S are contained in $\bigcup_{i=0}^{k} S_k$ and $\phi(S_k) \in [U : V]R$ for some $V \in \mathcal{U}_k$. We provide the inductive step and highlight the base case $k = 0$ in the process. Assume $0 \leq k \leq N - 1$. Given $S_0, \ldots, S_{k-1}$, let T_k be the (possibly empty) set of depth k points in $R_k := S - \bigsqcup_{i=0}^{k-1} S_i$ where $R_k = S$ if $k = 0$. By construction, R_k is W-invariant compact open subset of S and depth of R_k is at least k. By Lemma 3.4.4, $T_k \subset R_k$ is closed, whence compact. For each $x \in T_k$, let $Y_x \subset R_k$ be a (W, V_x)-smooth neighbourhood of x such that ϕ is constant on Y_x. Then $T_k \subset \bigsqcup_{x \in T_k} Y_x$ has a finite subcover, whose union we take to be S_k. Clearly S_k is W-invariant compact open since Y_x are and depth k points of R_k are in S_k, whence depth k points of S are in $\bigcup_{i=0}^{k} S_k$. Our assumption on $\phi(x)$ for $x \in T_k$ and local constancy of $\phi \in Y_x$ implies that $\phi(S_k) \in [U : V]$ with $\operatorname{dep} U = k$. This completes the inductive step.

Now for each $0 \leq k \leq N$, we may partition $S_k = \bigsqcup_{U \in \mathcal{U}_k} S_U$ by compact open subsets such that each point $x \in S_U$ is (W, U)-smooth since any point S_k is (W, U)-smooth for a unique $U \in \mathcal{U}_k$. We therefore obtain a partition $S = \bigsqcup_{U \in \mathcal{U}} S_U$ such that $\phi(S_U) \in [U : V]R$ and each $x \in S_U$ is (W, U)-smooth. By Lemma 3.4.2, there are U-invariant compact open subsets $Z_U \subset X$ such that $S_U = \bigsqcup_{\gamma \in W/U} \gamma Z_U$. We define $\psi(x)$ to be $[U : V]^{-1}\phi(x)$ if $x \in Z_U$ and zero elsewhere. Then ψ is easily seen to be the desired element. □

3.5 Gluing zeta elements

In this section, we model the situation of taking the tensor product of two componentwise zeta elements when the various data at hand is given as a product of two analogous ones. One may extend this situation to any finite number of components.

Suppose for all of this section only that in the set up of §3.1, $G = G_\alpha \times G_\beta, H = H_\alpha \times H_\beta$, and for $\ell \in \{\alpha, \beta\}$,

- $\mu_H = \mu_{H_\alpha} \times \mu_{H_\beta}$ where μ_{H_ℓ} are Haar measures on H_ℓ,
- $\iota = \iota_\alpha \times \iota_\beta$ where $\iota_\ell : H_\ell \to G_\ell$ are embeddings
- $K = K_\alpha \times K_\beta$ where $K_\ell \subset G_\ell$
- $U = U_\alpha \times U_\beta$ where $U_\ell \subset H_\ell$,
- $L = L_\alpha \times L_\beta, L_\ell \subset K_\ell$.
- $\Upsilon_{H_\ell}, \Upsilon_{G_\ell}$ pullback of Υ_H, Υ_G to H_ℓ, G_ℓ respectively
- $\mathfrak{H} = \mathfrak{H}_\alpha \otimes \mathfrak{H}_\beta$ where $\mathfrak{H}_\ell \in \mathcal{H}_{\mathcal{O}}(K_\ell \backslash G_\ell / K_\ell)$

Abusing notation, we denote by $\mathfrak{H}_\alpha$ the $\mathcal{O}$-linear map $\mathfrak{H}_\alpha \otimes \mathrm{ch}(K'_\beta) : M_{G,\mathcal{O}}(K_\alpha \times K'_\beta) \to M_{G,\mathcal{O}}(K_\alpha \times K'_\beta)$ for any fixed $K'_\beta \in \Upsilon_{G_\beta}$, and similarly for $\mathfrak{H}_\beta$. Let

$$M_{H_\alpha,\mathcal{O}}(-) = M_H\left(- \times U_\beta\right), \quad M_{G_\alpha,\mathcal{O}}(-) = M_G\left(- \times K_\beta\right)$$

be the functors on $\Upsilon_{H_\alpha}, \Upsilon_{G_\alpha}$ respectively obtained by restriction and define $M_{H_\beta,\mathcal{O}}, M_{G_\beta,\mathcal{O}}$ similarly. Consider U_ℓ, K_ℓ as bottom levels of M_{H_ℓ}, M_{G_ℓ} and $x_{U_\ell} := x_U$ as the bottom class. To ease notation, we omit the multiplication signs from above i.e. we write $K_\alpha L_\beta$ instead of $K_\alpha \times L_\beta$ etc. By Lemma 2.4.3, $\mathfrak{H} = \mathfrak{H}_\alpha \circ \mathfrak{H}_\beta = \mathfrak{H}_\beta \circ \mathfrak{H}_\alpha$ as endomorphisms on $M_G(K_\alpha K_\beta)$. We let $\mathcal{M}_\ell$ denote the Φ-vector spaces for $\ell = \alpha, \beta$ defined before Definition 3.1.4.

Definition 3.5.1. Suppose $\zeta_\ell = \sum_{i=1}^{r_\ell} b_{i,\ell}\left(x_{i,\ell} \otimes g_{i,\ell} K_\ell\right)$ are zeta elements for $(x_{U_\ell}, \mathfrak{H}_\ell, \mathcal{L}_\ell)$, with $\eta_{i,\ell}, V_{i,\ell}$, $x_{V_{t,\ell}}$ and $\theta_{i,\ell}$ the associated data such that (Z1)-(Z4) are satisfied. We say that (the chosen data of) $\zeta_\alpha, \zeta_\beta$ are *compatibly parametrized* if the following conditions are satisfied:

- there exists functors $N_\ell : H_\ell \to \mathcal{O}$-Mod, a morphism $\varphi : N_\alpha \otimes N_\beta \to M_{H,\mathcal{O}}$ and classes $\phi_{V_{i,\ell}} \in N_{H_\ell}\left(V_{i,\ell}\right), \phi_{U_\ell} \in N_{H_\ell}\left(U_\ell\right)$ for $i = 1, \ldots, r_\ell, \ell = \alpha, \beta$ such that

  $$\varphi\left(\phi_{U_\alpha} \otimes \phi_{U_\beta}\right) = x_U, \quad \varphi\left(\phi_{V_{i,\alpha}} \otimes \phi_{U_\beta}\right) = x_{V_{i,\alpha}}, \quad \varphi\left(\phi_{U_\alpha} \otimes \phi_{V_{j,\beta}}\right) = x_{V_{j,\beta}}$$

  i.e. the collection $\left\{x_{V_{i,\ell}}, x_U\right\}$ is compatibly parametrized (Definition 2.6.8) by $\left\{\phi_{V_{i,\ell}}, \phi_{U_\ell}\right\}$.

- for any $\phi_\alpha \in \widehat{N}_{\alpha,\Phi}, \phi_\beta \in \widehat{N}_{\beta,\Phi}$

  $$\eta_{i,\alpha} \cdot \widehat{\varphi}\left(\hat{\phi}_{U_\alpha} \otimes \phi_\beta\right) = \frac{b_{i,\alpha}\theta_{i,\alpha}}{\deg\left(\theta_{i,\alpha}\right)} \cdot \widehat{\varphi}\left(\hat{\phi}_{V_{i,\alpha}} \otimes \phi_\beta\right), \quad \eta_{j,\beta} \cdot \widehat{\varphi}\left(\phi_\alpha \otimes \hat{\phi}_{U_\beta}\right) = \frac{b_{j,\beta}\theta_{j,\beta}}{\deg\left(\theta_{j,\beta}\right)} \cdot \widehat{\varphi}\left(\phi_\alpha \otimes \hat{\varphi}_{V_{j,\beta}}\right)$$

  where $\widehat{\varphi} : \widehat{N}_{\alpha,\Phi} \otimes \widehat{N}_{\beta,\Phi} \to M_{H,\Phi}$ denotes the induced map and $\hat{\phi}_{U_\ell}, \hat{\phi}_{V_{i,\ell}}$ denote the images of ϕ_{U_ℓ}, $\phi_{V_{i,\ell}}$ respectively in $\widehat{N}_{H,\Phi}$.

Lemma 3.5.2. *Suppose that there exist compatibly parametrized zeta elements ζ_ℓ for $\left(x_U, \mathfrak{H}_\ell, L_\ell\right)$. Then there exists a zeta element ζ for $\left(x_U, \mathfrak{H}, L\right)$. Moreover, if $y_{L_\alpha} \in M_G\left(L_\alpha K_\beta\right), y_{L_\beta} \in M_G\left(K_\alpha L_\beta\right), y_L \in M_{G,\mathcal{O}}(L)$ are the classes associtead to $\zeta_\alpha, \zeta_\beta, \zeta$ respectively, then the classes*

- $\mathfrak{H}_\beta\left(y_{L_\alpha}\right) - \mathrm{pr}_{L,L_\alpha K_\beta,*}\left(y_L\right) \in M_{G,\mathcal{O}}\left(L_\alpha K_\beta\right)$,
- $\mathfrak{H}_\alpha\left(y_{L_\beta}\right) - \mathrm{pr}_{L,K_\alpha L_\beta,*}\left(y_L\right) \in M_{G,\mathcal{O}}\left(K_\alpha L_\beta\right)$

are $\mathcal{O}$-torsion.

Proof. Suppose that $\zeta_\ell = \sum_{i=1}^{r_\ell} b_{i,\ell}\left(x_{i,\ell} \otimes g_{i,\ell} K_\ell\right)$ and $\eta_{i,\ell} \in \Phi\left[H_\ell\right], V_{i,\ell} \subset g_{i,\ell} L_\ell g_{i,\ell}^{-1} \cap H_\ell, x_{V_{i,\ell}} \in M_{H_\ell,\mathcal{O}}\left(V_{i,\ell}\right)$ and $\theta_{i,\ell} \in \Phi\left[H_\ell\right]$ objects associated to ζ_ℓ by $(Z1) - (Z4)$. Let $N_\ell : H_\ell \to \mathcal{O}$-Mod, $\varphi : N_\alpha \otimes N_\beta \to M_{H,\mathcal{O}}$ and $\phi_{U_\ell} \in N_\ell\left(U_\ell\right), \phi_{V_{i,\ell}} \in N_\ell\left(V_{i,\ell}\right)$ be the data of parametrization. For $i = 1, \ldots, \ldots r_\alpha, j = 1, \ldots, r_\beta$

- $g_{i,j} = (g_{i,\alpha}, g_{j,\beta}) \in G$,
- $b_{i,j} = b_{i,\alpha} b_{j,\beta}$,
- $\eta_{i,j} = \eta_{i,\alpha} \cdot \eta_{j,\beta} \in \Phi[H]$,
- $V_{i,j} = V_{i,\alpha} V_{j,\beta}$,
- $x_{V_{i,j}} = \varphi\left(\phi_{i,\alpha} \otimes \phi_{j,\beta}\right) \in M_{H,\mathcal{O}}\left(V_{i,j}\right)$,
- $\theta_{i,j} = \theta_{i,\alpha}\theta_{j,\beta} \in \Phi[H]$
- $x_{i,j} = j_{V_{i,j}}\left(x_{V_{i,j}}\right) \in \widehat{M}_{H,\Phi}$
- $a_{i,j} = b_{i,j}\mu_H\left(V_{i,j}\right)/\mu_H(U)$

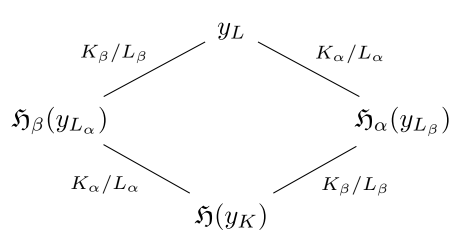
We claim that $\zeta = \sum_{i,j} b_{i,j}\left(x_{i,j} \otimes g_{i,j}K\right)$ is a zeta element. (Z1) holds for ζ since $\mathcal{M}_\alpha \otimes \mathcal{M}_\beta \cong \mathcal{M}$ via $(h_\alpha U_\alpha \otimes g_\alpha K_\alpha) \otimes (h_\beta U_\beta \otimes g_\beta K_\beta) \mapsto h_\alpha h_\beta U \otimes g_\alpha g_\beta K$ and $U \otimes \mathfrak{H}^t = (U_\alpha \otimes \mathfrak{H}^t_\alpha) \otimes \left(U_\beta \otimes \mathfrak{H}^t_\beta\right)$, so one can apply the operations of ζ_ℓ componentwise to arrive at $\sum_{i,j} \eta_{i,j} U \otimes g_{i,j}K$. (Z2) holds since

$$a_{i,j} = b_{i,\alpha}\mu_{H_\alpha}\left(V_{i,\alpha}\right)/\mu_{H_\alpha}\left(U_\alpha\right) \cdot b_{j,\beta}\mu_{H_\beta}\left(V_{j,\beta}\right)/\mu_{H_\beta}\left(U_\beta\right) \in \mathcal{O}$$

(Z3) holds by definition. Finally, (Z4) holds since

$$\begin{aligned}
\eta_{i,j} \cdot j_U\left(x_U\right) &= \eta_{i,\alpha}\eta_{j,\beta} \cdot \widehat{\varphi}\left(\hat{\phi}_{U_\alpha} \otimes \hat{\phi}_{U_\beta}\right) \\
&= b_{i,\alpha}b_{j,\beta}\left(\theta_{i,\alpha}/\deg\left(\theta_{i,\alpha}\right)\right) \cdot \left(\theta_{j,\beta}/\deg\left(\theta_{j,\beta}\right)\right) \cdot \widehat{\varphi}\left(\hat{\phi}_{V_{i,\alpha}} \otimes \hat{\phi}_{V_{j,\beta}}\right) \\
&= b_{i,j}\left(\theta_{i,j}/\deg\left(\theta_{i,j}\right)\right) \cdot j_{V_{i,j}}\left(x_{V_{i,j}}\right) = b_{i,j}\left(\theta_{i,j}/\deg\left(\theta_{i,j}\right)\right) \cdot x_{i,j}
\end{aligned}$$

where $\widehat{\varphi} : \widehat{N}_{\alpha,\Phi} \otimes \widehat{N}_{\beta,\Phi} \to \widehat{M}_{H_\ell,\Phi}$ denotes the the map induced by φ and $\hat{\phi}_{U_\alpha}$ denotes the image of ϕ_{U_α} in the limit etc.



Now let $y_L = \sum_{i,j} a_{i,j}\left[V_{i,j}g_{i,j}L\right]_*\left(x_{V_{i,j}}\right)$ be the class associated to ζ. Then

$$\begin{aligned}
j_{L_\alpha K_\beta}\left(\mathfrak{H}_\beta y_{L_\alpha}\right) &= \sum_{i=1}^{r_\alpha} b_{i,\alpha} \cdot \left(\theta_{i,\alpha}/\deg\left(\theta_{i,\alpha}\right)\right)\hat{\imath}_*\left(\widehat{\varphi}\left(\hat{\phi}_{V_{i,\alpha}} \otimes \hat{\phi}_{U_\beta}\right) \otimes \left(\mathfrak{H}_\beta \cdot \mathrm{ch}\left(g_{i,\alpha}K\right)\right)\right) \\
&= \sum_{i,j} b_{i,j} \cdot \left(\theta_{i,j}/\deg\left(\theta_{i,j}\right)\right) \cdot \hat{\imath}_*\left(\widehat{\varphi}\left(\hat{\phi}_{V_{i,\alpha}} \otimes \hat{\phi}_{V_{j,\beta}}\right) \otimes \mathrm{ch}\left(g_{i,j}K\right)\right) \\
&= j_{L_\alpha K_\beta} \circ \mathrm{pr}_{L,L_\alpha K_\beta,*}\left(y_L\right)
\end{aligned}$$

which shows that $\mathfrak{H}_\beta\left(y_{L_\alpha}\right) - \mathrm{pr}_{L,L_\alpha K_\beta}\left(y_L\right)$ is $\mathcal{O}$-torsion. The proof for the other claim is similar. □

3.6 Two prototypical examples

The following are two warm up examples of zeta elements on modular curves. In both cases, the target group is not GL_2, but $\mathrm{GL}_2 \times \mathbf{T}$ where $\mathbf{T}$ is a torus. As also already pointed out, the reason is that our Hecke polynomials lie in the Hecke algebra of the latter and this will be a theme that will persist in higher dimensional examples. The presence of the torus is also in a sense to allow 'access' to the Frobenius action, which is apriori abstract but which is explicit (given by matrices) on the classes that we push by Shimura reciprocity laws and class field theory.

The first example is based on the split case of CM points on modular curves as expounded in §0.2. The main was point was also discussed in the 1.1.1. Here we are going to record for purposes of illustrate how the criteria derived in §3.2 apply. The second example is based off of Kato's Siegel units, and is worked out in [Col03, §1] in a language that more or less already fits our formalism. Here, we use the example to illustrate Definition 3.1.4 in gory detail.

3.6.1 CM points

Consider again the embedding $\iota : \mathbf{H} \to \mathrm{GL}_2$ of introduced in §0.1 Let U_1 be the torus over $\mathbb{Q}$ whose R points for a $\mathbb{Q}$-algebra R norm one elements of $(E \otimes R)^{\times}$. Recall that the norm map $(E \otimes R)^{\times}$ on simple tensors as $e \otimes r \mapsto e\gamma(e)r^2$ where $\gamma \in \mathrm{Gal}(E/F)$ denotes the non-trivial element. We will denote $\mathbf{T} := \mathrm{U}_1$. Next, let

$$\begin{aligned} \nu : \mathbf{H} &\to \mathbf{T} & \iota_\nu : \mathbf{H} &\to \mathrm{GL}_2 \times \mathbf{T} \\ h &\mapsto h\gamma(h)^{-1} & h &\mapsto (h, \nu(h)) \end{aligned}$$

Then ι_ν is also a Shimura datum. Let us note in passing that the reciprocity law (the map 0.1.7) for $\mathbf{T}$ corresponds to the so-called 'anticyclotomic' extensions. As one may also verify using this law, the behaviour of Frobenius in unramified subextensions of this extension is peculiar: the Frobenii at inert prime are trivial and the geometric Frobenii at one prime above a split prime of $\mathbb{Q}$ equals the inverse of the arithmetic Frobenius at the other.

Let G_f, H_f, T_f denote the $\mathbf{A}_f$ points of GL_2, $\mathbf{H}$, $\mathbf{T}$ respectively and let $\mathcal{G}_f := G_f \times T_f$. Let $\Upsilon_{\mathcal{G}_f}$ denote the collection of all neat compact open subgroups of $\mathcal{G}_f$ of the form $K \times C$ where $K \subset \mathcal{G}_f$, $C \subset T_f$ and Υ_{H_f} denote the collection of all neat compact open subgroups of H_f. Then the mappings

$$\begin{aligned} N_{\mathbb{Z}_p} : \Upsilon_{H_f} &\to \mathbb{Z}_p\text{-Mod} & M_{\mathbb{Z}_p} : \Upsilon_{\mathcal{G}_F} &\to \mathbb{Z}_p\text{-Mod} \\ U &\mapsto \mathrm{H}^0_{\text{ét}}\big(\mathrm{Sh}(U), \mathbb{Z}_p\big) & \mathcal{K} &\mapsto \mathrm{H}^2_{\text{ét}}\big(\mathrm{Sh}(\mathcal{K}) \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\, E, \mathbb{Z}_p(1)\big) \end{aligned}$$

that send each compact open subgroup to the corresponding arithmetic étale cohomology of the corresponding varieties over E constitute CoMack functors. The embedding $\iota : \mathbf{H} \to \mathbf{G}$ induces a pushforward $\iota : N_{\mathbb{Z}_p} \to M_{\mathbb{Z}_p}$ of RIC functors. For each U, $N_{\mathbb{Z}_p}(U)$ is the free $\mathbb{Z}_p$-module on the class of $1_{\mathrm{Sh}_H(U)}$ and $N_{\mathbb{Z}_p}$ is the trivial functor on Υ_{H_f}. The pushforward ι_* sends this class to the corresponding divisor class on the modular curve in $M_{\mathbb{Z}_p}$.

We pick a split prime ℓ, so that $\mathbf{H}_{\mathbb{Q}_\ell} \simeq \mathbb{G}_m \times \mathbb{G}_m$, $\mathbf{T}_{\mathbb{Q}_\ell} \simeq \mathbb{G}_m$. The isomorphisms are chosen so that $(h_1, h_2) \in \mathbf{H}_{\mathbb{Q}_\ell}$ map to $h_2/h_1 \in \mathbf{T}_{\mathbb{Q}_\ell}$. The particular choice is so that the action of uniformizer $\varpi \in \mathbb{Q}_\ell^\times \simeq \mathbf{T}(\mathbb{Q}_\ell)^\times$ is identified with the action of the arithmetic Frobenius Frob_λ and the λ corresponding to the first component (which is then our 'preferred' prime above ℓ as in 0.2). We moreover fix $U\Upsilon_{H_f}$, $K \times C \in \Upsilon_{\mathcal{G}_f}$ such that $\mathcal{K} = K^\ell K_\ell \times C^\ell C_\ell$, $U = U^\ell U_\ell$ where K_ℓ, C_ℓ, U_ℓ are local groups at ℓ. We assume that all these groups are unramified at ℓ, so that $K_\ell = \mathrm{GL}_2(\mathbb{Z}_\ell)$, $C_\ell \simeq \mathbb{Z}_\ell^\times$, $U_\ell \simeq \mathbb{Z}_\ell^\times \times \mathbb{Z}_\ell^\times$. Let $\mathfrak{H}_\ell(X)$ denote the polynomial introduced in Example 0.2.5. Then we can think of

$$\mathfrak{H}_\ell = \mathfrak{H}_\ell^t(\mathrm{Frob}_\lambda) = \mathfrak{H}_\ell^t(\varpi C)$$

as an endormophism of $M_{\mathbb{Z}_p}$. Let $D = C^\ell D_\ell$ where $D_\ell = 1 + \ell\mathbb{Z}_\ell$ and let $x_U = 1_{\mathrm{Sh}(U)}$. Set $\mathcal{L} = K \times D$. We may now ask whether there is a zeta element $(x_U, \mathfrak{H}_\ell, \mathcal{L})$. Recall that such an element would solve the corresponding question posed in 3.1.1.

It is clear that this checking can be done locally at the prime ℓ and the resulting zeta element would be compatibly parametrized as we are working with the trivial functor. As was noted in 0.2.5, the local embedding of $H_\ell \hookrightarrow \mathcal{G}_\ell$ is not diagonal one on the $\mathrm{GL}_2(\mathbb{Q}_\ell)$ copy. We may however conjugate this embedding by an appropriate element of K_ℓ and pretend for simplicity that the embedding is indeed diagonal. As in 1.1.1, let

$$\sigma_0 = \begin{pmatrix} \ell & \\ & \ell \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} \ell & \\ & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \ell & 1 \\ & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & \ell \end{pmatrix}.$$

Let $J := \{(1,1), (\sigma_1, \varpi^{-1}), (\sigma_2, \varpi^{-1}), (\sigma_3, \varpi^{-1}), (\sigma_0, \varpi^2)\} \subset \mathcal{G}_\ell$. Then

$$\mathfrak{H}_\ell^t = \mathfrak{H}_\ell(\varpi^{-1}C) = \sum_{\sigma \in J} c_\sigma \mathrm{ch}(U_\ell \sigma \mathcal{K}_\ell)$$

where $c_{(1,1)} = \ell$, $c_{(\sigma_i, \varpi)} = -1$ for $i = 1,2,3$ and $c_{(\sigma_0, \varpi^{-2})} = 1$. Let

$$g_0 := (1,1), \qquad g_1 := (\sigma_1^{-1}\sigma_2, 1), \qquad g_2 := (1, \varpi^{-2})$$

Then we can partition $J = J_0 \sqcup J_1 \sqcup J_1$ so elements of J_i are in the same H-orbit. One easily sees that

$$J_0 = \{(1,1), (\sigma_1, \varpi^{-1})\}, \quad J_1 = \{(\sigma_2, \varpi^{-1})\}, \quad J_2 = \{(\sigma_0, \varpi^{-2}), (\sigma_3, \varpi^{-1})\}.$$

Thus Corollary 3.2.2 would imply that a zeta element exist if d_i divides $\sum_{\sigma \in J_i} c_\sigma$ for $i = 0,1,2$. The condition holds automatically for $i = 1$. For $i = 0,2$, these sums are $\ell - 1$ and 0 respectively. Thus a zeta element in spanned by g_0, g_1, g_2. But the coefficient of g_2 vanishes which means that the twists of our zeta element are only g_0, g_1 (see Corollary 3.2.2).

Remark 3.6.1. The 'phantom twist' g_2 does not contribute to zeta element for the particular normalization chosen for the Hecke polynomial but does if this normalization is altered e.g. it does contribute if instead $\mathfrak{H}_\ell(X)$ was taken to be

$$\mathrm{ch}(K_\ell) - \mathrm{ch}(K_\ell \sigma_\ell K_\ell)X + \ell \cdot \mathrm{ch}(K\tau_\ell K_\ell)X^2$$

in the notations of §0.2.2. The same phenomenon persists in the situation considered in Chapter 9 whose $m = 1$ case is closely related to the split case of CM points on modular curves: for a particular normalization (i.e. the choice of s in Definition 4.3.2) of the Hecke polynomial, there will be fewer twists contributing to zeta element. See Remark 9.5.2.

3.6.2 Siegel units

Let $c > 1$ be an integer relatively prime to 6 and p a prime. Let $\mathbb{A}_f^{(c)}$ denote the adeles of $\mathbb{Q}$ away from places dividing c, $K_c = \prod_{\ell | c} \mathrm{GL}_2(\mathbb{Z}_\ell)$, $C_c = \prod_{\ell | c} \mathbb{Z}_\ell^c$ and $\mathcal{S}(X, \mathbb{Z}_p)$ the $\mathbb{Z}_p$-submodule of locally constant compactly supported functions on $X =: \mathbb{A}_f^2 - (0,0)$ which are of the form $\phi^c \phi_c$ with $\phi_c = \mathrm{ch}(\prod_{\ell|c} \mathbb{Z}_\ell)$. We view elements of X as row vectors. There is a smooth right action $X \times \mathrm{GL}_2(\mathbb{A}_f^{(c)}) \to X$ which is given by right matrix multiplication. Then $\mathcal{S}_{\mathbb{Z}_p}(X)$ becomes a smooth left representation.

Let $\mathcal{S}_{\mathbb{Z}_p}(X \times X) = \mathcal{S}_{\mathbb{Z}_p}(X) \otimes_{\mathbb{Z}_p} \mathcal{S}_{\mathbb{Z}_p}(X)$ and let $\mathrm{GL}_2(\mathbb{A}_f^{(c)})$ act diagonally on it. Equivalently, we may view elements of $X \times X$ as the set of 2×2 matrices in which the rows correspond to the two copies of X and $\mathrm{GL}_2(\mathbb{A}_f^c)$ acts by right matrix multiplication.

Let $H = G := \mathrm{GL}_2(\mathbb{A}_f^{(c)})$, $\iota : H \to G$ the identity map, $T := \mathbb{G}_m(\mathbb{A}_f^c)$ and $\mathcal{G} := G \times T$. Let $\nu : H \to T$ determinant map, $\iota_\nu := \iota \times \nu$, $\Upsilon_G = \Upsilon_H$, the collection of all compact open subgroups $K = K^c \subset \mathrm{GL}_2(\mathbb{A}_f^{(c)})$ such that $K^c K_c$ is a neat compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$ and Υ_T the collection of all compact open subgroups of T. Set $\Upsilon_{\mathcal{G}} = \Upsilon_G \times \Upsilon_T$. Let $M_{\mathbb{Z}_p} : \Upsilon_{\mathcal{G}} \to \mathbb{Z}_p$-Mod be the functor given by

$$K \times C \mapsto \mathrm{H}^2_{\text{ét}}\left(\mathrm{Sh}_{\mathrm{GL}_2}(KK_c) \times_{\mathrm{Spec}\mathbb{Q}} \mathrm{Spec}\, E_C, \mathbb{Z}_p(2)\right)$$

where $\mathrm{Sh}_{\mathrm{GL}_2}(KK_c)$ is the modular curve of level K and E_C is the field extension determined by $\mathbb{Q}^\times \backslash \mathbb{A}_f^\times / CC_c$. Let $N_{\mathbb{Z}_p}$ be the functor sending K to the K-invariants of $\mathcal{S}$, Both $M_{H,\mathbb{Z}_p}, M_{\mathcal{G},\mathbb{Z}_p}$ are CoMack and $M_{H,\mathbb{Z}_p}$ is also Galois. The modular curve of level KK_c is a disjoint union of geometrically connected modular curves with level structure given by congruence subgroup and which are permuted among themselves by the action of $\mathrm{GL}_2(\mathbb{A}_f)$ on $\mathbb{Q}^\times \backslash T_f / \det(KK_c)$. Combining this observation with [Col03, Proposition 1.7 (i)] and the pushforward obtained by the morphism of Shimura data $\iota_\nu : \mathrm{GL}_2 \to \mathrm{GL}_2 \times \mathbb{G}_m$, one obtains a pushforward $\iota_{\nu,*} : N_{\mathbb{Z}_p} \to M_{\mathbb{Z}_p}$.

Now fix a prime $\ell \nmid 6pc$ and fix $U = U^{\ell} U_{\ell}$ where $U_{\ell} := \mathrm{GL}_2(\mathbb{Z}_{\ell})$, $\mathcal{K} = K \times C$ where $K = K^{\ell} K_{\ell}, K_{\ell} := \mathrm{GL}_2(\mathbb{Z}_{\ell})$ and $C = C_{\ell} C^{\ell}, C_{\ell} := \mathbb{Z}_{\ell}^{\times}$. Let $\mathcal{L} = K \times D$ where $D = D^{\ell} D_{\ell}, D_{\ell} := 1 + \ell \mathbb{Z}_{\ell}$. Denote by $x_U = \phi^{\ell} \otimes \phi_{\ell} \in M_{H, \mathbb{Z}_p}(U)$ where

$$\phi_{\ell} = \mathrm{ch}\begin{pmatrix} \mathbb{Z}_{\ell} & \mathbb{Z}_{\ell} \\ \mathbb{Z}_{\ell} & \mathbb{Z}_{\ell} \end{pmatrix}.$$

Let $\mathfrak{H}_{\ell}(X) = \mathrm{ch}(K_{\ell}) - \mathrm{ch}(K_{\ell} \sigma_{\ell} K_{\ell}) + \ell \mathrm{ch}(K_{\ell} \tau_{\ell} K_{\ell})$ where σ_{ℓ}, τ_{ℓ} are as in 0.2.2. Let $\mathrm{Frob}_{\ell} = \mathrm{ch}(\varpi C)$ and let $\mathfrak{H}_{\ell} = \mathfrak{H}_{\ell}^{t}(\mathrm{Frob}_{\ell})$. We ask if there is a zeta element for $(x_U, \mathfrak{H}, \mathcal{L})$. Recall that this would solve the norm relation problem 3.1.1 posed for Siegel units. In what follows, the pushforward ι_* will not be needed.

Denote

$$\varsigma = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} + \sum_{i=1}^{\ell} \begin{pmatrix} 1 & i \\ & 1 \end{pmatrix} \in \mathbb{Z}_p[U_{\ell}].$$

Let $\eta = 1 - \varsigma \sigma_{\ell} + \ell \tau_{\ell} \in \mathbb{Z}_p[H_{\ell}]$ and $b = \ell - 1$. Let $V = U^{\ell} V_{\ell}$, $W = U^{\ell} W_{\ell}$ where

$$V_{\ell} = \iota_{\nu,\ell}^{-1}(\mathcal{L}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_{\ell}) \mid ad - bc \in 1 + \ell \mathbb{Z}_{\ell} \right\}$$

$$W_{\ell} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_{\ell}) \mid a-1, b, c, d-1 \equiv 0 (\mathrm{mod} \ell), \right\}$$

Furthermore, let

$$\psi_{\ell} = \mathrm{ch}\begin{pmatrix} 1 + \ell \mathbb{Z}_{\ell} & \ell \mathbb{Z}_{\ell} \\ \ell \mathbb{Z}_{\ell} & 1 + \ell \mathbb{Z}_{\ell} \end{pmatrix}$$

and denote by $x_W = \phi^{\ell} \psi_{\ell} \in M_{H, \mathbb{Z}_p}(W)$, $x_V = \mathrm{pr}_{W,V,*}(x_V)$ and $x = j_V(x_V)$. Finally, let $\theta = \sum_{\gamma \in U/V} \gamma$

We claim that $\zeta = b(x \otimes \mathcal{K})$ is a (optimal, trace-like) zeta element whose data is (η, V, x_V, θ). (Z1) is checking using the decomposition of $\mathrm{ch}(K)$. (Z2) is immediate, since $[U : V] = \ell - 1$. (Z3) holds by definition. To verify (Z4), first note that $\mathrm{pr}_{V,U,*}(x_V) = \mathrm{pr}_{W,U,*}(x_W) = \phi^{\ell} \otimes \mathrm{ch}(\mathrm{GL}_2(\mathbb{Z}_{\ell}))$. Indeed, since $U_{\ell}/W_{\ell} \cong \mathrm{GL}_2(\mathbb{F}_{\ell})$ where $\mathbb{F}_{\ell} = \mathbb{Z}/\ell\mathbb{Z}$, the trace is obtained as the orbit of the free and transitive action on the set of basis elements of $\mathbb{F}_{\ell}^2$. Now one can verify that $\eta \cdot \phi_{\ell} = \mathrm{ch}(\mathrm{GL}_2(\mathbb{Z}_{\ell}))$[5] by showing that the function $\eta \cdot \phi_{\ell}$ vanishes on non-integral matrices and integral matrices whose determinant has a positive valuation. We therefore have

$$\begin{aligned} \eta \cdot j_U(x_U) &= \phi^{\ell} \otimes (\eta \cdot \phi_{\ell}) \\ &= j_U\left(\mathrm{pr}_{V,U,*}(x_V)\right) \\ &= \theta \cdot j_V(x_V) = b\left(\frac{\theta}{\ell - 1}\right) x \end{aligned}$$

the third equality using Mackey property for M_{H, Z_p}.

[5]see the end of the proof in [Col03, Proposition 1.10] for a more general calculation relating characteristic functions on $\mathrm{GL}_2(\mathbb{Z}_{\ell})$ and 2×2 matrices

Remark 3.6.2. Since the source functor $M_{H,\mathbb{Z}_p}$ is an adelic Schwartz space, it itself provides a compatible parametrization for the zeta elements at each place $\ell \nmid 6pc$, and therefore one can use Lemma 3.5.2 along any finite number of places to create an Euler system going up $\mathbb{Q}^{\mathrm{ab}}$. Moreover, one obtains equality in norm relations by Lemma 3.3.2, since $\iota_{\nu,*}$ is Mackey and the zeta element we constructed is uniform.

Chapter 4

L-factors and Hecke polynomials

In this chapter, we describe the Hecke algebra valued polynomials associated to a representations of the Langlands dual of a reductive group and record some techniques that can be used to compute them. These are the central objects of interest in this thesis.

Notation 4.0.1. Throughout this chapter, we let F denote a local field, $\mathscr{O}_F$ its ring of integers, ϖ a uniformizer, $\Bbbk = \mathscr{O}_F/\varpi$ its residue field, $q = |\Bbbk|$ the cardinality of $\Bbbk$ and $\mathrm{ord} : F \to \mathbb{Z} \cup \{\infty\}$ the additive valuation assigning 1 to ϖ. We pick once and for all $[\Bbbk] \subset \mathscr{O}_F$ a fixed choice of representatives for $\Bbbk$. We let $\bar{F}$ denote a separable closure of F and let $F^{\mathrm{unr}} \subset \bar{F}$ denote the maximal unramified subextension.

4.1 Preliminaries

Let $\mathbf{G}$ be an unramified[1] reductive group over F, $\mathbf{A}$ a maximal F-split torus, $\mathbf{P} \supset \mathbf{A}$ a F-Borel subgroup and $\mathbf{N}$ the unipotent radical of $\mathbf{P}$. Let $\mathbf{M} := \mathbf{Z}(\mathbf{A})$ the centralizer of $\mathbf{A}$ which is a maximal torus in $\mathbf{G}$. We let $X^*(\mathbf{M})$ (resp. $X_*(\mathbf{M})$) denote group of characters (resp. cocharacters) of $\mathbf{M}$ and let

$$\langle -, - \rangle : X_*(\mathbf{M}) \times X^*(\mathbf{M}) \to \mathbb{Z} \tag{4.1.1}$$

denote the natural pairing. We let $\Phi_{\bar{F}} \subset X^*(\mathbf{M})$ denote the set of *absolute roots* of $\mathbf{G}$ with respect to $\mathbf{M}$ and $\Phi^+_{\bar{F}} \subset \Phi_{\bar{F}}$ the set of positive roots associated with $\mathbf{P}$ and $\Delta_{\bar{F}}$ a base. For $\alpha \in \Phi_{\bar{F}}$, we denote by $\alpha^\vee \in X_*(\mathbf{M})$ the corresponding coroot and denote their set by $\Phi^\vee_{\bar{F}} \subset X_*(\mathbf{M})$. Since $\Phi_{\bar{F}}$ is reduced, $\Delta^\vee_{\bar{F}} = \{\alpha^\vee \,|\, \alpha \in \Delta\}$ is a base for the positive (co)roots in $\Phi^\vee_{\bar{F}}$. We let $W_M = N_G(M)/M$ denote the absolute Weyl group of $\mathbf{G}$. For each $\alpha \in \Phi_{\bar{F}}$, the reflection $s_\alpha \in W$ acts on $\lambda \in X_*(\mathbf{M})$

$$\lambda \mapsto \lambda - \langle \lambda, \alpha \rangle \alpha^\vee \tag{4.1.2}$$

[1] i.e. quasi split and split over an unramified extension of F

The pair $\left(W_M, \{s_\alpha\}_{\alpha\in\Delta_{\bar{F}}}\right)$ is a Coxeter system and we denote by $\ell_{\bar{F}} : W_M \to \mathbb{Z}$ the resulting length function. The root lattice $Q(\Phi_{\bar{F}})$ is defined to be the $\mathbb{Z}$-span of $\Phi_{\bar{F}}$ and the weight lattice is defined to be

$$P(\Phi_{\bar{F}}) = \{\mu \in Q(\Phi_{\bar{F}}) \otimes_{\mathbb{Z}} \mathbb{Q} \,|\, \langle \alpha^\vee, \mu\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi_{\bar{F}}\}\,^{2}. \tag{4.1.3}$$

We similarly define $Q(\Phi_{\bar{F}}^\vee)$, $P(\Phi_{\bar{F}}^\vee)$ and refer to them the coroot and coweight lattices respectively. We set $X_0 = X_0(\Phi_{\bar{F}}) = \{\chi \in X^*(\mathbf{M}) \,|\, \langle \alpha^\vee, \chi\rangle = 0 \text{ for all } \alpha^\vee \in \Phi_{\bar{F}}\}$ and define $X_0^\vee = X_0(\Phi_{\bar{F}}^\vee)$ similarly. We have $X_*(\mathbf{M}) \subset P(\Phi_{\bar{F}}) + X_0^\vee$ and $P(\Phi_{\bar{F}}) \cap X_0^\vee = 0$. Here, we are thinking of $X_*(\mathbf{M})$, $P(\Phi_{\bar{F}})$ and $X_0^\vee$ as subgroups of $X_*(\mathbf{M}) \otimes_{\mathbb{Z}} \mathbb{Q}$. In what follows, we will denote the natural extension of (4.1.1) to $P(\Phi_{\bar{F}})$ etc by $\langle -, - \rangle$ as well. Let $\delta = \frac{1}{2}\sum_{\alpha\in\Phi_{\bar{F}}^+} \alpha \in$ be the half sum of positive (absolute) roots of $\mathbf{M}$. It is an element of $P(\Phi_{\bar{F}})$ [Hum78, §13.3 Lemma A].

Let $X^*(\mathbf{A})$, $X_*(\mathbf{A})$ denote respectively the set of characters, cocharacters of $\mathbf{A}$. We let $\mathrm{res} : X^*(\mathbf{M}) \to X^*(\mathbf{A})$ and $\mathrm{cores} : X_*(\mathbf{A}) \to X_*(\mathbf{M})$ denote the natural maps induced by the inclusion $\mathbf{A} \hookrightarrow \mathbf{M}$. Let $\Gamma = \mathrm{Gal}(F^{\mathrm{unr}}/F) \simeq \widehat{\mathbb{Z}}$ denote the unramified Galois group of F. Then Γ acts on $X^*(\mathbf{M})$ via $(\gamma, \chi) \mapsto \gamma\chi\gamma^{-1}(x)$ where $\gamma \in \Gamma$, $\chi \in X^*(\mathbf{M})$ and $x \in \mathbf{M}(F^{\mathrm{unr}})$. Similarly Γ acts on $X_*(\mathbf{M})$. Then

$$X^*(\mathbf{M})_\Gamma \overset{\mathrm{res}}{\simeq} X^*(\mathbf{A}), \qquad X_*(\mathbf{A}) \overset{\mathrm{cores}}{\simeq} X_*(\mathbf{M})^\Gamma$$

where $\Gamma = \widehat{\mathbb{Z}}$ denotes the unramified Galois group of F. Moreover, the pairing

$$\langle -, - \rangle : X_*(\mathbf{A}) \times X^*(\mathbf{A}) \to \mathbb{Z} \tag{4.1.4}$$

is compatible with (4.1.1) with respect to res and cores. The set $\Phi_F \subset X^*(\mathbf{A})$ of non-trivial restrictions of $\Phi_{\bar{F}}$ to $\mathbf{A}$ is the set of *relative roots* for $\mathbf{G}$. If $\Delta_{\bar{F}} \subset \Phi_{\bar{F}}$ denotes a base, then elements of $\Delta_{\bar{F}}$ in the same Γ-orbit restrict to the same element of Φ_F. The set of of such restrictions that are non-zero constitutes a base Δ_F for Φ_F (see [BT65, Proposition 6.8] or [Spr98, Proposition 15.5.3(iii)]). To each root $\alpha \in \Phi_F$, there is associated a relative coroot $\alpha^\vee \in X_*(\mathbf{A})$ such that $\mathrm{cores}(\alpha^\vee) = \sum_\beta \beta^\vee$ where the sum runs over $\beta \in \Phi_{\bar{F}}$ such that $\mathrm{res}(\beta) = \alpha$. The set of relative coroots is denoted $\Phi_F^\vee$. The set $\left\{\alpha^\vee \,|\alpha \in \Phi_F^+\right\}$ is a system of positive (co)roots for $\Phi_F^\vee$ and we let $\Delta_F^\vee$ denote its base. The quadruplet $(X^*(\mathbf{A}), X_*(\mathbf{A}), \Phi_F, \Phi_F^\vee)$ is then a root datum [Spr98, Theorem 15.3.8].

We will say that a cocharacter $\lambda \in X_*(\mathbf{A})$ is *dominant* (resp. *antidominant*) if for all $\alpha \in \Delta_F$, $\langle \lambda, \alpha\rangle \geq 0$ (resp. $\langle \lambda, \alpha\rangle \leq 0$). We will denote the set of dominant (resp. antidominant) cocharacters by $X^*(\mathbf{A})^+$ (resp. $X_*(\mathbf{A})^-$). We similarly define $X_*(\mathbf{M})^+$, $X_*(\mathbf{M})^-$. If $\lambda \in X_*(\mathbf{A})^+$, then $\mathrm{cores}(\lambda) \in X_*(\mathbf{M})^+$. There also exists a partial ordering $\succeq$ on $X_*(\mathbf{A})$ defined by declaring $\lambda \succeq \mu$ for $\lambda, \mu \in X_*(\mathbf{A})$ if

$$\lambda - \mu = \sum_{\alpha^\vee \in \Delta_F^\vee} n_{\alpha^\vee} \alpha^\vee$$

[2]As any element in $\Phi_{\bar{F}}$ is an integral linear cominbation of $\Delta_{\bar{F}}$, we may define the weight lattice using $\Delta_{\bar{F}}$ instead of $\Phi_{\bar{F}}$

for some non-negative integers $n_{\alpha^\vee}$. We will say that λ is *positive* if $\lambda \succeq 0$. We similarly define a partial order $\succeq_M$ on $X_*(\mathbf{M})$ using $\Delta_{\bar{F}}$ and the subset of positive cocharacters in $X_*(\mathbf{M})$.

In general, a dominant (absolute or relative) cocharacter need not be positive and a positive cocharacter need not be dominant (cf. the 'dangerous bend' in [Bou02, Ch. VI §1 n°6]). We however have the following

Lemma 4.1.5. *$\lambda \in X_*(\mathbf{M})$ is dominant if and only if for all $w \in W_M$, $\lambda \succeq_M w\lambda$. Similarly for $X_*(\mathbf{A})$.*

Proof. Since $W_M\lambda$ contains a unique dominant cocharacter, it suffices to establish the only if direction. For $w = s_\alpha$ for $\alpha \in \Phi_{\bar{F}}$, $\lambda - w\lambda = \langle \lambda, \alpha \rangle \alpha^\vee$ by (4.1.2). One then proceeds by an induction argument on the length of w similar to [Bou02, Ch. VI §1 n°6 Prop. 18(iii)] or [Hum78, §13.2 Lemma A] ☐

Denote by G, A, P, M, N the corresponding F points of $\mathbf{G}, \mathbf{A}, \mathbf{P}, \mathbf{M}, \mathbf{N}$ respectively. We fix throughout a smooth reductive group scheme $\mathscr{G}$ over $\mathscr{O}_F$ such that $\mathbf{G} = \mathscr{G}_F$, and let

$$K := \mathscr{G}(\mathscr{O}_F), \quad A^\circ := A \cap K, \quad M^\circ := M \cap K, \quad \Lambda := A/A^\circ.$$

Then K is a *hyperspecial* maximal compact subgroup and A°, M° are the unique maximal compact subgroups of A, M respectively. In particular, A°, M° do not depend on $\mathscr{G}$. In what follows, we will denote $X_*(\mathbf{A})$ by Λ. For $\lambda \in \Lambda$, we let $\varpi^\lambda := \lambda(\varpi) \in A$. We have isomorphisms $X_*(\mathbf{A}) \xrightarrow{\sim} A/A^\circ \xrightarrow{\sim} M/M^\circ$ induced respectively by $\lambda \mapsto \varpi^\lambda A^\circ, A \hookrightarrow M$ (see [Bor79, §9.5]), and we will identify these henceforth. We denote by $v : A/A^\circ \to \Lambda$ the inverse of the negative[3] isomorphism $\Lambda \to A/A^\circ$ given by $\lambda \mapsto \varpi^{-\lambda} A^\circ$.

4.1.1 Weyl groups

Let $N_G(A)$ denote the normalizer of A in G. The quotient $W := N_G(A)/M$ is called the *relative Weyl group* of G. It is the subgroup of W_M which stabilize $A \subset M$. The group W also coincides with the set of fixed points of Γ acting on W_M and we have $W = N_G(A^\circ)/M^\circ$. Moreover, the pair (W, S) for $S := \{s_\alpha \,|\, \alpha \in \Delta_F\})$ forms a Coxeter system. We denote by $\ell = \ell_F : W \to \mathbb{Z}$ the resulting length function. The *longest Weyl element* $w_\circ \in W$ is defined to be the unique element which attains the maximum length in W. It is also the unique element in W such that $w_\circ \cdot \Delta_F = -\Delta_F$ (as a set). We have $w_\circ^2 = \mathrm{id}$. For each $\lambda \in \Lambda^+$, $\lambda^{\mathrm{opp}} := w_\circ \lambda$ is the unique element in the Weyl orbit of λ that lies in Λ^- and we have

$$\lambda \succeq \mu \Longleftrightarrow -\lambda^{\mathrm{opp}} \succeq -\mu^{\mathrm{opp}} \tag{4.1.6}$$

for $\lambda, \mu \in \Lambda$. The quotient $N_G(A)/A^\circ \simeq A/A^\circ \rtimes W$ is called the *Iwahori Weyl group* of G. It is isomorphic to $\Lambda \rtimes W$ via v. Let $Q_F^\vee = Q(\Phi_F^\vee)$ denote the relative coroot lattice. The subgroup $W_{\mathrm{aff}} := Q_F^\vee \rtimes W \subset \Lambda \rtimes W$ is called the (relative) *affine Weyl group*. The group W_{aff} acts on the vector space $Q_F^\vee \otimes \mathbb{R}$ by translations

[3]the negative sign is introduced to match the conventions of [Tit79]. See §5.3 for more details.

and it is customary to denote the element $(\lambda, 1) \in W_{\text{aff}}$ by $t(\lambda)$ and $Q_F^\vee$ by $t(Q_F^\vee)$. More generally, we denote the element $(\lambda, 1) \in \Lambda \rtimes W$ by $t(\lambda)$ and consider it as a translation of $\Lambda \otimes \mathbb{R}$. If Φ_F is irreducible, $\alpha_0 \in \Phi_F$ is the highest root and $s_{\alpha_0} \in W$ denotes the reflection associated with α_0, W_{aff} is a Coxeter group with generators $S_{\text{aff}} := S \sqcup \{t(\alpha_0^\vee)s_{\alpha_0}\}$. In general, it is a Coxeter group whose set of generators S_{aff} is obtained by extending the set S by the reflections associated to the simple affine root of each irreducible component of Φ_F. In particular, its rank (as a Coxeter group) is the number of irreducible components of Φ_F added to the rank of W. We denote by $\ell : W_{\text{aff}} \to \mathbb{Z}$ the extension of $\ell : W \to \mathbb{Z}$ and by $\geq$ the strong Bruhat order induced by the set S_{aff}.

Via the isormorphism $W_I \xrightarrow[\sim]{v} \Lambda \rtimes W$, the element $\varpi^\lambda A^\circ \in W_I$ for $\lambda \in \Lambda$ is identified with $t(-\lambda)$. The quotient $\Omega := W_I/W_{\text{aff}}$ acts on W_{aff} by automorphisms (of Coxeter groups) and one has an isomorphism $W_I \simeq W_{\text{aff}} \rtimes \Omega$. One extends the length function $\ell : W_I \to \mathbb{Z}$ by declaring the length of elements of Ω to be 0. Similarly, the Bruhat ordering on W_{aff} is extended to W_I by declaring $w\rho \geq w'\rho'$ for $w, w' \in W_{\text{aff}}$, $\rho, \rho' \in \Omega$ if $w \geq w'$

Remark 4.1.7. See [Car79, §3.5] and [Tit79, Ch. 1] for the role of buildings in defining these groups. Buildings will be briefly used in §5.3.

4.2 The Satake transform

Fix a Haar measure μ_G on G such that $\mu_G(K) = 1$. For a ring R, let $\mathcal{H}_R(K\backslash G/K)$ be the Hecke algebra of level K with coefficients in R (Definition 2.3.1) and $R\langle G/K\rangle$ be the set of finite R-linear combinations on cosets in G/K. For $\sigma \in G$, we denote by $\text{ch}(K\sigma K) \in \mathcal{H}_R(K\backslash G/K)$ the characteristic function of $K\sigma K$ which we will occasionally also write simply as $(K\sigma K)$. For $\lambda \in \Lambda$, denote by e^λ the element corresponding to λ in the group algebra $\mathbb{Z}[\Lambda]$ and $e^{W\lambda}$ the (formal) sum $\sum_{\mu \in W\lambda} e^\mu$. This allows one to convert from additive to multiplicative notation for cocharacters. For $\lambda \in \Lambda$, let $\langle \lambda, \delta\rangle$ denote the quantity $\langle \text{cores}(\lambda), \delta\rangle = \langle \lambda, \text{res}(\delta)\rangle$.

Let $\mathcal{R} = \mathcal{R}_q$ denote the ring $\mathbb{Z}[q^{\pm\frac{1}{2}}] \subset \mathbb{C}$ where $q^{\frac{1}{2}} \in \mathbb{C}$ denotes a root of $x^2 - q$ and $q^{-\frac{1}{2}}$ denotes its inverse. Denote by $p : G/K \to K\backslash G/K$ the natural map and $p^* : \mathcal{H}_{\mathcal{R}}(K\backslash G/K) \to \mathcal{R}\langle G/K\rangle$ the induced map that sends the characteristic function of $K\sigma K$ to the formal sum (with coefficients 1) of left cosets γK contained in $K\sigma K$. Let $\mathscr{I} : \mathcal{R}\langle G/K\rangle \to \mathcal{R}[\Lambda]$ denote the $\mathcal{R}$-linear map defined by $\text{ch}(\varpi^\lambda n K) \mapsto q^{-\langle\lambda,\delta\rangle} e^\lambda$ for $\lambda \in \Lambda, n \in N$. This is well defined since $G = PK, P = M \ltimes N$ and $MK/K \simeq M/M^\circ \simeq \Lambda$. The composition

$$\mathscr{S} : \mathcal{H}_R(K\backslash G/K) \xrightarrow{p^*} \mathcal{R}\langle G/K\rangle \xrightarrow{\mathscr{I}} \mathcal{R}[\Lambda] \tag{4.2.1}$$

is then a homomorphism of $\mathcal{R}$-algebras known as the *Satake transform*. Its image lies in the Weyl invariants

$\mathcal{R}[\Lambda]^W$. By [Car79, Theorem 4.1] or [Sat63, Theorem 3]), the induced map $\mathscr{S}_\mathbb{C}$ over $\mathbb{C}$ is an isomorphism onto $\mathbb{C}[\Lambda]^W$.

We note that $\{(K\varpi^\lambda K) \mid \lambda \in \Lambda^+\}$ is a basis for $\mathcal{H}_\mathcal{R}(K\backslash G/K)$ by Cartan decomposition. We are therefore interested in the Satake transform of such functions. For $\lambda \in \Lambda^+$, write

$$\mathscr{S}(K\varpi^\lambda K) = \sum_{\mu\in\Lambda} q^{-\langle \mu,\delta\rangle} a_\lambda(\mu) e^\mu \tag{4.2.2}$$

where $a_\lambda(\mu) \in \mathbb{Z}_{\geq 0}$. By definition, $a_\lambda(\mu)$ is equal to the number of distinct left cosets $\varpi^\mu nK$ for $n \in N$ such that $\varpi^\mu nK \subset K\varpi^\lambda K$. The W-invariance of $\mathscr{S}$ implies that $q^{-\langle\mu_1,\delta\rangle}a_\lambda(\mu) = q^{-\langle\mu_2,\delta\rangle}a_\lambda(\mu_2)$ for all $\mu_1, \mu_2 \in \Lambda$ such that $W\mu_1 = W\mu_2$.

Proposition 4.2.3. *For $\lambda, \mu \in \Lambda^+$, $a_\lambda(\mu) \neq 0$ only if $\lambda \succeq \mu$. Moreover, $a_\lambda(\lambda^{\mathrm{opp}}) = 1$.*

Proof. Set $\kappa = \lambda^{\mathrm{opp}}$ and $\nu := \mu^{\mathrm{opp}}$. Then $-\kappa, -\nu \in \Lambda^+$. Since the image of $\mathscr{S}$ is W-invariant, $a_\lambda(\mu) \neq 0$ if and only if $a_\lambda(\nu) \neq 0$. By definition, this is equivalent to $\varpi^\nu NK \cap K\varpi^\lambda K \neq \varnothing$. Now $\varpi^\nu N = N\varpi^\nu$ as A normalizes N and $K\varpi^\lambda K = K\varpi^\kappa K$ as $K \cap N_G(A)$ surjects onto W. Thus

$$a_\lambda(\mu) \neq 0 \iff N\varpi^\nu K \cap K\varpi^\kappa K \neq \varnothing.$$

By [BT72, Proposition 4.4.1 (i)], we get that $N\varpi^\nu K \cap K\varpi^\kappa K \neq \varnothing \implies -\kappa \succeq -\nu$. But the last condition is the same as $\lambda \succeq \mu$ (see 4.1.6), which establishes the first part. By [BT72, Proposition 4.4.1 (ii)], $N\varpi^\kappa K \cap K\varpi^\kappa K = \varpi^\kappa K$ i.e. the only coset of the form $\varpi^\kappa nK$ where $n \in N$ such that $\varpi^\kappa nK \subset K\varpi^\lambda K$ is $\varpi^\kappa K$. The second claim follows. □

Corollary 4.2.4. *, For $\lambda \in \Lambda^+$, $\mathscr{S}(K\varpi^\lambda K) - q^{\langle\lambda,\delta\rangle}e^{W\lambda}$ lies in is in the $\mathcal{R}$-span of $\{e^{W\mu} \mid \mu \in \Lambda^+, \mu \precneqq \lambda\}$.*

Proof. Since $\lambda^{\mathrm{opp}} = w_\circ \cdot \lambda$ and $w_\circ$ sends Φ_F^+ to Φ_F^-, $w_\circ \cdot \mathrm{res}(\delta) = -\delta$. Therefore,

$$\langle \lambda^{\mathrm{opp}}, \delta\rangle = \langle \lambda, w_\circ \cdot \mathrm{res}(\delta)\rangle = -\langle\lambda,\delta\rangle.$$

The second part of Proposition 4.2.3 therefore implies that $q^{-\langle\lambda^{\mathrm{opp}},\delta\rangle}a_\lambda(\lambda^{\mathrm{opp}}) = q^{\langle\lambda,\delta\rangle}$ and the claim here follows by the first part of Proposition. □

Corollary 4.2.5. *The Satake transform induces an isomorphism $\mathcal{H}_\mathcal{R}(K\backslash G/K) \simeq \mathcal{R}[\Lambda]^W$ of $\mathcal{R}$-algebras.*

Proof. Let $\lambda \in \Lambda^+$. We show that $e^{W\lambda}$ is in the image of $\mathscr{S}$. By Corollary 4.2.4, $f := \mathscr{S}\big(q^{-\langle\lambda,\delta\rangle}(K\varpi^\lambda K)\big) - e^{W\lambda} \in \mathcal{R}[\Lambda]^W$ is equal to $\sum c_\lambda(\mu)e^{W\mu}$ where the sum runs over the finite set $\{\mu \in \Lambda^+ \mid \mu \precneqq \lambda\}$ and $c_\lambda(\mu)$ are elements in $\mathcal{R}$. Let $\mu_1, \ldots, \mu_k \in \Lambda^+$, $\mu_i \precneqq \lambda$ be pairwise incomparable elements such that for any $\mu \in \Lambda^+$, $\mu \precneqq \lambda \implies \mu \preceq \mu_i$ for some i. By Proposition 4.2.3 again,

$$\mathscr{S}\left(q^{-\langle\lambda,\delta\rangle}(K\varpi^\lambda K) - \sum_{i=1}^k q^{-\langle\mu_i,\delta\rangle}c_\lambda(\mu_i)(K\varpi^{\mu_i}K)\right) = e^{W\lambda} + g$$

where $g \in \mathcal{R}[\Lambda]^W$ is a linear combination of $e^{W\mu} \in \mathcal{R}[\Lambda]^W$ for $\mu \in \Lambda^+$ such that $\mu \prec \mu_i$ for some i and $\mu \notin \{\mu_1, \ldots, \mu_k\}$. Repeating this process, we can make all terms appearing in f. $\square$

Definition 4.2.6. For $\lambda \in \Lambda^+$, we call the element $q^{\langle \lambda, \delta \rangle} e^{W\lambda} \in \mathcal{R}[\Lambda]^W$ the *leading term* of the Satake transform of $(K\varpi^\lambda K)$. If $gK \subset K\varpi^\lambda K$ is a coset, we call the unique cocharacter $\mu \in \Lambda$ such that $gK = \varpi^\mu n K$ for some $n \in N$ the *shape* of the coset gK.

By Proposition 4.2.3 and Lemma 4.1.5, the shape μ of any $gK \subset K\varpi^\lambda K$ for $\lambda \in \Lambda^+$ satisfies $\lambda \succeq \mu$.

Remark 4.2.7. Proposition 4.2.3 and its corollaries may be found several places in literature ([Kot84], [Gro98]). However, as the proofs of these results are often only sketched, we have chosen to include them here. The conventions introduced here will also be helpful in computations later on. Cf. [FP21, §3.2].

Remark 4.2.8. One can strengthen Proposition 4.2.3 to $a_\lambda(\mu) \neq 0 \iff \lambda \succeq \mu$. See [Rap00, Theorem 1.1].

4.2.1 Examples

In this subsection, we provide a few examples of Satake transform computations for GL_2. More example may be found in Part II of this thesis.

Let $\mathbf{G} = \mathrm{GL}_{2,F}$, $\mathbf{A} = \mathbb{G}_m \times \mathbb{G}_m \hookrightarrow \mathbf{G}$ be the standard diagonal torus and $K = \mathrm{GL}_2(\mathscr{O}_F)$. For $i = 1, 2$, let $e_i : \mathbf{A} \to \mathbb{G}_m$ for $i = 1, 2$ be the characters given by $\mathrm{diag}(u_1, u_2) \mapsto u_i$, $i = 1, 2$ and $f_i : \mathbb{G}_m \to \mathbf{A}$ be the cocharacters that insert u into the i-th component. Then $\Phi = \{\pm(e_1 - e_2)\}$ and $\Lambda = \mathbb{Z}f_1 \oplus \mathbb{Z}f_2$. We will denote $\lambda = a_1 f_1 + a_2 f_2 \in \Lambda$ by (a_1, a_2). We take $\chi := e_1 - e_2 \in X^*(\mathbf{A})$ as the positive root, so that $\delta = \frac{\chi}{2}$ and Λ^+ is the set (a_1, a_2) such that $a_1 \geq a_2$. Let $\alpha := e^{f_1}$, $\beta := e^{f_2}$ considered as elements of the group algebra $\mathbb{Z}[\Lambda]$. Then $\mathcal{R}[\Lambda]^W = \mathcal{R}[\alpha^\pm, \beta^\pm]^{S_2}$ where the non-trivial element of S_2 acts via $\alpha \leftrightarrow \beta$.

Example 4.2.1. Let $\lambda = f_1 \in \Lambda^+$. Then $\lambda^{\mathrm{opp}} = f_2$. As in Lemma 0.2.18, we have

$$K\varpi^\lambda K = \begin{pmatrix} 1 & \\ & \varpi \end{pmatrix} K \sqcup \bigsqcup_{\kappa \in [\mathscr{k}]} \begin{pmatrix} \varpi & \kappa \\ & 1 \end{pmatrix} K.$$

In this decomposition, there is 1 coset of shape f_2 and q cosets of shape f_1. Therefore, we obtain

$$\mathscr{S}(K\varpi^\lambda K) = q^{\frac{1}{2}}\beta + q \cdot q^{-\frac{1}{2}}\alpha = q^{\frac{1}{2}}(\alpha + \beta) \in \mathcal{R}[\Lambda]^W.$$

Example 4.2.2. . Let $\lambda = 2f_1 \in \Lambda^+$. Then $\lambda^{\mathrm{opp}} = 2f_2$. Using row and column operations in conjunction with Iwasawa decomposition, one obtains

$$K\varpi^\lambda K = \begin{pmatrix} 1 & \\ & \varpi^2 \end{pmatrix} K \sqcup \bigsqcup_{\kappa \in [\mathscr{k}] \setminus \{0\}} \begin{pmatrix} \varpi & \kappa \\ & \varpi \end{pmatrix} K \sqcup \bigsqcup_{\kappa_1, \kappa_2 \in [\mathscr{k}]} \begin{pmatrix} \varpi^2 & \kappa_1 + \varpi\kappa_2 \\ & 1 \end{pmatrix} K.$$

In this decomposition, there is one coset of shape f_2, $q-1$ cosets of shape f_1+f_2 and q^2 of shape f_1. So,

$$\begin{aligned}\mathscr{S}(K\varpi^\lambda K) &= q\beta^2 + (q-1)\cdot\alpha\beta + q^2\cdot q^{-1}\alpha^2 \\ &= q(\alpha^2+\beta^2) + (q-1)\alpha\beta \in \mathcal{R}[\Lambda]^W.\end{aligned}$$

Remark 4.2.9. One can in fact write an explicit formula for $\mathscr{S}(K\varpi^\lambda K)$ for any $\lambda \in \Lambda$. See [Cas17, §2 p.20] for a formula[4] in terms of $\mathcal{R}$-basis $\alpha^m\beta^n$ of $\mathcal{R}[\Lambda]$.

4.2.2 Macdonald's formula

The Satake transform is not explicit in the sense that the coefficients of the non-leading terms are not explicit. In general, the coefficients can be quite cumbersome expressions in q. There is however the following formula due to I.G. Macdonald [Mac71] (see also [HKP10, Theorem 5.6.1]).

Theorem 4.2.10 (Macdonald)**.** *Suppose* $\mathbf{G}$ *is split and* $\Phi_{\bar{F}} = \Phi_F$ *is irreducible. Then for any* $\lambda \in \Lambda^+$,

$$\mathscr{S}(K\varpi^\lambda K) = \frac{q^{\langle\lambda,\delta\rangle}}{W_\lambda(q^{-1})}\sum_{w\in W}\prod_{\alpha\in\Phi^+} e^{w\lambda}\cdot\frac{1-q^{-1}e^{-w\alpha^\vee}}{1-e^{-w\alpha^\vee}}$$

where $W_\lambda(x) := \sum_{w\in W^\lambda} x^{\ell(w)}$ *denotes the Poincaré polynomial of the stabilizer* $W^\lambda \subset W$ *of* λ

For arbitrary reductive groups, there is a similar but slightly more complicated expression as it takes into account divisible/multipliable roots and different contributions of root group filtrations. We refer the reader to [Cas80, Theorem 4.2] and [Car79, §3.7] for details. These formulas however will not be needed in this thesis.

Remark 4.2.11. Let us observe that since [Cas80, p. 3] considers a single 'dominant' (i.e. highest) root in order to define the set of simple affine reflections, the underlying root system is seemingly assumed to be irreducible. However, this is not explicitly stated anywhere. The final expression in Theorem 4.2 of *op. cit.* might therefore be valid only when the irreducibility condition is assumed.

Example 4.2.3. Retain the notations of §4.2.1. We have $e^{-\chi^\vee} = \alpha^{-1}\beta$ and

$$\frac{1-q^{-1}e^{-\chi^\vee}}{1-e^{-\chi^\vee}} = \frac{\alpha-q^{-1}\beta}{\alpha-\beta}, \qquad \frac{1-q^{-1}e^{\chi^\vee}}{1-e^{\chi^\vee}} = \frac{\beta-q^{-1}\alpha}{\beta-\alpha}$$

[4]The expression $\tau_{n,n}$ is not defined in [Cas17] but is meant to denote $\alpha^n\beta^n$ in the notations of *loc. cit.*

For $\lambda = 2f_1$, we compute

$$\begin{aligned}\mathscr{S}(K\varpi^\lambda K) &= q\left(\alpha^2\cdot\frac{\alpha-q^{-1}\beta}{\alpha-\beta}+\beta^2\cdot\frac{\beta-q^{-1}\alpha}{\beta-\alpha}\right)\\ &= q\left(\frac{(\alpha^3-\beta^3)-q^{-1}\alpha\beta(\alpha-\beta)}{\alpha-\beta}\right)\\ &= q(\alpha^2+\alpha\beta+\beta^2)-\alpha\beta\\ &= q(\alpha^2+\beta^2)+(q-1)\alpha\beta\end{aligned}$$

which agrees with Example 4.2.2.

4.3 Representations of Langlands dual

Let $\hat{\mathbf{G}}$ denote the dual group of $\mathbf{G}$ considered as a split reductive group over $\mathbb{Q}$. Let $\hat{\mathbf{M}} \subset \hat{\mathbf{G}}$ denote the maximal torus such that $X_*(\hat{\mathbf{M}}) = X^*(\mathbf{M})$. We let $\hat{\mathbf{P}}$ be the Borel subgroup of $\mathbf{G}$ corresponding to $\hat{\Phi}^+_F := \Phi^\vee_F \subset X^*(\hat{\mathbf{M}}) = X_*(\mathbf{M})$. The action of Γ on based root datum of $\mathbf{G}$ together with a choice of pinning determines an action of Γ on $\hat{\mathbf{G}}$ which is unique upto an inner automorphism by $\hat{\mathbf{T}}$. We define the *Langlands dual* to be ${}^L\mathbf{G} = {}^L\mathbf{G}_F := \hat{\mathbf{G}} \rtimes \Gamma$ considered as an algebraic group over $\mathbb{Q}$. We refer the reader to [Bor79, Ch. I-III] for a detailed treatment of this group. See also [BR94, §1].

Remark 4.3.1. The subscript F in the notation ${}^L\mathbf{G}_F$ is not meant to suggest base change of algebraic groups but rather the fixed field for the Galois group Γ. If E/F is an unramified field extension, and ${}^L\mathbf{G}_E$ denotes the subgroup $\hat{\mathbf{G}} \rtimes \Gamma_E$ of ${}^L\mathbf{G}_F$.

Since the weights of algebraic representations of $\hat{\mathbf{G}}$ are elements of $X^*(\hat{\mathbf{M}}) = X_*(\mathbf{M})$, we also refer to elements of $X_*(\mathbf{M})$ as *coweights*[5] of $\mathbf{G}$. For each dominant coweight $\lambda \in X_*(\mathbf{M})^+$, there exists a simple representation (π, V_λ) of $\hat{\mathbf{G}}$ unique upto isomorphism such that $\lambda \succeq_M \mu$ for any coweight μ appearing $V(\lambda)$ ([Mil17, Theorem 22.2]). Since $\hat{\mathbf{G}}$ is defined over $\mathbb{Q}$, so is the representation V_λ ([Mil17, 22.5]). For μ is a coweight of V_λ, we denote by V_λ^μ the corresponding (co)weight space.

Let $\varphi : \hat{\mathbf{G}} \to \hat{\mathbf{G}}$ be an endomorphism that sends $\hat{\mathbf{M}}$ and $\hat{\mathbf{B}}$ to themselves. Then the representation of $\hat{\mathbf{G}}$ obtained via the composition $\pi \circ \varphi$ also has dominant coweight λ and is therefore isomorphic to V_λ. Since $\mathrm{End}(V_\lambda) \simeq \mathbb{Q}$ ([Mil17, 22.3]), there is a unique isomorphism $T_\varphi : (\pi, V_\lambda) \xrightarrow{\sim} (\pi \circ \varphi, V_\lambda)$ of $\hat{\mathbf{G}}$-representations such that T_φ is identity on the highest weight space $(V_\lambda)^\lambda$. If $\Xi \subset \mathrm{Aut}(\hat{\mathbf{G}})$ is a subgroup of automorphisms preserving $\mathbf{B}$ and $\mathbf{T}$, then the construction just described determines an action of $\mathbf{G} \rtimes \Xi$ on V_λ extending that of $\mathbf{G}$. Since Γ acts on $\hat{\mathbf{G}}$ by automorphisms that preserve $\mathbf{B}$ and $\mathbf{T}$, one can extend the action of $\hat{\mathbf{G}}$ to

[5] In the terminology of [Bou02, Ch. VI], the elements of $Q(\Phi^\vee_F)$ would be called *radical weights*.

${}^L\mathbf{G}_F$. We will henceforth consider $(\pi, V(\lambda))$ as a representation of ${}^L\mathbf{G}_F$ under the conventions introduced above.

Let γ be a topological generator of Γ. Then the trace of $\hat{\mathbf{T}} \rtimes \gamma$ on V_λ is an element of $\mathbb{Z}[\Lambda]^W$ which is moreover independent of the choice of γ [FP21, Lemma 3.1] (cf. [Bor79, §6-7]). Similar arguments as in *loc. cit.* or [Bor79, Lemma 6.4] show that the trace of $\hat{\mathbf{T}} \rtimes \gamma$ on $\bigwedge^i V_\lambda$ for any $i \geq 0$ also lies in $\mathbb{Z}[\Lambda]^W$.

Definition 4.3.2. Let $\lambda \in \Lambda^+$. The *Satake polynomial* $\mathfrak{S}_\lambda(X) \in \mathbb{Z}[\Lambda]^W[X]$ is defined to be the reverse characteristic polynomial of $\hat{\mathbf{T}} \rtimes \gamma$ acting on V_λ. For $s \in \frac{1}{2}\mathbb{Z}$, the *Hecke polynomial* $\mathfrak{H}_{\lambda,s}(X) \in \mathcal{H}_{\mathcal{R}}(K\backslash G/K)[X]$ *centered at* s is defined to be the unique polynomial that satisfies $\mathscr{S}(\mathfrak{H}_{\lambda,s}(X)) = \mathfrak{S}_\lambda(q^{-s}X) \in \mathcal{R}[\Lambda]^W$.

In other words, $\mathfrak{S}_\lambda(X) \in \mathbb{Z}[\Lambda]^W[X]$ is the polynomial of degree $d = \dim_{\mathbb{Q}} V_\lambda$ such that the coefficient of X^k is $(-1)^k$ times the trace of $\hat{\mathbf{T}} \rtimes \gamma$ on $\bigwedge^k V(\lambda)$ and $\mathfrak{H}_{\lambda,s}$ is the polynomial such that the the Satake transform of the coefficient of X^k in $\mathfrak{H}_{\lambda,s}(X)$ is q^{-ks} times the coefficient of X^k in $\mathfrak{S}_\lambda(X)$.

4.3.1 Miniscule coweights

The representations of ${}^L\mathbf{G}_F$ that will interest will be associated to certain dominant cocharacters that arise out of a Shimura data. Such cocharacters satisfy the special condition of being 'miniscule'. In this subsection, we recall this notion and establish some results scattered over several exercises of [Bou02, Ch. VI §1-2]. The reader may consult [Bou02, Chapter VI §1 n° 6-9], [Bou05, Ch. VIII §7 n°3] and [Hum78, §13] for general reference of the material provided here. Cf. [Kot84, §2.3].

Definition 4.3.3. We say that $\lambda \in P(\Phi_{\bar{F}})^\vee$ is *minuscule* if $\langle \lambda, \alpha \rangle \in \{1, 0, -1\}$ for all $\alpha \in \Phi_{\bar{F}}$.

We similarly define minuscule cocharacters in $X_*(\mathbf{M})$. One easily sees that $\lambda \in X_*(\mathbf{M})$ is miniscule if the image of λ in $X^*(\mathbf{M})/X_0^\vee \hookrightarrow P(\Phi_{\bar{F}}^\vee)$ is so.

Definition 4.3.4. A set $S \subset P(\Phi_{\bar{F}}^\vee)$ is said to be *saturated* or $\Phi_{\bar{F}}$*-saturated* if for all $x \in S$, $\alpha \in \Phi_{\bar{F}}$ and i between 0 and $\langle x, \alpha \rangle$, $x - i\alpha \in S$. For $\lambda \in P(\Phi_{\bar{F}}^\vee)$, we define $S(\lambda)$ to be the smallest saturated subset of $P(\Phi_{\bar{F}})$ containing λ obtained as the intersection of all saturated subsets containing λ.

We similarly define saturated subsets of $X_*(\mathbf{M})$ and $S(\lambda)$ for $\lambda \in X_*(\mathbf{M})$. Then $S \subset X_*(\mathbf{M})$ is saturated if $(S + X_0^\vee)/X_0^\vee \subset P(\Phi_{\bar{F}}^\vee)$ is. Similarly, if $\lambda \in X_*(\mathbf{M})$ and $\lambda = \lambda_1 + \lambda_0$ for unique $\lambda_1 \in P(\Phi_{\bar{F}}^\vee)$, $\lambda_0 \in X_0^\vee$, $(S(\lambda) + X_0^\vee)/X_0^\vee \subset P(\Phi_{\bar{F}}^\vee)$ identifies with $S(\lambda_1)$ If $S \subset P(\Phi_{\bar{F}})$ or $X_*(\mathbf{M})$ is saturated, then $s_\alpha(x) = x - \langle x, \alpha \rangle \alpha^\vee \in S$ for all $x \in S$, $\alpha \in \Phi_{\bar{F}}$. Thus any saturated set is W_M-stable. In particular $W_M\lambda \subset S(\lambda)$ for $\lambda \in X_*(\mathbf{M})$, $P(\Phi_{\bar{F}})^\vee$. We define the set of dominant elements $P(\Phi_{\bar{F}}^\vee)^+$ in the same manner as $X_*(\mathbf{M})^+$.

Proposition 4.3.5. *$\lambda \in X_*(\mathbf{M})^+$ is minsucule if and only if $S(\lambda) = W_M\lambda$. A similar claim holds for elements of $P(\Phi_{\bar{F}}^\vee)^+$.*

Proof. Write $\lambda = \lambda_1 + \lambda_0$ with $\lambda_1 \in P^\vee = P(\Phi_{\bar{F}}^\vee)$, $\lambda_0 \in X_0^\vee$. Since $\lambda_0 \in X_*(\mathbf{M})$, so does λ_1 and λ_1 is dominant. Moreover, λ is minuscule if and only if λ_1 is and $S(\lambda) = W_M \cdot \lambda$ if and only if $S(\lambda_1) = W_M \lambda_1$. It therefore suffices to establish the claim for $\lambda \in (P^\vee)^+$.

Let $V^\vee = P^\vee \otimes \mathbb{Q}$, $V = Q \otimes \mathbb{Q}$ where $Q := Q(\Phi_{\bar{F}})$. Then V, $V^\vee$ are in duality and $P^\vee \subset V^\vee$, $Q \subset V$ are dual lattices. Let $V^\vee \times V^\vee \to \mathbb{R}$ be a W-invariant pairing. Then V is identified with $V^\vee$, $\langle -, - \rangle$ with $(-,-)$ and $\alpha \in \Phi_{\bar{F}}$ with $2\alpha^\vee/(\alpha^\vee, \alpha^\vee)$. In particular,

$$(\lambda, \alpha^\vee) = \frac{\langle \lambda, \alpha \rangle}{2} \cdot (\alpha^\vee, \alpha^\vee).$$

Note that $\langle \lambda, \alpha \rangle$ and therefore $(\lambda, \alpha^\vee)$ are non-negative for $\alpha \in \Phi_{\bar{F}}^+$ as λ is dominant.

($\Longleftarrow$) Suppose $S(\lambda) = W_M \lambda$ and suppose moreover for the sake of contradiction that λ is not minuscule. Then there exists $\alpha \in \Phi_{\bar{F}}$ such that $k := \langle \lambda, \alpha \rangle > 1$. Then $(\lambda, \alpha^\vee) = \frac{k}{2}(\alpha^\vee, \alpha^\vee)$. Set $\mu := \lambda - \alpha^\vee \in P^\vee$. Then $\mu \in S(\lambda)$ by definition. Now

$$(\mu, \mu) = (\lambda, \lambda) - k(\alpha^\vee, \alpha^\vee) + (\alpha^\vee, \alpha^\vee) < (\lambda, \lambda).$$

Since elements of $W_M \lambda$ must have the same length with respect to $(-,-)$, $\mu \notin W_M \lambda = S(\lambda)$, a contradiction. Therefore $k \in \{0, 1\}$ and we deduce that λ is minuscule.

($\Longrightarrow$) Suppose that λ is minuscule. For all $w \in W_M$, $\langle w\lambda, \alpha \rangle = \langle \lambda, w^{-1}\alpha \rangle \in \{1, 0, -1\}$ which implies that $w\lambda - t\alpha \in \{w\lambda, s_\alpha(w\lambda)\}$. Thus $W\lambda$ is saturated and therefore $W\lambda = S(\lambda)$. □

Corollary 4.3.6. *Every non-empty saturated subset of the coweight lattice contains a minuscule element.*

Proof. Retain the notations in the proof of Proposition 4.3.5. Let $X \subset P^\vee$ be a saturated subset. Let $\lambda \in X$ be the shortest element i.e. $\|\lambda\| := (\lambda, \lambda)^{\frac{1}{2}}$ is minimal possible for $\lambda \in X$. We claim that λ is minuscule. Suppose on the contrary that there exist $\alpha \in \Phi_{\bar{F}}$ such that $\langle \lambda, \alpha \rangle \notin \{1, 0, -1\}$. Replacing α with $-\alpha$ if necessary, we may assume that $\langle \lambda, \alpha \rangle > 1$. Then $\lambda - \alpha \in X$ by definition and the length calculation in the proof of 4.3.5 shows that $\lambda - \alpha$ is a shorter element. □

Under additional assumptions, one can describe the minuscule cocharacters more explicitly. Suppose $\Phi_{\bar{F}}$ is irreducible for the rest of this subsection. Say $\Delta_{\bar{F}} = \{\alpha_1, \ldots, \alpha_n\}$ and let $\bar{\omega}_1, \ldots, \bar{\omega}_n \in P(\Phi_{\bar{F}}^\vee)$ denote the basis dual to the basis $\Delta_{\bar{F}}$ of Q. The elements $\bar{\omega}_i$ are referred to as the *fundamental coweights* of $\Phi_{\bar{F}}$. Since $\Phi_{\bar{F}}$ is irreducible, there exists a highest root ([Bou02, Ch. VI §1 n°8])

$$\tilde{\alpha} = \sum_{j=1}^{n} m_{\alpha_j} \alpha_j \in \Phi_{\bar{F}}$$

where $m_{\alpha_j} \geq 1$ are integers. Let $J \subset \{1, \ldots, n\}$ be the subset of indices j such that $m_{\alpha_j} = 1$.

Lemma 4.3.7. *$\{\bar{\omega}_j\}_{j\in J}$ is the set of all minuscule elements in $P(\Phi_{\bar{F}}^\vee)^+$. These elements form a system of representatives for non-zero classes in $P(\Phi_{\bar{F}}^\vee)/Q(\Phi_{\bar{F}}^\vee)$.*

Proof. Let $\lambda \in P(\Phi_{\bar{F}}^\vee)^+$ be non-zero. Since $\bar{\omega}_1, \ldots, \bar{\omega}_n$ is a basis of $P^\vee$, we can write $\lambda = a_1\bar{\omega}_1 + \ldots + a_n\bar{\omega}_n$ uniquely. Since λ is dominant and non-zero, we have $a_1, \ldots, a_n \geq 0$ with at least one positive, say a_k. Now λ is minuscule only if $a_1 m_{\alpha_1} + \ldots + a_n m_{\alpha_n} = \langle \lambda, \tilde{\alpha} \rangle = 1$ which can only occur if $a_k = 1$ and $k \in J$. Thus minuscule elements of $P(\Phi_{\bar{F}})^\vee - \{0\}$ are contained in $\{\bar{\omega}_j\}_{j\in J}$. Since $\tilde{\alpha}$ is highest, any root $\sum_{j=1}^n p_{\alpha_j}\alpha_j \in \Phi_F$ satisfies $m_{\alpha_j} \geq p_{\alpha_j}$ and one easily sees that $\bar{\omega}_j$ for $j \in J$ is minuscule. The second claim follows by Corollary of Prposition 6 in [Bou02, Ch. VI §2 n°3] □

For $\lambda \in X_*(\mathbf{M})^+$, set $\Sigma(\lambda) := \{\mu \in X_*(\mathbf{M}) \,|\, w\mu \preceq_M \lambda \text{ for all } w \in W_M\}$. Then $\lambda \in \Sigma(\lambda)$ by Lemma 4.1.5 and $\Sigma(\lambda)$ is easily seen to be saturated. We have $W_M\lambda \subset S(\lambda) \subset \Sigma(\lambda)$.

Corollary 4.3.8. *$\lambda \in X_*(\mathbf{M})^+$ is minuscule only if $\Sigma(\lambda) = W_M\lambda$. The converse holds if $\Phi_{\bar{F}}$ is irreducible, in which case and the image of λ in $P(\Phi_{\bar{F}}^\vee)$ is outside the root lattice unless $\lambda \in X_0^\vee$.*

Proof. Retain the notations introduced in the proof of 4.3.5. By similar arguments, we can reduce to the case of $\lambda \in X_*(\mathbf{M}) \cap P^\vee$. We have $W_M\lambda \subset S(\lambda) \subset \Sigma(\lambda)$. Therefore $\Sigma(\lambda) = W_M\lambda \implies S(\lambda) = W_M\lambda$ which by Proposition 4.3.5 implies that λ is minuscule.

Assume now that $\Phi_{\bar{F}}$ is irreducible and suppose λ is minuscule. Then $\lambda \in Q(\Phi_{\bar{F}}^\vee) \implies \lambda \in X_0^\vee$ since the only dominant coweights that are minuscule are fundamental weights. We may therefore also assume that $\lambda \notin Q(\Phi_{\bar{F}}^\vee)$. Suppose moreover for the sake of contradiction that there exists a $\mu \in \Sigma(\lambda) - W_M\lambda_M$. We may assume μ is dominant since $\Sigma(\lambda) - W_M\lambda$ is stable under W_M. Since $\Sigma(\lambda)$ is saturated and contains μ, $\Sigma(\lambda) \supset S(\mu)$. By Corollary 4.3.6 $S(\mu)$ contains a minuscule element λ_1. Since all elements of $W_M\lambda_1$ are minuscule and $S(\mu)$ is W_M-stable, we may take λ_1 to be dominant. Since $S(\mu) \subset \Sigma(\lambda)$, $\lambda \succeq_M \lambda_1$. In particular, $\lambda - \lambda_1 \in Q(\Phi_{\bar{F}}^\vee)$. Thus λ and λ_1 are distinct non-zero dominant coweights represent the same non-zero class in $P^\vee/Q^\vee$, which contradicts the second part of Lemma 4.3.7. Hence $\Sigma(\lambda)$ must equal $W_M\lambda$. The final claim is immediate. □

Let V_λ be the irreducible representation of the highest weight $\lambda \in \Lambda^+$. For each $\mu \in X_*(\mathbf{M})^+$ such that $\mu \preceq_M \lambda$, the dimension (as a vector space over $\mathbb{Q}$) of the coweight space V_λ^μ is called the *multiplicity* of μ in $V(\lambda)$. Corollary 4.3.8 implies that when λ is minuscule and $\Phi_{\bar{F}}$ is irreducible, the set of coweights is just the Weyl orbit $W_M\lambda$ and the multiplicities of these coweights are 1. Since γ acts trivially on such coweight spaces, we get

Corollary 4.3.9. *If $\lambda \in \Lambda^+$ is minuscule and $\Phi_{\bar{F}}$ is irreducible, then $\mathfrak{S}_\lambda(X) = \prod_{\mu \in W_M\lambda}(1 - e^\mu X) \in \mathcal{R}[\Lambda]^W$.*

Remark 4.3.10. The content of this subsection is developed in Exercises 24, 25 of §1 and Exercise 5 of §2 in [Bou02, Ch. VI]. While the results are well-known, all sources we have encountered seem to cite the aforementioned exercises and written proofs are harder to find. Since we are unable to find a better reference, we have included full proofs here.

4.4 Kazhdan-Lusztig theory

We conclude this chapter by recording an important property of the coefficients of Satake transform when taken modulo $q-1$. Since we will only need the results of this section in Ch. 13, we assume for all of this section that $\mathbf{G}$ is split and $\Phi_{\bar{F}} = \Phi_F$ is irreducible. We refer the reader to [Hum90, §7.9], [HKP10, §7] and [Kat82] for the material presented here in addition to the original article [KL79]. See also [Kno05] for generalization to non-split case.

The *Hecke algebra* $\mathcal{H}_{\mathcal{R}}(W_I)$ of W_I is the unital associative $\mathcal{R}$-algebra with $\mathcal{R}$-basis $\{T_w\}_{w\in W_I}$ subject to the relations

$$T_s^2 = (q-1)T_s + qT_e \qquad \text{for } s \in S_{\text{aff}}$$
$$T_w T_{w'} = T_{ww'} \qquad \text{if } \ell(w) + \ell(w') = \ell(ww')$$

Each element T_w possesses an inverse in $\mathcal{H}_{\mathcal{R}}(W_I)$. Explicitly, $T_s^{-1} = q^{-1}T_s - (1-q^{-1})T_e$. The $\mathbb{Z}$-linear map $\iota : \mathcal{H}_{\mathcal{R}}(W_I) \to \mathcal{H}_{\mathcal{R}}(W_I)$ induced by $T_w \mapsto (T_{w^{-1}})^{-1}$ and $q^{\frac{1}{2}} \mapsto q^{-\frac{1}{2}}$ induces a ring automorphism of order two known as the *Kazhdan-Lusztig involution.*

Definition 4.4.1. For each $y, w \in W_I$ such that $x \le w$ in Bruhat ordering, the *Kazhdan-Lusztig polynomial* $P_{x,w}(q) \in \mathbb{Z}[q]$ (considering q as an indeterminate) are uniquely characterized by the following three properties:

- $\iota\big(q^{-\ell(w)/2} \sum_{x\le w} P_{x,w}(q)T_x\big) = q^{\ell(w)/2} \sum_{x\le w} P_{x,w}(q)T_x$,
- $P_{x,w}(q)$ is a polynomial in degree $\le (\ell(w) - \ell(x) - 1)/2$ if $x \lneq w$,
- $P_{w,w}(q) = 1$.

If $x \not\le w$, we extend the definition of these polynomials by setting $P_{x,w}(q) = 0$. We will refer to $P_{x,w}$ for any $x, w \in W_I$ as KL-polynomials.

For any $\lambda \in \Lambda$, there is a unique element denoted w_λ which has the longest possible length in the double coset $Wt(\lambda)W \subset W_I$. When $\lambda \in \Lambda^+$, this element is $t(\lambda)w_\circ$ and $\ell(t(\lambda)w_\circ) = \ell(t(\lambda) + \ell(w_\circ) = 2\langle \lambda, \delta\rangle + \ell(w_\circ)$. For any $\mu \in \Lambda^+$, we have $\mu \preceq \lambda$ if and only if $w_\mu \le w_\lambda$.

Theorem 4.4.2 (Kato-Lusztig)**.** *Let $\lambda \in \Lambda^+$ and $\chi_\lambda \in \mathbb{Z}[\Lambda]^W$ denote the trace of $\hat{\mathbf{T}}$ on V_λ. Then $\chi_\lambda = \sum_{\mu \preceq \lambda} q^{-\langle \lambda, \delta \rangle} P_{w_\mu, w_\lambda}(q) \mathscr{S}(K\varpi^\mu K)$.*

Proof. See [HKP10, §7]. We also note that the proof provided in [Kat82] carries over with minor changes. □

Corollary 4.4.3. $\chi_\lambda = \sum_{\mu \preceq \lambda} P_{w_\mu, w_\lambda}(1) e^{W\lambda}$.

Proof. Using Macdonald's formula 4.2.10 for the expression $\mathscr{S}(K\varpi^\mu K)$ in the Kato-Lusztig formula 4.4.2, we obtain an expression for χ_λ as a linear combination in $e^{W\mu}$ which has coefficients in $\mathcal{R}_q$ (see [Kat82, Theorem 1.5]). Since the χ_λ is independent of q, we can formally replace q with 1 which yields the expression above. □

Let $\mathcal{I} = \mathcal{I}_q \subset \mathcal{R}_q$ denote the ideal generated by $q^{\frac{1}{2}} - 1$ and let $\mathcal{S} = \mathcal{S}_q := \mathcal{R}/\mathcal{I}$. For $f \in \mathcal{R}[\Lambda]^W$, we let $[f] \in \mathcal{S}[\Lambda]^W$ denote the image of f. Similarly, for $\xi \in \mathcal{H}_{\mathcal{R}}(K\backslash G/K)$, we let $[\xi] \in \mathcal{H}_{\mathcal{S}}(K\backslash G/K)$ denote the class of ξ. For $f = \sum_{\lambda \in \Lambda^+} c_\lambda e^{W\lambda} \in \mathcal{R}[\Lambda]^W$, let

$$\xi_f := \sum_{\lambda \in \Lambda^+} c_\lambda (K\varpi^\lambda K) \in \mathcal{H}_{\mathcal{R}}(K\backslash G/K).$$

Corollary 4.4.4. *Let $f \in \mathcal{R}[\Lambda]^W$ and $\xi = \mathscr{S}^{-1}(f)$. Then $[\xi] = [\xi_f]$.*

Proof. Since χ_λ form a $\mathbb{Z}$-basis for $\mathbb{Z}[\Lambda]^W$, it suffices to establish the claim for $f = \chi_\lambda$. But this follows by Kato-Lusztig formula and Corollary 4.4.3. □

Chapter 5

Decompositions of double cosets

In this chapter, we derive using the elementary theory of Tits systems a recipe for decomposing certain double cosets into their constituent left cosets. Invoking the existence of a such a system on the universal covering of the derived group of reductive group over a local field, we obtain a recipe for decomposing Hecke operators arising out of double cosets of what are known as 'parahoric' subgroups. The method used here for decomposing such double cosets is based on the one introduced in [Lan01] in the setting of split Chevalley groups. Proposition 5.2.8 – which is the main result of this chapter – is our primary tool for studying zeta element problems in Part II of this thesis.

5.1 Coxeter systems

Let (W, S) be a Coxeter system. Given $X \subset S$, we let $W_X \subset W$ be the group generated by X. Then (W_X, X) is a Coxeter system itself and $W_X \cap S = X$. We refer to subgroups of W obtained in this manner as *standard parabolic subgroups*. Let $\ell : W \to \mathbb{Z}$ denote the length function. Then $\ell_{|W_X}$ is the length function on W_X. Given $X, Y \subset W$ and $a \in W$, there is a unique element $w \in W_X a W_Y$ of minimal possible length and every $w' \in W$ can be written as $w' = xwy$ with $x \in W_X$, $y \in W_Y$ such that

$$\ell(w') = \ell(x) + \ell(w) + \ell(y).$$

We refer to the unique element w as the (X, Y)-*reduced element* of $W_X a W_Y$. We denote the set of (X, Y)-reduced elements of W by $[W_X \backslash W / W_Y]$. If $w \in W$ is $(X, \varnothing)$-reduced, then we have $\ell(xw) = \ell(x) + \ell(w)$ for *all* elements $x \in W_X$ and every element $\sigma \in W$ may be written *uniquely* as $\sigma = xw$ where $x \in W_X$ and w is $(X, \varnothing)$ reduced. Similarly for $(\varnothing, Y)$ elements. An element is (X, Y)-reduced if and only if it is $(X, \varnothing)$-reduced and $(\varnothing, Y)$-reduced. If σ is (X, Y)-reduced, then $W_X \cap \sigma W_Y \sigma^{-1}$ is a standard parabolic subgroup W_X generated by its intersection with X and we have

$$\ell(\tau\sigma w) = \ell(\tau\sigma) + \ell(\upsilon) = \ell(\tau) + \ell(\sigma) + \ell(\upsilon)$$

for any $\tau \in [W_X/W_X \cap \sigma W_Y \sigma^{-1}]$, $\upsilon \in W_Y$.

Suppose now that Ω is a group, $\Omega \times W \to W$ is a left action that keeps S (as a set) fixed. Then we may form the extension $\tilde{W} := W \rtimes \Omega$ and extend the length function $\ell : \tilde{W} \to \mathbb{Z}$ by declaring $\ell(\sigma\rho) = \ell(\sigma)$ for $\sigma \in W$, $\rho \in \Omega$. We refer to elements of Ω as length zero elements. Given $A \subset W$, we denote by A^ρ the set $\rho A \rho^{-1} \subset W$. Then $\rho W_X \rho^{-1} = W_{X^\rho} \subset W$ for $X \subset S$. Given $X, Y \subset S$, $b = a\rho \in \tilde{W}$ where $a \in W$, $\rho \in \Omega$, there is again a unique element $w \in W_X b W_Y$ of minimal possible length given by $w = \sigma\rho$ where σ is the (X, Y^ρ)-reduced element in $W_X a W_{Y^\rho}$. Moreover $W_X \cap w W_Y w^{-1} = W_X \cap \sigma(W_{Y^\rho})\sigma^{-1}$ is still a standard parabolic subgroup of W_X with respect to X and the length property above can be extended to $\sigma\rho \in \tilde{W}$. We continue to call $\sigma\rho$ as the (X, Y)-reduced elements in $W_X b W_Y \subset \tilde{W}$ and denote their collection by $[W_X \backslash \tilde{W} / W_Y]$.

Remark 5.1.1. Most of the content of the first paragraph is subsumed in [Bou02, §1 Ex. 3] from which we have also borrowed the terminology of an element being reduced in a double coset. Detailed proofs of all claims may be found in [Lan01, §4] and although the group W is assumed to be an affine Weyl group in *op. cit.*, the arguments are completely general. The square bracket notation for reduced elements is also borrowed from *op. cit.* See also [Hum90, §1.10, §5.12]. Groups $\tilde{W}$ as above are sometimes called 'quasi-Coxeter groups'.

5.2 Tits systems

Definition 5.2.1. A *Tits system* $\mathcal{T}$ is a quadruple (G, B, N, S) where G is a group, B, N are two subgroups of G and S is a subset of $N/(B \cap N)$ such that the following conditions are satisfied

(T1) $B \cup N$ generates G and $T = B \cap N$ is a normal subgroup of N

(T2) S generates the group $W = N/T$ and consists of elements of order 2

(T3) $sBw \subset BwB \cup BswB$ for all $s \in S$, $w \in W$.

(T4) $sBs \neq B$ for all $s \in S$

We call W the *Weyl group* of the system and let $\nu : N \to W$ denote the natural map.

Remark 5.2.2. For any $v, w \in W$, the products wB, Bvw, vBw etc are well-defined since if, say, $n_w \in N$ is a representative of w, then any other is given by $n_w t$ for $t \in T \subset B$ and one has $n_w t = t' n_w$ for some $t' \in T$ by normality of T in $B \cap N$.

For any such system, the pair (W, S) forms a Coxeter system. We denote by $\ell : W \to \mathbb{Z}$ the corresponding length function. The set S is uniquely determined by the groups (G, B, N) and the axioms (T1)-(T4). We therefore also say that (G, B, N) is a Tits system or that (B, N) constitutes a Tits system for G. The subsets $BwB \subset G$ for $w \in W$ are called *Bruhat cells* which provide a decomposition

$$G = \bigsqcup_{w \in W} BwB$$

called the *Bruhat-Tits decomposition*. If $w = s_1 \cdots s_{\ell(w)}$ is a reduced decomposition of W, then $BwB = Bs_1B \cdot Bs_2B \cdots Bs_{\ell(w)}B$. A subgroup of G that contains B is called a *standard parabolic*. There is a bijection between such subgroups of G and subsets X of S given in one direction as follows: given $X \subset S$, we let $K_X := BW_XB \supset B$ where $W_X \subset W$ is the group generated by X. Then K_X is the standard parabolic subgroup associated with X. In particular, $K_\varnothing = B$, $K_S = G$. If N' is a subgroup of N such that $\nu(N') = W_X$, then (G_X, B, N_X, X) is a Tits system itself. If $X, Y \subset S$, there is a bijection

$$K_X \backslash G / K_Y \cong W_X \backslash W / W_Y$$

given by sending $K_X w K_Y \mapsto W_X w W_Y$. For Z a normal subgroup of G contained in B, denote $G' = G/Z$ and let $B' = B/Z$, $N' = N/Z$ denote the images of B, N in G. Set $W' = B'/(B' \cap N')$ and S' the image of S under $W \to W'$. Then $(W, S) \to (W', S')$ is an isomorphism of Coxeter groups and (G', B', N', S') is a Tits system which is said to be *induced* by (G, B, N, S).

Definition 5.2.3. Let (G, B, N, S) be a Tits system. We say that the system is *commensurable* if BsB/B is finite for all $s \in S$. Then BwB/B is finite for all $w \in W$. We let q_w denote the quantity $|BwB/B| = [B : B \cap wBw^{-1}]$.

Lemma 5.2.4 (Braid Relations)**.** *Let (G, B, N, S) be a commensurable Tits system. For any $\sigma, \tau \in W$ such that $\ell(\sigma) + \ell(\tau) = \ell(\sigma\tau)$, $q_{\tau\sigma} = q_\tau q_\sigma$*

Proof. Since BwB/B is finite for all $w \in W$, one may form the convolution algebra $\mathcal{H}_{\mathbb{Z}}(B\backslash G/B)$ with product $\mathrm{ch}(BwB) * \mathrm{ch}(BvB)$ given as in 2.3.1. The linear map ind : $\mathcal{H}_{\mathbb{Z}}(B\backslash G/B) \to \mathbb{Z}$ given by $\mathrm{ch}(BwB) \mapsto q_w$ is then a homomorphism of rings. If $s \in S$, $w \in W$. Then $\mathrm{ch}(BwB) * \mathrm{ch}(BsB) = \sum_{u \in W} c^u_{w,s} \mathrm{ch}(BuB)$ with $c^\mu_{w,s} = |(BwB \cap uBsB)/B|$. Moreover, $c^u_{w,s} \neq 0 \iff BuB \subset BwBsB$. Suppose that $\ell(w) + \ell(s) = \ell(s)$. Then $BwBsB = BwsB$, whence $c^\mu_{w,s} = 0$ for $u \neq ws$. Since

$$wsBsB \subset wBsBsB = w(B \cup BsB) = wB \cup BwsB$$

have $BwB \cap (wsBsB) \subset BwB \cap (wB \cup BwsB) = wB$ as BwB, $BwsB$ are disjoint. Therefore $c^{ws}_{w,s} = 1$ and $\mathrm{ch}(BwB) * \mathrm{ch}(BsB) = \mathrm{ch}(BwsB)$. Repeating this argument by writing $\sigma = ws = w's's$, we see that

$\mathrm{ch}(B\sigma B) = \mathrm{ch}(Bs_1B) * \cdots * \mathrm{ch}(Bs_{\ell(w)}B)$ where $\sigma = s_1 \cdots s_{\ell(w)}$ is a reduced decomposition. Since ind is a homomorphism, we see that $q_\sigma = q_{s_1} \cdots q_{s_{\ell(w)}}$ and similarly for q_τ. The claim follows. □

Definition 5.2.5. Let (G, B, N, S) be a Tits system and $\varphi : G \to \tilde{G}$ be a homomorphism of groups. Then φ is said to be (B, N)-*adapted* if

(i) $\ker \varphi \subset B$,

(ii) for all $g \in \tilde{G}$, there is $h \in G$ such that $g\varphi(B)g^{-1} = \varphi(hBh^{-1})$ and $g\varphi(N)g^{-1} = \varphi(hNh^{-1})$.

For any such map, $\varphi(G) \triangleleft \tilde{G}$ and the induced map $G/\ker(\varphi) \hookrightarrow \tilde{G}$ is adapted with respect to the induced Tits system on $G/\ker\varphi$.

Let $\varphi : G \to \tilde{G}$ as above be injective and consider G a subgroup of $\tilde{G}$. Denote by $T = B \cap N$, $W = N/T$ be the Weyl groups. Let $\hat{B}$, $\hat{N}$ denote respectively the normalizers of B, N in $\tilde{G}$ and let $\Gamma = \hat{B} \cap \hat{N}$, Then $\tilde{G} := \Gamma G$ and the groups $\Omega := \Gamma/(\Gamma \cap B)$, $\tilde{G}/G$ and $\hat{B}/B$ are canonically isomorphic. Let $\tilde{N} = \Gamma B$, $\tilde{T} = (\Gamma \cap B) \cdot T$. The inclusion of N (resp. Γ) in $\tilde{N}$ allows us to identify W (resp. Ω) as a subgroup of W. The group Ω normalizes W and induces automorphism of (W, S) (i.e. Ω permutes S) and we have $\tilde{W} = W \rtimes \Omega$. We then get a *generalized Bruhat-Tits decomposition*

$$\tilde{G} = \bigsqcup_{w \in \tilde{W}} BwB.$$

Similarly, if $X, Y \subset S$, $K_X, K_Y \subset G$ denote the corresponding groups, we have $K_X \backslash \tilde{G}/K_Y \cong W_X \backslash \tilde{W}/W_Y$.

Remark 5.2.6. For the general theory of Tits systems, we refer the reader to [Bou02, Ch. 4 §2]. The material on (B, N)-adapted morphisms and commensurable Tits systems is developed in Exercises 2, 8, 22, 23, 24 of *op. cit.* While most of these exercises are straightforward, we found 24(b) (on braid relations) particularly tricky. As we were unable to find a better reference[1], we have included it as Lemma 5.2.4. The terminology of Definition 5.2.5 is taken from [BT72, Ch. I §2.13]. This notion is referred to as *generalized Tits systems* in [Iwa66].

Assume for the rest of this section that (G, B, N, S) is a commensurable Tits system and $\varphi : G \hookrightarrow \tilde{G}$ is a (B, N)-adapted inclusion. Retain also the notations W, $\tilde{W}$, Ω introduced above. For each $s \in S$, let $\hslash_s \subset G$ denote a set of representatives of $B/(B \cap sBs^{-1})$ (so $|\hslash_s| = q_s$) and let $\tilde{s}$ denote a lift of s to N under ν (so that $\nu(\tilde{s}) = s$). Define

$$g_s : \hslash_s \to G, \quad \hslash_s \ni \kappa \mapsto \kappa\tilde{s}$$

considered as a map of sets. Fix a $w = \sigma\rho \in \tilde{W}$ where $\sigma \in W$, $\rho \in \Omega$ and let $m = m_w := \ell(\sigma)$ the length of σ, $r(\sigma) = (s_1, \ldots, s_m)$ a reduced word decomposition of σ. Denote by $\hslash_{r(\sigma)}$ the product $\hslash_{s_1} \times \hslash_{s_2} \times \cdots \times \hslash_{s_m}$.

[1]they do seem to be rare – see the comment in [Vig16, Proposition 3.38]

Lemma 5.2.7. *$BwB = \bigsqcup_{\vec{\kappa}\in\mathscr{k}_{r(\sigma)}} g_{s_1}(\kappa_1)g_{s_2}(\kappa_2)\cdots g_{s_m}(\kappa_m)\rho B$ where κ_i denotes the i-th component of $\vec{\kappa}$.*

Proof. We have $BwB = B\sigma B\rho = Bs_1B\cdots Bs_kB\rho$. Now

$$\begin{aligned} B\sigma B &= \bigcup_{\kappa_1\in\mathscr{k}_1} g_{s_1}(\kappa_1)Bs_2B\cdots Bs_mB \\ &= \bigcup_{\substack{(\kappa_1,\kappa_2)\\ \in\mathscr{k}_1\times\mathscr{k}_2}} g_{s_1}(\kappa_1)g_{s_2}(\kappa_2)Bs_3B\cdots Bs_mB = \cdots = \bigcup_{\vec{\kappa}\in\mathscr{k}_{r(w)}} g_{s_1}(\kappa_1)\cdots g_{s_m}(\kappa_m)B \end{aligned}$$

As $|B\sigma B/B| = q_\sigma = q_{s_1}\cdots q_{s_m}$, the union above is disjoint. Multiplying each coset in the decomposition above on the right by ρ and moving it inside, we get the desired decomposition of BwB. ☐

Retain the notations $w, \sigma, \rho,\ m$. For $K\subset B$, we denote by $\mathcal{X}_{r(\sigma),\,\rho,\,K} : \mathscr{k}_{r(\sigma)} \to G/K$ the map $\vec{\kappa}\mapsto g_{s_1}(\kappa_1)g_{s_2}(\kappa_2)\cdots g_{s_m}(\kappa_m)\tilde{\rho}B$ (where we have suppressed the dependency on various choices). Then in this notation,

$$BwB = \bigsqcup_{\vec{\kappa}\in\mathscr{k}_{r(\sigma)}} \mathcal{X}_{r(\sigma),\rho,B}(\vec{\kappa}).$$

In particular, the image of $\mathcal{X}_{r(\sigma),\,\rho,\,K}$ in G/K is independent of all the choices involved. Since we will only be interested in the image of $\mathcal{X}_{r(\sigma),\,\rho,\,K}$, we will abuse our notation and denote this map simply by $\mathcal{X}_{w,K}$ or even by $\mathcal{X}_w$ if it is understood that these are representatives of left cosets of some fixed K. Similarly, we will denote $\mathscr{k}_{r(\sigma)}$ by $\mathscr{k}_w$.

Proposition 5.2.8. *Let $X, Y\subset S$, W_X, W_Y be the subgroups of W generated by X, Y respectively and let $K_X = BW_XB$, $K_Y = BW_YB$. For $w\in[W_X\backslash\tilde{W}/W_Y]$,*

$$K_XwK_Y = \bigsqcup_{\tau}\bigsqcup_{\vec{\kappa}\in\mathscr{k}_{\tau w}} \mathcal{X}_{\tau w}(\vec{\kappa})K_Y$$

where τ runs over $[W_X/(W_X\cap wW_Yw^{-1})]$. In particular, $|K_XwK_Y/K_Y| = \sum_\tau |\mathscr{k}_{\tau w}|$.

Proof. First note that $K_XwK_Y = \bigcup_{x\in W_X} BxBwBK_Y = \bigcup_{x\in W_X} BxwK_Y$, the second equality following since $\ell(xw) = \ell(x)+\ell(w)$. Now $BxwK_Y = Bx'wK_Y$ if and only if $xwW_Y = x'wW_Y$, since $B\backslash\tilde{G}/K_Y$ is in bijection with $\tilde{W}/W_Y$. Therefore,

$$K_XwK_Y = \bigsqcup_\tau B\tau wK_Y$$

with τ running over a set of representatives of $W_X/(W_X\cap wW_Yw^{-1})$ which are free to take to be in the set $A := [W_X/(W_X\cap wW_Yw^{-1})]\subset W_X$. Fix $\tau\in A$. We have $B\tau wK_Y = B\tau wBK_Y = \bigcup_{\kappa\in\mathscr{k}_{\tau w}}\mathcal{X}_{\tau w}(\vec{\kappa})K_Y$. Say $\vec{\kappa}_1,\vec{\kappa}_2\in\mathscr{k}_{\tau w}$ are such that $g_1K_Y = g_2K_Y$ where $g_i := \mathcal{X}_{\tau w}(\vec{\kappa}_i)\in\tilde{G}$ for $i = 1, 2$. As $K_Y = BW_YB$, we have $g_1K_Y = \bigsqcup_{y\in W_Y}\bigsqcup_{\vec{\kappa}\in\mathscr{k}_y} g_1\mathcal{X}_y(\vec{\kappa})B$ by Lemma 5.2.7. As $g_2B\subset g_2K_Y = g_1K_Y$, there exists $y\in W_Y$, $\vec{\kappa}_y\in\mathscr{k}_y$, such that

$$g_1\mathcal{X}_y(\vec{\kappa}_y)B = g_2B.$$

Now $Bg_1\mathcal{X}_y(\vec{\kappa}_y)B = B\tau w\mathcal{X}_y(\vec{\kappa}_y)B \subset B\tau wByB$ and $B\tau wByB = B\tau wyB$ since $\ell(\tau wy) = \ell(\tau w) + \ell(y)$ by reducedness properties of τ, w (see §5.1). Therefore, $g_2B = g_1\mathcal{X}_y(\kappa_y)B \subset B\tau wyB$, and since g_2B is also contained in $B\tau wB$, we see that $B\tau wB = B\tau wyB$. This can only happen if $y = 1_{W_Y}$ which in particular means that $\mathscr{k}_y$ is a singleton and $\mathcal{X}_y(\vec{\kappa}_y)B = B$. We therefore have $g_1B = g_1\mathcal{X}_y(\vec{\kappa})B = g_2B$ which in turn implies that $\kappa_1 = \kappa_2$. The upshot is that $B\tau wK_Y = \bigsqcup_{\vec{\kappa}\in\mathscr{k}_{\tau w}} \mathcal{X}_{\tau w}(\vec{\kappa})K_Y$ is a disjoint union for all $\tau \in A$, whence

$$K_XwK_Y = \bigsqcup_{\tau\in A} B\tau wK_Y = \bigsqcup_{\tau\in A}\bigsqcup_{\vec{\kappa}\in\mathscr{k}_{\tau w}} \mathcal{X}_{\tau w,K_Y}(\vec{\kappa})$$

which completes the proof. □

Remark 5.2.9. The proof of Proposition 5.2.8 is inspired by [Lan01, Theorem 5.2].

5.3 Reductive groups

In this section, we recall the relevant results from the theory of Bruhat-Tits buildings. The reader may consult [Cas80, §1] and [SS97, §2] as reference for the material provided here.

We retain the notations introduced in Ch. 4. Additionally, we let $\tilde{\mathbf{G}}$ be the simply connected covering of the derived group $\mathbf{G}^{\mathrm{der}}$ of $\mathbf{G}$ and let $\psi : \tilde{\mathbf{G}} \to \mathbf{G}$ denote the resulting map. For a group $\mathbf{H} \subset \mathbf{G}$, we denote by $\tilde{\mathbf{H}} \subset \tilde{\mathbf{G}}$ the pre-image of $\mathbf{H}$ under ψ. Let $\mathscr{B}$ be the Bruhat-Tits building of $\tilde{G} = \tilde{\mathbf{G}}(F)$ and let $\mathscr{A} \subset \mathscr{B}$ be the apartment stabilized (as a subset) by $\tilde{A} = \tilde{\mathbf{A}}(F)$. By definition $\mathscr{A}$ is an affine space under the real vector space $\tilde{V} := X_*(\tilde{\mathbf{A}}) \otimes \mathbb{R}$. Let $\nu : \tilde{M} \to V$ be the unique morphism determined by the condition

$$\chi(\nu(a)) = -\mathrm{ord}\big(\chi(a)\big)$$

for all $a \in \tilde{A}$, χ a F-rational cocharacter of $\tilde{\mathbf{M}}$. The action of $m \in \tilde{M} := \tilde{\mathbf{M}}(F) \subset \tilde{G}$ on $x \in \mathscr{A}$ is then given by the translation $x \mapsto \nu(m) + x$. The kernel of ν is a maximal compact open subgroup $\tilde{M}^\circ$. Set $\tilde{A}^\circ := \tilde{A} \cap \tilde{M}^\circ$. Then $\tilde{A}/\tilde{A}^\circ = \tilde{M}/\tilde{M}^\circ$ via the inclusion $\tilde{A} \hookrightarrow \tilde{M}$ and the image $\tilde{\Lambda} \subset \tilde{V}$ of ν is identified with $X_*(\tilde{\mathbf{A}})$. Let $\tilde{N}$ denote the stabilizer of $\mathscr{A}$ (as a subset of $\mathscr{B}$). The map ν admits a unique extension $\tilde{N} \to \mathrm{Aut}(\mathscr{A})$ where $\mathrm{Aut}(\mathscr{A})$ denotes the group of affine automorphisms of $\mathscr{A}$. The action of $\tilde{G}$ on $\mathscr{B}$ is then uniquely determined by this extension.

Fix $x_0 \in \mathscr{A}$ a hyperspecial point via which we identify $\tilde{V}$ with $\mathscr{A}$. Then ν identifies $\tilde{N}/\tilde{M}^\circ$ with $W_{\mathrm{aff}} = \tilde{\Lambda} \rtimes W$. Let $C \subset \mathscr{A}$ be an alcove (affine Weyl chamber) containing x_0 such that the set S_{aff} chosen in Ch. 4 is identified with the set of reflections in the walls of C and let $\tilde{B}$ be the (pointwise) stabilizer of C in $\tilde{G}$. Then $(\tilde{G}, \tilde{B}, \tilde{N})$ is a Tits system with Weyl group W_{aff} and the morphism $\psi : \tilde{G} \to G$ is $(\tilde{B}, \tilde{N})$-adapted.

The action of G on $\tilde{G}$ induced by the natural map $\mathbf{G} \to \mathrm{Aut}(\tilde{\mathbf{G}})$ determines an action of G on $\mathscr{B}$. Let

$$G^1 := \{g \in G \,|\, \chi(g) = 1 \text{ for } \chi : \mathbf{G} \to \mathbb{G}_m\} .$$

Then $G^1/\psi(\tilde{G})$ is compact. Let $B \subset G^1$ be the compact open sub-group of elements that stabilizes C (as a subset of $\mathscr{B}$) and $K \subset G^1$ the compact open sub-group of elements stabilizing x_0. Then B is a Iwahori subgroup of G and K a hyperspecial subgroup. In particular, $K = \bigsqcup BwB$ for $w \in W$. In what follows, we will assume that the group scheme $\mathscr{G}$ in §4.2 is chosen so that $\mathscr{G}(\mathscr{O}_F) = K$.

Finally, let G_0 be the intersection of G^1 with the subgroup of G that preserves the alcove C and let $N_0 = G_0 \cap N$. Then $(G_0, B, N_0, S_{\mathrm{aff}})$ is a Tits system with Weyl group W_{aff} and the inclusion $G_0 \to G$ is (B, N_0) adapted. One may therefore apply the result of Proposition 5.2.8 to the inclusion $G_0 \to G$.

Remark 5.3.1. If $s \in W_{\mathrm{aff}}$ denotes the reflection in a wall of the alcove, $B/(B \cap sBs)$ has cardinality $q^{d(s)}$ for some $d(s) \in \mathbb{Z}$ and a set of representatives can be taken in the F points of the root group U_α where $\alpha \in \Phi_F$ is the vector part of the corresponding affine root associated with s. The precise description of $d(s)$ is given in terms of the root group filtrations and is recorded on the corresponding *local index* which is the Coxeter diagarm of W_{aff} with additional data. When $\mathbf{G}$ is split, $d(s) = 1$. We refer the reader to [Tit79] for more details.

Convention 5.3.2. In what follows we denote the Iwahori subgroup $B \subset G$ by the letter I.

5.4 Decomposition in practice

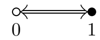
Let us illustrate the recipe of Proposition 5.2.8 by a few examples for GL_2. More examples can be found in Part II of this thesis.

5.4.1 Decompositions for GL_2

Retain the notations introduced in §4.2.1. Let $\chi^\vee = f_1 - f_2$ denote the coroot associated with χ and $s = s_\chi$ denote the unique non-trivial element in W. Let

$$w_0 = \begin{pmatrix} & \frac{1}{\varpi} \\ \varpi & \end{pmatrix}, \quad w_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \rho = \begin{pmatrix} & 1 \\ \varpi & \end{pmatrix}$$

Then w_0, w_1, ρ normalize A and $\rho w_0 \rho^{-1} = w_1$, $\rho w_1 \rho^{-1} = w_0$. Under the conventions introduced, the matrices w_0, w_1 represent the two simple reflections $S_{\mathrm{aff}} = \{t(\chi^\vee)s, s\}$ of the affine Weyl group $\mathbb{Z}\langle f_1 - f_2\rangle \rtimes W$. The element ρ represents $t(-f_2)s_\chi \in \Lambda \rtimes W$ and is a generator of $\Omega = W_I/W_{\mathrm{aff}}$. The action of ρ on $\mathscr{B}$ preserves the alcove C and permutes the two walls corresponding to w_0, w_1. We say that ρ induces an automorphism of the Coxeter-Dynkin diagram



given by switching the two nodes. Let I denote the Iwahori subgroup corresponding to the set of affine roots χ and $\chi+1$ (considered as functions on the space $\Lambda\otimes\mathbb{R}$). Then I is the usual Iwahori subgroup of $\mathrm{GL}_2(\mathscr{O}_F)$ given by matrices that reduce to upper triangular matrices modulo ϖ. Let $x_0, x_1 : \mathbb{G}_a \to \mathrm{GL}_2$ denote the following 'root group' maps

$$x_0 : u \mapsto \begin{pmatrix} 1 & \\ \varpi u & 1 \end{pmatrix}, \qquad x_1 : u \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

and let $[\mathscr{k}] \subset \mathscr{O}_F$ denote a set of representatives of $\mathscr{k}$. Then $x_i([\kappa])$ constitute a set of representatives for $I/(I\cap w_i I w_i)$ for $i = 0, 1$.

Let $g_{w_i} : [\mathscr{k}] \to G$ be the maps $\kappa \mapsto x_i(\kappa)w_i$. If $w = s_{w_1}\cdots s_{w,\ell(w)}\rho_w \in W_I$ is a reduced word decomposition (where $s_{w,i} \in S_{\mathrm{aff}}$, $\rho_w \in \Omega = \rho^{\mathbb{Z}}$), let

$$\mathcal{X}_w : [\mathscr{k}]^{\ell(w)} \to G/K$$
$$(\kappa_1,\ldots,\kappa_{\ell(w)}) \mapsto g_{s_{w,1}}(\kappa_1)\cdots g_{s_{w,\ell(w)}}(\kappa_{\ell(w)})\rho_w K$$

The maps $\mathcal{X}_w$ may be thought of as parametrizing certain *Schubert cells*[2] and will be referred to as such. Proposition 5.2.8 provides a decomposition of double cosets $K\varpi^\lambda K$ for $\lambda\in\Lambda$ in terms of these maps. Let us illustrate the decomposition with a few simple examples.

Example 5.4.1. Let $\lambda = f_1$. Then $K\varpi^\lambda K = K\varpi^{\lambda^{\mathrm{opp}}}K = K\rho K$. The decomposition reads $K\varpi^\lambda K = \mathrm{im}(\mathcal{X}_\rho)\sqcup\mathrm{im}(\mathcal{X}_{w_1\rho})$. Explicitly,

$$\mathrm{im}(\mathcal{X}_\rho) = \left\{\begin{pmatrix} 1 & \\ & \varpi\end{pmatrix}K\right\}, \qquad \mathrm{im}(\mathcal{X}_{w_1\rho}) = \left\{\begin{pmatrix} \varpi & \kappa \\ & 1\end{pmatrix}K \;\middle|\; \kappa\in[\mathscr{k}]\right\}.$$

There are $q+1$ left cosets contained in the double coset $K\varpi^{f_2}K$.

Example 5.4.2. Let $\lambda = 2f_2$. Then $K\varpi^\lambda K = Kw_0\rho^2K$ and $w = w_0\rho^2 \in W_I$ if of shortest possible length. The decomposition therefore reads $K\varpi^\lambda K = \mathrm{im}(\mathcal{X}_{w_0\rho^2})\sqcup\mathrm{im}(\mathcal{X}_{w_1w_0\rho^2})$. Explicitly,

$$\mathrm{im}(\mathcal{X}_{w_0\rho^2}) = \left\{\begin{pmatrix} 1 & \\ \kappa\varpi & \varpi^2\end{pmatrix}K \;\middle|\; \kappa\in[\mathscr{k}]\right\}, \qquad \mathrm{im}(\mathcal{X}_{w_1w_0\rho^2}) = \left\{\begin{pmatrix} \varpi^2 & \kappa_1\varpi+\kappa_2 \\ & 1\end{pmatrix}K \;\middle|\; \kappa_1,\kappa_2\in[\mathscr{k}]\right\}.$$

There are $q(q+1)$ cosets contained in $K\varpi^\lambda K$. Compare the decomposition of Example 4.2.2.

As seen from the examples, the Schubert cell maps $\mathcal{X}_w$ are recursive in nature and going from one Schubert cell to the 'next' amounts to applying a reflection operation on rows and adding multiple of one

[2]See §9.3 that makes the connection with classical Schubert cells of Grassmannians more precise.

row to another. We also note that the actual product of matrices in $\mathcal{X}_w$ in the example above may not necessarily be upper or lower triangular as displayed e.g. $\mathcal{X}_{w_0\rho^2}(\kappa) = g_{w_0}(\kappa)\rho^2$ equals

$$\begin{pmatrix} & 1 \\ \varpi^2 & \kappa\varpi \end{pmatrix}.$$

However, since we are only interested left K-coset representatives, we can replace $\mathcal{X}_{w_0\rho^2}(\kappa)$ with $\mathcal{X}_{w_0\rho^2}(\kappa)\gamma$ for any $\gamma \in K$. In general, multiplying by a reflection matrix on the left has the effect of 'jumbling up' the diagonal entries of the matrix. While performing these computations, it is desirable to keep the 'cocharacter' entries on the diagonal and one may do so by applying a corresponding reflection operation on columns using elements of K. In the computations done in Part II of this thesis, this will be done without any comment.

Remark 5.4.1. In computing $\mathcal{X}_w$, one can establish certain 'rules' specific to the group at hand that dictate where the entries of the a particular cell are supposed to be written depending on the permutation of λ described by the word. For instance, the rule of filling a Schubert cell

$$\begin{pmatrix} \varpi^a & \square \\ \diamond & \varpi^b \end{pmatrix}$$

as displayed above is as follows:

- if $a \geq b$, the $\diamond$ entry is zero and the $\square$ entry runs over a set of representatives of $\varpi^a \mathscr{O}_F / \varpi^b \mathscr{O}_F$
- if $a < b$, then $\square$ entry is zero, and the $\diamond$ entry runs over representatives of $\varpi^b \mathscr{O}_F / \varpi^{a+1} \mathscr{O}_F$.

A decomposition of this type is proved in Proposition 9.3.3. Another instance where such a rule is used is in the proof of Lemma 13.4.3(d). We however note that such rules are not always possible (or even useful) to write down.

5.4.2 Reduced words

Retain the notations introduced in Ch. 4. Let $\lambda \in \Lambda^+$. The recipe of Proposition 5.2.8 requires writing the reduced decomposition of the word $w \in W_I$ of minimal possible length such that $K\varpi^\lambda K = KwK$. This is of course the same for $K\varpi^{\lambda^{\mathrm{opp}}}K$. We may equivalently think of W_I as $t(\Lambda) \rtimes W$ and the length we seek is the minimal possible length of elements in $Wt(-\lambda^{\mathrm{opp}})W \subset t(\Lambda) \rtimes W$ which is the same as the minimal possible length $\ell_{\min}(t(\lambda))$ of elements in $Wt(\lambda)W$. Let $\Psi = \Phi_F^{\mathrm{red}} \subset \Phi_F$ denote the subset of non-divisible roots and let $\Psi^+ = \Psi \cap \Phi_F^+$.

Lemma 5.4.2. *For any $\lambda \in \Lambda$, the minimal possible length of elements in $t(\lambda)W \subset W_I$ is achieved by a unique element. The length of this element is given by*

$$\sum_{\alpha \in \Psi_\lambda} |\langle \lambda, \alpha \rangle| + \sum_{\alpha \in \Psi^\lambda} (\langle \lambda, \alpha \rangle - 1)$$

where $\Psi_\lambda = \{\alpha \in \Psi^+ \,|\, \langle \lambda, \alpha \rangle \le 0\}$*,* $\Psi^\lambda = \{\alpha \in \Psi^+ \,|\, \langle \lambda, \alpha \rangle > 0\}$*. If moreover* $\lambda \in \Lambda^+$*, this length equals* $\ell_{\min}(t(\lambda))$*.*

Proof. See [IM65, §1]. □

Once the minimal possible length is known, the word w can be found by trial and error. Let us illustrate by a simple example.

Example 5.4.3. Retain the notation of §4.2.1. Let $\lambda = 5f_1 \in \Lambda^+$. Then

$$\ell_{\min}(t(\lambda)) = \langle 5f_1, e_1 - e_2 \rangle - 1 = 5 - 1 = 4.$$

Say $w \in W_I$ is of length 4 and $K\varpi^\lambda K = KwK$. Since $\det(\varpi^\lambda) = 5$, we may assume that $w = v\rho^5$ where v is a word on $S_{\text{aff}} = \{w_0, w_1\}$. Now the final letter of v cannot be w_0, since $\rho w_0 \rho^{-1} = w_1 \in K$. Thus we may assume that $v = v' w_1$. Since we can only place w_0 next to w_1 for a reduced word, we see that the only possible choice is $w = w_0 w_1 w_0 w_1 \rho^3$.

5.4.3 Weyl orbit diagrams

Retain the notations introduced in Ch. 4. Besides the usual Bruhat order $\ge$ on the Weyl group W, there is another partial order that will be useful to us. We say that $w \succeq x$ for $w, x \in W$ if there exists a reduced word decomposition for x which appears as a consecutive string on the left of some reduced word for w. The pair $(W, \succeq)$ is then a lattice [BB05, Chapter 3] and is known as the *weak (left) Bruhat order*.

Definition 5.4.3. Let $\lambda \in \Lambda^+$. The *Weyl orbit diagram* of λ is the Hasse diagram on the set of representatives of W/W^λ of minimal possible length with respect to order $\succeq$.

Let $\lambda \in \Lambda^+$ and suppose that $K\varpi^\lambda K = Kw_\lambda K$ for a unique $w_\lambda \in W_I$ of minimal possible length. Then $w = \varpi^{\lambda^{\text{opp}}} \sigma_\lambda$ for a unique $\sigma_\lambda \in W$ and $W \cap w_\lambda W w_\lambda^{-1}$ is just the stabilizer of λ^{opp} in W. We may thus make the identification

$$[W/(W \cap w_\lambda W w_\lambda^{-1})] \simeq W/W^\lambda$$

But we also have $W/W^\lambda \simeq W\lambda$ via $v \mapsto v \cdot \lambda^{\text{opp}} \in W\lambda$ for $v \in W/W^\lambda$. Thus the Weyl orbit diagram may also be seen as a diagram on the set of permutations of λ. The decomposition of $K\varpi^\lambda K/K$ as described by Proposition 5.2.8 can then be seen as a collection of Schubert cells $\mathcal{X}_\mu$, one for each $\mu \in W\lambda$.

The diagram can also be used to read of the cardinalities of each Schubert cell. If q_s denotes the cardinality $|IsI/I|$ for $s \in S_{\text{aff}}$, then the cardinality of the cell $\mathcal{X}_v$ is the product of q_s taken one for each letter in a reduced word decomposition for v. In particular, if $\mathbf{G}$ is split (so that $q_s = q$ for all $s \in S_{\text{aff}}$), this cardinality is equal to $q^{\ell(v)}$. If $\mu \in W\lambda$ is the cocharacter corresponding to $v \in [W/W^\lambda]$, the length $\ell(v)$ is the minimal possible length of elements in $t(\mu)W$ which may be computed using Lemma 5.4.2.

In the following, we adapt the convention of drawing the Weyl orbit diagrams of $\lambda \in \Lambda^+$ from left to right, starting from the anti-dominant cocharacter and ending in the dominant one. The permutation of λ corresponding to the node is then the shape of (the matrices in) the corresponding Schubert cell. For example, in the notations of §5.4.1, the Weyl orbit diagram of f_1 is

$$f_2 \xrightarrow{s_\chi} f_1$$

and the matrices in $\mathrm{im}(\mathcal{X}_\rho)$, $\mathrm{im}(\mathcal{X}_{w_1\rho})$ in Example 5.4.1 have 'diagonal entries' given by ϖ^{f_2}, ϖ^{f_1} respectively. We will often omit the explicit cocharacters on the nodes in these diagrams and only display the labels of the arrows. See Part II of this thesis (e.g. diagram 10.1) for more illustrative examples.

Remark 5.4.4. Let us note however that the shape (in the sense of Definition 4.2.6) of the matrices in these cells may not match the corresponding cocharacter. In Example 4.2.2, the shape of the matrices that appear in the decomposition of $K\varpi^{2f_1}K$ can be $2f_1$, $2f_2$ or $f_1 + f_2$. Note also that $2f_1 \succ f_1 + f_2$ as guaranteed by Proposition 4.2.3.

5.5 Miscellaneous results

The following assortment of results will be useful in determining the structure of mixed double cosets in Part II of this thesis.

Lemma 5.5.1. *Suppose G is a group, $X, Y \subset G$ are subgroups. Then for $\sigma, \tau \in G$, $X\sigma Y = X\tau Y$ only if $X \cap \sigma Y \sigma^{-1}$ and $X \cap \tau Y \tau^{-1}$ and X-conjugate.*

Proof. $X\sigma Y = X\tau Y \iff \sigma = x\tau y$ for $x \in X$, $y \in Y \implies X \cap \sigma Y \sigma^{-1} = x(X \cap \tau Y \tau^{-1})x^{-1}$. □

Lemma 5.5.2. *Let $\iota : H \hookrightarrow G$ be an inclusion of groups, $K \subset G$ a subgroup and $U = K \cap H$. Then for any $t_1, t_2 \in H$, $g \in G$, $Ut_1gK = Ut_2gK$ if and only if $Ut_1H_g = Ut_2H_g$ where $H_g = H \cap gKg^{-1}$. Moreover for any $t \in H$, the index $[H \cap tgK(tg)^{-1} : U \cap tgK(tg)^{-1}]$ is equal to $[H_g : H_g \cap tUt^{-1}]$.*

Proof. The map (of sets) $H \twoheadrightarrow HgK/K$, $h \mapsto hgK$ induces a H-equivariant bijection $H/H_g \xrightarrow{\sim} HgK/K$ where H acts by left multiplication. Thus the orbits of U on the two coset spaces are identified i.e. $U\backslash H/H_g \xrightarrow{\sim} U\backslash HgK/K$ which proves the first claim. For $t \in H$, let $\mathrm{inn}_{t^{-1}} : G \to G$ denote conjugation by $\iota(t^{-1})$. Then $\mathrm{inn}_{t^{-1}}(H \cap tgK(tg)^{-1}) = H \cap gKg^{-1}$ and $\mathrm{inn}_{t^{-1}}(U \cap tgK(tg)^{-1}) = tUt^{-1} \cap H_g$. Since $\mathrm{inn}(t) : G \to G$ is a bijection, the second claim follows. □

Proposition 5.5.3. *Let H be a group, $\sigma \in H$ an element and U, U_1, X be subgroups of H such that $U_1\sigma U/U$, XU_1/U_1 are finite sets and $U_2 = XU_1$ is a group. Then $U_2\sigma U/U$ is finite and*

$$e \cdot \mathrm{ch}(U_2\sigma U) = \sum\nolimits_{\delta} \mathrm{ch}(\delta U_1 \sigma U)$$

where $\mathrm{ch}(Y) : H \to \mathbb{Z}$ *denotes the characteristic of $Y \subset H$, $\delta \in X$ run over representatives of $X/(X \cap U_1)$ and $e = [U_2 \cap \sigma U \sigma^{-1} : U_1 \cap \sigma U \sigma^{-1}]$. Moreover, if $U_2 \cap \sigma U\sigma^{-1}$ is equal to the product of $X \cap \sigma U \sigma^{-1}$ and $U_1 \cap \sigma U \sigma^{-1}$, then $e = [X \cap \sigma U \sigma^{-1} : X \cap U_1 \cap \sigma U \sigma^{-1}]$.*

Proof. Let $W_i := U_i \cap \sigma U \sigma^{-1}$ for $i = 1, 2$, $Z := X \cap U_1$ and let $\gamma_1, \ldots, \gamma_m \in U_1$ be representatives of U_1/W_1, $\delta_1, \cdots, \delta_n \in X$ be representatives of X/Z. We first show that $\delta_j\gamma_i$ form a complete set of distinct representatives of the coset space U_2/W_1. Let $x \in X$, $u \in U_1$. Then there exists a $z \in Z$, $w \in W_1$ and (necessarily unique) integers i, j such that $xz = \delta_j$, $z^{-1}uw = \gamma_i$. In other words, $xuW_1 = (xz)(z^{-1}uw)W_1 = \delta_j\gamma_iW_1$. Therefore, every element of U_2/W_1 is of the form $\delta_j\gamma_iW_1$ and so

$$U_2 = \bigcup_{j=1}^{n}\bigcup_{i=1}^{m} \delta_j\gamma_i W_1$$

We claim that this union is disjoint. Suppose $x, y \in X$, $u, v \in U_1$ are such that $xuW_1 = yvW_1$. Then $v^{-1}y^{-1}xu \in W_1$. Since U_2 is a group containing both $v^{-1} \in U_1$ and $y^{-1}x \in X$, $v^{-1}y^{-1}x \in U_2$. Since U_2 is equal to $X \cdot U_1$, there exists $x_1 \in X$, $u_1 \in U_1$ such that $v^{-1}y^{-1}x = x_1u_1$ or equivalently, $y^{-1}x = vx_1u_1$. Now

$$\begin{aligned} v^{-1}y^{-1}xu \in W_1 &\implies x_1u_1u \in W_1 \subset U_1 \\ &\implies x_1 \in U_1 \\ &\implies y^{-1}x = vx_1u_1 \in U_1 \implies xZ = yZ \end{aligned}$$

Thus if x, y are distinct modulo Z, xuW_1, yvW_1 are distinct left W_1-cosets for *any* $u, v \in U_1$. Thus, in the union above, different j correspond to necessarily distinct W_1-cosets. It is clear that $\delta_j\gamma_{i_1}W_1 = \delta_j\gamma_{i_2}W_1$ iff $i_1 = i_2$. Thus the union above is disjoint as both δ_j and γ_i vary.

Now we prove the first claim. Let $p : U_2/W_1 \to U_2/W_2$ be the natural projection map. Since U_2/W_1 is finite, so is U_2/W_2 and therefore $U_2\sigma U/U$. Moreover, as $W_2/W_1 \hookrightarrow U_2/W_1$, $e = [W_2 : W_1]$ is finite. Let $y = aW_2 \in U_2/W_2$ be a W_2-coset of U_2. Then $p^{-1}(y) = \{awW_1 | w \in W_2\}$ and we have

$$|p^{-1}(y)| = p^{-1}(W_2) = [W_2 : W_1] = e.$$

Thus, in the list of mn left W_2-cosets given by $\delta_1\gamma_1W_2, \delta_1\gamma_2W_2, \ldots, \delta_n\gamma_mW_2$, each element of U_2/W_2 appears exactly e times. Equivalently, among the mn left U-cosets $\delta_1\gamma_1\sigma U$, $\delta_1\gamma_2\sigma U, \ldots, \delta_m\gamma_n\sigma U$, each element of $U_2\sigma U/U$ appears exactly e times. Since $U_1\sigma U = \bigsqcup_{i=1}^{m} \gamma_i\sigma U$, we see that

$$e \cdot \mathrm{ch}(U_2\sigma U) = \sum\nolimits_{i,j} \mathrm{ch}(\delta_j\gamma_i\sigma U) = \sum\nolimits_{j} \mathrm{ch}(\delta_j U_1 \sigma U)$$

and the first claim is proved. The second claim follows since $W_2 = (X \cap \sigma U \sigma^{-1})W_1$ implies that $W_2/W_1 = (X \cap \sigma U \sigma^{-1})W_1/W_1 = (X \cap \sigma U \sigma^{-1})/(X \cap W_1)$. $\square$

Chapter 6

Cohomology of Shimura varieties

A typical example of cohomological Mackey functor for a locally profinite group G is obtained by taking functions on the quotients (by compact open subgroups) of a space X on which G acts. In this chapter, we record a similar result for the cohomology of Shimura varieties. In particular, the results of Ch. 2-3 apply to them. We begin by reviewing relevant results in Jannsen's continuous étale cohomology as expounded upon in [GS21, Appendix A]. These will also be used for permorming various constructions in Ch. 7.

6.1 Pushforwards in étale cohomology

Let S be a separated scheme of finite type over a Noetherian regular scheme of dimension at most 1. We denote by Sch_S the category of schemes over S, $\mathscr{S}(S_{\text{ét}})$ the category of étale sheaves of abelian groups on the small étale site $S_{\text{ét}}$ of S and $\mathscr{S}^{\text{pro}}(S_{\text{ét}})$ the category of pro-systems of étale sheaves on S indexed by $\mathbf{N}$ ([AM86, Appendix 1]). For a rational prime p, we let $\text{Ét}(S)_{\mathbb{Z}_p}$ denote the category of constructible $\mathbb{Z}_p$-sheaves on S. If $\mathscr{F} = (\mathscr{F})_{n\geq 1} \in \text{Ét}(S)_{\mathbb{Z}_p}$, we denote by $\mathscr{F} \otimes \mathbb{Q}_p$ the corresponding object in the isogeny category $\text{Ét}(S)_{\mathbb{Q}_p}$ whose objects are referred to as $\mathbb{Q}_p$-sheaves. We let $\Gamma_S : \mathscr{S}^{\text{pro}}(S_{\text{ét}}) \to \mathbf{Ab}$ denote the functor $\mathscr{F} = (\mathscr{F}_n) \mapsto \varprojlim_n H^0(S, \mathscr{F}_n)$. Then, following [Jan88, §3], the continuous étale cohomology of $\mathscr{F}$ is defined to be

$$\mathrm{H}^j_{\text{ét}}(S, \mathscr{F}) := R^j\Gamma_S(\mathscr{F}).$$

We let $\mathbf{D}(S) = \mathbf{D}(\text{Ét}(S)_{\mathbb{Z}_p})$ denote the 'derived category' of constructible sheaves in the sense of [Eke90]. It is a triangulated category with t-structure whose heart is $\text{Ét}(S)_{\mathbb{Z}_p}$. There is a full six functor formalism on it. In general, the morphism of this category do not compute the Ext groups. However, $\mathscr{F}, \mathscr{G} \in \text{Ét}(S)_{\mathbb{Z}_p}$,

$$\mathrm{Hom}_{\text{Ét}(S)_{\mathbb{Z}_p}}(\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_{\mathbf{D}(S)}(\mathscr{F}, \mathscr{G}), \qquad \mathrm{Ext}^1_{\text{Ét}(S)_{\mathbb{Z}_p}}(\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_{\mathbf{D}(S)}(\mathscr{F}, \mathscr{G}[1]).$$

By [Hub97, 4.1], $\mathrm{Hom}_{\mathbf{D}(S)}(\mathbb{Z}_p, \mathscr{F}[j])$ coincides with $\mathrm{H}^j_{\text{ét}}(S, \mathscr{F})$. If $f : T \to S$ denotes a morphism, then we have the direct image and pullback functors

$$Rf^*, Rf^! : \mathbf{D}(S) \to \mathbf{D}(T) \qquad Rf_*, Rf_! : \mathbf{D}(T) \to \mathbf{D}(S).$$

6.1.1 Purity for unipotent sheaves

Let $\pi : X \to S$ be a separated morphism of finite type. A $\mathbb{Z}_p$-sheaf $\mathscr{F} \in \text{Ét}(X)_{\mathbb{Z}_p}$ is said to be *unipotent of length* k if there is a decreasing filtration

$$\mathscr{F} = \mathscr{F}^0 \supset \mathscr{F}^1 \supset \ldots \supset \mathscr{F}^n \supset \mathscr{F}^{n+1} = 0$$

such that the graded pieces $\mathscr{F}^{i+1}/\mathscr{F}^i$ are isomorphic to $\pi^*\mathscr{G}^i$ for a $\mathbb{Z}_p$ sheaf $\mathscr{G}^i \in \text{Ét}(S)$. We can similarly define unipotence of sheaves over Λ_r, $\mathbb{Q}_p$.

Lemma 6.1.1. *Suppose $\pi_i : X_i \to S$ for $i = 1, 2$ are as above such that π_i is smooth of relative dimension d_i. Let $f : X_1 \to X_2$ be any S-morphism. Then for any unipotent sheaf $\mathscr{F}$ (of some finite length),*

$$f^!\mathscr{F} \simeq f^*\mathscr{F}(d_1 - d_2)[2d_1 - 2d_2]$$

Proof. This is [HK18, Lemma 2.8.1]. □

6.1.2 Pushforwards and pullbacks

Definition 6.1.2. Let X, Y be schemes, $\mathscr{F} \in \mathscr{S}^{\text{pro}}(X_{\text{ét}})$ and $\mathscr{G} \in \mathscr{S}^{\text{pro}}(Y_{\text{ét}})$. A *pushforward*

$$(f, \phi)_* \colon (X, \mathscr{F}) \to (Y, \mathscr{G})$$

is a morphism $f \colon X \to Y$ of schemes and a morphism $\phi : \mathscr{F} \to f^*\mathscr{G}$ of sheaves on X.

Definition 6.1.3. Let S be a scheme. An *étale smooth S-pair of codimension c* is a morphism $f \colon X \to Y$ of smooth quasi-compact quasi-separated S-schemes satisfying the following condition: there exists a smooth S-scheme $\bar{Y}$ and a factorization $f : X \xrightarrow{h} \bar{Y} \xrightarrow{g} Y$ such that h is a closed immersion with fibers over each point of S of pure codimension c in $\bar{Y}$ and g is étale. If $f' \colon X' \to Y'$ is another such pair, then a *morphism* from f to f' is a pair of étale maps $p \colon X' \to X$ and $q \colon Y' \to Y$ that commute with f, f'.

Proposition 6.1.4. *Let $f \colon X \to Y$ be an étale smooth S-pair of codimension c and let $\mathscr{F}$, $\mathscr{G}$ be étale $\mathbb{Z}_p$-sheaves on X, Y respectively. Assume that p is invertible on S. Then for any pushforward $(f, \phi)_* : (X, \mathscr{F}) \to (Y, \mathscr{G})$ and any $j \in \mathbb{Z}_{\geq 0}$, there is an induced "pushforward" on cohomology*

$$(f, \phi)_* \colon \mathrm{H}^j_{\text{ét}}(X, \mathscr{F}) \to \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c))$$

that is functorial and Cartesian:

- *(Functoriality) if $f'\colon X' \to Y'$ is another such pair and $(p,q)\colon f \to f'$ is a morphism, then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^j_{\text{ét}}(X', p^*\mathscr{F}) & \xrightarrow{(f',\psi)_*} & \mathrm{H}^{j+2c}_{\text{ét}}(Y', q^*\mathscr{G}(c)) \\ \downarrow{\scriptstyle \mathrm{Tr}_p} & & \downarrow{\scriptstyle \mathrm{Tr}_q} \\ \mathrm{H}^j_{\text{ét}}(X, \mathscr{F}) & \xrightarrow{(f,\phi)_*} & \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c)) \end{array}$$

where $\psi = p^*\phi$.

- *(Cartesian) if $q : Y' \to Y$ is any finite étale morphism, $X' = X \times_Y Y'$, $f' : X' \to Y'$, $p : X' \to X$ are the natural morphisms, then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}^j_{\text{ét}}(X', p^*\mathscr{F}) & \xrightarrow{(f',\psi)_*} & \mathrm{H}^{j+2c}_{\text{ét}}(Y', q^*\mathscr{G}(c)) \\ \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle q^*} \\ \mathrm{H}^j_{\text{ét}}(X, \mathscr{F}) & \xrightarrow{(f,\phi)_*} & \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c)) \end{array}$$

where $\psi = p^*\phi$.

When f is a closed immersion, $(f,\phi)_*$ is defined in [Jan88, Theorem 3.17]. The proof below is along similar lines except that we account for the independence of the factorization of f.

Proof. Let $f = g \circ h : X \xrightarrow{h} \bar{Y} \xrightarrow{g} Y$ be a factorization. As g is étale, $g^! = g^*$. By Lemma 6.1.1, $Rh^!\mathbb{Z}_p(c) = \mathbb{Z}_p[-2c]$ canonically. Thus, for any $\bar{\mathscr{G}} \in \mathscr{S}^{\mathrm{pro}}(\bar{Y}_{\text{ét}})$, we have $Rh^!\bar{\mathscr{G}}(c) = h^*\bar{\mathscr{G}}(c) \otimes Rh^!\mathbb{Z}_p(c) = h^*\bar{\mathscr{G}}[-2c] \in \mathbf{D}(X)$. Applying this to $\bar{\mathscr{G}} = g^*\mathscr{G}$, we see that

$$Rf^!\mathscr{G}(c) = Rh^! \circ Rg^!\,\mathscr{G}(c) = Rh^!(g^*\mathscr{G}(c)) = f^*\mathscr{G}[-2c]$$

Deriving $\mathrm{Tr}_f \colon \Gamma_X f^! \to \Gamma_Y$, we obtain a natural transformation $R(\mathrm{Tr}_f) : R(\Gamma_X f^!) = R\Gamma_X \circ Rf^! \to R\Gamma_Y$ between the two derived functors. Evaluating $R(\mathrm{Tr}_f)$ at $\mathscr{G}^*(c)$, we obtain an induced map $\mathrm{H}^j_{\text{ét}}(X, f^*\mathscr{G}) \cong R^{j+2c}(\Gamma_X f^!)(\mathscr{G}(c)) \xrightarrow{\mathrm{Tr_f}} \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c))$ via passage to cohomology. The pushforward map is now defined to be

$$(f,\phi)_* = \mathrm{H}^j_{\text{ét}}(X,\mathscr{F}) \xrightarrow{\phi} \mathrm{H}^j_{\text{ét}}(X, f^*\mathscr{G}) \xrightarrow{\mathrm{Tr}_f} \mathrm{H}^{j+2c}_{\text{ét}}(Y, \mathscr{G}(c)).$$

The commutativity of upper shrieks then gives the independence of this map on the choice of $\bar{Y}$. The functoriality of these pushforwards follows from that of the trace, i.e. $R\mathrm{Tr}_f \circ R\mathrm{Tr}_p = R\mathrm{Tr}_q \circ R\mathrm{Tr}_{f'}$. The Cartesian property follows by applying proper base change applied to $q : Y' \to Y$. □

6.2 Shimura varieties

In this section, we show that the cohomlogy of Shimura varieties constitutes a cohomological Mackey functor. We begin by recalling the notion of Shimura-Deligne data as introduced in [GS21, Appendix B].

6.2.1 Recollections

We fix an algebraic closure $\mathbb{C}$ of $\mathbb{R}$, and take $\overline{\mathbb{Q}}$ to be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We denote the Deligne torus by $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ and let $w\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$ defined via the *inverse* of the inclusion $\mathbb{R}^\times \hookrightarrow \mathbb{C}^\times$. We fix the identification $\mathbb{S}_\mathbb{C} \cong \mathbb{G}_m \times \mathbb{G}_m$ such that the inclusion $\mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C})$ is given by $z \mapsto (z, \overline{z})$, and we take $\mu\colon \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_\mathbb{C}$ to be the cocharacter $z \mapsto (z, 1)$. For an algebraic group $\mathbf{G}$, we denote by $\mathbf{Z}_\mathbf{G}$ its centre, $\mathbf{G}^{\mathrm{der}}$ its derived group, and $\mathbf{G}^{\mathrm{ad}}$ its quotient by $\mathbf{Z}_\mathbf{G}$.

Definition 6.2.1. A *Shimura–Deligne datum* is a pair $(\mathbf{G}, X)$ consisting of a connected reductive algebraic group $\mathbf{G}$ over $\mathbb{Q}$ and a $\mathbf{G}(\mathbb{R})$-conjugacy class X of homomorphisms $h\colon \mathbb{S} \to \mathbf{G}_\mathbb{R}$ satisfying

(SD1) For all $h \in X$, the Hodge bigrading of the complex vector space $\mathrm{Lie}(\mathbf{G})_\mathbb{C}$ under the adjoint action of $\mathbb{S}_\mathbb{C}$ is of type $\{(-1,1),(0,0),(1,-1)\}$. In particular, the cocharacter $h \circ w : \mathbb{G}_m \to \mathbf{G}_\mathbb{R}$ is central and independent of h.

(SD2) For any $h \in X$, $\mathrm{ad}\big(h(\sqrt{-1})\big) : \mathbf{G}_\mathbb{R} \to \mathbf{G}_\mathbb{R}$ (which is an involution by (SD1)) is a Cartan involution of $\mathbf{G}^{\mathrm{der}}_\mathbb{R}$, i.e. the real Lie group

$$\big\{g \in \mathbf{G}^{\mathrm{der}}(\mathbb{C}) \,|\, h(\sqrt{-1})\overline{g}h(\sqrt{-1})^{-1} = g\big\}$$

is compact.

A *morphism* $(\mathbf{G}_1, X_1) \to (\mathbf{G}_2, X_2)$ of Shimura–Deligne datum is a homomorphism $u\colon \mathbf{G}_1 \to \mathbf{G}_2$ such that $u(X_1) \subset X_2$. We say that such a morphism is *injective* if u is.

One occasionally imposes the following additional axioms:

(SD3) $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$ has no $\mathbb{Q}$-simple factors that are $\mathbb{R}$-anisotropic.

(SD4) The *weight morphism* $w_X := w \circ h\colon \mathbb{G}_{m,\mathbb{R}} \to \mathbf{G}_\mathbb{R}$, a priori defined over $\mathbb{R}$, is defined over $\mathbb{Q}$.

(SD5) $\mathbf{Z}_\mathbf{G}(\mathbb{Q})$ is discrete in $\mathbf{Z}_\mathbf{G}(\mathbb{A}_f)$.

If (SD1)-(SD3) are all satisfied, then we recover the usual notion of Shimura data.

Given a Shimura–Deligne datum $(\mathbf{G}, X)$, and a compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, we define the *Shimura–Deligne variety* to be the double quotient

$$\mathbf{G}(\mathbb{Q})\backslash[X \times (\mathbf{G}(\mathbb{A}_f)/K)]$$

which we denote by $\mathrm{Sh}_\mathbf{G}(X, K)(\mathbb{C})$ or just $\mathrm{Sh}_\mathbf{G}(K)(\mathbb{C})$. For any $\mathbb{Q}$-algebra R, the group $\mathbf{G}(R)$ acts on the left on the space $\mathrm{Hom}_R(\mathbb{G}_{m,R}, \mathbf{G}_R)$ of algebraic group homomorphisms over R, by conjugation on the target. Let $Y = Y_\mathbf{G}$ be the (fppf sheafification of) the functor

$$\mathbb{Q}\text{-algebras} \to \mathbf{Sets} \qquad R \mapsto \mathbf{G}(R)\backslash \mathrm{Hom}_R(\mathbb{G}_{m,R}, \mathbf{G}_R).$$

If $F/\mathbb{Q}$ is a finite Galois extension over which $\mathbf{G}$ splits, then restricted to F-algebras, Y is a constant functor, hence representable as $\bigsqcup \operatorname{Spec} F$. By Galois descent, one sees that it is representable by an étale $\mathbb{Q}$-scheme. Since the cocharacters $h_{\mathbb{C}} \circ \mu \colon \mathbb{G}_{m,\mathbb{C}} \to \mathbf{G}_{\mathbb{C}}$ for $h \in X$ lie in the same $\mathbf{G}(\mathbb{C})$-conjugacy class (as h lies in a single $\mathbf{G}(\mathbb{R})$-conjugacy class), one obtains a geometric point $\mu_X \in Y(\mathbb{C}) = Y(\overline{\mathbb{Q}}) \subset Y_{\overline{\mathbb{Q}}}$. The *reflex field* $E(\mathbf{G}, X)$ of a Shimura–Deligne datum $(\mathbf{G}, X)$ is the field of definition of the point $\mu_X \in Y_{\mathbf{G}}(\overline{\mathbb{Q}})$.

A *canonical model* of $\mathrm{Sh}_{\mathbf{G},\mathbb{C}}$ is a model $\mathrm{Sh}_{\mathbf{G},E}$ defined over the reflex field $E = E(\mathbf{G}, X)$ such that for every Shimura–Deligne datum $(\mathbf{T}, X')$ with reflex field E', where $\mathbf{T}$ is a torus, and any injective morphism $(\mathbf{T}, X') \to (\mathbf{G}, X)$, the induced map $\mathrm{Sh}_{\mathbf{T},\mathbb{C}} \to \mathrm{Sh}_{\mathbf{G},\mathbb{C}}$ is the pullback of a morphism $\mathrm{Sh}_{\mathbf{T},E'} \to \mathrm{Sh}_{\mathbf{G},E} \times_{\operatorname{Spec} E} \operatorname{Spec} E'$. If canonical models exist, they are unique up to a unique isomorphism. By [GS21, Corollary B.15], a canonical model exists if there is an embedding of $(\mathbf{G}, X)$ into another such pair for which canonical modes exist, in particular those that satisfies (SD3).

6.2.2 Cohomology

In [GS21, Lemma B.18], the following result was established assuming (SD5).

Lemma 6.2.2. *Let $K, K' \subset \mathbf{G}(\mathbb{A}_f)$ be neat compact open subgroups $K \cap \mathbf{Z}(\mathbb{Q}) = K' \cap \mathbf{Z}(\mathbb{Q})$. Then the map $\varphi_{K',K} : \mathrm{Sh}_{\mathbf{G}}(K')(\mathbb{C}) \twoheadrightarrow \mathrm{Sh}_{\mathbf{G}}(K)(\mathbb{C})$ of smooth $\mathbb{C}$-manifolds is a covering map of degree $[K : K']$.*

Proof. Suppose that there exists $x \in X$, $g \in \mathbf{G}(\mathbb{A}_f)$, $k \in K$ such that

$$[x, g]_{K'} = [x, gk]_{K'}$$

as elements of $\mathrm{Sh}_{\mathbf{G}}(K')(\mathbb{C})$. Let K_∞ denote the stabilizer of x in $\mathbf{G}(\mathbb{R})$. By definition, there exists $\gamma \in \mathbf{G}(\mathbb{Q}) \cap K_\infty$ such that $gk = \gamma g k'$ for some $k' \in K'$. Thus γ is an element of $\Gamma := \mathbf{G}(\mathbb{Q}) \cap gKg^{-1}$. Since $\mathbf{G}(\mathbb{Q})$ is discrete in $\mathbf{G}(\mathbb{A})$, Γ is discrete in $\mathbf{G}(\mathbb{R})$, whence the group $\Gamma \cap K_\infty$ is discrete in K_∞ and therefore so is the group $C := \langle \gamma \rangle$ generated by γ. By [GS21, Lemma B.5], the quotient $K_\infty/(\mathbf{Z}(\mathbb{R}) \cap K_\infty)$ is a compact group and since $C/(\mathbf{Z}(\mathbb{R}) \cap C)$ is a (necessarily closed) discrete subgroup, it must be finite. There is therefore a $n > 0$ such that

$$\gamma^n \in C \cap \mathbf{Z}(\mathbb{R}) \subset \mathbf{Z}(\mathbb{Q}).$$

Since $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is neat, it's image under $\varphi : \mathbf{G} \to \mathbf{G}^{\mathrm{ad}}$ is also neat [Bor19, 17.3] i.e. $\Gamma/(\mathbf{Z}(\mathbb{Q}) \cap \Gamma)$ is neat and in particular torsion free. Thus $\gamma \in \mathbf{Z}(\mathbb{Q})$ and $k = \gamma k'$ by our hypothesis and $\gamma \in K$. Since $K \cap \mathbf{Z}(\mathbb{Q}) = K' \cap \mathbf{Z}(\mathbb{Q})$ by our assumption on K, K', we have

$$k = \gamma k' \in K'.$$

The upshot of the argument is that the fiber of $\varphi_{K',K}$ above $[x, g]_K$ is of cardinality $[K : K']$. Let $L \subset K'$ be normal in K. Applying the same argument to L, we see that $\mathrm{Sh}_{\mathbf{G}}(K)(\mathbb{C})$ is a quotient of $\mathrm{Sh}_{\mathbf{G}}(L)(\mathbb{C})$ by

K/L with trivial stabilizers. This implies that $\mathrm{Sh}_{\mathbf{G}}(L)(\mathbb{C})$ is a cover of $\mathrm{Sh}_{\mathbf{G}}(K')(\mathbb{C})$, $\mathrm{Sh}_{\mathbf{G}}(K)(\mathbb{C})$ of degree $[K' : L]$, $[K : L]$, whence the claim. ☐

If canonical models exist for $\mathrm{Sh}_{\mathbf{G}}(K)$, then the following lemma allows us to descend the Cartesian property of diagrams of Shimura varieties.

Lemma 6.2.3. *Let W, X, Y, Z be geometrically reduced locally of finite type schemes over a field k such that*

$$\begin{array}{ccc} W & \xrightarrow{a} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{b} & Y \end{array}$$

is a commutative diagram where f, g are étale. Suppose for each closed point $z \in Z$ the map $a : W \to X$ is injective on the pre-image $g^{-1}(z)$ and surjects onto the pre-image of $f^{-1}(b(z))$. Then the diagram above is Cartesian in the category of k-schemes.

Proof. Suppose that $\mathcal{W} = X \times_Y Z$ is a pullback and $p_X : \mathcal{W} \to X$, $p_Y : \mathcal{W} \to Z$ are the natural projection maps. By the universal property of $\mathcal{W}$, there is an induced map $\gamma : W \to \mathcal{W}$. As f is étale, so is p_Z and since $p_Z \circ \gamma = g$ is étale, so is γ. Let $\bar{k}$ denote the algebraic closure of k. Since $\mathcal{W}(\bar{k}) = \{(x, z) \in X(\bar{k}) \times Z(\bar{k}) \mid f(x) = b(z)\}$, the condition on closed points (i.e. $\bar{k}$-points) implies that $\gamma : W(\bar{k}) \to \mathcal{W}(\bar{k})$ is a bijection. A bijective étale morphism is an isomorphism. ☐

Proposition 6.2.4. *Suppose that $\mathrm{Sh}_{\mathbf{G}}(K)$ have canonical models and $\{\mathscr{F}_K \in \text{Ét}(\mathrm{Sh}_{\mathbf{G}}(K))_{\mathbb{Z}_p} \mid K \subset \mathbf{G}(\mathbb{A}_f)\}$ is a collection of constructible $\mathbb{Z}_p$ sheaves that are compatible under pullbacks induced by $\mathrm{Sh}_{\mathbf{G}}(L) \to \mathrm{Sh}_{\mathbf{G}}(K)$ for $L \subset K$. Fix a neat compact open subgroup K_0 and let Υ be any collection of neat compact open subgroups of $\mathbf{G}(\mathbb{A}_f)$ satisfying (T1)-(T3) of §2.1 such that $K \cap \mathbf{Z}(\mathbb{Q}) = K_0 \cap \mathbf{Z}(\mathbb{Q})$ for any $K \in \Upsilon$. Then for any fixed non-negative integer i, the mapping*

$$\Upsilon \to \mathbb{Z}_p\text{-}Mod, \qquad K \mapsto \mathrm{H}^i_{\text{ét}}(\mathrm{Sh}_{\mathbf{G}}(K), \mathscr{F}_K)$$

constitutes a cohomological Mackey functor.

Proof. Let $M : \Upsilon \to \mathbb{Z}_p$-Mod denote the mapping. Then M is a RIC functor by with respect to induced pullback and trace maps. By Lemma 6.2.2 and see [AGV73, Tome 3, Expose IX, §5]), we see that M is cohomological. Let $K, L, L' \in \Upsilon$ with $L, L' \subset K$. By Lemma 6.2.2 and Lemma 6.2.3, we have a pullback diagram in the category of smooth schemes over the reflex field of $(\mathbf{G}, X)$

$$\begin{array}{ccc} \bigsqcup_\gamma \mathrm{Sh}(L_\gamma) & \longrightarrow & \mathrm{Sh}(L) \\ \downarrow & & \downarrow \\ \mathrm{Sh}(L) & \longrightarrow & \mathrm{Sh}(K) \end{array}$$

where $\gamma \in K$ runs over representatives of $L\backslash K/L'$ and $L_\gamma = L \cap \gamma L' \gamma^{-1}$. The second part of Proposition 6.1.4 implies the claim. □

Remark 6.2.5. Using similar arguments, one may establish that an injective morphism $(\mathbf{H}, Y) \hookrightarrow (\mathbf{G}, X)$ of Shimura-Deligne data and a collection on sheaves of the two sets of varieties that are compatible under all possible pullbacks induce a Mackey pushforward on the corresponding cohomology of varieties over the reflex field of $(\mathbf{H}, Y)$.

Chapter 7

Analytical parametrizations of arithmetic Eisenstein classes

In this chapter, we provide a Schwartz space parametrization (in the sense of Definition 2.6.7) of certain integral elements in the arithmetic étale cohomology of symplectic Shimura varieties known as *Eisenstein classes*. The classes in question are obtained as specializations of the so-called *polylogarithms* whose integral versions were recently constructed by Kings in [Kin16]. These constructions generalize classical ones on modular curves such as Beĭlinson's Eisenstein symbol [Beĭ86] and Kato's universal norm compatible units on elliptic curves [Kat04, §1]. The parametrization that we provide is a function theoretic characterization of the so-called 'distribution relations' of Eisenstein classes analogous to those for Kato's Siegel units ([Kat04, Lemma 1.7]). For purposes of this thesis, these classes serve as an interesting source of zeta elements for the cohomology of Shimura varieties in degrees apart from 0, but they are also generally expected to encode important arithmetic information about special values of appropriate L-functions. An application of the results of this section to Galois representations in the middle degree cohomology of Siegel sixfolds by means of Euler systems is discussed in Chapter 8 and the corresponding horizontal norm relations are proved in Chapter 13.

This chapter is organized as follows. In §7.1, we recall the construction of polylogarithms on arbitrary abelian schemes on suitable base schemes and use them to define rational Eisenstein classes. More crucial to us is the p-adic interpolation of these Eisenstein classes by the so-called *Iwasawa-Eisenstein classes* which in particular establish the integrality of these classes. This is based on the relatively recent work of Kings in [Kin16, §4-§6] and is recalled in §7.2. Since the content of these two sections is primarily an exposition of the results of [Kin16] we skip most proofs and refer the reader to *op. cit.* for details. Note however that since [Kin16] works with arbitrary commutative schemes, the expressions found here will be slightly different (and simpler) than the ones found in *op. cit.* as we can replace lower shrieks with the usual direct image functors in all instances. To make up for the apparent discrepancy, we have expanded on some details in the exposition.

In §7.3.3, we study these classes in the context of symplectic Shimura varieties of and show that when appropriately normalized, they admit parametrization by certain adelic Schwartz spaces over $\mathbb{Z}_p$ in the spirit of [Col03, Proposition 1.7]. The parametrization in question arises as a result of varying linear combinations of torsion sections of universal abelian schemes over appropriate level structures and invoking base change functoriality of the logarithm prosheaf used to define such classes. We refer the reader to [Kin15] for a thorough discussion of these classes in the setting of modular curves and in particular their relationship with classical constructions. See also [HK99, §1]

7.1 Rational polylogarithms and Eisenstien classes

Throughout the first two sections, we let S denote a separated scheme of finite type over a Noetherian regular scheme of dimension at most 1. The conventions introduced at the beginnging of §6.1 are maintained. For $r \geq 1$, we will write $\Lambda_r := \mathbb{Z}/p^r\mathbb{Z}$.

7.1.1 The $\mathbb{Q}_p$-logarithm pro-sheaf

Let $\pi : A \to S$ denote an abelian scheme of relative dimension d i.e. A is a group scheme and π is a smooth proper morphism with connected geometric fibers of dimension d. The unit section is denoted by $e : S \to A$. The *p-adic Tate module* of π is defined to be first relative homology

$$\mathscr{H}_{\mathbb{Z}_p} := \mathscr{H}om_S(R^1\pi_*\mathbb{Z}_p, \mathbb{Z}_p) = R^{-1}\pi_*\pi^!\mathbb{Z}_p = R^{2d-1}\pi_*\mathbb{Z}_p(d) \in \text{Ét}(S)_{\mathbb{Z}_p} \tag{7.1.1}$$

of G with respect to S. We let $\mathscr{H}_{\mathbb{Q}_p} := \mathscr{H}_{\mathbb{Z}_p} \otimes \mathbb{Q}_p \in \text{Ét}(S)_{\mathbb{Q}_p}$ denote the corresponding $\mathbb{Q}_p$-sheaf. For $r \geq 1$, we similarly define $\mathscr{H}_{\Lambda_r} := R^{2d-1}\pi_*\Lambda_r(d)$. Then

$$A[p^r] \simeq \mathscr{H}_{\Lambda_r} \simeq \mathscr{H}_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^r\mathbb{Z} \tag{7.1.2}$$

([Kin16, Lemma 2.1.2]). In what follows, we denote $\mathscr{H}_{\mathbb{Z}_p}$ simply by $\mathscr{H}$ if no confusion can arise.

The low term exact sequence arising from the Leray spectral sequence associated with the composition $\mathscr{H}om_S(\mathbb{Z}_p, -) \circ \pi_*$ applied to $\pi^*\mathscr{H}$ gives

$$\begin{aligned} 0 \to \text{Ext}^1_S(\mathbb{Z}_p, \mathscr{H}) \xrightarrow{\pi^*} \text{Ext}^1_A(\mathbb{Z}_p, \pi^*\mathscr{H}) \to \text{Hom}_S(\mathbb{Z}_p, R^1\pi_*\pi^*\mathscr{H}) \\ \to \text{Ext}^2_S(\mathbb{Z}_p, \mathscr{H}) \xrightarrow{\pi^*} \text{Ext}^2_A(\mathbb{Z}_p, \pi^*\mathscr{H}) \end{aligned}$$

The maps π^* are necessarily injective as $e^* \circ \pi^* = (\pi \circ e)^* = \mathrm{id}$ and the morphism to the second line above therefore 0. By the projection formula, $R^1\pi_*\pi^*\mathscr{H} \simeq R^1\pi_*\mathbb{Z}_p \otimes \mathscr{H}$. Since $R^1\pi_*\mathbb{Z}_p \simeq \mathscr{H}om(\mathscr{H}, \mathbb{Z}_p) =: \mathscr{H}^\vee$ and $\mathscr{H}^\vee \otimes \mathscr{H} \simeq \mathscr{H}om(\mathscr{H}, \mathscr{H})$, we get a short exact sequence

$$0 \to \mathrm{Ext}^1_S(\mathbb{Z}_p, \mathscr{H}) \to \mathrm{Ext}^1_A(\mathbb{Z}_p, \pi^*\mathscr{H}) \to \mathrm{Hom}_S(\mathscr{H}, \mathscr{H}) \to 0 \tag{7.1.3}$$

Note that the pullback of $\pi^*\mathscr{H}$ along e is $\mathscr{H}$.

Definition 7.1.4. The *first logarithm sheaf* is defined to be the pair $(\mathcal{L}og^{(1)}_{\mathbb{Z}_p}, \mathbf{1}^{(1)})$ where $\mathcal{L}og^{(1)}_{\mathbb{Z}_p} \in \acute{\mathrm{E}}\mathrm{t}(A)_{\mathbb{Z}_p}$ arises from an extension class

$$0 \to \pi^*\mathscr{H} \to \mathcal{L}og^{(1)}_{\mathbb{Z}_p} \xrightarrow{\delta} \mathbb{Z}_p \to 0 \tag{7.1.5}$$

in $\mathrm{Ext}^1_A(\mathbb{Z}_p, \pi^*\mathscr{H})$ whose image in $\mathrm{Hom}_S(\mathscr{H}, \mathscr{H})$ under (7.1.3) is identity and $\mathbf{1}^{(1)} : \mathbb{Z}_p \to e^*\mathcal{L}og^{(1)}$ is a fixed splitting of the pullback $0 \to \mathscr{H} \to e^*\mathcal{L}og^{(1)}_{\mathbb{Z}_p} \to \mathbb{Z}_p \to 0$ of (7.1.5). The pair $(\mathcal{L}og^{(1)}_{\mathbb{Z}_p}, \mathbf{1}^{(1)})$ is then unique up to a unique isomorphism. We denote by $\mathcal{L}og^{(1)}_{\mathbb{Q}_p}$ the associated $\mathbb{Q}_p$-sheaf.

It is clear that $\mathcal{L}og^{(1)}$ is unipotent of length 1. One defines $\mathcal{L}og^{(1)}_{\Lambda_r}$ for $r \geq 1$ in the same way as $\mathscr{H}_{\Lambda_r}$ using the coefficients Λ_r. Then $\mathcal{L}og^{(1)}_{\Lambda_r} = \mathcal{L}og^{(1)}_{\mathbb{Z}_p} \otimes \Lambda_r$ and $\mathcal{L}og^{(1)}_{\mathbb{Z}_p} = (\mathcal{L}og^{(1)}_{\Lambda_r})_{r \geq 1}$. If no confusion can arise, we will denote $\mathcal{L}og^{(1)}_{\mathbb{Z}_p}$.

Definition 7.1.6. For $k \geq 1$, the *k-th $\mathbb{Q}_p$-logarithm sheaf* is the pair $(\mathcal{L}og^{(k)}_{\mathbb{Q}_p}, \mathbf{1}^{(1)})$ where $\mathcal{L}og^{(k)}_{\mathbb{Q}_p} := \mathrm{Sym}^k\big(\mathcal{L}og^{(1)}_{\mathbb{Q}_p}\big) \in \acute{\mathrm{E}}\mathrm{t}(S)_{\mathbb{Q}_p}$ and

$$\mathbf{1}^{(k)} := \frac{1}{k!}\mathrm{Sym}^{(k)}(\mathbf{1}^{(1)}) : \mathbb{Q}_p \to e^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p}$$

is the splitting induced by $\mathbf{1}^{(1)}$ on the symmetric power.

The $\mathbb{Q}_p$-logarithms for $k \geq 1$ and their canonical splittings fit into an inverse system as follows. Let $\beta = \delta \oplus \mathrm{id} : \mathcal{L}og^{(1)}_{\mathbb{Q}_p} \to \mathbb{Q}_p \oplus \mathcal{L}og^{(1)}_{\mathbb{Q}_p}$ denote the diagonal map induced by the projection map δ in Definition 7.1.4 and identity. For all $k \geq 2$, define *transition maps* $u^k : \mathcal{L}og^{(k)}_{\mathbb{Q}_p} \to \mathcal{L}og^{k-1}_{\mathbb{Q}_p}$ via

$$\begin{aligned}\mathcal{L}og^{(k)}_{\mathbb{Q}_p} = \mathrm{Sym}^k(\mathcal{L}og^{(1)}_{\mathbb{Q}_p}) &\xrightarrow{\mathrm{Sym}^k\beta} \mathrm{Sym}^k(\mathbb{Q}_p \oplus \mathcal{L}og^{(1)}) \\ &\simeq \bigoplus\nolimits_{i+j=k} \mathrm{Sym}^i(\mathbb{Q}_p) \otimes \mathrm{Sym}^j(\mathcal{L}og^{(1)}_{\mathbb{Q}_p}) \\ &\to \mathrm{Sym}^1(\mathbb{Q}_p) \otimes \mathrm{Sym}^{k-1}(\mathcal{L}og^{(1)}_{\mathbb{Q}_p}) \simeq \mathcal{L}og^{k-1}_{\mathbb{Q}_p}\end{aligned}$$

For $k \geq 2$, $u^k \circ \mathbf{1}^{(k)} : \mathbb{Q}_p \to e^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p}$ is equal to $\mathbf{1}^{(k-1)}$ where we are abusing notation to denote e^*u^k by u^k etc. Indeed, since $\beta \circ \mathbf{1}^{(1)} = \mathrm{id} \oplus \mathbf{1}^{(1)} : \mathbb{Q}_p \to \mathbb{Q}_p \oplus \mathcal{L}og^{(1)}_{\mathbb{Q}_p}$, the composition of symmetric powers $\mathrm{Sym}^k\beta$ and

$\mathrm{Sym}(\mathbf{1}^{(1)})$ is given by

$$\begin{aligned}\mathrm{Sym}^k(\beta)\circ\mathrm{Sym}^{(k)}(\mathbf{1}^{(1)}) &= \mathrm{Sym}^{(k)}(\mathrm{id}\oplus\mathbf{1}^{(1)})\\ &\simeq \bigoplus_{i+j=k}\binom{k}{i}\left(\mathrm{Sym}^i(\mathrm{id})\otimes\mathrm{Sym}^j(\mathbf{1}^{(1)})\right)\\ &= \bigoplus_{i+j=k}\frac{k!}{i!}\left(\mathrm{Sym}^i(\mathrm{id})\otimes\mathbf{1}^{(j)}\right).\end{aligned}$$

Now the projection of the last sum above to the summand at $i=1$ is equal to $k!\cdot\mathbf{1}^{(k-1)}$. Therefore, $k!\cdot(u^k\circ\mathbf{1}^{(k)}) = u^k\circ\mathrm{Sym}^k(\mathbf{1}^{(1)}) = k!\cdot\mathbf{1}^{(k-1)}$. If we take $\mathcal{L}og^{(0)}_{\mathbb{Q}_p} := \mathbb{Q}_p$, the identity map $\mathbf{1}^{(0)} = \mathrm{id}:\mathbb{Q}_p\to e^*\mathcal{L}og^{(0)}_{\mathbb{Q}_p}$ as a section and $u_1 := \delta : \mathcal{L}og^{(1)}_{(p)}\to\mathcal{L}og^{(0)}_{\mathbb{Q}_p}$, then we still have $u_1\circ\mathbf{1}^{(1)} = \mathbf{1}^{(0)}$.

Definition 7.1.7. The $\mathbb{Q}_p$-*logarithm* $(\mathcal{L}og_{\mathbb{Q}_p},\mathbf{1})$ is the pro-system $(\mathcal{L}og^{(k)}_{\mathbb{Q}_p},\mathbf{1}^{(k)})_{k\geq 0}$ where transitions are given by u^k.

For each $k\geq 0$, the transition maps give an exact sequence $\mathbf{1}^{(k)}$ induces an exact sequence

$$0\to\pi^*\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_p}\to\mathcal{L}og^k_{\mathbb{Q}_p}\xrightarrow{u^k}\mathcal{L}og^{k-1}_{\mathbb{Q}_p}\to 0 \tag{7.1.8}$$

and therefore the splitting $\mathbf{1}^{(k)}$ gives an identification $e^*\mathcal{L}og^{(k)}_{\mathbb{Q}_p} = \prod_{i=0}^k\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_p}$ (as one may verify by induction on k). The isomorphism is uniquely determined by the requirement that $e^*u^k:\mathcal{L}og^{(k)}_{\mathbb{Q}_p}\to\mathcal{L}og^{(k-1)}_{\mathbb{Q}_p}$ is identified with the projection map $\prod_{i=0}^k\mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_p}\to\prod_{i=0}^{k-1}\mathrm{Sym}^{k-1}\mathrm{H}_{\mathbb{Q}_p}$ that forgets the k-th component. Using 7.1.8, one sees by induction that $\mathcal{L}og^{(k)}_{\mathbb{Q}_p}$ is unipotent of length k. The $\mathbb{Q}_p$-logarithm satisfies several other important properties. Below, we record the ones that will be needed later.

Proposition 7.1.9 (Base Change Compatibility)**.** *Suppose that $f:T\to S$ is a morphism, $A_T := A\times_S T$ denote the pullback of of A to T and $g:A_T\to A$ denote the natural map. Then for all $k\geq 0$,*

$$g^*\left(\mathcal{L}og^{(k)}_{A,\mathbb{Q}_p}\right)\simeq\mathcal{L}og^{(k)}_{A_T,\mathbb{Q}_p}$$

and $g^*(\mathbf{1}^{(k)})$ defines a splitting of the pullback along identity section. Moreover, these isomorphisms are compatible with respect to the transition maps.

Proposition 7.1.10 (Splitting principle)**.** *Let A_i for $i=1,2$ be abelian schemes over S, $e_i:S\to A_i$ their identity sections and $\varphi:A_1\to A_2$ be an isogeny over S. For any torsion section $t:S\to A_1$ contained in* $\ker\varphi$ *and $k\geq 0$, there are natural isormophisms*

$$\varrho_t^k : t^*\mathcal{L}og^{(k)}_{A_1,\mathbb{Q}_p}\xrightarrow{\sim}e_1^*\mathcal{L}og^{(k)}_{A_1,\mathbb{Q}_p}\simeq\prod_{i=0}^k\mathrm{Sym}^i(\mathscr{H}_{A,\mathbb{Q}_p})$$

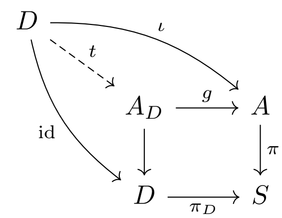
that commute with the transition maps. In particular, $t^*\mathcal{L}og^{(k)}_{A_1,\mathbb{Q}_p}\simeq e_2^*\varphi^*\mathcal{L}og^{(k)}_{A_2,\mathbb{Q}_p}$.

Corollary 7.1.11. *Let $\varphi : A \to A_1$ be an isogeny of abelian schemes over S. Denote $D := \ker\varphi$, $\iota : D \to A$ the natural map and $\pi_D := \pi \circ \iota$ the structure map. Then for each $k \geq 0$, there are natural isomorphisms*

$$\iota^* \mathcal{L}og^{(k)} \xrightarrow{\sim} \prod_{i=1}^{k} \pi_D^* \mathrm{Sym}^i(\mathscr{H}_{A,\mathbb{Q}_p})$$

that commute with the transition maps.

Proof. Let $A_D := A \times_S D$ denote the pullback abelian scheme over D, $r : A_D \to D$, $g : A_D \to A$ be the natural maps and $e_D : D \to A_D$ the identity section. Let $t : D \to A_D$ be the torsion section determined the diagram below.



By Proposition 7.1.9 and 7.1.10, we have

$$\iota^* \mathcal{L}og^{(k)}_{A,\mathbb{Q}_p} \simeq t^* g^* \big(\mathcal{L}og^{(k)}_{A,\mathbb{Q}_p}\big) \simeq t^* \mathcal{L}og^{(k)}_{A_D,\mathbb{Q}_p} \simeq e_D^* \mathcal{L}og^{(k)}_{A_D,\mathbb{Q}_p} \simeq e_D^* g^* \mathcal{L}og^{(k)}_{A,\mathbb{Q}_p}$$

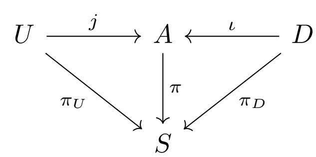
But $e_D^* g^* \mathcal{L}og^{(k)}_{A,\mathbb{Q}_p} \simeq \pi_D^* e^* \mathcal{L}og^{(k)}_{A,\mathbb{Q}_p} \simeq \pi_D^* \mathrm{Sym}^i(\mathscr{H}_{A,\mathbb{Q}_p})$. □

Proposition 7.1.12 (Vanishing of cohomology)**.** *There exist natrual isomorphisms $R^{2d}\pi_*(\mathcal{L}og^{(k)}_{\mathbb{Q}_p}) \simeq \mathbb{Q}_p(-d)$ for all $k \geq 0$ that commute with the transition maps. For $i = 0, \ldots, 2d-1$, the induced maps $R^i u^k : R^i\pi_* \mathcal{L}og^{(k)}_{\mathbb{Q}_p} \to R^i\pi_* \mathcal{L}og^{(k-1)}_{\mathbb{Q}_p}$ are zero. In particular,*

$$\mathrm{H}^{2d}_{\text{ét}}\big(A, \mathcal{L}og_{\mathbb{Q}_p}(d)\big) \simeq \begin{cases} 0 & \text{if } i < 2d \\ \mathrm{H}^0_{\text{ét}}(S, \mathbb{Q}_p) & \text{if } i = 2d \end{cases}$$

7.1.2 Residues along torsion subschemes

Fix $c > 1$ an integer invertible on S. Denote by $D := A[c]$ the c-torsion on A and $U = U_D := A \setminus A[c]$ the complement of $A[c]$ in A. Consider the diagram.



(7.1.13)

where $j = j_D$ and $\iota = \iota_D$ are natural inclusions and $\pi_D := \pi \circ \iota$, $\pi_U := \pi \circ j$ denote the structural map. For any $\mathbb{Q}_p$-sheaf $\mathscr{F} \in \text{Ét}(A)_{\mathbb{Q}_p}$, we have a distinguished triangle $\iota_*\iota^!\mathscr{F} \to \mathscr{F} \to Rj_*j^* \to \iota_*\iota^!\mathscr{F}[1] \in \mathbf{D}(A)_{\mathbb{Q}_p}$

known as the *localization* triangle. Applying $R\pi_*$ to the localization triangle with $\mathscr{F} = \mathcal{L}og_{\mathbb{Q}_p}(d)$, we get a distinguished triangle

$$R\pi_{D,*}\iota^!\mathcal{L}og_{\mathbb{Q}_p}(d) \to R\pi_*\mathcal{L}og_{\mathbb{Q}_p}(d) \to R\pi_{U,*}j^*\mathcal{L}og_{\mathbb{Q}_p}(d) \to R\pi_{D,*}\iota^!\mathcal{L}og_{\mathbb{Q}_p}(d)[1] \in \mathbf{D}(S)_{\mathbb{Q}_p} \tag{7.1.14}$$

Using Lemma 6.1.1 and the fact that $\mathcal{L}og^{(k)}_{\mathbb{Q}_p}$ is unipotent for each k, we see that $\iota^!\mathcal{L}og_{\mathbb{Q}_p}(d) = \iota^*\mathcal{L}og_{\mathbb{Q}_p}[-2d]$ and therefore $R\pi_{D,*}\iota^!\mathcal{L}og_{\mathbb{Q}_p}(d)[1] = R\pi_{D,*}\iota^*\mathcal{L}og_{\mathbb{Q}_p}[1-2d]$. Applying $R\Gamma$ and passing to cohomology, we obtain a (long) exact sequence

$$\begin{aligned} \cdots \to \mathrm{H}^{2d-1}_{\text{ét}}\big(A, \mathcal{L}og_{\mathbb{Q}_p}(d)\big) &\to \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og_{U,\mathbb{Q}_p}(d)\big) \\ &\to \mathrm{H}^0_{\text{ét}}\big(D, \iota^*\mathcal{L}og_{\mathbb{Q}_p}\big) \to \mathrm{H}^{2d}_{\text{ét}}\big(A, \mathcal{L}og(d)\big) \to \cdots \end{aligned} \tag{7.1.15}$$

where $\mathcal{L}og_{U,\mathbb{Q}_p}$ denotes the pullback of $\mathcal{L}og_{\mathbb{Q}_p}$ to the open subset U. The splitting principle (Corollary 7.1.11) applied to $\varphi = [c] : A \to A$ implies that $\iota^*\mathcal{L}og_{\mathbb{Q}_p} \simeq \prod_{i=0}\pi_D^*\mathrm{Sym}^i(\mathscr{H}_{\mathbb{Q}_p})$. Combining this observation with Proposition 7.1.12, we obtain from (7.1.15) an exact sequence

$$0 \to \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og_{U,\mathbb{Q}_p}(d)\big) \to \mathrm{H}^0_{\text{ét}}\big(D, \prod_{i=0}\pi_D^*\mathrm{Sym}^i(\mathscr{H}_{\mathbb{Q}_p})\big) \to \mathrm{H}^0_{\text{ét}}(S, \mathbb{Q}_p) \tag{7.1.16}$$

Definition 7.1.17. The *residue of $\mathcal{L}og_{\mathbb{Q}_p}$ along D* is defined to be the connecting homomorphism

$$\mathrm{res} : \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og_{U,\mathbb{Q}_p}(d)\big) \to \mathrm{H}^0_{\text{ét}}\big(D, \prod\nolimits_{i=0}\pi_D^*\mathrm{Sym}^i(\mathscr{H}_{\mathbb{Q}_p})\big)$$

appearing in (7.1.16).

Remark 7.1.18. One can also define the residue map for the Ext group of the extensions by the pullback of sheaf $\mathscr{F} \in \acute{\mathrm{E}}\mathrm{t}(S)_{\mathbb{Q}_p}$ in degree $2d-1$ via

$$\mathrm{res} : \mathrm{Ext}^{2d-1}_A\big(\pi_U^*\mathscr{F}, \mathcal{L}og_{U,\mathbb{Q}_p}(d)\big) \to \mathrm{Hom}_D\big(\pi_D^*\mathscr{F}, \prod\nolimits_{i=0}\pi_D^*\mathrm{Sym}^i(\mathscr{H}_{\mathbb{Q}_p})\big)$$

obtained by applying $R\mathscr{H}om_S(\mathscr{F}, -)$ to the triangle (7.1.14) and then passing to cohomology.

7.1.3 The cohomology classes

Let $\mathbb{Q}_p[D]^0$ denote the kernel of the map $\mathrm{H}^0_{\text{ét}}(D, \mathbb{Q}_p) \simeq \mathrm{H}^0_{\text{ét}}(S, \pi_{D,*}\mathbb{Q}_p) \xrightarrow{\mathrm{tr}} \mathrm{H}^0_{\text{ét}}(S, \mathbb{Q}_p)$ induced by the trace morphism $\mathrm{tr}_{\pi_D} : \pi_{D,!}\pi_D^!\mathbb{Q}_p = \pi_{D,*}\mathbb{Q}_p$[1] $\to \mathbb{Q}_p$. Then

$$\mathbb{Q}_p[D]^0 \subset \mathrm{H}^0_{\text{ét}}(D, \mathbb{Q}_p) \hookrightarrow \mathrm{H}^0_{\text{ét}}\big(D, \prod_{i=0}\pi_D^*\mathrm{Sym}^i(\mathscr{H}_{\mathbb{Q}_p})\big)$$

is in the image of the residue map. Indeed, since $\mathrm{H}^0\big(D, \iota^*\mathcal{L}og_{\mathbb{Q}_p}\big) \to \mathrm{H}^{2d}_{\text{ét}}\big(A, \mathcal{L}og_{\mathbb{Q}_p}(d)\big)$ was defined using the triangle (7.1.14), it coincides with the trace map in the component of 0-th symmetric power and therefore

[1] using Lemma 6.1.1 and properness of π_D

maps $\mathbb{Q}_p[D]^0$ to zero. We use the residue conditions determined by non-zero elements $\alpha \in \mathbb{Q}[D]^0$ to define cohomology classes on U (and on S via pullback along torsion sections) in degree $2d-1$.

Definition 7.1.19. Let $\alpha \in \mathbb{Q}_p[D]^0$. The *polylogarithm class with residue* α is the unique cohomology class

$$ {}_{\alpha}\mathrm{pol}_{\mathbb{Q}_p} \in \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og(d)_{\mathbb{Q}_p}\big) $$

such that $\mathrm{res}\big({}_{\alpha}\mathrm{pol}_{\mathbb{Q}_p}\big) = \alpha$. We define ${}_{\alpha}\mathrm{pol}^k_{\mathbb{Q}_p} \in \mathrm{H}^{2d}_{\text{ét}}\big(U, \mathcal{L}og^{(k)}(d)_{\mathbb{Q}_p}\big)$ to be the image of ${}_{\alpha}\mathrm{pol}$ under the augmentation map $\mathcal{L}og_{\mathbb{Q}_p} \to \mathcal{L}og^{(k)}_{\mathbb{Q}_p}$.

Now let $N > 1$ be an integer invertible on S and such that $(N, c) = 1$. Let $t : S \to U$ be a N-torsion section i.e. $[N] \circ j \circ t : S \to A$ is the identity section $e : S \to A$. The adjunction $\mathrm{id} \to Rt_*t^*$ associated with the section $t : S \to U$ of $\pi_U : U \to S$ gives a morphism

$$ \begin{aligned} R\pi_{U,*}\mathcal{L}og_{\mathbb{Q}_p} &\to R\pi_{U,*}Rt_*t^*\mathcal{L}og_{U,\mathbb{Q}_p}(d) \\ &\simeq R(\pi_U \circ t)_*t^*\mathcal{L}og_{\mathbb{Q}_p}(d) = t^*\mathcal{L}og_{U,\mathbb{Q}_p}(d) \end{aligned} $$

which in turn induces the pullback map $t^* : \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og_{U,\mathbb{Q}_p}(d)\big) \to \mathrm{H}^{2d-1}_{\text{ét}}\big(S, t^*\mathcal{L}og(d)\big)$ on cohomology. By Proposition 7.1.10, we have a map

$$ \mathrm{H}^{2d-1}_{\text{ét}}\big(S, t^*\mathcal{L}og_{\mathbb{Q}_p}(d)\big) \xrightarrow[\sim]{\varrho_t} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \prod_{i=0} \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_p}(d)\big) \xrightarrow{\mathrm{pr}^k} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_p}(d)\big) \tag{7.1.20} $$

where pr^k is the projection on the k-th symmetric power.

Definition 7.1.21. Let $\alpha \in \mathbb{Q}_p[D]^0$, $t : S \to U$ be a N-torsion section as above and $k \geq 0$ be an integer. The k-th *(rational) Eisenstein class*

$$ {}_{\alpha}\mathrm{Eis}^k_{\mathbb{Q}_p}(t) \in \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_p}(d)\big) $$

associated with section t *and residue* α is defined to be the image of ${}_{\alpha}\mathrm{pol}^k_{\mathbb{Q}_p} \in \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}og^{(k)}_{U,\mathbb{Q}_p}(d)\big)$ under the composition $\mathrm{pr}^k \circ \varrho_t \circ t^*$.

Let α, t be as above. Let $f : T \to S$ be a morphism and $A_T := A \times_S T$ and $D_T = A_T[c]$. Let $f^*t : T \to A_T$ denote the tautological section induced by t and universal property of A_T.

$$ \begin{array}{ccc} A_T & \xrightarrow{g} & A \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{f} & S \end{array} $$

Denote also by f^* the maps

$$ \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_p}(d)\big) \to \mathrm{H}^{2d-1}_{\text{ét}}\big(T, \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_p}(d)\big), \qquad \mathrm{H}^0_{\text{ét}}(D, \mathbb{Q}_p) \to \mathrm{H}^0_{\text{ét}}(D_T, \mathbb{Q}_p) $$

induced by $Rf_*f^* \to \mathrm{id}$. Then $f^*(\alpha)$ lies in the image of the residue map for T. Abusing notation, we denote $f^*(\alpha)$ by α.

Lemma 7.1.22. $f^*{}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t) = {}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(f^*t)$.

Proof. Let $U_T = A_T \setminus D_T$ and ${}_\alpha\mathrm{pol}_{T,\mathbb{Q}_p} \in \mathrm{H}^{2d-1}_{\text{ét}}\big(U_T, \mathcal{L}og_{A_T,\mathbb{Q}_p}(d)\big)$ denote the polylogarithm class for T. Then 7.1.9 implies that $g^*{}_\alpha\mathrm{pol}_{\mathbb{Q}_p} = \alpha\mathrm{pol}_{T,\mathbb{Q}_p}$. Since $f^* \circ t^* = f^*t \circ g^*$ and α are compatible under pullbacks, the same holds for Eisenstein classes. □

7.2 p-adic interpolation

In this section, we provide an exposition of King's results on the interpolation of the Eisenstein class in 7.1.21. We skip all proofs and refer the reader to [Kin16, §4-6].

7.2.1 The moment map

Let $\pi : A \to S$ be as in §7.1.1. Recall from (7.1.2) that for each non-negative integer r, $\mathscr{H}_r := \mathscr{H}_{\Lambda_r} \in \mathscr{S}(S_{\text{ét}})$ is isomorphic to the representable sheaf associated with p^r-torsion subscheme $X_r := A[p^r]$. Let $\mathrm{pr}_r : X_r \to S$ denote the structure map and set

$$\Lambda_r[\mathscr{H}_r] = \Lambda_r[X_r] := \mathrm{pr}_{r,*}\Lambda_r \in \mathscr{S}(S_{\text{ét}}) \tag{7.2.1}$$

Then $\Lambda_r[\mathscr{H}_r]$ is a sheaf of (abelian) group algebras over Λ_r with product $\mathscr{H}_r \times \mathscr{H}_r \to \mathscr{H}_r$ induced by the group structure on X_r. More generally, let $p_r : A_r \to A \in A_{\text{ét}}$ denote the finite étale cover $A_r := A \xrightarrow{[p^r]} A$, $t : S \to A$ be a torsion section and let $X_r\langle t\rangle \in S_{\text{ét}}$ denote the pullback

$$\begin{array}{ccc} X_r\langle t\rangle & \longrightarrow & A_r \\ {\scriptstyle \mathrm{pr}_r}\downarrow & & \downarrow{\scriptstyle p_r} \\ S & \xrightarrow{\;t\;} & A \end{array}$$

and let $\mathrm{pr}_r = \mathrm{pr}_r\langle t\rangle : X_r\langle t\rangle \to S$ denote the structure map. We denote by $\mathscr{H}_r\langle t\rangle \in \mathscr{S}(S_{\text{ét}})$ the corresponding representable sheaf. Then we define

$$\Lambda_r[\mathscr{H}_r\langle t\rangle] = \Lambda_r[X_r\langle t\rangle] := \mathrm{pr}_{r,*}\Lambda_r \in \mathscr{S}(S_{\text{ét}}) \tag{7.2.2}$$

Then $\Lambda_r[\mathscr{H}_r\langle e\rangle] = \Lambda_r[\mathscr{H}_r]$. Since $A_r \xrightarrow{p_r} A \in A_{\text{ét}}$ is a X_r-torsor via $0 \to X_r \to A_r \xrightarrow{p_r} A \to 0$ on A, $X_r\langle t\rangle \to S$ is a X_r-torsor over S. Thus $\Lambda_r[\mathscr{H}_r\langle t\rangle]$ is a module over $\Lambda_r[\mathscr{H}_r]$.

For $r \geq 0$, let $\lambda_r : X_{r+1} \to X_r$ denote map induced by multiplication by p. Then $\mathrm{pr}_{r+1} = \mathrm{pr}_r \circ \lambda_r$ and we may view λ_r as the transition map in the pro-system $(X_r \to S)_{r\geq 0}$ of finie etale schemes on S. Similarly, the maps $\lambda_r = \lambda_r\langle t\rangle : X_{r+1}\langle t\rangle \to X_r\langle t\rangle$ be the map induced by the universal property of $X_r\langle t\rangle$ and the maps $X_{r+1}\langle t\rangle \to A_{r+1} \xrightarrow{[p]} A_r$, the structure map $X_{r+1}\langle t\rangle$ give rise to a pro-system $(X_r\langle t\rangle \to S)_{r\geq 0}$. The adjunction $\lambda_{r,*} = \lambda_{r,!} \dashv R\lambda_r^! = \lambda_r^*$ gives us a map

$$\lambda_{r,*}\Lambda_{r+1} = \lambda_{r,*}\lambda_r^*\Lambda_{r+1} = \lambda_{r,!}\lambda_r^!\Lambda_{r+1} \to \Lambda_{r+1}.$$

Applying $\mathrm{pr}_{r,*}$ to it gives us a map $\Lambda_{r+1}[\mathscr{H}_{r+1}\langle t\rangle] = \mathrm{pr}_{r,*}\lambda_{r,*}\Lambda_{r+1} \to \mathrm{pr}_{r,*}\Lambda_{r+1}$. Composing this map with reduction modulo p^r, we obtain an induced 'trace' map

$$\mathrm{Tr}_r : \Lambda_{r+1}[\mathscr{H}_{r+1}\langle t\rangle] \to \Lambda_r[\mathscr{H}_r\langle t\rangle] \tag{7.2.3}$$

Then the maps Tr_r are compatible with the module structure.

Definition 7.2.4. The *sheaf of Iwasawa algebras of* $\mathscr{H}$ *on* S is defined to be the pro-sheaf $\Lambda(\mathscr{H}) := (\Lambda_r[\mathscr{H}_r])_{r\geq 0}$ with transition maps given by (7.2.3) for $t = e$. The *sheaf of Iwasawa modules associated with* t is defined to be the pro-system $(\Lambda_r[\mathscr{H}_r\langle t\rangle])_{t\geq 0}$.

For each non-negative integer k, let $\Gamma_k(\mathscr{H}_r)$ denote the sheafification of the presheaf that sends an open subscheme $U \subset S$ to the k-th divided power algebra $\Gamma_k(\mathscr{H}_r(U))$. Then the reduction maps $\mathscr{H}_{r+1} \to \mathscr{H}_r$ induce isomorphism $\Gamma_k(\mathscr{H}_r)\otimes_{\mathbb{Z}/p^r\mathbb{Z}}\mathbb{Z}/p^{r-1}\mathbb{Z} \simeq \Gamma_k(\mathscr{H}_{r-1})$ and we obtain a pro-sheaf $\Gamma_k(\mathscr{H}) := (\Gamma_k(\mathscr{H}_r)_{r\geq 0} \in \text{Ét}(S)_{\mathbb{Z}_p}$. There is a canonical map

$$\gamma_k : \mathrm{Sym}^k(\mathscr{H}) \to \Gamma_k(\mathscr{H}) \tag{7.2.5}$$

induced by $m^{\otimes k} \in \mathrm{Sym}^k(\mathscr{H})$ to $k!m^{[k]}$ for m a section of $\mathscr{H}$. It induces an isomoprhism $\mathrm{Sym}^k() \simeq \mathrm{Sym}^k(\mathscr{H}) \otimes \mathbb{Q}_p \to \Gamma_k(\mathscr{H}) \otimes \mathbb{Q}_p \simeq \Gamma_k(\mathscr{H}_{\mathbb{Q}_p})$ the corresponding $\mathbb{Q}_p$-sheaves.

The sheaf $\mathscr{H}_r \in \mathscr{S}(S_{\text{ét}})$ possesses over $X_r \in S_{\text{ét}}$ a tautological section $\tau_r \in \Gamma(X_r, \mathscr{H}_r) = \mathscr{H}_r(X_r) = \mathrm{Hom}_S(X_r, X_r)$ corresponding to the identity map $X_r \to X_r$. It's k-th divided power gives rise to a section $\tau_r^{[k]} \in \Gamma(X_r, \Gamma_k(\mathscr{H}_r))$. Let $\Gamma_k(\mathscr{H}_r)_{|X_r} := \mathrm{pr}_r^*\Gamma_k(\mathscr{H}_r)$ denote the restriction of $\Gamma_k(\mathscr{H}_r)$ to $X_r \in S_{\text{ét}}$. Then

$$\begin{aligned}\Gamma(X_r, \Gamma_k(\mathscr{H}_r)) = \mathrm{Hom}_{X_r}(\Lambda_r, \Gamma_k(\mathscr{H}_r)_{|X_r}) &\simeq \mathrm{Hom}_S(\mathrm{pr}_{r,!}\Lambda_r, \Gamma_k(\mathscr{H}_r)) \\ &\simeq \mathrm{Hom}_S(\Lambda_r[\mathscr{H}_r], \Gamma_k(\mathscr{H}_r))\end{aligned}$$

where the penultimate isomorphism follows via the adjunction $\mathrm{pr}_{r,!} \dashv R\,\mathrm{pr}_r^! = \mathrm{pr}_r^*$ (as pr_r is étale) and the last by $\mathrm{pr}_{r,!} = \mathrm{pr}_{r,*}$ (as pr_r is proper). Thus $\tau_r^{[k]}$ corresponds to a map

$$\mathrm{mom}_r^k : \Lambda_r[\mathscr{H}_r] \to \Gamma_k(\mathscr{H}_r). \tag{7.2.6}$$

For fixed k and varying r, the maps mom_r^k are compatible with respect to $\mathrm{Tr}_r : \Lambda_r[\mathscr{H}_r] \to \Lambda_{r-1}[\mathscr{H}_{r-1}]$ and the reduction maps $\Gamma_k(\mathscr{H}_r) \to \Gamma_k(\mathscr{H}_r) \otimes_{\Lambda_r} \Lambda_{r-1} \simeq \Gamma_k(\mathscr{H}_{r-1})$ ([Kin16, Lemma 4.5.1]).

Definition 7.2.7. The *k-th moment map* is defined to be the morphism of pro-sheaves $\mathrm{mom}^k : \Lambda(\mathscr{H}) \to \Gamma_k(\mathscr{H})$ obtained by the compatible system $(\mathrm{mom}_r^k)_{r\geq 0}$ given in (7.2.6).

Remark 7.2.8. According to [Kin16, §4.2], the moment map is the sheaf theoretic version of the Laplace transform on the Iwasawa algebra of a free module H that takes values in the divided power algebra of H.

7.2.2 Integral logarithm pro-sheaf

Parallel to the construction of $\Lambda(\mathscr{H}) \in \mathscr{S}(S_{\text{ét}})$ is the construction of a pro-sheaf on A that is the integral analogue of the $\mathbb{Q}_p$-logarithm pro-sheaf. Using this sheaf, one defines classes degree $2d-1$ cohomology of S with coefficients in sheaf of Iwasawa algebras $\Lambda(\mathscr{H})$. Via the moment map of Definition 7.2.7, one defines Eisenstein classes in cohomology with coefficients in $\Gamma_k(\mathscr{H})$ which are the integral analogue of Eisenstein classes defined in 7.1.21.

Let $p_r : A_r \to A$ be the X_r-torsor over A as in §7.2.1. Denote by $\lambda_r : A_r \to A_r$ the transition map induced $[p] : A \to A$. Then $p_{r+1} = p_r \circ \lambda_r$ and we have a pro-system $(A_r \to A)_{r\geq 0}$ of étale schemes on A. For each non-negative integer s, let

$$\Lambda_s[A_r] := p_{r,*}\Lambda_s \in \mathscr{S}(A_{\text{ét}}).$$

Using the adjunction $\lambda_{r,*} \dashv \lambda_r^*$ (as λ_r is again finite étale), we get a map $p_{r,s} : \Lambda_s[A_{r+1}] \to \Lambda_s[A_r]$. If $s = r$, then composition of this map with reduction modulo p^r gives a map $\mathrm{Tr}_r : \Lambda_{r+1}[A_{r+1}] \to \Lambda_r[A_r]$ as in (7.2.3).

Definition 7.2.9. The Λ_s *logarithm sheaf* $\mathcal{L}_{\Lambda_s}$ is the pro-sheaf $(\Lambda_s[A_r])_{r\geq 0}$ with transition maps induced by $p_{r,s}$. The *integral logarithm sheaf* $\mathcal{L}$ is defined to be the pro-sheaf $(\Lambda_r[A_r])_{r\geq 0}$ with transition maps given by Tr_r. We have $\mathcal{L} = (\mathcal{L}_{\Lambda_s})_{s\geq 0}$.

Since $\mathcal{L}_{\Lambda_s} \otimes_{\Lambda_s} \Lambda_{s-1} \simeq \mathcal{L}_{\Lambda_{s-1}}$ by construction, we have $\mathcal{L} \in \text{Ét}(A)_{\mathbb{Z}_p}$. Via the X_r torsor $p_r : A_r \to A$, $\mathcal{L}$ becomes a free rank 1 module over $\pi^*\Lambda(\mathscr{H})$. The base change compatiblity of the construction of $\Lambda_r[A_r]$ implies that $\mathcal{L}$ are compatible with arbitrary base change. Consequently, there is an isomorphism

$$\varsigma_t : t^*\mathcal{L} \simeq \Lambda(\mathscr{H}\langle t\rangle) \tag{7.2.10}$$

as sheaf of modules over $\Lambda(\mathscr{H})$. In particular, $e^*\mathcal{L} \simeq \Lambda(\mathscr{H})$.

The integral logarithm sheaf enjoys properties similar to those of $\mathcal{L}og_{\mathbb{Q}_p}$. Below we record the ones needed in the next subsection.

Proposition 7.2.11 (Splitting principle)**.** *Let n be a positive integer and $t : S \to A$ be a n torsion section. Then there exists a canonical homomorphism $[n]_\# : t^*\mathcal{L} \to \Lambda(\mathscr{H})$ which is an isomorphism if n is relatively prime to p.*

Corollary 7.2.12. *Let c be an integer prime to p, $D = A[c]$. Then there exists a canonical isomorphism $\iota_D^*\mathcal{L} \simeq \pi_D^*\Lambda(\mathscr{H})$ where ι_D, π_D are as in 7.1.13.*

Proposition 7.2.13 (Vanishing of Cohomology)**.** *Let r, s be non-negative integers. There exist natural isomorphisms $R^{2d}\pi_*(\Lambda_s[A_r]) \simeq \Lambda_s(-d)$ which are compatible with respect to $p_{r,s}$ and reduction modulo p^{s-1}. For each $i = 0, \ldots, 2d-1$, there exist a sufficiently large integer r' such that $R^i\pi_*\Lambda_s[A_{r'}] \to R^i\pi_*\Lambda_s[A_r]$ is zero. In particular,*

$$\mathrm{H}^{2d}_{\text{ét}}\big(A, \mathcal{L}(d)\big) \simeq \begin{cases} 0 & \text{if } i < 2d \\ \mathrm{H}^0_{\text{ét}}(S, \mathbb{Z}_p) & \text{if } i = 2d \end{cases}$$

7.2.3 Iwasawa-Eisenstein classes

Let $c > 1$ be an integer invertible on S and prime to p and retain the notations 7.1.13. Repeating the the same argument as in 7.1.2 and invoking Proposition 7.2.13, we find an exact sequence

$$0 \to \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}(d)\big) \xrightarrow{\text{res}} \mathrm{H}^0_{\text{ét}}\big(D, \iota^*\mathcal{L}\big) \to \mathrm{H}^0_{\text{ét}}(S, \mathbb{Q}_p) \tag{7.2.14}$$

where res is again referred to as the *residue* map. By Corollary 7.2.12, we may replace $\mathrm{H}^0_{\text{ét}}(D, \iota^*\mathcal{L})$ with $\mathrm{H}^0_{\text{ét}}\big(D, \pi_D^*\Lambda(\mathscr{H})\big)$. Let $\mathbb{Z}_p[D]^0$ denote the kernel of the map $\mathrm{H}^0_{\text{ét}}(D, \mathbb{Z}_p) \to H^0_{\text{ét}}(S, \mathbb{Z}_p)$ induced by the trace map $\pi_{D,*}\mathbb{Z}_p = \pi_{D,!}\pi_D^!\mathbb{Z}_p \to \mathbb{Z}_p$. Then $\mathbb{Z}_p[D]^0$ lies in the image of residue map for the same reasons as in §7.1.3.

Definition 7.2.15. Let $\alpha \in \mathbb{Z}_p[D]^0$. The *integral étale polylogarithm with residue* α is the unique class

$$_\alpha\mathrm{pol}_{\mathbb{Z}_p} \in \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}(d)\big)$$

such that $\mathrm{res}\big(_\alpha\mathrm{pol}_{\mathbb{Z}_p}\big) = \alpha$.

Let $N > 1$ be an integer prime to c and $t : A \to U_D$ be a N-torsion section of $\pi_U : U_D \to S$. By Proposition 7.2.11, we have an induced map $[N]_\# : \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Lambda(\mathscr{H}\langle t\rangle)\big) \to \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Lambda(\mathscr{H})\big)$ which is an isomorphism if $(N, p) = 1$. The adjunction $\mathrm{id} \to Rt_*t^*$ yields a pullback map $t^* : \mathrm{H}^{2d-1}_{\text{ét}}\big(U, \mathcal{L}(d)\big) \to \mathrm{H}^{2d-1}_{\text{ét}}(S, t^*\mathcal{L}(d))$. Since $\varsigma_t : t^*\mathcal{L}(d) \simeq \Lambda[\mathscr{H}\langle t\rangle]$ is an isomoprhism, (7.2.10) , we obtain a chain of maps

$$\begin{aligned} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, t^*\mathcal{L}(d)\big) \xrightarrow[\sim]{\varsigma_t} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathscr{H}\langle t\rangle\big) \xrightarrow{[N]_\#} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Lambda(\mathscr{H})\big) \\ \xrightarrow{\mathrm{mom}^k} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Gamma_k(\mathscr{H})\big). \end{aligned} \tag{7.2.16}$$

where the last map is induced by the moment map $\mathrm{mom}^k : \Lambda(\mathscr{H}) \to \Gamma_k(\mathscr{H})$ (Definition 7.2.6).

Definition 7.2.17. The *Iwasawa-Eisenstien class* ${}_\alpha\mathcal{EI}_N(t) \in \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Lambda(\mathscr{H})\big)$ *associated with* t, N *and* α is defined to be image of ${}_\alpha\mathrm{pol}_{\mathbb{Z}_p}$ under the composition $[N]_\# \circ \varsigma_t \circ t^*$. The *k-th integral Eisenstein class*

$$ {}_\alpha\mathrm{Eis}^k_N(t) \in \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Gamma_k(\mathscr{H})\big) $$

associated with k, N and t is defined to be the image of ${}_\alpha\mathcal{EI}_N(t)$ under the map mom^k.

The Eisenstein class ${}_\alpha\mathrm{Eis}_N(t)$ depends on N (in addition to α and t). The following interpolation result of Kings makes this dependence rather precise. Let $\mathrm{mom}^k_{\mathbb{Q}_p}$ denote the composition

$$ \begin{aligned} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Lambda(\mathscr{H})\big) \xrightarrow{\mathrm{mom}^k} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Gamma_k(\mathscr{H})\big) &\xrightarrow{-\otimes\mathbb{Q}_p} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Gamma_k(\mathscr{H})\big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\ &\xrightarrow[\sim]{\gamma_k} \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k(\mathscr{H}_{\mathbb{Q}_p})\big) \end{aligned} $$

where γ_k denotes the isomorphism induced by (7.2.5).

Theorem 7.2.18 ([Kin16, Theorem 6.3.3])**.** *For* $\alpha \in \mathbb{Z}_p[D]^0$ *and* $N > 1$ *an integer relative prime to* c, *the* k*-th rational Eisenstein class* ${}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t)$ *associated with a* N*-torsion section* $t : S \to U_D$ *satisfies*

$$ \mathrm{mom}^k_{\mathbb{Q}_p}\big({}_\alpha\mathcal{EI}_N(t)\big) = N^k\,{}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t). $$

In particular, if $(N, p) = 1$, *the class* ${}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t) \in \mathrm{H}^{2d-1}_{\text{ét}}\big(S, \mathrm{Sym}^k(\mathscr{H}_{\mathbb{Q}_p})\big)$ *lies in the* $\mathbb{Z}_p$*-submodule given by the image of* $\mathrm{H}^{2d-1}_{\text{ét}}\big(S, \Gamma_k(\mathscr{H})\big)$ *under the composition* $\gamma_k \circ (- \otimes \mathbb{Q}_p)$.

7.3 Eisenstein classes for Shimura varieties

Since (PEL type) Shimura varieties are moduli spaces of abelian varieties with additional structures, they admit universal abelian families on them. The torsion sections of these abelian families can therefore be used to create Eisenstein classes in the cohomology of Shimura varieties. In this section, we study these classes for Siegel modular varieties.

7.3.1 Siegel modular varieties

Let $g \geq 1$ be an integer. Denote by

$$ J = J_g = \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix} \in \mathrm{Mat}_{2g\times 2g}(\mathbb{Q}) $$

the standard symplectic matrix. Let $\mathbf{G} = \mathrm{GSp}_{2g}$ denote the reductive group over $\mathbb{Q}$ whose R points are given by $\mathbf{G}(R) = \{g \in \mathrm{GL}_{2g}(R) \,|\, g^t \in Jg = c(g)J \text{ for } c(g) \in R^\times\}$. The induced map $c : \mathrm{GSp}_{2g} \to \mathbb{G}_m$ is called the *similitude*. Let $\mathbb{S}$ denote the Deligne torus and consider

$$ h_{\text{std}} : \mathbb{S} \to \mathbf{G}_{\mathbb{R}} \qquad (a + b\sqrt{-1}) \mapsto \begin{pmatrix} aI_g & bI_g \\ -bI_g & aI_g \end{pmatrix} $$

and let $\mathcal{X}$ denote the $\mathbf{G}(\mathbb{R})$-conjugacy class of h_{std}. Then $(\mathbf{G}, X)$ constitutes of a Shimura datum [Mil03, §6] satisfying axioms SV1-SV6 of *op. cit.* It's reflex field is $\mathbb{Q}$. For a neat compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, let $\mathrm{Sh}_K = \mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ denote the corresponding $\mathbb{Q}$-variety. It is smooth of dimension $g(g+1)/2$.

Let (V, ψ) denote the standard symplectic $\mathbb{Q}$-vector space where $V = \mathbb{Q}^{2g}$ and $\psi : V \times V \to \mathbb{Q}$ is the pairing induced by J. We let $e_1, \ldots, e_{2g}$ denote the standard basis of V. Let $V_{\mathbb{Z}}$ denote the standard symplectic lattice obtained by the $\mathbb{Z}$-span of e_i, $V_{\hat{\mathbb{Z}}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and $V_f = V \otimes_{\mathbb{Q}} \mathbb{A}_f$. Fix $N \geq 3$ a positive integer and let $K_n \subset \mathbf{G}(\mathbb{A}_f)$ denote the subgroup $\left\{ g \in \mathbf{G}(\mathbb{A}_f) \,|\, (g-1)V_{\hat{\mathbb{Z}}} \subset NV_{\hat{\mathbb{Z}}} \right\}$. Then $\mathrm{Sh}(K_N)$ is the $\mathbb{Q}$-scheme representing the following moduli problem. Let $\mathrm{Sch}_{\mathbb{Q}}$ denote the category of $\mathbb{Q}$ schemes and let

$$\mathfrak{M}_N : \mathrm{Sch}_{\mathbb{Q}} \to \mathrm{Sets}$$

denote the contravariant functor that sends a $\mathbb{Q}$-scheme S isomorphism classes of triples (A, λ, η) where

- A is an abelian scheme over S of relative dimension g,
- $\lambda : A \xrightarrow{\sim} A^{\vee}$ is a principal polarization
- $\eta : A[N] \xrightarrow{\sim} (V_{\mathbb{Z}}/NV_{\mathbb{Z}})_S$ is a symplectic similitude of group schemes over S i.e. η is an isomorphism of S-group schemes such that the Weil pairing on $A[N]$ corresponds to a multiple of the pairing ψ.

The isomorphism η is also referred to as a 'principal level N-structure'. Given a morphism $f : T \to S$ of schemes, the morphism $\mathfrak{M}_N(S) \to \mathfrak{M}_N(T)$ is given by pullback of families. To say that $\mathfrak{M}_N$ is represented by $\mathrm{Sh}(K_N)$ is to say that there exists a natural transformation $\Psi_N : \mathfrak{M}_N \to \mathrm{Hom}_{\mathrm{Sch}_{\mathbb{Q}}}(-, \mathrm{Sh}(K_N))$. We denote by $\mathrm{id}_N \in \mathfrak{M}_N(\mathrm{Sh}(K_N))$ the element corresponding to the identity map under Ψ_N. By definition of $\mathfrak{M}_N$, there is an abelian scheme $\pi_N : \mathcal{A}_N \to \mathrm{Sh}(K_N)$ with polarization and principal level N structure associated with id_N. It is referred to as the *universal family* on $\mathrm{Sh}(K_N)$. We denote by $\{\varepsilon_{i,N} \,|\, i = 1, \ldots, 2g\}$ the canonical ordered basis of N-torsion sections of π_N where $\varepsilon_{i,N}$ corresponds to the section $e_i : S \to (V_{\mathbb{Z}}/NV_{\mathbb{Z}})_S = \bigsqcup_i S$ is the identity map on the i-th copy of S in the disjoint union.

Given M, N such that $M|N$, there is a natural forgetful transformation $\Phi_{N,M} : \mathfrak{M}_N \to \mathfrak{M}_M$ which sends (A_N, λ_N, η_N) to (A_M, λ_M, η_M) where $A_M := A_N$, $\lambda_M := \lambda_M$ and η_M is the map completing the diagram

$$\begin{array}{ccc} A[N] & \xrightarrow{\eta_N} & \left(V_{\mathbb{Z}}/NV_{\mathbb{Z}}\right)_S \\ {\scriptstyle [N/M]}\downarrow & & \downarrow \\ A[M] & \xrightarrow{\eta_M} & \left(V_{\mathbb{Z}}/MV_{\mathbb{Z}}\right)_S \end{array}$$

where $[N/M] : A[N] \to A[M]$ is induced by the isogeny $[N/M] : A \to A$ and $V_{\mathbb{Z}}/NV_{\mathbb{Z}} \to V_{\mathbb{Z}}/MV_{\mathbb{Z}}$ is the reduction modulo M map. The corresponding map on Hom functors identifies via Ψ_N with the one induced by the the finite étale morphism $\iota_{N,M} : \mathrm{Sh}(K_N) \to \mathrm{Sh}(K_M)$.

We can consider $\iota_{M,N} \in \mathfrak{M}_M\big(\mathrm{Sh}(K_N)\big)$ via Ψ_M^{-1}. By definition of $\mathfrak{M}_M$, there is a corresponding to an abelian scheme $\pi_N : \mathcal{A}'_N \to \mathrm{Sh}(K_N)$ with polarization λ'_M and principal level M-structure η'_M obtained by pulling back the universal family on $\mathrm{Sh}(K_M)$ along $\iota_{M,N}$. But $\iota_{M,N}$ is the image of $\mathrm{id}_N \in \mathfrak{M}_N\big(\mathrm{Sh}(K_N)\big)$ under $\Phi_{N,M} : \mathfrak{M}_N\big(\mathrm{Sh}(K_N)\big) \to \mathfrak{M}_M\big(\mathrm{Sh}(K_N)\big)$ which implies that the class of $(\mathcal{A}'_N, \lambda'_M, \eta'_M)$ coincides with the one obtained by the universal family on $\mathrm{Sh}(K_N)$ by 'forgetting' the data of level structure. The upshot is that we have a pullback diagram

$$\begin{array}{ccc} \mathcal{A}_N & \xrightarrow{\pi_N} & \mathcal{A}_M \\ {\scriptstyle \pi_N}\downarrow & & \downarrow{\scriptstyle \pi_M} \\ \mathrm{Sh}(K_N) & \longrightarrow & \mathrm{Sh}(K_M) \end{array}$$

and that $\iota^*_{N,M}(\varepsilon_{i,M})$ is equal to $[N/M] \circ \varepsilon_{i,N} : S \to \mathcal{A}_N$.

7.3.2 Adelic Schwartz spaces

Fix $c > 1$ a positive integer with $(c,p) = 1$. Let

$$S := \{K_N \mid N \geq 3 \text{ an integer such that } (N,c) = (N,p) = 1\}$$

and let Υ be the collection of all compact open subgroups that are contained in a conjugate of K_3. Then Υ satisfies (T1)-(T3) of §2.1 and S is a base for Υ (Definition 2.6.5).

Set $X := V_f \backslash \{\vec{0}\}$. We view the elements of X as row vectors. There is a smooth right action $X \times \mathbf{G}(\mathbb{A}_f) \to X$ given by right matrix multiplication. If $\xi : X \to \mathbb{Z}$ is a function, we let $g \cdot \xi$ denote the function $x \mapsto \xi(xg)$. Let $\mathcal{S}_{\mathbb{Z}}(X)$ denote the set of all locally constant and compactly supported functions. Then the action $\mathbf{G}(\mathbb{A}_f) \times \mathcal{S}_{\mathbb{Z}}(X) \to \mathcal{S}_{\mathbb{Z}}(X)$ given by $(g,\xi) \mapsto g \cdot \xi$ makes $\mathcal{S}_{\mathbb{Z}}(X)$ a smooth left representation of $\mathbf{G}(\mathbb{A}_f)$. We let $\mathcal{S} : \Upsilon \to \mathbb{Z}$-Mod denote the functor that sends $K \in \Upsilon$ to the K-invariants $\mathcal{S}_{\mathbb{Z}}(X)^K$.

For each $K_N \in S$, let $B_{\mathbb{Z}_p}(K_N) \subset \mathcal{S}_{\mathbb{Z}}(X)$ denote the set of all functions $\xi : X \to \mathbb{Z}$ that are supported on $V_{\hat{\mathbb{Z}}} \cap X$ and invariant under the translation by $NV_{\hat{\mathbb{Z}}} \cap X$. In other words, elements of $B(K_N)$ are $\mathbb{Z}$-valued maps (of sets) on

$$V_{\hat{\mathbb{Z}}}/NV_{\hat{\mathbb{Z}}} \setminus \{\vec{0}\} = (\mathbb{Z}/N\mathbb{Z})^{2g} \setminus \{\vec{0}\}.$$

For $K \in \Upsilon$, let $\mathcal{T}_{\mathbb{Z}}(K) \subset \mathcal{S}_{\mathbb{Z}}(K)$ denote the $\mathbb{Z}$-submodule of all functions $\xi : X \to \mathbb{Z}$ such that ξ equals a finite sum $\sum_{i \in I} c_i [K\sigma_i K_{N_i}](\xi_i)$ for some $\sigma_i \in \mathbf{G}(\mathbb{A}_f)$, $K_{N_i} \in S$, $c_i \in \mathbb{Z}$ and $\xi_i \in B_{\mathbb{Z}}(K_{N_i})$.

Lemma 7.3.1. *$\mathcal{T}_{\mathbb{Z}}$ is a cohomological Mackey subfunctor of $\mathcal{S}_{\mathbb{Z}}(K)$ for all $K \in \Upsilon$.*

Proof. Since pullbacks and induction are Hecke correspondences and composition of two Hecke correspondences is one itself (Corollary 2.5.7), we see that $\mathcal{T}_{\mathbb{Z}}(K)$ is a RIC functor. Since $\mathcal{S}_{\mathbb{Z}}(K)$ is CoMack, so is $\mathcal{T}_{\mathbb{Z}}(K)$. □

7.3.3 Parametrization

Fix a $k \geq 0$ and $\alpha \in \mathbb{Z}_p[D]^0$. Let $M_{\mathbb{Z}_p} : \Upsilon \to \mathbb{Z}_p$-Mod denote the map given by $\Upsilon \ni K \mapsto \operatorname{im}(\gamma_k \circ \jmath)$ where γ_k and $\jmath$ are given by

$$\mathrm{H}^{2g-1}_{\text{ét}}\big(\mathrm{Sh}(K), \Gamma_k(\mathscr{H}_{\mathbb{Z}_p})\big) \xrightarrow{\jmath} \mathrm{H}^{2g-1}_{\text{ét}}\big(\mathrm{Sh}(K), \Gamma_k(\mathscr{H}_{\mathbb{Z}_p})\big) \otimes \mathbb{Q}_p \xrightarrow[\sim]{\gamma_k} \mathrm{H}^{2g-1}_{\text{ét}}\big(\mathrm{Sh}(K), \mathrm{Sym}^k(\mathscr{H}_{\mathbb{Q}_p})\big)$$

are as in §7.2.3. Then $M_{\mathbb{Z}_p}$ is a CoMack functor (6.2.4). For $K_N \in S$, let $B_{\mathbb{Z}_p}(K_N) := B_{\mathbb{Z}}(K_N) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We think of elements $\xi \in B_{\mathbb{Z}_p}(K_N)$ as $\mathbb{Z}_p$ valued functions on the set of $N^{2g}-1$ non-zero torsion sections $T(N)$ $t : S \to \mathcal{A}_N$ determined by the universal level-N structure η_N. Let $\mathscr{F} = \big\{\varphi_K : B_{\mathbb{Z}_p}(K_N) \to M_{\mathbb{Z}_p}(K_N)\big\}_S$ be the family of $\mathbb{Z}_p$-module homomorphisms given by

$$\varphi_K : \xi = \sum_{t \in T(N)} \xi(t) t \mapsto \xi(t) \cdot {}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t).$$

By Theorem 7.2.18, this is well-defined. Let $\mathcal{T}_{\mathbb{Z}_p} : G \to \mathbb{Z}_p$-Mod denote the functor $\mathcal{T}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

Proposition 7.3.2. *There exists a unique morphism $\varphi : \mathcal{T}_{\mathbb{Z}_p} \to M_{\mathbb{Z}_p}$ of RIC functors such that for each $K \in S$, the restriction $\varphi(K)_{|B_{\mathbb{Z}_p}(K_N)}$ equals φ_K.*

Proof. Since $M_{\mathbb{Z}_p}$ has injective restrictions, it suffices by Proposition 2.6.6 to establish that $\mathscr{F}$ is compatible under pullbacks (Definition 2.6.2). Since no two K_N, $K_{N'}$ are conjugate under g, we only need to check compatibility under restrictions $\mathrm{pr}^*_{K_N,K_M}$ for $M|N$. Since $B_{\mathbb{Z}_p}(K_M)$ has a basis given by $\mathrm{ch}(x + MV_{\hat{\mathbb{Z}}})$ for $x \in V_{\hat{\mathbb{Z}}}$ a set of representatives for $V_{\hat{\mathbb{Z}}}/MV_{\hat{\mathbb{Z}}} \setminus \{0\}$, it suffices to verify that

$$\varphi_{K_N} \circ \mathrm{pr}^*_{K_N,K_M}\big(\mathrm{ch}(x + MV_{\hat{\mathbb{Z}}})\big) = \mathrm{pr}^*_{K_N,K_M} \circ \varphi_{K_M}\big(\mathrm{ch}(x + MV_{\hat{\mathbb{Z}}})\big).$$

Let $t_{x,M} : S \to \mathcal{A}_M$ (resp. $t_{x,N} : S \to \mathcal{A}_N$) given by the class of x in $V_{\hat{\mathbb{Z}}}/MV_{\hat{\mathbb{Z}}}$ (resp. $V_{\hat{\mathbb{Z}}}/NV_{\hat{\mathbb{Z}}}$) via η_M (resp. η_N). Since pr_{K_N,K_M} on $\mathcal{T}_{\mathbb{Z}_p}$ is the natural inclusion, the left hand side ${}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}\big([N/M] \circ t_{x,N}\big)$ by (7.3.1). Since $\mathrm{pr}^*_{K_N,K_M}$ on $M_{\mathbb{Z}_p}$ is the pullback map $\iota^*_{N,M}$, the right hand side is $\iota^*_{N,M}\big({}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{x,M})\big)$. Since $\iota^*_{N,M}(t_{x,M}) = [N/M] \circ t_{x,N}$ (see the discussion before (7.3.1)), the equality above is equivalent to

$$\iota^*_{N,M}\big({}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}(t_{x,M})\big) = {}_\alpha\mathrm{Eis}^k_{\mathbb{Q}_p}\big(\iota^*_{N,M} t_{x,M}\big)$$

But this holds by Lemma 7.1.22. ☐

Part II

Examples

Chapter 8

Various embeddings of Shimura data

In this chapter, we collect together various examples of Shimura data and their embeddings that motivate the zeta element problems considered in later chapters. We refer the reader to [BR94], [Lan79] and [Kot92] for descriptions[1] of Galois representations in the cohomology of varieties considered here.

8.1 Unitary Shimura varieties

Let $E \subset \mathbb{C}$ be an imaginary quadratic number field, $\gamma \in \mathrm{Gal}(E/\mathbb{Q})$ denote the non-trivial automorphism. Let $J = \mathrm{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q})$. Then $\gamma(J)^t = J$ i.e. J is $E/\mathbb{Q}$-hermitian. Let $\mathbf{G} = \mathrm{GU}_{p,q}$ denote the algebraic group over $\mathbb{Q}$ whose R points for a $\mathbb{Q}$-algebra R are given by

$$\mathrm{GU}_{p,q}(R) := \left\{ g \in \mathrm{GL}_{p,q}(R) \,|\, \gamma(g)^t J \gamma(g) = c(g) J \text{ for some } c(g) \in R^\times \right\}.$$

Then $c : \mathrm{GU}_{p,q} \to \mathbb{G}_m$ is a character called the *similitude*. Let

$$\begin{aligned} h : \mathbb{S} &\to \mathbf{G}_{\mathbb{R}} \\ z &\mapsto \mathrm{diag}(\underbrace{z,\ldots,z}_{p}, \underbrace{\bar{z},\ldots,\bar{z}}_{q}) \end{aligned}$$

and let X be the $\mathbf{G}(\mathbb{R})$-conjugacy class of h. Then $(\mathbf{G}, X)$ constitutes a Shimura-Deligne that satisfies (SD3) if $p, q \neq 0$. The dimension of the associated Shimura varieties $p \cdot q$. There is an identification $\mathbf{G}_E \simeq \mathbb{G}_{m,E} \times \mathrm{GL}_{p+q,E}$ induced by the isomoprhism of E-algebras $E \otimes R \simeq R^\times \times R^\times$, $(e, r) \mapsto (er, \gamma(e)r)$ for any E-algebra R. The cocharacter $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{C}} \times \mathrm{GL}_{p+q,\mathbb{C}}$ associated with h is given by $z \mapsto \big(z, \mathrm{diag}(\underbrace{z,\ldots,z}_{p}, \underbrace{1,\ldots,1}_{q})\big)$ which is defined over E. It γ-conjugate is $z \mapsto \big(z, \mathrm{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{z,\ldots,z}_{q})\big)$. The reflex field is therefore E if $p \neq q$ and $\mathbb{Q}$ otherwise.

[1] in some cases, conjectural

We are interested in studying the degree $(p+q)$ (the middle degree) cohomology of such Shimura varieties or the products thereof. By subdividing $p = p_1 + p_2$, $q = q_1 + q_2$ for $p_i, q_i \geq 0$ positive integers, one obtains embeddings of Shimura-Deligne data

$$\mathrm{GU}_{p_1,q_1} \times_c \mathrm{GU}_{p_2,q_2} \hookrightarrow \mathrm{GU}_{p,q}$$

which we can assume to be given by contiguous diagonal blocks by replacing J with

$$\mathrm{diag}(\underbrace{1,\ldots,1}_{p_1},\underbrace{-1,\ldots,-1}_{q_1},\underbrace{1,\ldots,1}_{p_2},\underbrace{-1,\ldots,-1}_{q_2}).$$

8.1.1 Embeddings in $\mathrm{GU}_{1,2m-1}$

Let $\mathbf{H} = \mathrm{GU}_{1,m-1} \times_c \mathrm{GU}_{0,m}$. Then codimension of Shimura varieties is $2m - 1 - (m - 1) = m$. Thus the pushforwards of cycle classes land in degree $2m$ arithmetic étale cohomology which is one more than the middle degree $2m - 1$. Our source functor here will be the trivial one.

The cocharacter μ_h corresponds to the representation of ${}^L\mathbf{G}_E$ which is trivial on the factor $\mathscr{W}_E$. Thus at a choice of a split prime λ of E above ℓ, we are interested in the Hecke polynomial of the standard representation of GL_n. As the dimension is odd, the normalization of the Satake polynomial will be at one half of odd integers. This is the content of Ch. 9. When ℓ is inert, we are interested in the base change of the standard L factor. This setup is the studied in Ch. 11 below. While there are many choice of tori that admit a map from $\mathbf{H}$, it is $\mathbf{U}_1$ defined by that yields a zeta element. See Remark 9.5.4 for an explanation.

Remark 8.1.1. The split case was studied in [GS21] by alternative methods.

8.1.2 Embeddings in $\mathrm{GU}_{2,m} \times \mathrm{GU}_{1,1}$

In this case, the target variety has dimension $2m + 1$. Consider the embedding

$$\mathrm{GU}_{1,m-1} \times_c \mathrm{GU}_{1,1} \hookrightarrow \mathrm{GU}_{2,m} \times_c \mathrm{GU}_{1,1}$$

where the $\mathrm{GU}_{1,1}$ factor of the source is embedded diagonally. The source variety is dimension $(m-1)+1 = m$ and we therefore have a codimension $m + 1$ setup. Pushing cycle classes allows us to land in degree $2m + 2$ arithmetic étale cohomology, one more than the middle degree $2m+1$. The expected dimension of the Galois representation in degree $2m+1$ of the target would $(m+2)(m+1)$. For the case $m = 2$, all Shimura varieties would have reflex fields $\mathbb{Q}$. The degree of the Hecke polynomial to consider in this case would be 12 and given as a convolution of exterior square L-factor and standard L-factor. To keep matters simple, we instead opt for the embedding

$$\mathrm{GU}_{1,1} \times \mathrm{GU}_{1,1} \hookrightarrow \mathrm{GU}_{2,2}$$

though note that cycles in this case do not land in middle degree. This simpler example is studied in Ch. 10. The torus we have used is the one that sends each copy to their common similitude.

8.2 Siegel modular varieties

The data for these varieties was introduced in 7.3.1. We recall that GSp_{2g}-Shimura varieties are $g(g+1)/2$-dimensional.

8.2.1 Embedding in GSp_4

Consider the embedding $\mathrm{GL}_2 \times_{\mathbb{G}_m} \mathrm{GL}_2 \hookrightarrow \mathrm{GSp}_4$ which corresponds at the level of moduli problems to taking product of two elliptic curves. This is a codimension one setup. Thus pushing classes from H^2 gives classes in H^4 of the target, one more than the desired middle degree. One may take the cup product of two Eisenstein classes in the H^1 of each modular curve to construct a source of zeta elements in degree 2. The expected L-factor for the Galois representation is the spin L-factor of degree 4. The torus we use is given by the common determinant of the two GL_2 copies, and corresponds to cyclotomic extensions of $\mathbb{Q}$.

Remark 8.2.1. This setup was studied in [LSZ22] by alternative means.

8.2.2 Embeddings in GSp_6

Consider the embedding $\mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2 \hookrightarrow \mathrm{GSp}_6$. This is a codimension $6-4=2$ setup. Thus pushing classes from H^3 of the source gives classes in H^7 of the target, one more than the desired middle degree. We may use Eisenstein classes in H^3 of the GSp_4 factor as our source of zeta elements. The expected L-factor for these Galois representations is the spin L-factor of degree 8. Again the torus that we use for the zeta element problem is given by the similitude, which corresponds to cyclotomic extensions.

Remark 8.2.2. The vertical norm relations of a closely related setup have been explored [CRJ20]. The horizontal norm relations in this setting are however completely unexplored in literature so far.

Chapter 9

Standard L-factor of $\mathbb{G}_m \times \mathrm{GL}_{2m}$

In this chapter, we study the HNR problem for the split case of the embedding discussed in §8.1.1. The source functor in this scenario will parametrize fundametal classes of the source Shimura variety.

Notation. The symbols F, $\mathscr{O}_F$, ϖ, $\Bbbk$, q and $[\Bbbk]$ have the same meaning as in Notation 4.0.1. The letter $\mathbf{G}$ will denote the group scheme $\mathbb{G}_m \times \mathrm{GL}_n$ over $\mathscr{O}_F$ where n is a positive integer and is assumed to be even in from §9.4 onwards. We will denote $G := \mathbf{G}(F)$ and $K := \mathbf{G}(\mathscr{O}_F)$. For a ring R, we let $\mathcal{H}_R = \mathcal{H}_R(K\backslash G/K)$ denote the Hecke algebra of G of level K with coefficients in R with respect to a Haar measure μ_G such that $\mu_G(K) = 1$. For simplicity, we will sometimes denote $\mathrm{ch}(K\sigma K) \in \mathcal{H}_R$ as $(K\sigma K)$.

9.1 Desiderata

Let $\mathbf{A} = \mathbb{G}_m^{n+1}$ and $\mathrm{dis} : \mathbf{A} \to \mathbf{G}$ be the embedding given by

$$(u_0, u_1, \ldots, u_n) \mapsto \big(u_0, \mathrm{diag}(u_1, \ldots, u_n)\big).$$

Then dis identifies $\mathbf{A}$ with a maximal split torus in $\mathbf{G}$. We denote $A := \mathbf{A}(F)$ the F-points of A and $A^\circ := A \cap K$ the unique maximal compact subgroup. For $i = 0, \ldots n$, let $e_i : \mathbf{A} \to \mathbb{G}_m$ be the projection on the i-th component and $f_i : \mathbb{G}_m \to \mathbf{A}$ be the cocharacter inserting u in the i-th component of $\mathbf{A}$. We will denote by Λ the cocharacter lattice $\mathbb{Z}f_0 \oplus \cdots \oplus \mathbb{Z}f_n$. An element $a_0 f_0 + \ldots + a_n f_n \in \Lambda$ will also be denoted by $(a_0, \ldots, a_n)$ The set $\Phi \subset X^*(\mathbf{A})$ of roots of $\mathbf{G}$ are $\pm(e_i - e_j)$ for $1 \le i < j \le n$ which constitutes an irreducible root system of type A_{n-1}. We let $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\} \subset \Phi$ where

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \;\; \ldots, \;\; \alpha_{2n} = e_{n-1} - e_n.$$

Then Δ constitutes a base for Φ. We let $\Phi^+ \subset \Phi$ denote the set of resulting positive roots. The half sum of positive roots is then

$$\delta := \frac{1}{2}\sum_{k=1}^{n}(n-2k+1)e_k \tag{9.1.1}$$

With respect to the ordering induced by Δ, the highest root is $\alpha_0 = e_1 - e_{2n}$. We let I_G be the standard Iwahori subgroup of G, which corresponds to the alcove determined by the simple affine roots $\alpha_1 + 0$, $\alpha_2 + 0$, $\ldots$, $\alpha_{n-1} + 0$, $-\alpha_0 + 1$. The coroots corresponding to α_i are

$$\alpha_0^\vee = f_1 - f_n, \quad \alpha_1^\vee = f_1 - f_2, \quad \alpha_2^\vee = f_2 - f_3, \ \ldots, \ \alpha_{n-1}^\vee = f_{n-1} - f_n$$

and their $\mathbb{Z}$ span in Λ is denoted by $Q^\vee$. An element $\lambda = (a_0, \ldots, a_n) \in \Lambda$ is dominant iff $a_1 \geq a_2 \geq \ldots \geq a_n$ and anti-dominant if all these inequalities hold in reverse. We denote the set of dominant cocharacters by Λ^+. The translation action of $\lambda \in \Lambda$ on $\Lambda \otimes \mathbb{R}$ via $x \mapsto x + \lambda$ is denoted by $t(\lambda)$. We denote $\varpi^\lambda \in A$ the element $\lambda(\varpi)$ for $\lambda \in \Lambda$ and $v : A/A^\circ \to \Lambda$ be the inverse of the map $\Lambda \to A/A^\circ$, $\lambda \mapsto \varpi^{-\lambda}A^\circ$. Let s_i be the reflection associated with α_i for $i = 0, \ldots, n$. The action of s_i on Λ is given explicitly as follows:

- s_i acts by the transposition $f_i \leftrightarrow f_{i+1}$ for $i = 1, 2, \ldots, n-1$
- s_0 acts by transposition $f_1 \leftrightarrow f_n$.

For $\lambda \in \Lambda$, we let $e^\lambda \in \mathbb{Z}[\Lambda]$ denote the element corresponding to λ and $e^{W\lambda} \in \mathbb{Z}[\Lambda]$ denote the element obtained by the formal sum of elements in the orbit $W\lambda$. Let $S_{\text{aff}} = \{s_1, s_2, \ldots, s_{n-1}, t(\alpha_0^\vee)s_0\}$ and W, W_{aff}, W_I be the Weyl, affine Weyl and Iwahori Weyl groups respectively determined by A. We consider W_{aff} as a subgroup of affine transformations of $\Lambda \otimes \mathbb{R}$. We have

- $W = \langle s_1, \ldots, s_{n-1}\rangle \cong S_{n-1}$,
- $W_{\text{aff}} = t(Q^\vee) \rtimes W$
- $W_I = N_G(A)/A^\circ = A/A^\circ \rtimes W \overset{v}{\simeq} \Lambda \rtimes W$,

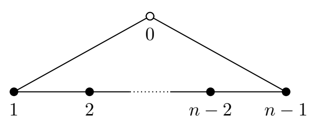
The pair $(W_{\text{aff}}, S_{\text{aff}})$ forms a Coxeter system of type $\tilde{A}_{n-1}$. We consider W_{aff} a subgroup of W_I via $W_{\text{aff}} \simeq Q^\vee \rtimes W \hookrightarrow \Lambda \rtimes W \overset{v}{\simeq} W_I$. The natural action of W_{aff} on $\Lambda \otimes \mathbb{R}$ then extends to W_I with $\lambda \in \Lambda$ acting as a translation $t(\lambda)$. We set $\Omega := W_I/W_{\text{aff}}$, which is a free abelian group on two generators and we have $W_I \cong W_{\text{aff}} \rtimes \Omega$. We let $\ell : W_I \to \mathbb{Z}$ denote the induced length function with respect S_{aff}. Given $\lambda \in \Lambda$, the minimal length of elements in $t(\lambda)W$ is achieved by a unique element. This length is given by

$$\ell_{\min}(t(\lambda)) := \sum_{\alpha \in \Phi_\lambda} |\langle \lambda, \alpha\rangle| + \sum_{\alpha \in \Phi^\lambda} (\langle \lambda, \alpha \rangle - 1) \tag{9.1.2}$$

where $\Phi_\lambda = \{\alpha \in \Phi^+ \mid \langle \lambda, \alpha \rangle \leq 0\}$, $\Phi^\lambda = \{\alpha \in \Phi^+ \mid \langle \lambda, \alpha\rangle > 0\}$. When λ is dominant, this is also the minimal length of elements in $Wt(\lambda)W$. We let

- $w_1 := \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$, $w_2 := \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$, $\ldots$, $w_{n-1} := \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$,
- $w_0 := \begin{pmatrix} 0 & & & & & \frac{1}{\varpi} \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ \varpi & & & & & 0 \end{pmatrix}$, $\rho = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ \varpi & & & & & 0 \end{pmatrix}$

which we consider as elements of $N_G(A)$ (the normalizer of A in G) whose component in $\mathbb{G}_m$ is 1. The classes of $w_0, w_1, \ldots, w_{n-1}$ in W_I represent $t(\alpha_0^\vee)s_0, s_1, \ldots, s_{n-1}$ respectively and the class of ρ is a generator of $\Omega/\langle t(f_0)\rangle$. The reflection s_0 in α_0 is then represented by $w_{\alpha_0} := \varpi^{f_1} w_0$. We will henceforth use the letters w_i, ρ to denote both the matrices and the their classes in W_I if no confusion can arise. We note that conjugation by ω on W_I acts by cycling the (classes of) generators via $w_{n-1} \to w_{n-2} \to \ldots \to w_1 \to w_0 \to w_{n-1}$, thereby inducing an automoprhism of the extended Coxeter-Dynkin diagram



where the labels below the vertices correspond to the index of w_i. Note also that $\rho^n = \varpi^{(1,1,\ldots,1)} \in A$ is central. For $i = 0, 1, \ldots, n-1$, let $x_i : \mathbb{G}_a \to \mathbf{G}$ be the root group maps defined by

$$x_1 : u \mapsto \begin{pmatrix} 1 & u & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \; x_2 : u \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & u & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & & 1 \end{pmatrix}, \; \ldots, \; x_{n-1} : u \mapsto \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & u \\ & & & & 1 \end{pmatrix},$$

$$x_0 : u \mapsto \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \varpi u & & & & 1 \end{pmatrix}$$

where again the matrices are considered as elements of $\mathbf{G}$ with 1 in the $\mathbb{G}_m$ component. Let $g_{w_i} : [\bbkappa] \to G$ be the maps $\kappa \mapsto x_i(\kappa) w_i$. For $w \in W_I$ with $w = s_{w,1} s_{w,2} \cdots s_{w,\ell(w)} \rho_w$, where $s_{w,i} \in S_{\text{aff}}$, $\rho_w \in \Omega$, a reduced word decomposition, let

$$\begin{aligned} \mathcal{X}_w : [\bbkappa]^{\ell(w)} &\to G/K \\ (\kappa_1, \ldots, \kappa_{\ell(w)}) &\mapsto g_{s_{w,1}}(\kappa_1) \cdots g_{s_w,\ell(w)}(\kappa_{\ell(w)}) \rho_w K \end{aligned} \tag{9.1.3}$$

where we have suppressed the dependence on the decomposition chosen in the notation. By Proposition 5.2.8, the image of $\mathcal{X}_w$ is independent of the choice of decomposition and we have $\#\operatorname{im}(\mathcal{X}_w) = q^{\ell(w)}$. We note that $\ell(w) = \ell_{\min}(t(-\lambda_w))$ where $\lambda_w \in \Lambda$ is the unique cocharacter such that $wK = \varpi^{\lambda_w} K$.

Remark 9.1.4. Cf. the matrices in [Iwa66, p. 75].

9.2 Standard Hecke polynomial

Let $\mathcal{R} = \mathcal{R}_q$ denote the ring $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ and let $y_i := e^{f_i} \in \mathcal{R}[\Lambda]$ the element corresponding to f_i. Then $\mathcal{R}[\Lambda] = \mathcal{R}[y_0^{\pm}, \cdots, y_n^{\pm}]$. We are interested in the characteristic polynomial of the standard representation of the dual group $\widehat{\mathbf{G}}_F = \mathbb{G}_m \times \mathrm{GL}_n$ whose highest coweights are $\mu_{\mathrm{std}} = f_0 + f_1$ which is the cocharacter obtained from the data in 8.1.1. Since μ_{std} is minuscule, the (co)weights of the associated representation are the elements in the Weyl orbit of μ_{std}. These are $f_0 + f_1$, $f_0 + f_2, \ldots, f_0 + f_n$. The Satake polynomial (see Definition 4.3.2) for μ_{std} is therefore

$$\mathfrak{S}_{\mathrm{std}}(X) = (1 - y_0 y_1 X)\,(1 - y_0 y_2 X) \cdots (1 - y_0 y_n X) \in \mathbb{Z}[\Lambda]^W[X]$$

As in §4.2, we let $\mathscr{S} : \mathcal{H}_{\mathcal{R}} \to \mathcal{R}[\Lambda]^W$ denote the Satake isomorphism.

Definition 9.2.1. The polynomial $\mathfrak{H}_{\mathrm{std},c}(X) \in \mathcal{H}_{\mathcal{R}}[X]$ is defined so that $\mathscr{S}\left(\mathfrak{H}_{\mathrm{std},c}(X)\right) = \mathfrak{S}_{\mathrm{std}}\left(q^{-\frac{c}{2}} X\right)$ for any $c \in \mathbb{Z}$.

Proposition 9.2.2 (Tamagawa)**.** *Let $\varrho = \varpi^{f_0}\rho \in N_G(A)$. Then for $c \in \mathbb{Z} - 2\mathbb{Z}$,*

$$\mathfrak{H}_{\mathrm{std},c}(X) = \sum_{k=0}^{n} (-1)^k q^{-k(n-k+c)/2} \left(K\varrho^k K\right) X^k \in \mathcal{H}_{\mathbb{Z}_{(q)}}[X].$$

Proof. Let $p_k = p_k(y_1, \ldots, y_n) \in \mathbb{Z}[\Lambda]^W$ denote the k-th elementary symmetric polynomial in $y_1, \ldots, y_n$. Then $\mathfrak{S}_{\mathrm{std}}(X) = \sum_{k=0}^{n} (-1)^k x_0^k p_k X^k$. So it suffices to establish that

$$\mathscr{S}(K\varrho^k K) = q^{k(n-k)/2} x_0^k p_k.$$

For $k \geq 1$, set $\mu_k := f_0 + f_1 + \ldots + f_k \in \Lambda^+$. Then, $K\varrho^k K = K\varpi^{\mu_k} K$ as double cosets. But μ_k are themselves minuscule. Therefore, Corollary 4.2.4 and the second part of Corollary 4.3.8 together imply that $\mathscr{S}(K\varpi^{\mu_k} K)$ is supported on $x_0^k p_k$ and that the coefficient of $x_0 p_k$ is $q^{\langle \mu_k, \delta \rangle}$ where δ is as in (9.1.1). One easily calculates that $\langle \lambda_k, \delta \rangle = k(n-k)/2$. ☐

Remark 9.2.3. The formula for $\mathfrak{H}_{\mathrm{std},c}$ was first obtained by Tamagawa [Tam63, Theorem 3] and the case $n = 2$ is due to Hecke [Hec37], hence the terminology 'Hecke polynomial' – see the note at the bottom of [Shi94, p. 62]. See also the historical commentary in §4, §8 of [Cas17]. Cf. [Gro98, eq. (3.14)].

Remark 9.2.4. An alternate proof of Proposition 9.2.2 that does not use Corollary 4.2.4 may be obtained using the decomposition of $K\rho^k K$ described in Proposition 9.3.3 which is closer in spirit to the proof by Tamagawa.

9.3 Decomposition of minuscule operators

In this section, we study the decomposition of Hecke operators $K\varrho^k K$ into individual left cosets. As σ_0^k is central, it suffices to describe the decomposition $K\rho^k K$, so that the left coset representatives γ will have 1 in the $\mathbb{G}_m$-component.

Definition 9.3.1. Let $k \geq 1$ be an integer, $k \leq n$. A *Schubert symbol* of length k is a k-element subset $\mathbf{j}$ of $[n] := \{1, \ldots, n\}$. We write the elements of $\mathbf{j} = \{j_1, \ldots, j_k\}$ such that $j_1 < \cdots < j_n$. The *dimension* of $\mathbf{j}$ is defined to be $\|\mathbf{j}\| = j_1 + \ldots + j_k - \binom{k+1}{2}$. The set of Schubert symbols of length k is denoted by J_k. We have $|J_k| = \binom{n}{k}$.

We define a partial order $\preceq$ on J_k by declaring $\mathbf{j} \preceq \mathbf{j}'$ for symbols $\mathbf{j} = \{j_1, \ldots, j_k\}$, $\mathbf{j}' = \{j'_1, \ldots, j'_k\}$ if $j_i \leq j'_i$ for all $i = 1, \ldots, k$. Then $(J, \preceq)$ is a lattice (in the sense of order theory). The smallest element and largest elements of J_k are $\{1, \ldots, k\}$ and $\{n-k+1, \ldots, n\}$ and we assign a grading to J_k so that the smallest element has length is 0.

Definition 9.3.2. For $\mathbf{j} \in J_k$, the *Schubert cell* $\mathcal{C}_\mathbf{j}$ is the finite subset of $\mathrm{Mat}_{n\times k}(F)$ consisting of all $n \times k$ matrices C such that

- M has 1 in (j_i, i)-entry, which are referred to as *pivots*.
- the entries of M that are below or to the right of a pivot are zero,
- M has entries in $[\Bbbk] \subset \mathscr{O}_F$ elsewhere.

Then $|\mathcal{C}_\mathbf{j}| = q^{\|\mathbf{j}\|}$. Given $C \in \mathcal{C}_\mathbf{j}$, we let $\varphi_\mathbf{j}(C) \in \mathrm{GL}_n(\mathscr{O}_F)$ be the $n \times n$ matrix such that by inserting the i-th column of $\mathcal{C}_\mathbf{j}$ for the j_i-th column of $\varphi_\mathbf{j}(C)$, making the rest of the diagonal entries ϖ and inserting zero elsewhere. We let $\mathcal{X}_\mathbf{j} \subset \mathrm{GL}_n(F)$ denote the image of $\mathcal{C}_\mathbf{j}$ and consider $\mathcal{X}_\mathbf{j} \subset G$ by taking 1 in the $\mathbb{G}_m$-component.

Example 9.3.1. Let $n = 4$, , $k = 2$. Then the Schubert cells are

$$\mathcal{C}_{\{1,2\}} = \begin{pmatrix} 1 & \\ & 1 \\ & \\ & \end{pmatrix}, \qquad \mathcal{C}_{\{1,3\}} = \begin{pmatrix} 1 & \\ & * \\ & 1 \\ & \end{pmatrix}, \qquad \mathcal{C}_{\{2,3\}} = \begin{pmatrix} * & * \\ 1 & \\ & 1 \\ & \end{pmatrix}$$

$$\mathcal{C}_{\{1,4\}} = \begin{pmatrix} 1 & \\ & * \\ & * \\ & 1 \end{pmatrix}, \qquad \mathcal{C}_{\{2,4\}} = \begin{pmatrix} * & * \\ 1 & \\ & * \\ & 1 \end{pmatrix}, \qquad \mathcal{C}_{\{3,4\}} = \begin{pmatrix} * & * \\ * & * \\ 1 & \\ & 1 \end{pmatrix}$$

where the star entries are elements of $[\not{k}]$ and zeros are omitted. The corresponding collections $\mathcal{X}_{\mathbf{j}}$ are

$$\mathcal{X}_{\{1,2\}} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varpi & \\ & & & \varpi \end{pmatrix}, \ \mathcal{X}_{\{1,3\}} = \begin{pmatrix} 1 & & & \\ & \varpi & * & \\ & & 1 & \\ & & & \varpi \end{pmatrix}, \ \mathcal{X}_{\{2,3\}} = \begin{pmatrix} \varpi & * & * & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{pmatrix},$$

$$\mathcal{X}_{\{1,4\}} = \begin{pmatrix} 1 & & & \\ & \varpi & & * \\ & & \varpi & * \\ & & & 1 \end{pmatrix}, \ \mathcal{X}_{\{2,4\}} = \begin{pmatrix} \varpi & * & & * \\ & 1 & & \\ & & \varpi & * \\ & & & 1 \end{pmatrix}, \ \mathcal{X}_{\{3,4\}} = \begin{pmatrix} \varpi & & * & * \\ & \varpi & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

We have a total of $1 + q + q^2 + q^2 + q^3 + q^4$ matrices in these six sets.

Proposition 9.3.3. *For $1 \leq k \leq n$, $K\rho^k K = \bigsqcup_{\mathbf{j} \in J_k} \bigsqcup_{\gamma \in \mathcal{X}_{\mathbf{j}}} \gamma K$.*

Proof. Let $\lambda_k = \sum_{i=1}^{k} f_{n-k+i} \in \Lambda^-$. We have $\rho^k W \rho^{-k} = \langle S_{\text{aff}} \setminus w_{n-k} \rangle$ and therefore $W \cap \rho^k W \rho^{-k} = \text{Stab}_W(\lambda_k)$. By Proposition 5.2.8,

$$K\rho^k K = \bigsqcup_{w \in [W/W^{\lambda_k}]} \text{im}\big(\mathcal{X}_{w\rho^k}\big).$$

where $W^{\lambda_k} := \text{Stab}_W(\lambda_k)$ and $[W/W^{\lambda_k}]$ denotes the set of representatives in W of W/W^{λ_k} of minimal possible length. For $\lambda \in W\lambda_k$, let $\mathbf{j}(\lambda) \in J_{n-k}$ be the Schubert symbol consisting of integers $1 \leq j_1 < \ldots < j_{n-k} \leq n$ such that the coefficient f_{j_i} in λ is 0. If $w \in [W/W^{\lambda_k}]$ and $\lambda = w\lambda_k \in W\lambda_k$, we let $\mathbf{j}(w) := \mathbf{j}(w\lambda_k)$. We let $\preceq$ denote the left (weak) Bruhat order on W with respect to S. Then $(W, \preceq)$ is a graded lattice with grading given by length.

Claim 1. Then map $w \mapsto \mathbf{j}(w)$ sets up an order preserving bijection $[W/W^{\lambda_k}] \xrightarrow{\sim} J_{n-k}$.

The set W/W^{λ_k} is in one-to-one correspondence with the orbit $W\lambda_k \subset \Lambda$. The orbit consists of the $\binom{n}{k}$ permutations of the cocharacter $\lambda_k = f_{n-k+1} + \cdots + f_n$. Picking a permuation of λ_k in turn is the same thing as choosing $n-k$ integers $1 \leq j_1 < \ldots < j_{n-k} \leq n$ such that $f_{j_1}, \ldots, f_{j_{n-k}}$ have coefficient zero in the permutation of λ_k. This establishes the bijectivity of $w \mapsto \mathbf{j}(w)$. The identity element is mapped to $\{1, \ldots, k\}$ and one establishes by induction on the length that the mapping preserves the orders.

Claim 2. For all $w \in [W/W^{\lambda_k}]$, $\text{im}(\mathcal{X}_{w\rho}) = \big\{\gamma K \,|\, \gamma \in \mathcal{X}_{\mathbf{j}(w)}\big\}$.

We proceed by induction on the the length of w. If w is of length 0, then w is the identity element and $\mathbf{j} = \mathbf{j}_{\lambda_k} = \{1, \ldots, n-k\}$. Now $\text{im}(\mathcal{X}_{\rho^k}) = \big\{\rho^k K\big\}$ is a singleton and $\mathcal{X}_{\mathbf{j}} = \varpi^{\lambda_k} K$. As $\rho^k K = \varpi^{\lambda_k} K$, the base case holds. Now suppose that the claim holds for all $w \in [W/W^{\lambda_k}]$ of length m. Let $v = sw$

where $s \in \{s_1, \ldots, s_{n-1}\}$, $w \in [W/W^{\lambda_k}]$ such that $\ell(v) = \ell(w) + 1$ and $\ell(w) = m$. Let $\mathbf{j}_v$, $\mathbf{j}_w$ be the Schubert symbols corresponding to v, w respectively. By Claim 1, there exists a unique $j \in \{1, \ldots, n-1\}$ such that $i \in \mathbf{j}_w$, $j+1 \in \mathbf{j}_v$ and $\mathbf{j}_w \setminus \{j+1\} = \mathbf{j}_v \setminus \{j\}$. If $\sigma K \in \mathcal{X}_{w\rho}$, then by induction hypothesis $\sigma K = \varphi_{\mathbf{j}(w)}(C)K$ for some $C \in \mathcal{C}_{\mathbf{j}_w}$. Denote $\tau := \varphi_{\mathbf{j}(w)}(C)$. By definition, $\tau(j,j) = 1$, $\tau(j+1, j+1) = \varpi$ and $\tau(j, j_1) = \tau(j_2, j) = \tau(j+1, j_3) = 0$ for $j_1, j_2 > j$, $j_3 \neq j+1$.

$$\tau = \begin{pmatrix} \ddots & & & & \\ & 1 & 0 & \cdots & 0 \\ & 0 & \varpi & & \\ & \vdots & & & \\ & \underbrace{0}_{j} & & & \ddots \end{pmatrix} \}j$$

Then $g_{w_j}(\kappa)\tau K = x_j(\kappa) w_j \tau w_j K$ i.e. the the effect of multipliying τK by $g_{w_j} x_j$ is to switch the rows and columns in indices j and $j+1$ and then adding κ times the $j+1$-st row to the j-th row. Clearly, $x_j(\kappa) w_j \tau w_j \in \mathcal{X}_{\mathbf{j}(v)}$. Since gK was arbitrary, we see that $\operatorname{im}(\mathcal{X}_{w\rho^k}) = \left\{\gamma K \,|\, \gamma \in \mathcal{X}_{\mathbf{j}(w)}\right\}$ for $w \in [W/W^{\lambda_k}]$ with $\ell(w) = m+1$. By induction, we get the claim. □

9.4 Mixed coset structures

From now on, let $n = 2m$ be even. If $g \in \mathrm{GL}_{2m}(F)$, we will denote by $A_g, B_g, C_g, D_g \in \mathrm{Mat}_{m\times m}(F)$ so that

$$g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}.$$

If $g \in G$, then A_g, B_g, C_g, D_g denote the matrices associated with the $\mathrm{GL}_{2m}(F)$ component of G. Moreover, we adapt the following

Convention 9.4.1. An element of $\mathrm{GL}_{2m}(F)$ is considered as an element of G via the embedding $\mathrm{GL}_{2m}(F) \hookrightarrow G$ in the second component.

Let $\iota : \mathbf{H} \hookrightarrow \mathbf{G}$ be the subgroup generated $\mathbf{A}$ and root groups of $\Delta \setminus \{\alpha_m\}$. Then $\mathbf{H} \simeq \mathbb{G}_m \times \mathrm{GL}_m \times \mathrm{GL}_m$ embedding block diagonally in $\mathbf{G}$. We denote $H = \mathbf{H}(F)$, $U = H \cap K$ and $H_1 = H_2 \simeq \mathrm{GL}_n$ the two components so that $H = F^\times \times H_1 \times H_2$. If $h \in H$, we denote by h_1, h_2 the components of H in H_1, H_2 respectively. We let $W_H \simeq S_m \times S_m$ denote the Weyl group of W which we consider as a subgroup of W generated by $s_1, \ldots, s_{m-1}, s_{m+1}, \ldots, s_{2m}$. The roots of H are denoted by Φ_H. These are $\pm(e_i - e_j)$ for $1 \le i, j \le m$ and for $m+1 \le i, j, \le 2m$ and we have a partition $\Phi_H = \Phi_{H_1} \sqcup \Phi_{H_2}$ into a union of two root systems isomorphic to A_{m-1}. For $\alpha = e_i - e_j \in \Phi_H$ and $k \in \mathbb{Z}$, we let $U_{\alpha,k}$ denote the unipotent subgroup of H with 1's on diagonal and zeros elsewhere except for the (i,j) entry, which is required to have ϖ-adic valuation less than or equal to k.

9.4.1 Mixed decompositions

For $k = 0, \dots, 2m$, let P_k denote the set of pairs (k_1, k_2) of non-negative integers such that $k_1 + k_2 = k$ and $k_1, k_2 \le m$. For $\kappa = (k_1, k_2) \in P_k$, denote $l(\kappa) := \min(k_1, m - k_2)$ and let

$$\lambda_\kappa := \sum_{i=1}^{k_1} f_i + \sum_{j=m-k_2+1}^{m} f_{m+j} \in \Lambda \tag{9.4.2}$$

For $i = 0, \dots, m$, let $t_i := \mathrm{diag}(\underbrace{\varpi^{-1} \dots, \varpi^{-1}}_{i}, \underbrace{0, \dots, 0}_{m-i}) \in \mathrm{Mat}_{m\times m}(F)$ and

$$\tau_i := \begin{pmatrix} I_m & t_i \\ & I_m \end{pmatrix} \in \mathrm{GL}_{2m}(F). \tag{9.4.3}$$

Set $H_{\tau_i} := H \cap \tau_i K \tau_i^{-1}$. For $g \in G$, let $U\varpi^\lambda g K$ denote the set of all double cosets $U\varpi^\lambda g K$ for $\lambda \in \Lambda$.

Lemma 9.4.4. *For $i = 0, \dots, m$, the collections $U\varpi^\Lambda \tau_i K$ are disjoint.*

Proof. It suffices to show that $H\tau_i K$ are distinct double cosets. Suppose for the sake of contradiction that that $\tau_i \in H\tau_j K$ for $i \neq j$. Then $\tau_i^{-1} h \tau_j \in K$. Say $h = (u, h_1, h_2)$. Now

$$\tau_i^{-1}(h_1, h_2)\tau_j = \begin{pmatrix} h_1 & h_1 t_j - t_i h_2 \\ & h_2 \end{pmatrix}$$

and therefore $\tau_i^{-1} h \tau_j \in K$ implies that $h_1, h_2 \in \mathrm{GL}_m(\mathscr{O}_F)$ and $h_1 t_j - t_i h_2 \in \mathrm{Mat}_{m\times m}(\mathscr{O}_F)$. But the second condition implies that the reduction modulo ϖ of one of h_1, h_2 is singular (the determinant vanishes modulo ϖ), which contradicts the first condition. ∎

Proposition 9.4.5. *For $0 \le k \le m$, $\mathrm{ch}(K\rho^k K) = \sum_{\kappa \in P_k} \sum_{i=0}^{l(\kappa)} \mathrm{ch}(U\varpi^{\lambda_\kappa}\tau_i K)$.*

Proof. We first claim that for each fixed k, the double cosets $U\varpi^{\lambda_\kappa}\tau_i K$ are pairwise disjoint for distinct choices of κ and i. By Lemma 9.4.4, two such cosets are disjoint for distinct i, so it suffices to distinguish the cosets for different κ but a fixed i. By Lemma 5.5.2, it suffices to show that $U\varpi^{\lambda_\kappa} H_{\tau_i}$ are pairwise disjoint for $\kappa \in P_k$. Since $H_{\tau_i} \subset U$, it in turn suffices to show that $U\varpi^{\lambda_\kappa} U$ are pairwise disjoint for $\kappa \in P_k$. But this follows by Cartan decomposition for H.

Now let k be fixed. For $\kappa = (k_1, k_2) \in P_k$, let $\mathbf{j} = \{1, \dots, k_1\} \cup \{m - k_2 + 1, \dots, 2m\}$. From the description of $\mathcal{X}_\mathbf{j}$ and Proposition 9.3.3, we see $\varpi^{\lambda_\kappa}\tau_i K \subset K\rho^k K$ (and therefore $U\varpi^{\lambda_\kappa}\tau_i K$) for all $\kappa \in P_k$, $0 \le i \le l(\kappa)$.

So to prove the claim of proposition, it suffices to show that for any $\gamma \in G$ such that $\gamma K \subset K\rho^k K$, there exist κ and i such that $U\gamma K = U\varpi^{\lambda_\kappa}\tau_i K$. By Proposition 9.3.3, any such γ can be replaced by an element of $\mathcal{X}_{\mathbf{j}}$ for some $\mathbf{j}$. Furthermore, since any $\gamma \in \mathcal{X}_{\mathbf{j}}$ has non-zero non-diagonal entries only above a pivot and these entries are in $\mathscr{O}_F$, we can replace γ by $\sigma \in U$ such that A_σ, D_σ are diagonal matrices and $U\gamma K = U\sigma K$. We therefore define a set $\mathcal{Y}_{\mathbf{j}} \subset \mathrm{GL}_n(\mathscr{O}_F)$ that contains all such σ as follows. An element $g \in G$ lies in $\mathcal{Y}_{\mathbf{j}}$ if

- the diagonal of g has 1 (referred to as pivots) in positions (j, j) for $j \in \mathbf{j}$ and ϖ if $j \notin \mathbf{j}$,
- A_g, D_g are diagonal matrices and $C_g = 0$,
- B_g has non-zero entries only in columns of H that contain a pivot and rows that do not.

For any $\mathbf{j} \in J_k$, let $\mathbf{j}_1$ (resp. $\mathbf{j}_2$) denote the subset of elements not greater than m (resp. strictly greater than m) and let $\kappa(\mathbf{j}) := (|\mathbf{j}_1|, |\mathbf{j}_2|) \in P_k$. We establish the following

Claim. For any $\mathbf{j} \in J_k$ and $\gamma \in \mathcal{Y}_{\mathbf{j}}$ there exists an integer i between 0 and $l(\kappa(\mathbf{j}))$ such that $U\gamma K = U\varpi^{\lambda_\kappa}\tau_i K$.

We prove this by induction on m. The case $m = 1$ is straightforward. Assume the truth of the claim for some positive integer $m - 1 \geq 1$. If $\mathbf{j}_1 = \varnothing$, then $A_\gamma = I_m$, $B_\gamma = C_\gamma = 0$ and D_γ is diagonal. Since $w_{m+1}, \ldots, w_{2m}$ lie in both U and K, one can put all the $k \leq m$ pivots in the top diagonal entries of D_γ and we are done. We can similarly rule out the case $\mathbf{j}_2 = \{m+1, \ldots, 2m\}$. Finally, if $B_\gamma = 0$, we can again use reflections in H to rearrange the A_γ and D_γ diagonal entries to match $\varpi^{\lambda_{l,k}}$.

So suppose that $k_1 := |\mathbf{j}_1| > 0$, $k_2 := |\mathbf{j}_2| < m$ and $B_\gamma \neq 0$. Pick $j_1 \in \mathbf{j}_1$ such that the j_1-th row of B_γ is non-zero and let $j_2 \notin \mathbf{j}_2$, $m + 1 \leq j_2 \leq 2m$ be such that the (j_1, j_2) entry of γ in B_γ is not 0. If $j_1 \neq 1$, then using row and columns operations, one can switch the first and j_1-th row and columns to obtain a new matrix γ'. Clearly, γ' is an element of $\mathcal{Y}_{\mathbf{j}'}$ for some new $\mathbf{j}'$, $U\gamma K = U\gamma' K$ and the $(1, j_2)$ entry of γ' is non-zero. Similarly if $j_2 \neq m + 1$, we can produce a matrix using row and columns operations so that $(m + 1, m + 1)$ diagonal entry of the new matrix is 1 and the class of this matrix in $U\backslash G/K$ is the same as γ. The upshot is that we may safely assume that $j_1 = 1$, $j_2 = m + 1$ (so in particular, $1 \in \mathbf{j}$, $m + 1 \notin \mathbf{j}$).

$$\gamma = \left(\begin{array}{cccc|cccc} \bigcirc & & & & & & & \\ & \ddots & & & & & & \\ & & \varpi & & & * & & \\ & & & \ddots & & & & \\ \hline & & & & \square & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & \underbrace{}_{j_1} & & & \underbrace{}_{j_2} & \ddots \end{array}\right)\Big\}j_1 \quad \rightsquigarrow \quad \gamma' = \left(\begin{array}{cccc|cccc} \varpi & & & & * & & & \\ & \ddots & & & & & & \\ & & \bigcirc & & & & & \\ & & & \ddots & & & & \\ \hline & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & \square & \\ & & \underbrace{}_{j_1} & & & & \underbrace{}_{j_2} & \ddots \end{array}\right)\Big\}j_1$$

Since the top left diagonal entry of B_γ is non-zero, we can use elementary operations for rows and columns with labels in $\mathbf{j}_2$[1] to make all the other entries of the first row of B_γ zero and keep D_γ a diagonal matrix. The column operations may change the other rows of B_γ but the new matrix still belongs to $\mathcal{Y}_{\mathbf{j}}$ and has same class in $U\backslash G/K$. Similarly, we can use elementary operations for rows and columns with labels in $\{1,\dots,m\}\setminus \mathbf{j}_1$ to make all the entries below $(1,m+1)$ in B_γ equal to zero, while keeping A_γ a diagonal matrix. Finally, conjugating by an appropriate element of the compact diagonal $A^\circ \subset U$, we can also assume that the top left entry of B_γ is 1.

In summary, we have arrived at a matrix that has the same class in $U\backslash G/K$ as the original γ and has zeros in rows and columns labelled 1, $m+1$ except for the diagonal entries in positions $(1,1)$, $(m+1,m+1)$, $(1,m+1)$ which are ϖ, 1, 1 respectively. The submatrix obtained by deleting the first and $(m+1)$-th rows and columns is a $(2m-2)\times(2m-2)$ matrix in $\mathcal{Y}_{\mathbf{j}'}$ for some $\mathbf{j}'$ of cardinality $k-1$. By induction, this matrix can be put into the desired form using the groups U and K associated with $\mathbb{G}_m \times \mathrm{GL}_{2m-2}$. The possible value of i that can appear from this submatrix have to be at most $\max(k_1-1, m-1+k_2)$ by induction hypothesis and therefore the bound for possible i holds for m as well. This completes the proof. $\square$

9.4.2 Mixed degrees

For $1 \le r \le m$, let $\mathscr{X}_r := \mathrm{GL}_k(F)$. We have inclusions $\mathscr{X}_1 \hookrightarrow \mathscr{X}_2 \hookrightarrow \ldots \hookrightarrow \mathscr{X}_m$ obtained by a considering a matrix $\sigma \in \mathscr{X}_r$ as a $(r+1)\times(r+1)$ matrix whose top left $r\times r$ submatrix is σ, has 1 in last diagonal entry and zeros elsewhere. Let

$$\jmath_r : \mathscr{X}_r \to G \qquad \sigma \mapsto \iota(\sigma,\sigma) = \begin{pmatrix} \sigma & \\ & \sigma \end{pmatrix} \in G$$

where σ is considered as an element of H_1, H_2 as above, so that j_r factorizes as $\mathscr{X}_r \hookrightarrow \mathscr{X}_m \xrightarrow{j_m} G$. We henceforth consider all $\mathscr{X}_r$ as subgroups of G and omit $\jmath$ unless necessary. We denote $\mathscr{X}_r^\circ = \mathscr{X} \cap K \simeq \mathrm{GL}_r(\mathscr{O}_F)$. Now let $\psi_r : \Phi_H \to \mathbb{Z}$ be the function

$$\psi_s(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \{e_i - e_j \in \Phi_H \,|\, \text{either } 1 \le j \le r \text{ or } m+1 \le i \le m+r\} \\ 0 & \text{otherwise} \end{cases}$$

and let H_{ψ_r} be the subgroup generated by $U_{\alpha,\psi_r(\alpha)}$ and $A \cap \tau_r K \tau_r^{-1}$ More explicitly, H_{ψ_r} is the subgroup of elements $(v, h_1, h_2) \in U$ satisfying the three conditions below:

- all the non-diagonal entries in the first r columns of h_1 are divisible by ϖ,
- all non-diagonal entries in the first r rows of h_2 are divisible by ϖ,

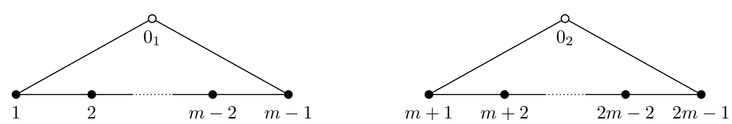
[1]the non-zero columns of B_γ are above a pivot of γ

- the difference of the j and $j+m$−th diagonal entries of $h=(h_1,h_2)\in G$ is divisible by ϖ for all $j=1,\dots,r$.

Lemma 9.4.6. *$H_{\tau_r}=\mathscr{X}_r^\circ H_{\psi_r}=H_{\psi_r}\mathscr{X}_r^\circ$ for $r=1,\dots,m$.*

Proof. The $\mathbb{G}_m$ component on both sides are $\mathscr{O}_F^\times$ and we may therefore ignore it. Let $h=(h_1,h_2)\in H$. Then $h\in H_{\tau_r}$ and if and only if $h\in U$ and $h_1t_r-t_rh_2\in \mathrm{Mat}_{m\times m}(\varpi\mathscr{O}_F)$ (see the calculation in Lemma 9.4.4). It is then clear that $H_{\tau_r}\supset \mathscr{X}_r^\circ\cdot H_{\psi_r}$. Let $h=(h_1,h_2)\in H_{\tau_r}$. From the description of H_{τ_r}, we see that the $r\times r$ submatrix σ formed by first r rows and columns of h_1 must be invertible (and similarly for h_2). Then $\jmath_r(\sigma^{-1})\cdot h$ has the top $r\times r$ block equal the identity matrix. Since this matrix lies in H_{τ_r}, we see again from the description of elements of H_{τ_r} that $\jmath_r(\sigma^{-1})h\in H_{\psi_r}$. This implies the reverse inclusion $H_{\tau_r}\subset\mathscr{X}_r^\circ H_{\psi_r}$. Since the product of $\mathscr{X}_r^\circ$ and H_{ψ_r} is a group, $\mathscr{X}_r^\circ H_{\psi_r}=H_{\psi_r}\mathscr{X}_i^\circ$. ☐

Recall that $\Phi_H=\Phi_H\sqcup\Phi_{H_2}$ respectively. Let $\alpha_1,\dots,\alpha_m\in\Phi_{H_1}$ and $-\alpha_{m+1},\dots,-\alpha_{2m}\in\Phi_{H_2}$ be the set of positive roots. Then $\alpha_{1,0}:=e_1-e_m\in\Phi_{H_1}$, $\alpha_{2,0}=e_{2m}-e_{m+1}\in\Phi_{H_2}$ are the highest roots. Let $s_{1,0}$, $s_{2,0}\in W_H$ denote the reflections associated with $\alpha_{1,0}$, $\alpha_{2,0}$ respectively. Then the affine Weyl group $W_{H,\mathrm{aff}}$ (as a subgroup of W_{aff}) is generated by $S_{H,\mathrm{aff}}=\left\{t(\alpha_{1,0}^\vee)s_{1,0},s_1,\dots,s_{m-1}\right\}\sqcup\left\{t(\alpha_{2,0}^\vee)s_{2,0},s_{m+1},\dots,s_{2m-1}\right\}$ and $(W_{H_{\mathrm{aff}}},S_{H,\mathrm{aff}})$ is a Coxeter system of type $\tilde{A}_{m-1}\times\tilde{A}_{m-1}$. We denote by $\ell_H:W_H\to\mathbb{Z}$ the resulting length function. The extended Coxeter-Dynkin has two components



(9.4.7)

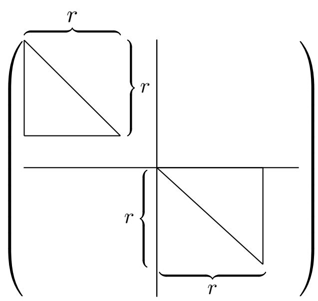
where the labels 0_1, 0_2 correspond to the two affine reflections corresponding to $\alpha_{0,1}$, $\alpha_{0,2}$.

Now let I_{H_1} (resp. I_{H_2}) be the Iwahori subgroup of H_1 (resp. H_2) consisting of integral matices that reduce modulo ϖ to upper triangular (resp. lower triangular) matrices and set $I_H:=\mathscr{O}_F^\times\times I_{H_1}\times I_{H_2}$. Then I_H is the Iwahori subgroup associated with alcove determined by $S_{H,\mathrm{aff}}$. If we let

$$\rho_1:=\begin{pmatrix}0&1&&&&\\&0&1&&&\\&&0&1&&\\&&&\ddots&\ddots&\\&&&&0&1\\\varpi&&&&&0\end{pmatrix}\in H_1\qquad \rho_2:=\begin{pmatrix}0&&&&&\varpi\\1&0&&&&\\&1&0&&&\\&&1&0&&\\&&&\ddots&\ddots&\\&&&&1&0\end{pmatrix}\in H_2$$

(so we have $\rho_1=\rho_2^t$). Both ρ_1, ρ_2 normalize I_H and the effect of conjugation $w\mapsto\rho_1w\rho_1^{-1}$ (resp. $w\mapsto\rho_2w\rho_2^{-1}$) is by cycling in clockwise (resp. counterclockwise) direction the left (resp. right) component of the diagram displayed (9.4.7). we set $\rho_H:=(\rho_1,\rho_2)\in H$ and for $\kappa=(k_1,k_2)\in\mathbb{Z}^2$, we denote by ρ_H^κ the element $(\rho_1^{k_1},\rho_2^{k_2})\in H$. We will denote $-\kappa$ the pair $(-k_1,-k_2)$.

Definition 9.4.8. For $r = 0, \dots, m$, let $I_{H,r}$ denote the subgroup of H which contains I_H and whose Weyl group $W_{H,r} \subset W_H$ is generated by $S_{H,r} := \{s_r \dots, s_{m-1}, s_{m+r}, \dots, s_{2m-1}\}$. More explicitly, $I_{H,r}$ is the subgroup of U consisting of all matrices as below



such that the non-diagonal entries inside the two triangles are divisible by ϖ.

Lemma 9.4.9. *For $\kappa \in P_k$ and $r = 0, \dots, l(\kappa)$, we have $H_{\tau_r} \varpi^{-\lambda_\kappa} U = I_{H,r}\, \rho_H^{-\kappa} U$.*

Proof. Since $r \leq l(\kappa)$, $\varpi^{-\lambda_\kappa}$ commutes with $\mathscr{X}_r^\circ$ and therefore $H_{\tau_r} \varpi^{-\lambda_k} U = H_{\psi_r} \varpi^{-\lambda_\kappa} U$. It is also easily seen that

$$I_{H,r} = H_{\psi_r} \cdot A^\circ \cdot \prod_{\alpha \in \Phi^+_{H,r}} U_{\alpha,0}$$

where $\Phi^+_{H,r} := \big\{ e_i - e_j \in \Phi^+_H \,|\, \text{either } 1 \leq j \leq r \text{ or } m+1 \leq i \leq m+r \big\}$. Since ϖ^{λ_κ} commutes with A° and $U_{\alpha,0}$ for $\alpha \in S_r$, we see that $H_{\psi_r} \varpi^{-\lambda_\kappa} U$. Since $\varpi^{\lambda_\kappa} U = (\rho_1^{k_1}, \rho_2^{-k_2}) U = \rho_H^{-\kappa} U$, the claim follows. □

For $\kappa = (k_1, k_2) \in P_k$, $r = 0, \dots, l(\kappa)$, let $W_{\kappa,r} \subset W_{H,r}$ denote the subgroup generated by $S_{H,r} \setminus \{s_{k_1}, s_{2m-k_2}\}$. Then $W_{\kappa,r}$ is a Coxeter subgroup of $W_{H,r}$. Let

$$P_{\kappa,r} := \sum_{w \in [W_{H,r}/W_{\kappa,r}]} q^{\ell_H(w)}$$

denote the Poincaré polynomial of $[W_{H,r}/W_{\kappa,r}] \subset W_H$.

Proposition 9.4.10. *For $\kappa \in P_k$ and $r = 0, \dots, l(\kappa)$, we have* $\deg\, [U\varpi^{\lambda_\kappa}\tau_r K]_* = P_{\kappa,r}(q)$.

Proof. We have $\deg\, [U\varpi^{\lambda_\kappa} K]_* = \deg[H_{\tau_r}\varpi^{-\lambda_\kappa} U]$ which is by definition the cardinality of $H_{\tau_r}\varpi^{-\lambda_\kappa} U/U$. By Lemma 9.4.6, we have $H_{\tau_r}\varpi^{-\lambda_\kappa} U/U = I_{H,r}\, \rho_H^{-\kappa} U/U$. Proposition 5.2.8 therefore implies that $\deg\, [U\varpi^{\lambda}_{\kappa} K]_*$ is the Poincaré polynomial of $[W_{H,r}/(W_{H,r} \cap \rho^{-\kappa} W_H \rho^\kappa)]$. Now $\rho^{-\kappa} W_H \rho^\kappa$ is the subgroup of $W_{I,H}$ generated by

$$S_{H,\text{aff}} \setminus \rho_H^{-\kappa}\, \{s_{1,0}, s_{2,0}\}\, \rho_H^\kappa = S_{H,\text{aff}} \setminus \{s_{k_1}, s_{2m-k_2}\}$$

where the equality follows since $\rho_1^{-1} s_{1,0} \rho_1 = s_1$ and $\rho_2^{-1} s_{2,0} \rho_2 = s_{2m}$ (see above for the description of the action of ρ_1, ρ_2 on (9.4.7)). Thus, we have $W_{H,r} \cap \rho^{-k} W_H \rho^k = W_{\kappa,r}$ and the claim follows. □

Corollary 9.4.11. $\deg\, [U\varpi^{\lambda_\kappa}\tau_r K]_* \equiv \binom{m-r}{m-k_1}\binom{m-r}{k_2} \pmod{q-1}$.

Proof. $|W_{H,r}| = (m-r)! \cdot (m-r)!$ since $W_{H,r}$ is the product of the groups generated $s_r, \ldots, s_{m-1}$ and $s_{m+r}, \ldots, s_{2m-1}$, each of which have cardinality $(m-r)!$. Similarly, $W_{\kappa,r}$ is the product of four groups generated by four sets of reflections labelled

$$r+1, \ldots, k_1 - 1, \qquad k_1+1, \ldots, m-1, \qquad m+r+1, \ldots, 2m-k_2-1, \qquad 2m-k_2+1, \ldots, 2m-1$$

which have sizes $(k_1-r)!$, $(m-k_1)!$, $(m-k_2-r)!$ and $k_2!$ respectively. □

9.5 Zeta elements

We now formulate the zeta element problem relevant to the situation of 8.1.1 and use the work done in the previous sections to establish an affirmative answer. Let $T := F^\times$, $C = \mathscr{O}_F^\times \subset T$ the unique maximal compact subgroup, $D = 1 + \varpi\, \mathscr{O}_F$ a subgroup of index $q-1$ and $\nu H \to T$ be the map given by $(h_1, h_2) \mapsto \det(h_2)/\det h_1$. Let $\mathcal{O}$ be any integral domain containing $\mathbb{Z}[q, q^{-1}]$. Set

- $\mathcal{G} = G \times T$, $\Upsilon_\mathcal{G}$ collection of all compact open subgroups of $\mathcal{G}$,
- $\iota_\nu = \iota \times \nu : H \to \mathcal{G}$,
- U, $\mathcal{K} := K \times C$ as bottom levels
- $M_{H,\mathcal{O}} = M_{H,\mathcal{O},\mathrm{triv}}$ the trivial functor,
- $x_U = 1_\mathcal{O} \in M_{H,\mathcal{O}}(U)$ the source bottom class,
- $\mathcal{L} = K \times D$ the compactum of field extension of degree $q-1$,
- $\mathfrak{H}_c = \mathfrak{H}^t_{\mathrm{std},c}(\mathrm{Frob}) \in \mathcal{C}_\mathcal{O}(\mathcal{K}\backslash\mathcal{G}/\mathcal{K})$ where $\mathrm{Frob} := \mathrm{ch}(\varpi C)$ is the 'arithmetic Frobenius'.

Theorem 9.5.1. *There exists a zeta element for $(x_U, \mathfrak{H}_c, \mathcal{L})$ for all $c \in \mathbb{Z} - 2\mathbb{Z}$.*

Proof. For each $k_2 = 0, \ldots, m$ and i an integer such that $0 \le i \le m - k_2$, let $g_{i,k_2} := (1, \tau_i, \varpi^{-2k_2}) \in \mathcal{G}$ and $J_{i,k_2} := \{(k_1, k_2) \,|\, i \le k_1 \le m, k_1 \in \mathbb{Z}\}$. We show that there is a zeta element with twists g_{i,k_2} by verifying the criteria of Corollary 3.2.2

For each i, k_2 as above, let $d_{i,k_2} := [H \cap g_{i,k_2}\mathcal{K}g_{i,k_2}^{-1} : H \cap g_{i,k_2}\mathcal{L}g_{i,k_2}^{-1}]$. By Lemma 3.2.7 (iii), $d_{i,k_2} = [H_{\tau_i} \cap \tau_i K \tau_i^{-1} : \nu^{-1}(D)]$ and we therefore write d_i for d_{i,k_2}. Since $\nu(H \cap \tau_i K \tau_i^{-1}) = C$ for $i = 0, \ldots, m-1$, we have $d_i = q-1$. Now if $(h_1, h_2) \in H_{\tau_m}$, then $h_1 - h_2 \in \varpi \cdot \mathrm{Mat}_{m\times m}(\mathscr{O}_F)$. Thus, $\nu(H_{\tau_m}) \subset D$ (if fact, equal) and $H_{\tau_m} = \nu^{-1}(D)$. Therefore $d_m = 1$.

Next, for each (i, k_2) as above and $j = (k_1, k_2) \in J_{i,k_2}$, denote $h_j := (\varpi^{-kf_0}, \varpi^{-\lambda_j}) \in H$ and $\sigma_j = \iota_\nu(h_j^{-1}) \cdot g_{i,k_2} = (\varpi^{kf_0}, \varpi^{\lambda_j}, \varpi^{-k}) \in \mathcal{G}$ where k in these expressions denotes $k_1 + k_2$. Denote by J the disjoint union of $J_{i,v}$ for all possible i, v as above. By Proposition 9.2.2, Proposition 9.4.5 and first part Lemma 3.2.7 (i), we see that

$$\mathfrak{H}_c^t = \sum_{j \in J} c_j \operatorname{ch}(U\sigma_j\mathcal{K})$$

where $c_j \in \mathbb{Z}_{(q)}$ for $j = (k_1, k_2) \in J_{i,k_2}$ is given by $(-1)^k q^{-k(2m-k+c)/2}$, $k = k_1 + k_2$. In particular, $c_j \equiv (-1)^k \pmod{q-1}$. By Corollary 9.4.11 and Lemma 3.2.7 (ii) ,

$$\begin{aligned}\sum_{j \in J_{i,k_2}} c_j \deg [U\sigma_j\mathcal{K}]_* &\equiv \sum_{k_1=i}^{m} (-1)^{k_1+k_2} \binom{m-i}{m-k_1}\binom{m-i}{k_2} \pmod{q-1} \\ &= (-1)^{k_2}\binom{m-i}{k_2} \cdot (-1)^i (1-1)^{m-i} = 0\end{aligned}$$

for all i, k_2 such that $i < m$. Since $d_m = 1$, the criteria of Corollary 3.2.2 is satisfied for all J_{i,k_2}. The claim follows. □

Remark 9.5.2. For $m = 2$, the coefficients $\sum_{j \in J_{i,k_2}} c_j \deg [U\sigma\mathcal{K}]_*$ as follows

- $1 - q^{-\frac{c+3}{2}}(q+1) + q^{-(c+2)}$ for $g_{0,0}$,
- $q^{-(c+2)} - q^{-\frac{3c+1}{2}}$ for $g_{1,0}$,
- $q^{-(c+2)}$ for $g_{2,0}$
- $(q+1)\left(q^{-(c+2)}(q+1) - q^{-\frac{3}{2}(c+1)} - q^{\frac{-(c+3)}{2}}\right)$ for $g_{0,1}$,
- $q^{-(c+2)} - q^{-\frac{3}{2}(c+1)}(q+1) + q^{-2c}$ for $g_{1,2}$,
- $q^{-(c+2)} - q^{-\frac{3}{2}(c+1)}$ for $g_{1,1}$.

When $c = 1$, the twists $g_{0,1}, g_{1,2}, g_{1,1}$ do not contribute to the zeta element, since their corresponding coefficients all vanish. A somewhat tedious induction argument shows that for $c = 1$, the only non-trivial twists of the zeta element are $g_{i,0}$. We will however not need this.

Remark 9.5.3. The normalization at $c = 1$ is relevant for the setting [GS21, §7] (corresponding to the L-value at $s = \frac{1}{2}$), and the twists/coefficients we obtain match those of [GS21, Theorem 7.1]. More precisely, the coefficient denoted 'b_i' in Theorem 7.1 (b) of *loc.cit* of the twist g_i is the coefficient computed in the proof above multiplied with $\frac{q}{q-1}\mu_H(U)/\mu_H(V_i)$ (after replacing ℓ in *loc.cit* with q). Note also that what we denote by V_i here is denoted '$V_{1,i}$' in *loc.cit*. One of the chief advantages of the approach here is that one does not need to compute the measures $\mu_H(V_i)$ in Definition 3.1.4 for which there does not seem to be a general recipe.

Remark 9.5.4. Notice that twist $g_{m,0}$ in the decomposition Prosposition 9.4.5 only arises from a single Hecke operator $K\varrho^m K$. One can show using Lemma 3.1.7(c) that a zeta element exists only if the degree d_m is 1. In Theorem 9.5.1, this was guaranteed by the choice of ν and T. If say, ν is replaced by the product of determinants of H_1, H_2, a zeta element fails to exists. One therefore one can only hope to make 'anticyclotomic' zeta elements in this setting.

Chapter 10

Exterior square L-factor of $\mathbb{G}_m \times \mathrm{GL}_4$

The standard L-factor of $\mathbb{G}_m \times \mathrm{GL}_n$ enjoys the special property its Hecke operators can be written in terms of length zero elements. One may wonder if this property played a role in the existence of a zeta element, especially in the divisibility of various sums of degrees of mixed Hecke operators. In this chapter, we study the exterior square L-factor of $\mathbb{G}_m \times \mathrm{GL}_4$ which does not enjoy these properties but still retains the features encountered in the proof of Theorem 9.5.1. See §8.1.2 for a closely related example that motivates the one studied here. The source functor is the trivial one.

Notation. The symbols F, $\mathscr{O}_F$, ϖ, $\mathscr{k}$, q and $[\mathscr{k}]$ have the same meaning as in Notation 4.0.1. The letter $\mathbf{G}$ will denote the group scheme $\mathbb{G}_m \times \mathrm{GL}_4$ over $\mathscr{O}_F$. We will denote $G := \mathbf{G}(F)$ and $K := \mathbf{G}(\mathscr{O}_F)$. For a ring R, we let $\mathcal{H}_R = \mathcal{H}_R(K\backslash G/K)$ denote the Hecke algebra of G of level K with coefficients in R with respect to a Haar measure μ_G such that $\mu_G(K) = 1$. For simplicity, we sometimes denote the characteristic function $\mathrm{ch}(K\sigma K) \in \mathcal{H}_R$ by $(K\sigma K)$. We will also freely use the notations introduced in §9.1 and the first paragraph of §9.2 for the case $n = 4$.

10.1 The Hecke polynomial

We are interested in the representation of the Langlands dual whose highest coweight is $\mu_{\mathrm{ext}} = f_0 + f_1 + f_2$. As it is minuscule and Φ is irreducible, the coweight of this representation are the Weyl orbits of the highest weight by Corollary 4.3.8. These are $f_0 + f_i + f_j$, $1 \leq i < j \leq 4$. The Satake polynomial (see Definition 4.3.2) is therefore

$$\mathfrak{S}_{\mathrm{ext}}(X) := (1 - x_0x_1x_2X)\,(1 - x_0x_1x_3X)\,(1 - x_0x_1x_4X)\,(1 - x_0x_2x_4X)\,(1 - x_0x_3x_4X)$$

viewed as an element of $\mathcal{R}[\Lambda]^W$. Let $\mathscr{S} : \mathcal{H}_{\mathcal{R}} \to \mathcal{R}[\Lambda]^W$ denote the Satake isomorphism.

Definition 10.1.1. The polynomial $\mathfrak{H}_{\mathrm{ext},c}(X) \in \mathcal{H}_{\mathcal{R}}[X]$ is defined so that $\mathscr{S}\left(\mathfrak{H}_{\mathrm{ext},c}(X)\right) = \mathfrak{S}_{\mathrm{ext}}\left(q^{-c}X\right)$ for any $c \in \mathbb{Z}$.

Proposition 10.1.2. *Let $\varrho = \varpi^{f_0}\rho^2 \in N_G(A)$. Then for $c \in \mathbb{Z}$,*

$$\begin{aligned}\mathfrak{H}_{\mathrm{ext},c}(X) = 1 &- q^{-(2+c)}(K\varrho K)X \\ &+ q^{-(3+2c)}\big((Kw_0\varrho^2K) + (q^2+q+1)(K\varrho^2K)\big)X^2 \\ &- q^{-(3+3c)}\big((Kw_0w_1w_2\varrho^3K) + (Kw_0w_3w_2\varrho^3K) + 2(K\varrho^3K)\big)X^3 \\ &+ q^{-(3+4c)}\big((Kw_0\varrho^4K) + (q^2+q+1)(K\varrho^4K)\big)X^4 \\ &+ q^{-(2+5c)}(K\varrho^5K)X^5 + q^{-6c}(K\varrho^6K)X^6 \in \mathcal{H}_{\mathbb{Z}_{(q)}}[X]\end{aligned}$$

Proof. We have

$$\begin{aligned}\mathfrak{S}_{\mathrm{ext}}(X) = 1 &- e^{W(1,1,0,0)}X + \left(e^{W(2,1,1,0)} + 3e^{W(1,1,1,1)}\right)X^2 \\ &+ \left(e^{W(3,1,1,1)} + e^{W(2,2,2,0)} + 2e^{W(2,2,1,1)}\right)X^3 \\ &+ \left(e^{W(3,2,2,1)} + 3e^{W(2,2,2,2)}\right)X^4 + e^{W(3,3,2,2)}X^5 + e^{(3,3,3,3)}X^6\end{aligned}$$

The coefficients of X^k in $\mathfrak{H}_{\mathrm{ext},c}(X)$ are therefore supported on $K\varpi^\lambda K$ for those λ for which $e^{W\lambda}$ appears in the expansion above. To write $K\varpi^\lambda K = KwK$ with w of minimal possible length in $Wt(\lambda)W$ for the various dominant λ, one first computes the length using (9.1.2) and the corresponding words found by trial and error. For instance, the minimal length of elements from $Wt(\lambda)W$ for $\lambda = (3,1,1,1)$ is three and the determinant of ϖ^λ is ϖ^6. This means the power of ρ is 6 which is last letter of the word w. Since ρ^6 conjugates of w_0, w_1, w_3 lie in K, the penultimate letter of w is w_2 and similarly the first letter of w needs to be w_0. From this one easily finds $K\varpi^\lambda K = Kw_0w_1w_2\rho^6K$. The coefficients of the leading terms (see 4.2.6) are $q^{\langle\lambda,\delta\rangle - c}$ by Corollary 4.2.4.

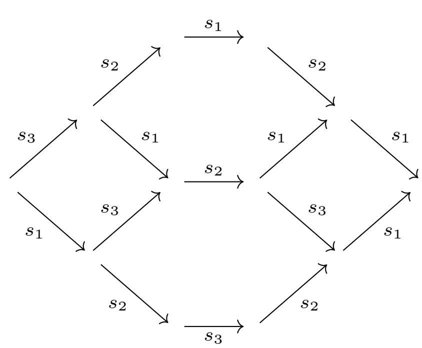


Figure 10.1: Weyl orbit diagram of $(2,1,1,0)$

To compute the non-leading terms e.g. for $K\rho^4K$ in X^2 term, we proceed as follows. Let $\lambda = (2,1,1,0)$, so that $\lambda^{\mathrm{opp}} = (0,1,1,2)$. We see from the Weyl orbit diagram above that the total number of left cosets in

$Kw_0\rho^4K/K$ is

$$q\left(q^5+2q^4+3q^3+3q^3+2q+1\right)$$

while the total number of left cosets which are leading (i.e. of shape a permutation of $\lambda=(2,1,1,0)$) are $\sum_{\mu\in W\lambda}q^{\langle\lambda+\mu,\delta\rangle}=q^6+2q^5+3q^4+3q^2+2q+1$. Thus the coefficient of $K\rho^4K$ is q^{-2c} times $3-q^{-3}\left(3q^3-q^2-q-1\right)$ which equals $q^{-3}(q^2+q+1)$. The other coefficients are obtained in a similar fashion.

□

Remark 10.1.3. Let $\lambda=(2,1,1,0)$. The Weyl dimension formula applied to $\hat{\mathbf{G}}\simeq\mathrm{GL}_4$ implies that the dimension of the representation of highest coweight λ is 15 and therefore coincides with the exterior square of the exterior square representation of GL_4. Thus, the coefficient of $K\rho^4K$ in the coefficient of X^2 is the KL-polynomial $P_{w_\lambda,w_\mu}(q)$ where $\mu=(1,1,1,1)$ and $w_\lambda=w_\circ t(\lambda)$, $w_\mu=w_\circ t(\mu)$ (see §4.4). Now $w_\circ=w_3w_2w_1w_3w_2w_3$, $t(\lambda)=w_0w_1w_2w_3w_2w_1$ and $t(\mu)$ can be replaced with 1 as it is central. The following Sage Math code

```
sage: W = WeylGroup("A3~",prefix="s")
sage: [w0,w1,w2,w3] = W.simple_reflections()
sage: R.<q> = LaurentPolynomialRing(QQ)
sage: KL = KazhdanLusztigPolynomial(W,q)
sage: KL.P(w3*w2*w1*w3*w2*w3,w3*w2*w1*w3*w2*w3*w0*w1*w2*w3*w2*w1)
```

returns $1+q+q^2$ which agrees with our computation in Proposition 10.1.2.

10.1.1 Index calculations for GL_2

Let $x_+,x_-:\mathbb{G}_a\to\mathrm{GL}_2$ denote the standard root group maps

$$x_+:u\mapsto\begin{pmatrix}1&u\\&1\end{pmatrix},\quad x_-:u\mapsto\begin{pmatrix}1&\\u&1\end{pmatrix}\tag{10.1.4}$$

Denote $X:=\mathrm{GL}_2(F)$. For integers $a,b\in\mathbb{Z}$ such that $a+b\geq 0$, let $X_{a,b}\subset X$ be the subgroup generated by $\mathrm{diag}\left(\mathscr{O}_F^\times\times\mathscr{O}_F^\times\right)$ $x_+\left(\varpi^a\mathscr{O}_F\right)$ and $x_-\left(\varpi^b\mathscr{O}_F\right)$. Then $X_{a,b}$ are compact open subgroups of X satisfying the following properties:

- $X_{a,b}\cap x_\pm(F)=x_\pm\left(\varpi^a\mathscr{O}_F\right)$,
- $X_{a,b}\cap X_{a',b'}=X_{\max(a,a'),\max(b,b')}$,
- $t_zX_{a,b}t_z^{-1}=X_{a+z,b-z}$ where $t_z=\mathrm{diag}\left(\varpi^z,1\right)$, $z\in\mathbb{Z}$.

For $a\in\mathbb{Z},\epsilon\in\mathbb{Z}_{\geq0}$, set $U_{a,\epsilon}:=X_{a,\epsilon-a}$, $V_{a,\epsilon}:=X_{\max(a,0),\max(\epsilon-a,0)}$. Then $U_{a,\epsilon}\supset V_{a,\epsilon}$ has finite index. Let $P_{a,\epsilon}(q):=[U_{a,\epsilon}:V_{a,\epsilon}]$.

Lemma 10.1.5. *We have*

$$P_{a,\epsilon}(q) = \begin{cases} q^{|a|-1}(q+1) & \textit{if} \quad 1 \le |a|, \epsilon = 0 \\ 1 & \textit{if} \quad 0 \le a \le \epsilon \\ q^{-a} & \textit{if} \quad a < 0 < \epsilon \\ q^{a-\epsilon} & \textit{if} \quad 0 < \epsilon < a \end{cases}$$

Proof. Say $a < 0$. Then $\epsilon - a > 0$, whence $V_{a,\epsilon} = X_{0,\epsilon-a}$ so that $P_{a,\epsilon}(q) = [X_{a,\epsilon-a} : X_{0,\epsilon-a}]$ and $[X_{a,\epsilon-a} : X_{0,\epsilon-c}] = [X_{\epsilon,0} : X_{\epsilon-a,0}]$ by conjugating with $t_{\epsilon-a}$. Now note that

$$[X_{0,0} : X_{\epsilon-a,0}] = [X_{0,0} : X_{\epsilon,0}] \cdot [X_{\epsilon,0} : X_{\epsilon-a,0}]$$

Since $X_{0,0} = \mathrm{GL}_2(\mathscr{O}_F)$ and $X_{b,0} = X_{0,0} \cap t_b X_{0,0} t_b^{-1}$ for $b \ge 0$, we easily compute using the recipe of §5 that $[X_{0,0} : X_{\epsilon-a,0}] = q^{\epsilon-a-1}(q+1)$, $[X_{0,0} : X_{\epsilon,0}] = 1$ or $q^{\epsilon-1}(q+1)$ depending on whether $\epsilon = 0$ or $\epsilon > 0$ respectively. Taking there ratio establishes the third case and part of the first case. The other cases are established in a similar fashion. □

10.2 Mixed coset structures

Let $\mathbf{H} \hookrightarrow \mathbf{G}$ be the algebraic subgroup generated by root groups of $\pm\alpha_1, \pm\alpha_3$ and the torus $\mathbf{A}$. Then $\mathbf{H} \cong \mathbb{G}_m \times \mathrm{GL}_2 \times \mathrm{GL}_2$ and is defined over $\mathscr{O}_F$. Let $H := H(F)$, $U := \mathbf{H}(\mathscr{O}_F)$ and $W_H := \langle s_1, s_3 \rangle \cong S_2 \times S_2$ be the Weyl group of H. Let $\tau_0 = 1_G$ and

- $\tau_1 = \begin{pmatrix} 1 & & \frac{1}{\varpi} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 1 & & \frac{1}{\varpi^2} & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & & \frac{1}{\varpi} & \\ & 1 & & \frac{1}{\varpi} \\ & & 1 & \\ & & & 1 \end{pmatrix}$

considered as elements of G by inserting 1 in the $\mathbb{G}_m$ component. Set $H_{\tau_\epsilon} = H \cap \tau_\epsilon K \tau_\epsilon^{-1}$ for $\epsilon = 0, \ldots, 3$. For $\tau \in G$, let $U\varpi^\Lambda \tau K$ denote the collection of double cosets $U\varpi^\lambda \tau K$ for $\lambda \in \Lambda$.

Lemma 10.2.1. *Let* $\lambda = (a_1, a_2, a_3, a_4)$.

(a) $\deg[U\varpi^\lambda \tau_\epsilon K]_* = P_{a_1-a_2,\epsilon}(q) P_{a_4-a_3,\epsilon}(q)$ *for* $\epsilon = 0, 1, 2$.

(b) $\deg[U\varpi^\lambda \tau_3] = 1$ *if* $a_1 = a_2, a_3 = a_4$,

(c) $\deg[U\varpi^{(1,2,0,1)} \tau_3 K]_* = q+1$.

Proof. (a) For $\epsilon = 0, 1, 2$, $f_\epsilon, g_\epsilon : \{\pm\alpha_1, \pm\alpha_3\} \to \mathbb{Z}$ be the function given by

$$f_\epsilon(\alpha) = \begin{cases} a_1 - a_2 & \text{if } \alpha = \alpha_1 \\ a_2 - a_1 + \epsilon & \text{if } \alpha = -\alpha_1 \\ a_3 - a_4 + \epsilon & \text{if } \alpha = \alpha_3 \\ a_4 - a_3 & \text{if } \alpha = -\alpha_3 \end{cases} \quad g_\epsilon(\alpha) = \begin{cases} \max\{a_1 - a_2, 0\} & \text{if } \alpha = \alpha_1 \\ \max\{a_2 - a_1 + \epsilon, 0\} & \text{if } \alpha = -\alpha_1 \\ \max\{a_3 - a_4 + \epsilon, 0\} & \text{if } \alpha = \alpha_3 \\ \max\{a_4 - a_3, 0\} & \text{if } \alpha = -\alpha_3 \end{cases}.$$

Let A_ϵ° be the subgroup of elements $(u_i)_{i=0,\dots,5} \in A^\circ$ such that $u_1 - u_2 \in \varpi^\epsilon \mathscr{O}_F$ and let $U_{f_\epsilon}, U_{g_\epsilon}$ be the subgroups of H generated by root groups

$$U_{\alpha, f_\epsilon(\alpha)} := \operatorname{im} x_\alpha \left(\varpi^{f_\epsilon(\alpha)}\right), \quad U_{\alpha, g_\epsilon(\alpha)} := \operatorname{im} x_\alpha \left(\varpi^{g_\epsilon(\alpha)}\right)$$

where x_β denotes the standard root-group maps as in (10.1.4) for $\beta \in \{\pm\alpha_1, \pm\alpha_3\}$ in each copy of GL_2 in H. Then $H \cap \left(\varpi^\lambda \tau_\epsilon\right) K \left(\varpi^\lambda \tau_\epsilon\right)^{-1}$ (resp. $U \cap \left(\varpi^\lambda \tau_\epsilon\right) K \left(\varpi^\lambda \tau_\epsilon\right)^{-1}$) is the group $A_\epsilon^\circ U_{f_\epsilon}$ (resp. $A_\epsilon^\circ U_{g_\epsilon}$). Therefore, $\deg\,[U\varpi^\lambda \tau_\epsilon K]_* = [A^\circ U_{f_\epsilon} : A^\circ U_{g_\epsilon}]$. The task is therefore reduced to computing indices of compact open subgroups in each copy of $\mathrm{GL}_2(F)$ of H which we did in Lemma 10.1.5.

(b) & (c) Let H_{ψ_2}, $j_2 : \mathscr{X}_2^\circ \to G$ be as in §9.4.2. so that $H_{\tau_3} = H_{\psi_2}\mathscr{X}_2^\circ = \mathscr{X}_2^\circ H_{\psi_2}$. Then (b) follows since ϖ^λ commutes with both H_{ψ_2} and $\mathscr{X}_2^\circ$ for $a_1 = a_2$, $a_3 = a_4$. For similar reasons, $\deg\,[U\varpi^{(1,2,0,1)}\tau_3 K]_* = \deg\,[U\varpi^{(0,1,0,1)}\tau_3 K]_*$, which is then equal to $\deg[H_{\tau_3}\varpi^{-(1,0,1,0)}K]$ by the second part of Lemma 5.5.2. Now $H_{\tau_3}\varpi^{-(0,1,0,1)}K = \mathscr{X}_2^\circ \eta U$ where $\eta := j_2\big(\mathrm{diag}(1, \varpi^{-1})\big)$, since $\eta^{-1} H_{\psi_2} \eta$ is contained in U. Now Lemma 5.5.2 again implies that $\deg\,[\mathscr{X}_2^\circ \eta U]$ is equal to cardinality of $\mathscr{X}_2^\circ j_2^{-1}(\eta)\mathscr{X}_2^\circ / \mathscr{X}_2^\circ$ which is $q+1$. ☐

Lemma 10.2.2. *$U\varpi^\Lambda \tau_i K$ are pairwise disjoint collections for $i = 0, 1, 2, 3$,*

Proof. By Lemma 9.4.4, $H\tau_i K$ are distinct for $i = 0, 1, 3$. One easily sees that $H_{\tau_2} \subsetneq H_{\tau_1} \subsetneq U$, Lemma 5.5.1 differentiates that $H\tau_2 K$ from U, $H\tau_1 K$. Since the map $H \to F^\times$, $(h_1, h_2) \mapsto \det(h_1) / \det h_2$ sends H_{τ_2} (resp. H_{τ_3}) to $1 + \pi\mathscr{O}_F$ (resp. $\mathscr{O}_F^\times$), H_{τ_2}, H_{τ_3} cannot be H-conjugate and Lemma 5.5.1 distinguishes $H\tau_2 K$ and $H\tau_3 K$. ☐

Let $\Lambda \to U\varpi^\Lambda K, U\varpi^\Lambda \tau_1 K$ denote the maps $\lambda \mapsto U\varpi^\lambda K$, $U\varpi^\lambda \tau_1 K$ respectively. Since U and K contain w_1, w_2 the first map factors to give $W_H \backslash \Lambda \to U\varpi^\Lambda K$.

Lemma 10.2.3. *The maps $W_H \backslash \Lambda \to U\varpi^\Lambda K$ and $\Lambda \to U\varpi^\Lambda \tau_1 K$ are bijections.*

Proof. Both maps are surjective by definition. That the first map is a bijection follows by Lemma 5.5.2 and Cartan decomosition for H. For the second, we note that H_{τ_3} is contained in the Iwahori subgroup $I_{H,2} \subset H$ defined in 9.4.8. Now $I_{H,2} \backslash H / U = W_I / W_H \simeq \Lambda$ which implies that the second map is injective. ☐

Proposition 10.2.4. *Let*

- $\mathfrak{T}(\rho^2) = \left\{\varpi^{(0,0,1,1)}, \varpi^{(1,0,0,1)}, \varpi^{(1,0,0,1)}\tau_1, \varpi^{(1,1,0,0)}, \varpi^{(1,1,0,0)}\tau_1, \varpi^{(1,1,0,0)}\tau_3\right\}$
- $\mathfrak{T}(w_0 w_1 w_2 \rho^6) = \left\{\varpi^{(1,1,1,3)}, \varpi^{(2,1,2,1)}\tau_1, \varpi^{(3,1,1,1)}, \varpi^{(3,1,1,1)}\tau_1, \varpi^{(3,1,1,1)}\tau_2\right\}$
- $\mathfrak{T}(w_0 w_3 w_2 \rho^6) = \left\{\varpi^{(0,2,2,2)}, \varpi^{(1,2,1,2)}\tau_1, \varpi^{(2,2,0,2)}, \varpi^{(2,2,0,2)}\tau_1, \varpi^{(2,2,0,2)}\tau_2\right\}$

- $\mathfrak{T}(w_0\rho^4) = \left\{ \begin{array}{l} \varpi^{(0,1,1,2)},\ \varpi^{(1,1,0,2)},\ \varpi^{(1,1,1,1)}\tau_1,\ \varpi^{(1,1,0,2)}\tau_1,\ \varpi^{(2,1,0,1)}\tau_1,\ \varpi^{(2,0,1,1)}\tau_1, \\ \varpi^{(2,1,0,1)},\ \varpi^{(2,0,1,1)},\ \varpi^{(2,1,0,1)}\tau_2,\ \varpi^{(1,2,0,1)}\tau_1,\ \varpi^{(1,2,0,1)}\tau_3,\ \varpi^{(2,1,1,0)}\tau_1 \end{array} \right\}$

Then for $w \in \{\rho^2, w_0\rho^4, w_0w_1w_2\rho^2, w_0w_3w_2\rho^2\}$, $\mathrm{ch}(KwK) = \sum_{\sigma\in\mathfrak{T}(w)} \mathrm{ch}(U\sigma K)$.

Proof. The case $w = \rho^2$ was handled in 9.4.5. For each of the remaining w, let λ_w denote the unique anti-dominant cocharacter such that $wK = \varpi^{\lambda_w}K$. In each case, we will draw a Weyl orbit diagram (see §5.4 for terminology) originating in λ_w. For $w = w_0\rho^4$, this was drawn in 10.1. In these diagrams, we pick the first vertex and the vertices that only have one incoming arrow labelled s_2. Corresponding to each such node, there is a 'Schubert cell' (i.e. the image of the map 9.1.3) and we need to study the left action of U on the representatives contained in the cell. The goal is to show that for all γK appearing in these Schubert cells satisfy $U\gamma K = U\tau K$ for $\tau \in \mathfrak{T}(w)$ By Proposition 5.2.8, the representatives would give a decomposition of KwK. That the claimed representatives in each case are distinct cosets follows from Lemma 10.2.2 and 10.2.3.

- $w = w_0w_1w_2\rho^2$. We have $\lambda_w = (1, 1, 1, 3)$. The Weyl orbit diagram is as follows

$$\xrightarrow{s_3} \quad \xrightarrow{s_2} \quad \xrightarrow{s_1}$$

and we need to analyze the cells correspond to first and third nodes (which have lengths 3 and 5 respectively). We record the analysis of the cell corresponding to third nodes, which is labelled by $\sigma = w_2w_3w_0w_1w_2$ and leave the other for the reader. We have

$$\mathrm{im}\,(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi & & & \\ \varpi^2 x & \varpi^3 & a\varpi^2 + b_1\varpi & b\varpi^2 + c\varpi \\ & & \varpi & \\ & & & \varpi \end{pmatrix} K \;\middle|\; x, a, b, b_1, c \in [\Bbbk] \right\}.$$

We can eliminate the $\varpi^2 x$ term by a row operation. The third and fourth row and column operations and conjugation by elements of A° allow us to compute the greatest common divisor of the entries $a\varpi^2 + b_1\varpi$ and $b\varpi^2 + c\varpi$. This greatest common divisor could be 0, ϖ or ϖ^2. Conjugating by w_3 if necessary, we may assume that this entry is in second row and third column. Conjugating by w_1, we obtain three of the desired representatives.

- $w = w_0w_3w_2\rho^2$. This is entirely similar to the previous case, so we skip it.

- $w = w_0\rho^2$. This is the most involved case. From the diagram (10.1), we see that there are four nodes to be analyzed. One of these have length 1, two have length 3 and one has length 4. We record the analysis of

$\sigma = w_2 w_1 w_0 \rho^2$ and $\sigma' = w_2 w_1 w_3 w_0 \rho^2$. We have

$$\mathrm{im}(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi & & a & \\ & \varpi & b & \\ & & 1 & \\ & & \varpi x & \varpi^2 \end{pmatrix} K \,\middle|\, x, a, b \in [\mathscr{k}] \right\}.$$

We can eliminate ϖx entry using a row operation. Since the top two diagonal entries are equal, we can eliminate a or b using row and column operations if one of them is not zero e.g. if $b \neq 0$, we can add b^{-1} times the second row to the first and then b^{-1} times the first column to the second. Conjugating by w_1 and an element of A° if necessary, we can make the $(1,3)$ entry equal to 1 and we obtain $\varpi^{(1,1,0,2)}\tau_1$ as a representative. If however both $a = b = 0$, then we obtain $\varpi^{(1,1,2,1)}$ as representatives. In summary, the orbits of U acting on $\mathrm{im}(\mathcal{X}_\sigma)$ contain exactly one of

$$\varpi^{(1,1,2,1)}, \quad \varpi^{(1,1,0,2)}.$$

For σ', we have

$$\mathrm{im}(\mathcal{X}_{\sigma'}) = \left\{ \begin{pmatrix} \varpi & & a & \\ & \varpi^2 & b + \varpi x & c\varpi \\ & & 1 & \\ & & & \varpi \end{pmatrix} K \,\middle|\, x, a, b, b_1, c \in [\mathscr{k}] \right\}.$$

If $b \neq 0$, then we can eliminate a and $c\varpi$ using row and column operations while keeping the first and fourth 2×2 blocks diagonal e.g. we can $b^{-1}a$ times the second row to the first, add $b^{-1}a\varpi$ (resp. $b^{-1}a\varpi$) times the first column to the second (resp. third) column to eliminate a. We can conjugate by w_1 to make the diagonal $\varpi^{(2,1,0,1)}$ and since $b + \varpi x$ is a unit, we can noramlize to obtain $\varpi^{(2,1,0,1)}\tau_1$. If $b = 0$ and $a \neq 0$, we can eliminate $b + \varpi x = \varpi x$. Depending on wether c is zero or not, we obtain $\varpi^{(1,2,0,1)}\tau_1$ and $\varpi^{(1,2,0,1)}\tau_3$ as representatives. Next, if $a = b = 0$ and $c \neq 0$ gives $\varpi^{(2,1,1,0)}\tau_1$ and $a = b = c = 0$, $x \neq 0$ gives $\varpi^{(2,1,0,1)}\tau_2$. Finally, $a = b = c = x = 0$ gives $\varpi^{(2,1,1,0)}$. In summary, the orbits U acting on $\mathrm{im}(\mathcal{X}_{\sigma'})$ fall contain exactly one of

$$\varpi^{(2,1,0,1)}\tau_1, \quad \varpi^{(1,2,0,1)}\tau_1, \quad \varpi^{(1,2,0,1)}\tau_3, \quad \varpi^{(2,1,1,0)}\tau_1 \quad \varpi^{(2,1,0,1)}\tau_2, \quad \varpi^{(2,1,1,0)}$$

All of these are in $\mathfrak{T}(w_0\rho^2)$. ☐

10.3 Zeta elements

We now pose a zeta element problem inspired by the situation of 8.1.2 and use the work done so far to show the existence of such an element. Let $T = F^\times$, $C = \mathscr{O}_F^\times$ the unique maximal compact subgroup, $D = 1 + \varpi\mathscr{O}_F$ a subgroup of index $q - 1$ and $\nu : H \to T$ be the map given by $(z, h_1, h_2) \mapsto z$. Let $\mathcal{O}$ be any integral domain containing $\mathbb{Z}[q, q^{-1}]$. Set

- $\mathcal{G} = G \times T$ the target group,
- $\iota_\nu = \iota \times \nu : H \to \mathcal{G}$,
- $M_{H,\mathcal{O}} = M_{H,\mathcal{O},\mathrm{triv}}$
- U (resp. $\mathcal{K} = K \times C$) as source (resp. target) bottom level,
- $\mathcal{L} = K \times D$ the compactum of field extension of degree $q-1$,
- $x_U = 1_{\mathcal{O}} \in M_{H,\mathcal{O}}(U)$ source bottom class,
- $\mathfrak{H}_c = \mathfrak{H}^t_{\mathrm{ext},c}(\mathrm{Frob})$ Hecke polynomial where $\mathrm{Frob} = \mathrm{ch}(\varpi^{-2}C)$ denotes the 'arithmetic Frobenius'.

Theorem 10.3.1. *There exists a zeta element for $(x_U, \mathfrak{H}_c, \mathcal{L})$ for all $c \in \mathbb{Z}$.*

Proof. We claim that the $g_i = (\tau_i, 1) \in \mathcal{G}$ for $i = 0,1,2,3$ span a zeta element. Since we are working the trivial functor, we show the existence of a zeta element by verifying the criteria of Corollary 3.2.2. Let λ_0 denote the central cocharacter $f_1 + \ldots + f_4$. For $P \subset \Lambda$ and $\mu \in \Lambda$, we let $P(\mu)$ denote $\{\lambda + \mu \,|\, \mu \in P\}$. Set

- $J_{0,0} := \{(0,0,0,0)\}$
- $J_{0,1} := \{(0,0,1,1), (1,0,0,1), (1,1,0,0)\}$
- $J_{0,2} := \{(0,1,1,2), (1,1,0,2), (2,1,0,1), (2,0,1,1), (1,1,1,1)\}$
- $J_{0,3} := \{(1,1,1,3), (3,1,1,1), (0,2,2,2), (2,2,0,2)\} \sqcup J_{0,1}(\lambda_0)$

and for $k = 1,2,3$, $J_{0,3+k} := J_{0,3-k}(k\lambda_0)$. Next let

- $J_{1,1} := \{(1,0,0,1), (1,1,0,0)\}$,
- $J_{1,2} = \{(1,1,0,1), (1,1,0,2), (2,1,0,1), (2,0,1,1), (1,2,0,1), (2,1,1,0)\}$,
- $J_{1,3} := \{(2,1,2,1), (3,1,1,1), (1,2,1,2), (2,2,0,2)\} \sqcup J_{1,1}(\lambda_0)$,

for $k = 1,2$, let $J_{1,3+k} := J_{0,3-k}(k\lambda_0)$. Next, we let

- $J_{2,2} = \{(2,1,0,1),\}$, $J_{2,3} = \{(3,1,1,1), (2,2,0,2)\}$,
- $J_{2,4} = J_{2,2}(\lambda_0)$.

Finally, we let

- $J_{3,1} := \{(1,1,0,0)\}$, $J_{3,2} := \{(1,2,0,1), (1,1,0,0)\}$,
- $J_{3,4} = J_{3,2}(\lambda_0)$, $J_{3,5} = J_{3,1}(2\lambda_0)$.

We define

$$J_0 = \bigsqcup_{k=0}^{6} J_{0,k}, \quad J_1 = \bigsqcup_{k=1}^{5} J_{1,k}, \quad J_2 := \bigsqcup_{k=2}^{4} \qquad J_3 = \bigsqcup_{\substack{1\le k\le 5\\ k\neq 3}} J_{3,k}$$

and J denote the disjoint unition of J_i indexed by i i.e. elements of J are pairs (i,j) for $j \in J_i$, $i = 0,1,2,3$. For $j = (a_1,\dots,a_4) \in J_i$, we define $h_j = (\varpi^{-i} f_0, \varpi^{-j}) \in H$ and $\sigma_j := \iota_\nu(h_j^{-1}) g_i = (\varpi^{i f_0}, \varpi^j \tau_i, \varpi^{2i})$. By Proposition 10.1.2, Proposition 10.2.4 and Lemma 3.2.7(i) (and that ρ^2 is central), we see that

$$\mathfrak{H}^t = \sum_{j\in J} c_j \mathrm{ch}(U\sigma_j K)$$

for some $c_j \in \mathbb{Z}_{(q)}$ independent of i e.g. if $j = (2,2,1,1) \in J_{1,3} \subset J_1$ (so that $i = 1$), then $c_j = 2q^{-(3+3c)}$. By Corollary 3.2.2, it suffices to show the following

Claim. For $i = 0,\dots,3$, we have $\sum_{j\in J_i} c_j \deg[U\sigma_j\mathcal{K}]_* \equiv 0 \pmod{q-1}$.

By Lemma 3.2.7 (ii), we have $\deg\,[U\sigma_j\mathcal{K}]_* = \deg\,[U\varpi^j\tau_i K]_*$ which were computed in Lemma 10.2.1. We record these below. The degrees of $\deg[U\varpi^j K]_*$ for $j \in J_{0,k}$ are

- 1 for $k = 0,6$,
- $1, (q+1)^2, 1$ for $k = 1,5$,
- $(q+1)^2, q(q+1), q(q+1)^2$, $q(q+1)$ for the four elements in $J_{0,k}$ for $k = 2,6$,
- $q(q+1)$ for all four elements in $J_{0,3}$,

The sum of the degrees for each $k = 0,1,\dots,6$ is $1,12,6,8,6,12,1$ modulo $q-1$ respectively. Since

$$1-(6)+(12+3)-(4+4+2\cdot 6)+(12+3)-(6)+1 = 0$$

we have the divisibility for $i = 0$. For the twist $i = 1$, the degrees are always congruent to 1 modulo $q-1$, so we only need to record the coefficient c_j for $j \in J_1$ modulo $q-1$. Now c_j is congrent to

- 1 for both elements of $J_{1,k}$ for $k = 1,5$,
- 1 for each of the six elements of $J_{1,k}$ for $k = 2,4$,
- 1 (resp. 2) for four (resp. two) elements of $J_{1,k}$ for $k = 3$.

Since we have

$$0-2+(6+0)-(4+2\cdot 2)+(6+0)-2+0 = 0$$

the divisibility of degrees holds for $i = 1$. The argument for $i = 2$ is similar to $i = 1$, as all indices are again 1 modulo $(q-1)$. Finally, for $i = 3$, the degrees are

- 1 for both singletons $J_{3,1}$, $J_{3,5}$,

- 1 and $q+1$ for both $J_{3,2},\ J_{3,4}$

Since we have

$$0 - 1 + (2+0) - (0+0+2\cdot 1) + (2+0) - 1 + 0 = 0$$

we have the divisibility for $i = 3$. We have proved the claim. $\square$

Chapter 11

Base change L-factor of GU_4

In this chapter, we study the inert case of the embedding discussed in §8.1.1. The source functor will be the trivial one, parametrizing fundamental classes of the source Shimura variety. We first collect some generalities on unitary group GU_4. Let E/F be separable extension of of degree 2, $\Gamma := \mathrm{Gal}(E/F)$, $\gamma \in \Gamma$ the non-trivial element. Let

$$J = \begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix}$$

where I_2 is the 2×2 identity matrix. Then $J = \gamma(J)^t$ is Hermitian. We let $\mathbf{G} = \mathrm{GU}_4$ be the reductive group over F given whose R points for a F-algebra R are given by

$$\mathbf{G}(R) = \left\{ g \in \mathrm{GL}_4(E \otimes R) \mid \gamma({}^t g)Jg = \mu(g)J, \mu(g) \in R^\times \right\}.$$

Then $\mathbf{G}$ is the unique quasi-split unitary similitude group of split rank 3. It's derived group is a special unitary group whose Tits index (see [Tit66]) is ${}^2A^{(1)}_{3,2}$. The mapping $\mathbf{G} \to \mathbb{G}_m$, $g \mapsto \mu(g)$ is called the similitude character and we will denote it by μ_{sim}. The determinant map $\det : \mathbf{G} \to \mathrm{Res}_{E/F}\mathbb{G}_m$ then satisfies $\gamma \circ \det \cdot \det = \mu^4_{\mathrm{sim}}$.

Let R be a E-algebra. We let $\gamma_R : E \otimes R \to E \otimes R, x \otimes r \mapsto \gamma(x) \otimes r$ the map induced by γ and $i_R : E \otimes R \to R \times R$ the isomorphism $x \otimes r \mapsto (xr, \gamma(x)r)$, where $x \in E, r \in R$. We let $\pi_1, \pi_2 : E \otimes R \to R$ the projections of i_R to the first and second component respectively. We have an induced action $\gamma_R : \mathrm{GL}_4(E \otimes R) \to \mathrm{GL}_4(E \otimes R)$ and an induced isomorphism $i_R : \mathrm{GL}_4(E \otimes R) \to \mathrm{GL}_4(R) \times \mathrm{GL}_4(R), (g_{i,j}) \mapsto (\pi_1(g_{i,j}), \pi_2(g_{i,j}))$ Under the identification i_R, the group $\mathbf{G}(R) \subset \mathrm{GL}_4(E \otimes R)$ is identified with the subgroup of elements $(g, h) \in \mathrm{GL}_4(R) \times \mathrm{GL}_4(R)$ such that

$$\left({}^t h, {}^t g\right) \cdot (J, J) \cdot (g, h) = (rJ, rJ).$$

We thus have functorial isomorphisms $\psi_R : (c_R : \mathrm{pr}_1 \circ i_R) : \mathbf{G}(R) \xrightarrow{\sim} \mathbb{G}_m \times \mathrm{GL}_4(R)$ via which we identify $\mathbf{G}_E \xrightarrow{\sim} \mathbb{G}_m \times \mathrm{GL}_4$ (as group schemes over E) canonically.

Notation. The symbols $F, \mathscr{O}_F, \varpi, \Bbbk = \Bbbk_F, q = q_F$ have the same meaning as in 4. We let E/F denote an unramified quadratic extension and set $q_E = |\Bbbk|_E = q^2$ where $\Bbbk_E$ is the residue field of E. We denote by $[\Bbbk_F]$, $[\Bbbk_E]$ a fixed choice of representatives in $\mathscr{O}_F$, $\mathscr{O}_E$ of elements of $\Bbbk_F$, $\Bbbk_E$ respectively. We let $\mathbf{G}$ be the group defined above, $G = \mathbf{G}(F)$, $G_E = \mathbf{G}(E) \overset{\psi}{=} E^\times \times \mathrm{GL}_n(E)$, $K_E = \mathbf{G}_E(\mathscr{O}_F) \overset{\psi}{=} \mathscr{O}_E^\times \times \mathrm{GL}_4(\mathscr{O}_E)$, $K = K_E \cap \mathbf{G}(F)$. For a ring R, we let $\mathcal{H}_R$, $\mathcal{H}_{R,E}$ denote the Hecke algebras $\mathcal{H}_R\left(K\backslash G/K\right), \mathcal{H}_R(K_E\backslash G_E/K_E)$ over R respectively. For simplicity, we will denote the characteristic function $\mathrm{ch}(K\sigma K) \in \mathcal{H}_R$ by $(K\sigma K)$ and similarly for $\mathcal{H}_{R,E}$.

11.1 Desiderata

Let $\mathbf{A} = \mathbb{G}_m^3$, $\mathrm{dis} : \mathbf{A} \to \mathbf{G}$ be the map

$$(u_0, u_1, u_2) \mapsto \begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \frac{u_0}{u_1} & \\ & & & \frac{u_0}{u_2} \end{pmatrix}$$

which identifies $\mathbf{A}$ with the maximal split torus of $\mathbf{G}$. Let $\mathbf{M}$ be the normalizer of $\mathbf{A}$. Then $\psi : \mathbf{M}_E \xrightarrow{\sim} \mathbb{G}_{m,E}^5$ and we consider $\mathbb{G}_{m,E}^5$ as a maximal torus of $\mathbf{G}_E \overset{\psi}{=} \mathbb{G}_{m,E} \times \mathrm{GL}_{4,E}$ via $(u_0, \ldots, u_4) \mapsto (u_0, \mathrm{diag}\,(u_1, \ldots, u_4))$. We will denote $A := \mathbf{A}(F)$, $M := \mathbf{M}(F)$. We have $X^*(\mathbf{M}) = \mathbb{Z}e_0 \oplus \cdots \oplus \mathbb{Z}e_4, X_*(\mathbf{M}) = \mathbb{Z}f_0 \oplus \cdots \oplus \mathbb{Z}f_4$, where f_i, e_i are as in §9.1. The Galois action Γ on $X_*(\mathbf{M}), X^*(\mathbf{M})$, is as follows:

$$\gamma \cdot e_i = \begin{cases} e_0 & \text{if } i = 0 \\ e_0 - e_{i+2} & \text{if } i = 1, \ldots, 4 \end{cases} \qquad \gamma \cdot f_i = \begin{cases} f_0 + \cdots + f_4 & \text{if } i = 0 \\ -f_{i+2} & \text{if } i = 1, \ldots, 4 \end{cases}$$

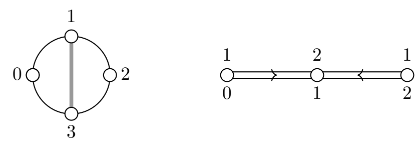
where $e_i = e_{i-4}, f_i = f_{i-4}$ if $i > 4$. For $i = 0, 1, 2$, let

- $\phi_i : \mathbb{G}_m \to \mathbf{A}$ by sending u to the j-th component,
- $\varepsilon_i : \mathbb{G}_m \to \mathbf{A}, \mathrm{dis}\,(u_0, u_1, u_2) \mapsto u_i$

Then $X^*(\mathbf{A}) = \mathbb{Z}\varepsilon_0 \oplus \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, $X_*(\mathbf{A}) = \mathbb{Z}\phi_0 \oplus \mathbb{Z}\phi_1 \oplus \mathbb{Z}\phi_2$. Let $\mathrm{res} : X^*(\mathbf{M}) \to X^*(\mathbf{A})$, $\mathrm{cores} : X_*(\mathbf{A}) \to X_*(\mathbf{M})$ be the maps obtained by restriction and inclusion respectively. Then

$$\mathrm{res}\,(e_i) = \begin{cases} \varepsilon_i & \text{if } i = 0, 1, 2 \\ \varepsilon_0 - \varepsilon_{i-2} & \text{if } i = 3, 4 \end{cases} \qquad \mathrm{cores}\,(\phi_i) = \begin{cases} f_0 + f_3 + f_4 & \text{if } j = 0 \\ f_i - f_{i+2} & \text{if } j = 1, 2 \end{cases}$$

We let Φ_E denote the set of absolute roots of $\mathbf{G}_E$ as in §5, and Φ_F denote the set of relative roots obtained as restrictions of Φ_E to $\mathbf{A}$. Then $\Phi_F = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 + \varepsilon_2 - \varepsilon_0), \pm(2\varepsilon_1 - \varepsilon_0), \pm(2\varepsilon_2 - \varepsilon_0)\}$ constitutes a root system of type C_2. The associated coroots are We let $\beta_1 = e_1 - e_2$, $\beta_2 = e_2 - e_4$, $\beta_3 = e_4 - e_3$, $\Delta_E = \{\beta_1, \beta_2, \beta_3\}$. Then Δ_E is a set of simple roots for Φ_E and with respect to this ordering, $\beta_0 = e_1 - e_3$ is the highest root. The set Δ_E and β_0 are invariant under Γ, and the labeling is chosen so that (absolute) local Dynkin diagram (with the bar showing the Galois orbits) is the diagram on the left



The set of corresponding relative simple roots is therefore $\Delta_F = \{\alpha_1, \alpha_2\}$ where $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = 2\varepsilon_2 - \varepsilon_0$. With this ordering, the highest root is $\alpha_0 = 2\varepsilon_1 - \varepsilon_0$. The associated simple coroots are $\alpha_0^\vee = \phi_1$, $\alpha_1^\vee = \phi_1 - \phi_2$, $\alpha_2^\vee = \phi_2$ and we denote by $Q^\vee$ their span in Λ. Executing the recipe provided in §1.11 of [Tit79] on the absolute diagram above, we find that the *local index* or *relative local Dynkin diagram* (see §4 of *loc.cit*) is the diagram on the right above. Here, the indices below the diagram correspond to the affine roots $-\alpha_0 + 1, \alpha_1, \alpha_2$ and the indices above the diagram are half the number of roots of a semi-simple group of relative rank 1 whose absolute Dynkin-diagram is the corresponding Galois orbit in the diagram on the left. The endpoints of the diagram on the right, and in particular the one labelled 0, are hyperspecial and hence so is the subgroup K by construction. The diagrams above can be found in the fourth row of the table on p. 62 of *op.cit.*

From now on, we denote by Λ the cocharacter lattice $X^*(\mathbf{A})$ and denote by t the translation action of Λ on $\Lambda \otimes \mathbb{R}$. An element $\lambda = a_0\phi_0 + a_1\phi_1 + a_2\phi_2 \in \Lambda$ will be denoted by (a_0, a_1, a_2) and ϖ^λ denotes the element $\lambda(\varpi) \in A$. Let s_i, $i = 0, 1, 2$ denote the simple reflections associated α_i. The action of s_i on Λ is given explicitly as follows:

- s_1 acts as a transposition $\phi_1 \leftrightarrow \phi_2$,
- s_2 acts by sending $\phi_0 \mapsto \phi_0 + \phi_2, \phi_1 \mapsto \phi_1, \phi_2 \mapsto -\phi_2$
- $s_0 = s_1 s_2 s_1$ acts by sending $\phi_0 \mapsto \phi_0 + \phi_1$, $\phi_1 \mapsto -\phi_1$, $\phi_2 \mapsto \phi_2$.

We let e^λ (resp. $e^{W\lambda}$) denote the element in the group algebra $\mathbb{Z}[\Lambda]$ corresponding to λ (resp. the formal sum of elements of $W\lambda$). Let $S_{\text{aff}} = \{s_1, s_2, t(\alpha_0^\vee)s_0\}$ and W, W_{aff} and W_I denote the Weyl, affine Weyl, Iwahori Weyl groups respectively. We consider W_{aff} as a group of affine transformations of $\Lambda \otimes \mathbb{R}$. We have

- $W \cong (\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2$,
- $W_{\text{aff}} = t(Q)^\vee \rtimes W$ the affine Weyl group

- $W_I = A/A^\circ \rtimes W \xrightarrow[\sim]{v} \Lambda \rtimes W$,

The pair $(W_{\text{aff}}, S_{\text{aff}})$ is a Coxeter system of type $\tilde{C}_2$ and we consider $W_{\text{aff}} \subset W_I$ via v. Then $W_I = W_{\text{aff}} \rtimes \Omega$. Given $\lambda \in \Lambda$, the minimal possible length of elements in $t(\lambda)W$ is obtained by a unique element. This length is given by

$$\ell_{\min}(\lambda) = \sum_{\lambda \in \Phi_\lambda^1} |\langle \lambda, \alpha \rangle| + \sum_{\alpha \in \Phi_\lambda^2} (\langle \lambda, \alpha \rangle - 1) \tag{11.1.1}$$

where $\Phi_\lambda^1 = \left\{\alpha \in \Phi_F^+ \,|\, \langle \lambda, \alpha \rangle \leq 0\right\}$, $\Phi_\lambda^2 = \left\{\alpha \in \Phi_F^+ \,|\, \langle \lambda, \alpha > 0\right\}$. When λ is dominant, the first sum is zero, and the length is then also minimal among elements of $Wt(\lambda)W$. We let

$$w_0 = \begin{pmatrix} & & \frac{1}{\varpi} & \\ & 1 & & \\ \varpi & & & \\ & & & 1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{pmatrix}, \quad \rho = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \varpi & & \\ \varpi & & & \end{pmatrix}$$

be elements of $N_G(A)$. Then the classes of w_0, w_1, w_2 represent $t\left(\alpha_0^\vee\right) s_0, s_1, s_2$ in W_I and ρ represents $t\left(-\phi_0\right) s_2 s_1 s_2$, which is a generator of $\Omega \cong \mathbb{Z}$. The conjugation action of ρ switches w_0, w_2 and keeps w_1 fixed, inducing an automorphism of the extended Coxeter diagram $\underset{0}{\circ}\overset{4}{-}\underset{1}{\bullet}\overset{4}{-}\underset{2}{\bullet}$.

Let $\xi \in \mathscr{O}_E^\times$ be an element of trace 0 i.e. $\xi + \gamma(\xi) = 0$. Let $x_1 : \operatorname{Res}_{E/F} \mathbb{G}_a \to \mathbf{G}$ and $x_i : \mathbb{G}_a \to \mathbf{G}$ for $i = 0, 2$ be the root group maps

$$x_0 : u \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ \varpi \xi u & & 1 & \\ & & & 1 \end{pmatrix}, \quad x_1 : u \mapsto \begin{pmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{u} & 1 \end{pmatrix}, \quad x_2 : u \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \xi u \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where $\bar{u} := \gamma(u)$. We let $\Bbbk_{w_i} = \Bbbk_{w_2} := \Bbbk_F$, $\Bbbk_{w_1} := \Bbbk_E$ and for $i = 0, 1, 2$, we denote by $g_{w_i} : [\Bbbk_{w_i}] \to G$ the map $u \mapsto x_i(u) w_i$. For $w \in W_I$ and $w = s_{w,1} s_{w,2} \cdots s_{w,\ell(w)} \rho_w$, where $s_{w,i} \in S_{\text{aff}}$, $\rho_w \in \Omega$, a reduced word decomposition, we let

$$\mathcal{X}_w : \prod_{i=1}^{\ell(w)} [\Bbbk_{w_i}] \to G \tag{11.1.2}$$

$$(\kappa_1, \ldots, \kappa_{\ell(w)}) \mapsto g_{s_{w,1}}(\kappa_1) \cdots g_{s_{w,\ell(w)}}(\kappa_{\ell(w)}) \rho_w$$

11.2 Base change Hecke polynomial

Let $y_i := e^{\phi_i} \in \mathbb{Z}[\Lambda]$, so that $\mathbb{Z}[\Lambda] = \mathbb{Z}[y_0^\pm, y_1^\pm, y_2^\pm]$ and let $\mathcal{R}_q := \mathbb{Z}[q^{\pm \frac{1}{2}}]$, $\mathcal{R}_{q^2} := \mathbb{Z}[q^{-1}]$. The abelian group homomorphism $1 + \gamma : X_*(\mathbf{M}) \to X_*(\mathbf{M})$ given by $f \mapsto f + \gamma \cdot f$ has image in $\Lambda = X_*(\mathbf{M})^\Gamma$ and hence induces a map $e^{1+\gamma}$ on $\mathcal{R}_{q^2}$-algebras

$$\begin{array}{ccc} \mathcal{H}_{\mathcal{R}_{q^2}}(G_E) & \xrightarrow{\mathscr{S}_E} & \mathcal{R}_{q^2}[X_*(\mathbf{M})]^{W_E} \\ \text{BC}\downarrow & & \downarrow e^{1+\gamma} \\ \mathcal{H}_{\mathcal{R}_q}(G) & \xrightarrow{\mathscr{S}_F} & \mathcal{R}_q[\Lambda]^{W_F} \end{array}$$

corresponding to which we have what is called the *base change map* BC. The Satake polynomial that we consider here is the base change of the Satake polynomial of $\mathbf{G}_E$ associated with the standard representation considered in §9.2. This polynomial is

$$\mathfrak{S}_{\mathrm{bc}}(X) = (1 - yy_1X)\left(1 - yy_1^{-1}X\right)(1 - yy_2X)\left(1 - yy_2^{-1}X\right) \in \mathbb{Z}[\Lambda]^{W_F}[X].$$

where $y = y_0^2y_1y_2$.

Remark 11.2.1. We have an embedding (of reductive groups) ${}^L\mathbf{G}_E \hookrightarrow {}^L\mathbf{G}_F$. Given an unramified L-parameter $\varphi : \mathscr{W}_F \to {}^L\mathbf{G}_F$, let $\hat{t} \rtimes \mathrm{Frob}_F^{-1} := \varphi(\mathrm{Frob}_F^{-1})$, where $\mathrm{Frob}_F \in \mathscr{W}_F$ denotes a lift of the arithmetic frobenius, we have $\varphi(\mathrm{Frob}_E^{-1}) = (\hat{t} \rtimes \mathrm{Frob}_F^{-1})^2 = \hat{t}\gamma(\hat{t}) \rtimes \mathrm{Frob}_E^{-1} \in {}^L\mathbf{G}_E$. If we think of $\hat{t}$ as the Satake parameters of an unramified representation π_F of $\mathbf{G}(F)$, then $\hat{t}\gamma(\hat{t})$ are the Satake parameters of an unramified representation π_E of $\mathbf{G}(E)$ which is called the *base change* of π_F. The base change map BC above can then also be characterized as in [Kot84, §2.2].

Definition 11.2.2. The polynomial $\mathfrak{H}_{\mathrm{bc},c}(X)$ to be the image of $\mathfrak{H}_{\mathrm{std},c}(X)$ under the map BC for c an *odd integer*. Equivalently, $\mathfrak{H}_{\mathrm{bc},c}(X)$ is the polynomial such that $\mathscr{S}_F\left(\mathfrak{H}_{\mathrm{bc},c}(X)\right) = \mathfrak{S}_{\mathrm{bc}}(q^{-c}X)$.

Proposition 11.2.3. *We have*

$$\begin{aligned}\mathfrak{H}_{\mathrm{bc,c}}(X) = (K) & \\ & - q^{-(c+3)}\left((Kw_0\rho^2K) + (q^2+1)(1-q)(K\rho^2K)\right)X \\ & + q^{-(2c+4)}\left((Kw_0w_1w_0\rho^4K) + (q^2+1)(1+q^4-q^5)(K\rho^4K)\right)X^2 \\ & - q^{-(3c+3)}\left((Kw_0\rho^6K) + (q^2+1)(1-q)(K\rho^6K)\right)X^3 \\ & + q^{4c}(K\rho^8K)X^4\end{aligned}$$

Proof. We have

$$\mathfrak{S}_{\mathrm{bc},c}(X) = 1 - e^{W(2,2,1)}X + \left(e^{W(4,3,3)} + 2e^{W(4,2,2)}\right)X^2 - e^{W(6,4,3)}X^3 + e^{(8,4,4)}X^4$$

One computes the length of the cocharacters appearing in the exapnsion above using the formula using (11.1.1) and the corresponding words are found easily. The leading coefficients (see 4.2.6 for terminology) are $q^{-\langle\lambda,\delta\rangle}$ for the various λ occuring in $\mathfrak{S}_{\mathrm{bc},c}$ as exponents. To compute the non-leading coefficients, say of $\mathrm{ch}(K\rho^2K)$ appearing in X, we proceed as follows. Note that since $(2,2,1) - (2,1,1) = \alpha_1^\vee + \alpha_2^\vee$,

$(2,2,1) \succ (2,1,1)$ and that $(2,1,1)$ is the only dominant cocharacter which $(2,2,1)$ succeeds. Let $w = w_0\rho^2$, so that $KwK = K\varpi^{(2,1,1)}K$. We see from the the Weyl orbit diagram

$$(2,0,1) \xrightarrow{s_1} (2,1,0) \xrightarrow{s_2} (2,1,2) \xrightarrow{s_1} (2,2,1)$$

that $|Kw_0\rho^2K/K| = q + q^3 + q^4 + q^6$. Of these, the number of cosets of shape a permutation of $(2,2,1)$ is $\sum_{\mu \in W(2,2,1)} q^{\langle \lambda+\mu,\delta\rangle} = 1 + q^2 + q^4 + q^6$, whence the coefficient of $K\rho^2K$ is the difference of the two times $q^{-(c+3)}$. This also gives the coefficient of $\mathrm{ch}(K\rho^6K)$. The computation of the coefficient of $\mathrm{ch}(K\rho^4K)$ in the X^2 term is done in a similar fashion. □

Remark 11.2.4. For $\lambda = (a_0, a_1, a_2)$, the pairing $\langle\lambda,\delta\rangle$ can be computed by pairing λ with $\mathrm{res}(\delta) = -2\varepsilon_0 + 3\varepsilon_1 + \varepsilon_2$ and equals $-2a_0 + 3a_1 + a_2$. Note also that

$$2 \cdot \mathrm{res}(\delta) = 2(\varepsilon_1 - \varepsilon_2) + 2(\varepsilon_1 + \varepsilon_2 - \varepsilon_0) + (2\varepsilon_1 - \varepsilon_0) + (2\varepsilon_2 - \varepsilon_0)$$

is a weighted sum of the positive roots in Δ_F, with the weights given by the degree of the splitting field of the corresponding root.

11.3 Mixed coset structures

Let $\tau_0 = 1_G$,

$$\tau_1 = \begin{pmatrix} 1 & & & -\frac{1}{\varpi} \\ & 1 & \frac{1}{\varpi} & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & & & -\frac{1}{\varpi^2} \\ & 1 & \frac{1}{\varpi^2} & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & \frac{1}{\varpi} & \frac{1}{\varpi^2} & -\frac{1}{\varpi} \\ & 1 & \frac{1}{\varpi} & \\ & & 1 & \\ & & -\frac{1}{\varpi} & 1 \end{pmatrix}$$

be elements of G. Let $\mathbf{H}$ be the subgroup of $\mathbf{G}$ generated by the maximal torus $\mathbf{M}$, and the root groups corresponding to $\pm\alpha_0$, $\pm\alpha_2$. Then $\mathbf{H} \cong \mathrm{GU}_2 \times_\mu \mathrm{GU}_2$. Here GU_2 is the reductive group over F whose R points for a F-algebra R are given by $\mathrm{GU}_2(R) = \{g \in \mathrm{GL}_2(E \otimes R) \,|\, \gamma({}^t g)J_2 g = \mu(g)J_2, \mu(g) \in R^\times\}$ where

$$J_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

and the fiber product in $\mathbf{H}$ is over the similitude character of the two copies of GU_2.

- $H = \mathbf{H}(F)$, $\iota : H \to G$ the natural embedding,
- $U = H \cap K$,
- $W_H = \langle s_0, s_2\rangle \cong S_2 \times S_2$ the Weyl group of $\mathbf{H}$,

Let U_τ, H_τ denotes the sbugroups $U \cap \tau K\tau^{-1}$, $H \cap \tau K \tau^{-1}$ for $\tau \in G$ and $U\varpi^\Lambda\tau K$ denote the collection of cosets $U\varpi^\lambda\tau K$ for $\lambda \in \Lambda$.

Lemma 11.3.1. *We have*

- $\deg\,[U\varpi^{(2,2,1)}K]_* = (q+1)$,
- $\deg\,[U\varpi^{(4,3,3)}K]_* = (q+1)^2$,

Proof. Since $\varpi^\Lambda \subset H$, we see that $\deg\,[U\varpi^\lambda K]_* = [U : U \cap \varpi^{-\lambda}U\varpi^\lambda]$ for any $\lambda \in \Lambda$. These latter indices are then computed using Proposition 5.2.8. ☐

Lemma 11.3.2. *We have*

- $U\varpi^\Lambda\tau_i K$ *and* $U\varpi^\Lambda\tau_j K$ *are disjoint if* $1 \le i < j \le 2$ *and for* $i = 0, 1,\ j = 3$ *if moreover* $q \neq 2$.
- $U\varpi^{(2,2,1)}K$ *and* $U\varpi^{(2,1,2)}K$ *are disjoint,*
- $U\varpi^{(2,2,1)}\tau_1 K$ *and* $\varpi^{(2,1,2)}\tau_1 K$ *are disjoint.*

Proof. For part (a), we show that $H\tau_i K \neq H\tau_j K$ for various pairs i, j. It suffices to show that the measure of $H \cap \tau_i K \tau_i^{-1}$ are different, since then the said groups cannot be conjugate under H as H is unimodular. By inspecting the entries of $\tau_i^{-1}H \cap \tau_i \cap K$, we deduce that $U = H \cap K \supsetneq H \cap \tau_1 K \tau_1^{-1} \subsetneq H \cap \tau_2 K \tau_2^{-1}$, and that $H \cap \tau_1 K \tau_1^{-1} \supsetneq H \cap \tau_3 K \tau_3^{-1}$ when $q \neq 2$, which proves part (a). Let $\mathrm{pr}_i : H \to \mathrm{GU}_2(F)$ be the projection maps for $i = 1, 2$. Then $\mathrm{pr}_1(U \cap \varpi^{(2,2,1)}K\varpi^{(-2,-2,-1)}) \supsetneq \mathrm{pr}_1(U \cap \varpi^{(2,1,2)}K\varpi^{(-2,-1,-2)})$, whence the projections of the two intersections cannot be conjugate under U. Similarly for part (c). ☐

From now on, we assume that $q \neq 2$.

Lemma 11.3.3. *Let*

- $\mathfrak{T}(w_0\rho^2) = \left\{\varpi^{(2,2,1)},\ \varpi^{(2,2,1)}\tau_1,\ \varpi^{(2,2,1)}\tau_3,\ \varpi^{(2,1,2)},\ \varpi^{(2,1,2)}\tau_1\right\}$
- $\mathfrak{T}(w_0 w_1 w_0 \rho^4) = \left\{\varpi^{(4,3,3)},\ \varpi^{(4,3,3)}\tau_1,\ \varpi^{(4,3,3)}\tau_2\right\}$.

Then for $w \in \left\{w_0\rho^2, w_0w_1w_0\rho^4\right\}$, $\mathrm{ch}(KwK) = \sum_{\sigma \in \mathfrak{T}(w)} \mathrm{ch}(U\sigma K)$.

Proof. Let $w = w_0\rho^2$. Then $KwK/K = \mathrm{im}(\mathcal{X}_w) \sqcup \mathrm{im}(\mathcal{X}_{w_1 w}) \sqcup \mathrm{im}(\mathcal{X}_{w_2 w_1 w}) \sqcup \mathrm{im}(\mathcal{X}_{w_1 w_2 w_1 w})$ of which we need to analyze the first, the second and the fourth. We analyze the last, leaving the other two cases for the reader. We have

$$\mathrm{im}(\mathcal{X}_{w_1 w_2 w_1 w}) = \left\{ \begin{pmatrix} \varpi^2 & a_1\,\varpi & a\,a_1 + \xi\,y + \varpi\,x\,\xi & -\varpi\,\bar{a} \\ & \varpi & a & \\ & & 1 & \\ & & -\bar{a_1} & \varpi \end{pmatrix} K \,\middle|\, a, a_1 \in [\hbar_E] \in, x, y \in [\hbar_F] \right\}$$

We can get rid of $\xi(y + \varpi x)$ in the third column using a row operation (as U contains $x_{\alpha_0}(\mathscr{O}_F)$, $x_{\alpha_0} := w_1 x_2 w_1$). If $a_1 = 0, a = 0$, then we obtain the representatives $\varpi^{(2,2,1)}$. If $a_1 = 0,\ a \neq 0$, we can conjugate

$\mathrm{diag}(\bar{a}^{-1},1,a,1) \in M^\circ$ to obtain the representative $\varpi^{(2,2,1)}\tau_1$. Finally, if $a_1 \neq 0$, we can conjugate by $\mathrm{diag}(a_1^{-1},1,\bar{a}_1,1)$ to obtain a matrix

$$\begin{pmatrix} \varpi^2 & \varpi & u & -\varpi\bar{u} \\ & \varpi & u & \\ & & 1 & \\ & & -1 & \varpi \end{pmatrix}$$

where $u = a/\bar{a}_1$. We can assume $u \in \mathscr{O}_F$ by applying row and column operations. If $u = 0$, we can conjugate by w_2 and $\mathrm{diag}(1,1,-1,-1)$ to obtain the reprsentative $\varpi^{(2,2,1)}\tau_1$, and if $u \neq 0$, then conjugating by $\mathrm{diag}(1,1,u,u)$ gives us the representative $\varpi^{(2,2,1)}\tau_3$. That these are distinct follows by Lemma 11.3.2.

Let $w = w_0w_1w_0\rho^4$. Then $KwK/K = \mathrm{im}(\mathcal{X}_w) \sqcup \mathrm{im}(\mathcal{X}_{w_2w_1}) \sqcup \mathrm{im}(\mathcal{X}_{w_1w_2w}) \sqcup \mathrm{im}(\mathcal{X}_{w_2w_1w_2})$. Of these we need to analyze the first and the third. We analyze the third, leaving the other for the reader. We have

$$\mathrm{im}(\mathcal{X}_{w_1w_2w}) = \left\{ \begin{pmatrix} \varpi^3 & a_1\,\varpi - \varpi^2\,\bar{a} & x\,\xi\,\varpi^2 + \xi\,y\,\varpi & \\ & \varpi & & \\ & & \varpi & \\ & \varpi^2\,x_1\,\xi & a\,\varpi^2 - \varpi\,\bar{a_1} & \varpi^3 \end{pmatrix} K \,\middle|\, x,y \in \mathscr{k}_F, a, a_1 \in \mathscr{k}_E \right\}$$

We can get rid of $\varpi^2x_1\xi$, $x\xi\varpi^2 + \xi y\varpi$ using row operations. Conjugating by w_2 makes the diagonal $\varpi^{(4,3,3)}$ and puts the entry $a_1\varpi - \varpi^2\bar{a}$ and its conjugate on the top right anti-diagonal. A case analysis of whether a, a_1 are zero or not gives us all the three possible entries. That these are distinct follows by Lemma 11.3.2. □

11.4 Zeta elements

Let U_1 be the F-torus whose R points over a F-algebra R are given by $\mathrm{U}_1(R) = \{z \in E^\times \mid z\gamma(z) = 1\}$. Then $\mathrm{U}_1(F) \subset \mathscr{O}_E^\times$ is compact. There is a homomorphism of F-tori given $\mathscr{N} : \mathrm{Res}_{E/F}\mathbb{G}_m \to \mathrm{U}_1$ given by $z \mapsto z/\gamma(z)$ with kernel $\mathbb{G}_m$. An application Hilbert's Theorem 90 gives us that $\mathscr{N}$ is surjective, inducing isomorphism $\mathscr{O}_E^\times / \mathscr{O}_F^\times = E^\times/F^\times \xrightarrow{\sim} \mathrm{U}_1(F)$. Now let $T = C := \mathrm{U}_1(F)$, $D = \mathscr{N}(\mathscr{O}_F^\times + \varpi\,\mathscr{O}_E)$, and let $\nu : H \to T$ be the map $(h_1, h_2) \mapsto \det h_2 / \det h_1$. For the zeta element problem, we take

- $\mathcal{G} := G \times T$ the target group,
- $\iota_\nu := \iota \times \nu : H \to \mathcal{G}$ the embedding,
- $\mathcal{O}$ an integral domain containing $\mathbb{Z}[q^{-1}]$, Φ its field of fractions,
- $M_{H,\mathcal{O}}$ the trivial functor,

- $U, \mathcal{K} := K \times C$ as bottom levels,
- $x_U = 1_{\mathcal{O}} \in M_{H,\mathcal{O}}(U)$ the source bottom class,
- $\mathcal{L} = K \times D$ as compactum of field extension of degree $d = q + 1$,
- $\mathfrak{H}_c = \mathfrak{H}^t_{\mathrm{bc},c}(\mathrm{Frob}) \in \mathcal{C}_{\mathcal{O}}(\mathcal{K}\backslash\mathcal{G}/\mathcal{K})$ Hecke polynomial where $\mathrm{Frob} = \mathrm{ch}(C)$.

Theorem 11.4.1. *There exists a zeta element for $(x_U, \mathfrak{H}_c, \mathcal{L})$ for all $c \in \mathbb{Z} - 2\mathbb{Z}$.*

Proof. We claim that $g_i = (\tau_i, 1_T) \in \mathcal{G}$ span a zeta element. Note first that one can decompose $\mathfrak{H}^t = \sum_\sigma \mathrm{ch}(U\sigma\mathcal{K})$ using Lemma 11.3.3, since $\nu(U) \subset C$ and $\mathrm{ch}(Kw_0\rho^2K)^t = \rho^{-4}\mathrm{ch}(Kw_0\rho^2K)$ etc. Since $\varpi^\Lambda \subset H$, it is easily seen that $U \otimes \mathfrak{H}^t_c$ is equal to a linear combination with twists g_i in $\mathcal{M}$.

Since we are working with $M_{H,\mathcal{O},\mathrm{triv}}$, we can apply Lemma 3.2.2. Observe that $\deg\,[U(\sigma, 1_T)\mathcal{K}]_* = \deg\,[U\sigma K]_* = \deg\,[U\sigma\rho^k K]_*$ for any $\sigma \in G$, $k \in 2\mathbb{Z}$, since $C = T$ is compact, $H_\sigma = H_{(\sigma,1_T)}$, $U_\sigma = U_{(\sigma,1_T)}$ and ρ^k is central for k even.

Next note that $d_i = 1$ for $i = 1, 2, 3$. Indeed, we have $\mathbf{H}(F) = \mathbf{M}(F) \cdot \mathbf{H}^{\mathrm{der}}(F)$ and $\mathbf{H}^{\mathrm{der}}(F) \subset \ker \nu$, and thus it suffices to show that $\nu(M \cap \tau_i K \tau_i^{-1}) \subset D$ which is straightforward to check. Thus we only need to check the divisibility for d_0. By Lemma 11.3.1, we see that all degrees are either 1 or divisible by $q + 1$. It therefore suffices to check the divisibility of

$$1 - (q^2 + 1)(1 - q) + (q^2 + 1)(1 + q^4 - q^5) - (q^2 + 1)(1 - q) + 1$$

by $q + 1$. Modulo $q + 1$, this is congruent to $1 - 4 + 6 - 4 + 1 = 0$. ☐

Chapter 12

Spin L-factor of GSp_4

In this chapter, we study the HNR problem for pushforwards of cup products of Eisenstein classes on the product of modular curves into the cohomology of Siegel three folds and show that a zeta element exists in this scenario. In particular, we strengthen [LSZ17, Corollary 3.10.5] to Hecke algebra valued norm relations that holds before passage to cohomology. We also note that since the underlying (relative) root datum of GU_4 and GSp_4 coincide, the combinatorics of their Hecke operators share many similarities and we have maintained notation to emphasize this similarity.

Remark 12.0.1. As is the case in other examples, we will encounter decompositions of various Hecke operators relevant to the degree 4 spin L-factor. The reader is invited to compare the decompositions provided here with the ones appearing in [And87, Ch. 3], [Tay88, §2.4], [RS07, §6.1] etc.

Notation. The symbols F, $\mathscr{O}_F$, ϖ, $\mathscr{k}$, q and $[\mathscr{k}]$ have the same meaning as in Notation 4.0.1. Let $\mathbf{G}$ be the reductive over F whose R points for a F-algebra R are $\{g \in \mathrm{GL}_4(R) \,|\, {}^t gJg = \mu(g)J \text{ for } \mu(g) \in R^\times\}$ where

$$J = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}$$

is the standard symplectic matrix. We call $g \mapsto \mu(g)$ the similitude character. We let $G = \mathbf{G}(F)$, $K = G \cap \mathrm{GL}_4(\mathscr{O}_F)$. We will denote $G := \mathbf{G}(F)$ and $K := \mathbf{G}(\mathscr{O}_F)$. For a ring R, we let $\mathcal{H}_R = \mathcal{H}_R(K\backslash G/K)$ denote the Hecke algebra of G of level K with coefficients in R with respect to a Haar measure μ_G such that $\mu_G(K) = 1$. For simplicity, we will sometimes denote $\mathrm{ch}(K\sigma K) \in \mathcal{H}_R$ as $(K\sigma K)$.

12.1 Desiderata

Let $\mathbf{A} = \mathbb{G}_m^3$, $\mathrm{dis} : \mathbf{A} \to \mathbf{G}$ be the map $(u_0, u_1, u_2) \mapsto \mathrm{diag}(u_1, u_2, u_0u_1^{-1}, u_0u_2^{-1})$. Then dis identifies $\mathbf{A}$ with the maximal torus in $\mathbf{G}$. We let $A = \mathbf{A}(F)$ and $A^\circ = A \cap K$ denote the unique maximal compact subgroup.

For $i = 0, 1, 2$, let ϕ_i, ε_i be the maps defined in §11.1. We let Λ denote the cocharacter lattice, and the conventions about elements of Λ established in *ibid.* are maintained.

Let $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = 2\varepsilon_2 - \varepsilon_0$ the choice of simple roots and $\alpha_0 = 2\varepsilon_1 - \varepsilon_0$ the highest root. We let s_i denote the associated reflections. The groups W, W_{aff}, W_I, Ω, the set S_{aff} are analogous to the ones defined in 11.1 and conventions established about these groups in *ibid.* are maintained. We denote $\ell : W_I \to \mathbb{Z}$ denotes the length. The minimal length of elements in $t(\lambda)W \subset W_I$ can be computed by the formula

$$\ell_{\min}(t(\lambda)) := \sum_{\alpha \in \Phi_\lambda} |\langle \lambda, \alpha \rangle| + \sum_{\alpha \in \Phi^\lambda} (\langle \lambda, \alpha \rangle - 1) \tag{12.1.1}$$

where $\Phi_\lambda = \{\alpha \in \Phi^+ \,|\, \langle \lambda, \alpha \rangle \leq 0\}$, $\Phi^\lambda = \{\alpha \in \Phi^+ \,, |\langle \lambda, \alpha \rangle > 0\}$. We set

$$w_0 = \begin{pmatrix} & & \frac{1}{\varpi} & \\ & 1 & & \\ \varpi & & & \\ & & & -1 \end{pmatrix}, \; w_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \; w_2 = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & -1 & \\ & 1 & & \end{pmatrix}, \; \rho = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \varpi & & \\ \varpi & & & \end{pmatrix}$$

be elements of $N_G(A)$. These then represent the same elements in W_I as the corresponding elements of *ibid.* We moreover let $w_{\alpha_0} := w_1 w_2 w_1 = \varpi^{\phi_1} w_0$ the matrix representing s_{α_0}. For $i = 0, 1, 2$, let $x_i : \mathbb{G}_a \to \mathbf{G}$ be the root group maps

$$x_0 : u \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & \\ \varpi u & & 1 & \\ & & & 1 \end{pmatrix}, \; x_1 : u \mapsto \begin{pmatrix} 1 & u & & \\ & 1 & & \\ & & 1 & \\ & & -u & 1 \end{pmatrix}, \; x_2 : u \mapsto \begin{pmatrix} 1 & & & \\ & 1 & & u \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{12.1.2}$$

and let $g_i : \Bbbk \to G$ be the map $\kappa \mapsto x_i(\kappa) w_i$. For $w \in W_I$ with $w = s_{w,1} \cdots s_{w,\ell(w)} \rho_w$, where $s_{w,i} \in S_{\text{aff}}$, $\rho_w \in \Omega$, a reduced word decomposition, let

$$\begin{aligned} \mathcal{X}_w : [\Bbbk]^{\ell(w)} &\to G/K \\ (\kappa_1, \dots, \kappa_{\ell(w)}) &\mapsto g_{s_{w,1}}(\kappa_1) \cdots g_{s_{w,\ell(w)}}(\kappa_{\ell(w)}) \rho_w K \end{aligned} \tag{12.1.3}$$

Then $\operatorname{im}(\mathcal{X}_w)$ is independent of the choice of decomposition of w by Proposition 5.2.8.

12.2 Spin Hecke polynomial

The dual group of $\mathbf{G}$ is GSpin_5 which has a four dimensional representation called the *spin representation*. The highest coweight of this representation is $\phi_0 + \phi_1 + \phi_2$ (see §8.2.1 for arithmetic motivation) which is minuscule. By Corollary 4.3.8, the coweights are $\frac{2\phi_0 + \phi_1 + \phi_2}{2} \pm \frac{\phi_1}{2} \pm \frac{\phi_2}{2}$. The Satake polynoimal is therefore

$$\mathfrak{S}_{\text{spin}}(X) = (1 - y_0 X)(1 - y_0 y_1 X)(1 - y_0 y_2 X)(1 - y_0 y_1 y_2 X) \in \mathbb{Z}[\Lambda]^W[X]$$

where $y_i = e^{\phi_i} \in \mathbb{Z}[\Lambda]$. Let $\mathcal{R} = \mathbb{Z}[q^{\pm \frac{1}{2}}]$, and $\mathscr{S} : \mathcal{H}_{\mathcal{R}}(K \backslash G / K) \to \mathcal{R}[\Lambda]^W$ denote the Satake isomorphism. For $c \in \mathbb{Z} - 2\mathbb{Z}$, the polynomial $\mathfrak{H}_{\text{spin},c}(X)$ is defined so that $\mathscr{S}(\mathfrak{H}_{\text{spin},c}(X)) = \mathfrak{S}_{\text{spin}}(q^{-c/2} X)$.

Proposition 12.2.1. *For $c \in \mathbb{Z} - 2\mathbb{Z}$, we have*

$$\begin{aligned}\mathfrak{H}_{\mathrm{spin},c}(X) = (K) &- q^{-\frac{c+3}{2}}(K\rho K)X \\ &+ q^{-(c+2)}\left((Kw_0\rho^2K) + (q^2+1)(K\rho^2K)\right)X^2 \\ &- q^{-\frac{3c+3}{2}}(K\rho^3K)X^3 + q^{-2c}(K\rho^4K)X^4 \in \mathcal{H}_{\mathbb{Z}_{(q)}}(K\backslash G/K)[X]\end{aligned}$$

Proof. We have

$$\mathfrak{S}_{\mathrm{spin}}(X) = 1 - e^{W(1,1,1)}X + \left(e^{W(2,2,1)} + 2e^{W(2,1,1)}\right)X^2 - e^{W(3,2,2)}X^3 + e^{(4,2,2)}X^4.$$

The lengths of the cocharacters appearing in the exponents is computed using the formula , and the leading coefficients (Definition 4.2.6) of $K\varpi^\lambda K$ for $\lambda \in \Lambda$ dominant appearing as exponent of $\mathfrak{S}_{\mathrm{spin}}(X)$ are $q^{-\langle \lambda, \delta \rangle}$ (Corollary 4.2.4) shifted by an appropriate power of $q^{-c/2}$. The coefficient of the non-leading term $K\rho^2K$ is computed as follows.

$$(2,0,1) \xrightarrow{s_1} (2,1,0) \xrightarrow{s_2} (2,1,2) \xrightarrow{s_1} (2,2,1) \tag{12.2.2}$$

From the Weyl orbit diagram above (and that the minimal length of elements $t(2,2,1)W$ is 1), we see that $|Kw_0\rho^2K/K| = q + q^2 + q^3 + q^4$. Of these, the number of cosets of shape a permutation of $(2,2,1)$ are $\sum_{\mu \in W(2,2,1)} q^{-\langle (2,2,1)+\mu, \delta \rangle} = 1 + q + q^3 + q^4$. Thus the required coefficient is q^{-c} times $2 - q^{-2}(q + q^2 + q^3 + q^4 - (1 + q + q^3 + q^4)) = q^{-2}(q^2+1)$. □

Remark 12.2.3. See [And87, Proposition 3.3.35] where $\mathfrak{H}_{\mathrm{spin},-3}(X)$ makes an appearance. Cf. the polynomial in [Tay88, §2]. The dual group of $\mathbf{G}$ also has a 5 dimensional representation called the *standard representation*. It's highest coweight is ϕ_1 and it's Satake polynomial is

$$\mathfrak{S}_{\mathrm{std}}(X) = (1-X)(1-y_1^{-1}X)(1-y_1X)(1-y_2^{-1}X)(1-y_2X).$$

Cf. the polynomial $\mathfrak{S}_{\mathrm{bc}}(X)$ of §11.2. See [AS01] for a discussion of this L-factor.

12.3 Mixed coset decompositions

Let $\mathbf{H}$ be the subgroup of $\mathbf{G}$ generated by $\mathbf{A}$ and the root groups of $\pm\alpha_0$, $\pm\alpha_2$. Then $\mathbf{H} \cong \mathrm{GL}_2 \times_{\det} \mathrm{GL}_2$, the fiber product being over the determinant map. Set $H = \mathbf{H}(F)$, $U = H \cap K$, $W_H = \langle s_0, s_2 \rangle \cong S_2 \times S_2$ the Weyl group of H, $\Phi_H := \{\pm\alpha_0, \pm\alpha_2\}$ the set of roots of $\mathbf{H}$ and $\iota : H \to G$ denote the natural embedding. Let $H_1, H_2 := \mathrm{GL}_2(F)$ and $\mathrm{pr}_i : H \to H_i$ for $i = 1, 2$ denote the natural projection maps corresponding to

$\pm\alpha_0$, $\pm\alpha_2$ respectively. Let $\jmath \in \mathrm{GL}_2(F)$, $\tau \in H$ be the matrices displayed below on the left

$$\jmath := \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \ \tau := \begin{pmatrix} 1 & & & \frac{1}{\varpi} \\ & 1 & \frac{1}{\varpi} & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \jmath : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & & b & \\ & d & & c \\ c & & d & \\ & b & & a \end{pmatrix},$$

and $\jmath : \mathrm{GL}_2(F) \hookrightarrow H$ be the embedding displayed above on the right. Then $\jmath : \mathrm{GL}_2(F) \to H \simeq H_1 \times_{\det} H_2$ is given by $h \mapsto (h, \jmath h \jmath)$. Set $t_0 := \varpi^{(1,1,1)} \in A$, $\sigma_0 := t_0\tau \in H$,

$$H_{\sigma_0} := H \cap \sigma_0 K \sigma_0^{-1}, \quad \mathscr{X} := \operatorname{im} \jmath \subset H, \quad \mathscr{X}^\circ := \jmath\big(\mathrm{GL}_2(\mathscr{O}_F)\big).$$

Let $x_{\alpha_2} := x_2$, $x_{-\alpha_2} := w_2 x_2 w_2$, $x_{\alpha_0} := w_1 x_2 w_1$, $x_{-\alpha_0} = w_1 x_{-\alpha_2} w_1$ the 'standard' root group maps of H as in (10.1.4) and let H' be the compact open subgroup of H generated by $x_\alpha\big(\varpi\,\mathscr{O}_F\big)$ for $\alpha \in \Phi_H$ and $A_\tau := A \cap \tau K \tau^{-1}$.

Lemma 12.3.1. $H_{\sigma_0} = \mathscr{X}^\circ H'$.

Proof. Let $h = (h_1, h_2) \in H$ and say $h_i := \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ where $a_i, b_i, c_i, d_i \in F$. Then $h \in H_{\sigma_0} \iff \tau^{-1} h \tau \in K$ which is equivalent to

$$a_1, a_2, c_1, c_2, d_1, d_2 \in \mathscr{O}_F, \quad a_1 - d_2, a_2 - d_1, b_1 - c_2, b_2 - c_1 \in \varpi\,\mathscr{O}_F$$

It is then clear that $H_{\sigma_0} \supset \mathscr{X}^\circ H'$. Now let $h = (h_1, h_2) \in H_{\sigma_0}$ and consider the element $(h_1', h_2') \in \jmath(h_1^{-1})h \in H_{\sigma_0}$. By definition, we have $h_1' = 1_{H_1}$. The condition $(1_{H_1}, h_2') \in H_{\sigma_0}$ then imply that $(1_{H_1}, h_2') \in H'$. ☐

Let $\jmath_{\sigma_0} := \sigma_0^{-1}\jmath(\jmath)\sigma_0 \in K$ and let $\theta : \Lambda \to \Lambda$ denote the involution $\lambda \mapsto f_1 + f_2 + s_0 s_2(\lambda)$. Then for all $\lambda \in \Lambda$, we have

$$U\varpi^\lambda \tau K = U\varpi^\lambda \tau \jmath_{\sigma_0} K = U\varpi^\lambda t_0^{-1}\jmath(\jmath) t_0 \tau K = U\varpi^{\theta(\lambda)}\tau K, .$$

In particular, $U\varpi^{(2,2,1)}\tau K = U\varpi^{(2,1,2)}\tau K$.

Proposition 12.3.2. *Let*

- $\mathfrak{T}(\rho) = \big\{\varpi^{(1,1,1)}, \varpi^{(1,1,1)}\tau\big\}$,
- $\mathfrak{T}(w_0\rho^2) = \big\{\varpi^{(2,2,1)}, \varpi^{(2,1,2)}, \varpi^{(2,2,1)}\tau\big\}$.

Then for $w \in \big\{\rho, w_0\rho^2\big\}$, $\mathrm{ch}(KwK) = \sum_{\sigma \in \mathfrak{T}(w)} \mathrm{ch}(U\sigma K)$.

Proof. Let $w = \rho$. We have $KwK/K = \bigsqcup_\sigma \operatorname{im}(\mathcal{X}_\sigma)$ for $\sigma \in \{w, w_2 w, w_1 w_2 w, w_2 w_1 w\}$. To obtain the mixed representatives, we need to analyze the U-action on the cells corresponding to w, $w_1 w_2 w$. The first is a singleton and gives $\varpi^{(1,1,1)}$ (after conjugating by $w_{\alpha_0} w_2$). For $\sigma = w_1 w_2 w$, we see that

$$\mathrm{im}(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi & a & y & \\ & 1 & & \\ & & 1 & \\ & & -a & \varpi \end{pmatrix} K \,\middle|\, a, y \in [\Bbbk] \right\}$$

We can eliminate y by a row operation from U, and conjugating by w_2 gives us a matrix with diagonal $\varpi^{(1,1,1)}$. If $a = 0$, we obtain $\varpi^{(1,1,1)}$ and if $a \neq 0$, we conjugate by $\mathrm{diag}(1,1,a,a)$ to obtain $\varpi^{(1,1,1)}\tau$. These are distinct since $U\varpi^{(1,1,1)}K \subset HK$, $U\varpi^{(1,1,1)}\tau K \subset H\tau K$ and HK, $H\tau K$ are disjoint[1].

Let $w = w_0 \rho^2$. From diagram 12.2.2, we have $KwK/K = \bigsqcup_\sigma \mathrm{im}(\mathcal{X}_w)$ for $\sigma \in \{w, w_1 w, w_2 w_1 w, w_1 w_2 w_1 w\}$ and it suffices to analyze the cells corresponding to $w, w_1 w, w_1 w_2 w_1 w$. We analyze $\sigma = w_1 w_2 w_1 w$, and leave the other two for the reader. We have

$$\mathrm{im}(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi^2 & a_1 \varpi & y + \varpi x & a\varpi \\ & \varpi & a & \\ & & 1 & \\ & & -a_1 & \varpi \end{pmatrix} K \,\middle|\, a, a_1, x, y \in [\Bbbk] \right\}$$

A row operation from U eliminates $y + \varpi x$. Next note that conjugation by w_2 swaps a_1 and a. Using row and column operations, we can assume that $a_1 = 0$. If $a = 0$, we get $\varpi^{(2,2,1)}$ and if not, then conjugation by $\mathrm{diag}(1,1,a,a)$ gives us $\varpi^{(2,2,1)}\tau$. The same argument as above distinguishes $U\varpi^{(2,2,1)}K, U\varpi^{(2,1,2)}K$ from $\varpi^{(2,2,1)}\tau$. Since $U\backslash HK/K \simeq U\backslash H/U$, Cartan decomposition for U implies that $U\varpi^{(2,2,1)}K$, $U\varpi^{(2,1,2)}K$ are distinct. $\square$

12.4 Schwartz space computations

Let $X := F^2 \oplus F^2$ considered topological vector space over F. Elements of X are viewed as pairs of row vectors. Then $H_1 \times H_2$ act on as via

$$(u, v) \cdot (h_1, h_2) \mapsto (uh_1, vh_2) \qquad v_1, v_2 \in F^2, h_1 \in H_1, h_2 \in H_2.$$

We then let H act on X via the natural embedding $H \hookrightarrow H_1 \times H_2$. Let $\mathcal{O}$ be an integral domain that contains $\mathbb{Z}[q^{-1}]$ and let $\mathcal{S} = \mathcal{S}_{\mathcal{O}}(X)$ be the space of functions $\xi : X \to \mathcal{O}$ which are locally constant, compactly supported on X. Then $\mathcal{S}$ has an induced left action $\mathcal{S} \times H \to \mathcal{S}$ via $(h, \xi) \mapsto \xi((-)h)$, which makes $\mathcal{S}$ a smooth representation of H. Let Υ_H be the collection of all compact open subgroups of H and $M_{H,\mathcal{O}} : \Upsilon \to \mathcal{O}\text{-Mod}$ denote the functor $V \mapsto \mathcal{S}^V$ associated with $\mathcal{S}$ (see Definition 2.2.1). For $a \in \mathbb{Z}$, let $Y_a := \varpi^a \mathscr{O}_F \subset F$ and for $a, b, c, d \in \mathbb{Z}$, let $Y_{a,b} = Y_a \times Y_b \subset F^2$, $Y_{a,b,c,d} := Y_a \times Y_b \times Y_c \times Y_d \subset X$.

[1]alternatively, one may note that $H \cap \sigma_0 K \sigma_0^{-1} \subsetneq U$ and use Lemma 5.5.1

We denote $\phi_{(a,b,c,d)} := \mathrm{ch}(Y_{u,v,w,x}) \in \mathcal{S}$ where $\mathrm{ch}(Y)$ denotes the characteristic function of $Y \subset X$ and set set $\phi := \phi_{(0,0,0,0)}$. The element ϕ will serve as the bottom class of the zeta element. We note that if $U_1 = U_2 := \mathrm{GL}_2(\mathscr{O}_F)$, $\lambda = (a_0, a_1, a_2) \in \Lambda$,

$$[U\varpi^{\lambda}U](\phi_{(u,v,w,x)}) = [U_1 t_1 U_1](\phi_{(u,v)}) \otimes [U_1 t_2 U_2](\phi_{(w,x)})$$

where $t_i = \mathrm{diag}(\varpi^{a_i}, \varpi^{a_0 - a_i})$ and $\phi_{(a,b)} : F^2 \to \mathcal{O}$ is the characteristic function of $Y_a \times Y_b$ for $a, b, \in \mathbb{Z}$.

Lemma 12.4.1. *We have*

(a) $[U\varpi^{(1,1,1)}U]_*(\phi) = \phi + q(\phi_{(1,1,0,0)} + \phi_{(0,0,1,1)}) + q^2\phi_{(1,1,1,1)}$,

(b) $[U\varpi^{(2,2,1)}U]_*(\phi) = \phi_{(0,0,1,1)} + (q-1)\phi_{(1,1,1,1)} + q^2\phi_{(2,2,1,1)}$,

(c) $[U\varpi^{(2,1,2)}U]_*(\phi) = \phi_{(1,1,0,0)} + (q-1)\phi_{(1,1,1,1)} + q^2\phi_{(1,1,2,2)}$.

Proof. Let $\rho_i := \mathrm{diag}(1, \varpi) \cdot \jmath \in H_i$. We consider $w_0, w_{\alpha_0}, w_2 = w_{\alpha_2}$ as elements in H_1, H_2 via $\mathrm{pr}_i : H \to H_i$. Similar remarks apply to the maps $x_0, x_2, x_{\pm\alpha_0}, x_{\pm\alpha_2} : \mathbb{G}_a \to \mathbf{H}$ etc.

(a) We have $[U\varpi^{-(1,1,1)}U](\phi) = [U_1\rho_1^{-1}U_1](\phi_{(0,0)}) \otimes [U_2\rho_2^{-1}U_2](\phi_{(0,0)})$ and $\xi := [U_i\rho_i^{-1}U_i](\phi_{(0,0)})$ is independent of $i = 1, 2$. By Proposition 5.2.8, we easily see that $\xi(a,b)$ for $a, b \in F$ is the number of integral vectors in the following list of $q+1$ vectors

$$\left(\frac{a}{\varpi}, b\right) \quad \left(a, \frac{b + a\kappa}{\varpi}\right) \kappa \in [\hbar]$$

It is clear that $\mathrm{Supp}(\xi) \in \mathscr{O}_F^2$. If $a \in \mathscr{O}_F^\times$ or if $a \in \varpi\mathscr{O}_F$, $b \in \mathscr{O}_F^\times$, then there is a unique integral vector. If however both $a, b \in \varpi\mathscr{O}_F$, then all $q+1$ vectors are integral. So, $\xi = \phi_{(0,0)} + q\phi_{(1,1)}$ and $\xi \otimes \xi$ is the required function.

(b) & (c) We have $[U\varpi^{-(1,1,1)}U](\phi) = [U_1 w_0 \rho_1^{-2} U_1](\phi_{(0,0)}) \otimes [U_2\rho_2^{-2}U_2](\phi_{(0,0)})$ and $[U_2\rho_2^{-2}U_2](\phi_{(0,0)}) = \phi_{(1,1)}$. Let $\xi := [U_2 w_0 \rho_2^{-2} U_2](\phi_{(0,0)}) = [U_1 w_0 U_1](\phi_{(1,1)})$. By Proposition 5.2.8, $U_1 w_0 U_1/U_1 = \bigsqcup_{j\in[\hbar]} x_0(j) w_0 U_1 \sqcup \bigsqcup_{\kappa_1,\kappa_2\in[\hbar]} x_{\alpha_0}(\kappa_2) w_{\alpha_0} x_0(\kappa_2) w_0 U_2$. Let $a, b \in \mathscr{O}_F$. From the decomposition of $U_1 w_0 U_1/U_1$, we see that the $\xi(a,b)$ equals the number of integral vectors in the following list of $q^2 + q$ vectors

$$\left(\frac{a + bj\varpi}{\varpi^2}, b\right) \quad \left(a, \frac{b + a(\kappa_1 + \kappa_2\varpi)}{\varpi^2}\right)$$

where $\kappa_1, \kappa_2 \in [\hbar]$. One then argues as in part (a) to conclude that $\xi = \phi_{(0,0)} + (q-1)\phi_{(1,1)} + q^2\phi_{(2,2)}$ and (b) follows easily. Part (c) is obtained by switching the roles H_1, H_2. ☐

Let $\mathrm{Mat}_{2\times 2}(F)$ be the F-vector space 2×2 matrices over F. We let $\jmath : X \to \mathrm{Mat}_{2\times 2}(F)$ the inverse of F-linear isomorphism $X \to \mathrm{Mat}_{2\times 2}(F)$ given by

$$(u, v) \mapsto \begin{pmatrix} u_1 & u_2 \\ v_2 & v_1 \end{pmatrix}.$$

where $u = (u_1, u_2)$, $v = (v_1, v_2) \in X$. Then for $h \in \mathrm{GL}_2(F)$, $(u, v) \cdot \jmath(h) = \jmath(u, v) \cdot h$ where the last product is matrix multiplication. Let $\psi \in \mathcal{S}$ denote the element such that $\jmath(\psi) = \mathrm{ch}(\mathrm{GL}_2(\mathscr{O}_F))$, considered as a function on $\mathrm{Mat}_{2\times 2}(F)$. We recall that σ_0 denotes $t_0\tau$. Let

$$\mathfrak{h} = \mathfrak{h}_{\sigma_0} := [UH_{\sigma_0}]_* - [U\varpi^{(1,1,0)}H_{\sigma_0}]_* + q[U\varpi^{(2,1,1)}H_{\sigma_0}]_* \tag{12.4.2}$$

considered as an element of $\mathrm{Hom}_{\mathcal{O}}(M_{H,\mathcal{O}}(U), M_{H,\mathcal{O}}(H_{\sigma_0}))$.

Lemma 12.4.3. *$\mathfrak{h}(\phi) = \psi$.*

Proof. We note that $H_{\sigma_0} \subset U$, whence the degrees of $[UH_{\sigma_0}]_*$, $[U\varpi^{(2,1,1)}H_{\sigma_0}]_*$ are 1. Let $t_1 = \varpi^{(1,1,0)}$. By Lemma 12.3.1, $H_{\sigma_0}t_1^{-1}U = \mathscr{X}^\circ H' t_1^{-1}U = \mathscr{X}^\circ t_1^{-1}U$ since $t_1 H' t^{-1} \subset U$. Moreover, $\mathscr{X}^\circ \cap t_1^{-1}Ut_1 = \jmath(I_{\mathrm{GL}_2})$ where I_{GL_2} is the Iwahori subgroup of $\mathrm{GL}_2(F)$ generated by $\mathrm{diag}(\mathscr{O}_F^\times, \mathscr{O}_F^\times)$, $x_+(\mathscr{O}_F)$, $x_-(\varpi\,\mathscr{O}_F)$. Thus $\mathscr{X}^\circ t_1^{-1}U/U = t_1^{-1}U \cup \bigsqcup_{\kappa \in [\hbar]} \jmath(x_+(\kappa)\jmath)t_1^{-1}U$. We therefore see that $\mathfrak{h}(\phi) = \jmath^{-1}\big(\jmath(\phi) - T_\varpi^t\,\jmath(\phi) + q\,S_\varpi^t\,\jmath(\phi)\big)$ where T_ϖ, S_ϖ are the Hecke operators of $\mathrm{GL}_2(F)$ given by the $\mathrm{GL}_2(\mathscr{O}_F)$-double cosets of $\mathrm{diag}(1,\varpi)$, $\mathrm{diag}(\varpi,\varpi)$ respectively. By Example, 3.6.2, we get the claim. ☐

Remark 12.4.4. One may think of $\mathfrak{h}$ as a 'mixed' factor of $\mathfrak{H}^t_{\mathrm{spin},-1}(X)$ for the following reason. Let

$$\begin{aligned}\mathfrak{s}(X) = \mathfrak{s}_{\sigma_0}(X) &:= (1 - y_0y_1X)(1 - y_0y_2X) \\ &= 1 - y_0(y_1 + y_2)X + y_0^2y_1y_2X^2\end{aligned}$$

Then $\mathfrak{s}(X)$ is invariant under the action of s_0s_2 and the lift $\jmath \in H$ of this element is contained in H_{σ_0}. The polynomial $\mathfrak{h}_{\sigma_0}$ then 'corresponds' to $\mathfrak{s}_{\sigma_0}$ in the same sense $\mathfrak{H}_{\mathrm{spin}}(X)$ corresponds to $\mathfrak{S}_{\mathrm{spin}}(X)$. We also note that $[U\varpi^{(1,1,1)}H_\tau]_* = [UH_{\sigma_0}]_* \circ c_{t_1}$, $[U\varpi^{(2,2,1)}H_\tau]_* = [U\varpi^{(1,1,0)}H_{\sigma_0}]_* \circ c_{t_1}$ where c_{t_1} denotes conjugation by $t_1 = \varpi^{(1,1,1)}$. Therefore in the decomposition of $\mathfrak{H}_{\mathrm{spin},-1}(1)$ into a sum of mixed Hecke correspondences, the mixed representatives that have a τ in them (see 12.3.2) are collected together into $\mathfrak{h} \circ c_{t_0}$. This is the motivation for Lemma 12.4.3 which we will invoke in Theorem 12.5.1.

12.5 Zeta elements

Let $T = F^\times$, $C = \mathscr{O}_F^\times$, $D = 1 + \varpi\,\mathscr{O}_F$, $\nu = \mu_{\mathrm{sim}} \circ \iota : H \to T$ be the map that sends (h_1, h_2) to the common determinant of h_1, h_2. For the zeta element problem, we set

- $\mathcal{G} = G \times T$, $\Upsilon_{\mathcal{G}}$ collection of all compact open subgroups of $\mathcal{G}$,
- $\iota_\nu = \iota \times \nu : H \to \mathcal{G}$,
- U, $\mathcal{K} := K \times C$ as bottom levels
- $x_U = \phi_{0,0} \in M_{H,\mathcal{O}}(U)$ as the the source bottom class

- $\mathcal{L} = K \times D$ field extension of degree $q-1$,
- $\mathfrak{H}_c = \mathfrak{H}^t_{\mathrm{spin},c}(\mathrm{Frob}) \in \mathcal{C}_{\mathcal{O}}(\mathcal{K}\backslash\mathcal{G}/\mathcal{K})$ where $\mathrm{Frob} = \mathrm{ch}(\varpi^{-1}C)$ denotes the 'arithmetic Frobenius'

Theorem 12.5.1. *There exists a uniform zeta element for $(x_U, \mathfrak{H}_c, \mathcal{L})$ for all $c \in \mathbb{Z} - 2\mathbb{Z}$.*

Proof. Let $g_0 := (1_G, 1_T)$, $g_1 := (\sigma_0, 1_T)$. We show that a uniform zeta element with twists g_i exists. Let

- $J_0 = \{(0,0,0), (1,1,1), (2,2,1), (2,2,1), (3,2,2), (4,2,2)\}$,
- $J_1 = \{(0,0,0), (1,1,0), (2,1,1)\}$,
- $J = J_1 \sqcup J_2$.

For $j = (a_0, a_1, a_2) \in J_i$, we let $h_j := \varpi^{-(a_0,a_1,a_2)} \in H$ and let $\sigma_j = \iota_\nu(h_j^{-1})g_i = (\varpi^{(a_0,a_1,a_2)}\sigma_0, \varpi^{a_0}) \in \mathcal{G}$. Using Proposition 12.2.1, Proposition 12.3.2, Lemma 3.2.7(i) and that ρ^2 is central, we see that

$$\mathfrak{H}^t = \sum_{j \in J} c_j \mathrm{ch}(U\sigma_j\mathcal{K})$$

for some $c_j \in \mathbb{Z}_{(q)}$ e.g. $c_{(1,1,1)} = q^{-\frac{c+3}{2}}$. We note that $H_{g_1} = H_{\sigma_0}$ and $H_{g_0} = U$ i.e. the twisted intersections are the same for K and $\mathcal{K}$. Let $W_j = h_j U_{\sigma_j} h_j^{-1}$, $\varsigma_j = \sum_{\gamma \in H_{g_i}/W_j} \gamma \in \Phi[H_{g_i}]$ for $j \in J$ and $V_i = H \cap g_i \mathcal{L} g_i^{-1}$ for $i = 0, 1$. For convenience in retrieving the computation below, we record the powers of q in the coefficients (even though we need them modulo $q-1$) and for the readability of expressions, we let $c = -(2k+1)$ for $k \in \mathbb{Z}$. To show a zeta element exists, it suffices by Proposition 3.2.1 to show the following two claims.

Claim 1. There exists $x_{V_0} \in M_{H,\mathcal{O}}(V_0)$ such that $\sum_{j \in J_0} c_j \varsigma_j h_j \cdot j_U(x_U) = j_{H_{g_0}}(\mathrm{pr}_{V_0, H_{g_0},*}(x_{V_0}))$.
By Proposition 12.2.1 and Lemma 12.4.1 (a)-(c), we see that

$$\begin{aligned}\sum\nolimits_{j \in J_0} c_j \varsigma_j h_j \cdot j_U(x_U) &= (1 - q^{k-1})(\phi) + (q^{2k-1} - q^k)(\phi_{(0,0,1,1)} + \phi_{(1,1,0,0)}) \\ &+ \left(q^{2k-1}(2q - 2 + q^2 + 1) - q^{3k} - q^{k+1}\right)\phi_{(1,1,1,1)} \\ &+ (q^{2k+1} - q^{3k+1})(\phi_{(1,1,2,2)} + \phi_{(2,2,1,1)}) \\ &+ (q^{4k+2} - q^{3k+2})\phi_{(2,2,2,2)}\end{aligned}$$

which is an element of $(q-1) \cdot M_{H,\mathcal{O}}(U)$. Invoking Corollary 3.2.3 or Proposition 3.3.1, this guarantees the existence of x_{V_0}.

Claim 2. There exists $x_{V_1} \in M_{H,\mathcal{O}}(V_1)$ such that $\sum_{j \in J_1} c_j \varsigma_j h_j \cdot j_U(x_U) = j_{H_{g_1}}(\mathrm{pr}_{V_1, H_{g_1},*}(x_{V_1}))$.
First assume $c = -1$, so that $k = 0$. By Proposition 12.2.1 and Lemma 12.4.3, we see that

$$\sum\nolimits_{j \in J_1} c_j \varsigma_j h_j \cdot j_U(x_U) = -q^{-1}\psi$$

One then easily verifies that all points in the support of ψ are (H_{σ_0}, V_1)-smooth, whence x_{V_1} exists by Proposition 3.4.5. Now for k arbitrary, the sum above on the left is congruent modulo $q-1$ to the corresponding sum for $k=0$ and the claim follows by the second part of Corollary 3.3.2 (or just the property (Co)).

□

Remark 12.5.2. We note that zeta element problem should strictly speaking be posed for the local sub-functor $\mathcal{T}_{\mathcal{O}}$ generated by Hecke correspondences applied to $\phi_{0,0}$. It is however clear that the classes x_{V_i} are contained in this sub-functor. Since we suspect that $\mathcal{T}_{\mathcal{O}} = \mathcal{S}_{\mathcal{O}}$, we have chosen to work with $\mathcal{S}_{\mathcal{O}}$ instead.

Chapter 13

Spin L-factor of GSp_6

Notation. The symbols F, $\mathscr{O}_F$, ϖ, $\Bbbk$ and q retain their meaning as Notation 4.0.1 and $[\Bbbk] \subset \mathscr{O}_F$ denotes a set of representatives of $\Bbbk$. Let

$$J = \begin{pmatrix} & I_3 \\ -I_3 & \end{pmatrix}$$

where I_3 is the 3×3 identity matrix. We let $\mathbf{G}$ be the reductive group scheme over F whose R points for a F-algebra R are given by $\{g \in \mathrm{GL}_6(R) \,|\, {}^t gJg = \mu(g)J, \mu(g) \in R^\times\}$. We let $G = \mathbf{G}(F)$, $K = G(F) \cap \mathrm{GL}_6(\mathscr{O}_F)$. For a ring R, we let $\mathcal{H}_R(G)$ denote the Hecke algebra of G of level K with coefficients in R with respect to the Haar measure μ_G on G that gives K measure 1.

13.1 Desiderata

Let $\mathbf{A} = \mathbb{G}_m^4$ and $\mathrm{dis} : \mathbf{A} \to \mathbf{G}$ given by

$$(u_0, u_1, u_2, u_3) \mapsto \mathrm{diag}(u_1, u_2, u_3, u_0 u_1^{-1}, u_0 u_2^{-1}, u_0 u_3^{-1}).$$

Then dis identifies $\mathbf{A}$ with a maximal torus in $\mathbf{G}$. Let $A = \mathbf{A}(F)$, $A^\circ = A \cap K$. Let $e_i : \mathbf{A} \to \mathbb{G}_m$ be the projection on the i-th component, $f_i : \mathbb{G}_m \to \mathbf{A}$ be the cocharacter inserting u in the i-th component. We will denote by Λ the cocharacter lattice $\mathbb{Z}f_0 \oplus \cdots \oplus \mathbb{Z}f_3$. An element $a_0 f_0 + \ldots + a_3 f_3 \in \Lambda$ will also be denoted by $(a_0, \ldots, a_3)$ The set $\Phi \subset X^*(\mathbf{A})$ of roots of $\mathbf{G}$ are

- $\pm(e_i - e_j)$ for $1 \leq i < j \leq 3$,
- $\pm(e_i + e_j - e_0)$ for $1 \leq i < j \leq 3$
- $\pm(2e_i - e_0)$ for $i = 1, 2, 3$

which makes an irreducible root system of type C_3. We pick $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = 2e_3 - e_0$ as simple roots and denote $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ the base, Φ^+ the set of positive roots. With respect to the ordering induced by Δ, the highest root is $\alpha_0 = 2e_1 - e_0$. We let I_G be the standard Iwahori subgroup of G, which corresponds to the alcove determined by the simple affine roots $\alpha_1, \alpha_2, -\alpha_0 + 1$. The coroots corresponding to α_i are $\alpha_0^\vee = f_1, \alpha_1^\vee = f_1 - f_2, \alpha_2^\vee = f_2 - f_3$, $\alpha_3^\vee = f_3$ and their $\mathbb{Z}$ span in Λ is denoted by $Q^\vee$. An element $\lambda = (a_0, \dots, a_3) \in \Lambda$ is then dominant iff $a_1 \geq a_2 \geq a_3$ and $2a_3 - a_0 \geq 0$ and anti-dominant if all these inequalities hold in reverse. We denote the set of dominant cocharacters by Λ^+. The translation action of $\lambda \in \Lambda$ on $\Lambda \otimes \mathbb{R}$ via $x \mapsto x + \lambda$ is denoted by $t(\lambda)$. We denote $\varpi^\lambda \in A$ the element $\lambda(\varpi)$ for $\lambda \in \Lambda$. We let $v : A/A^\circ \to \Lambda$ be the inverse of the map $\Lambda \to A/A^\circ$, $\lambda \mapsto \varpi^{-\lambda} A^\circ$. Let s_i be the reflection associated with α_i, $i = 0, \dots, 3$. The action of s_i on Λ is given explicitly as follows:

- s_i acts by switching $f_i \leftrightarrow f_{i+1}$ for $i = 1, 2$
- s_3 acts by sending $f_0 \mapsto f_0 + f_3$, $f_3 \mapsto -f_3$, keeping f_1, f_2 fixed,
- $s_0 = s_1 s_2 s_3 s_2 s_1$ acts by sending $f_0 \mapsto f_1$, $f_1 \mapsto -f_1$, keeping f_2, f_3 fixed.

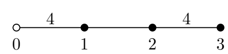
For $\lambda \in \Lambda$, we let $e^\lambda \in \mathbb{Z}[\Lambda]$ denote the element corresponding to λ and $e^{W\lambda} \in \mathbb{Z}[\Lambda]$ denote the element obtained by the formal sum of elements in the orbit $W\lambda$. Let $S_{\text{aff}} = \{s_1, s_2, s_3, t(\alpha_0^\vee) s_0\}$ and W, W_{aff}, W_I be the Weyl, affine Weyl and Iwahori Weyl groups respectively determined by A. We consider W_{aff} as a subgroup of affine transformations of $\Lambda \otimes \mathbb{R}$. We have

- $W = \langle s_1, s_2, s_3 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3 \rtimes S_3$,
- $W_{\text{aff}} = t(Q^\vee) \rtimes W$
- $W_I = N_G(A)/A^\circ = A/A^\circ \rtimes W \overset{v}{\simeq} \Lambda \rtimes W$,

The pair $(W_{\text{aff}}, S_{\text{aff}})$ forms a Coxeter system of type $\tilde{C}_3$. We consider W_{aff} a subgroup of W_I via $W_{\text{aff}} = t(Q^\vee) \rtimes W \hookrightarrow \Lambda \rtimes W \overset{v}{\simeq} W_I$. The natural action of W_{aff} on $\Lambda \otimes \mathbb{R}$ then extends to W_I with $\lambda \in \Lambda$ acting as a translation $t(\lambda)$. We set $\Omega := W_I / W_{\text{aff}}$, which is an infinite cyclic group and we have $W_I \cong W_{\text{aff}} \rtimes \Omega$. We let $\ell : W_I \to \mathbb{Z}$ denote the induced length function with respect S_{aff}. Given $\lambda \in \Lambda$, the minimal length of elements in $t(\lambda)W$ is achieved by a unique element. This length is given by $\ell_{\min}(t(\lambda)) := \sum_{\alpha \in \Phi_\lambda} |\langle \lambda, \alpha \rangle| + \sum_{\alpha \in \Phi^\lambda} (\langle \lambda, \alpha \rangle - 1)$ where $\Phi_\lambda = \{\alpha \in \Phi^+ \mid \langle \lambda, \alpha \rangle \leq 0\}$, $\Phi^\lambda = \{\alpha \in \Phi^+ \,, | \langle \lambda, \alpha \rangle > 0\}$. When λ is dominant, this is also the minimal length of elements in $W t(\lambda) W$. We let

- $w_1 := \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 1 & & & \\ & & & 0 & 1 & \\ & & & 1 & 0 & \\ & & & & & 1 \end{pmatrix}$, $w_2 := \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & 1 & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$, $w_3 := \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & 1 \\ & & & -1 & & \\ & & & & -1 & \\ & & 1 & & & 0 \end{pmatrix}$,
- $w_0 := \begin{pmatrix} 0 & & & \frac{1}{\varpi} & & \\ & 1 & & & & \\ & & 1 & & & \\ \varpi & & & 0 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix}$, $\rho = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & \varpi & & & \\ & \varpi & & & & \\ \varpi & & & & & \end{pmatrix}$

be the elements of $N_G(A)$, the normalizer of A in G. The classes of w_0, w_1, w_2, w_3 in W_I represent $t(\alpha_0^\vee)s_0, s_1, s_2, s_3$ respectively. The reflection s_0 in α_0 is then represented by $w_{\alpha_0} := \varpi^{f_1} w_0 = w_1 w_2 w_3 w_2 w_1$. The class of ρ represents $\omega := t(-f_0)s_3s_2s_3s_1s_2s_3$ which is a generator of Ω and the conjugation by ω acts by switching $s_0 \leftrightarrow s_3$, $s_1 \leftrightarrow s_2$ i.e. it induces an automoprhism of the extended Coxeter-Dynkin diagram



where the labels below the vertices correspond to w_i. Note also that $\rho^2 = \varpi^{(2,1,1,1)} \in A$ is central. We will henceforth use the letters w_i, ρ to denote both the matrices and the their classes in W_I if no confusion can arise. Let $x_i : \mathbb{G}_a \to \mathbf{G}$ for $i = 0, 1, 2, 3$, be the following root group maps

$$x_1 : u \mapsto \begin{pmatrix} 1 & u & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & -u & 1 & \\ & & & & & 1 \end{pmatrix}, \; x_2 : u \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & u & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & -u & 1 \end{pmatrix}, \; x_3 : u \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & u \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix},$$

$$x_0 : u \mapsto \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \varpi u & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix},$$

and let $g_i : [\cancel{k}] \to G$ be the maps $\kappa \mapsto x_i(\kappa)w_i$. For $w \in W_I$ with $w = s_{w,1}s_{w,2}\cdots s_{w,\ell(w)}\rho_w$, where $s_{w,i} \in S_{\text{aff}}$, $\rho_w \in \Omega$, a reduced word decomposition, let

$$\mathcal{X}_w : [\cancel{k}]^{\ell(w)} \to G/K$$
$$(\kappa_1, \ldots, \kappa_{\ell(w)}) \mapsto g_{s_{w,1}}(\kappa_1) \cdots g_{s_{w,\ell(w)}}(\kappa_{\ell(w)})\rho_w K$$

where we have suppressed the dependence on the decomposition chosen in the notation. The image of

$\mathcal{X}_w$ is independent of the choice of decomposition and we have $\#\operatorname{im}(\mathcal{X}_w) = q^{\ell(w)}$. We note that $\ell(w) = \ell_{\min}(t(-\lambda_w))$ where $\lambda_w \in \Lambda$ is the unique cocharacter such that $wK = \varpi^{\lambda_w}K$.

13.2 Spin Hecke polynomial

For this subsection only, we let x_i[1] $:= e^{f_i} \in \mathbb{Z}[\Lambda]$, so that $\mathbb{Z}[\Lambda] = \mathbb{Z}[x_0^{\pm}, \ldots, x_3^{\pm}]$, $\mathcal{R} = \mathcal{R}_q = \mathbb{Z}[q^{\pm\frac{1}{2}}]$. The dual group of $\mathbf{G}$ has a 8-dimensional representation called the *spin* representation. Its highest coweight is $f_0 + f_1 + f_2 + f_3$, and the coweights are $\frac{1}{2}(2f_0 + f_1 + f_2 + f_3) + \frac{1}{2}(\pm f_1 \pm f_2 \pm f_3)$. Its Satake polynomial is therefore

$$\begin{aligned}\mathfrak{S}_{\text{spin}}(X) =&(1 - x_0X)(1 - x_0x_1X)(1 - x_0x_2X)(1 - x_0x_3X)\\ &(1 - x_0x_1x_2X)(1 - x_0x_1x_3X)(1 - x_0x_2x_3X)(1 - x_0x_1x_2x_3X) \in \mathbb{Z}[\Lambda]^W(X)\end{aligned}$$

Let $\mathscr{S} : \mathcal{H}_{\mathcal{R}}(G) \to \mathcal{R}[\Lambda]^W$ denote the Satake isomorphism.

Definition 13.2.1. For $c \in \mathbb{Z}$, we define the (degree 8) *spin Hecke polynomial* $\mathfrak{H}_{\text{spin},c}(X) \in \mathcal{H}_{\mathcal{R}}(G)[X]$ to be such that $\mathscr{S}(\mathfrak{H}_{\text{spin},c}) = \mathfrak{S}_{\text{spin}}(q^{-c}X)$.

Our goal is now to 'compute' the Hecke polynomial associated with the Satake polynomial above. Since the eventual computation of zeta element will only need this polynomial modulo $q - 1$, we forgo the computation of the actual coefficients in this case. In the next subsection, we will also need to decompose the various Hecke operators appearing in the spin Hecke polynomial into mixed ones. We therefore record the operators in $\mathfrak{H}_{\text{spin},c}(X)$ in a form suitable for this purpose (5.4). We begin by recording the following

Lemma 13.2.2. *For each $\lambda \in \Lambda^+$ below, the element $w = w_\lambda \in W_I$ specified is the unique element in W_I of minimal possible length such that $K\varpi^\lambda K = KwK$.*

- $\lambda = (1, 1, 1, 1)$, $w = \rho$,
- $\lambda = (2, 2, 2, 1)$, $w = w_0w_1w_0\rho^2K$,
- $\lambda = (2, 2, 1, 1)$, $w = w_0\rho^2$,
- $\lambda = (3, 3, 2, 2)$, $w = w_0w_1w_2w_3\rho^3K$,
- $\lambda = (4, 4, 2, 2)$, $w = w_0w_1w_2w_3w_2w_1w_0\rho^4$,
- $\lambda = (4, 3, 3, 3)$, $w = w_0w_1w_0w_2w_1w_0\rho^4$.

Remark 13.2.3. We point out that the translation component of each w_λ above (i.e. the Λ-component of $v(w_\lambda) \in \Lambda \rtimes W$) is $t(-\lambda^{\text{opp}})$ where λ^{opp} is the anti-dominant element in the Weyl orbit $W\lambda$. The minimal

[1] this conflicts with the earlier notation for root group maps but its use is only confined to the current subsection

possible length in each case is computed using that $\ell(w_\lambda) = \ell_{\min}(t(-\lambda^{\mathrm{opp}})) = \ell_{\min}(t(\lambda))$. See §5.4 for some heuristics that can be used to find these words.

Notation 13.2.1. We denote $\upsilon_1 := w_0w_1w_2w_3w_2w_1w_0$, $\upsilon_2 := w_0w_1w_0w_2w_1w_0$.

Proposition 13.2.4. *We have*

$$\begin{aligned}\mathfrak{H}_{\mathrm{spin},0}(X) &= (K) - (K\rho K)X + \rho^2 T_1(q)X^2 - \rho^2 T_2(q)X^3 + \rho^4 T_3(q)X^4 \\ &\quad - \rho^4 T_2(q)X^5 + \rho^6 T_1(q)X^6 - (K\rho^7 K)X^7 + (K\rho^8 K)X^8 \in \mathcal{H}_{\mathbb{Z}_{(q)}}(G)[X]\end{aligned}$$

where $T_i(x) \in \mathcal{H}_{\mathbb{Z}}(G)[x, x^{-1}]$ *are such that*

- $T_1(1) = (Kw_0w_1w_0K) + 2(Kw_0K) + 4(K)$,
- $T_2(1) = (Kw_0w_1w_2w_3\rho K) + 4(K\rho K)$,
- $T_3(1) = (K\upsilon_1 K) + (K\upsilon_2 K) + 2(Kw_0w_1w_0K) + 4(Kw_0K) + 8(K)$

Moreover, $\mathfrak{H}_{\mathrm{spin},c}(X) \equiv \mathfrak{H}_{\mathrm{spin},0}(X)$ *modulo* $q-1$ *for all* $c \in \mathbb{Z}$.

Proof. Solving the plethysm problem for exterior powers of the spin representation by combining i choices of coweights $\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + \left(0, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$ for $0 \le i \le 8$ or simply by expanding $\mathfrak{S}_{\mathrm{spin}}(X)$, we see that

$$\begin{aligned}\mathfrak{S}_{\mathrm{spin}}(X) &= 1 - e^{W(1,1,1,1)}X \\ &\quad + \left(e^{W(2,2,2,1)} + 2e^{W(2,2,1,1)} + 4e^{(2,1,1,1)}\right)X^2 - \left(e^{W(3,3,2,2)} + 4e^{W(3,2,2,2)}\right)X^3 \\ &\quad + \left(e^{W(4,4,2,2)} + e^{W(4,3,3,3)} + 2e^{W(4,3,3,2)} + 4e^{W(4,3,2,2)} + 8e^{(4,2,2,2)}\right)X^4 \\ &\quad - \left(e^{W(5,4,3,3)} + 4e^{W(5,3,3,3)}\right)X^5 + \left(e^{W(6,4,4,3)} + 2e^{W(6,4,3,3)} + 4e^{(6,3,3,3)}\right)X^6 \\ &\quad - e^{W(7,4,4,4)}X^7 + e^{(8,4,4,4)}X^8\end{aligned}$$

Since $\delta \in X^*(\mathbf{A})$, the claim is a cosequence Corollary 4.4.4. □

Remark 13.2.5. To obtain the operators $T_i(q)$ exactly, one can use Sage Math to compute appropriate Kazhdan-Lusztig polynomials $P_{\sigma,\tau}(q)$ for $\sigma, \tau \in W_I$. Let us provide some ingredients required for it. The element $(w_1w_3)(w_2)$ is a Coxeter element, written as a product of commuting simple reflections. Then $w_\circ = (w_1w_3w_2)^3$ is the longest element in W. For $\lambda \in \Lambda^+$, the element $\sigma_\lambda \in t(\lambda)W$ with the longest length (necessarily unique) has length $\ell(w_\circ) + \langle \lambda, \delta \rangle = 9 + \langle \lambda, \delta \rangle$. Using this and a bit of trial and error, one computes the reduced decompositions for words σ_λ, σ_μ for various $\mu \le \lambda$, $\mu, \lambda \in \Lambda^+$ and uses these as input for the polynomials $P_{\sigma_\lambda,\sigma_\mu}(q)$. See §4.4 for more details.

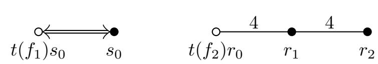
13.3 Mixed coset decompositions

Let $\mathbf{H}$ be the subgroup of $\mathbf{G}$ generated by $\mathbf{A}$ and the root groups associated with $\pm\alpha_0, \pm\alpha_2, \pm\alpha_3$. Then $\mathbf{H} \cong \mathrm{GSp}_4 \times_{\mathbb{G}_m} \mathrm{GL}_2$ where the fiber product is over the similitude $\mathrm{sim} : \mathrm{GSp}_4 \to \mathbb{G}_m$ and the determinant $\det : \mathrm{GL}_2 \to \mathbb{G}_m$. We set $H := \mathbf{H}(F)$, $U := H \cap K$, $H_1 = \mathrm{GL}_2(F)$, $H_2 := \mathrm{GSp}_4(F)$ and $\mathrm{pr}_i : H \to H_i$ denote the natural projection maps. We will informally refer to H_1, H_2 as the 'components' of $H = H_1 \times_{\mathbb{G}_m} H_2$. The set of roots of $\mathbf{H}$ is

$$\Phi_H := \{\pm\alpha_0, \pm\alpha_2, \pm\alpha_3, \pm(\alpha_2 + \alpha_3), \pm(2\alpha_2 + \alpha_3)\}$$

The Weyl group W_H of H is generated by s_0, s_2, s_3 and $W_H \cong S_2 \times \big((\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2\big)$. We denote by $N_H(A) = N_G(A) \cap H$ the normalizer of A in H, $W_{I,H} = N_H(A)/A^\circ \subset W_I$ the Iwahori Weyl group of H and $I_H = I_G \cap H$ the Iwahori subgroup of H determined by I_G. The positive simple roots are corresponding to I_H are $\Delta_H = \{\alpha_0, \alpha_2, \alpha_3\}$ and we let $\Phi_H = \Phi_H^+ \sqcup \Phi_H^-$ denote the partition into positive and negative roots. For each $\alpha \in \Phi_H$, we let $\mathbf{U}_\alpha$ denote the root group corresponding to α and for each $k \in \mathbb{Z}$, we let $U_{\alpha,k}$ denote the subgroup of elements $h \in \mathbf{U}_\alpha(F) \subset H$ such that the valuation of the non-diagonal entries of the matrix h is at least k. For example, we have $U_{\alpha_2,k} = x_2(\varpi^k \mathscr{O}_F)$.

For notational convenience in referring to the roots of GSp_4-component of $\mathbf{H}$ later on, we will denote $\beta_1 := \alpha_2$, $\beta_2 := \alpha_3$, $\beta_0 := 2\alpha_2 + \alpha_3$ and let r_0, r_1, r_2 denote the reflections associated with $\beta_0, \beta_1, \beta_2$ respectively. In this notation, the generators of the affine Weyl group $W_{\mathrm{aff},H}$ of H are $S_{\mathrm{aff},H} = \{s_0, t(f_1)s_0, r_1, r_2, t(f_2)r_0\}$ and $W_{I,H}$ is a semidirect product of $W_{\mathrm{aff},H}$ with the cyclic group $\Omega_H \subset W_I$ generated by $\omega_H := t(-f_0)s_0 r_2 r_1 r_2 \in W_{I,G}$. The action of ω_H on $S_{\mathrm{aff},H}$ is given via $s_0 \leftrightarrow t(f_1)s_0$, $r_2 \leftrightarrow t(f_2)r_0$ fixing r_1 and induces an order 2 automorphism of the extended Coxeter-Dynkin diagram



(13.3.1)

We let $\ell_H : W_{I,H} \to \mathbb{Z}$ denote the induced length function by declaring $\ell_H(\omega_H) = 0$. We denote $v_0 := w_1 w_0 w_1, v_1 := w_2, v_2 := w_3 \in N_H(A)$ the lifts of simple (affine) reflections $t(f_2)r_0, r_1, r_2$ respectively. We let $y_i : \mathbb{G}_a \to \mathbf{H}$ for $i = 0, 1, 2, 3$ denote the root group maps $y_0 := w_1 x_0 w_1$, $y_1 := x_2$ and $y_2 := x_3$. Then y_i correspond to maps denoted 'x_i' in 12.1.2 after projection to H_2. We moreover let $v_{\beta_0} := w_2 w_3 w_2 = \varpi^{f_2} v_0$ which represents $s_2 = r_1 \in W_{I,H}$, $y_{\beta_0} := w_2 x_3 w_2$, $y_{-\beta_0} := v_{\beta_0} y_{\beta_0} v_{\beta_0}$ and similarly $x_{\alpha_0} := w_1 y_{\beta_0} w_1$, $x_{-\alpha_0} := w_{\alpha_0} x_{\alpha_0} w_{\alpha_0}$. Then $x_{\pm\alpha_0}, y_{\pm\beta_0}$ are the 'standard' root group maps for $\pm\alpha_0$, $\pm\beta_0$ respectively. To be clear, we have $x_{-\alpha_0}(u\varpi) = x_0(u)$ and $y_{-\beta_0}(u\varpi) = y_0(u)$ for $u \in \mathscr{O}_F$. Finally, let

- $\tau_0 = 1_G$,

- $\tau_1 = \begin{pmatrix} 1 & & & & \frac{1}{\varpi} & \\ & 1 & & \frac{1}{\varpi} & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 1 & & & & \frac{1}{\varpi^2} & \\ & 1 & & \frac{1}{\varpi^2} & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$

Our goal now is to show that the various Hecke operators appearing in the spin Hecke polynomial of the previous subsection can be decomposed into mixed ones (with respect to U) using representatives in $A\tau_\epsilon$ for $\epsilon = 0, 1, 2$. This is executed in Proposition 13.3.12 below. We however need to do some preliminary work that will also be useful for the computations of the next subsection. Let $U\varpi^\Lambda\tau_i K$ denote the collection of all double cosets $U\varpi^\lambda\tau_i K$ for $\lambda \in \Lambda$. For $g \in G$, we denote $H_g = H_{g,K} := H \cap gKg^{-1}$. In particular, $H_{\tau_i} = H \cap \tau_i K \tau_i^{-1}$.

Remark 13.3.2. In some of the results, the reader will need to write out explicit matrices in H in order to follow some of the proofs.

Lemma 13.3.3. *$U\varpi^\Lambda K$, $U\varpi^\Lambda\tau_1 K$ and $U\varpi^\Lambda\tau_2 K$ are pairwise disjoint collections.*

Proof. Since $U\varpi^\lambda\tau_i K \subset H\tau_i K$, it suffices to show that the double cosets $H\tau_i K$ are distinct (and hence disjoint) for $i = 0, 1, 2$. Now $H\tau_i K = H\tau_j K$ for $i \neq j$ implies that $\tau_i^{-1} h \tau_j \in K$. Requiring the entries of $k := \tau_i^{-1} h \tau_j$ to be in $\mathscr{O}_F$, one easily deduces that $\det(k) \in \varpi\,\mathscr{O}_F^\times$ (e.g. the first column of k necessarily becomes a multiple of ϖ), which is a contradiction. □

We will need to know the structure of the groups H_{τ_ϵ} for $\epsilon = 1, 2$. To this end, let $\mathscr{X} := \mathrm{GL}_2(F)$, $\varepsilon_i : \mathrm{diag}(u_1, u_2) \mapsto u_i$, $i = 1, 2$ the characters of the diagonal torus and $\pm\varkappa := \pm(\varepsilon_1 - \varepsilon_2)$ be the root, $x_\pm : F \to \mathscr{X}$ denote the standard root group maps 10.1.4 and $\jmath$ the reflection matrix

$$\jmath = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \in \mathscr{X}.$$

We let $\jmath : \mathscr{X} \hookrightarrow H$ be embedding determined by $\jmath : \mathrm{diag}_2(u_1, u_2) \mapsto \mathrm{diag}(u_1, u_2, 1, u_2, u_1, u_1 u_2)$, $x_+(u) \mapsto x_{\alpha_0}(u) y_{-\beta_0}(u)$ and $x_-(u) \mapsto x_{-\alpha_0}(u) y_{\beta_0}(u)$. Then $\jmath$ is the identity map on the H_1-component of H and is embedded into the copy of GL_2 in H_2 determined by $\pm\beta_0$ after conjugation with reflection $\jmath$. It is the analogue of the map $\jmath$ defined in §12. We will henceforth consider $\mathscr{X}$ as a subgroup of H via $\jmath$. For $a, b \in \mathbb{Z}$ with $a + b \geq 0$, we let $\mathscr{X}_{a,b} \hookrightarrow \mathscr{X}$ denote the subgroup generated by $x_+(\varpi^a\,\mathscr{O}_F)$, $x_-(\varpi^b\,\mathscr{O}_F)$ and $\mathrm{diag}(\mathscr{O}_F^\times, \mathscr{O}_F^\times)$ (so e.g. $\mathscr{X}_{0,0} = \mathrm{GL}_2(\mathscr{O}_F)$).

Remark 13.3.4. One may informally think of $\varkappa_\pm$ as 'paired' or 'mixed' roots of $\mathbf{H}$ i.e. $\varkappa_\pm = \pm\{\alpha_0, -\beta_0\}$.

Definition 13.3.5. Let $\varphi, \psi_1, \psi_2 : \Phi_H \to \mathbb{Z}$ be functions given by

$$\varphi(\alpha) = \begin{cases} -1 & \text{if } \alpha \in \{\alpha_0, \beta_0\} \\ 0 & \text{if } \alpha \in \{\beta_1, \pm\beta_2, \beta_1 + \beta_2\} \\ 1 & \text{if } \alpha \in \Phi_H^- \setminus \{-\beta_2\} \end{cases} \qquad \psi_\epsilon(\alpha) = \begin{cases} 0 & \text{if } \alpha \in \Phi_H^+ \cup \{-\beta_2\} \\ \epsilon & \text{if } \alpha \in \{-\beta_1, -(\beta_1 + \beta_2)\} \\ 2\epsilon & \text{if } \alpha \in \{-\alpha_0, -\beta_0\} \end{cases}$$

for $\epsilon = 1, 2$. We let H_φ be the compact open subgroup of H generated by A° and $U_{\alpha,\varphi(\alpha)}$ for $\alpha \in \Phi_H$. For $\epsilon = 1, 2$, we let H_{ψ_ϵ} (resp. H'_{ψ_ϵ}) be the groups generated by A° (resp. $A \cap \tau_\epsilon K \tau_\epsilon^{-1}$) and $U_{\alpha,\psi_\epsilon(\alpha)}$ for $\alpha \in \Phi_H$.

We note that for $\epsilon = 1, 2$, $H'_{\psi_\epsilon}, \subset H'_{\psi_\epsilon} A^\circ = H_{\psi_\epsilon} \subset U$ and H_{ψ_ϵ} is normalized by $t_\epsilon w_{\alpha_0} t_\epsilon^{-1}$, $t_\epsilon v_{\alpha_0} t_\epsilon^{-1}$. Moreover, we have $H_\varphi \supset I_H$, whence $H_\varphi = I_H W_\varphi I_H$ where $W_\varphi \subset W_{I,H}$ is generated by a subset of $S_{H,\mathrm{aff}}$ and a power of ω_H (see §5.2.8). It is then easily verified that

$$W_\varphi = \langle t(f_1)s_0, r_2, t(f_2)r_0 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^3 \tag{13.3.6}$$

and we have a bijection $p_\varphi : W_H \backslash W_{I,H} / W_\varphi \simeq U \varpi^\Lambda H_\varphi$ (see §5.2). Denote $t_1 := \varpi^{(1,0,0,0)}$, $t_2 := \varpi^{(2,0,0,1)}$.

Lemma 13.3.7. *For $\epsilon = 1, 2$, H_{τ_ϵ} is equal to the product $t_\epsilon \mathscr{X}_{0,0} t_\epsilon^{-1} \cdot H'_{\psi_\epsilon}$ and $\mathscr{X}_{0,0} \cap t_\epsilon^{-1} H'_{\psi_1} t_\epsilon = \mathscr{X}_{\epsilon,\epsilon}$. For any $\lambda \in \Lambda$, the intersection of H_{τ_ϵ} and $\varpi^\lambda U \varpi^{-\lambda}$ is equal to the product of the intersections of $\mathscr{X}_{0,0}$ and H'_{ψ_ϵ} with $\varpi^\lambda U \varpi^{-\lambda}$. Finally, H_{τ_1} is contained in H_φ.*

Proof. All claims follow by elementary matrix computations analogous to those done in Lemma 12.3.1. □

Remark 13.3.8. Only the containments $t_\epsilon \jmath t_\epsilon^{-1} \in H_{\tau_\epsilon}$ and $H_{\tau_1} \subset H_\varphi$ are needed for this subsection.

Since U, K both contain $w_{\alpha_0}, v_1 = w_2$, $v_2 = w_3$, the natural map $p_0 : \Lambda \to U \varpi^\Lambda K$ induces a map $W_H \backslash \Lambda \to U \varpi^\Lambda K$ given by $W_H \lambda \mapsto U \varpi^\lambda K$. For $\epsilon = 1, 2$, let $\theta_\epsilon := t_\epsilon \jmath t_\epsilon^{-1} \in H_{\tau_\epsilon}$. Then θ_ϵ represents $t(\epsilon f_1) s_0 t(\epsilon f_2) r_0$ in $W_{I,H}$ and $\theta_1 U = w_0 v_0 U$. Let $\vartheta_\epsilon := \tau_\epsilon^{-1} \theta_\epsilon \tau_\epsilon \in K$. Then for $\lambda \in \Lambda$,

$$U \varpi^\lambda \tau_\epsilon K = U \varpi^\lambda \tau_\epsilon \vartheta_\epsilon K = U \varpi^\lambda \theta_\epsilon \tau_\epsilon K = U \varpi^{\theta_\epsilon(\lambda)} \tau_\epsilon K \tag{13.3.9}$$

where $\theta_\epsilon : \Lambda \to \Lambda$ denotes the involution $\lambda \mapsto \epsilon(f_1 + f_2) + s_0 r_0(\lambda)$. As $w_3 = v_2$ commutes with τ_1, we also have $U \varpi^\lambda \tau_\epsilon K = U \varpi^{s_3(\lambda)} \tau_\epsilon K$. Let $p_1 : \Lambda \to U \varpi^\Lambda \tau_1 K$ denote the natural map.

Lemma 13.3.10. *The map p_0 induces a bijection $W_H \backslash U \simeq U \varpi^\Lambda K$. The map p_1 is injective on $\mathrm{im}(\imath)$ where $\imath : W_\varphi \backslash \Lambda \to \Lambda$ is any section of the natural projection map $\Lambda \to W_\varphi \backslash \Lambda$. In other words, if $\lambda_1, \lambda_2 \in \Lambda$ are in distinct W_φ-orbits, then $U \varpi^{\lambda_1} \tau_1 K$ is distinct from $U \varpi^{\lambda_2} \tau_1 K$.*

Proof. By Lemma 5.5.2, the collections $U \varpi^\Lambda K$, $U \varpi^\Lambda \tau_1 K$ are in one to one correspondence with the double coset collections $U \varpi^\Lambda U$, $U \varpi^\Lambda H_{\tau_1}$ respectively. Now $U \varpi^\Lambda U \simeq W_H \backslash \Lambda$ by Cartan decomposition for U, whence the first claim. Since $H_{\tau_1} \subset H_\varphi$, the second claim follows since p_φ is a bijection. □

Remark 13.3.11. One may even show (using e.g. [BT72, Proposition 4.2.1]) that $\langle s_3, \gamma_\epsilon \rangle \backslash \Lambda \simeq U\varpi^\Lambda \tau_\epsilon K$. For our purposes however, Lemma 13.3.10 (together with Lemma 13.3.3) suffices to distinguish the various mixed coset representatives we are going to run into from each other.

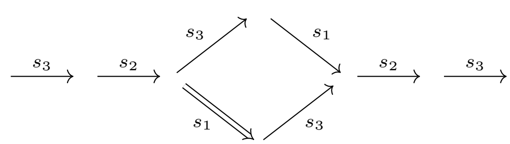
Proposition 13.3.12. *Let*

- $\mathfrak{T}(\rho) = \{\varpi^{(1,1,1,1)}, \varpi^{(1,1,1,1)}\tau_1\}$,
- $\mathfrak{T}(w_0\rho^2) = \{\varpi^{(2,2,1,1)}, \varpi^{(2,2,1,1)}\tau_1, \varpi^{(2,1,2,1)}\}$,
- $\mathfrak{T}(w_0w_1w_0\rho^2K) = \{\varpi^{(2,2,2,1)}, \varpi^{(2,2,2,1)}\tau_1, \varpi^{(2,2,2,1)}\tau_2, \varpi^{(2,1,2,2)}, \varpi^{(2,1,2,2)}\tau_1\}$,
- $\mathfrak{T}(\upsilon_1\rho^4) = \{\varpi^{(4,4,2,2)}, \varpi^{(4,4,2,2)}\tau_1, \varpi^{(4,4,2,2)}\tau_2, \varpi^{(4,2,4,2)}, \varpi^{(4,2,4,2)}\tau_1\}$,
- $\mathfrak{T}(\upsilon_2\rho^4) = \{\varpi^{(4,3,3,3)}, \varpi^{(4,3,3,3)}\tau_1, \varpi^{(4,3,3,3)}\tau_2\}$
- $\mathfrak{T}(w_0w_1w_2w_3\rho^3) = \left\{ \begin{matrix} \varpi^{(3,3,2,2)}, & \varpi^{(3,3,2,2)}\tau_1, \ \varpi^{(3,3,2,2)}\tau_2, \ \varpi^{(3,2,3,2)}, \ \varpi^{(3,2,3,2)}\tau_1, \\ \varpi^{(3,3,1,2)}\tau_1, & \varpi^{(3,2,2,3)}\tau_1 \end{matrix} \right\}$.

Then for all $w \in \{\rho, w_0\rho^2, w_0w_1w_0\rho^2, \upsilon_1\rho^4, \upsilon_2\rho^4, w_0w_1w_2w_3\rho^3\}$, *we have* $\mathrm{ch}(KwK) = \sum_{\sigma \in \mathfrak{T}(w)} \mathrm{ch}(U\sigma K)$.

Proof. For each of the words w, we will draw the Weyl orbit diagram of the anti-dominant cocharacter λ_w associated with w (see the conventions of §5.4). In these diagrams, we pick the first vertex and the vertices that only have one incoming arrow labelled s_1. Corresponding to each such node, there is a 'Schubert cell' and we study the action of U on the representatives contained in the cell and show that all of them have in their orbit one of the claimed representatives. The totality of these will give the set of representatives for mixed Hecke operators in each case. That the claimed representatives form distinct double cosets follows from Lemma 13.3.3, which distinguishes representatives with different τ_ϵ and Lemma 13.3.10, which distinguishes representatives that have τ_0 (resp. τ_1) among themselves since none of the cocharacters appearing in the representatives in each of these lists falls in the same W_H (resp. W_φ) orbit.

- $w = \rho$. The Weyl orbit diagram of $\lambda_w = (1,0,0,0)$ is below.



Thus there are two cells of interests, one of length 0 and one of length 3 with an incoming s_1 arrow. The first corresponds to ρ which obviously gives the representative $\varpi^{(1,1,1,1)}$ after conjugating by $w_{\alpha_0}, w_2, w_3 \in U$.

The other corresponds to $\sigma = w_1 w_2 w_3 \rho$. We have

$$\operatorname{im}(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi & a & c & z & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & -a & \varpi & \\ & & & -c & & \varpi \end{pmatrix} K \;\middle|\; a, c, z \in [\not{k}] \right\}$$

We can eliminate z via a row operation and then conjugate by reflections w_3, $v_2 = w_2 w_3 w_2$ to make the diagonal $\varpi^{(1,1,1,1)}$ which puts the entries a, c in the top right 3×3 block. Conjugation by w_1 switches a, c and one execute Euclidean algorithm (using row/column operations) to make one of these two entries zero. Conjugating by an appropriate element of A° if necessary, we get $\varpi^{(1,1,1,1)}$ $\varpi^{(1,1,1,1)}\tau_1$ as representatives from this cell.

• $w = w_0\rho^2$. The Weyl orbit diagram of $\lambda_w = (2, 0, 1, 1)$ is

$$\xRightarrow{s_1} \quad \xrightarrow{s_2} \quad \xrightarrow{s_3} \quad \xrightarrow{s_2} \quad \xRightarrow{s_1}$$

There are three cells of interests with length 1, 2, 6. We claim that

(i) length 1 cell gives $\varpi^{(2,2,1,1)}$,

(ii) length 2 cell gives $\varpi^{(2,1,2,1)}$, $\varpi^{(2,2,1,1)}\tau_1$,

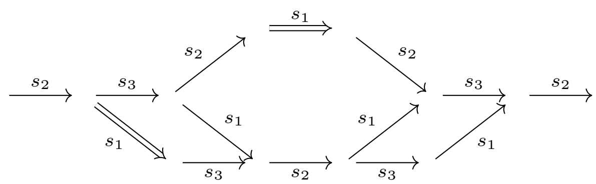
(iii) length 6 cells gives us $\varpi^{(2,2,1,1)}$, $\varpi^{(2,2,1,1)}\tau_1$.

We verify case (iii) which corresponds to $\sigma = w_1 w_2 w_3 w_2 w_1 w$. We have

$$\operatorname{im}(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi^2 & a_1\varpi & c_1\varpi & z + \varpi x & a\,\varpi & c\,\varpi \\ & \varpi & & a & & \\ & & \varpi & c & & \\ & & & 1 & & \\ & & & -a_1 & \varpi & \\ & & & -c_1 & & \varpi \end{pmatrix} K \;\middle|\; \begin{matrix} a, a_1, c, c_1, \\ x, z \in [\not{k}] \end{matrix} \right\}$$

We can eliminate the entry $z + \varpi x$ using a row operation. Conjugation by $w_3 \in U$ (resp. $w_2 w_3 w_2 \in U$) switches a_1, a (resp. c_1, c) and keeps the diagonal $\varpi^{(2,2,1,1)}$. Using row/column operations, we may apply Euclidean algorithm to make one a, a_1 (resp. c, c_1) zero while still keeping the diagonal $\varpi^{(2,2,1,1)}$. Without loss of generality, we may assume a_1, c_1 are zero. Conjugation by $w_2 \in U$ switches a, c and we may again apply Euclidean algorithm to make one of a, c zero. Normalizing by an appropriate entry of A°, we get the representative $\varpi^{(2,2,1,1)}$, $\varpi^{(2,2,1,1)}\tau_1$.

• Let $w = w_0 w_1 w_0 \rho^2$. The Weyl orbit diagram of $\lambda_w = (2, 0, 0, 1)$ is



We need to analyze three Schubert cells of length 3, 5 and 7. We claim that

(i) length 3 cell gives $\varpi^{(2,2,2,1)}$, $\varpi^{(2,2,2,1)}\tau_1$,

(ii) length 5 cell gives $\varpi^{(2,1,2,2)}$, $\varpi^{(2,1,2,2)}\tau_1$,

(iii) length 7 cell gives $\varpi^{(2,2,2,1)}, \varpi^{(2,2,2,1)}\tau_1, \varpi^{(2,2,2,1)}\tau_2$.

We verify case (iii) which corresponds to $\sigma = w_1 w_2 w_3 w_2 w$. We have

$$\mathrm{im}(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi^2 & a_1 + a\,\varpi & c_1\,\varpi & z + \varpi\,x & & c\,\varpi \\ & 1 & & & & \\ & & \varpi & c & & \\ & & & 1 & & \\ & \varpi\,x_1 & & -(a_1 + a\,\varpi) & \varpi^2 & \\ & & & -c_1 & & \varpi \end{pmatrix} K \;\middle|\; \begin{matrix} a, a_1, c, c_1, \\ x, z \in [\bar{k}] \end{matrix} \right\}$$

We can eliminate $z + \varpi x$, ϖx_1 using row operations. Conjugation by $w_3 \in U$ switches c_1, c while keeping the diagonal $\varpi^{(2,2,0,1)}$ and we can apply Euclidean algorithm to make one of these zero, say $c_1 = 0$. Conjugating by $v_2 = w_2 w_3 w_2 \in U$, we see that any element of $\mathrm{im}(\mathcal{X}_\sigma)$ falls in the U-orbit of

$$\begin{pmatrix} \varpi^2 & & & & a_1 + a\,\varpi & c\,\varpi \\ & \varpi^2 & & a_1 + a\,\varpi & & \\ & & \varpi & c & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \varpi \end{pmatrix}$$

for some $a, a_1, c \in [\bar{k}]$. We now divide in two case. Suppose first that c is zero. Then $\mathrm{ord}(a_1 + a\varpi) \in \{0, 1, \infty\}$ and we can normalize by conjugating with an element of A° to obtain representatives $\varpi^{(2,2,2,1)}$, $\varpi^{(2,2,2,1)}\tau_1$, $\varpi^{(2,2,2,1)}\tau_2$. In the second case, if $c \neq 0$, then we may assume $a = 0$ by row-column operations. If now $a_1 \neq 0$, we may make $c = 0$ and normalizing by A° gives us $\varpi^{(2,2,2,1)}\tau_1$. If $a_1 = 0$ however, then conjugating by w_2 and normalizing by A° gives us $\varpi^{(2,2,1,2)}\tau_1$. Now $U\varpi^{(2,2,1,2)}\tau_1 K = U\varpi^{(2,1,2,2)}\tau_1 K$ as $\vartheta_1 \in K$, whence the claimed representatives.

• $w = v_1\rho^4$. The Weyl orbit diagram for $\lambda_w = (4, 0, 2, 2)$ is the same as for $(2, 0, 1, 1)$. We need to analyze the images of Schubert cells of length 7, 8 and 12. We claim that

(i) length 7 cell gives $\varpi^{(4,4,2,2)}$, $\varpi^{(4,4,2,2)}\tau_1$,

(ii) length 8 cell gives $\varpi^{(4,2,4,2)}$, $\varpi^{(4,2,4,2)}\tau_1$, $\varpi^{(4,4,2,2)}\tau_2$

(iii) length 12 element gives $\varpi^{(4,4,2,2)}$, $\varpi^{(4,4,2,2)}\tau_1$,

We verify case (ii) which corresponds to $\sigma = w_1 w$. We have

$$\mathrm{im}(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi^2 & a_2 + a\,\varpi & & & & \\ & 1 & & & & \\ & c\,\varpi & \varpi^2 & & & \\ & a_1\,\varpi & & \varpi^2 & & \\ a_1\,\varpi^3 & y\,\varpi & c_1\,\varpi^3 & -\varpi^2(a_2 + a\,\varpi) & \varpi^4 & -c\,\varpi^3 \\ & c_1\,\varpi & & & & \varpi^2 \end{pmatrix} K \;\middle|\; \begin{array}{c} y = x_1 + z\,\varpi + x\,\varpi^2 \\ \text{and} \\ a, a_1, a_2, c, c_1, x, x_1, z \\ \in [\Bbbk] \end{array} \right\}$$

We can eliminate the entry $\varpi(x_1 + z\varpi + x\varpi^2)$ and entries involving c, c_1 using row operations. If $a_1 = 0$, then conjugating $w_2w_3w_2$ and normalizing by an appropriate element of A°, we obtain the representatives $\varpi^{(4,2,4,2)}$, $\varpi^{(4,2,4,2)}\tau_1$, $\varpi^{(4,2,4,2)}\tau_2$ depending on the valuation of $a_2 + a\varpi$. If however $a_1 \neq 0$, then a can be made zero via row-column operations. We then have two subcases. If $a_2 = 0$, then we can conjugate by s_{α_0} and normalize by A° to obtain $\varpi^{(4,2,4,2)}\tau_1$. On the other hand, if $a_2 \neq 0$, then a_1 can be made zero so that normalizing by A° gives $\varpi^{(4,2,4,2)}\tau_2$. Now $U\varpi^{(4,2,4,2)}\tau_2 K = U\varpi^{(4,4,2,2)}\tau_2 K$, whence we only get the claimed representatives.

• $w = \upsilon_2\rho^4$. The Weyl orbit diagram for $\lambda_w = (4,1,1,1)$ is the same as for $(1,0,0,0)$ and so we have to analyze cells of length 6 and 9. We claim that

(i) length 6 cell gives $\varpi^{(4,3,3,3)}$, $\varpi^{(4,3,3,3)}\tau_1$,

(ii) length 9 cell gives $\varpi^{(4,3,3,3)}$, $\varpi^{(4,3,3,3)}\tau_1$, $\varpi^{(4,3,3,3)}\tau_2$,

We verify case (ii) which corresponds to $\sigma = w_1w_2w_3w_2w_1w$. We have

$$\mathrm{im}(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi^2 & a_2 + c\,\varpi & c_1 + a\,\varpi & z + x\,\varpi & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & x_2\,\varpi & a_1\varpi & -(a_2 + c\,\varpi) & \varpi^2 & \\ & a_1\varpi & x_1\,\varpi & -(c_1 + a\,\varpi) & & \varpi^2 \end{pmatrix} \rho^2 K \;\middle|\; \begin{array}{c} a, a_1, a_2, c, \\ c_1, x, x_1, z \\ \in [\Bbbk] \end{array} \right\}$$

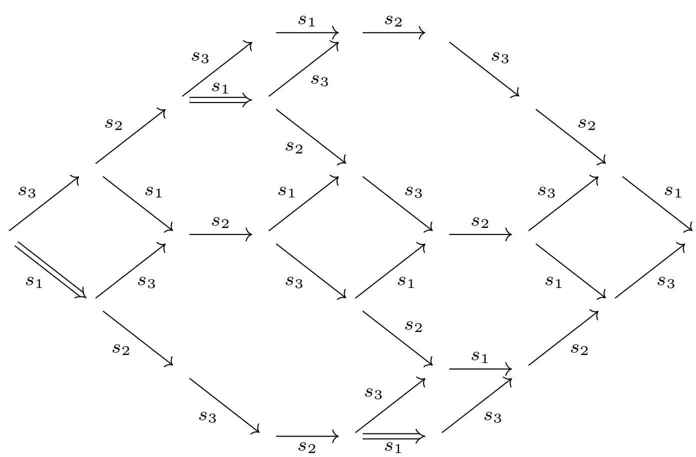
We can eliminate the entries involving a_1, x, x_1, x_2, z using row operations. Conjugating by w_3 and $w_2w_3w_2$

gives us

$$\begin{pmatrix} \varpi^2 & & & & a_2 + c\,\varpi & c_1 + a\,\varpi \\ & \varpi^2 & & a_2 + c\,\varpi & & \\ & & \varpi^2 & c_1 + a\,\varpi & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix} \rho^2$$

and one can apply Euclidean algorithm to the entries $c_1 + a\varpi$, $a_2 + c\varpi$ to replace one of them with 0 and the other by the greatest common divisor which is either $0, 1$ or ϖ. Conjugating by w_2 and normalizing by A° if necessary, we obtain the three representatives.

- $w = w_0 w_1 w_2 w_3 \rho^3$. The Weyl orbit diagram of $\lambda_w = (3, 0, 1, 1)$ is



There are four cells to analyze, corresponding to words of length 4, 5, 7 and 9 determined by the bold arrows labelled s_1 in the diagram above. We claim that

(i) length 4 cell gives $\varpi^{(3,3,2,2)}$, $\varpi^{(3,3,2,2)}\tau_1$,

(ii) length 5 cell gives $\varpi^{(3,2,3,2)}$, $\varpi^{(3,2,3,2)}\tau_1$,

(iii) length 7 cell gives $\varpi^{(3,2,3,2)}$, $\varpi^{(3,2,3,2)}\tau_1$, $\varpi^{(3,3,2,2)}\tau_2$, $\varpi^{(3,2,2,3)}\tau_1$,

(iv) length 9 cell gives $\varpi^{(3,3,2,2)}$, $\varpi^{(3,3,2,2)}\tau_1$, $\varpi^{(3,3,2,2)}\tau_2$, $\varpi^{(3,3,1,2)}\tau_1$.

We verify claim (iv) which corresponds to $\sigma := w_1 w_2 w_3 w_2 w_1 w$. Then,

$$\mathrm{im}(\mathcal{X}_\sigma) = \left\{ \begin{pmatrix} \varpi^3 & a\,\varpi^2 + a_2\,\varpi & c\,\varpi^2 + c_2\,\varpi & y & a_1\,\varpi^2 & c_1\,\varpi^2 \\ & \varpi & & a_1 & & \\ & & \varpi & c_1 & & \\ & & & 1 & & \\ & & & -(a_2 + a\,\varpi) & \varpi^2 & \\ & & & -(c_2 + c\,\varpi) & & \varpi^2 \end{pmatrix} K \;\middle|\; \begin{matrix} y = z_1 + x\varpi + z\varpi^2 \\ \text{and} \\ a, a_1, a_2, c, c_1, c_2, x \\ z, z_1 \in [\Bbbk] \end{matrix} \right\}$$

We can eliminate the entry y by a row operation. Now note that if a_1 (resp. c_1) is not zero, then we can assume a (resp. c) is zero by row column operations. Moreover, conjugation by w_2 switches the places of a, a_1, a_2 by c, c_1, c_2 respectively and keeps the diagonal $\varpi^{(3,3,1,1)}$. We have three cases to discuss.

Case 1. Suppose $a_1 = c_1 = 0$. We can now apply row column operations to replace $a\varpi^2 + a_2\varpi$, $c\varpi^2 + c_2\varpi$ by their greatest common divisor (with the other entry being zero). We may normalize the gcd by an element of A° so that the greatest common divisor is $0, \varpi$ or ϖ^2. Then conjugating by $w_2w_3w_2$ and $w_3 \in H$ makes the diagonal $\varpi^{(3,3,2,2)}$. Conjugating by w_2 if necessary, we obtain the representatives $\varpi^{(3,3,2,2)}\tau_1$, $\varpi^{(3,3,2,2)}\tau_2$.

Case 2. Suppose exactly one of a_1, c_1 is non-zero. Without loss of generality, we may assume $a_1 \neq 0$, $c_1 = 0$. Then we may assume $a = 0$. Assume first that $a_2 \neq 0$. We may then assume a_1, c, c_2 are zero. Conjugating by $w_2w_3w_2$, w_3 and normalizing by A° gives us $\varpi^{(3,3,2,2)}\tau_2$. If however $a_2 = 0$, then we can conjugate by w_3 to make the diagonal $\varpi^{(3,3,1,2)}$ while moving the $c\varpi^2 + c_2\varpi$ entry corresponding to the root group of $e_1 + e_3 - e_0$. As $a_1 \neq 0$, we may make $c_1 = 0$ using row and column operations. There are now two further sub-cases. If $c_2 = 0$, we obtain the representative $\varpi^{(3,3,1,2)}\tau_1$ after normalizing by an element of A°. If however $c_2 \neq 0$, we can make $a_1 = 0$. Conjugating by w_2, w_3 and normalizing by A° gives us $\varpi^{(3,3,2,2)}\tau_2$.

Case 3. Suppose both a_1, c_1 are non-zero. Then we may assume a, c are zero. If a_2 (resp. c_2) is not zero, we can eliminate entries containing a_1 (resp. c_1). Then an argument similar to Case 2 yields $\varpi^{(3,3,2,2)}\tau_2$, $\varpi^{(3,3,1,2)}\tau_1$ as representatives. □

13.4 Schwartz space computations

The zeta element we are going to compute in the next subsection will have as its source bottom class (see §3.1 for terminology) an element in a Schwartz space. In this subsection, we compute the action of various Hecke operators of H constructed out of the representatives in Lemma 13.3.12 on this bottom class. We will record this action only modulo $q - 1$, since the zeta element we construct in Theorem 13.5.1 is uniform (Corollary 3.3.2). Doing so also simplifies our expressions considerably which otherwise have cumbersome powers of q appearing in them.

Let $X := F^4$ considered as a topological vector space over F. Elements of X are to be seen as row vectors. Recall that we denote

$$H_1 = \mathrm{GL}_2(F), \quad H_2 = \mathrm{GSp}_4(F)$$

and $\mathrm{pr}_i : H = H_1 \times_{\mathbb{G}_m} H_2 \to H_i$ the projection maps. We moreover set $U_i := \mathrm{pr}_i(U)$. We have a continuous right-action $X \times H_2 \to X$ induced by right matrix multiplication. and via pr_2, a continuous right action

$X \times H \to X$ induced by right matrix multiplication.

Let $\mathcal{O}$ be an integral domain containing $\mathbb{Z}[q^{-1}]$ and $\mathcal{I}$ denote the ideal generated by $q-1$. Let $\mathcal{S} = \mathcal{S}_{\mathcal{O}}(X)$ be the space of all functions $\xi : X \to \mathcal{O}$ which are locally constant, compactly supported on X. If $\xi \in \mathcal{S}$, we let $[\xi] : X \to \mathcal{O}/\mathcal{I}$ the reduction of ξ modulo $\mathcal{I}$ and for $\xi_1, \xi_2 \in \mathcal{S}$, we write $\xi_1 \equiv \xi_2$ if $[\xi_1] = [\xi_2]$. We let $h \in H$ acts on $\xi \in S$ via $(h, \xi) \mapsto \xi((-) \cdot h)$. Then $\mathcal{S}$ is a smooth (left) H-representation. Let Υ_H denote the collection of all compact open subgroups of H and let $M_{H,\mathcal{O}} : \Upsilon_H \to \mathcal{O}\text{-Mod}$, $V \mapsto \mathcal{S}^V$ be the associated CoMack functor (Definition 2.2.1). For $m \in \mathbb{Z}$, we let $Y_m := \varpi^m \mathscr{O}_F$ and for $u, v, w, x \in \mathbb{Z}$, let

$$Y_{u,v,w,x} = Y_u \times Y_v \times Y_w \times Y_x \subset X.$$

We denote $\phi_{(u,v,w,x)} := \mathrm{ch}(Y_{u,v,w,x})$ the characteristic function of $Y_{u,v,w,x} \subset X$. For $m \in \mathbb{Z}$, we denote $Y_{(m)} := Y_{m,m,m,m}$ and $\phi_{(m)} := \mathrm{ch}(Y_{(m)})$, $\phi^{\times}_{(m)} := \phi_{(m)} - \phi_{(m+1)}$. If $m = 0$, we will omit the subscript and denote $\phi_{(0)}$ by just ϕ. The element $\phi = \phi_{(0)} \in M_{H,\mathcal{O}}(U)$ will serve as the bottom class of our zeta element[2].

In order to compute the action of Hecke operators of H on ϕ, we will need to decompose H-double cosets. For this purpose, we fix some notations/conventions first. We set

$$\rho_1 := \begin{pmatrix} & 1 \\ \varpi & \end{pmatrix} \in H_1 \qquad \rho_2 := \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \varpi & & \\ \varpi & & & \end{pmatrix} \in H_2, \qquad \rho_H = (\rho_1, \rho_2) \in H.$$

Then $\rho_H \in N_H(A)$ is a lift of $\omega_H \in W_{I,H}$. For $\lambda = (a_0, a_1, a_2, a_3) \in \Lambda$, we let $\lambda_1 := a_0 f_0 + a_1 f_1$, $\lambda_2 := a_0 f_1 + a_2 f_2 + a_3 f_3$. We denote by $\varpi^{(a_0,a_1)} \in H_1$, $\varpi^{(a_0,a_1,a_2)} \in H_2$ the elements $\mathrm{pr}_1(\varpi^{\lambda_1})$, $\mathrm{pr}_2(\varpi^{\lambda_2})$ respectively. Since the action of H on X factors via the action $X \times (H_1 \times H_2)$, we will able to compute the action 'decomposable' operators of H componentwise. More precisely, let $V_i \subset H_i$ be compact open subgroups, and $V := V_1 \times_{\mathbb{G}_m} V_2 \subset H$. Let $\lambda \in \Lambda$ be such that $V\varpi^{\lambda}V/V = \bigsqcup_{\gamma\delta} (\gamma, \delta)V$ where $\gamma \in H_1$, $\delta \in H_2$ are such that $V_1\varpi^{\lambda_1}V_1 = \bigsqcup \gamma V_1$, $V_2\varpi^{\lambda_2}V_2/V_2 = \bigsqcup_{\delta} \delta V_2$. Then

$$[U\varpi^{\lambda}V]_*(\phi) = \deg\left([U_1\varpi^{\lambda_1}V_1]_*\right) \cdot [U_2\varpi^{\lambda_2}V_2]_*(\phi) \tag{13.4.1}$$

where $[U_2\varpi^{\lambda_2}V_2]_*(\phi)$ is the function $\sum_{\delta} \delta \cdot \phi$ for $\gamma \in H_2$ running over *any* set of representatives of $V_2\varpi^{-\lambda_2}U_2/U_2$, even those for which $(1, \delta) \notin H$. We will use this observation for groups V_1, V_2 such that $\mathrm{pr}_i(A^{\circ}) \subset V_i$ and therefore freely use reprsentatives γ whose similitude in $\mathscr{O}_F^{\times}$ is not necessarily 1.

13.4.1 Action of $[U\varpi^{\lambda}H_{\tau_0}]_*$

We first compute the action of Hecke operators in which V appearing in (13.4.1) is $H_{\tau_0} = U$. These computations will correspond to verification for the identity twist of our zeta element in Theorem 13.5.1.

[2] A remark similar to 12.5.2 applies here

Since the degree of $U_1\varpi^{\lambda_1}U_1$ is, modulo $q-1$, 1 or 2 depending on whether ϖ^{λ_1} is central in H_1 or not, it suffices to restrict attention to the second component. In what follows, we will think of v_0, v_1, v_2 as elements H_2 and similarly, consider y_0, y_1, y_2 as maps from $\mathbb{G}_a$ to H_2. Let W_{I,H_2} denote the Iwahori Weyl group of H_2. For $w \in W_{I,H_2}$, we denote $\mathcal{Y}_w : [\not{k}]^{\ell_2(w)} \to H_2/U_2$ denote the map denoted '$\mathcal{X}_w$' of 12.1.3 in the conventions of this section (so we will now write 12.1.3 using r_i (resp. y_i) in place of 'x_i' (resp. 's_i') of §12).

Remark 13.4.2. In the computations that follow, the degree of the operators, modulo $q-1$, can be read off form the value of the function recorded at $(0,0,0,0) \in X$ e.g. $\deg([U_2\varpi^{(1,1,1,1)}U_2]) \equiv 4$. This is also the number of permutations under W_{H_2} of the cocharacter displayed.

Lemma 13.4.3. *We have*

(a) $[U_2\varpi^{(1,1,1)}U_2]_*(\phi) \equiv 2\phi + 2\phi_{(1)}$,

(b) $[U_2\varpi^{(2,2,1)}U_2]_*(\phi) \equiv \phi + 2\phi_{(1)} + \phi_{(2)}$,

(c) $[U_2\varpi^{(2,2,2)}U_2]_*(\phi) \equiv 2\phi + 2\phi_{(2)}$,

(d) $[U_2\varpi^{(3,3,2)}U_2]_*(\phi) \equiv 2\phi + 2\phi_{(1)} + 2\phi_{(2)} + 2\phi_{(3)}$,

(e) $[U_2\varpi^{(4,4,2)}U_2]_*(\phi) \equiv \phi + 2\phi_{(2)} + \phi_{(4)}$.

For each $\lambda = (a_0, a_2, a_3) \in \Lambda_2 := \mathbb{Z}f_0 \oplus \mathbb{Z}f_2 \oplus \mathbb{Z}f_3$ below, we denote $\xi = \xi_\lambda := [U_2\varpi^\lambda U_2]_*(\phi)$ the function to be computed. Then $\xi(\vec{v})$ for $\vec{v} = (a,b,c,d) \in X$ counts the number of distinct $\gamma U_2 \subset U_2\varpi^{-\lambda}U_2/U_2$ such that $\vec{v}\cdot\gamma \in \mathscr{O}_F^4$. This number will depend on the valuations of the coordinates of $\vec{v}$. We compute these numbers by applying what we call the argument of *pivots and free variables* which we briefly outline. The representatives matrices γ that we pick are, as per the recipe of 5.2.8, parametrized by Schubert cells, which in turn are parametrized by certain number of variables in $[\not{k}]$ that serve as the entries of these matrices in each cell. For each cell, we will have at least one column that has no variable entry. The coordinates of $\vec{v}$ corresponding to those columns then become the 'pivots' i.e. all other coordinates in $\vec{v}\cdot\gamma$ will appear as linear combinations of the pivots. This will allow us to deduce that $\mathrm{supp}(\xi) \subset \mathscr{O}_F^4$. We will then consider for each $m = 0, \ldots, a_0 - 1$, the restriction of ξ on the set

$$\varpi^m\,\mathscr{O}_F^\times \times Y_m \times Y_m \times Y_m.$$

For each m, we will be able to freely pick certain variable entries (the 'free variables') in the matrices γ such that once the free variables are fixed in $[\not{k}]$, the condition $\vec{v}\cdot\gamma \in \mathscr{O}_F^4$ will be true for a unique such γ i.e. the other variables are uniquely determined in terms of free variables and the value of ξ is thus the number of free variables. Since ξ is U_2-invariant and all lifts of W_{H_2} are in U_2, we know that e.g. $\xi(a,b,c,d) = v_1\cdot\xi(a,b,c,d) = \xi(b,a,d,c)$. Thus if ξ turns out to be constant on $\varpi^m\,\mathscr{O}_F^\times \times Y_m \times Y_m \times Y_m$

(which it will), it will be constant on $Y_{(m)} \setminus Y_{(m+1)}$ and we will therefore be able to write ξ (uniquely) as a linear combination of $\phi^{\times}_{(m)}$ for $m = 0, 1, \ldots, a_0 - 1$ and $\phi_{(a_0)}$. This argument is illustrated in detail in parts (a), (b) where we explicitly list all of the vectors and then employed somewhat abstractly in part (d) where there are 8 Schubert cells to work with. We leave parts (c) and (e) for the reader to verify.

Proof. (a) Let $\lambda = (1,1,1)$. We have $U_2\varpi^{-\lambda}U_2 = U_2\rho_2^{-1}U_2$. The Weyl orbit diagram of $\lambda_{\rho_2^{-1}} = -\lambda$ is

$$-(1,1,1) \xrightarrow{r_2} -(1,1,0) \xrightarrow{r_1} -(1,0,1) \xrightarrow{r_2} (-1,0,0)$$

Proposition 5.2.8 therefore implies that $U_2\rho_2^{-1}U_2 = \bigsqcup \operatorname{im}(\mathcal{Y}_\sigma)$ for $\sigma \in \{\rho_2^{-1}, v_2\rho_2^{-1}, v_1v_2\rho_2^{-1}, v_1v_2v_1\rho_2^{-1}\}$. Using the explicit decomposition of $U_2\rho_2^{-1}U_2$ (cf. 12.3.2(a)), we see that $\xi(a,b,c,d)$ for $a,b,c,d \in F^4$ is the number of integral vectors in the following four lists

$$\left(\frac{a}{\varpi}, \frac{b}{\varpi}, c, d\right), \quad \left(\frac{a}{\varpi}, b, c, \frac{d+bi}{\varpi}\right)_{i\in[\mathscr{k}]}, \quad \left(a, \frac{b+aj_2}{\varpi}, \frac{c-dj_2+aj_1}{\varpi}, d\right)_{j_1,j_2\in[\mathscr{k}]}$$
$$\left(a, b, \frac{c-ak_1+bk_2}{\varpi}, \frac{d+ak_2+bk_3}{\varpi}\right)_{k_1,k_2,k_3\in[\mathscr{k}]}$$

where repetitions are allowed e.g. if a,b,c,d are all 0, we count $1+q+q^2+q^3$. Clearly, all of a,b,c,d need to be integral for any of these vectors to be integral and thus $\operatorname{Supp}(\xi) \subset \mathscr{O}_F^4$. So assume that $a,b,c,d \in \mathscr{O}_F$. Now observe that there are exactly $q+1$ such vectors in each of the following cases: $a \in \mathscr{O}_F^{\times}$[3] or $a \in \varpi\,\mathscr{O}_F$, $b \in \mathscr{O}_F^{\times}$, or $a,b \in \varpi\,\mathscr{O}_F$, $d \in \mathscr{O}_F^{\times}$ or $a,b,d \in \varpi\,\mathscr{O}_F$, $c \in \mathscr{O}_F^{\times}$. Thus, if at least one of $a,b,c,d \in \mathscr{O}_F$ is a unit, $\xi(a,b,c,d) = q+1$. Moreover, $\xi(a,b,c,d) = 1+q+q^2+q^3$ if $a,b,c,d \in \varpi\,\mathscr{O}_F$. We therefore see that

$$\xi = (q+1)\phi^{\times}_{(0)} + (q^3+q^2+q+1)\phi_{(1)}.$$

Since $\phi^{\times}_{(0)} + \phi_{(1)} = \phi$, we see that $\xi = (q+1)\cdot\phi + (q^3+q^2)\cdot\phi_{(1)}$ from which the claim easily follows.

(b) Let $\lambda = (2,2,1)$. We have $U_2\varpi^{-\lambda}U_2 = U_2vU_2$ where $v = v_0\rho_2^{-1}$. From Weyl orbit diagram of λ_{v_0} (see 12.2.2) and Proposition 5.2.8, we see that $U_2vU_2 = \bigsqcup_\sigma \operatorname{im}(\mathcal{Y}_\sigma)$ for $\sigma \in \{v, v_1v, v_2v_1, v_1v_2v_1v\}$. Using the decomposition of U_2vU_2 (cf. 12.3.2(b)), we see that $\xi(a,b,c,d)$ for $a,b,c,d \in F^4$ is the number of integral vectors among the four lists

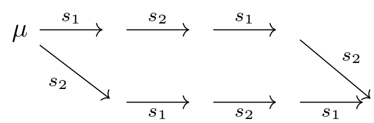
$$\left(\frac{a+ci\varpi}{\varpi^2}, \frac{b}{\varpi}, c, \frac{d}{\varpi}\right)_{i\in[\mathscr{k}]}$$
$$\left(\frac{a}{\varpi}, \frac{b+ai_1+d\,i_2\varpi}{\varpi^2}, \frac{c-di_1}{\varpi}, d\right)_{i_1,i_2\in[\mathscr{k}]}$$
$$\left(\frac{a}{\varpi}, b, \frac{c+bj_1}{\varpi}, \frac{d+aj_1+b(j_2+j_3\varpi)}{\varpi^2}\right)_{j_1,j_2,j_3\in[\mathscr{k}]}$$
$$\left(a, \frac{b+a\,k_1}{\varpi}, \frac{c+bk_2-dk_3+a(k_4+k_4\varpi)}{\varpi^2}, \frac{d+ak_2}{\varpi}\right)_{k_1,\ldots,k_5\in[\mathscr{k}]}$$

[3] k_3 is a free variable in this case: for any given $k_3 \in [\mathscr{k}]$, there is a unique $k_2 \in [\mathscr{k}]$ and thus a unique $k_1 \in [\mathscr{k}]$ for which rightmost vector is integral

of a total of $q+q^2+q^3+q^4$ vectors where as above, repetitions are allowed. Applying an argument similar to part (a), we see that $\xi = \phi^\times_{(0)} + (q+q^2+q^3)\phi^\times_{(1)} + (q+q^2+q^3+q^4)\phi_{(2)}$, from which (b) easily follows.

(c) Let $\lambda = (2,2,2)$. We have $U_2\varpi^{-\lambda}U_2 = U_2vU_2$ where $v = v_0v_1v_0\rho_H^{-2}$. The Weyl orbit diagram of $\lambda_v = -\lambda$ is the same as in part (a) and we see that $U_2vU_2 = \bigsqcup_\sigma \operatorname{im}(\mathcal{Y}_\sigma)$ for $\sigma \in \{v, v_2v, v_1v_2v, v_2v_1v_2v\}$. We leave the rest of the rest of calculation to the reader to deduce that $\xi = (q+q^2)\left(\phi^\times_{(0)} + q^2\phi^\times_{(1)} + (q^2+q^4)\phi^\times_{(2)}\right)$.

(d) Let $\lambda = (3,3,2)$ and $\mu := (3,0,1) = (6,3,3) - \lambda$. Then $U\varpi^\mu U_2 = U_2v_0v_1v_2\rho_2^3U_2$. It's Weyl orbit diagram is as follows.



Using Proposition 5.2.8, one can write the following 'skeletons' (or Schubert cells) of matrices that will serve as representatives of $U_2\varpi^\mu U_2$ where $\mu \in W_H\lambda$ is a permutation of λ.

$$
\begin{pmatrix} 1 & & & \\ & \varpi & & \\ \square & * & \varpi^3 & \\ \hat{*} & & & \varpi^2 \end{pmatrix}
\begin{pmatrix} \varpi & \circ & & \\ & 1 & & \\ & * & \varpi^2 & \\ \hat{*} & \diamond & -\hat{\circ} & \varpi^3 \end{pmatrix}
\begin{pmatrix} \varpi & & & * \\ \circ & \varpi^3 & \hat{*} & \diamond \\ & & \varpi^2 & -\hat{\circ} \\ & & & 1 \end{pmatrix}
\begin{pmatrix} \varpi^3 & \circ & \square & * \\ & \varpi & \hat{*} & \\ & & 1 & \\ & & -\hat{\circ} & \varpi^2 \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & & & \\ \circ & \varpi^2 & & \diamond \\ \square & & \varpi^3 & -\hat{\circ} \\ & & & \varpi \end{pmatrix}
\begin{pmatrix} \varpi^2 & \circ & \square & \\ & 1 & & \\ & & \varpi & \\ & \diamond & -\hat{\circ} & \varpi^3 \end{pmatrix}
\begin{pmatrix} \varpi^2 & & \square & * \\ & \varpi^3 & \hat{*} & \diamond \\ & & \varpi & \\ & & & 1 \end{pmatrix}
\begin{pmatrix} \varpi^3 & \circ & \square & * \\ & \varpi^2 & \bar{*} & \diamond \\ & & 1 & \\ & & -\circ & \varpi \end{pmatrix}
$$

Here, matrices in the top (resp. bottom) row correspond to the top (resp. bottom) row of the Weyl orbit diagram. The various symbols are to be replaced with entries of $\mathscr{O}_F$ by a recipe that we describe in a moment. In these 'skeletons', the symbols $\circ$, $\hat{\circ}$ and $*$, $\hat{*}$, $\bar{*}$ signify that the position of these entries corresponds to the root groups of the short roots $\pm(\beta_1 - \beta_2)$, $\pm(\beta_1+\beta_2)$ respectively. In the same sense, the symbols $\square$, $\diamond$ are to correspond to the long roots $\pm\beta_0, \pm\beta_2$ respectively.

We now describe how each cell is to be 'filled' with entries of $\mathscr{O}_F$. Suppose a cell is chosen with diagonal ϖ^ν where ν is the permutation of μ corresponding to that cell. Choose $\beta \in \Phi_{H_2}$. If β is positive, then a symbol occurs in the corresponding root group entry iff $n := \langle \nu, \beta\rangle > 0$. In that case, the symbol is allowed to vary over q^n elements of $\varpi^a \cdot [\not{k}]_n$ where a is the power of ϖ on the diagonal in the column of that entry. Symbols $*$, $\hat{*}$ (resp. $\circ$, $\hat{\circ}$) are therefore given proportional entries with ratio a power of ϖ. For instance, in the third cell on the first row, $*$ (resp. $\hat{*}$) are to be replaced with κ (resp. $\varpi^2\kappa$) for $\kappa \in [\not{k}]$. If on the other hand β is negative, then a symbol occurs iff $n = \langle \nu, \beta\rangle \geq 2$. In that case, the corresponding symbol is allowed to vary over q^{n-1} elements of $\varpi^a \cdot [\not{k}]_{n-1}$ where a is 1 plus the power of ϖ on the diagonal in the column of that symbol. In the special case of $\nu = (3,3,2)$, all the rules apply except that there is no $\hat{*}$ and we instead have $\bar{*} = * - \frac{\circ\diamond}{\varpi^3}$. The cardinality of each cell (i.e. the number of matrices that each 'skeleton' gives us) is given by the minimial possible length of elements in $t(-\nu)W_{H_2}$ which one may compute using the formula labelled '$\ell_{\min}$' in 12. In total, we have $q^3(1+2q+2q^2+2q^3+q^4)$ matrices coming from 8 Schubert cells.

Now let's compute $\xi = [U\varpi^{-\lambda}U_2](\phi) = [U\varpi^{\mu}U](\varpi^{-3}\cdot\phi)$ (modulo $q-1$) from the above skeleta. We first argue that $\mathrm{supp}(\xi) \subset Y_{(0)} = \mathscr{O}_F^4$. Given $\vec{v} = (a,b,c,d) \in F^4$, the value $\xi(\vec{v})$ is the number of matrices γ appearing in these cells above such that

$$\frac{1}{\varpi^3}\cdot(a,b,c,d)\cdot\gamma \in \mathscr{O}_F^4 \qquad (\dagger)$$

In all cells, there is at least one column that has no symbol in it. We call letter a,b,c,d corresponding to such column a 'pivot'. The requirement ($\dagger$) above tells us that the non-pivotal coordinates of $\vec{v}$ are linear combinations of the pivotal ones and elements of $\mathscr{O}_F$. Since the pivots are necessarily in $\mathscr{O}_F$, we see that $\mathrm{supp}(\xi) \subset \mathscr{O}_F^4$. We may therefore assume that $(a,b,c,d) \in \mathscr{O}_F^4 = Y_{(0)}$ and compute the value of ξ on various subsets of $Y_{(0)}$.

So assume first that $a \in \mathscr{O}_F^\times$. From the description of the cells above, we see that ($\dagger$) is satisfied (by examining the first coordinate of $(a,b,c,d)\cdot\gamma$ for various γ) only in the cases where the first column of γ is $(\varpi^3,0,0,0)^t$ i.e. the rightmost ones shown above, one with the diagonal $\varpi^{(3,3,1)}$ and the other with $\varpi^{(3,3,2)}$. Let us note in the passing that the first has no 'free variables' i.e. there is a unique choice for $\circ, *$ and hence $\square$ entries such that the condition ($\dagger$) is satisfied. The second has one free variable coming from the '$\diamond$' entry. Thus $\xi = (q+1)\cdot\phi_{(0)}^\times$ on $Y_{(0)} - Y_{(1)}$ since the condition is symmetric in the coordinates of $\vec{v}$. In the argument we just provided, if we instead work modulo $q-1$, the task is reduced to counting the number of skeleta only, since the number of free variables in each cell only affect the value of ξ by a power of q.

Now assume in the second case that $a \in \varpi\,\mathscr{O}_F^\times$, $b,c,d \in \varpi\,\mathscr{O}_F$. Then the number of cells that can contribute to the support of ξ on $Y_{(1)} - Y_{(2,1,1,1)}$ is 4, and therefore $\xi \equiv 4\phi_{(1)}^\times$ on $Y_{(1)} - Y_{(2)}$ because of symmetry in the four coordinates. Continuing in this fashion, we deduce

$$\xi \equiv 2\phi_{(0)}^\times + 4\phi_{(1)}^\times + 6\phi_{(2)}^\times + 8\phi_{(3)}$$

and (d) following by recombining terms as in previous parts.

(e) Let $\lambda = (4,4,2)$. Then $U_2\varpi^{-\lambda}U_2 = U_2vU_2](\rho_2^{-4}\cdot\phi)$ where $v = v_0v_1v_2v_1v_0\rho_2^{-4}$ and $v \in [W_H\backslash W_{I,H}/W_H]$. The Weyl orbit diagram of $\lambda_v = -\lambda$ is the same as in part (b). Proposition 5.2.8 implies that we have $U_2vU_2 = \bigsqcup_\sigma \mathrm{im}(\mathcal{Y}_\sigma)$ for $\sigma \in \{v, v_1v, v_2v_1v, v_1v_2v_1v\}$. The argument of pivots and free variables applies and we leave it to the reader to deduce that

$$\xi = \phi_{(0)}^\times + q^3\phi_{(1)}^\times + (1+q+q^2)(q^4\phi_{(2)}^\times + q^5\phi_{(3)}^\times) + q^5(1+q+q^2+q^3)\phi_{(4)}$$

The claim follows by reducing modulo $q-1$ and recombining terms. $\square$

13.4.2 Action of $[U\varpi^{\lambda}H_{\tau_1}]_*$

We now turn our attention to Hecke operators relevant to the (equivalence class of) the second twist 'τ_1' of the zeta element. This requires describing $H_{\tau_1}\varpi^{\lambda}U/U$ for $\lambda \in \Lambda$. However, the methods of §5.2 do not generally apply to these double cosets (at least directly) owing to the 'mixed' structure of the group H_{τ_1} as described in Lemma 13.3.7. The idea is to 'replace' the these double cosets with those whose action ϕ is the same but to which Proposition 5.2.8 does apply. For some λ, this would be rather straightforward. For others, it would be more convenient to proceed in two steps by first utilizing the structure of the group H_{ψ_1} appearing in the decomposition 13.3.7 and then invoking Lemma 5.5.3 with 'X' taken to be $t_1\mathscr{X}_{0,0}t_1^{-1}$.

For illustration of the various aspects of the techniques involved, let us first do the more straightforward computations in Lemma 13.4.4 below. Let $\mathscr{U}$, $\mathscr{U}_{a,b}$ for $a, b \in \mathbb{Z}$ denote the t_1 conjugates of $\mathscr{X}$, $\mathscr{X}_{a,b}$ and let $\mathscr{u}_{\pm} : \mathbb{G}_a \to \mathscr{U}$ be given be the t_1 conjugates of $\mathscr{x}_{\pm}$. By Lemma 13.3.7, $\mathscr{U}_{0,0} \cap H_{\psi_1} = \mathscr{U}_{1,1}$. We have

$$\mathscr{U}_{0,0}/\mathscr{U}_{0,1} = \mathscr{U}_{0,1} \sqcup \bigsqcup_{i\in[\Bbbk]} \mathscr{u}_+(\kappa)\theta_1\mathscr{U}_{0,1} \qquad \mathscr{U}_{0,1}/\mathscr{U}_{1,1} = \bigsqcup_{j\in[\Bbbk]} \mathscr{u}_+(\kappa)\mathscr{U}_{1,1}$$

and we obtain $\{\mathscr{u}_+(j_1), \mathscr{u}_+(i)\theta_1\mathscr{u}_+(j) \mid i_1, j_1, j_2 \in [\Bbbk]\}$ as a set representatives for $\mathscr{U}_{0,0}/\mathscr{U}_{1,1}$ by multiplying the representatives of $\mathscr{U}_{0,0}/\mathscr{U}_{0,1}$ (on the left) with those of $\mathscr{U}_{0,1}/\mathscr{U}_{1,1}$ on the right. Recall that $\varrho = \varrho_H$ denotes the inverse of ρ_H.

Lemma 13.4.4. *We have*

(a) $[U\varpi^{(1,1,1,1)}H_{\tau_1}]_*(\phi) \equiv (1+\varrho)\cdot\phi_{(1,0,0,0)}$

(b) $[U\varpi^{(2,2,1,1)}H_{\tau_1}]_*(\phi) \equiv (\varrho+\varrho^2)\cdot\phi_{(1,0,0,0)}$,

(c) $[U\varpi^{(2,2,2,1)}H_{\tau_1}]_*(\phi) \equiv (\varrho+\varrho^2)\cdot\phi_{(1,0,0,0)}$,

(d) $[U\varpi^{(3,2,2,3)}H_{\tau_1}]_*(\phi) \equiv (1-\varrho+\varrho^2+\varrho^3-\varrho^4+\varrho^5)\cdot\phi_{(1,0,0,0)}$

(e) $[U\varpi^{(4,4,2,2)}H_{\tau_1}]_*(\phi) \equiv (\varrho^3+\varrho^4)\cdot\phi_{(1,0,0,0)}$.

Proof. Let us point out that cocharacters λ appearing in part (a) and (c) are stable under θ_1, while the rest are stable under $r_2 = s_3$. This will dictate whether the action of $\mathscr{U}_{0,0}$ needs to be taken into account. In latter cases, both reflections contribute to the action of operator and hence the need for an intermediate step. Let us observe that since $\theta_1, v_2 \in H_{\tau_1}$ and the action of θ_1 on ϕ is via $\mathrm{pr}_2(\theta_1) = v_0$, the functions we compute are invariant under v_0 and v_2. We have

$$v_0\cdot\phi_{(u,v,w,x)} = \phi_{(w+1,v,u-1,x)}, \qquad v_2\cdot\phi_{(u,v,w,x)} = \phi_{(u,x,w,v)}$$

where $u, v, w, x \in \mathbb{Z}$. This provides a check on our computations e.g. both $\phi_{(1,0,0,0)}$ and $\varrho\cdot\phi_{(1,0,0,0)} = \phi_{(1,1,0,1)}$ are stable under these actions.

(a) We have $H_{\tau_1}\varpi^{-(1,1,1,1)}U = H_{\tau_1}\varrho U \subset H_\varphi \varrho U$. Since $\varrho = \rho_H^{-1}$ is length 0, $\varrho \in [W_\varphi\backslash W_{I,H}/W_H]$. Since $W_\varphi \cap \varrho W_H \varrho^{-1} = \langle t(f_1)s_0, t(f_2)r_0\rangle$, we have $[W_\varphi/(W_\varphi \cap \varrho W_H \varrho^{-1})] = \{1, s_3\}$. By Proposition 5.2.8, we have $H_\varphi \varrho U = \bigsqcup_\sigma \operatorname{im}(\mathcal{Y}_\sigma)$ for $\sigma \in \{\varrho, s_3\varrho\}$. Thus $H_{\tau_1}\varrho U/U$ has at most $1+q$ representatives. On the other hand, since H_{τ_1} contains w_3, $x_3(\kappa)$ for $\kappa \in [\Bbbk]$, $H_{\tau_1}\varrho U$ contains $\operatorname{im}(\mathcal{Y}_\sigma)$ for both $\sigma = \varrho, s_3\varrho$ i.e. $H_{\tau_1}\varrho = H_\varphi \varrho U$. Thus the value of $\xi := [H_{\tau_1}\varpi^{-\lambda}U](\phi)$ at $(a,b,c,d) \in X$ is the number of integral vectors in the following list of $q+1$ vectors:

$$\left(\frac{a}{\varpi}, \frac{b}{\varpi}, c, d\right), \quad \left(\frac{a}{\varpi}, b, c, \frac{d+b\kappa}{\varpi}\right)_{\kappa\in[\Bbbk]}$$

where as usual repetitions are allowed. Clearly, $\operatorname{supp}(\xi) \subset Y_{(1,0,0,0)}$. The usual argument gives us that ξ is 1 on $Y_{(1,0,0,0)} \setminus Y_{(1,1,0,1)}$, $q+1$ on $Y_{(1,1,0,1)}$ and zero everywhere else. Therefore $\xi = \phi_{(1,0,0,0)} + q\phi_{(1,1,0,1)} = (1+q\varrho)\cdot\phi_{(1,0,0,0)}$. The claim follows.

(b) We have $H_{\tau_1}\varpi^{-(2,2,1,1)}U = H_{\tau_1}w_0\varrho^2 U$. Since $w_0 H'_{\psi_1} w_0 \subset w_0 H_{\psi_1} w_0 = H_{\psi_1}$ is contained in U, Lemma 13.3.7 implies that $H_{\tau_1}w_0U = \mathscr{U}_{0,0}w_0U$. Now $\mathscr{U}_{0,0} \cap (w_0Uw_0) = \mathscr{U}_{0,1}$ (as may be conveniently seen by comparing $\mathscr{X}_{0,0} \hookrightarrow H$ with the $\varpi^{-(1,1,0,0)}$ conjugates of the root groups $U_{\pm\alpha_0,0}, U_{\pm\beta_0,0}$). Using the decomposition of $\mathscr{U}_{0,0}/\mathscr{U}_{0,1}$ given above and that $\theta_1 w_0 U = v_0 U$, we see that $H_{\tau_1}w_0U = w_0U \sqcup \bigsqcup_{\kappa\in[\Bbbk]} \mathcal{u}_+(\kappa)v_0U$. Denote $\xi := [H_{\tau_1}w_0U](\varrho^2\cdot\phi)$. The decomposition of $H_{\tau_1}w_0U$ above gives us that $\xi(a,b,c,d)$ for $a,b,c,d \in F^4$ is the number of integral vectors in the following list

$$\left(\frac{a}{\varpi^2}, \frac{b}{\varpi}, c, \frac{d}{\varpi}\right), \quad \left(\frac{a}{\varpi}, \frac{b}{\varpi}, \frac{c\varpi + a\kappa}{\varpi^2}, \frac{d}{\varpi}\right)_{\kappa\in[\Bbbk]}$$

of $1+q$ vectors, with repetitions allowed as usual. The rest is routine counting argument.

(c) We have $H_{\tau_1}\varpi^{-(2,2,2,1)}U = H_{\tau_1}w_0v_0\varrho U = H_{\tau_1}\theta_1\varrho U$ As $\theta_1 H'_{\psi_1}\theta_1 \subset \theta_1 H_{\psi_1}\theta_1 = H_{\psi_1}$ is contained in U, Lemma 13.3.7 implies that $H_{\tau_1}\theta_1 U = \mathscr{U}_{0,0}\theta_1 U = \mathscr{U}_{0,0}\theta_1 U = \mathscr{U}_{0,0}U$. One checks that $\mathscr{U}_{0,0} \cap U = \mathscr{U}_{1,1}$ and therefore the function is computed by taking the sum of translates of $\varrho\cdot\phi$ under $\mathscr{U}_{0,0}/\mathscr{U}_{1,1}$. We note that since $\operatorname{pr}_2(w_0)$, $\operatorname{pr}_2(\mathcal{u}_+(j))$ for $j \in [\Bbbk]$ are in U_2, they stabilize ϕ. Summing over the $\mathscr{U}_{0,1}/\mathscr{U}_{0,0}$ representatives first, we see that

$$[H_{\tau_1}\theta_1\varrho U](\phi) = \sum\nolimits_{\gamma\in\mathscr{U}_{0,0}/\mathscr{U}_{0,1}} \varrho\gamma\cdot\Big(\sum_{\kappa\in[\Bbbk]} \mathcal{u}_+(\kappa)\cdot\phi\Big) = [H_{\tau_1}w_0\varrho U](q\cdot\phi)$$

and therefore modulo $q-1$, we get the same function as in (b).

(d) This is similar to part (a) though a little more involved. We have $H_{\tau_1}\varpi^{-(3,2,2,3)}U_2 = H_{\tau_1}vU \subset H_\varphi vU$ where $v = v_1v_0v_1v_2\varrho^3$. The word v is $(\varnothing, W_H)$-reduced as the minimal possible length for $t(3,2,3)W_{H_2}$ is four, and the lengths of $v'v$ are all at least four for all $v' \in W_\varphi$. i.e. v is W_φ-reduced. Thus $v \in [W_\varphi\backslash W_{I,H}/W_H]$. As the v conjugates of $s_0, r_1, r_2 \in W_H$ are $t(f_1)s_0, t(f_3-f_2), t(f_2)r_0$ respectively, we have

$W_\varphi \cap vW_Hv^{-1} = \langle t(f_1)s_0, t(f_2)r_0\rangle$. We see by Proposition 5.2.8 that $H_\varphi vU = \bigsqcup_\sigma \mathrm{im}(\mathcal{Y}_\sigma)$ for $\sigma \in \{v, v_2v\}$. Since the v_1 (resp. v_1v_0, resp. $v_1v_0v_1$) conjugates of $y_0(\mathscr{O}_F)$ (resp. $y_1(\mathscr{O}_F)$, resp. $y_2(\mathscr{O}_F)$) are contained in H_{τ_1}, $H_{\tau_1}vU \supset \mathrm{im}(\mathcal{Y}_v)$. Similarly, $H_{\tau_1}vU \supset \mathrm{im}(\mathcal{Y}_{v_2v})$ i.e. $H_{\tau_1}vU = H_\varphi vU$. Using the decomposition of $H_\varphi vU$, we see that $\xi := [H_{\tau_1}\varpi^{-(3,2,2,3)}U](\phi)$ is the function whose value at $(a,b,c,d) \in X$ is the number of integral vectors in the following list of q^4+q^5 vectors

$$\left(\frac{a+d\,i_1\varpi}{\varpi^2}, \frac{b+a\,i_2+c\,i_1\varpi+d\,\varpi(i_3+i_4\varpi)}{\varpi^3}, \frac{c-di_2}{\varpi}, d\right)_{i_1,\ldots,i_4\in[\Bbbk]}$$
$$\left(\frac{a+b\,j_1\varpi}{\varpi^2}, b, \frac{c+b\,j_2}{\varpi}, \frac{d+a\,j_2-c\,j_1\varpi+b(j_3+j_4\varpi+j_5\varpi^2)}{\varpi^3}\right)_{j_1\ldots,j_5\in[\Bbbk]}$$

with repetitions allowed as usual. The routine argument of restricting ξ to various starata (e.g. ξ is a 1 on $Y_{(1,0,0,0)} \setminus Y_{(1,1,0,1)}$, q^3 on $Y_{(2,1,1,1)} \setminus Y_{(2,2,1,2)}$ etc) gives the claim.

(e) Here we start needing the two steps alluded to above. We have $H_{\tau_1}\varpi^{-(4,4,2,2)}U = \mathscr{U}_{0,0}H'_{\psi_1}\sigma U$ where $\sigma = \varpi^{-2f_1}\varrho^2$. We first compute $[H'_{\psi_1}\sigma U](\phi)$. Note first that $H'_{\psi_1}\sigma U = H_{\psi_1}\sigma U$ since $A^\circ \subset U$. Then note that $H_{\psi_1} = V_1 \times_{\mathbb{G}_m} V_2$ where $V_i := \mathrm{pr}_i(H_{\psi_1})$ and the observation of 13.4.1 applies to these groups. Therefore

$$[H_{\psi_1}\sigma U](\phi) = d\varrho^2\cdot\phi = d\cdot\phi_{(2,2,2,2)}$$

where $d = [V_1 : V_1 \cap \varpi^{-2f_1}V_1\varpi^{2f_1}]$ is the degree of $[U_1\varpi^{2f_1}V_1]_*$. Since V_1 contained in an Iwahori subgroup of $H_1 = \mathrm{GL}_2(F)$, d is a power of q and therefore $d = 1$ modulo $q-1$. We now execute the second step. As in part (b), one notes that $\mathscr{U}_{0,0} \cap \sigma U\sigma^{-1} = \mathscr{U}_{0,3}$, $\mathscr{U}_{0,0} \cap H'_{\psi_1} \cap \sigma U\sigma^{-1} = \mathscr{U}_{1,1} \cap \mathscr{U}_{0,3} = \mathscr{U}_{1,3}$. Thus $e = [\mathscr{U}_{0,3} : \mathscr{U}_{1,3}] = q$ which is 1 modulo $q-1$. Applying Lemma 5.5.3, we see

$$[H_{\tau_1}\sigma U](\phi) \equiv \sum\nolimits_{\gamma\in\mathscr{U}_{0,0}/\mathscr{U}_{1,1}} \gamma\cdot\phi_{(2,2,2,2)}$$

and the right hand side above is ϱ^2 times the function computed in part (c) (or (b)). □

Let us now outline the general procedure. By Proposition 5.5.3 applied by taking the groups 'X', 'U_1' in the statement to be $\mathscr{U}_{0,0}$, H'_{ψ_1} respectively, we see that for $\sigma \in H$,

$$e\cdot[H_{\tau_1}\sigma U](\phi) = \sum\nolimits_\delta \delta\cdot\left([H'_{\psi_1}\sigma U](\phi)\right) \tag{13.4.5}$$

where δ runs over representatives of $\mathscr{U}_{0,0}/\mathscr{U}_{1,1}$, and $e = e_\sigma$ is the index of $H'_{\psi_1} \cap \sigma U\sigma^{-1}$ in $H_{\tau_1} \cap \sigma U\sigma^{-1}$. If $\sigma = \varpi^{-\lambda}$ for $\lambda \in \Lambda$, then the second parts of Lemma 13.3.7 & Lemma 5.5.3 together imply that $e = [\mathscr{U}_{0,0} \cap \sigma U\sigma^{-1} : \mathscr{U}_{1,1} \cap \sigma U\sigma^{-1}]$, and we denote this quantity by e_λ. If $\lambda = (a_0,\ldots,a_3)$, then one sees from the embedding $\mathscr{X}_{0,0} \hookrightarrow H$ that $\mathscr{U}_{0,0} \cap \sigma U\sigma^{-1} = \mathscr{U}_{a_\lambda,-b_\lambda}$ where a_λ (resp. b_λ) is the maximum (resp. minimum) in the set

$$\{0, 2a_2-a_0-1, 1+a_0-2a_1\}$$

For instance, $\mathscr{U}_{0,0} \cap t_1 U t_1^{-1} = \mathscr{U}_{0,0}$. Thus $e_\lambda = 1$ modulo $q-1$ if $a_\lambda \neq b_\lambda$, since then $\mathscr{U}_{a_\lambda, b_\lambda}$ is contained in a (conjugate of) an Iwahori subgroup of $\mathscr{U} \simeq \mathrm{GL}_2(F)$. The equality is achieved only for $a_\lambda = b_\lambda = 0$ i.e. if $\theta_1 \cdot \lambda = \lambda$ or equivalently, if $\varpi^\lambda \mathscr{U}_{0,0} \varpi^{-\lambda} \subset U$. In the rest of our computations, θ does not fix λ and we can ignore e appearing in eq. 13.4.5. The task at hand is therefore reduced to computing $[H'_{\psi_1} \sigma U](\phi)$ and then computing the translate of the resulting function under $\mathscr{U}_{0,0}/\mathscr{U}_{1,1}$. We compute 13.4.5 with $\sigma = \varpi^{-\lambda}$ for the remaining six λ's by executing these two steps.

The first step for the remaining λ's, which is to compute $[H'_{\psi_1} \varpi^{-\lambda} U](\phi)$, is executed in Lemma 13.4.7 below. Let us briefly elaborate on the techniques involved. We note that $H'_{\psi_1} \varpi^{-\lambda} U = H_{\psi_1} \varpi^{-\lambda} U$ since $A^\circ \subset U$, $\varpi^{-\lambda}$ normalizes A° and H_{ψ_1}[4] $= H'_{\psi_1} A^\circ$. For $i = 1, 2$, let $H_{i,\psi_1}, := \mathrm{pr}_i(H_{\psi_1})$. Then

$$H_{\psi_1} = H_{1,\psi_1} \times_{\mathbb{G}_m} H_{2,\psi_1}$$

and one the observation of 13.4.1 is applicable for all $\lambda \in \Lambda$. As H_{1,ψ_1} is contained in a Iwahori subgroup of H_1, the degree of $[U_1 \varpi^{\lambda_1} H_{1,\psi_1}]_*$ is always a power of q for all $\lambda \in \Lambda$ and we will be able to ignore it when working modulo $q-1$ (the reader will be reminded of this). To compute $[U_2 \varpi^{\lambda_2} H_{2,\psi_1}]_*(\phi)$, we will compare H_{2,ψ_1} with certain 'parahoric' subgroup of H_2. For $S \subset S_{H,\mathrm{aff}}$, let W_S be the Coxeter group generated by S and $I_{H,S} = I_H W_{H,S} I_H$ be the corresponding subgroup of H. Then $I_{H,S}$ is compact open iff S does not contain all the reflections in each of the two components of the Dynkin diagram displayed in 13.3.1. In particular, we let $I_{H,r_2} := I_H \langle v_2 \rangle I_H$ which contains H_{ψ_1}. We let $I_2 = I_{H_2} := \mathrm{pr}_2(I_H)$ be the 'standard' Iwahori subgroup of H_2 and let $I_{2,r_2} = \mathrm{pr}_2(I_{H,r_2}) = I_2 \langle v_2 \rangle I_2$. It contains the component H_{2,ψ_1}. More crucially, we have

$$I_{H,r_2} = H_{\psi_1} \cdot U_{-\alpha_0,1} U_{-\beta_0,1}, \qquad I_{2,r_2} = H_{2,\psi_1} \cdot U_{-\beta_0,1}$$

where in the second equality, we are considering $U_{-\beta_0,1}$ as subgroups of H_2. So if $\lambda = (a_0, \ldots, a_3) \in \Lambda$ as above and λ_2 denotes $(a_0, a_2, a_3) = a_0 f_0 + a_2 f_2 + a_3 f_3$, then $H_{2,\psi_1} \varpi^{-\lambda_2} U_2 = I_{2,r_2} \varpi^{-\lambda_2} U_2$ if the $\varpi^{-\lambda}$ conjugate of $U_{-\beta_0,0}$ (which is $U_{-\beta_0,\langle \beta_0, \lambda \rangle}$), contains $U_{-\beta_0,1}$ i.e. if $2a_2 - a_0$ is at most 1. Similarly, if both $2a_1 - a_0, 2a_2 - a_0$ are at most 1, then $H'_{\psi_1} \varpi^{-\lambda} U = H_{\psi_1} \varpi^{-\lambda} U = I_{H,r_2} \varpi^{-\lambda} U$ etc.

Remark 13.4.6. Using similar arguments, one can show that $H_{\tau_1} \varpi^{-\lambda} U = H_\varphi \varpi^{-\lambda} U$ if both $2a_2 - a_0$, $2a_1 - a_0$ are equal to 1 i.e. $\varpi^\lambda v_0 \varpi^{-\lambda} \in U$. This applies to parts (a) and (d) of Lemma 13.4.4.

Lemma 13.4.7. *We have*

(a) $[U_2 \varpi^{(2,1,2)} H_{2,\psi_1}]_*(\phi) \equiv 2\phi_{(1)} + (1 - \varrho^2) \cdot \big(\phi_{(1,0,0,0)} - \phi_{(1,1,0,1)}\big)$,

(b) $[U_2 \varpi^{(2,0,1)} H_{2,\psi_1}]_*(\phi) \equiv (1 + \varrho^2 + \varrho^4) \cdot \phi_{(0)} - \big(\phi_{(1,0,0,0)} + \varrho^2 \cdot \phi_{(1,1,0,1)}\big)$,

(c) $[U_2 \varpi^{(1,0,1)} H_{2,\psi_1}]_*(\phi) \equiv (2 + 2\varrho^2) \cdot \phi_{(0)} - \big(\phi_{(1,0,0,0)} + \phi_{(1,1,0,1)}\big)$.

[4]As $H_{\tau_1} A^\circ = \mathscr{U}_{0,0} H_{\psi_1}$ is not a group, one cannot apriori replace H'_{ψ_1} with H_{ψ_1} in 13.4.5 which invokes Lemma 5.5.3

(d) $[U_2\varpi^{(1,1,1)}H_{2,\psi_1}]_*(\phi) \equiv \phi_{(1,0,0,0)} + \phi_{(1,1,0,1)}$,

Proof. Let $\lambda = (a_0, 0, a_2, a_3) \in \Lambda$ and let $\xi := [U_2\varpi^\lambda H_{2,\psi_1}]_*$ the function to be computed. Since $2a_2 - a_0 \le 1$ in all cases, we can replace H_{2,ψ_1} with I_{2,r_2} by the discussion above. In what follows, V_2 will denote the group I_{2,r_2} and W_{V_2} the Weyl group $\{1, r_2\} \subset W_{I,H_2}$ of I_{2,r_2}. As I_{2,r_2} contains v_2, ξ will be invariant under the action of v_2 which corresponds to switching second and fourth coordinates of a vector in X. This provides a check on our computations. In each part, we will need to know the minimal possible length (with respect to $S_{H_2,\mathrm{aff}} = \{v_1, v_2, v_0\}$) of elements in $W_{V_2}t(-\lambda)W_{H_2} \subset W_{I,H_2}$. This can be done systematically as follows. We first find the minimal possible length of elements in $t(-\lambda)U_2$ using the formula given by '$\ell_{\min}$' in 12.1.1. If $v \in t(-\lambda)W_{H_2}$ is such a word, then either v or v_2v is $(W_{V_2}, \varnothing)$-reduced, the latter case only occurring if the first letter of v is v_2. If v is $(W_{V_2}, \varnothing)$-reduced we are done, and if not, we repeat the process for μ such that $v_2vW_{H_2} = t(\mu)W_{H_2}$. In the cases we consider below, finding such v is more or less straightforward even without this algorithm. (a) We have $V_2\varpi^{-(2,1,2)}U_2 = V_2vU_2$ where $v := v_1v_0\rho_2^{-2}$ and we easily see that v is both $(W_{V_2}, \varnothing)$ and $(\varnothing, W_{H_2})$ reduced i.e. $v \in [W_{V_2}\backslash W_{I,H_2}/W_{H_2}]$. Moreover, $W_{V_2} \cap vW_Hv^{-1}$ is trivial. By Proposition 5.2.8, we see that $V_2vU_2 = \bigsqcup_\sigma \mathrm{im}(\mathcal{Y}_\sigma)$ for $\sigma \in \{v_1v_0, v_2v_1v_0\}$. Thus $\xi(a,b,c,d)$ for $(a,b,c,d) \in F^4$ is the number of integral vectors in the following list of $q^2 + q^3$ vectors given by

$$\left(\frac{a}{\varpi}, \frac{b + a\,i_1 + d\,i_2\varpi}{\varpi^2}, \frac{c - d\,i_1}{\varpi}, d\right)_{i_1,i_2\in[\Bbbk]} \quad \text{and} \quad \left(\frac{a}{\varpi}, b, \frac{c + bj_1}{\varpi}, \frac{d + a\,j_1 + b(j_2 + j_3\varpi)}{\varpi^2}\right)_{j_1,j_2,j_3\in[\Bbbk]}$$

where repetitions are allowed. We have $\mathrm{Supp}(\xi) \subset \mathscr{O}_F^4$ by the usual argument of pivots. Then we count the integral vectors by counting free variables. First note that in every case, $a \in \varpi\,\mathscr{O}_F$ for these vectors to be integral. If one of b or d is a unit, there is a unique such vector, and therefore ξ is 1 on $Y_{(1,0,0,0)} \setminus Y_{(1,1,0,1)}$. If both $b, d \in \varpi\,\mathscr{O}_F$, then so must c. In that case, if $a \in \varpi\,\mathscr{O}_F^\times$, there are $q+q^2$ integral vectors and so ξ is $q+q^2$ on $Y_{(1,1,1,1)} \setminus Y_{(2,1,1,1)}$. We then restrict ξ to $Y_{(2,1,1,1)}$ and show that it is equal to q^2 on $Y_{(2,1,1,1)} \setminus Y_{(2,2,1,2)}$ and $q^2 + q^3$ on $Y_{(2,2,1,2)}$. Thus,

$$\begin{aligned}\xi = &\left(\phi_{(1,0,0,0)} - \phi_{(1,1,0,1)}\right) + (q + q^2)\left(\phi_{(1,1,1,1)} - \phi_{(2,1,1,1)}\right)\\ &+ q\left(\phi_{(2,1,1,1)} - \phi_{(2,2,1,2)}\right) + (q^2 + q^3)\phi_{(2,2,1,2)}.\end{aligned}$$

The claim follows.

(b) $V_2\varpi^{-(2,0,1)}U_2 = V_2vU_2$ where $v = v_1v_2v_1v_0\rho_2^{-2}$. One checks that $v \in [W_{V_2}\backslash W_{I,H_2}/W_{H_2}]$ and $W_{V_2} \cap vW_{H_2}v^{-1} = W_{V_2}$. Thus $V_2vU_2 = \mathrm{im}(\mathcal{Y}_v)$ by Proposition 5.2.8. Therefore $\xi(a,b,c,d)$ is the number of integral vectors in the following list q^4 vectors given by

$$\left(a, \frac{b + a\,i_1}{\varpi}, \frac{c + b\,i_2 - d\,i_1 + a(i_3 + i_4\varpi)}{\varpi^2}, \frac{d + a\,i_2}{\varpi}\right)_{i_1,\ldots,i_4\in[\Bbbk]}$$

where as usual, repetitions are allowed. Then the usual argument gives us that ξ is 1 on $Y_{(0)} - Y_{(1,0,0,0)}$, q^3 on $Y_{(1)} - Y_{(2,2,1,2)}$, q^4 on $Y_{(2)}$ and zero everywhere elsewhere. The claim follows.

(c) $V_2\varpi^{-(1,0,1)}U_2 = U_2vU_2$ where $v = v_1v_2\rho_2^{-1}$ and one shows that $V_2 = \bigsqcup_\sigma \operatorname{im}(\mathcal{Y}_\sigma)$ for $\sigma \in \{v, v_2v\}$ with a total of $q^2 + q^3$ vectors. We leave this computation to the reader.

(d) $V_2\varpi^{-(1,1,1)}U_2 = V_2\rho_2^{-1}U_2$. As ρ_2^{-1} is length 0 in W_{I,H_2}, $\rho_2^{-1} \in [W_{V_2}\backslash W_{I,H_2}/W_{H_2}]$ and $W_{V_2} \cap \rho_2^{-1}W_{H_2}\rho_2$ is trivial. Therefore, $V_2\rho_2^{-1}U_2 = \bigsqcup_\sigma \operatorname{im}(\mathcal{Y}_\sigma)$ for $\sigma \in \left\{\rho_2^{-1}, v_2\rho_2^{-1}\right\}$ with a total of $q+1$ representatives. The lists that are relevant here are the first two on the left in Lemma 13.4.3(a). From those lists, we see that

$$\xi = \phi_{(1,1,0,0)} + \left(\phi_{(1,0,0,0)} - \phi_{(1,1,0,0)}\right) + q\phi_{(1,1,0,1)}.$$

Reducing modulo $q-1$ gives the claim. Cf. the function of Lemma 13.4.4(a). □

We now execute the second step.

Lemma 13.4.8. *We have*

(a) $[U\varpi^{(2,1,2,2)}H_{\tau_1}]_*(\phi) \equiv 2(1+\varrho^3)\cdot\phi_{(1,0,0,0)}$,

(b) $[U\varpi^{(3,3,2,2)}H_{\tau_1}]_*(\phi) \equiv 2(\varrho^2+\varrho^3)\cdot\phi_{(1,0,0,0)}$

(c) $[U\varpi^{(3,2,3,2)}H_{\tau_1}]_*(\phi) \equiv 2(\varrho+\varrho^4)\cdot\phi_{(1,0,0,0)}$,

(d) $[U\varpi^{(3,3,1,2)}H_{\tau_1}]_*(\phi) \equiv 2(\varrho+\varrho^4)\cdot\phi_{(1,0,0,0)}$,

(e) $[U\varpi^{(4,2,4,2)}H_{\tau_1}]_*(\phi) \equiv (\varrho-\varrho^2+\varrho^3+\varrho^4-\varrho^5+\varrho^6)\cdot\phi_{(1,0,0,0)}$,

(f) $[U\varpi^{(4,3,3,3)}H_{\tau_1}]_*(\phi) \equiv 2(\varrho^2+\varrho^5)\cdot\phi_{(1,0,0,0)}$.

Proof. In each part, let $\xi = \xi_\lambda := [U\varpi^\lambda H_{\tau_1}]_*(\phi)$ denote the function to be computed. By 13.4.5 and the following discussion, it will suffice in each part to compute the translates of certain H'_{ψ_1}-invariant functions under $\mathscr{U}_{0,0}/\mathscr{U}_{1,1}$ that we have computed in Lemma 13.4.7. Let us make a few observations in order to do so efficiently. Set $\eta_1 := \sum_{\kappa\in[\hbar]} u_+(\kappa) \in \mathbb{Z}[\mathscr{U}_{0,0}]$ and $\eta_0 := 1 + \eta_1\theta_1$. Then $\eta_0\eta_1 \in \mathbb{Z}[\mathscr{U}_{0,0}]$ represents the sum over a set of representatives of $\mathscr{U}_{0,0}/\mathscr{U}_{1,1}$. Thus, if $\xi : X \to \mathcal{O}$ is a H'_{ψ_1}-invariant function, the sum of translates $\sum_\gamma \gamma\cdot\xi$ under $\gamma \in \mathscr{U}_{0,0}/\mathscr{U}_{1,1}$ is equal to $\eta_0\eta_1\xi = \eta_1\eta_2\cdot\xi$. Note that the action of $\mathscr{U}$ on $\vec{v} \in X$ only affects the first and third component. One easily checks that

$$\eta_1\cdot\phi_{(u+z,v,u,x)} = q\cdot\phi_{(u+z,v,u,x)} \tag{13.4.9}$$

$$\eta_1\cdot\phi_{(u+1,v,u-1,x)} = \phi_{(u,v,u-1,x)} - \phi_{(u,v,u,x)} + q\phi_{(u+1,v,u,x)} \tag{13.4.10}$$

where $u, v, w, x, z \in \mathbb{Z}$, $z \le 1$. Since $\theta_1 \in \mathscr{U}_{0,0}$ acts by $\mathrm{pr}_2(\theta_1) = v_0$, these will suffice to compute the action on various functions we have computed. For example, $\eta_0\cdot\phi_{(1)} = \phi_{(1,1,0,1)} + q\phi_{(2,1,1,1)}$ etc.

(a) We have $H_{\tau_1}\varpi^{-(2,1,2,2)}U = \mathscr{U}_{0,0}H'_{\psi_1}v_0\varpi^{-(2,1,1,2)}U$. Since v_0 normalizes H_{ψ_1} and $H_{\psi_1} = H'_{\psi_1}A^\circ$, we have $[H'_{\psi_1}v_0\varpi^{-(2,1,1,2)}U](\phi) = v_0 \cdot [U\varpi^{(2,1,1,2)}H_{\psi_1}](\phi)$. Using the observation of 13.4.1, we see that

$$[U\varpi^{(2,1,1,2)}H_\psi](\phi) = \deg\left([U_1\varpi^{(2,1)}H_{1,\psi_1}]_*\right) \cdot [U_2\varpi^{(2,1,2)}H_{2,\psi_1}](\phi).$$

As noted earlier, the degree of $[U_1\varpi^{(a_0,a_1)}H_{1,\psi_1}]_*$ is 1 modulo $q-1$ for all $a_0f_0 + a_1f_1 \in \Lambda$. Invoking eq. 13.4.5 (and noting that the value of $e = e_{(2,1,2,2)}$ in this case is 1 modulo $q-1$), we see that ξ is equivalent to $\eta_0\eta_1v_0 \cdot [U_2\varpi^{(2,1,2)}H_{2,\psi_1}]_*(\phi)$. From the two actions computed in 13.4.9 and 13.4.10, we easily deduce that $\eta_0\eta_1v_0 \cdot \phi_{(1)} = q \cdot \phi_{(1,1,0,1)} + q^2 \cdot \phi_{(2,1,1,1)}$ and that the action of $\eta_0\eta_1$ on both $\phi_{(1,1,0,1)}$, $\phi_{(1,0,0,0)}$ is via multiplication by $(1+q)q$. Using these observations on the summands occuring in the function computed in 13.4.7(a), we get the claim.

(b) We have $H_{\tau_1}\varpi^{-(3,3,2,2)}U = \mathscr{U}_{0,0}H'_{\psi_1}w_0\varpi^{-(1,0,1,1)}\varrho^2U$. The same argument as in part (a) gives us that $[H'_{\psi_1}w_0\varpi^{-(1,0,1,1)}\varrho^2U](\phi) = w_0\varrho^2 \cdot [U\varpi^{(1,0,1,1)}H_{\psi_1}]_*(\phi)$. Since w_0 acts trivially on ϕ, it suffices by eq. 13.4.5 to compute $\varrho^2 \cdot [U\varpi^{(1,0,1,1)}H_{\psi_1}]_*(\phi)$. By analogous arguments, $[U\varpi^{(1,0,1,1)}H_{\psi_1}](\phi) \equiv [U_2\varpi^{(1,1,1)}H_{2,\psi_1}]_*(\phi)$. The claim therefore follows by applying 13.4.5 and invoking Lemma 13.4.7(d) i.e. by computing the action of $\eta_0\eta_1\varrho^2$ on the function $\phi_{(1,0,0,0)} + \phi_{(1,1,0,1)}$, which is just multiplication by $q+1$.

(c) We have $H_{\tau_1}\varpi^{-(3,2,3,2)}U = \mathscr{U}_{0,0}H'_{\psi_1}v_0\varpi^{-(1,1,0,1)}\varrho^2U$. The argument proceeds similarly as in part (a) i.e. one computes the action $\eta_1\eta_0v_0\varrho^2$ on the function computed in Lemma 13.4.7(c).

We outline the steps required for the remaining parts which are entirely analogous to the ones above. In part (d), one shows that ξ is equivalent to $\eta_0\eta_1\varrho^2$ times the function computed Lemma 13.4.7(c). Part (e) reduces to computing $\eta_0\eta_1\varrho^2$ times the function in 13.4.7 (b) and part (f) is reduced to computing ϱ^2 of the function computed in part (a) above. ☐

13.4.3 Action of $[U\varpi^\lambda H_{\tau_2}]_*$

The last set of operators that we need to study are those involving the element τ_2 in Lemma 13.3.5. As with the τ_1-case, we let $\mathscr{V}, \mathscr{V}_{0,0}$ denote the t_2 conjugates of $\mathscr{X}, \mathscr{X}_{0,0}$ respectively and $\mathscr{v}_\pm : F^\times \to \mathscr{V}$ denote the t_2 conjugates of $\mathscr{x}_\pm : F^\times \to H$. We recall that $H_{\psi_2} = H'_{\psi_2}A^\circ$. By Lemma 5.5.3, we have

$$e_\lambda \cdot [U\varpi^\lambda H_{\tau_2}]_*(\phi) = \sum\nolimits_\delta \delta \cdot [U\varpi^\lambda H_{\psi_2}]_*(\phi) \tag{13.4.11}$$

where δ runs over a set of representatives of $\mathscr{V}_{0,0}/\mathscr{V}_{2,2}$ and $e_\lambda = [\mathscr{V}_{0,0} \cap \sigma U\sigma^{-1} : \mathscr{V}_{2,2} \cap \sigma U\sigma^{-1}]$, σ denoting $\varpi^{-\lambda}$. We have $e_\lambda = 1$ modulo $q-1$ unless $\theta_2 \cdot \lambda \neq \lambda$. Explicitly, if $\lambda = (a_0, \ldots, a_3)$, the condition $\theta_2 \cdot \lambda = \lambda$ means that $2a_1 - a_0 = 2a_2 - a_2 = 2$ or what amounts to the same thing as $\varpi^\lambda\mathscr{V}_{0,0}\varpi^{-\lambda} \subset U$. We will use that H_{ψ_2} is normalized by $t_2 w_{\alpha_0} t_2^{-1} = \varpi^{-2f_2} w_{\alpha_0}$, $t_2 v_{\beta_0} t_2^{-1} = \varpi^{-2f_3} v_{\beta_0}$. Additionally, we will require a factorization of the group H_{ψ_2} via the embedding of $\mathscr{X} = \mathrm{GL}_2(F)$ along the 'standard' root groups $U_{\pm\beta_2}$. More precisely, let $\jmath_{\beta_2} : \mathscr{X} \to H$ be the embedding determined by

$$\mathrm{diag}(u_1, u_2) \mapsto \mathrm{dis}(u_1 u_2, 1, 1, u_1) \quad x_{\pm}(u) \mapsto y_{\pm\beta_2}(u)$$

where $y_{\beta_2} := y_2$, $y_{-\beta_2} := v_2 y_2 v_2$. We let $\mathscr{Y}$, $\mathscr{Y}_{u,v} \subset H$ for $u, v \in \mathbb{Z}$ denote the images of $\mathscr{X}$, $\mathscr{X}_{u,v}$ respectively. Let $\psi_3 : \Phi_H \to \mathbb{Z}$ be the be the mapping that is equal to ψ_2 on $\Phi_H \setminus \{\pm\beta_2\}$ and $\psi_3(\beta_2) = 0$, $\psi_3(-\beta_2) = 2$. Let H_{ψ_3} be the group generated by $U_{\alpha,\psi_3(\alpha)}$ for $\alpha \in \Phi_H$ and A°. Then

$$H_{\psi_2} = \mathscr{Y}_{0,0} H_{\psi_3} \tag{13.4.12}$$

Lemma 13.4.13. *We have*

(a) $[U\varpi^{(2,2,2,1)} H_{\tau_2}]_*(\phi) \equiv \phi_{(2,1,0,1)}$.

(b) $[U\varpi^{(3,3,2,2)} H_{\tau_2}]_*(\phi) \equiv \phi_{(2,1,0,1)} + \phi_{(3,1,1,1)} + \phi_{(2,2,0,2)} + \phi_{(3,2,1,2)}$,

(c) $[U\varpi^{(4,4,2,2)} H_{\tau_2}]_*(\phi) \equiv \phi_{(2,2,0,2)} + \phi_{(4,2,2,2)}$,

(d) $[U\varpi^{(4,3,3,3)} H_{\tau_2}]_*(\phi) \equiv \phi_{(3,1,1,1)} + \phi_{(3,3,1,3)}$.

Proof. For $m \geq 1, n$ integers, let $\eta_{\pm}(m,n) := \sum_{\kappa \in \varpi^m [\not{k}]_n} v_{\pm}(\kappa) \in \mathbb{Z}[\mathscr{V}]$. Set

$$\eta_0 := 1 + \eta_+(0,1)\theta_1, \qquad \eta_1 := \eta_-(1,1), \qquad \eta_2 := \eta_+(0,2).$$

Then $\eta_0\eta_1\eta_2 \in \mathbb{Z}[\mathscr{V}]$ represents a sum over representatives of $\mathscr{V}_{0,0}/\mathscr{V}_{2,2}$.

(a) As θ_2 stabilizer $\lambda = (2,2,2,1)$, $H_{\tau_2}\varpi^{-\lambda}U = H'_{\psi_2}\mathscr{V}_{0,0}\varpi^{-\lambda}U = H_{\psi_2}\theta_1\varrho^2 U$. As θ_1 normalizes H_{ψ_1} and $U \supset H_{\psi_1} \supset H_{\psi_2}$, we have $H_{\psi_2}\theta_1\varrho^2 U = \theta_1\varrho^2 U = \varpi^{-(2,2,2,1)}U$. The claim follows.

(b) & (c) Let $w = t_2 w_{\alpha_0} t_2^{-1} \in N_H(A)$. We have $H_{\tau_2}\varpi^{-(3,3,2,2)}U = \mathscr{V}_{0,0}H_{\psi_2}w\varrho^3 U$ and $H_{\psi_2}w\varrho^3 U = w\varrho^2 H_{\psi_2}\varrho U$ as w normalizes H_{ψ_2}. Now $H_{\psi_2}\varrho U \subset H_\varphi \varrho U$ and from the decomposition of $H_\varphi\varrho U/U$ obtained in Lemma 13.4.4 (a), we easily see that $H_{\psi_2}\varrho U = H_\varphi\varrho U$. By the same result, we have $[H_\varphi\varrho U](\phi) = (1+\varrho)\cdot\phi_{(1,0,0,0)}$. Applying 13.4.11, we see that $[U\varpi^{(3,3,2,2)}H_{\tau_2}]_*(\phi)$ is congruent to the function

$$\eta_0\eta_1\eta_2 w\varrho^2(1+\varrho)\cdot\phi_{(1,0,0,0)} = \eta_0\eta_1\eta_2\cdot\left(\phi_{(2,1,1,1)} + \phi_{(2,2,1,2)}\right).$$

Since the action of $\mathscr{V}_{0,0}$ on X only affects the first and third coordinates of a vector $\vec{v} \in X$, it suffices to compute the action on $\phi_{(2,1,1,1)}$, as analogous result would hold for $\phi_{(2,2,1,2)}$. It is easy to see that η_1, η_2 act $\phi_{(2,1,1,1)}$ via multiplication by q, q^2 respectively and that $\eta_0\cdot\phi_{(2,1,1,1)} = \phi_{(2,1,0,1)} + q\phi_{(3,1,1,1)}$. From these, (b) easily follows. Similarly, we have $H_{\tau_2}\varpi^{-(4,4,2,2)}U = \mathscr{V}_{0,0}wH_{\psi_2}\varrho^4 U = \mathscr{V}_{0,0}w\varrho^4 U$. As $\mathscr{V}_{0,0}\cap wUw^{-1} = \mathscr{V}_{0,2}$,we see that $[U\varpi^{(4,4,2,2)}H_{\tau_2}]_*(\phi) = \eta_0\eta_1\varrho^4\cdot\phi$. This is again easily computed.

(d) Let λ denote $(4,3,3,3)$. As θ stabilizes λ, we have $H_{\tau_2}\varpi^{-\lambda}U = H_{\psi_2}\varpi^{-\lambda}U$ (since the ϖ^{λ} conjugate of $\mathscr{V}_{0,0}$ is contained in U). As $H_{\psi_2} = \mathscr{Y}_{0,0}H_{\psi_3}$ (13.4.12) and ϖ^{λ} conjugate of H_{ψ_3} is contained in U, we also have

$$H_{\psi_2}\varpi^{-\lambda}U = \varpi^{f_3-\lambda}\mathscr{Y}_{0,0}\varpi^{-f_3}U = \varpi^{f_3-\lambda}\left(\mathscr{Y}_{0,0}\varpi^{-f_3}\mathscr{Y}_{0,0}\right)U.$$

As $U \supset \mathscr{Y}_{0,0}$, the natural map $\mathscr{Y}_{0,0}\varpi^{-f_3}\mathscr{Y}_{0,0} \to H_{\psi_2}\varpi^{-\lambda}U/U$ is a bijection. A set of representattives for $\mathscr{Y}_{0,0}\varpi^{-f_3}\mathscr{Y}_{0,0}$ may be taken to be $y_{-\beta_0}(\kappa)\varpi^{-f_3}$, $y_{\beta_0}(\kappa_1+\varpi\kappa_2)\upsilon_2\varpi^{-f_3}$ where $\kappa,\kappa_1,\kappa_2 \in [\mathscr{k}]$. From this, we easily verify the claim. $\square$

13.5 Zeta elements

Let $T = F^{\times}$, $C = \mathscr{O}_F^{\times}$, $D = 1+\varpi\,\mathscr{O}_F$, $\nu : \iota : H \to T$ be the map that sends (h_1,h_2) to $\det(h_1) = \mu_{\mathrm{sim}}(h_2)$. For the zeta element problem, we set

- $\mathcal{G} = G \times T$, $\Upsilon_{\mathcal{G}}$ collection of all compact open subgroups of $\mathcal{G}$,
- $\iota_{\nu} = \iota \times \nu : H \to \mathcal{G}$,
- U, $\mathcal{K} := K \times C$ as bottom levels
- $x_U = \phi = \phi_{(0,0,0,0)} \in M_{H,\mathcal{O}}(U)$ as the source bottom class
- $\mathcal{L} = K \times D$ as the compactum of field extension of degree $q-1$,
- $\mathfrak{H}_c = \mathfrak{H}^t_{\mathrm{spin},c}(\mathrm{Frob}) \in \mathcal{C}_{\mathcal{O}}(\mathcal{K}\backslash\mathcal{G}/\mathcal{K})$ where $\mathrm{Frob} := \mathrm{ch}(\varpi^{-1}C)$ is the 'arithmetic Frobenius'.

Theorem 13.5.1. *There exists a uniform zeta element for $(x_U, \mathfrak{H}_c, \mathcal{L})$ for all $c \in \mathbb{Z}$.*

This is mostly going to be a matter of bookkeeping and putting together the computations of §13.4. We are going to verify that three linear combinations (which correspond to the twists 'τ_i' for $i = 0,1,2$) of various functions that we have computed in the previous subsection are zero modulo $q-1$ i.e. everything cancels out. The coefficients for theses linear combinations are obtained from the coefficients appearing in $\mathfrak{H}_{\mathrm{spin},0}(X)$ obtained in 13.2.4 (as per the recipe of 3.2) which modulo $q-1$ are the same as those of $\mathfrak{S}_{\mathrm{spin}}(X)$. For $i = 0,1$, this cancellation will amount to verifying that certain polynomials in ϱ vanish, as all functions we computed were written in terms of polynomial actions of ϱ on the bottom class.

Proof. Let $g_0 := (1_G, 1_T)$, $g_1 := (\tau_1, 1_T)$ and $g_2 = (\tau_2, 1_T)$. We show that a uniform zeta element exists with g_i as its twists. By the second part of Corollary 3.3.2, it suffices to consider the case $c = 0$. Let λ_0 denote the central cocharacter $(2,1,1,1) \in \Lambda$ and for a subset $S \subset \Lambda$, cocharacter $\mu \in \Lambda$, let $S(\mu)$ denote $\{\mu+\lambda \,|\, \lambda \in S\} \subset \Lambda$. Set

- $J_{0,0} := \{(0,0,0,0)\}$,

- $J_{0,1} := \{(1,1,1,1)\}$,
- $J_{0,2} := \{(2,2,2,1),\ (2,2,1,2),\ (2,2,1,1),\ (2,1,2,1),\ (2,1,1,1)\}$
- $J_{0,3} := \{(3,3,2,2),\ (3,2,3,2),\ (3,2,2,2)\}$
- $J_{0,4} := \{(4,4,2,2),\ (4,2,4,2),\ (4,3,3,3)\} \sqcup J_{0,2}(\lambda_0) \sqcup J_{0,0}(2\lambda_0)$,

and for $n = 1, \ldots, 4$, $J_{0,4+n} := J_{0,4-n}(n\lambda_0)$. Next, we set

- $J_{1,1} := J_{0,1}$,
- $J_{1,2} := J_{0,2} \setminus \{(2,1,1,1)\}$,
- $J_{1,3} := \{(3,3,2,2),\ (3,2,3,2),\ (3,3,1,2), (3,2,2,3),\ (3,2,2,2)\}$,
- $J_{1,4} := \{(4,4,2,2),\ (4,2,4,2),\ (4,3,3,3),\ \} \sqcup J_{0,2}(\lambda_0)$

and for $n = 1,2,3$, $J_{1,4+n} := J_{0,4-n}(n\lambda_0)$. Finally, set

- $J_{2,2} := \{(2,2,2,1)\}$,
- $J_{2,3} := \{(3,3,2,2)\}$,
- $J_{2,4} := \{(4,4,2,2), (4,3,3,3)\} \sqcup J_{(2,2)}(\lambda_0)$,

and for $n = 2,3$, we set $J_{0,4+n} = J_{0,4-n}(n\lambda_0)$. Let

$$J_0 := \bigsqcup_{n=0}^{8} J_{0,n}, \quad J_1 = \bigsqcup_{n=1}^{7} J_{1,n}, \quad J_2 := \bigsqcup_{n=2}^{6} J_{0,n}.$$

and J denote the disjoint union of J_i indexed by i i.e. elements f J are pairs (i,j) for $i = 0,1,2$, $j \in J_i$. For $j = (a_0, \ldots, a_4) \in J_i$, we define $h_j := \varpi^{-j} \in H$ and $\sigma_j := \iota_\nu(h_j^{-1})g_i = (\varpi^j \tau_i, \varpi^{a_0}) \in \mathcal{G}$. Proposition 13.2.4, Proposition 13.3.12 and Lemma 3.2.7 (and that $\rho^2 \in G$ is central) together imply that

$$\mathfrak{H}^t = \sum_{j \in J} c_j \mathrm{ch}(U\sigma_j\mathcal{K}).$$

for some $c_j \in \mathbb{Z}_{(q)}$. Then c_j are easily seen to be independent of i and can be read off modulo $q-1$ from the coefficients of $\mathfrak{S}_{\mathrm{spin}}(X)$ e.g. $c_{(4,3,3,2)} \equiv 2$, $c_{(3,2,2,2)} \equiv 4$ etc. We note that $H_{g_i} = H_{\tau_i}$ i.e. the twisted intersections are the same whether one considers H as a subgroup of G or $\mathcal{G}$. For $i = 0,1,2$ and $j \in J_i$, let $W_j := h_j U_{\sigma_j} h_j^{-1}$, $\varsigma_j = \sum_{\gamma \in H_{g_i}/W_j} \gamma \in \Phi[H_{g_i}]$. For $i = 0,1,2$, let $V_i = H \cap g_i \mathcal{L} g_i^{-1}$ for $i = 0,1$. Recall that $\varrho_H \in H$ denotes the inverse of $\rho_H \in H$ which we will henceforth denote simply by ϱ. To show a (uniform) zeta element exists, it suffices by Corollary 3.2.3 to verify the following three claims.

Claim 1. $\sum_{j \in J_0} c_j \varsigma_j h_j \cdot \phi$ lies in $(q-1) \cdot M_{H,\mathcal{O}}(U)$.

We recall that $\varrho = \varrho_H$ denotes the inverse of $\rho_H \in H$. Denote $z = \varrho^2 \in H$. We define the following elements of the group algebra $\mathbb{Z}[H]$.

- $p_0 = 1$,
- $p_1 = 4(1+z)$,
- $p_2 = 6z^2 + 16z + 6$,
- $p_3 = 4z^3 + 24z^2 + 24z + 4$,
- $p_4 = z^4 + 16z^3 + 36z^2 + 16z + 1$,

and for $n = 1, 2, 3, 4$, let $p_{4+n} = z^n \cdot p_{4-n}$. By Lemma 13.4.3 and observation 13.4.1, we see $\sum_{j \in J_{0,n}} c_j \varsigma_j h_j \cdot \phi \equiv p_n \cdot \phi$ modulo $q-1$ for all $n = 0, 1, \ldots 8$. Since $p_0 + p_2 + p_4 + p_6 + p_8 - (p_1 + p_3 + p_5 + p_7) = 0$, the claim follows.

Claim 2. $\sum_{j \in J_1} c_j \varsigma_j h_j \cdot \phi$ lies in $(q-1) \cdot M_{H,\mathcal{O}}(H_{\tau_1})$.

Again, we define certain polynomials in ϱ. Let

- $q_1 = 1 + \varrho$,
- $q_2 = 2\varrho^3 + 3\varrho^2 + 3\varrho + 2$,
- $q_3 = \varrho^5 + 3\varrho^4 + 7\varrho^3 + 7\varrho^2 + 3\varrho + 1$,
- $q_4 = \varrho^6 + 5\varrho^5 + 8\varrho^4 + 8\varrho^3 + 5\varrho^2 + \varrho$

and for $n = 1, 2, 3$, let $q_{4+n} = \varrho^{2n} \cdot q_n$. By Lemma 13.4.4 and 13.4.8, we see that $\sum_{j \in J_{0,n}} c_j \varsigma_j h_j \phi \equiv q_n \cdot \phi$ for all $n = 1, \ldots, 7$. Since $q_2 + q_4 + q_6 + q_8 - (q_1 + q_3 + q_5 + q_7) = 0$, the claim follows.

Claim 3. $\sum_{j \in J_2} c_j \varsigma_j h_j \cdot \phi$ lies in $(q-1) \cdot M_{H,\mathcal{O}}(H_{\tau_2})$.

By Lemma 13.4.13, we see that

$$\begin{aligned}
\sum\nolimits_{j \in J_2} c_j \varsigma_j h_j \cdot \phi \equiv \phi_{(2,1,0,1)} & \\
& - \left(\phi_{(2,1,0,1)} + \phi_{(3,1,1,1)} + \phi_{(2,2,0,2)} + \phi_{(3,2,1,2)}\right) \\
& + \phi_{(2,2,0,2)} + \phi_{(4,2,2,2)} + \phi_{(3,1,1,1)} + \phi_{(3,3,1,3)} + 2 \cdot \phi_{(3,2,1,2)} \\
& - \left(\phi_{(3,2,1,2)} + \phi_{(4,2,2,2)} + \phi_{(3,3,1,3)} + \phi_{(4,3,2,3)}\right) \\
& + \phi_{(4,3,2,3)} = 0
\end{aligned}$$

The claim is proved. □

Remark 13.5.2. We would like to point out that the cancellation above is not at all guaranteed by anything in our setup and the reason for it remains somewhat mysterious to us. It however does seem to depend on the fact that polynomial $\mathfrak{S}_{\text{spin}}(X)$ is completely factored. If we rather computed the mixed degrees instead of their action on $\phi_{(0,0,0,0)}$, the linear combinations we see are expanded forms of certain (subfactors of) the Satake polynomial. We would like to make this precise in future.

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