A Note of Shimura's Paper
"Discontinuous Groups and Abelian Varieties"

D. Mumford

Recently, there has been considerable discussion of families of abelian varieties parametrized by quotients of bounded symmetric domains by arithmetic subgroups. An exposition of this material can be found in the papers of Shimura, Kuga, Satake and myself in [1]. Subsequently in vol. 168, p. 171 of this journal, Shimura analyzed closely certain families of this type and showed that the abelian varieties in these families are characterized, surprisingly, by the existence of certain non-holomorphic endomorphisms. Furthermore, he says at the end of his paper (p. 199) that it does not seem that these families fall into the class which I constructed in [1c], which have the following property: they can be grouped into finite sets of families, i.e., families over a base space which is a finite union of varieties $V_i = \Gamma_i \setminus D$, such that the abelian varieties $A_x$ in any one of these coarser families are exactly characterized by the rational $(p, p)$-forms on their powers $A_x \times \cdots \times A_x$. *

The purpose of this note is to clarify the relationship of the different types of families of abelian varieties. First, we recall the most general type $\mathcal{F} = \{X(x) \mid x \in V\}$, as defined by Kuga. Second, we note which of these families are "of Hodge type", i.e., defined by my procedure. Thirdly, we prove that one of Kuga's families is of Hodge type if and only if it contains one abelian variety $X(x_0)$ of CM-type. As a corollary, it follows that Shimura's families in [2] are indeed of Hodge type, contrary to his guess. Lastly, we will give another example of a family of Hodge type not characterized by its ring of endomorphisms.

§ 1. Kuga's Families

Actually, we slightly generalize his definitions, for the sake of our application. We start with:

1. a rational vector space $V$ of dimension $2g$,
2. a lattice $L \subset V$,
3. a non-degenerate skew-symmetric $A : V \times V \to \mathbb{Q}$, integral on $L \times L$.

* In the terminology of [1c], note that I showed that the Hodge group of an abelian variety was determined by these rational $(p, p)$-forms. But my families are given by the various different Hodge types, where a Hodge type is given by a Hodge group $G$, plus a conjugacy class of 1-dimensional tori $T \subset G$; where, however, the conditions on $T$ imply that only a finite number of conjugacy classes of $T$'s can be used to form a Hodge type.
(4) an algebraic group $G$, defined over $\mathbb{Q}$,
(5) a faithful symplectic representation
$$\varphi : G \rightarrow \text{Sp}(V, A)$$
defined over $\mathbb{Q}$.
(6) an arithmetic subgroup $\Gamma \subset G$ such that $\varphi(\Gamma)$ preserves the lattice $L$.

We are interested in complex structures on $V \otimes \mathbb{R}$ so as to make $V \otimes \mathbb{R}/L$ into a complex torus. We find it convenient to adopt the following method of describing possible complex structures: let $T = \{ z \in \mathbb{C} \mid |z| = 1 \}$, regarded as a 1-dimensional algebraic group over the reals $\mathbb{R}$. Then, all complex structures on $V \otimes \mathbb{R}$ define a homomorphism of algebraic groups over $\mathbb{R}$:
$$\varphi : T \rightarrow GL(V)$$
via $\varphi(e^{i\theta}) = \text{mult. by } e^{i\theta}$ in $V \otimes \mathbb{R}$. Conversely, any such homomorphism puts a complex structure on $V^0 \otimes \mathbb{R}$, hence makes $V \otimes \mathbb{R}/L$ into a complex torus, which we shall call $X_\varphi$. Furthermore, if $\varphi$ satisfies the Riemann conditions
(a) $\varphi(T) \subset \text{Sp}(V,A)$, and
(b) $A(x, \varphi(i) \cdot x) > 0$ all $x \in V$, $x \neq 0$, then $X_\varphi$ is an abelian variety, with a canonical polarization induced by $A$. Now we want a family of abelian varieties $X(x)$ parametrized by points $x \in V = \Gamma \backslash G^0_\mathbb{R}/K^0_\mathbb{R}$ for a certain maximal compact subgroup $K^0_\mathbb{R}$ of $G^0_\mathbb{R}$. In particular, the identity in $G^0_\mathbb{R}$ defines a double coset $\Gamma \cdot e \cdot K^0_\mathbb{R}$ which is a base point 0 of this base space $V$. Our family can be regarded as a family of perturbations of the “base abelian variety” $X(0)$ over 0. So as our next piece of data, we assume an arbitrary $X(0)$ is given:
(7) a complex structure $\varphi_0 : T \rightarrow \text{Sp}(V, A) \subset GL(V)$ such that $J_0 = \varphi_0(i)$ satisfies:
$$A(x, J_0 x) > 0, \text{ all } x \neq 0 \text{ in } V.$$ 

Now, to every point $g \in G^0_\mathbb{R}$, let
$$X(g) = X_{\varphi(g) \varphi_0 \cdot \varphi(g)^{-1}} = \left\{ \begin{array}{l}
\text{the polarized abelian variety } V \otimes \mathbb{R}/L, \\
\text{with complex structure } \varphi(g) \cdot \varphi_0 \cdot \varphi(g)^{-1}, \\
\text{Riemann form } A
\end{array} \right\}.$$ 

If $K^0_\mathbb{R} = \{ g \in G_\mathbb{R} \mid \varphi(g) \varphi_0 = \varphi_0 \varphi(g) \}$, then $X(g)$ depends obviously only on the image of $g$ in the coset space $G^0_\mathbb{R}/K^0_\mathbb{R}$. Moreover, the automorphism $\varphi(g)$ of $V$ sets up isomorphisms of the abelian variety $X(g)$ with $X(\gamma g)$ for all $g \in G^0_\mathbb{R}$. Therefore $X(g)$ depends only on the image $x$ of $g$ in $\Gamma \backslash G^0_\mathbb{R}/K^0_\mathbb{R}$ and we may write for simplicity $X(x)$, instead of $X(g)$.

We still want to ensure that $G^0_\mathbb{R}/K^0_\mathbb{R}$ is a bounded symmetric domain and that $\{ X(x) \}$ is a holomorphic family of abelian varieties. This is guaranteed by imposing on the data (1)—(7) the following condition (which is the integrated form of Satake’s original condition $(H_1)$):

$$(H^*_t) \quad \varphi(G) \text{ is normalized by } \varphi_0(T).$$ 

It follows easily from $(H^*_t)$ that $G$ is reductive, that $K^0_\mathbb{R}$ is a maximal compact subgroup of $G^0_\mathbb{R}$, that $G^0_\mathbb{R}/K^0_\mathbb{R}$ is a Hermitian symmetric space, and finally that
the map induced by $\tau$: \[ G_0^0 / K_0^0 \to \text{Sp}(V, A) \bigg/ \left( \text{centralizer of } \varphi_o \right) \cong \text{Siegel's upper half-plane} \]
is holomorphic. The resulting family \( \{ X(x) | x \in \Gamma \backslash G_0^0 / K_0^0 \} \) glued together into a complex analytic fibre system of abelian varieties over the base space \( \Gamma \backslash G_0^0 / K_0^0 \) will be denoted by \( \mathcal{X}(G, \varphi, \varphi_o) \).

Notice that the base point in Kuga's families is arbitrary. For any \( g_0 \in G_0^0 \), we can replace \( \varphi_o \) by \( \varphi'_o = \varphi(g_0) \cdot \varphi_o \cdot \varphi(g_0)^{-1} \), and obtain the same family as before, but with a different base point \( X(0) \).

§ 2. Hodge Groups and CM-Type

We continue to assume that the data \((V, L, A)\) is given. Suppose \( \varphi : T \to \text{Sp}(V, A) \) is any complex structure on \( V \otimes \mathbb{R} \) satisfying the Riemann positivity condition, so that the corresponding \( X_\varphi \) is an abelian variety.

**Definition.** The Hodge group \( \text{Hg}(X_\varphi) \) of \( X_\varphi \) is the smallest algebraic subgroup of \( \text{Sp}(V, A) \) defined over \( \mathbb{Q} \) and containing \( \varphi(T) \). Recall that \( \text{Hg}(X_\varphi) \) is always reductive, with compact center, and semi-simple part of Hermitian type.

**Definition.** If \( \text{id.} : \text{Hg}(X_\varphi) \to \text{Sp}(V, A) \) is just the inclusion map, then the families

\( \mathcal{X}(\text{Hg}(X_\varphi), \text{id.}, \varphi) \)

are called the families of Hodge type.

These are exactly the families constructed in [1 c].

**Definition.** An abelian variety \( X \) over \( \mathbb{C} \) is of CM-type if \( X \) is isogenous to a product \( X_1 \times \cdots \times X_k \) of simple abelian varieties and there are fields \( K_i \subset \text{Hom}(X_i, X_i) \otimes \mathbb{Q} \) such that \( [K_i : \mathbb{Q}] \geq 2 \dim X_i \) (in which case, \( [K_i : \mathbb{Q}] = 2 \dim X_i \) and \( K_i = \text{Hom}(X_i, X_i) \otimes \mathbb{Q} \)).

The following is well known ([3], § 5):

**Proposition.** \( X \) is of CM-type if and only if \( \text{Hom}(X, X) \otimes \mathbb{Q} \) contains a commutative semi-simple \( \mathbb{Q} \)-algebra \( R \) such that \( [R : \mathbb{Q}] \geq 2 \dim X \), and if \( R \) exists, \( [R : \mathbb{Q}] = 2 \dim X \).

Yet another characterization of CM-type is:

**Proposition.** \( X_\varphi \) is of CM-type if and only if \( \text{Hg}(X_\varphi) \) is a torus algebraic group.

**Proof.** Any endomorphism of \( X_\varphi \) has a natural representation as an endomorphism of \( V \) over \( \mathbb{Q} \), and it is easily seen that for any \( \varphi \),

\[ \text{Hom}(X_\varphi, X_\varphi) \otimes \mathbb{Q} \cong \{ g \in \text{Hom}(V, V) | g \varphi = \varphi g \} . \]
Since $Hg(X_\phi)$ is generated by $\text{Im}(\phi)$ and by its conjugates over $\mathbb{Q}$, it follows that:

$$\text{Hom}(X_\phi, X_\phi) \otimes \mathbb{Q} \cong \{ g \in \text{Hom}(V, V)_\mathbb{Q} \mid gg' = g'g, \quad \text{all} \quad g' \in Hg(X_\phi) \}.$$  \hspace{1cm} (\ast)

Now suppose that $X_\phi$ is of CM-type. Let $R$ be the commutative semi-simple $\mathbb{Q}$-algebra given by the previous proposition. Then via the above isomorphism it follows that $Hg(X_\phi)$ commutes with a maximal commutative semi-simple subalgebra $R'$ of $\text{Hom}(V, V)$. But therefore $Hg(X_\phi) \subseteq R'$, hence $Hg(X_\phi)$ is itself commutative, hence it is a torus algebraic group. Conversely, if $Hg(X_\phi)$ is a torus algebraic group, then as a subgroup of $GL(V)_\mathbb{Q}$ it is diagonalizable. Therefore, the commutator of $Hg(X_\phi)$ in $\text{Hom}(V, V)_\mathbb{Q}$ and hence in $\text{Hom}(V, V)_\mathbb{Q}$ contains maximal commutative semi-simple subalgebras $R'$. Since for all such $R'$, $[R':\mathbb{Q}] = \dim V = 2 \dim X_\phi$, this implies by ($\ast$) that $X_\phi$ is of CM-type.

QED.

\section{3. The Theorem}

\textbf{Theorem.} a) Every family $\mathcal{X}(Hg(X_\phi), \text{id}, \phi)$ of Hodge type contains abelian varieties of CM-type.

b) If $\mathcal{X}(G, \phi_0, \phi_0)$ contains a member of CM-type, then $\mathcal{X}(G, \phi, \phi_0)$ is isomorphic to a family of Hodge type.

\textbf{Proof.} It is well known that for any algebraic group $G$ over $\mathbb{Q}$, $G_\mathbb{Q}$ is dense in $G_\mathbb{R}$. This implies that for such groups $G$, every $G_\mathbb{R}$-conjugacy class of maximal algebraic tori in $G$, defined over $\mathbb{R}$, contains tori defined over $\mathbb{Q}$. Namely, let $T_1 \subseteq G$ be a maximal torus defined over $\mathbb{R}$. If $a \in (T_1)_\mathbb{R}$ is a regular element, then $T_1$ is the centralizer of $a$, and $a$ has an open neighborhood $U \subseteq G_\mathbb{R}$ such that the centralizer of any $a' \in U$ is a conjugate of $T_1$. If $a' \in U \cap G_\mathbb{Q}$, then the centralizer of $a'$ is a conjugate of $T_1$ defined over $\mathbb{Q}$, as required. Now apply these results to $Hg(X_\phi)$. Let $K = \text{centralizer of } \phi(T)$, regarded as an algebraic subgroup of $Hg(X_\phi)$ defined over $\mathbb{R}$. Then $K$ has at least one maximal algebraic torus $T_1 \subseteq K$ defined over $\mathbb{R}$ (1 1\epsilon, p. 26). Since $\phi(T) \subseteq \text{center}(K)$, $\phi(T) \subseteq T_1$. Moreover, if $T_1$ is any torus in $Hg(X_\phi)$ containing $T_1$, then $T_1'$ will centralize $\phi(T)$, hence $T_1' \subseteq K$, hence $T_1' = T_1$; i.e., $T_1$ is a maximal algebraic torus of $Hg(X_\phi)$ too. By our first remark, there are elements $g \in Hg(X_\phi)_\mathbb{R}$ such that $T_1 = gT_1g^{-1}$ is defined over $\mathbb{Q}$. But then $T_2 \supseteq g\phi(T)g^{-1}$. Therefore $T_2 \supseteq Hg(X_{\phi\phi^{-1}})$, and so $Hg(X_{\phi\phi^{-1}})$ must be an algebraic torus. Therefore $X_{\phi\phi^{-1}}$ is of CM-type and this proves (a).

To prove (b), first replace $\phi_0$ by $\phi(g)\phi_0\phi(g)^{-1}$ for a suitable $g \in G_\mathbb{Q}$, so that the abelian variety $X(0)$ is of CM-type. This does not alter the family $\mathcal{X}(G, \phi, \phi_0)$. Let $T_1 = Hg(X_{\phi0})$. This is an algebraic torus, defined over $\mathbb{Q}$. Since $T_1$ is generated by $\phi_0(T)$ and its $\mathbb{Q}$-conjugates, all of which normalize $\phi(G)$, it follows that $T_1$ normalizes $\phi(G)$. So $G^* = \phi(G)$. $T_1$ is an algebraic subgroup of $\text{Sp}(V, A)$ defined over $\mathbb{Q}$, with $\phi(G)$ as a normal subgroup. Note that $G^*$ is still reductive since $\phi(G)$ and $T_1$ are reductive. In particular, $G^* = (\text{semi-simple part}) \cdot (\text{central torus})$, hence $G^* = \phi(G) \cdot (\text{central torus})$. Therefore, the two collections of complex
structures:

(I) \{g(\varphi_0 g)(g)^{-1} \mid g \in G_{k}^0\},

(II) \{g \varphi_0 g^{-1} \mid g \in G_{k}^{*0}\}

are exactly the same. Moreover, \(G_{k}^{*0}/(\text{centr. of } \varphi_0)\) is canonically isomorphic to \(G_{k}^{*0}/(\text{centr. of } \varphi_0)\). Therefore the 2 families \(\mathcal{F}(G, \varphi, \varphi_0)\) and \(\mathcal{F}(G^{*}, \text{id.}, \varphi_0)\) (where id.: \(G^{*} \to \text{Sp}(V, A)\) is the inclusion map) are isomorphic. Finally, let \(\varphi = g \varphi_0 g^{-1}\) where \(g \in G_{k}^{*0}\) is a generic point of \(G^{*}\) over a field of definition of \(\varphi_0(T)\). Then \(Hg(X_{\varphi})\) is the smallest \(\mathbb{Q}\)-rational subgroup of \(G^{*}\) containing all the tori \(g \varphi_0 g^{-1}, g \in G^{*}, \sigma \in \text{Aut}(C)\). Hence \(Hg(X_{\varphi})\) is just the smallest subgroup of \(G^{*}\) (defined over any field) containing all the tori \(g T_1 g^{-1}, g \in G^{*}\). In particular \(Hg(X_{\varphi})\) is a normal subgroup of \(G^{*}\), defined over \(\mathbb{Q}\) and containing \(\varphi(T)\) and \(\varphi_0(T)\). Therefore \(G^{*} = Hg(X_{\varphi}) \cdot G_2\) where \(G_2\) commutes with \(Hg(X_{\varphi})\) and hence with \(\varphi(T)\) and with \(\varphi_0(T)\). It follows that the sets (I) and (II) of complex structures are equal to the set:

(III) \{g \varphi g^{-1} \mid g \in Hg(X_{\varphi})^0\}.

It also follows that \(G_{k}^{*0}/(\text{centr. of } \varphi_0)\) is isomorphic to \(Hg(X_{\varphi})^0/(\text{centr. of } \varphi_0)\), hence the 2 families \(\mathcal{F}(G^{*}, \text{id.}, \varphi_0)\) and \(\mathcal{F}(Hg(X_{\varphi}), \text{id.}, \varphi)\) are isomorphic. QED.

§ 4. An Example

Since families of abelian varieties which are not characterized by their endomorphism rings are fairly mysterious, it seems worth-while to present as an example what seems to be the only family of this type of 4-dimensional abelian varieties. (In dimensions 1, 2 and 3, all families are characterized by endomorphisms.)

To define this family, we need an apparently “well-known” construction for central simple algebras: if \(L \supset K\) is a finite separable extension of degree \(n\), and \(D\) is a central simple algebra over \(L\), with \([D : L] = e^2\), then there is canonical central simple algebra \(\text{Cor}_{L/K}(D)\) over \(K\), with \([\text{Cor}(D) : K] = e^{2n}\), and with a homomorphism

\[\text{Nm}: D^* \to \text{Cor}_{L/K}(D)^*\]

of units. It is simply the corestriction map in the cohomology theory of groups applied to the Brauer group. To construct it, let \(\Omega\) be a separable closure of \(K\) and let \(\sigma_1, \ldots, \sigma_n : L \to \Omega\) be the distinct \(K\)-isomorphisms from \(L\) to \(\Omega\). Let \(D^{(i)} = D \otimes_{\Omega}(\Omega, \sigma_i)\) be the central simple \(\Omega\)-algebra obtained by base change with respect to \(\sigma_i\). Then \(\text{Gal}(\Omega/K)\) acts on

\[E = D^{(1)} \otimes \cdots \otimes D^{(n)}\]

in a natural way, i.e. if \(\tau : \Omega \to \Omega\) is a \(K\)-isomorphism, then \(\tau \circ \sigma_i = \sigma_{\pi(i)}\) for some permutation \(\pi\), and the maps

\[d \otimes a \mapsto d \otimes \tau(a)\]
induce semi-linear-isomorphisms $D^{(i)} \cong D^{(i \cdot \mathbb{K})}$, hence a semi-linear automorphism of $E$. Let $\text{Cor}_{L/K}(D)$ be the subalgebra of $E$ left fixed by this action: this will be a central simple $K$-algebra such that $\text{Cor}_{L/K}(D) \otimes_K \Omega \cong E$. Finally define $\text{Nm}(d)$ to be the element

$$(d \otimes 1) \otimes \cdots \otimes (d \otimes 1)$$

of $E$, which is clearly in $\text{Cor}_{L/K}(D)$.

Now let $K$ be a totally real cubic number field, and let $D$ be a quaternion division algebra over $K$ such that

$$\text{Cor}_{K/\mathbb{Q}}(D) \text{ splits, i.e. } \cong M_3(\mathbb{Q}),$$

$$D \otimes_K \mathbb{R} \cong \mathbb{K} + \mathbb{K} + M_2(\mathbb{R}).$$

Then we get a natural homomorphism:

$$\text{Nm} : D^* \to GL(8, \mathbb{Q}).$$

Let $-\bar{\cdot}$ be the standard involution of $D$, and let

$$G = \{ x \in D^* \mid x \cdot \bar{x} = 1 \}.$$

Then $G$ is an algebraic group over $\mathbb{Q}$, which is $\mathbb{Q}$-simple, but which by (2), is a $\mathbb{Q}$-form of the $\mathbb{R}$-algebraic group $SU(2) \times SU(2) \times SL(2, \mathbb{R})$. Moreover, let $V = \mathbb{Q}^8$; then, via $\text{Nm}$, $G$ has an algebraic representation in $V$ defined over $\mathbb{Q}$, which is a $\mathbb{Q}$-form of $\mathbb{R}$-representation:

$$SU(2) \times SU(2) \times SL(2, \mathbb{R}) \to SO(4) \times SL(2, \mathbb{R})$$

acting on $\mathbb{R}^4 \otimes \mathbb{R}^2$

Over $\mathbb{R}$, this representation leaves invariant a unique symplectic form (up to scalars), so there is a unique symplectic form $A : V \times V \to \mathbb{Q}$ left fixed by our $\mathbb{Q}$-representation. Let $L \subset V$ be any lattice, and $\Gamma \subset G$ any arithmetic subgroup preserving $L$. Finally, let

$$\varphi_0 : T \to \text{Sp}(V, A)_{\mathbb{R}}$$

be the homomorphism:

$$\varphi_0 : T \to SO(4) \times SL(2, \mathbb{R}) \subset \text{Sp}(V, A)_{\mathbb{R}},$$

$$e^{i\theta} \to I_4 \otimes \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

This gives data (1)—(7) as in §1, and $(H^*_p)$ is obviously satisfied. Moreover, since $G$ is $\mathbb{Q}$-simple, $q(G)$ is exactly the Hodge group of a generic conjugate $q(d) \cdot \varphi_0 \cdot q(d)^{-1}$, so we have a family of Hodge type. Finally, $V$ is an absolutely irreducible representation of $G$, so whenever $\text{Hg}(X_p) = q(G)$, $X_p$ has no non-trivial endomorphisms.
References

   1a. Article of Shimura, p. 312.
   1b. Article of Kuga, p. 338.
   1c. Article of Mumford, p. 347.
   1d. Article of Satake, p. 352.

Professor D. Mumford
Department of Mathematics
Harvard University
Cambridge, Mass. 02138, USA

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