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AN ALGEBRAIC SURFACE WITH K AMPLE, \((K^2) = 9, \ p_g = q = 0\)

By D. Mumford

Severi raised the question of whether there existed an algebraic surface \(X\) homeomorphic to \(\mathbb{P}^2\) but not isomorphic to it (as a variety), and conjectured that such a surface did not exist. The essential problem in proving this is to eliminate the possibility that the canonical class \(K\), as a member of the infinite cyclic group \(H^2(X, \mathbb{Z})\) might be a positive multiple (in fact, 3) of the ample generator of \(H^2(X, \mathbb{Z})\) instead of a negative multiple (in fact, \(-3\)) as it ought to be. That this is the problem is clear from Castelnuovo’s criterion for rationality, and was analyzed and generalized to higher dimensions in the paper [3] of Hirzebruch and Kodaira where it was shown that in odd dimensions, \(\mathbb{P}^n\) is the only variety in its homeomorphism class. Severi’s question was finally answered by S. Yau [8] as a Corollary of his result that all varieties \(X\) on which \(K\) is ample carry a unique Kähler-Einstein metric. In fact, this result shows that when \(X\) is a surface on which \(K\) is ample, then the Chern numbers satisfy \(c_1^2 \leq 3c_2\), with equality if and only if \(X\) is isomorphic to \(D_2/\Gamma\) \((D_2 = \text{unit ball in } \mathbb{C}^2, \ \Gamma \subset SU(2, 1)/\text{(center)}\) a discrete torsion-free co-compact subgroup; Hirzebruch in [2] had much earlier shown that the surfaces \(D_2/\Gamma\) did satisfy \(c_1^2 = 3c_2\). However, the question arises: how close can we come to a surface with \(K\) ample which mimics the topology of \(\mathbb{P}^2\)? In particular, does there exist such a surface with the same Betti numbers as \(\mathbb{P}^2\)? By the standard results on algebraic surfaces, this means that we seek a surface \(X\) such that:

\[ p_g = q = 0, \ \text{hence } \chi(\mathcal{O}_X) = 1 \]

\[(c_1^2) = (K^2) = 9 \]

\[ B_0 = B_2 = B_4 = 1, \quad B_1 = B_3 = 0, \ \text{hence } c_2 = 3 \]

I shall exhibit one such surface. My method is not by complex uniformization, as used by Shavel [6] and Jenkins (unpublished) in the
construction of a surface with $K$ positive and the same Betti numbers as $\mathbb{P}^1 \times \mathbb{P}^1$, but by the $p$-adic uniformization introduced recently by Kurihara [1] and Mustafin [5]. After looking for such an example at some length, I would hazard the guess that there are, in fact, very few such surfaces (combining Yau's results with Weil's theorem [7] that discrete co-compact groups $\Gamma \subset SU(2, 1)$ are rigid, it follows that there are in any case only finitely many such surfaces). But it seems a difficult matter to find some way of enumerating all such surfaces.

1. **$p$-adic uniformizations in general.** In this section we wish to summarize and extend somewhat the results of Kurihara and Mustafin cited above, restricting ourselves however to the 2-dimensional case. Let $R$ be a complete discrete valuation ring with fraction field $K$ and residue field $k = R/\pi R$. We assume $k$ is finite. The basis of the construction is a beautiful scheme $\mathcal{X}$, locally of finite type over $R$, which may be described by charts as follows:

$$\mathcal{X} = \bigcup_{A \in GL(3, K)} \text{Spec} R \left[ \frac{l_0}{l_1}, \frac{l_1}{l_2}, \pi, \frac{l_2}{l_0} \right] - (C_0 \cup C_1 \cup C_2)$$

where $l_i = \sum_{j=0}^3 A_{ij} x_j$, $A = (A_{ij})$

$C_0 =$ set of curves

$$\pi = \frac{l_0}{l_1} = 0, \quad a \left( \frac{l_1}{l_2} \right) \left( \pi, \frac{l_2}{l_0} \right) + b \left( \pi, \frac{l_2}{l_0} \right) + c = 0$$

$a$, $b$, $c \in k$, $a \cdot c \neq 0$, plus the curves

$$\pi = \frac{l_0}{l_1} = 0, \quad \left( \pi, \frac{l_2}{l_0} \right) + c = 0$$

and

$$\pi = \frac{l_0}{l_1} = 0, \quad \frac{l_1}{l_2} + c = 0 \quad (c \in k^*).$$

$C_1$, $C_2 =$ similar sets of curves where the role of $l_0/l_1$, $l_1/l_2$, $\pi(l_2/l_0)$ are permuted cyclically.

Here the glueing represented by the union sign is induced by the re-
requirement that \( \mathcal{X} \) is irreducible and separated with function field

\[
K \left( \frac{X_1, X_2}{X_0} \right),
\]

which is the common fraction field of all affine rings.

The closed fibre \( \mathcal{X}_0 \) of \( \mathcal{X} \) can be represented graphically by means of the Bruhat-Tits building \( \Sigma \) attached to \( \text{PGL}(3, K) \). In fact, the 3 sets:

- **Components** \( E \) of \( \mathcal{X}_0 \)
- Free rank 3 \( R \)-submodules \( M \subset K \cdot X_0 \oplus K \cdot X_1 \oplus K \cdot X_2 \), modulo \( M \sim \pi^k \cdot M \)
- Vertices \( \nu \) of \( \Sigma \)

are isomorphic. Moreover, the components of \( \mathcal{X}_0 \) cross normally, and if \( E_i, M_i, \nu_i, i = 1, 2, 3 \), correspond as above, then:

a) \( E_1 \cap E_2 \) is a curve ⇔ \( M_1 \not\supseteq \pi^k M_2 \not\supseteq \pi M_1 \), some \( k \in \mathbb{Z} \)

⇒ \( \nu_1, \nu_2 \) are joined in \( \Sigma \) by a segment

b) \( E_1 \cap E_2 \cap E_3 \) is a triple point ⇔ \( M_1 \not\supseteq \pi^k M_2 \not\supseteq \pi^l M_3 \not\supseteq \pi M_1 \), some \( k, l \in \mathbb{Z} \) (or same with 2, 3 interchanged)

⇒ \( \nu_1, \nu_2, \nu_3 \) are the vertices of 2-simplex in \( \Sigma \).

To describe \( \mathcal{X} \) in a Zariski-open neighborhood of some component \( E \) of \( \mathcal{X}_0 \), we can proceed geometrically as follows: let \( E \) correspond to \( M \) and let \( Y_0, Y_1, Y_2 \) be an \( R \)-basis of \( M \). Start with \( \mathbb{P}_R^2 \) based on homogeneous coordinates \( Y_0, Y_1, Y_2 \) (hence with function field \( K(X_1/X_0, X_2/X_0) \) still). First, blow up all \( k \)-rational points of the closed fibre \( \mathbb{P}_k^2 \) of \( \mathbb{P}_R^2 \) (if \( k \) has \( q \) elements, there are \( q^2 + q + 1 \) of these). Second, blow up the proper transforms on this scheme of all \( k \)-rational lines on the original closed fibre \( \mathbb{P}_k^2 \) (again there are \( q^2 + q + 1 \) of these). Call this \( \mathcal{X}_M \) and let \( E_M \subset \mathcal{X}_M \) be the proper transform of \( \mathbb{P}_k^2 \). Then a suitable Zariski-neighborhood of \( E_M \) in \( \mathcal{X}_M \) is isomorphic to a neighborhood of \( E \) in \( \mathcal{X} \).

In particular, all surfaces \( E \) are rational surfaces gotten by blowing up \( \mathbb{P}_k^2(q^2 + q + 1) \) times and they meet the \( 2(q^2 + q + 1) \) adjacent components in rational curves. These rational curves are either exceptional curves \( C \) of the first kind, in which case \( (C^2) = -1 \), or proper trans-
forms of lines along which \( q + 1 \) points have been blown up, in which case:

\[
(C^2) = +1 - (q + 1) = -q.
\]

Thus geometrically, if \( C = E_1 \cap E_2 \), then \((C^2)_{E_i} = -1\) and \((C^2)_{E_i} = -q\) or vice versa; this asymmetry corresponds in (a) above to whether

\[
\dim_k(M_1/\pi^k M_2) = 1 \quad \text{or} \quad \dim_k(\pi^k M_2/\pi^k M_1) = 1
\]

and in \( \Sigma \) to the orientation on the segment from \( \nu_1 \) to \( \nu_2 \).

Now if \( \Gamma \subset \text{PGL}(3, K) \) is a discrete torsion-free co-compact group, we define first a formal scheme \( \mathfrak{X}/\Gamma \) over \( R \) by dividing the formal completion of \( \mathfrak{X} \) along \( \pi = 0 \) by \( \Gamma \) (this is possible because \( \Gamma \) acts freely and discontinuously in the Zariski-topology on \( \mathfrak{X}_0 \)). Secondly, one verifies that the dualizing sheaf \( \omega_{\mathfrak{X}} \) is ample on each component of \( \mathfrak{X}_0 \), hence it descends to an invertible sheaf \( \omega_{(\mathfrak{X}/\Gamma)} \) on \( \mathfrak{X}/\Gamma \) with the same property: this allows one to conclude that \( \mathfrak{X}/\Gamma \) can be algebraized to true projective scheme over \( R \), which, for simplicity, we denote \( \mathfrak{X}/\Gamma \). Since the generic fibre \( \mathfrak{X}_\eta \) of \( \mathfrak{X} \) is smooth over \( K \), the generic fibre \( (\mathfrak{X}/\Gamma)_\eta \) is also smooth over \( K \), hence

\[
\omega_{(\mathfrak{X}/\Gamma)_\eta} = \Omega_{(\mathfrak{X}/\Gamma)_\eta}^2
\]

hence \( (\mathfrak{X}/\Gamma)_\eta \) is a surface of general type without smooth rational curves \( C \) with \((C^2) = -1\) or \(-2\). It is not hard to compute the invariants of \( (\mathfrak{X}/\Gamma)_\eta \); to do this, note that \( (\mathfrak{X}/\Gamma)_0 \) consists of finite set of rational surfaces, crossing each other (possibly crossing themselves) transversally in rational double curves and triple points. Let

\[
E_\alpha = \text{normalizations of the components of } (\mathfrak{X}/\Gamma)_0, \quad 1 \leq \alpha \leq \nu_2
\]

\[
C_\beta = \text{normalizations of the double curves of } (\mathfrak{X}/\Gamma)_0, \quad 1 \leq \beta \leq \nu_1
\]

\[
P_\gamma = \text{triple points of } (\mathfrak{X}/\Gamma)_0, \quad 1 \leq \gamma \leq \nu_0
\]

We get an exact sequence:

\[
0 \to \mathcal{O}_{(\mathfrak{X}/\Gamma)_0} \to \bigoplus_{\alpha=1}^{\nu_1} \mathcal{O}_{E_\alpha} \to \bigoplus_{\beta=1}^{\nu_1} \mathcal{O}_{C_\beta} \to \bigoplus_{\gamma=1}^{\nu_0} \mathcal{O}_{P_\gamma} \to 0
\]
hence
\[ \chi(\mathcal{O}_{(\mathcal{X}/\Gamma)}_q) = \chi(\mathcal{O}_{(\mathcal{X}/\Gamma)_0}) = \sum_\alpha \chi(\mathcal{O}_{E_\alpha}) - \sum_\beta \chi(\mathcal{O}_{c_\beta}) + \sum_\gamma \chi(\mathcal{O}_{\nu_\gamma}) = \nu_2 - \nu_1 + \nu_0. \]

Let \( N \) be the number of orbits when \( \Gamma \) acts on the vertices of \( \Sigma \). Clearly \( N = \nu_2 \). But each \( E_\alpha \) contains \( 2(q^2 + q + 1) \) double curves, each on two \( E_\alpha \)'s, so
\[ \nu_1 = N(q^2 + q + 1). \]

And each double curve passes through \( (q + 1) \) triple points, each on three double curves, so
\[ \nu_0 = N \frac{(q^2 + q + 1)(q + 1)}{3}. \]

Thus
\[ \chi(\mathcal{O}_{(\mathcal{X}/\Gamma)_q}) = N \left[ 1 - (q^2 + q + 1) + \frac{(q^2 + q + 1)(q + 1)}{3} \right] = N \frac{(q - 1)^2(q + 1)}{3}. \]

Next:
\[ (c_{1, (\mathcal{X}/\Gamma)_q})^2 = (c_1(\omega_{(\mathcal{X}/\Gamma)_0})^2) = \sum_\alpha (\text{res}_{E_\alpha} c_1(\omega_{(\mathcal{X}/\Gamma)_0})^2) = \sum_\alpha ((c_1(\omega_{E_\alpha}) + \sum_{\beta \neq \alpha} E_\alpha \cap E_\beta)^2) \]

All \( E_\alpha \)'s are just \( B = \) (the blow-up of \( \mathbb{P}^2_k \) at all \( (q^2 + q + 1) \)-rational points). Let \( \pi : B \to \mathbb{P}^2_k \) be the blow-up map, let \( h \) be the divisor class of
a line on $\mathbb{P}_k^2$, let $e_i \subset B$ be the exceptional divisors and let $l_j \subset B$ be the proper transforms of the lines. Then $c_1(\omega_{E_\alpha}) + \Sigma_{\beta \neq \alpha} (E_\alpha \cap E_\beta)$ corresponds on $B$ to:

$$K_B + \Sigma e_i + \Sigma l_j \equiv (\pi^{-1}(-3h) + \Sigma e_i)$$

$$+ \left( \Sigma e_i \right) + \Sigma \left( \pi^{-1}(h) - \text{the } e_i \text{ meeting } l_j \right)$$

$$= \pi^{-1}((q^2 + q - 2)h) - (q - 1)\left( \Sigma e_i \right)$$

with self-intersection $3(q - 1)^2(q + 1)$. Thus

$$(c_{1,(\mathfrak{X}/\Gamma)_q})^2 = 3N(q - 1)^2(q + 1).$$

By Riemann-Roch,

$$c_{2,(\mathfrak{X}/\Gamma)_q} = N(q - 1)^2(q + 1).$$

To determine the irregularity of $(\mathfrak{X}/\Gamma)_q$, we can use the relative Picard scheme $\text{Pic}_{\mathfrak{X}/\Gamma}^0$: its closed fibre is $\text{Pic}_{(\mathfrak{X}/\Gamma)_0}^0$, and since $(\mathfrak{X}/\Gamma)_0$ has normal crossings and rational components, this is an algebraic torus. In particular points of finite order $l, p \nmid l$, are dense: these correspond to $l$-cyclic coverings of $(\mathfrak{X}/\Gamma)_0$ and such coverings lift to $(\mathfrak{X}/\Gamma)_q$. Therefore the points of finite order $l, p \nmid l$, of $(\text{Pic}_{\mathfrak{X}/\Gamma}^0)_0$ lift to points of $(\text{Pic}_{\mathfrak{X}/\Gamma}^0)_q$ and hence $\text{Pic}_{\mathfrak{X}/\Gamma}^0$ is flat over $R$. On the other hand, a line bundle on $(\mathfrak{X}/\Gamma)_0$ is a line bundle on $\mathfrak{X}_0$ with $\Gamma$ action: if it is in $\text{Pic}^0$, it is the trivial line bundle on $\mathfrak{X}_0$ and a $\Gamma$-action is just a homomorphism from $\Gamma$ to $\mathbb{G}_m$. Thus finally, using Kajdan's theorem [4] that $\Gamma/[\Gamma, \Gamma]$ is finite, we deduce

irregularity of $(\mathfrak{X}/\Gamma)_q = \dim (\text{Pic}_{\mathfrak{X}/\Gamma}^0)_q$

$$= \dim (\text{Pic}_{\mathfrak{X}/\Gamma}^0)_0$$

$$= \dim \text{Hom}(\Gamma, \mathbb{G}_m)$$

$$= rk_2\Gamma/[\Gamma, \Gamma]$$

$$= 0.$$
Thus the numbers $h^{p,q}$ of $(\mathcal{X}/\Gamma)_{q}$ fit into the pattern:

\[
\begin{array}{c|ccc}
q & M - 1 & 0 & 1 \\
0 & M & 0 & \\
1 & 0 & M - 1 & \\
\end{array}
\]

\[M = \frac{N(q - 1)^2(q + 1)}{3}\]

In particular, if $N = 1$, $q = 2$, then $M = 1$ and $(\mathcal{X}/\Gamma)_{q}$ is a surface of the desired type. In this case, in fact $(\mathcal{X}/\Gamma)_{0}$ is one rational surface, $\mathbb{P}^2$ blown up 7 times, crossing itself in 7 rational double curves, themselves crossing in 7 triple points. The confusion arising from trying to draw the result brings vividly to mind Lewis Carroll’s comment on the sandy shore—“If seven maids with seven brooms were to sweep it for half a year, do you suppose, the Walrus said, that they could get it clear?”

2. The Example. The example is based on the 7th roots of unity: fix the notation:

\[
\xi = e^{2\pi i/7}
\]

\[
\lambda = \xi + \xi^2 + \xi^4 = \left(\frac{-1 + \sqrt{-7}}{2}\right)
\]

\[
\bar{\lambda} = \xi^3 + \xi^5 + \xi^6 = \left(\frac{-1 - \sqrt{-7}}{2}\right)
\]

We have the fields:

\[
\begin{align*}
\mathbb{Q}(\xi) & \quad \text{deg 3} \\
\{\text{Galois group } \mathbb{Z}/3\mathbb{Z}, \text{ generator } \sigma, \sigma(\xi) = \xi^2\} \\
\mathbb{Q}(\lambda) & \quad \text{deg 2} \\
\{\text{Galois group } \mathbb{Z}/2\mathbb{Z}, \text{ generator } z \rightarrow \bar{z}\}
\end{align*}
\]

Note $\mathbb{Q}(\lambda)$ is a UFD, $2 = \lambda \cdot \bar{\lambda}$ is the prime factorization of 2 and 7 =
\(-(\sqrt{-7})^2\) is the prime factorization of 7. We set \(Q(\xi) = V\) and think of it only as a 3-dimensional vector space over \(Q(\lambda)\). We put the Hermitian form

\[h(x, y) = \text{tr}_{Q(\xi)/Q(\lambda)}(xy^*) = [xy^* + \sigma(xy^*) + \sigma^2(xy^*)]\]

on \(V\). Taking 1, \(\xi\), \(\xi^2\) as a basis of \(V\), we find that \(h\) has the matrix

\[
H = \begin{pmatrix}
3 & \bar{\lambda} & \bar{\lambda} \\
\lambda & 3 & \bar{\lambda} \\
\bar{\lambda} & \lambda & 3
\end{pmatrix}
\]

so that \(h\) is positive definite with determinant 7. Note that \(V\) contains the lattice \(L = Z[\xi]\), with basis 1, \(\xi\), \(\xi^2\) over \(Z[\lambda]\). Define

\[
\Gamma_1 = \text{Q}(\lambda)\text{-linear maps } \gamma: V \to V \text{ which preserve the form } h
\]

and map \(L[1/2]\) to \(L[1/2]\)

Since 2 splits in \(Q(\lambda)\), the \(\lambda\)-adic completion of \(Q(\lambda)\) is isomorphic to the 2-adic completion \(Q_2\) of \(Q\) (in fact, in \(Q_2\), we may take \(\lambda = (\text{unit}) \cdot 2, \bar{\lambda} = \text{unit}\)). So we have a canonical map \(V \to (\lambda\text{-adic completion of } V) \cong Q_2 \cdot 1 \oplus Q_2 \cdot \xi \oplus Q_2 \cdot \xi^2\) and a canonical homomorphism

\[
\Gamma_1 \to \text{GL}(3, Q_2) \to \text{PGL}(3, Q_2).
\]

From standard results on the theory of arithmetic groups*, the image \(\Gamma_1\) of \(\Gamma_1\) is discrete and co-compact. We introduce 3 elements of \(\Gamma_1\): the first is \(\sigma\) itself; the second is

\[
\tau(x) = \xi \cdot x.
\]

Note that \(\sigma^3 = e, \tau^7 = e\) and \(\sigma \tau \sigma^{-1} = \tau^2\), so together \(\sigma\) and \(\tau\) generate

---

*Consider \(U(V, h)\) as an algebraic group over \(Q\). \(\Gamma_1\) is its \(\mathbb{Z}[1/2]\)-rational points. Since \(U\) is compact at the infinite place, \(\Gamma_1\) is discrete and co-compact in \(U(V, h)(Q_2)\). But over \(Q_2\), \(U(V, h) \cong \text{GL}(3)\), so mod scalars \(\Gamma_1\) defines a discrete co-compact subgroup of \(\text{PGL}(3, K)\).
a subgroup $\Gamma_2 \subset \Gamma_1$ of order 21. The third is a map $\rho$ given by

$$\rho(1) = 1$$

$$\rho(\xi) = \xi$$

$$\rho(\xi^2) = \lambda - \frac{\lambda^2}{\lambda} \xi + \frac{\lambda}{\lambda} \xi^2$$

It can be checked easily that $\rho \in \Gamma_1$. It can also be checked that

$$(\rho \cdot \tau^i)^3 = \text{multiplication by } \lambda^{1/3}.$$ 

Note that the scalar matrices in $\Gamma_1$ are exactly

$$\pm (\lambda^{1/3})^k \cdot I_3$$

**Proposition.** $\rho$, $\sigma$, $\tau$ and $-I$ generate $\Gamma_1$. All torsion elements in $\Gamma_1$ are conjugate to either $\sigma^i \cdot \tau^j$ or to $(\rho \cdot \tau)^i$ (some $0 \leq i \leq 2$, $0 \leq j \leq 6$).

**Proof.** Consider the action of $\Gamma_1$ on $\Sigma_{0'} = [\text{the set of free rank 3}\mathbb{Z}_2\text{-submodules of } \mathbb{Q}_2^3]$. Let $M_0$ be the submodule $\mathbb{Z}_2 \cdot 1 \oplus \mathbb{Z}_2 \cdot \xi \oplus \mathbb{Z}_2 \cdot \xi^2$ or $\mathbb{Z}_2^3$ for short. If an element $\alpha \in \Gamma_1$ maps $M_0$ to itself, then back in $V$, $\alpha$ is given by a $3 \times 3$ matrix with coefficients in $\mathbb{Z}[\lambda][1/\lambda]$. Since $\alpha$ is $H$-unitary, its coefficients are also in $\mathbb{Z}[\lambda][1/\lambda]$, so $\alpha$ in fact has coefficients in $\mathbb{Z}[\lambda]$ and maps $L$ to itself. But in $L$, it is easy to see that $\{ \pm \xi^i \}$ are the only elements $x \in L$ with $h(x, x) = 3$. So $\alpha$ permutes them. Then $\pm \tau^i \circ \alpha$ also carries the element $1 \in L$ to itself. Now the equations

$$h(x, x) = 3$$

$$h(1, x) = \lambda$$

have only 3 solutions in $L$: $x = \xi$, $\xi^2$ or $\xi^4$. So $(\pm \tau^i \cdot \alpha)$ carries $\xi$ to $\xi$, $\xi^2$ or $\xi^4$. Then $(\pm \sigma^i \circ \tau^i \circ \alpha)$ fixes 1 and $\xi$ and it is easy to check that such a map must be the identity. Thus $\pm \Gamma_2$ is the stabilizer of $M_0$.

As in the Bruhat-Tits building, call $M, M' \in \Sigma_{0'}$ adjacent if $M \supset M'$ and $\dim_{\mathbb{Z}/2\mathbb{Z}} M/M' = 2$ or vice versa. Then $\rho(M_0) \subset M_0$ and is adjacent to $M_0$. Because $M/2M \equiv (\mathbb{Z}/2\mathbb{Z})^3$, there are only 7 modules $M' \subset M_0$ adjacent to $M_0$. One checks easily that these are the modules $\tau^i \rho(M_0)$, $0 \leq i \leq 6$. Thus $(\tau^i \rho)^{(1/3)}(M_0)$ is the set of all $M \in \Sigma_{0'}$ adjacent
to $M_0$. Since $\Sigma_0'$ is connected under adjacency, this shows that all elements of $\Sigma_0'$ can be expressed as:

$$(\tau^i \rho)^{\alpha_i} \cdots (\tau^k \rho)^{\epsilon_k}(M_0), \quad 0 \leq i \leq 5, \quad \epsilon_i = \pm 1.$$ 

Thus the subgroup of $\Gamma_1$ generated by $\rho, \sigma, \tau$ and $-I$ acts transitively as $\Gamma_1$ on $\Sigma_0'$ and $M_0$ has the same stabilizer in both groups, so they are equal.

Now let $\alpha \in \Gamma_1$ be torsion in $\overline{\Gamma}_1$. If $\alpha$ is torsion in $\Gamma_1$, then $\alpha$ fixes e.g. the module

$$M_1 = \sum_{i=1}^{\text{order}(\alpha)} \alpha^i(M_0)$$

and, if $M_1 = \beta(M_0)$, then $\beta^{-1} \alpha \beta$ fixes $M_0$. Thus $\beta^{-1} \alpha \beta \in \pm \Gamma_2$ and $\overline{\alpha}$ is conjugate to $\sigma^i \circ \tau^j$, some $i, j$. In general, we consider $\det \alpha$. Then $|\det \alpha|^2 = 1$ and $\det \alpha \in \mathbb{Z}[\lambda][^{1/2}]$, hence

$$\det \alpha = \pm (\lambda/\overline{\lambda})^i.$$

Replacing $\alpha$ by $\pm (\lambda/\overline{\lambda})^i \alpha^{\pm 1}$, we may assume $\det \alpha = \lambda/\overline{\lambda}$. Then

$$(\lambda/\overline{\lambda})^{-1} \alpha^3$$

has determinant 1 and is torsion in $\text{PGL}(3)$, so it is torsion in $\text{GL}(3)$. Now consider $\alpha$ and $\alpha^3/\lambda$ acting on $\mathbb{Q}_2^3$. Since $\mathbb{Z}_2[\alpha, \alpha^3/\lambda]$ is a finite ring over $\mathbb{Z}_2$, there is a free rank 3$\mathbb{Z}_2$-module $M \subset \Phi_2^3$ such that $\alpha(M) \subset M$, $\alpha^3/\lambda(M) \subset M$. Since $\alpha^3/\lambda(M)$ is torsion and $\overline{\lambda}$ is a $\lambda$-adic unit, it follows that

$$M \supseteq \alpha(M) \supseteq \alpha^2(M) \supseteq \alpha^3(M) = 2M.$$

As before, replacing $\alpha$ by a conjugate, we can assume $M = M_0$. But now it is easily checked that the 21 maps $\sigma^i \circ \tau^j$ act simply transitively on the flags

$$M_0/2M_0 \supseteq (2\text{-dim}^i \text{ subspace}) \supseteq (1 \text{ dim}^i \text{ subspace}).$$

So conjugating $\alpha$ by $\sigma^i \circ \tau^j$, we can assume

$$\alpha(M_0) = (\rho \circ \tau)(M_0)$$
\[ \alpha^2(M_0) = (\rho \circ \tau)^2(M_0). \]

Then \((\rho \circ \tau)^{-1} \circ \alpha\) carries \(M_0\) into itself and fixes a flag in \(M_0/2M_0\). The former implies that \((\rho \circ \tau)^{-1} \circ \alpha = \pm \sigma^i \circ \tau^j\), some \(i, j\), and the latter implies that \(i = j = 0\). Thus in \(\Gamma_1\), \(\alpha = \rho \circ \tau\). Q.E.D.

It remains to choose a suitable subgroup \(\Gamma \subset \Gamma_1\) of finite index such that:

a) \(\Gamma/\text{scalars}\) is torsion-free

b) \(\Gamma\) acts transitively on \(\Sigma_0\)', (hence \(\Gamma/\text{scalars}\) acts transitively on \(\Sigma_0\), the vertices of the Bruhat-Tits building).

It will then follow from the results of Section 1 that the corresponding surface \((X/\Gamma)_\eta\) is a surface of the desired type. To find \(\Gamma\), it is convenient to use a congruence subgroup for the prime 7. In fact, consider the maps:

\[ \mathbb{Z}[\lambda, \frac{1}{2}] \to \mathbb{Z}[\lambda, \frac{1}{2}]/(\sqrt{-7}) \cong \mathbb{Z}/7\mathbb{Z} \]

\[ \lambda, \bar{\lambda} \mapsto 3 \]

\[ L[\frac{1}{2}] \to L[\frac{1}{2}]/(\sqrt{-7})L[\frac{1}{2}] \cong (\mathbb{Z}/7\mathbb{Z})^3 \]

call this \(L_0\)

The induced form \(h_0\) on \(L_0\) is of rank 1 and has null-space \(L_1 \subset L_0\) spanned by \(\xi - 1, \xi^2 - 1\). Taking \(1, \xi - 1, (\xi - 1)^2\) as a basis of \(L_0\), it is easy to check that mod 7:

\[ \sigma \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \]

\[ \rho \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho \circ \tau \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 5 & 4 \\ 0 & 1 & 1 \end{pmatrix} \]

In particular, considering the action of \(\Gamma_1\) on \(L_1\), we get a homomor-
\pi: \Gamma_1 \to \text{GL}(2, \mathbb{Z}/7\mathbb{Z}) \cap \{X | \det X = \pm 1\} \overset{\text{def}}{=} G.

The group $G$ on the right has order $2^5 \cdot 3 \cdot 7$. Let $H$ be a 2-Sylow subgroup and define $\Gamma = \pi^{-1}(H)$. Since all 21 elements $\sigma \tau^i$ and all 3 elements $(\sigma \tau)^i$ except $e$ have non-zero images in $G$ of orders 3 or 7, $\Gamma$ is torsion-free. As the full group $\Gamma_1$ is set-theoretically $\Gamma \times \Gamma_2$, $\Gamma$ acts transitively on $\Sigma_0'$. This completes the construction.

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REFERENCES